Differential Cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams

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Abstract

We introduce a differential refinement of Cohomotopy cohomology theory, defined on Penrose diagram spacetimes, whose cocycle spaces are unordered configuration spaces of points. First we prove that brane charge quantization in this differential 4-Cohomotopy theory implies intersecting $p \perp (p+2)$-brane moduli given by ordered configurations of points in the transversal 3-space. Then we show that the higher (co-)observables on these brane moduli, conceived as the (co-)homology of the Cohomotopy cocycle space, are given by weight systems on horizontal chord diagrams and reflect a multitude of effects expected in the microscopic quantum theory of D$p \perp$ D$(p+2)$-brane intersections: condensation to stacks of coincident branes and their Chan-Paton factors, fuzzy funnel shapes and M2-brane 3-algebras, the Hanany-Witten rules, AdS$_3$-gravity observables, supersymmetric indices of Coulomb branches as well as gauge/gravity duality between all these. We discuss this in the context of the hypothesis that the M-theory C-field is charge-quantized in Cohomotopy theory.

Contents

1 Introduction and overview

The general open problem. The rich physics expected on coincident and intersecting branes (reviewed in [IU12]), which geometrically engineer non-perturbative quantum gauge field theories [KV97][HW97][GW08] (reviewed in [Kar98][GK99][Fa17]) close to quantum chromodynamics [Wi98][SSu04][SSu05] (reviewed in [Re14][Su16]) and model quantum microstates accounting for black hole entropy [SV96][CM96] (reviewed in [Kr06][Sen07]), has come to be center stage in string theory – or rather in the "theory formerly known as strings" [Du96]. Despite all that is known about D-branes from the two limiting cases of **(a)** string perturbation theory and **(b)** worldvolume gauge theory, an actual comprehensive theory of non-perturbative brane physics, namely an actual formulation of *M-theory* [Du99], still remains an open problem [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19] [1]. The lack of such a genuine theory of brane physics has recently surfaced in a debate about the validity of D3-brane constructions that had dominated the discussion in a large part of the community for the last 15 years; see [DvR18][Ba19, p. 14-22].

Hypothesis H. Based on a re-analysis of the super p-brane WZW terms from the point of view of homotopy theory [FSS13][FSS15][FSS16a][FSS16b][BSS18] (reviewed in [FSS19a]), we have recently formulated a concrete hypothesis about (at least part of) the mathematical nature of M-theory: This *Hypothesis H* [Sa13, 2.5][FSS19b] asserts that in M-theory the C-field of 11d supergravity [CJS78] is charge-quantized [Fr00][Sa10] in the non-abelian generalized cohomology theory called *J-twisted Cohomotopy theory*. This hypothesis turns out to imply [FSS19b][FSS19c][SS19a][SS19b] a wealth of subtle topological effects expected in string/M-theory. This suggests that it is a correct proposal about the mathematics underpinning M-theory, at least in the topological sector.

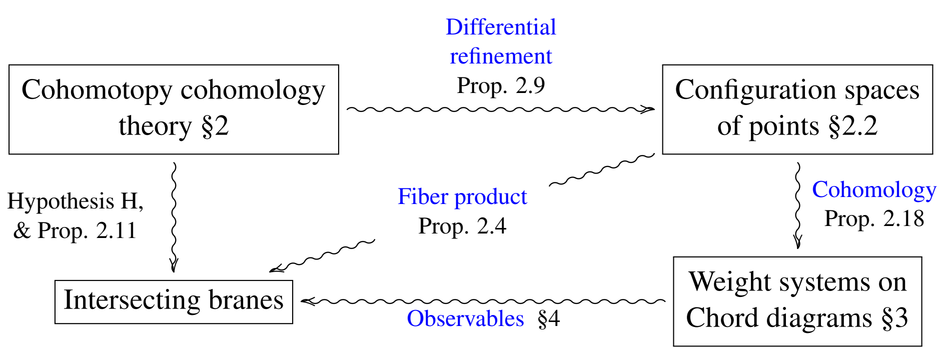
Differential refinement. In this article we take a step beyond the topological sector and investigate to which extent a geometrically ("differentially") refined form (cf. [FSS15]) of Hypothesis H leads to the emergence/derivation of expected phenomena on coincident and intersecting branes.

First, our main **mathematical observations** here are the following (§2):

(1) A differential refinement of Cohomotopy cohomology theory is given by *un-ordered configuration spaces of points*.

(2) The fiber product of such differentially refined Cohomotopy cocycle spaces describing D6 $\perp$ D8-brane intersections is homotopy-equivalent to the *ordered configuration space of points* in the transversal space.

(3) The higher observables on this moduli space are equivalently weight systems on horizontal chord diagrams.



Second, we make the **string-theoretic observation** (§4) that these weight systems on horizontal chord diagrams, when regarded as higher observables reflect a multitude of effects expected on brane intersections in string theory.

This leads to an understanding and clarification of relations among various physical concepts and points to a unifying theme, relying on constructions from seemingly distinct mathematical areas which are brought together – see Figure 1.

[1] [Wi19] at 21:15: "I actually believe that string/M-theory is on the right track toward a deeper explanation. But at a very fundamental level it's not well understood. And I'm not even confident that we have a good concept of what sort of thing is missing or where to find it."

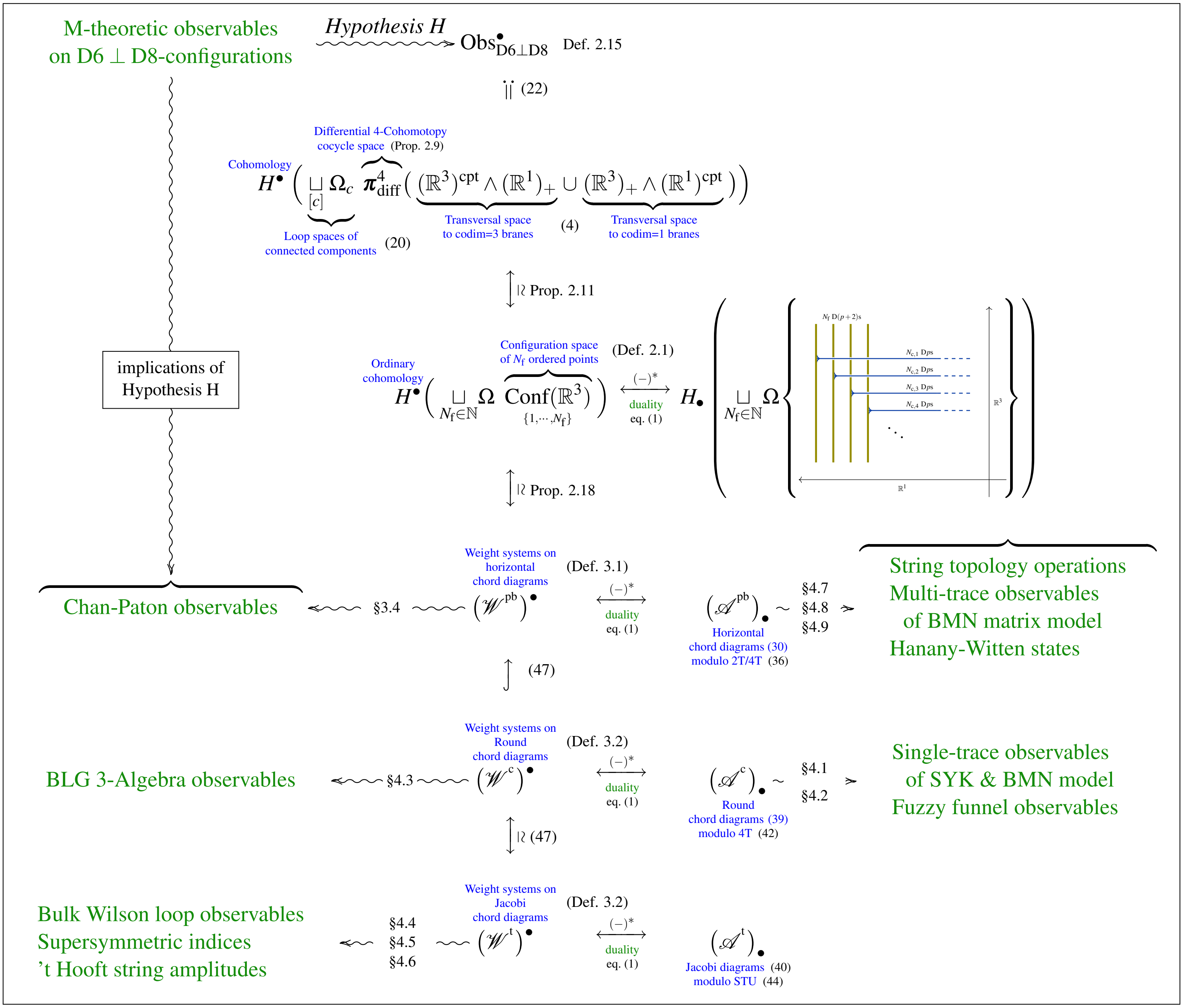


Figure 1 – Emergence of intersecting brane observables from a differentially refined version (§2) of *Hypothesis H*.

Top-down M-theory. We highlight that, assuming Hypothesis H, the analysis shown in Figure 1 is completely *top-down*: knowledge about gauge field theory and perturbative string theory is not used in deriving the algebras of observables of M-theory, but only to interpret them. See also Observation 4.2 on dualities.

While we suggest that the rich system of expected effects emerging in Figure 1, further supports the proposal that Hypothesis H is a correct proposal about the mathematical nature of M-theory, there must of course be more to M-theory than seen in Figure 1. But it is also clear that the differential refinement of Cohomotopy cohomology theory discussed here (in §2 below) is to be further refined, notably by enhancing it with super-differential flux form structure as in [FSS15][FSS16a], with ADE-equivariant structure as in [HSS19], and with fiberwise stabilization as in [BSS18]. This is to be discussed elsewhere.

Gauge/Gravity duality. Collecting the observables and states emerging in Figure 1, we observe that the *mathematical duality* [PT91][Co17] between higher (co-)observables (21) on D6 $\perp$ D8-branes by *Hypothesis H*

$$
\begin{array}{ccccccc}
\text{Higher observables} & \overset{\text{Cohomology}}{H^\bullet}\Big(\underbrace{\textstyle\bigsqcup_{N_f\in\mathbb{N}} \Omega\,\mathrm{Conf}_{\{1,\cdots,N_f\}}(\mathbb{R}^3)}\Big) & \simeq & \overset{\text{weight systems}}{\mathscr{W}} & = & (\mathscr{A})^* \\
\text{Prop. 2.16} & \text{phase space} & \text{Prop. 2.18} & & \text{dualization} & \\
\text{Higher co-observables} & \underset{\text{Homology}}{H_\bullet}\Big(\overbrace{\textstyle\bigsqcup_{N_f\in\mathbb{N}} \Omega\,\mathrm{Conf}_{\{1,\cdots,N_f\}}(\mathbb{R}^3)}\Big) & \simeq & \underset{\text{chord diagrams}}{\mathscr{A}} & = & (\mathscr{W})^*
\end{array}
\qquad (1)
$$

reflects the *gauge/gravity duality* (e.g. [DHMB15]) between observables/states of gauge theories and gravity theories on branes found in §4 – see Figure 2:

Gauge theory		
Observables:	chord diagrams	$\in \mathscr{A}$
	non-commutative/quantum	(Rem. 2.19)
States:	weight systems	$\in \mathscr{W}$

Gravity theory		
Observables:	weight systems	$\in \mathscr{W}$
	commutative/classical	
States:	chord diagrams	$\in \mathscr{A}$

BMN matrix model	§4.1, §4.8
Observables:	multi-trace observables encoded by Sullivan chord diagrams identified as horizontal chord diagrams
States:	fuzzy 2-sphere geometries encoded by $\mathfrak{su}(2)_{\mathbb{C}}$-representations V identified as Lie algebra weight systems

RW-twisted $D = 3$, $\mathcal{N} = 4$ SYM theory	§4.5
Observables:	e.g. supersymmetric index encoded by wheel chord diagrams
States:	Coulomb branches identified as RW weight systems

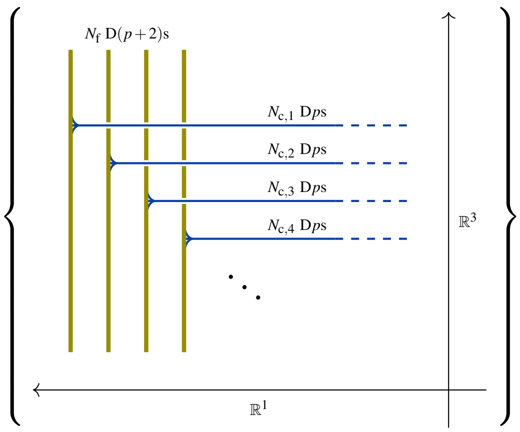
DBI theory on D$p \perp$ D$(p+2)$-branes	§4.2
Observables:	fuzzy funnel shape observables encoded by round chord diagrams
States:	fuzzy funnel geometries encoded by $\mathfrak{su}(2)_{\mathbb{C}}$-representations identified as Lie algebra weight systems

AdS$_3$-**gravity**	§4.4
Observables:	Wilson loop observables encoded by $\mathfrak{sl}(2,\mathbb{C})$-representations V identified as Lie algebra weight systems
States:	Hyperbolic 3-manifolds encoded by knots encoded by round chord diagrams

Black D$p \perp$ D$(p+2)$-branes	§4.9
Observables:	...
States:	Hanany-Witten brane intersections encoded by horizontal chord diagrams

Figure 2 – Emergence of gauge/gravity duality from a differentially refined version (§2) of *Hypothesis H*.

Configuration spaces of intersecting branes seen in Cohomotopy. The brane intersections arising this way from *Hypothesis H* & Prop. 2.11 are transversal $p \perp (p+2)$-brane intersections, specifically for $p = 6$ (Remark 2.14 below), where N_{f} $(p+2)$-branes are arranged along an axis and $N_{\mathrm{c}} = \sum_{i=1}^{N_{\mathrm{f}}} N_{\mathrm{c},i}$ semi-infinite p-branes transversally intersecting them, with $N_{\mathrm{c},i}$ of them coincident and ending on the ith $(p+2)$-brane. The p-branes move along the $\mathbb{R}^3$ inside the $(p+2)$-branes which is normal to their intersection locus.



$$\underset{\text{homotopy equivalence}}{\simeq} \quad \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3) \qquad (2)$$

Ordered configuration space of N_{f} points in $\mathbb{R}^3$

D$p \perp$ D$(p+2)$**-brane intersections as seen from nonabelian DBI theory.** Transversal D$p \perp$ D$(p+2)$-brane intersections have been discussed in the literature using the nonabelian DBI field theory [Ts97][My99] which is expected on the worldvolume of coincident D-branes; see [Kar98][GK99][Fa17] for review. The following table summarizes the main results of the traditional analysis (on the left) and indication of emergence from Cohomotopy (on the right):

Expected physics of D$p \perp$ D$(p+2)$-brane intersections	**Statement**	**Derivation from non-abelian DBI**	**Emergence from Hypothesis H**
Fuzzy funnel geometry	A single $N_{\mathrm{c},i}$ D$p \perp$ D$(p+2)$-intersection is described by non-commutative fuzzy funnel geometry, identified, via Nahm's equations, with Yang-Mills monopoles in the D$(p+2)$-brane worldvolume.	[Di97][CMT99] [HZ99][My01] [CL02][BB04] [RST04][MN06] [MPRS06]	§4.2
Hanany-Witten rules	The collection of all N_{f} D$p \perp$ D$(p+2)$-brane intersections is subject to combinatorial rules, such as the *s-rule* and the *ordering constraint*.	[HW97][BGS97] [BG98][HOO98] [GK99][GW08]	§4.9

The open problem of the Non-abelian DBI action. While these traditional discussions undoubtedly yield a compelling picture, it is worth recalling that (in contrast to the abelian case of non-coincident branes) there is to date *no derivation* from perturbative string theory of the nonabelian DBI-action for coincident D-branes, as highlighted in [TvR99, p. 1][Sc01, p. 2][Ch04, p. 5]. The commonly used symmetrized trace prescription of [Ts97][My99], is somewhat ad-hoc; and it is known not to be correct at higher orders [HT97][BBdRS01]. Some correction terms have been proposed in [TvR99], and different proposal for going about the non-abelian DBI-action has recently been made in [BFS19]. In contrast, here we find key expected consequences of non-abelian DBI-Lagrangians for intersecting brane physics emerge from Hypothesis H in a *non-Lagrangian* way altogether.

Outline: The paper is outlined as follows:
In §2 we introduce the *differential Hypothesis H* and show that it implies weight systems as higher observables.
In §3 we recall weight systems on chord diagrams, streamlined towards our applications.
In §4 we observe that weight system observables reflect a variety of effects in intersecting brane physics.

2 Intersecting brane charges in differential Cohomotopy theory

The open problem of formulating a genuine theory of brane physics. As indicated in the Introduction, despite all the discussion of (well-supported but conjectural) aspects of intersecting brane physics, an actual formulation of a non-perturbative quantum theory of branes, namely of *M-theory* [Du99], has remained an open problem [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19] [2] . The need for an identification of the non-perturbative theory has recently become manifest with the community no longer able to agree on the validity of brane constructions that have been discussed for many years [DvR18][Ba19, p. 14-22]. Even the very ingredients of such a theory have remained open.

Charge quantization in generalized cohomology theory. On the other hand, the low energy limit of M-theory is supposed to be $D = 11$ supergravity [CJS78], whose only ingredient, besides the field of gravity, is the *C-field*, the higher analog of the B-field in string theory, which in turn is the higher analog of the "A-field" in particle physics, namely of the Maxwell field, i.e., of the abelian Yang-Mills field. But a famous insight going back to Dirac (see [He18]) says that in its non-perturbative quantum theory, the Maxwell field becomes subject to a refinement known as *Dirac charge quantization* (see [Fr00] for a general treatment). In modern formulation this means that the flux density of the field (the Faraday tensor), which a priori seems to be just a differential 2-form, is promoted to a cocycle in differential ordinary 2-cohomology theory. Later, a directly analogous topological constraint has been argued to apply to the B-field in string theory, where up to some fine print, what naively looks like the flux density 3-form of the B-field is argued to really be regarded as being charge-quantized in differential ordinary 3-cohomology theory (see [Br93]). One might suspect an evident pattern here, which would seem to continue with the suggestion that the M-theory C-field needs to be regarded as charge-quantized in differential ordinary 3-cohomology theory, up to some fine print ([DFM03][HS05][Sa10][FSS14a]). On the other hand, and in contrast to the C-field in M-theory, the B-field in string theory does not exist in isolation; instead, it couples to the RR-field. The combination of the B-field and the RR-field has famously and widely been argued to be charge quantized in a differential *generalized* cohomology theory, namely in some version of twisted K-theory (see [GS19] and also [BSS18, 2] for pointers and discussion in our context).

Generalized cohomology theory for C-field charge quantization in M-theory. All this rich structure in string theory is – somehow – supposed to lift to just the metric field and the C-field in M-theory. This suggests that the M-theory C-field itself must be regarded as being charge-quantized in some rich generalized cohomology theory [Sa05a][Sa05b][Sa06][Sa10] such as Cohomotopy cohomology theory [Sa13, 2.5]. Based on a systematic analysis in super rational homotopy theory of the κ-symmetry super p-brane WZW terms [FSS13][FSS15][FSS16a][FSS16b] [BSS18] (see [FSS19a] for review), a concrete hypothesis for this generalized cohomological charge quantization of the C-field was formulated in [FSS19b]:

Hypothesis H. *The M-theory C-field is charge-quantized in J-twisted Cohomotopy theory.*

In a series of articles [FSS19b][FSS19c][SS19a][SS19b] various implications of this *Hypothesis H* have been checked to agree with various expected aspects of M-theory in the topological sector, i.e., in the approximation where only the homotopy type of spacetime is taken into account.

Differential Cohomotopy and intersecting branes. Here we consider a partial refinement of Cohomotopy cohomology theory to a *differential* cohomology theory, which is sensitive at least to the homeomorphism type of spacetime (Prop. 2.9 below). Then we prove (Prop. 2.11 below) that this charge quantization of the C-field in differential Cohomotopy theory implies that the cocycle space of intersecting D6-D8-brane charges is the ordered configuration space of points as in (2). This means that:

(1) The higher observables (22) in §3 and hence, by (27), the weight systems on chord diagrams in §3 are the quantum observables on intersecting brane moduli that are implied by Hypothesis H.

(2) Therefore, also the aspects of intersecting brane physics that are reflected in weight systems on chord diagrams according to the discussion in §4 are implications of Hypothesis H.

[2] [Wi19] at 21:15: "I actually believe that string/M-theory is on the right track toward a deeper explanation. But at a very fundamental level it's not well understood. And I'm not even confident that we have a good concept of what sort of thing is missing or where to find it."

2.1 Charges vanishing at infinity

Points at infinity. For the following definitions applied to physics, we are to think of all boundaries and base points as representing "points at infinity". We write $\mathbb{D}^n$ for the *closed* n-disk with boundary $\partial\mathbb{D}^n \simeq S^{n-1}$ and interior $\mathrm{Int}(\mathbb{D}^n) \simeq \mathbb{R}^n$. We write $(-)^{\mathrm{cpt}}$ for the one-point compactification of a topological space, so that

$$(\mathbb{R}^n)^{\mathrm{cpt}} \simeq \mathbb{D}^n/\partial\mathbb{D}^n \simeq S^n \tag{3}$$

and we write

$$\infty \in (\mathbb{R}^n)^{\mathrm{cpt}}$$

for the extra point. This is literally the *point at infinity*, and under the above equivalences, all points on the boundary of $\mathbb{D}^n$ get identified with it:

$$\begin{tikzcd}
 & \text{Euclidean n-space} & & \text{Closed n-disk} & & \text{$(n-1)$-sphere} \\
 & \mathbb{R}^n \arrow[rr, hook, "\text{interior}"] \arrow[d, hook] & & \mathbb{D}^n \arrow[d] & & S^{n-1} \arrow[ll, hook', "\text{boundary}"'] \arrow[d] \\
S^n \simeq & (\mathbb{R}^n)^{\mathrm{cpt}} \arrow[rr, "\simeq"'] & & \mathbb{D}^n/\partial\mathbb{D}^n & & \{\infty\} \arrow[ll, hook'] \\
\text{n-sphere} & \text{One-point compactification} & & \text{Boundary collapsed to one point} & & \text{Point at infinity}
\end{tikzcd}$$

We will thus regard one-point compactifications $(-)^{\mathrm{cpt}}$ as pointed topological spaces with the base point denoted "∞".

If for classifying spaces we instead denote the base point by "0", then pointed maps express exactly the idea of *cocycles vanishing at infinity*:

$$\begin{tikzcd}
 & \text{Space-time} & & \text{Classifying space} \\
\text{Pointed spaces} & X \arrow[rr, "\substack{\text{cocycle /} \\ \text{field configuration}}"] & & A \\
\text{Base points} & \{\infty\} \arrow[u, hook] \arrow[rr, "\text{vanishing at infinity}"'] & & \{0\} \arrow[u, hook] \\
 & \text{point at infinity} & & \text{vanishing field value}
\end{tikzcd}$$

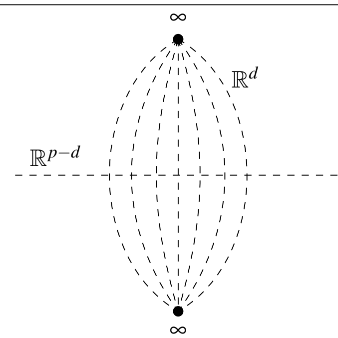
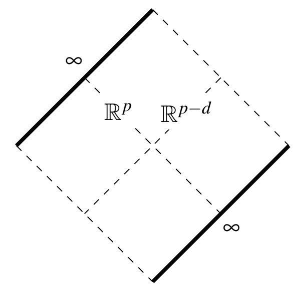
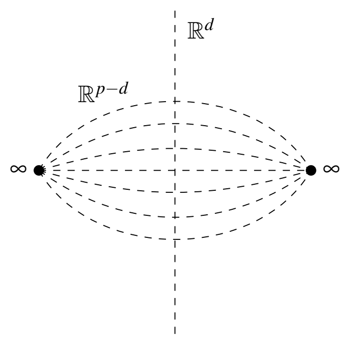
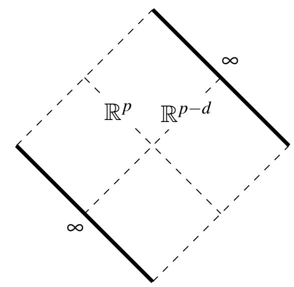
If we wish to consider $\mathbb{R}^d$ explicitly without the requirement that cocycles on it vanish at infinity, we instead add the "point at infinity" as a disjoint point

$$(\mathbb{R}^d)_+ := \mathbb{R}^d \sqcup \{\infty\}.$$

In summary:

$$\begin{tikzcd}
 & & \text{Euclidean space...} & \\
\text{Un-pointed space} & & \mathbb{R}^d \arrow[dl, hook'] \arrow[dr, hook] & \\
\text{Pointed spaces} & (\mathbb{R}^d)^{\mathrm{cpt}} & & (\mathbb{R}^d)_+ \\
 & \text{... such that cocycles vanish at infinity.} & & \text{...such that cocycles are unconstrained.}
\end{tikzcd}$$

Forming the smash product of these pointed spaces then yields Euclidean spaces on which cocycles have to vanish at infinity in some directions, but not necessarily in others:

Smash product of pointed topological spaces	Visualization with point at infinity	as Penrose diagram
$\big(\overbrace{\mathbb{R}^d}^{\text{cocycles vanish at infinity along these direction}}\big)^{\mathrm{cpt}} \wedge \big(\underbrace{\mathbb{R}^{p-d}}_{\text{...but not necessarily along these}}\big)_+$		
$\big(\underbrace{\mathbb{R}^d}_{\text{...but not necessarily along these}}\big)_+ \wedge \big(\overbrace{\mathbb{R}^{p-d}}^{\text{cocycles vanish at infinity along these direction}}\big)^{\mathrm{cpt}}$		

(4)

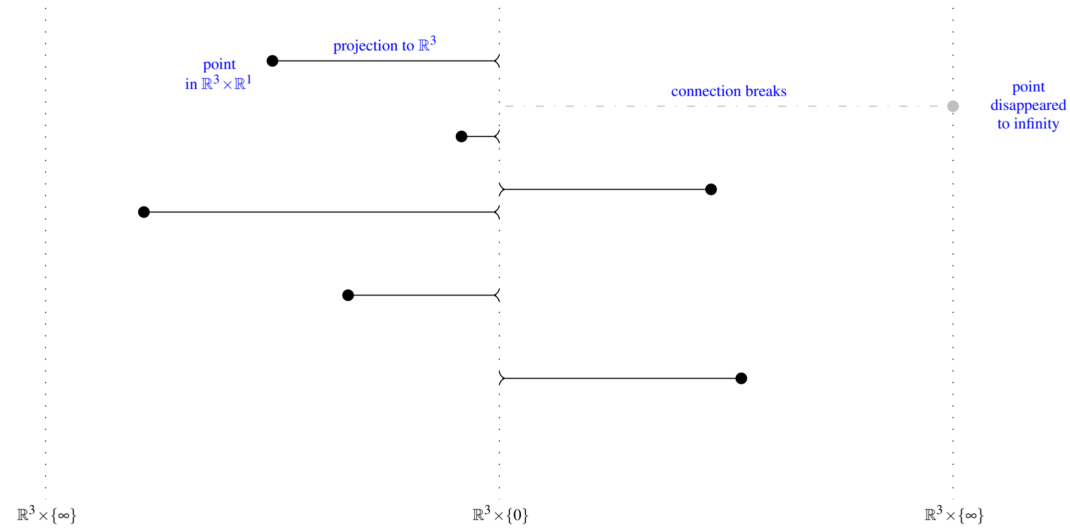
2.2 Configuration spaces of points

We now first recall, in Def. 2.1, the relevant definitions of configuration spaces of points (see e.g. [Bö87, 1]). Then we observe, in Prop. 2.4, a certain relation between un-ordered and ordered configuration spaces of points. This is the key to relating differential Cohomotopy to intersecting branes in §2.4.

Definition 2.1 (Configuration spaces of points)**.** Let Σ^D be a smooth manifold with (a possibly empty) boundary $\partial\Sigma^D \hookrightarrow \Sigma^D$. For $k \in \mathbb{N}$, with $\mathbb{D}^k$ denoting the closed k-disk, Δ the diagonal, and Sym_n the symmetric group of order n, we consider the following topological configuration spaces of points in Σ^D, possibly with labels in $\mathbb{D}^k$:

Symbol	Definition	Configuration space of...
$\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\Sigma^D) \;:=$	$(\Sigma^D)^{\times n} \setminus \underbrace{\Delta^n_{\Sigma^D}}_{\text{subspace of any coincident points}}$	n distinct and ordered points in Σ^D
$\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\Sigma^D, \mathbb{D}^k) \;:=$	$\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\Sigma^D)\Big) \times \underbrace{(\mathbb{D}^k)^{\times n}}_{\text{product of spaces of labels}}$	n distinct and ordered points in Σ^D each carrying a label in $\mathbb{D}^k$
$\mathrm{Conf}_n\big(\Sigma^D, \mathbb{D}^k\big) \;:=$	$\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\Sigma^D, \mathbb{D}^k)\Big) / \underbrace{\mathrm{Sym}(n)}_{\text{symmetric group acting diagonally}}$	n distinct un-ordered points in Σ^D each carrying a label in $\mathbb{D}^k$
$\mathrm{Conf}\big(\Sigma^D, \mathbb{D}^k\big) \;:=$	$\Big(\underbrace{\underset{n\in\mathbb{N}}{\cup}}_{\text{union over number of points}} \mathrm{Conf}_n(\Sigma^D, \mathbb{D}^k)\Big) \underbrace{/\sim}_{\text{points vanish at infinity}}$	Any number of distinct un-ordered points in Σ^D each carrying a label in $\mathbb{D}^k$ disregarded if at $\infty \in \big(\Sigma^D \times \mathbb{D}^k\big) / \partial\big(\Sigma^D \times \mathbb{D}^k\big)$
$\mathrm{Conf}\big(\Sigma^D\big) \;:=$	$\mathrm{Conf}\big(\Sigma^D, \underbrace{\mathbb{D}^0}_{\text{trivial labels}}\big)$	Any number of distinct un-ordered points in Σ^D disregarded if at $\infty \in \Sigma^D / \partial\Sigma^D$

Here is an illustration of a labelled and un-ordered configuration of points:



An element of the unordered $\mathbb{D}^1$-labeled configuration space $\mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big)$ according to Def. 2.1, is a set of points in $\mathbb{R}^3 \times \mathbb{R}^1$ with distinct projections to $\mathbb{R}^3 \times \{0\}$. The topology is such that points moving to infinity along $\mathbb{R}^1$ (i.e., to the boundary of $\mathbb{D}^1$) disappear.

In order to study all possible configurations, we introduce the following useful notion.

Definition 2.2 (Category of Penrose diagrams)**.** For $p \in \mathbb{N}$ we write

$$\mathrm{PenroseDiag}_p \;:=\; \left\{ \begin{array}{l} \text{Penrose-diagram spaces of dimension } p \\ \text{with continuous functions between them} \\ \text{which are injections away from infinity} \end{array} \right\} \tag{5}$$

for the category whose objects are the pointed topological spaces $(\mathbb{R}^p)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+$ from (4), for $0 \le d \le p$, and whose morphisms are the continuous functions between these that (co-)restrict to embeddings after removal of basepoints, as shown on the left of the following diagrams:

$$\begin{tikzcd}
\substack{\text{any} \\ \text{Penrose diagram}} & \substack{\text{its subspace} \\ \text{away from infinity}} & & & & \substack{\text{configuration space of points} \\ \text{in the Penrose diagram space} \\ \text{which may vanish towards infinity} \\ \text{along directions } \textbf{not} \text{ compactified}} & \\
(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+ & \mathbb{R}^p \arrow[l, hook'] \arrow[r, phantom, "\mapsto"] & {} & \mathbb{R}^d \times \mathbb{D}^{p-d}/\mathrm{bdry} \arrow[d, "i^*"'] \arrow[r, phantom, "\mapsto"] & {} & \mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{p-d}\big) \arrow[d, "(i^*)_*"'] & \text{for } d \ge 1 \\
(\mathbb{R}^{d'})^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d'})_+ \arrow[u, hook] \arrow[ur, phantom, "\text{continuous injections}"] & \mathbb{R}^p \arrow[l, hook'] \arrow[u, hook, "i"'] \arrow[r, phantom, "\mapsto"] & {} & \mathbb{R}^{d'} \times \mathbb{D}^{p-d'}/\mathrm{bdry} \arrow[r, phantom, "\mapsto"] & {} & \mathrm{Conf}\big(\mathbb{R}^{d'}, \mathbb{D}^{p-d'}\big) & \text{for } d' \ge 1
\end{tikzcd} \tag{6}$$

$$i(x) \mapsto x, \qquad y \notin \mathrm{Im}(i) \mapsto \infty$$

In the special case that the domain of the map is the Penrose diagram with no compactified dimensions, we set:

$$\begin{tikzcd}
(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+ & \mathbb{R}^p \arrow[l, hook'] \arrow[r, phantom, "\mapsto"] & {} & \mathbb{R}^d \times \mathbb{D}^{p-d}/\mathrm{bdry} \arrow[d, "i^*"'] \arrow[r, phantom, "\mapsto"] & {} & \mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{p-d}\big) \arrow[d, "(i^*)_*"'] & \text{for } d \ge 1 \\
\underbrace{(\mathbb{R}^0)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p})_+}_{=(\mathbb{R}^p)_+} \arrow[u, hook] & \mathbb{R}^p \arrow[l, hook'] \arrow[u, hook, "i"'] \arrow[r, phantom, "\mapsto"] & {} & \mathbb{D}^{p}/\mathrm{bdry} \arrow[r, phantom, "\mapsto"] & {} & \mathrm{Conf}\big(\mathbb{D}^{p}\big) & (\text{for } d' = 0)
\end{tikzcd} \tag{7}$$

$$i(x) \mapsto x, \qquad y \notin \mathrm{Im}(i) \mapsto \infty$$

in the special case that
no infinite directions are compactified...

...we assign this configuration space of any number of
points that may vanish towards infinity in any direction
(instead of $\mathrm{Conf}(\mathbb{R}^0, \mathbb{D}^p)$, whose configurations have at most one point)

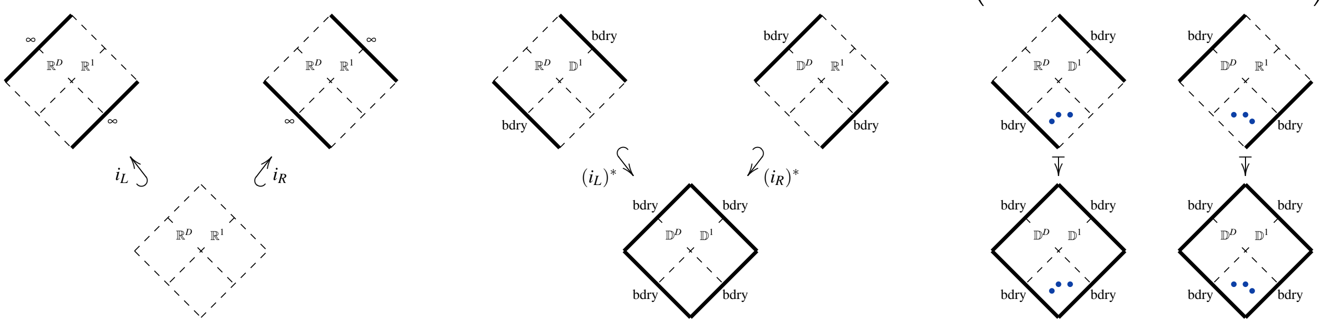
To this category (5) we extend the construction of configuration spaces from Def. 2.1 as a contravariant functor with values in pointed topological spaces,

$$\mathrm{Conf} \;:\; \mathrm{PenroseDiag}_p^{\mathrm{op}} \longrightarrow \underset{\substack{\text{1-category of actual}\\ \text{pointed topological spaces}\\ \text{(not up to homotopy)}}}{\mathrm{Top}^{*/}} \tag{8}$$

by defining its action on morphisms as shown on the right of the above diagrams (6) and (7).

Example 2.3 (Maps of configuration spaces for ordered fiber product)**.** We are going to be interested in the following pairs of maps of Penrose diagram spaces (4) and their induced maps of configuration spaces, according to Def. 2.2:

$$\left(\begin{tikzcd}[column sep=small] (\mathbb{R}^D)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ & & (\mathbb{R}^1)^{\mathrm{cpt}} \wedge (\mathbb{R}^D)_+ \\ & (\mathbb{R}^{D+1})_+ \arrow[ul, hook, "i_L"] \arrow[ur, hook', "i_R"'] & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd}[column sep=small] \mathbb{R}^D \times \mathbb{D}^1/\mathrm{bdry} \arrow[dr, hook, "(i_L)^*"'] & & \mathbb{R}^1 \times \mathbb{D}^D/\mathrm{bdry} \arrow[dl, hook', "(i_R)^*"] \\ & \mathbb{D}^D \times \mathbb{D}^1/\mathrm{bdry} & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd}[column sep=small] \mathrm{Conf}\big(\mathbb{R}^D, \mathbb{D}^1\big) \arrow[dr, hook, "((i_L)^*)_*"'] & & \mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^D\big) \arrow[dl, hook', "((i_R)^*)_*"] \\ & \mathrm{Conf}\big(\mathbb{D}^{D+1}\big) & \end{tikzcd}\right)$$



Proposition 2.4 (Ordered unlabeled configurations as fiber product of unordered labeled configurations)**.** *For $D \in \mathbb{N}$, there is a homotopy equivalence between the disjoint union of ordered unlabelled configuration spaces in $\mathbb{R}^D$ and the fiber product of unordered but labelled configuration spaces (Def. 2.1) as follows:*

$$\underset{\substack{\text{Ordered configurations}\\ \text{of points in } \mathbb{R}^D}}{\underbrace{\underset{n\in\mathbb{N}}{\sqcup}\;\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^D\big)}} \;\underset{\mathit{hmtpy}}{\simeq}\; \overset{\substack{\text{Un-ordered configurations}\\ \text{of points in } \mathbb{R}^D\\ \text{with labels in } \mathbb{D}^1}}{\overbrace{\mathrm{Conf}\big(\mathbb{R}^D, \mathbb{D}^1\big)}} \underset{\substack{\mathrm{Conf}(\mathbb{D}^{D+1})\\ \text{Un-ordered configurations}\\ \text{of points in } \mathbb{D}^{D+1}}}{\times} \overset{\substack{\text{Un-ordered configurations}\\ \text{of points in } \mathbb{R}^1\\ \text{with labels in } \mathbb{R}^D}}{\overbrace{\mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^D\big)}} \quad , \tag{9}$$

where the fiber product on the right is that induced from the maps in Example 2.3.

Proof. We compute as follows (where all topologies are the evident ones) – see Figure O for illustration of the logic behind the argument:

$$\begin{aligned}
&\mathrm{Conf}\big(\mathbb{R}^D, \mathbb{D}^1\big) \underset{\mathrm{Conf}(\mathbb{D}^{D+1})}{\times} \mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^D\big) \\
&\quad \underset{\text{homeo}}{\simeq} \underset{n\in\mathbb{N}}{\sqcup} \Big\{ \big\{ (\vec{x}_i, y_i) \in \mathbb{R}^D \times \mathbb{R}^1 \big\}_{i=1}^n \,\big|\, \underset{i\neq j}{\forall} \big(\vec{x}_i \neq \vec{x}_j \;\text{ and }\; y_i \neq y_j \big) \Big\} \big/ \mathrm{Sym}(n) \\
&\quad \underset{\text{homeo}}{\simeq} \underset{n\in\mathbb{N}}{\sqcup} \Big\{ \Big(\big\{ \vec{x}_i \in \mathbb{R}^D \big\}_{i=1}^n, \sigma \in \mathrm{Sym}(n), (d_0, d_1, \cdots, d_{n-1}) \in \mathbb{R}^1 \times (\mathbb{R}^1_+)^{n-1} \Big) \,\Big|\, \underset{i\neq j}{\forall} \big(\vec{x}_i \neq \vec{x}_j \big) \Big\} \big/ \mathrm{Sym}(n) \\
&\quad \underset{\text{hmtpy}}{\simeq} \underset{n\in\mathbb{N}}{\sqcup} \Big\{ \Big(\big\{ \vec{x}_i \mathbb{R}^D \big\}_{i=1}^n, \sigma \in \mathrm{Sym}(n) \Big) \,\Big|\, \underset{i\neq j}{\forall} \big(\vec{x}_i \neq \vec{x}_j \big) \Big\} \big/ \mathrm{Sym}(n) \\
&\quad \underset{\text{homeo}}{\simeq} \underset{n\in\mathbb{N}}{\sqcup} \Big\{ \big\{ \vec{x}_i \in \mathbb{R}^D \big\}_{i=1}^n \,\Big|\, \underset{i\neq j}{\forall} \big(\vec{x}_i \neq \vec{x}_j \big) \Big\} \\
&\quad = \underset{n\in\mathbb{N}}{\sqcup} \underset{\{1,\cdots,n\}}{\mathrm{Conf}} \big(\mathbb{R}^D \big) \,.
\end{aligned}$$

Here the first step just unwinds the definition of the fiber product. In the second step we encode an n-tuple of pairwise distinct real numbers $(y_1, y_2, \cdots, y_n)$ equivalently as a pair consisting of the permutation σ that puts them into linear order and the tuple $(d_0, d_1, \cdots, d_{n-1})$ of their relative positive distances:

$$
\begin{array}{ccccccccc}
y_{\sigma_1} & < & y_{\sigma_2} & < & y_{\sigma_3} & < & \cdots & < & y_{\sigma_n} \\
\| & & \| & & \| & & & & \| \\
d_0 & < & d_0 + d_1 & < & d_0 + d_1 + d_2 & < & \cdots & < & \sum\limits_{i=0}^{n-1} d_i
\end{array}
$$

In the third step we use that the space of these relative distances is, clearly, homotopy equivalent to the point: $\mathbb{R}^1 \times (\mathbb{R}^1)^{n-1} \underset{\text{hmtpy}}{\simeq} *$. In the fourth step we use that $(X \times G)/_{\text{diag}} G \underset{\text{homeo}}{\simeq} X$ for any G-space X. The last step recognizes the ordered configuration space according to Def. 2.1. □

The content of Prop. 2.4 is illustrated by the following graphics:

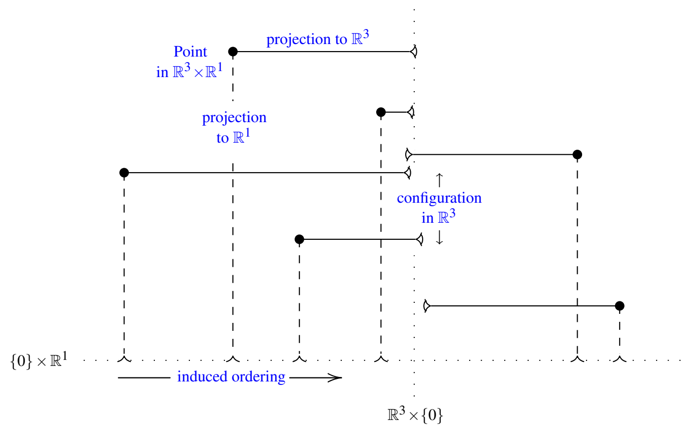


Figure O – The ordered unlabeled configuration space is a fiber product of unordered labeled configuration spaces according to Prop. 2.4: A linearly ordered configuration of points in $\mathbb{R}^3$ is the same as **(a)** an unordered configuration in $\mathbb{R}^3 \times \mathbb{R}^1$ which projects to **(b)** an unordered $\mathbb{D}^1$-labelled configuration in $\mathbb{R}^3$ as well as to **(c)** an unordered $\mathbb{D}^3$-labelled configuration in $\mathbb{R}^1$. Condition **(c)** equips the configuration from condition **(b)** with a linear ordering.

2.3 Differential Cohomotopy cocycle spaces

For the following, we take X to be a locally compact pointed topological space of the homotopy type of a CW-complex, for example one of the Penrose diagram spaces (4) discussed in §2.1.

Plain Cohomotopy cohomology theory. For $p \in \mathbb{N}$ a degree, the *cocycle space of p-Cohomotopy theory on X* is the pointed mapping space from X to the p-sphere:

$$\boldsymbol{\pi}^p(X) := \mathrm{Maps}^{*/}\big(X, S^p\big). \tag{10}$$

The set of connected component of this space is the actual *p-Cohomotopy set* of X:

$$\pi^p(X) := \pi_0\Big(\mathrm{Maps}^{*/}\big(X, S^p\big)\Big). \tag{11}$$

This implies that the homotopy type of $\boldsymbol{\pi}^n(X)$, and so in particular the isomorphism class of $\pi^n(X)$, depend only on the homotopy type of X. The resulting (contravariant) functorial assignment

$$\overbrace{X}^{\text{Space(-time)}} \longrightarrow \overbrace{\boldsymbol{\pi}^p(X)}^{\substack{\text{Cocycle space of} \\ p\text{-Cohomotopy cohomology theory} \\ \text{evaluated on } X}} \tag{12}$$

embodies *p-Cohomotopy theory* as non-abelian (unstable) generalized cohomology theory.

Differential cohomology. A refinement of Cohomotopy cohomology theory to a *differential* (or *geometric*) non-abelian generalized cohomology theory is an assignment

$$\overbrace{X}^{\text{Space(-time)}} \longrightarrow \overbrace{\boldsymbol{\pi}^p_{\mathrm{diff}}(X)}^{\substack{\text{Geometric cocycle space} \\ \text{of differential } p\text{-Cohomotopy theory} \\ \text{evaluated on } X}} \tag{13}$$

of a *geometric cocycle space* $\boldsymbol{\pi}^p_{\mathrm{diff}}(X)$, of sorts, which may depend on geometric data carried by X, but which is such that the underlying homotopy type $\int \boldsymbol{\pi}^p(X)$ of the geometric cocycle space is homotopy equivalent to that of the bare cocycle space (10):

$$\underbrace{\int\Big(\overbrace{\boldsymbol{\pi}^p_{\mathrm{diff}}(X)}^{\substack{\text{Geometric} \\ \text{cocycle space}}}\Big)}_{\substack{\text{Underlyng plain homotopy type} \\ \text{of geometric cocycle space}}} \underset{\text{hmtpy}}{\simeq} \underbrace{\pi^p(X)}_{\substack{\text{Plain Cohomotopy} \\ \text{cocycle space}}}. \tag{14}$$

In full generality, $\boldsymbol{\pi}^p(X)$ may be a *cohesive ∞-stack*, but for our purpose here it is sufficient to allow $\boldsymbol{\pi}^p(X)$ to be a manifold, or even just a topological space (understood up to homeomorphism, instead of up to homotopy equivalence), which is a special simple example of cohesive ∞-stacks. In this simple case the operation $\int(-)$ of computing underlying homotopy types is just the usual way of regarding a topological space as a representative of its homotopy type, and hence we will not further display it.

Configuration spaces as differential Cohomotopy cocycle spaces. The following statement provides a solution to the constraint (14) on a differential refinement of Cohomotopy cohomology theory, in the case when X is a Penrose diagram space (4). Applying the results from [May72, 2.7][Seg73, 3] in our setting leads us to the following.

Proposition 2.5 (Labelled configuration spaces via Cohomotopy cocycles)**.** *For any natural numbers $d < p \in \mathbb{N}$, the un-ordered configuration space* $\mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{p-d}\big)$ *of points in $\mathbb{R}^d$ with labels in $\mathbb{D}^{p-d}$ (Def. 2.1) has the homotopy type of the plain p-Cohomotopy cocycle space* (10) *of the one-point compactified d-dimensional Euclidean space* $(\mathbb{R}^d)^{\mathrm{cpt}}$ (3)*:*

$$\underset{\substack{\textit{Un-ordered configuration space} \\ \textit{of points in } \mathbb{R}^d \\ \textit{with labels in } \mathbb{D}^{p-d}}}{\mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{p-d}\big)} \xrightarrow[\simeq \atop \textit{hmtpy}]{\substack{\textit{send configuration of points} \\ \textit{to their Cohomotopy charge}}} \underset{\substack{\textit{Cocycle space of} \\ p\textit{-Cohomotopy cohomology theory} \\ \textit{on the one-point compactification} \\ \textit{of } d\textit{-dim Euclidean space}}}{\boldsymbol{\pi}^p\big((\mathbb{R}^d)^{\mathrm{cpt}}\big)} \qquad \textit{for } d < p. \tag{15}$$

Remark 2.6 (Cohomotopy charge map)**.** The *Cohomotopy charge map* (15) is described in detail in [SS19a], with many illustrations, and generalized to equivariant Cohomotopy of flat orbifolds. Notice that this map has originally been called the *electric field map* [Seg73], in an attempt to think of it as assigning a physical field sourced by a configuration of charged points. While this physics interpretation seems to superficially make sense for representative maps, it is incompatible with the passage to homotopy classes on the right side of (15) (which does not reflect the passage to gauge equivalence classes of electric fields). Instead, the claim of Hypothesis H is that the actual physics interpretation of the Cohomotopy charge map (15) is as assigning brane charge in M-theory.

Example 2.7 (Unlabeled from labeled)**.** The special case of Prop. 2.5 with $d = 0$ is evident:

$$\boldsymbol{\pi}^p\big(\underbrace{(\mathbb{R}^0)^{\mathrm{cpt}}}_{=S^0}\big) \xrightarrow[\text{hmtpy}]{\simeq} \mathrm{Conf}\big(\mathbb{R}^0, \mathbb{D}^p\big)$$

since now the left hand side is the space of maps from a single point to S^p, while right hand side is the space of labels in S^p carried by a single point. Both of these spaces are canonically homeomorphic to S^p itself.

But there is an alternative equivalence pertaining to this degenerate case, which is again non-trivial. Applying [Mc75, p. 95][Bö87, Example 11] to our setting we get the following.

Proposition 2.8 (Configurations vanishing at the boundary)**.** *There is a homotopy equivalence*

$$\boldsymbol{\pi}^p\big((\mathbb{R}^0)^{\mathrm{cpt}}\big) \xrightarrow[\simeq]{\text{hmtpy}} \mathrm{Conf}\big(\mathbb{D}^p, \mathbb{D}^0\big) \,.$$

Hence in the degenerate case of $d = 0$, the combination of Prop. 2.5 and Prop. 2.8 is the statement that we have a diagram of homotopy equivalences as follows:

$$\begin{tikzcd}[column sep=small]
 & \overset{\substack{\text{Cohomotopy cocycle space}\\ \text{of the point}}}{\boldsymbol{\pi}^p\big((\mathbb{R}^0)^{\mathrm{cpt}}\big)} \arrow[dl, "\underset{\text{hmtpy}}{\simeq}"'] \arrow[dr, "\underset{\text{hmtpy}}{\simeq}"] & \\
\underset{\substack{\text{Configuration space of}\\ \text{un-ordered points in } \mathbb{R}^0\\ \text{(which can be at most one point)}\\ \text{carrying a label in } S^p}}{\mathrm{Conf}(\mathbb{D}^0, \mathbb{D}^p)} \arrow[rr, "\underset{\text{hmtpy}}{\simeq}"'] & & \underset{\substack{\text{Configuration space of}\\ \text{unordered points in } \mathbb{D}^p\\ \text{carrying no label}\\ \text{but vanishing when at } \infty \in \mathbb{D}^p/\partial\mathbb{D}^0}}{\mathrm{Conf}(\mathbb{D}^p, \mathbb{D}^0) =: \mathrm{Conf}(\mathbb{D}^p)} \,.
\end{tikzcd} \tag{16}$$

With this we may finally state the main concept of this section, and prove its consistency:

Proposition 2.9 (Differential Cohomotopy on Penrose diagrams via configuration spaces)**.** *For any $p \in \mathbb{N}$, and for spacetimes in the category* (5) *of Penrose diagrams* (4)*, a consistent enhancement of plain p-Cohomotopy cohomology theory* (12) *to a geometric/differential cohomology theory* (13)*, hence satisfying the condition* (14)*, is given by the configuration space functor* (8)*:*

$$\underset{\substack{\textit{Geometric cocycle spaces}\\ \textit{of differential Cohomotopy}}}{\boldsymbol{\pi}^p_{\mathrm{diff}}} \;:=\; \underset{\substack{\textit{on Penrose diagram}\\ \textit{space(-times)}}}{\mathrm{PenroseDiag}_p^{\mathrm{op}}} \xrightarrow{\;\;\mathrm{Conf}\;\;} \underset{\substack{\textit{are configuration spaces of points}\\ \textit{regarded as actual topological spaces}}}{\mathrm{Top}^{*/}} .$$

Proof. The assignment $X \mapsto \boldsymbol{\pi}^p(X)$ of homotopy types of plain Cohomotopy cocycle spaces (10) is homotopy invariant in X. Hence the uncompactified factors $(\mathbb{R}^{p-d})_+$ in the Penrose diagrams (4), being homotopy-contractible, do not contribute to the homotopy type of the plain Cohomotopy cocycle spaces:

$$\boldsymbol{\pi}^p\big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+\big) \underset{\text{hmtpy}}{\simeq} \boldsymbol{\pi}^p\big((\mathbb{R}^d)^{\mathrm{cpt}}\big) .$$

With this, it follows that Prop. 2.5 implies that condition (14) is satisfied for $d \geq 1$

$$\text{Cohomotopy charge...} \quad \boldsymbol{\pi}^p\Big(\big(\overbrace{\mathbb{R}^d}^{\substack{\text{...vanishing at infinity}\\ \text{along these directions...}}}\big)^{\mathrm{cpt}} \wedge \big(\underbrace{\mathbb{R}^{p-d}}_{\substack{\text{...but not necessarily}\\ \text{along these directions...}}}\big)_+\Big) \xrightarrow[\underset{\text{hmtpy}}{\simeq}]{\text{...is equivalent to...}} \overset{\substack{\text{...configurations of charged points in } \mathbb{R}^p,\\ \text{which may disappear to infinity...}}}{\mathrm{Conf}\big(\mathbb{R}^d, \underbrace{\mathbb{D}^{p-d}}_{\substack{\text{... only along}\\ \text{the latter directions.}}}\big)} \qquad \text{for } d \geq 1$$

and Prop. 2.8 implies that condition (14) holds for $d = 0$:

$$\text{Cohomotopy charge...} \quad \boldsymbol{\pi}^p\Big(\overset{\substack{\text{.. not necessarily vanishing at infinity}\\ \text{in any direction...}}}{\underbrace{(\mathbb{R}^0)^{\mathrm{cpt}} \wedge (\mathbb{R}^p)_+}_{=(\mathbb{R}^p)_+}}\Big) \xrightarrow[\underset{\text{hmtpy}}{\simeq}]{\text{...is equivalent to...}} \overset{\substack{\text{...configurations of charged points in } \mathbb{R}^p,\\ \text{which may disappear to infinity...}}}{\mathrm{Conf}\big(\underbrace{\mathbb{D}^p}_{\text{...in any direction.}}\big)} \,.$$

In summary:

$$\boldsymbol{\pi}^p_{\mathrm{diff}} \;:\; \left\{ \begin{array}{lllll}
(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+ & \longmapsto & \mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{p-d}\big) \underset{\text{hmtpy}}{\simeq} & \boldsymbol{\pi}^p\big((\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{p-d})_+\big) & \text{for } d \geq 1 \\
\underbrace{(\mathbb{R}^0)^{\mathrm{cpt}} \wedge (\mathbb{R}^p)_+}_{=(\mathbb{R}^p)_+} & \longmapsto & \mathrm{Conf}\big(\mathbb{D}^p\big) \underset{\text{hmtpy}}{\simeq} & \boldsymbol{\pi}^p\big((\mathbb{R}^0)^{\mathrm{cpt}} \wedge (\mathbb{R}^p)_+\big) & \text{for } d = 0
\end{array} \right.$$

and hence condition (14) is verified. □

2.4 Intersecting brane charges in differential Cohomotopy

With Hypothesis H, we now assume that the differential 4-Cohomotopy theory of Prop. 2.9 reflects brane charges in string/M-theory on Penrose diagram spaces (4), and explore the consequences. By the discussion of charges vanishing at infinity in §2.1, we expect that the differential 4-Cohomotopy on the Penrose diagram space $(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+$ reflects charges of branes of codimension d. Indeed, for $d = 4$ we found an accurate picture of MK6-charges from Cohomotopy in [SS19a]. Now to speak about intersecting branes means to consider the Cohomotopy charge of unions of Penrose diagram spaces, which makes sense in the topological presheaf topos over the site of Penrose diagram 4-spaces from Def. 2.2.

Definition 2.10 (Union of Penrose diagram spaces)**.** For $0 \leq d \leq 4$ write

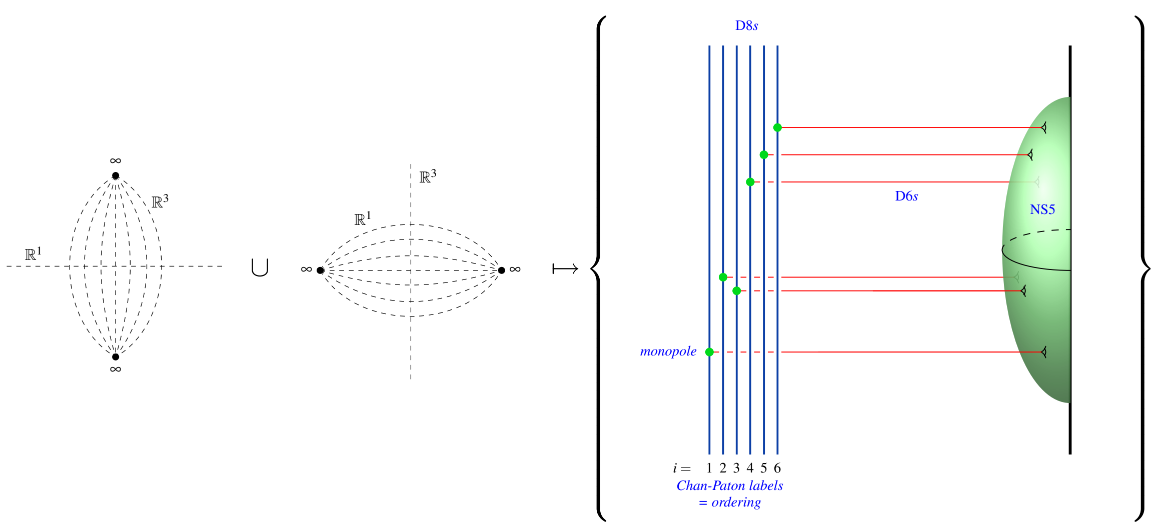
$$(\mathbb{R}^d)^{\mathrm{cpt}} \wedge (\mathbb{R}^{4-d})_+ \;\cup\; (\mathbb{R}^d)_+ \wedge (\mathbb{R}^{4-d})^{\mathrm{cpt}} \;\in\; \mathrm{Sh}\big(\mathrm{PenroseDiag}_4, \mathrm{Top}^{*/}\big) \tag{17}$$

(see the left half of (18) below) for the union, with respect to the canonical inclusion maps of Example 2.3, of Penrose diagram spaces (4), regarded as representables in the topological presheaf topos over the site (5).

By the discussion in §2.1, the generalized space (17) may be regarded as the transversal space to the intersection of charged objects of codimension-d with those of codimension-$(4-d)$. Indeed, we establish the following. We then have the following statement:

Proposition 2.11 (Differential Cohomotopy and configuration spaces)**.** *The geometric cocycle space* (13) *that the differential 4-Cohomotopy theory from Prop. 2.9 assigns to the transversal space* (17) *for $d = 3$ has the homotopy type of the ordered configuration space of points in $\mathbb{R}^3$ (Def. 2.1):*

$$\underset{\substack{\text{Transversal space}\\ \text{to 3-codim branes}\\ \text{hence to D6-branes}}}{\overset{}{(\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+}} \;\cup\; \underset{\substack{\text{Transversal space}\\ \text{to 1-codim branes}\\ \text{hence to D8-branes}}}{(\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}}} \quad \xmapsto[\substack{\text{Differential}\\ \text{Cohomotopy}}]{\;\boldsymbol{\pi}^4_{\mathrm{diff}}\;} \quad \underset{\substack{\text{Ordered}\\ \text{configuration space}}}{\underset{n\in\mathbb{N}}{\sqcup}\ \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^3\big)} \tag{18}$$



Proof. Being given by a contravariant functor (8), the assignment $\boldsymbol{\pi}^4_{\mathrm{diff}}$ takes the union (cofiber coproduct) of representable presheaves on the left to the intersection (fiber product) of its values on the cofactors. This fiber product is just the one appearing on the right of (9). Hence the statement follows by Prop. 2.4. □

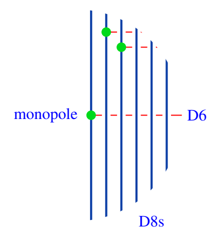
In conclusion, the following Remarks 2.12, 2.13, 2.14 highlight how, in the above discussion, the dimensions conspire, starting with the degree 4 of 4-Cohomotopy due to *Hypothesis H*:

Remark 2.12 (Distinguished system)**.** The case $d = 3$ (equivalently $d = 1$) in Def. 17, hence $p = 6$, is singled out as being the mathematically exceptional one: For $d \in \{0,2,4\}$ the corresponding analog of Prop. 2.11 produces a fiber product of unordered configuration spaces with fairly uninteresting cohomology. It is only in the case of codimensions $1 = 4-3$ that, via Prop. 2.4, a linear ordering on the points is induced, thus of Chan-Paton labels on the corresponding branes, leading to the rich observables found in §4.

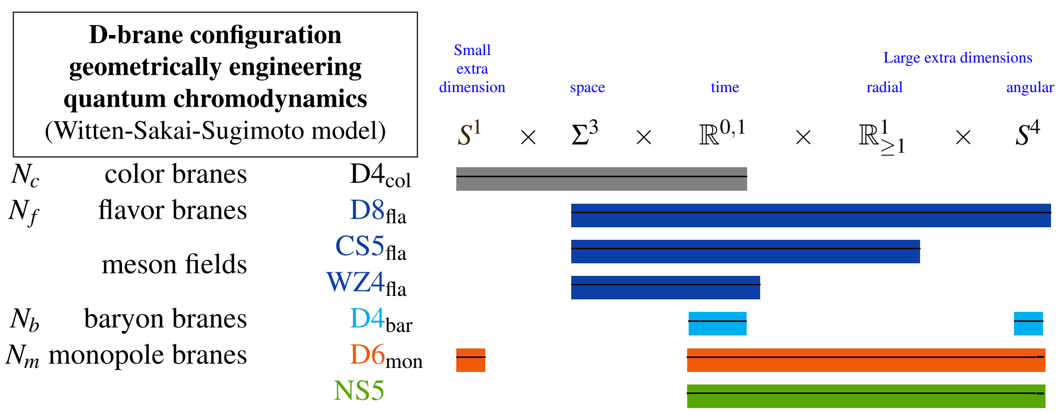
Remark 2.13 (Massive Type I')**.** Following the discussion of Hypothesis H in [FSS19d][SS19a], we are to think of Prop. 2.11 as applying to non-perturbative massive type I' string theory, hence to heterotic M-theory. With no equivariance considered here, the Hořava-Witten interval becomes invisible in homotopy theory and the codimensions 3 & 1 in Prop. 2.11 are those of D6 $\perp$ D8 brane intersections in massive type I', as shown.

Remark 2.14 (Geometric engineering of monopoles)**.** For any $p \in \{0,1,\cdots,6\}$ (at least) transversal D$p \perp$ D$(p+2)$-brane intersections geometrically engineer Yang-Mills monopoles (i.e. Donaldson-Atiyah-Hitchin-style monopoles [AH16][Do84] characterized by Nahm's equation) in the worldvolume theory of the D$(p+2)$-brane.

- For $p = 1$ this is due to [Di97], see also [HZ99, 2] [BB04][BB05].
- For $p = 2$ this is discussed in [GZZ09].
- For $p = 6$, which is the case of interest via Prop. 2.11 and by Remark 2.13, this is discussed in [HZ99] and [HLPY08, Sec. V].



In this case of $p = 6$, [HLPY08] observe that monopoles engineered as D6 $\perp$ D8-intersections yield the actual 4d monopoles of nuclear physics, through the Sakai-Sugimoto model for QCD [Wi98][SSu04][SSu05] (for review see [Re14][Su16]):



Under this identification and via Prop. 2.11, the statements about fuzzy funnel observables in §4.2 translate to statements about QCD monopoles.

2.5 Higher observables on intersecting brane configurations

Topological covariant phase spaces. We consider the following setting:

(i) Any assignment **Fields** of spaces of field configurations, such as the cohomotopically charge-quantized C-field $\mathbf{Fields} := \boldsymbol{\pi}^4_{\mathrm{diff}}$ of Prop. 2.9.

(ii) X a spatial slice of spacetime, hence with $\mathbf{Fields}(X)$ its field configuration space.

(iii) $c_{\mathrm{in}}, c_{\mathrm{out}} \in \mathbf{Fields}(X)$ two field configurations in the same connected component.

Then we may think of the the based path space

$$\overset{\text{Based path space in field configurations}}{P_{c_{\mathrm{in}}}^{c_{\mathrm{out}}}\mathbf{Fields}(X)} := \Big\{ c \in \mathrm{Maps}\big([0,1], \mathbf{Fields}(X)\big) \,\big|\, c(0) = c_{\mathrm{in}}, c(1) = c_{\mathrm{out}} \Big\} \underset{\text{hmtpy}}{\simeq} P_{c_{\mathrm{in}}}^{c_{\mathrm{in}}}\mathbf{Fields}(X) =: \overset{\text{Based loop space in field configurations}}{\Omega_{c_{\mathrm{in}}}\mathbf{Fields}(X)} \tag{19}$$

as an element of the *covariant phase space*, each of which represents a field history evolving from c_{in} to c_{out}. Any fixed choice of such field history induces (by evolving back along it) a homotopy equivalence to the based loop space of the cocycle space, as shown on the right in (20). This, in turn is independent, up to homotopy, from the choice of basepoint. Therefore we may regard the disjoint union of the construction (19) over the connected components $[c] \in \pi_0\big(\mathbf{Fields}(X)\big)$ of field configurations as the *topological covariant phase space*

$$\overset{\text{Topological covariant phase space}}{\mathbf{Phase}(X)} := \underbrace{\underset{[c]}{\sqcup}}_{\text{Disjoint union over connected components}} \overbrace{\Omega_c \mathbf{Fields}(X)}^{\text{Based loop space of field configuration space}} . \tag{20}$$

Without further equations of motion imposed on the field histories this would be the *off-shell* phase space; but for our purposes here all *topological* constraints on the fields, such as the "integral equation of motion" on the C-field [DMW00a][DMW00b], are enforced [FSS19b] by the cohomological charge quantization in the cohomology theory $\mathbf{Fields} = \boldsymbol{\pi}^4_{\mathrm{diff}}$, and therefore we do regard (20) as the topological sector of the full covariant phase space.

Higher order observables. The *observables* of a physical theory are traditionally taken to be $\mathbb{F}$-valued functions on the covariant phase space, hence functions with values in the given ground field. But to do justice to the homotopy-theoretic nature of fields charge-quantized in generalized cohomology theories, following [SS17], we here take *higher observables* to be $H\mathbb{F}$-valued functions on the topological covariant phase space (20), i.e., taking values in the Eilenberg-MacLane spectrum $H\mathbb{F}$ and its suspensions. After passage to gauge equivalence classes, these higher observables hence form the cohomology ring of the topological phase space (20):

$$\begin{array}{lllll} \text{Higher observables} & \mathrm{Obs}^\bullet(X) & := H^\bullet\big(\mathbf{Phase}(X)\big) & := & H^\bullet\Big(\underset{[c]}{\sqcup}\Omega_c\mathbf{Fields}(X)\Big) \\ \text{Higher co-observables} & \mathrm{Obs}_\bullet(X) & := H_\bullet\big(\mathbf{Phase}(X)\big) & := & H_\bullet\Big(\underset{[c]}{\sqcup}\Omega_c\mathbf{Fields}(X)\Big) \end{array} . \tag{21}$$

Higher observables on D6 $\perp$ D8**-brane configurations.** Specifying the higher observables (21) to the case where we consider, with *Hypothesis H*:

(i) $\mathbf{Fields} := \boldsymbol{\pi}^4_{\mathrm{diff}}$ to be the C-field moduli of Prop. 2.9;

(ii) $X := (\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \,\cup\, (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}}$ to be the transversal space of D6 $\perp$ D8-brane intersections according to Def. 17;

we are led to the following notion:

Definition 2.15. We take the algebra of *higher observables on configurations of* D6 $\perp$ D8*-brane intersections* (Remark 2.14) to be the ordinary cohomology ring (21) of the componentwise based loop space (20) of the differential 4-Cohomotopy cocycle space (Prop. 2.9) that is assigned to the transversal space for codim=3/codim=1 brane intersections (Def. 17):

$$\overset{\text{Higher observables on D6}\perp\text{D8-configurations by Hypothesis H}}{\mathrm{Obs}^\bullet_{\mathrm{D6}\perp\mathrm{D8}}} := H^\bullet\Big(\boldsymbol{\pi}^4_{\mathrm{diff}}\big((\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \,\cup\, (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}}\big)\Big). \tag{22}$$

With the results from §2 we may characterize these higher observables more concretely:

Proposition 2.16 (Higher observables as cohomology of looped configuration space)**.** *The algebra of higher observables on* D6 $\perp$ D8*-configurations* (22) *is isomorphic to the direct sum, over the number N_{f} of points, of the cohomology rings of the based loop spaces of configuration spaces (Def. 2.1) of N_{f} points in Euclidean 3-space:*

$$\mathrm{Obs}^\bullet_{\mathrm{D6}\perp\mathrm{D8}} \simeq \bigoplus_{N_{\mathrm{f}}\in\mathbb{N}} H^\bullet\Big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)\,. \tag{23}$$

Proof. Using the fact that ordinary cohomology is invariant under homotopy equivalences, this follows with Prop. 2.11. ☐

Remark 2.17. Via Prop. 2.16 the higher co-observables (21) are identified with the higher order OPEs of extended field theories as considered in [BBBDN18].

Higher observables on D6 $\perp$ D8 **are weight systems on chord diagrams.** Remarkably, there is a combinatorial model for the cohomology ring (22) of higher observables, namely in terms of *weight systems on chord diagrams*. The definitions of these are reviewed in detail in §3 below. The reader may wish to come back to the following Prop. 2.18 after looking through §3.

Proposition 2.18 (Cohomology of looped configuration space is horizontal weight systems)**.** *For any natural number $N_{\mathrm{f}}\in\mathbb{N}$ we have*[3] *for the based loop space of the ordered configuration space $\underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)$ of N_{f} points in $\mathbb{R}^3$ (Def. 2.1) that:*
(i) *Its homology Pontrjagin ring is isomorphic, as a graded Hopf algebra (see [Ha92]), to the algebra $\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}$* (36) *of horizontal chord diagrams* (30) *with N_{f} strands modulo the 2T-relations* (33) *and 4T relations* (34)*:*

$$\underset{\substack{\textit{Homology ring of}\\ \textit{based loop space of}\\ \textit{configuration space of}\\ N_{\mathrm{f}}\ \textit{ordered points in}\ \mathbb{R}^3}}{H_\bullet\Big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)} \simeq \underset{\substack{\textit{Concatenation algebra of}\\ \textit{horizontal chord diagrams}\\ \textit{with}\ N_{\mathrm{f}}\ \textit{strands}\\ \textit{modulo 2T- \& 4T-relations}}}{\big(\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)_\bullet} \tag{24}$$

(ii) *Its cohomology is isomorphic, as a graded vector space, to the space $\big(\mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)^\bullet$* (37) *of weight systems (Def. 3.1) on horizontal chord diagrams with N_{f} strands:*

$$\underset{\substack{\textit{Cohomology group of}\\ \textit{based loop space of}\\ \textit{configuration space of}\\ N_{\mathrm{f}}\ \textit{ordered points in}\ \mathbb{R}^3}}{H^\bullet\Big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)} \simeq \underset{\substack{\textit{Weight systems on}\\ \textit{horizontal chord diagrams}\\ \textit{with}\ N_{\mathrm{f}}\ \textit{strands}}}{\big(\mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)^\bullet} \tag{25}$$

(iii) *Hence the higher co-observables* (22) *are identified with horizontal chord diagrams of any number of strands*

$$\underset{\substack{\textit{Higher order co-observables on}\\ \mathrm{D6}\perp\mathrm{D8}\textit{-brane configurations}}}{\mathrm{Obs}_\bullet^{\mathrm{D6}\perp\mathrm{D8}}} \simeq \underset{\substack{\textit{Horizontal chord diagrams}\\ \textit{with any number of strands}}}{\big(\mathscr{A}^{\mathrm{pb}}\big)_\bullet} := \bigoplus_{N_{\mathrm{f}}\in\mathbb{N}} \big(\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)_\bullet \tag{26}$$

and the higher observables (22) *with the weight systems on these:*

$$\underset{\substack{\textit{Higher order observables on}\\ \mathrm{D6}\perp\mathrm{D8}\textit{-brane configurations}}}{\mathrm{Obs}^\bullet_{\mathrm{D6}\perp\mathrm{D8}}} \simeq \underset{\substack{\textit{Weight systems on}\\ \textit{horizontal chord diagrams}\\ \textit{with any number of strands}}}{\big(\mathscr{W}^{\mathrm{pb}}\big)^\bullet} := \bigoplus_{N_{\mathrm{f}}\in\mathbb{N}} \big(\mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)^\bullet \tag{27}$$

Proof. By [FH01, Thm. 2.2] (also [CG01, Thm. 4.1][CG02, Thm. 2.3]) we have an isomorphism

$$\underset{\substack{\text{Homology ring of}\\ \text{based loop space of}\\ \text{configuration space of points}}}{H_\bullet\Big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)} \simeq \underset{\substack{\text{Universal enveloping algebra of}\\ \text{infinitesimal braid Lie algebra}\\ \text{generated in degree 1}}}{\mathscr{U}\big(\mathscr{L}_{N_{\mathrm{f}}}(1)\big)} \tag{28}$$

identifying the homology ring of the looped configuration space with the universal enveloping algebra of the *infinitesimal braid Lie algebra* on N_{f} strands with generators in degree 1, hence of the Lie algebra freely defined by the infinitesimal braid relations (35). Using that these relations are equivalently the 2T-relations (33) and 4T-relations (34) on horizontal chord diagrams, direct inspection reveals that this universal enveloping algebra is

[3] This holds over any ground field $\mathbb{F}$ (such as the complex numbers), and in fact more generally over any commutative ring (such as the integers).

canonically isomorphic, as a graded associative algebra, to the concatenation algebra of horizontal chord diagrams (36):

$$\underset{\substack{\text{Universal enveloping algebra of}\\ \text{infinitesimal braid Lie algebra}}}{\mathscr{U}\big(\mathscr{L}_{N_{\mathrm{f}}}(1)\big)} \;\simeq\; \underset{\substack{\text{Concatenation algebra of}\\ \text{horizontal chord diagrams}}}{\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}} \tag{29}$$

The combination of (28) with (29) yields the first statement. The second statement then follows by direct dualization, using the universal coefficient theorem – see also the statement of [Koh02, Thm. 4.1]. With this, the third statement follows by Prop. 2.16. □

Remark 2.19 (Quantum algebra structure on higher co-observables). **(i)** The product operation on the homological Hopf algebras $H_\bullet\big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\big) \simeq \mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}$ in Prop. 2.18 is non-commutative (manifestly so from (36)) while its co-product is graded co-commutative (as it comes from the diagonal map on the space $\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)$.

(ii) Accordingly, for the cohomological Hopf algebras $H^\bullet\big(\Omega \underset{\{1,\cdots,N_{\mathrm{f}}\}}{\mathrm{Conf}}(\mathbb{R}^3)\big) \simeq \mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}$ in Prop. 2.18 it is the other way around: Here the product operation is graded-commutative (being the cup product on cohomology).

(iii) In this sense, when regarded as graded algebras of (co-)observables, weight systems $\mathscr{W}$ form an algebra of *classical observables*, while chord diagrams $\mathscr{A}$ form an algebra of *quantum observables*.

3 Weight systems on chord diagrams

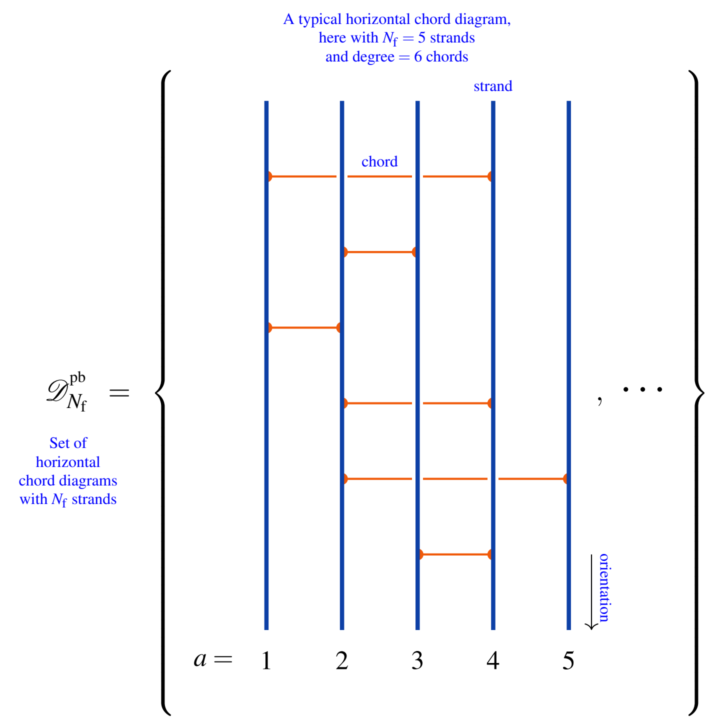
Here we lay out the key definitions and facts regarding weight systems on chord diagrams, streamlined towards our applications in §4. For round chord diagrams we follow [Bar95b], which has made it into textbook literature [CDM11, 4-6][JM19, 11,13-14]. For weight systems on horizontal chord diagrams, which we find to be of deeper relevance (see Theorem 3.4 and its interpretations in §4.7, §4.8, §4.9) we follow [BN96], which seems not to have found as much attention yet.

3.1 Horizontal chord diagrams

A *horizontal chord diagram* on N_f strands is a trivalent finite undirected graph with N_f disjoint, oriented lines embedded, the *strands*; all vertices lying on these strands, and the edges between the lines, the *chords*, ordered along the strands.

One traditionally writes $\mathscr{D}^{\mathrm{pb}}_{N_\mathrm{f}}$ for the set of horizontal chord diagrams, or just $\mathscr{D}^{\mathrm{pb}}$ if the number of strands is understood (the superscript "pb" alludes to *pure braids*, which are an equivalent way of presenting horizontal chord diagrams).

We let $\mathrm{Span}\big(\mathscr{D}^{\mathrm{pb}}\big)$ denote the linear span of this set, hence the vector space of formal finite linear combinations of horizontal chord diagrams. We regard this as a *graded vector space*, as such denoted $\mathrm{Span}\big(\mathscr{D}^{\mathrm{pb}}_{N_\mathrm{f}}\big)_\bullet$, where the degree of a horizontal chord diagram is its number of chords.

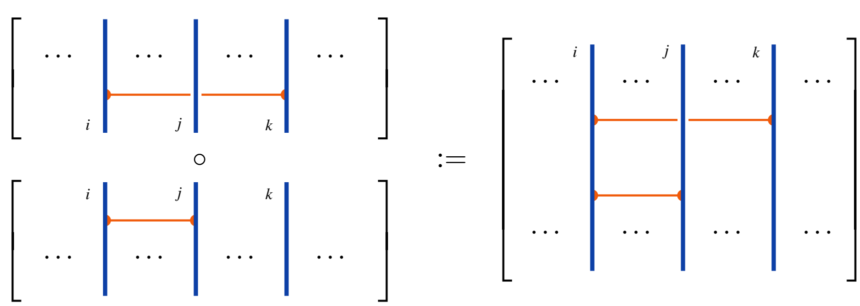


(30)

The linear span of the set (30) of horizontal chord diagrams is canonically a graded associative algebra under concatenation $\circ$ of strands:

$$\Big(\mathrm{Span}\big(\mathscr{D}^{\mathrm{pb}}_{N_\mathrm{f}}\big)_\bullet, \circ\Big) \qquad \text{e.g.:}$$

Graded associative algebra spanned by horizontal chord diagrams under concatenation of strands

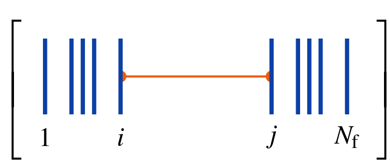


(31)

Hence if, for any $i < j \in \{1, \cdots, N_\mathrm{f}\}$, we write

$$t_{ij} = -t_{ji} \quad :=$$

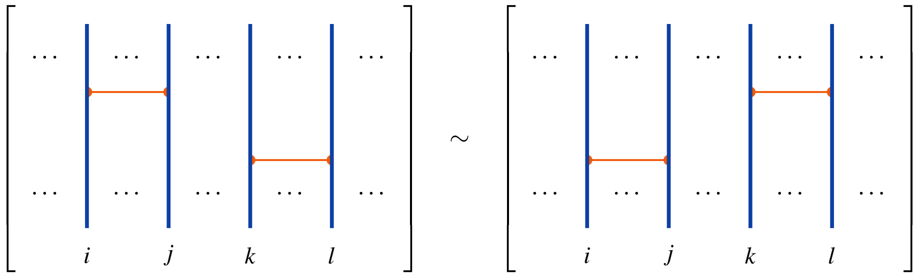
Generator



$$\in \mathrm{Span}\big(\mathscr{D}^{\mathrm{pb}}_{N_\mathrm{f}}\big)_1 \qquad (32)$$

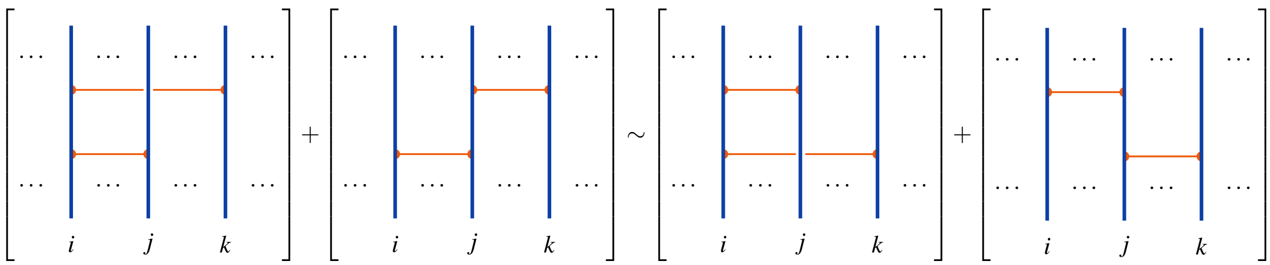
for the horizontal chord diagram with exactly one chord, which goes between the ith and the jth strand, then the algebra of horizontal chord diagrams is just the *free associative algebra* on these generators t_{ij} of degree 1.

On this free algebra consider the following relations:
(i) the *2T relations*:



(33)

(ii) the *4T relations* (42):



(34)

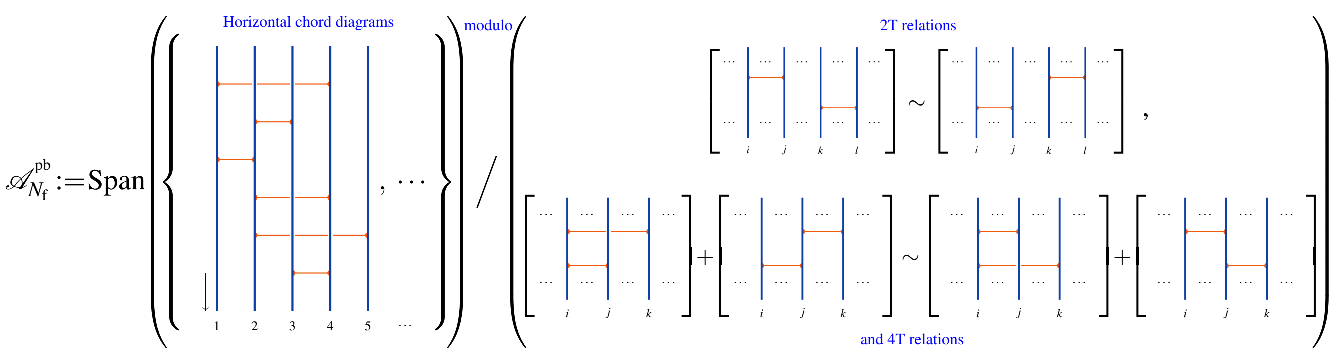
Expressed in terms of the algebra generators (32), these are equivalently the *infinitesimal braid relations* [Koh87, (1.1.4)]:

$$\left.\begin{array}{lll} \text{(2T)} & [t_{ij}, t_{jk}] & = 0 \\ \text{(4T)} & [t_{ik} + t_{jk}, t_{ij}] & = 0 \end{array}\right\} \quad \text{for all pairwise distinct } i,j,k,l \in \{1, \cdots, N_{\mathrm{f}}\}\,. \tag{35}$$

Now, the quotient of the graded algebra (31) of linear combinations of horizontal chord diagrams by these relations (35) is a graded associative algebra denoted

$$\begin{aligned} \mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}} &:= \mathrm{Span}\big(\mathscr{D}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)\big/(2\mathrm{T}, 4\mathrm{T}) \\ &= \mathrm{GradedAssoc}\Big(\big\{\underset{\deg = 1}{t_{ij}} = -t_{ji} \,|\, i < j \in \{1, \cdots N_{\mathrm{f}}\}\big\}\Big)\big/(2\mathrm{T}, 4\mathrm{T})\,. \end{aligned} \tag{36}$$

Hence:

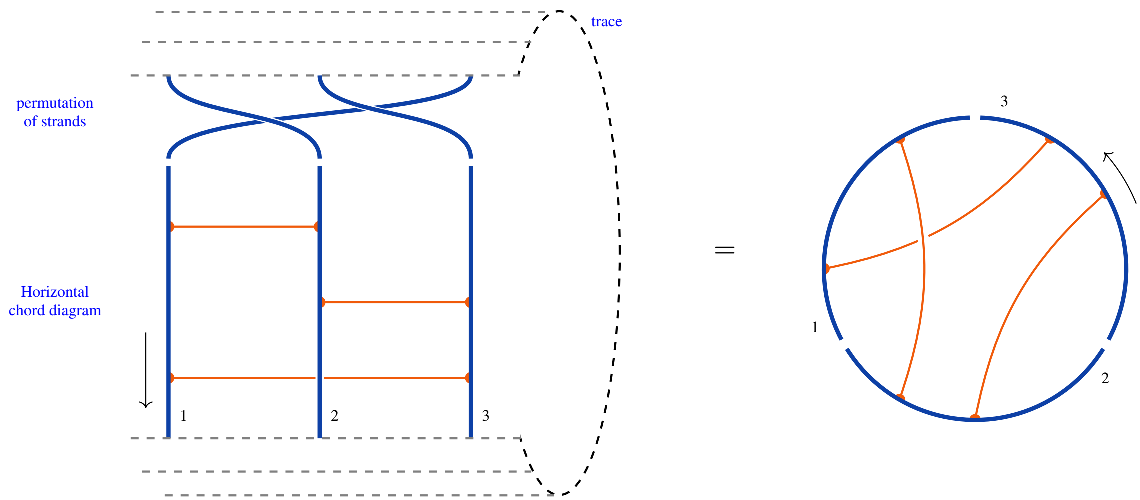


Definition 3.1. A *weight system on horizontal chord diagrams* is a linear function on the span of horizontal chord diagrams (31) modulo 2T- and 4T-relations (36). Hence the space of all weight systems is the graded linear dual space to the quotient space (36), to be denoted

$$\overset{\substack{\text{Space of weight systems} \\ \text{on horizontal chord diagrams} \\ \text{with } N_{\mathrm{f}} \text{ strands}}}{\big(\mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)^{\bullet}} \quad := \quad \overset{\substack{\text{Graded linear dual to span of} \\ \text{horizontal chord diagrams} \\ \text{modulo 2T- and 4T relations}}}{\Big(\big(\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}\big)_{\bullet}\Big)^{*}}. \tag{37}$$

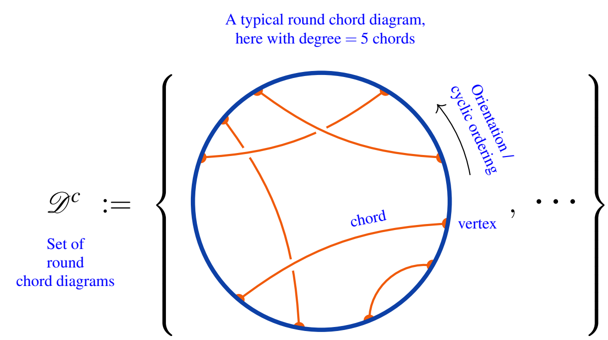
3.2 Round chord diagrams

Closing up horizontal chord diagrams. Given any permutation $\sigma \in \mathrm{Sym}(N_\mathrm{f})$ of N_f elements, there is an evident way to close a horizontal chord diagram (30) to a round chord diagram. For example, for $\sigma = (312)$ a cyclic permutation of three elements, we have:



(38)

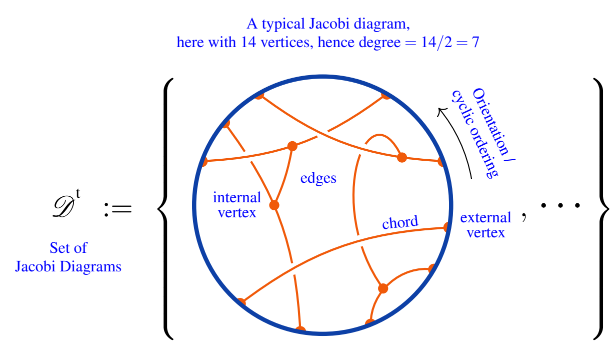
A *round chord diagram*, usually just called a *chord diagram*, is a trivalent and connected finite undirected graph with an embedded oriented circle and with all vertices being on that circle, regarded modulo cyclic permutation along the circle. The set of all round chord diagrams is traditionally denoted $\mathscr{D}^\mathrm{c}$ (the superscript is for *chords*). We write $\mathrm{Span}(\mathscr{D}^\mathrm{c})_\bullet$ for the linear span of this set, hence for the graded vector space of formal finite linear combinations of round chord diagrams, with degree half their number of vertices.



(39)

An evident generalization of round chord diagrams, needed below, is obtained by allowing internal vertices:

A *Jacobi diagram* is a trivalent connected finite undirected graph with an oriented embedded circle and with an orientation on each internal vertex (i.e. one not on the circle), regarded up to cyclic permutation of vertices. (These have also been called *Chinese character diagrams* and, for reasons discussed in §4.4, *Chern-Simons diagrams*.) The set of all Jacobi diagrams is traditionally denoted $\mathscr{D}^\mathrm{t}$ (the superscript is for *trivalent*). We write $\mathrm{Span}(\mathscr{D}^\mathrm{t})_\bullet$ for the linear span of this set, hence for the graded vector space of formal finite linear combinations of Jacobi diagrams, with degree half their number of vertices.

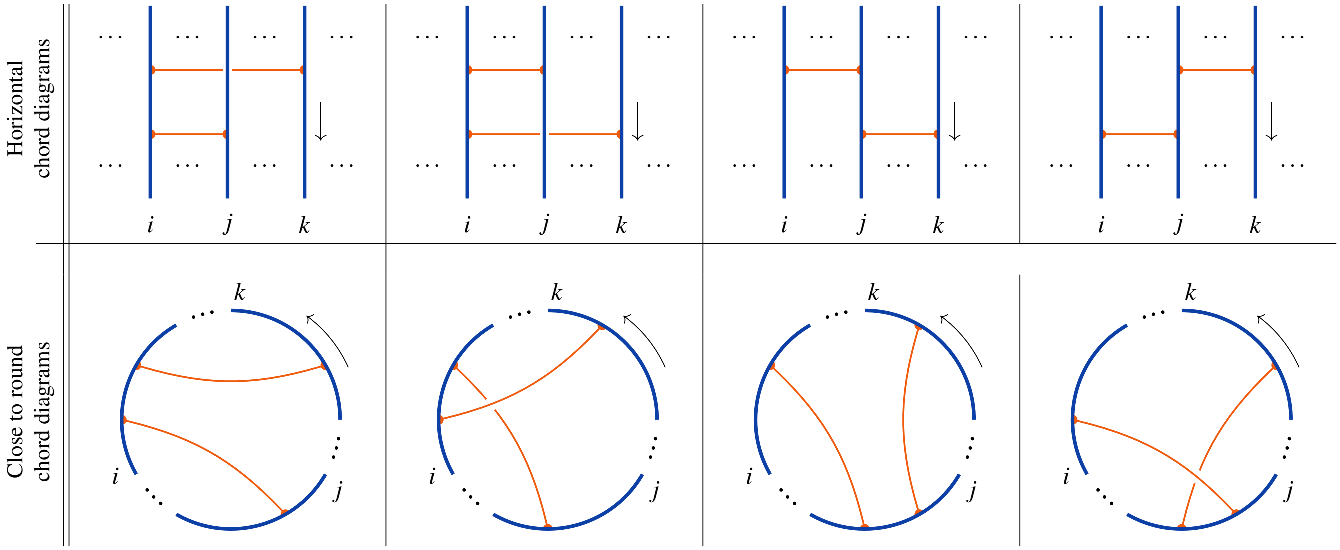


(40)

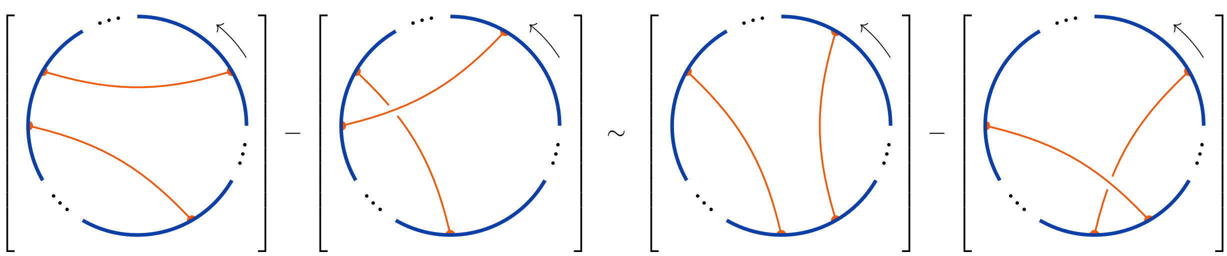
The closing operation as in (38) on the set of horizontal chord diagrams (30), together with the understanding of round chord diagrams (39) as special cases of Jacobi diagrams (40) gives functions of sets of our three types of diagrams, as follows:

$$\underset{\text{Set of horizontal chord diagrams}}{\mathscr{D}^{\mathrm{pb}}_{N_\mathrm{f}}} \xrightarrow{\mathrm{close}_{(N_\mathrm{f}12\cdots)}} \underset{\text{Set of round chord diagrams}}{\mathscr{D}^{\mathrm{c}}} \xhookrightarrow{\;\;i\;\;} \underset{\text{Set of Jacobi diagrams}}{\mathscr{D}^{\mathrm{t}}}. \tag{41}$$

Round closure of the 4T relations. Under the closing map (41), the four types of horizontal chord diagrams (30) that appear in the horizontal 4T relation (34) give the following four types of round chord diagrams (39):



This means that in order to make the closing operation on the left of (41) pass to the quotient space $\mathscr{A}^{\mathrm{pb}}$ (see (36)), we have to quotient the span of round chord diagrams by the following *round 4T relations* on $\mathrm{Span}\big(\mathscr{D}^c\big)$:



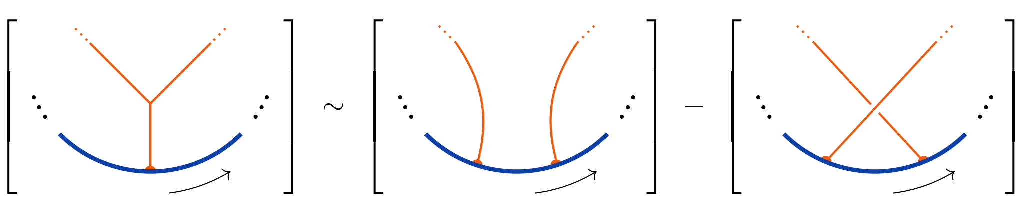
(42)

Hence, in direct analogy to Def. 36, we have:

Definition 3.2. Write
$$\mathscr{A}^{\mathrm{c}} := \mathrm{Span}\big(\mathscr{D}^{\mathrm{c}}\big)/(4\mathrm{T})$$
for the graded quotient vector space of the span of round chord diagrams (39) by the round 4T relations (42) (see first line of (46) below). A *weight system on round chord diagrams* [4] is a linear function on this space:

$$\underbrace{\big(\mathscr{W}^{\mathrm{c}}\big)^{\bullet}}_{\text{Space of weight systems on round chord diagrams}} := \underbrace{\big(\big(\mathscr{A}^{\mathrm{c}}\big)_{\bullet}\big)^{*}}_{\text{Graded linear dual to span of round chord diagrams modulo round 4T relations}} \tag{43}$$

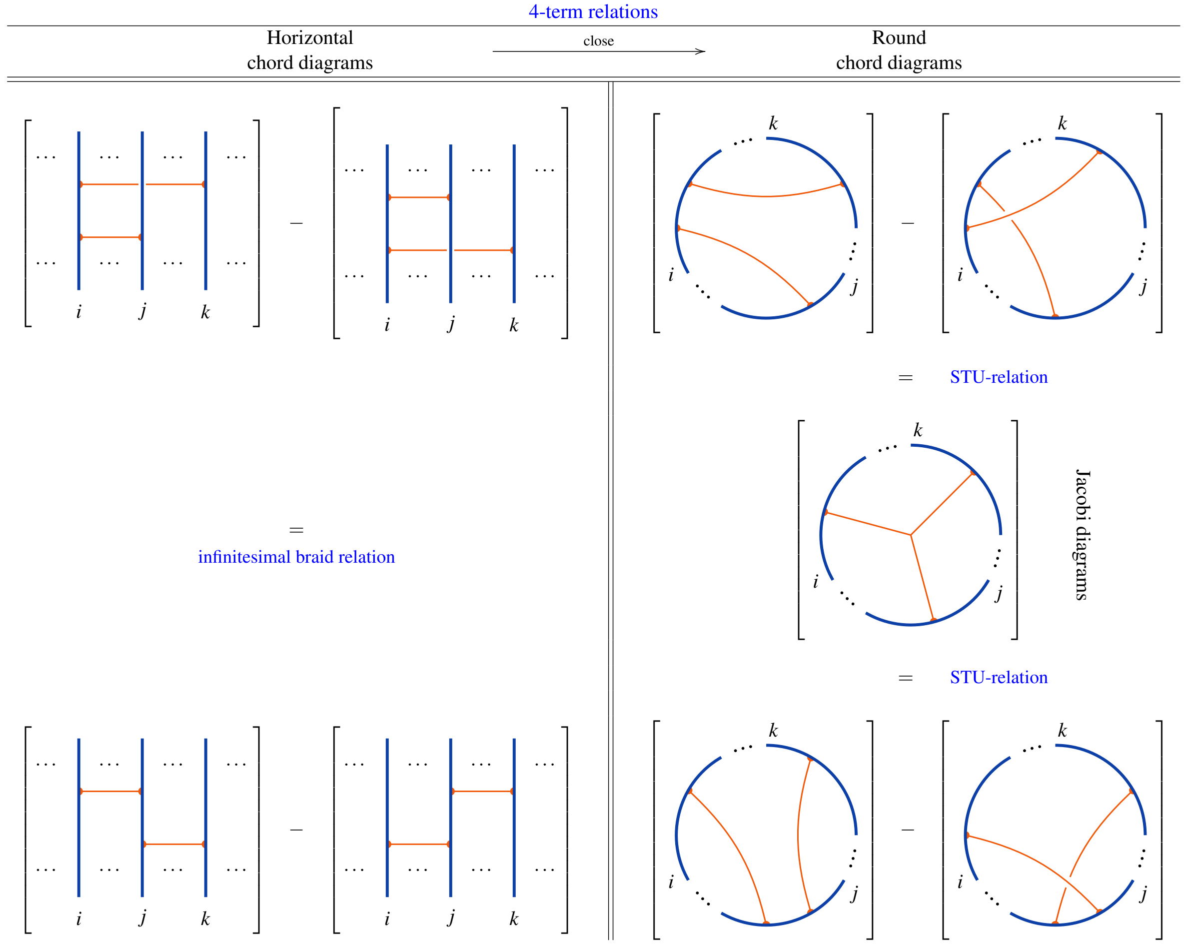
Resolution of round 4T- to STU-relations. We would like that also the injection i of round chord diagrams into Jacobi diagrams, on the right of (41), to pass to these quotients. For that we consider, moreover, the following relations on the linear span of Jacobi diagrams, called the *STU relations* on $\mathrm{Span}\big(\mathscr{D}^{\mathrm{t}}\big)$:



(44)

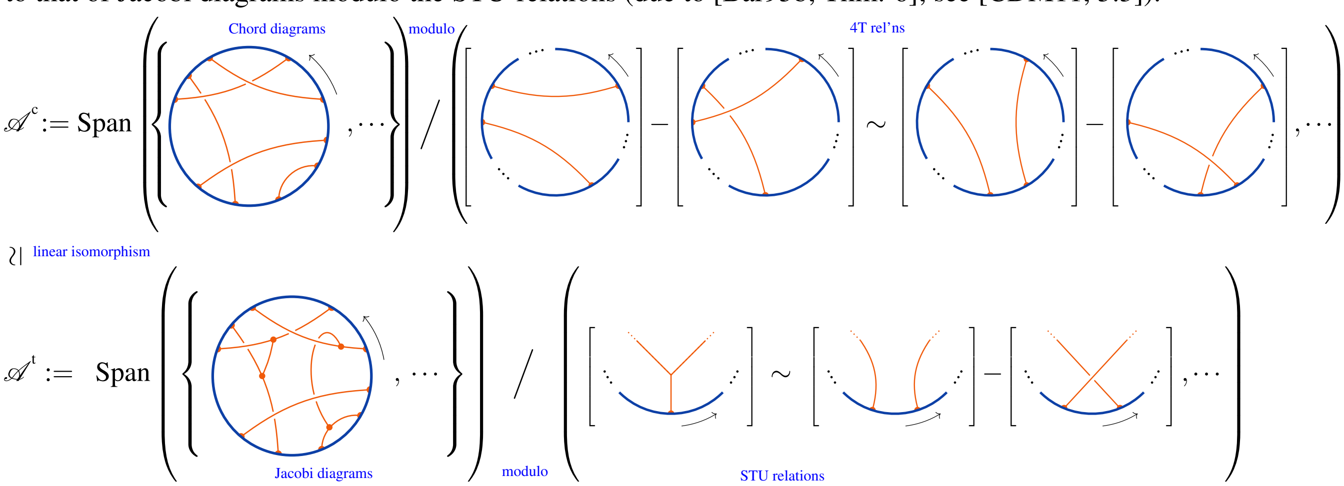
The *reason* behind these STU-relations is that on Jacobi diagrams they *resolve* the round 4T relations (42):

[4]Beware that some authors call these *framed* weight systems, since we do not impose the 1T relation.



(45)

Equivalence of chord diagrams and Jacobi diagrams. Using this factorization of the round 4T relation by the STU-relations, one proves that the linear span of round chord diagrams modulo the round 4T-relations is equivalent to that of Jacobi diagrams modulo the STU-relations (due to [Bar95b, Thm. 6]; see [CDM11, 5.3]):



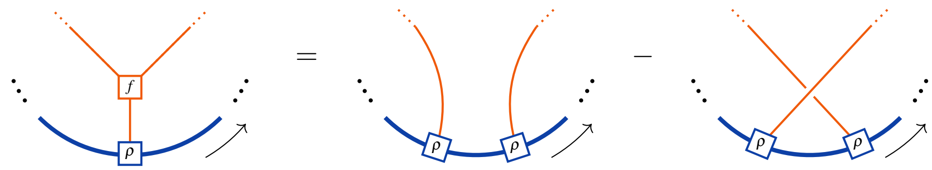
Hence: (46)

Proposition 3.3 (Relating weight systems)**.** *The maps* (41) *of sets of chord diagrams dualize to a linear bijection of weight systems on Jacobi diagrams (i.e., the graded linear dual of $\mathscr{A}^{\mathrm{t}}$) with weight systems on round chord diagrams* (43)*, followed by a linear injection of the latter into the space of weight systems on horizontal chord diagrams* (37)*:*

$$\underset{\substack{\textit{Weight systems on}\\ \textit{horizontal chord diagrams}}}{\mathscr{W}^{\mathrm{pb}}_{N_{\mathrm{f}}}} \xleftarrow[\textit{injection}]{(\mathrm{close}_{(N_{\mathrm{f}}12\cdots)})^*} \underset{\substack{\textit{Weight systems on}\\ \textit{round chord diagrams}}}{\mathscr{W}^{\mathrm{c}}} \xleftarrow[\simeq \;\; \textit{bijection}]{i^*} \underset{\substack{\textit{Weight systems on}\\ \textit{Jacobi diagrams}}}{\mathscr{W}^{\mathrm{t}}} . \tag{47}$$

3.3 Lie algebra weight systems

Metric Lie algebras appear. The equivalence (46) reveals that weight systems secretly encode Lie theoretic data. Indeed, the STU-relation (44) is manifestly the Jacobi identity, or more generally the Lie action property. This is expressed in *Penrose diagram notation* (reviewed in [PR84, appendix, p. 424-434]) also called *string diagram calculus* (reviewed in [Sel09]); see the big table on page 25 for the translation. Diagrammatically:



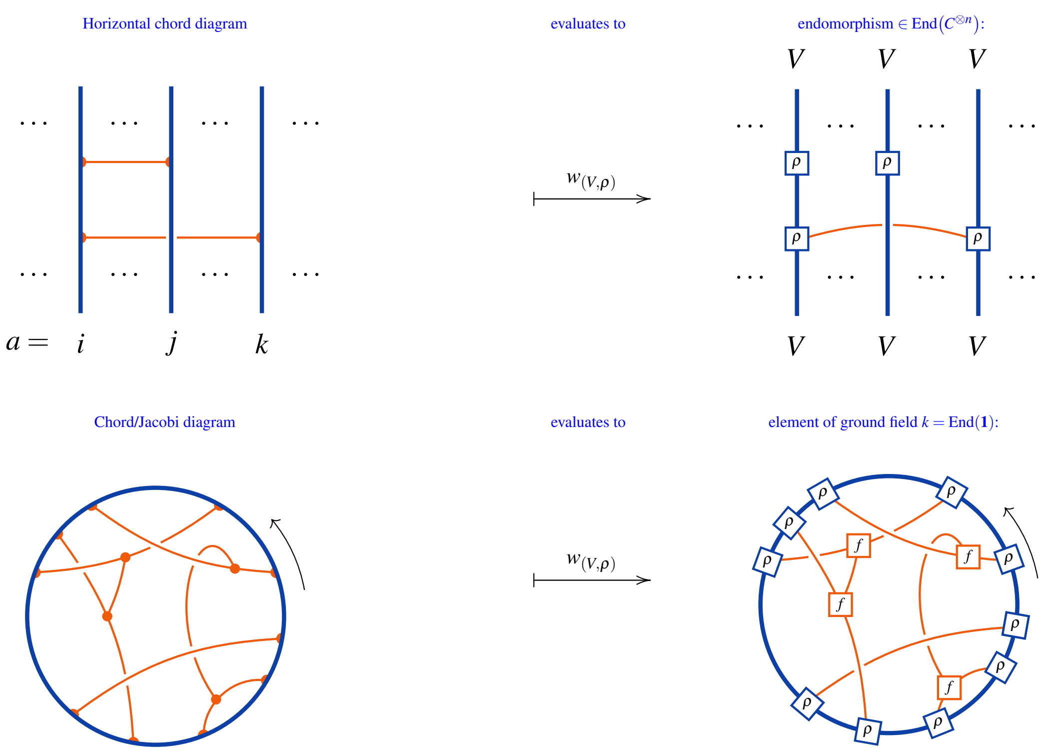
$$\rho(f(x,y),z) \qquad = \qquad \rho(y,\rho(x,z)) \qquad - \qquad \rho(x,\rho(y,z)) \tag{48}$$

With $f(x,y) = [x,y]$ a Lie bracket, this is the Lie action property on ρ. Moreover, with $\rho(x,z) = [x,z]$ the adjoint action, this is the Jacobi identity.

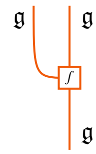
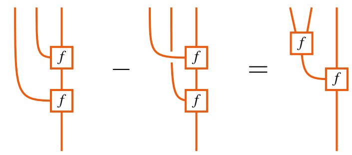
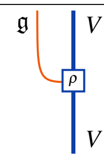
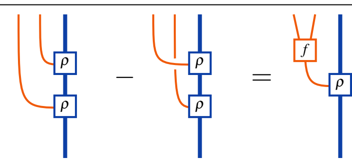
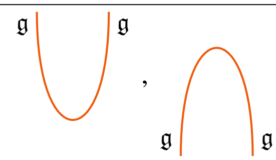
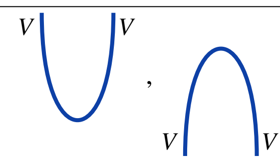
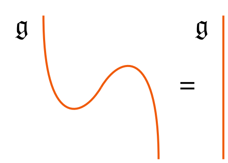
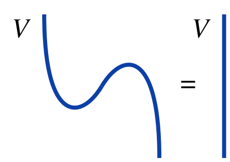
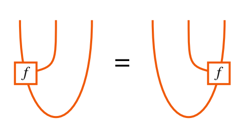
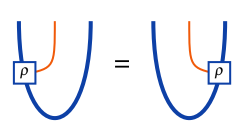
This means that metric Lie representations of metric Lie algebras internal to tensor categories induce weight systems (Def. 3.1) on chord diagrams. For ordinary Lie algebras this is due to [Bar95b, Sec. 2.4], while the general statement is made explicit in [RW06, Sec. 3], following observations in [Va94][Vo11]. We capture this as:

$$\textbf{Lie algebra weight system } w_{(V,\rho)} \text{ induced by } \left\{ \begin{array}{llcl} \text{Lie reprensentation} & V & \in & \mathscr{C} \\ \text{in tensor category} & \mathscr{C} & \in & \mathrm{TensorCat} \\ \text{with Lie action} & \rho & : & \mathfrak{g} \otimes V \to V \\ \text{by Lie algebra} & \mathfrak{g} & \in & \mathscr{C} \\ \text{with Lie bracket} & f & : & \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \\ \text{and compatible} & g & : & \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{1} \\ \text{metrics} & k & : & V \otimes V \to \mathbf{1} \end{array} \right.$$

whose effect on the corresponding chord diagrams is the following:



(49)

Data of metric Lie representation	Category notation	Penrose notation	Index notation
Lie bracket	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "f"] \\ \mathfrak{g} \end{tikzcd}$		$f_{ab}{}^{c}$
Jacobi identity	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "\mathrm{id}\otimes f - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213}\circ(\mathrm{id}\otimes f)"'] & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "f"] \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "f"'] & \mathfrak{g} \end{tikzcd}$		$f_{ae}{}^{d} f_{bc}{}^{e} - f_{be}{}^{d} f_{ac}{}^{e} = f_{ec}{}^{d} f_{ab}{}^{e}$
Lie action	$\begin{tikzcd} \mathfrak{g}\otimes V \arrow[d, "\rho"] \\ V \end{tikzcd}$		$\rho_a{}^{i}{}_{j}$
Lie action property	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes V \arrow[r, "\mathrm{id}\otimes\rho - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213}\circ(\mathrm{id}\otimes\rho)"'] & \mathfrak{g}\otimes V \arrow[d, "\rho"] \\ \mathfrak{g}\otimes V \arrow[r, "\rho"'] & V \end{tikzcd}$		$\rho_a{}^{j}{}_{l}\,\rho_b{}^{l}{}_{i} - \rho_b{}^{j}{}_{l}\,\rho_a{}^{l}{}_{i} = f_{ab}{}^{c}\,\rho_c{}^{j}{}_{i}$
Metric	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g"] \\ \mathbf{1} \end{tikzcd}$, $\begin{tikzcd} \mathbf{1} \arrow[d, "g^{-1}"] \\ \mathfrak{g}\otimes\mathfrak{g} \end{tikzcd}$		g_{ab} , g^{ab}
	$\begin{tikzcd} V\otimes V \arrow[d, "k"] \\ \mathbf{1} \end{tikzcd}$, $\begin{tikzcd} \mathbf{1} \arrow[d, "k^{-1}"] \\ V\otimes V \end{tikzcd}$		k_{ij} , k^{ij}
Metric property	$\begin{tikzcd} \mathfrak{g} \arrow[r, "\simeq"] \arrow[ddrr, "\mathrm{id}"'] & \mathfrak{g}\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes g^{-1}"] & \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g\otimes\mathrm{id}"] \\ & & \mathbf{1}\otimes\mathfrak{g} \arrow[d, "\simeq"] \\ & & \mathfrak{g} \end{tikzcd}$		$g_{ac}\,g^{cb} = \delta_a^b$
	$\begin{tikzcd} V \arrow[r, "\simeq"] \arrow[ddrr, "\mathrm{id}"'] & V\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes k^{-1}"] & V\otimes V\otimes V \arrow[d, "k\otimes\mathrm{id}"] \\ & & \mathbf{1}\otimes V \arrow[d, "\simeq"] \\ & & V \end{tikzcd}$		$k_{il}\,k^{lj} = \delta_i^j$
Metricity of Lie bracket	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "f\otimes\mathrm{id}"'] & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g"] \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "g"'] & \mathfrak{g} \end{tikzcd}$		$f_{ab}{}^{d}\,g_{dc} = f_{bc}{}^{d}\,g_{ad}$
Metricity of Lie action	$\begin{tikzcd} V\otimes\mathfrak{g}\otimes V \arrow[r, "\mathrm{id}\otimes\rho"] \arrow[d, "\rho\otimes\mathrm{id}"'] & V\otimes V \arrow[d, "k"] \\ V\otimes V \arrow[r, "k"'] & V \end{tikzcd}$		$\rho_a{}^{l}{}_{i}\,k_{lj} = \rho_a{}^{l}{}_{j}\,k_{li}$

Metric super Lie algebras appear. The relevance of tensor categories in (49) more general than that of plain vector spaces, is that by considering the tensor category of super vector spaces (e.g., [Va04, 3.1]), it immediately follows that metric representations of *super Lie algebras* [Kac77] or rather of *metric super Lie algebras* (as in [dMFMR09, 3.3]) are a source of weight systems on chord diagrams [Va94][FKV97][Vo11]; see [CDM11, 6.4]. Moreover, we observe that Deligne's theorem [De02] (see [Os04]) says that *all* reasonable tensor categories (satisfying just a mild set-theoretic size bound) are representation categories of algebraic super-groups, whence all reasonable Lie algebra weight systems on chord diagrams are induced by metric super Lie algebras, in general equivariant with respect to some super symmetry group. This means that the theory of weight systems on chord diagrams largely overlaps with that of metric representations of metric super Lie algebras. However, interestingly, weight systems see even one further datum, as we describe next.

3.4 On stacks of coincident strands

Stacks of coincident strands. We now consider horizontal chord diagrams $D \in \mathscr{D}^{\mathrm{pb}}$ that superficially have N_{f} strands as in (30), but where, on closer inspection, the ith strand is seen/resolved to consist of a stack of $N_{\mathrm{c},i}$ "coincident strands", for some tuple of natural numbers:

$$\vec{N}_{\mathrm{c}} = (N_{\mathrm{c},1}, \cdots, N_{\mathrm{c},N_{\mathrm{f}}}) \in \mathbb{N}^{N_{\mathrm{f}}} \quad \text{with} \quad N_{\mathrm{c}} := \sum_{i=1}^{N_{\mathrm{f}}} N_{\mathrm{c},i} \,. \tag{50}$$

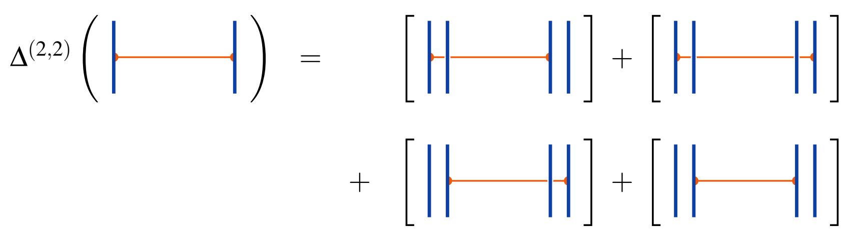
The following operation Δ (see [BN96, 2.2]) may be seen to make this idea precise:

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{pb}} \times \big(\bigoplus\limits_{\mathbb{N}} \mathbb{N} \big) & \xrightarrow{\qquad\qquad \Delta \qquad\qquad} & \mathscr{A}^{\mathrm{pb}} \\ \big(D, \vec{N}_{\mathrm{c}} \big) & \longmapsto & \underset{\substack{\text{chord diagram like } D \\ \text{but with stacks of} \\ \vec{N}_{\mathrm{c}} \text{ coincident strands}}}{\Delta^{\vec{N}_{\mathrm{c}}}(D)} := \left[\begin{array}{l} \text{Sum of horizontal chord diagrams} \\ \text{with } N_{\mathrm{c}} = N_{\mathrm{c}_1} + \cdots + N_{\mathrm{c},N_{\mathrm{f}}} \text{ strands} \\ \text{whose chords are the chords } t_{ij} \text{ of } D \\ \text{but re-attached in all } N_{\mathrm{c},i} \cdot N_{\mathrm{c},j} \text{ ways} \\ \text{to the } i\text{th and the } j\text{th stack of chords} \end{array} \right] \end{array} \tag{51}$$

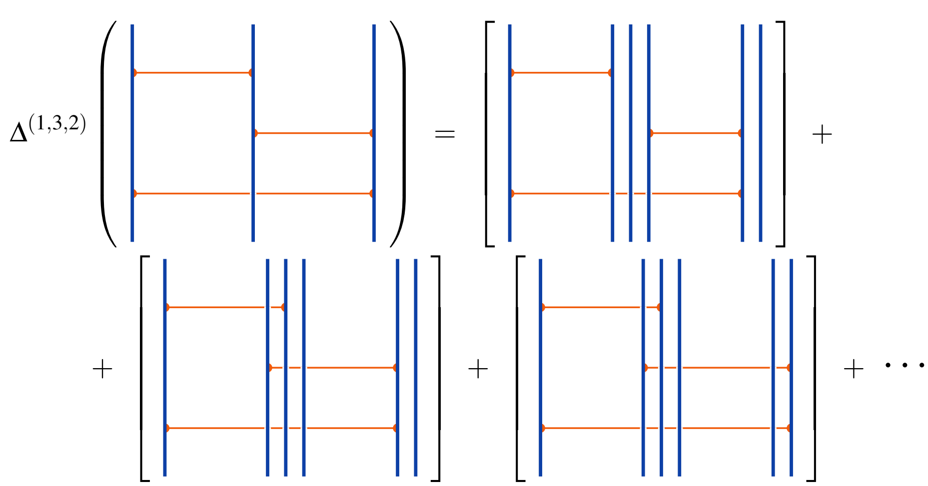
where now $\quad \mathscr{A}^{\mathrm{pb}} := \bigoplus\limits_{N_{\mathrm{f}} \in \mathbb{N}} \mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}, \qquad \mathscr{A}^{\mathrm{pb}} \xrightarrow{\;p_{N_{\mathrm{f}}}\;} \mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}} \xhookrightarrow{\;i_{N_{\mathrm{f}}}\;} \mathscr{A}^{\mathrm{pb}} \tag{52}$

denotes the direct sum of all spaces of horizontal chord diagrams (36) over the number N_{f} of strands.

For example:



More generic examples have many summands; for the one on the right we are showing just a few, for illustration:

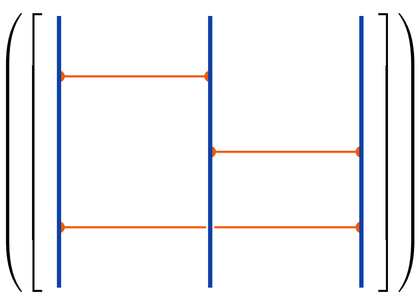


Lie algebra weight systems on horizontal chord diagrams with stacks of coincident strands. The construction Δ (51) of horizontal chord diagram with stacks of coincident strands passes from the plain set $\mathscr{D}^{\mathrm{pb}}$ of horizontal chord diagrams to a linear map on the vector space $\mathscr{A}^{\mathrm{pb}}$ in (52). This means that we obtain further weight systems (37) on horizontal chord diagrams by applying Lie algebra weight systems $w_{(V,\rho)}$ from (49) to a horizontal chord diagram D after "zooming in" to $\Delta^{\vec{N}_c}(\Delta)$, resolving their stacks of coincident strands:

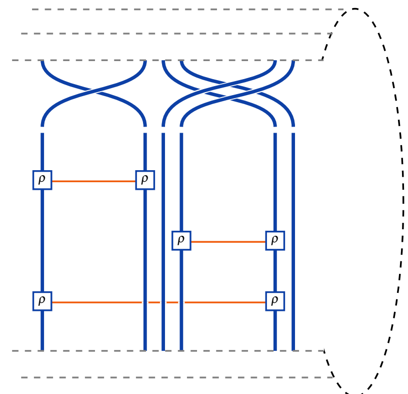
$$
\begin{array}{ccc}
\mathrm{Span}\Big(\big(\underbrace{\mathrm{MetLieMod}_{/\sim}}_{\text{Metric Lie representation}}\big) \times \underbrace{\big(\bigoplus_{\mathbb{N}} \mathbb{N}\big)}_{\text{Stacks of coincident strands}} \underset{\mathbb{N}}{\times} \underbrace{\big(\coprod_{N_{\mathrm{f}} \in \mathbb{N}} \mathrm{Sym}(N_{\mathrm{f}})\big)}_{\text{Winding monodromy}}\Big) & \xrightarrow[]{\overset{\text{Lie algebra weight systems on horizontal chord diagrams with stacks of coincident strands}}{\mathrm{tr}_{(-)} \circ w_{(-,-)} \circ \Delta^{(-)}(-)}} & \mathscr{W}^{\mathrm{pb}} \\
\big(\, \mathfrak{g} \otimes V \xrightarrow{\rho} V\,, \quad \vec{N}_{\mathrm{c}}\,, \quad \sigma \,\big) & \longmapsto & \Big(\underbrace{D}_{\text{Horizontal chord diagram}} \longmapsto \underbrace{\mathrm{tr}_{\sigma} \circ w_{(V,\rho)} \circ \Delta^{\vec{N}_{\mathrm{c}}}(D)}_{\sigma\text{-trace of value of } \rho \text{ Lie algebra weight system on } D\text{-shaped cord diagram with stacks of } \vec{N}_{\mathrm{c}} \text{ coincident strands}} \Big)
\end{array}
\tag{53}
$$

For example (see also (74)):

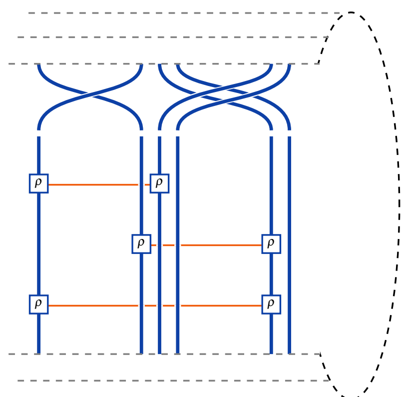
$$\mathrm{tr}_{(215634)} \circ w_{(\mathfrak{g} \otimes C \xrightarrow{\rho} C)} \circ \Delta^{(1,3,2)}$$



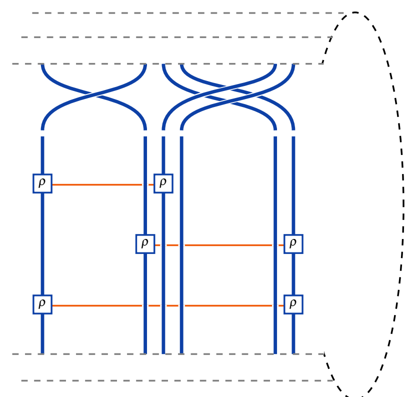
$=$



$+$



$+$



$+ \cdots$

Fundamental theorem on horizontal weight systems. With [BN96, Cor. 2.6] we now obtain:

Proposition 3.4. *All weight systems on horizontal chord diagrams (Def. 3.1) are Lie algebra weight systems with stacks of coincident strands* (53) *for (at least) any one of the following metric Lie algebras:*

$$\mathfrak{g} \in \big\{ \mathfrak{su}(2),\, \mathfrak{sl}(N, \mathbb{F}) \,\big|\, N \geq 2 \big\}\,,$$

in that for these Lie algebras the construction (53) *is surjective, hence in that on the quotient by its kernel it is a linear bijection. In particular,*

$$\mathrm{Span}\Big(\underbrace{\mathfrak{su}(2)\mathrm{MetMod}_{/\sim}}_{\mathfrak{su}(2)\text{-representations}} \times \underbrace{\big(\bigoplus_{\mathbb{N}} \mathbb{N}\big)}_{\text{Stacks of coincident strands}} \underset{\mathbb{N}}{\times} \underbrace{\big(\coprod_{N_{\mathrm{f}} \in \mathbb{N}} \mathrm{Sym}(N_{\mathrm{f}})\big)}_{\text{Winding monodromies}} \Big)_{/\sim} \xrightarrow[\mathrm{tr} \circ w \circ \Delta]{\text{bijection}} \underbrace{\mathscr{W}^{\mathrm{pb}}}_{\text{Weight systems on horizontal chord diagrams}}. \tag{54}$$

In summary: *The theory of weight systems on horizontal chord diagrams essentially is the theory of metric super Lie algebras and their metric representations, but enhanced by data of stacks of coincident strands.*

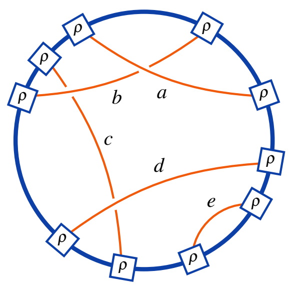
4 Weight systems as observables on intersecting branes

By the isomorphism (27), the higher observables (22) on the moduli space of D$p \perp$ D$(p+2)$-brane intersections, as described in diagram (2), are given by weight systems on horizontal chord diagrams, discussed in §3. Here we discuss how, under this interpretation, these weight systems from §3 turn out to capture various structures known from intersecting brane physics.

4.1 Lie algebra weight systems give matrix model single trace observables

Observation 4.1. By Prop. 3.3 and Prop 3.4 *all* weight systems (Def. 3.1, 3.2), on any of **(a)** horizontal chord diagrams (30), **(b)** round chord diagrams (39), and **(c)** Jacobi diagrams (40) evaluate, in the end, to a sum of circular string diagrams. The latter, in turn, by the rules of Penrose notation/string diagram calculus from §3.3, evaluate to a trace of a long product of matrices and summed over sets of pairs of matrices. For example the diagram on the left below evaluates to the trace expression shown on the right (with $(\rho_a \cdot \rho_b)^i{}_j = \rho_a{}^i{}_l \, \rho_b{}^l{}_j$ denoting the matrix product):

Typical value of a weight system in Penrose notation



$$= \; \mathrm{Tr}_V \big(\rho_a \cdot \rho_b \cdot \rho_c \cdot \rho^b \cdot \rho_d \cdot \rho^c \cdot \rho_a \cdot \rho^e \cdot \rho^d \big) \qquad (55)$$

Notice that the string diagram on the left may be, but *need not be*, the exact image of a round chord diagram of the same shape. In general it is the result a process of duplication and of reconnecting of strands, according to (53). However, the end result is always a sum over terms of this circular shape, hence is a sum of traces as on the right of (55). (This if the monodromy permutation in (53) has a single cycle, otherwise one gets traces along several connected circles, discussed in §4.7.)

Single trace observables subject to Wick's theorem are weight systems. Given a metric Lie representation $\mathfrak{g} \otimes V \xrightarrow{\rho} V$ as in §3.3, consider a quantum field or a random variable Z with values in $\mathfrak{g}$, hence with component expansion $Z = Z_a \rho^a$. A *single trace observable* in Z is an operator/random variable of the form

$$\mathscr{O} \; = \; \mathrm{Tr}\big(Z \cdot Z \cdot \cdots \cdot Z \big) \,. \qquad (56)$$

Assume then that the component fields Z_a are free quantum fields, or random variables of multivariate Gaussian distribution with covariance given by the metric k on V: $\langle Z_a Z_b \rangle = k_{ab}$. Then *Wick's theorem* says that the higher moments of Z are sums of contractions labelled by *linear chord diagrams*, as shown, by example, in the first two lines here:

$$\big\langle Z_a Z_b Z_c \, Z_d \big\rangle \;\underset{\text{Wick's theorem}}{=}\; k_{ab}\, k_{cd} \;+\; k_{ad}\, k_{bc} \;+\; k_{ac}\, k_{bd}$$

$$\big\langle Z_a Z_b Z_c Z_d \big\rangle \quad \rho^a \cdot \rho^b \cdot \rho^c \cdot \rho^d \;=\; \rho_a \cdot \rho^a \cdot \rho_b \cdot \rho^b \;+\; \rho_a \cdot \rho_b \cdot \rho^b \cdot \rho^a \;+\; \rho_a \cdot \rho_b \cdot \rho^a \cdot \rho^b$$

$$\underbrace{\Big\langle \mathrm{tr}\big(Z \cdot Z \cdot Z \cdot Z \big) \Big\rangle}_{\text{single trace observable}} \;=\; \big\langle Z_a Z_b Z_c Z_d \big\rangle \underbrace{\mathrm{tr}\big(\rho^a \cdot \rho^b \cdot \rho^c \cdot \rho^d \big)}_{\text{trace}} \;=\; \underbrace{2\, \mathrm{tr}\big(\rho_a \cdot \rho^a \cdot \rho_b \cdot \rho^b \big) \;+\; \mathrm{tr}\big(\rho_a \cdot \rho_b \cdot \rho^a \cdot \rho^b \big)}_{\text{Lie algebra weight system on round chord diagrams}} \qquad (57)$$

But then, as shown by example in the last line, the trace that defines the single trace observables closes up the resulting matrix product such that the terms that were previously controlled by linear chord diagrams are now labelled by round chord diagrams (39). Moreover, comparison with Observation 4.1 shows that the summands contributing to the expectation value of the single trace observables are exactly the values of Lie algebra weight systems on these round chord diagrams.

The SYK-model compactification of M5-branes. An observation along the lines of (57) (with emphasis on the appearance of chord diagrams, but without the identification of weight systems) was recently found to be crucial for the analysis of single trace observables in the SYK-model (review in [Ro18]) and analogous systems; see [GGJV18, Sec. 2.2][JV18, Sec. 4][BNS18, Sec. 2.1][BINT18, Sec. 2][Na19, 5-21].

Notce that from the point of view of string/M-theory, the SYK-model is the (near-)CFT which is the holographic dual to the full compactification of the M5-brane; see [LLL18, 4.1][BHT18].

The BMN matrix model and fuzzy sphere states. The *BFSS matrix model* famously is a (0+1)-dimensional super Yang-Mills theory which is thought to describe at least a sector of M-theory (but see [Mo14, [p. 43-44]) with un-wrapped M2-branes [NH98][DNP02], or equivalently strongly coupled type IIA string theory with stacks of unbound D0-branes [BFSS96], both on asymptotically Minkowski spacetime backgrounds (review in [Ba97][Ta01]). The *BMN matrix model* [BMN02, 5][DSJVR02], which is the KK-compactification over S^3 of $D = 4\mathcal{N} = 4$ super Yang-Mills theory [KKP03], generalizes this to asymptotically gravitational pp-wave backgrounds, which arise as Penrose limits of both the $\mathrm{Ads}_{4,7} \times S^{7,4}$ near horizon geometries of black M2-branes and M5-branes ([Bl04, 4.7]), and which deform the action functional of the BFSS model by a mass and a Chern-Simons term. These extra terms in the BMN model lift the notoriously problematic "flat directions" of the BFSS model ([dWLN89], see [BGR18]) thus leading to a well-defined quantum mechanics, which describes wrapped M2-branes (giant gravitons) or equivalently of $\mathrm{D}p \perp \mathrm{D}(p+2)$-brane bound states for $p = 0$ [Li04]:

The supersymmetric solutions are precisely [BMN02, (5.4)][DSJVR02, 4.2] those matrix configurations that constitute complex su(2)-representations $\mathfrak{su}(2) \otimes V \xrightarrow{\rho} V$, interpreted as systems of fuzzy 2-sphere geometries (discussed as such below in §4.2). This means by Observation 4.1 with (54) that:
The S^2-rotation invariant single-trace observables of the BMN matrix model are equivalently round chord diagrams D, evaluated on supersymmetric ground states $(\mathfrak{su}(2) \otimes V \xrightarrow{\rho} V)$ by the pairing them with the corresponding Lie algebra weight system (55). This generalizes to multi-trace observables, discussed in §4.7 below.

In view of this it may be worthwhile to briefly recall:

The general relevance of single trace observables in $\mathrm{AdS}/\mathrm{CFT}$**.** Single trace observables $\mathscr{O} = \mathrm{Tr}\big(Z \cdot Z \cdot \cdots \cdot Z\big)$ on the gauge theory side play a special role in the AdS/CFT correspondence. They map to single string excitations on the AdS side, in a way that identifies the string quite literally with the *string of characters* $Z \cdot Z \cdot Z \cdots$ in the expression of the single trace observables. An early account of the general mechanism is in [Po02], whose author already outlines the grand picture, indicating that space-time is gradually disappearing in the regions of large curvature, and the natural description is provided by a gauge theory in which the basic objects are the texts formed from the gauge-invariant words, and the theory provides us with the expectation values assigned to the various texts, words and sentences. The first concrete realization in $D = 4$, $\mathcal{N} = 4$ SYM is due to [BMN02], whose authors find that the "string of Zs" becomes the physical string and that each Z carries one unit of J which is one unit of momentum, and that locality along the worldsheet of the string comes from the fact that planar diagrams allow only contractions of neighboring operators. This led the authors to conclude that the Yang-Mills theory gives a string bit model where each bit is a Z operator. See also [GKP02] for similar arguments.

The correspondence between single trace operators in CFT and string excitations on AdS came to full fruition when it was realized that the single trace operators of a given length behave as integrable spin chains when the dilatation operator is regarded as their Hamiltonian. This led to the celebrated precision checks of $\mathrm{AdS}_5/\mathrm{CFT}_4$ starting with [BFST03], reviewed in [Bea10].

4.2 Lie algebra weight systems give fuzzy funnel observables

Fuzzy funnels of $\mathrm{D}p \perp \mathrm{D}(p+2)$ intersections. The configuration of N_c coincident $\mathrm{D}p$-branes ending on a $\mathrm{D}(p+2)$-brane is famously a noncommutative "fuzzy funnel" geometry [Di97][CMT99][My01, 4][GW08, 3.4.3], where the three $\mathfrak{su}(N_\mathrm{c})$-valued scalar fields $\{X^1, X^2, X^3\}$ on the $\mathrm{D}p$-branes solve Nahm's equation as

$$X^a(y) \;=\; \frac{1}{y}\frac{2}{\sqrt{N_\mathrm{c}^2 - 1}}\,\rho^a\,,$$

for y the transversal distance from the $\mathrm{D}(p+2)$-brane and $\{\rho_a\}$ the matrices of the N_c-dimensional representation of $\mathfrak{su}(2)$. Due to the Casimir relation

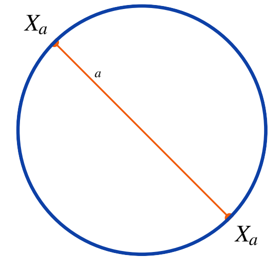
$$X_a \cdot X^a \;=\; \tfrac{1}{y^2}\,1_{N_\mathrm{c}\times N_\mathrm{c}}$$

this means that at fixed distance y the algebra of functions generated by the scalar fields is that on the fuzzy 2-sphere $S^2_{N_\mathrm{c}}$ [Mad92] of radius $R = 1/y$.

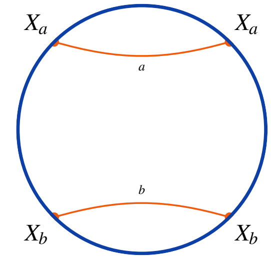
Shape observables on fuzzy 2-spheres. The fuzziness of the fuzzy 2-sphere $S^2_{N_\mathrm{c}}$ is reflected in the fact that functions of its radius R are not all constant, due to ordering ambiguity in the observables of the schematic form "R^{2k}". After averaging/integration over the fuzzy 2-sphere, hence under the trace operation, the remaining ordering ambiguities are fully reflected by round chord diagrams, as shown on the right by the first few examples. Hence these shape observables on the fuzzy 2-sphere are equivalently the values of $\mathfrak{su}(2)$-weight systems on round chord diagrams, as in §3.3, see Prop. 3.4. In fact, these fuzzy shape observables are instances of single trace observables as in §4.1.

$1/N_{\mathrm{N}_c}$-Corrections to the $\mathrm{D}p \perp \mathrm{D}(p+2)$-systems. In the large N_c limit the fuzzy 2-sphere $S^2_{N_\mathrm{c}}$ approaches the ordinary 2-sphere, and its fuzzy shape observables all converge to unity. This large N_c limit of the $\mathrm{D}p \perp \mathrm{D}(p+2)$-intersections had been studied in [CMT99]. But discussion of small N_c corrections, or even of the full matrix model mechanics of $\mathrm{D}p \perp \mathrm{D}(p+2)$-intersections requires handling the multitude of fuzzy shape observables as shown on the right. That and how these computations are crucially organized by round chord diagrams was noticed in [RST04, Sec. 3.2] for review see [MPRS06, A][MN06, 4][Pa06, p. 161-162].

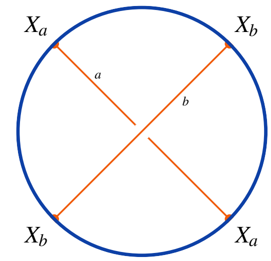
$$\int_{S^2_N} (R^2) = \tfrac{4\pi}{\sqrt{N^2-1}}\mathrm{Tr}\big(X_a \cdot X^a\big)$$



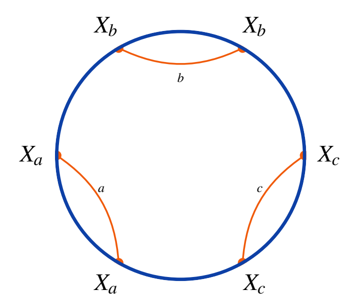
$$\int_{S^2_N} (R^2)^2 = \tfrac{4\pi}{\sqrt{N^2-1}}\mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X^b\big)$$



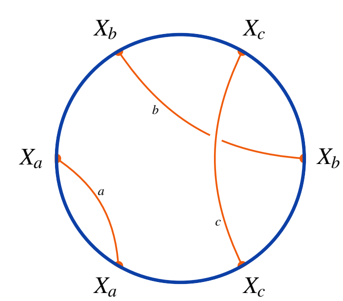
$$\int_{S^2_N} (R^2)^2 = \tfrac{4\pi}{\sqrt{N^2-1}}\mathrm{Tr}\big(X_a \cdot X_b \cdot X^a \cdot X^b\big)$$



$$\int_{S^2_N} (R^2)^3 = \tfrac{4\pi}{\sqrt{N^2-1}}\mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X^b \cdot X_c \cdot X^c\big)$$



$$\int_{S^2_N} (R^2)^3 = \tfrac{4\pi}{\sqrt{N^2-1}}\mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c\big)$$

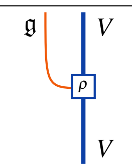
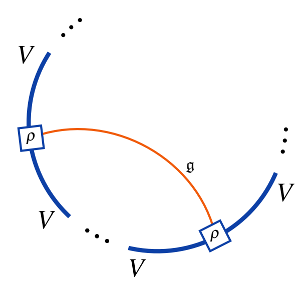
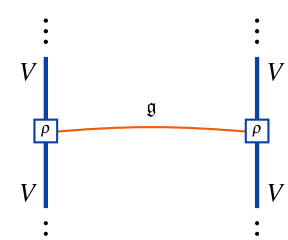


$\vdots$ $\vdots$

4.3 Lie algebra weight systems encode M2-brane 3-algebras

The value of a weight system on a single chord. We may observe that the value of a Lie algebra weight system (49), and hence (Prop. 3.3) of any weight system, on a chord diagram is a contraction of many copies of the one tensor in 4 copies of the representation space V that it assigns to any single chord. This assignment is shown in the three columns on the right of the following diagram, according to the rules discussed in §3.3 and for two choices of dualizing the 4 copies of V, as it generically appears for round chord diagrams (39) and for horizontal chord diagrams (30), respectively:

$$\text{M2-brane 3-algebra induced by} \left\{ \begin{array}{lccl} \text{Lie representation} & V & \in & \mathscr{C} \\ \text{in tensor category} & \mathscr{C} & \in & \mathrm{TensorCat} \\ \text{with Lie action} & \rho & : & \mathfrak{g} \otimes V \to V \\ \text{by Lie algebra} & \mathfrak{g} & \in & \mathscr{C} \\ \text{with Lie bracket} & f & : & \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \\ \text{and compatible} & g & : & \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{1} \\ \text{metrics} & \mathrm{k} & : & V \otimes V \to \mathbf{1} \end{array} \right.$$

Data of M2-brane 3-algebra	Category notation	Penrose notation	Index notation
Lie action	$\mathfrak{g} \otimes V \xrightarrow{\rho} V$		$\rho_a{}^i{}_j$
3-bracket	$V \otimes V \otimes V \xrightarrow{[-,-,-]} V$		$\rho_a{}^m{}_l \, \rho^a{}_{ji}$
Faulkner construction	$V \otimes V \xrightarrow{\Omega} V \otimes V$		$\rho_a{}^m{}_l \, \rho^{aj}{}_i$

(58)

M2-brane 3-algebras. Direct inspection shows that the tensor assigned by a Lie algebra weight system $w_{(\mathfrak{g} \otimes V \xrightarrow{\rho} V)}$ to a single chord is exactly the tensor considered in [dMFMR09, above Prop. 10 & (22)] (for the first case in the above table) or in [dMFMR09, (34)] (for the second case). By [dMFMR09, Prop. 10] these tensors are the 3-brackets constituting generalized M2-brane 3-algebras [BL06][CS08, 4][BLMP12, 3] and the Faulkner construction [Fa73], respectively, as shown above. In fact, by [dMFMR09, Theorem 11] this construction constitutes a bijective equivalence between (generalized real) M2-brane 3-algebras and metric Lie representations. Hence all M2-brane 3-algebras arise from Lie algebra weight systems on chord diagrams this way. The fact that fuzzy funnels from §4.2 and metric 3-algebras arise from the same underlying mathematical construction $\left(\mathscr{W}^{c}\right)^{\bullet}$ (see (43) and the big diagram in the Introduction) clarifies the relation between the two and highlights its origin.

4.4 Round weight systems are 3d gravity observables

We have seen in §4.1 and §4.2 that weight systems on round chord diagrams have the form of observables on (fully compactified) worldvolume theories of branes, where the circle in the chord diagram is what represents the worldvolume. Here we observe that generating functions of weight systems dually encode Chern-Simons amplitudes that may be thought of as propagating in a bulk spacetime away from these brane worldvolumes.

Write $(\mathscr{G}^\bullet, d_{\mathscr{G}})$ for Kontsevich's graph complex in its version with framed Wilson loops [Kon93, 5][AF96, 2][CCRL02, 7]. This is spanned by the graded set of Jacobi diagrams (40) together with analogous sets of graphs of valence higher than 3; the differential acts by sending any graph to the signed formal linear combination of the results of contracting any edge that is not a chord. We may then regard any Jacobi diagram (40) both as an element of $\mathscr{A} := \mathscr{A}^{\mathrm{t}} \simeq \mathscr{A}^{\mathrm{c}}$ (46) as before, or as an element of the graph complex, as shown on the right.

Graded set of Feynman diagrams for Chern-Simons theory with a Wilson loop

$$
\begin{tikzcd}
 & \mathscr{D}^{\mathrm{t}} \arrow[dl, "{[-]_{\mathscr{A}}}"'] \arrow[dr, "{[-]_{\mathscr{G}}}"] & \\
\mathscr{A}_\bullet & & \mathscr{G}^\bullet
\end{tikzcd}
$$

send graph to the respective vector that it represents

Graded linear span of Jacobi diagrams modulo STU-relations

Kontsevich graph complex for framed knots

Using [AF96, Thm. 1] we find that the sum over Jacobi diagrams of the tensor product of these two of their vector incarnations is a graph cocycle with coefficients in Jacobi diagrams in the linear dual of round weight systems (43)

$$
\begin{aligned}
\underbrace{\Big[(w,\mathscr{K}) \mapsto \big\langle \mathrm{Tr}_w \mathrm{Pexp}\big(\textstyle\int_{\mathscr{K}} A\big)\big\rangle \Big]}_{\text{Universal Wilson loop observable}} \;:=\; \sum_{n\in\mathbb{N}} \hbar^n \sum_{\Gamma \in (\mathscr{D}^{\mathrm{t}})_n} \Big(\tfrac{1}{|\mathrm{Aut}(\Gamma)|} [\Gamma]_{\mathscr{A}} \otimes [\Gamma]_{\mathscr{G}} \Big) \;&\in\; \overbrace{H^0\big(\mathscr{A}_\bullet \otimes \mathscr{G}^\bullet\big)}^{\substack{\text{0-cohomology of}\\ \text{graph complex with}\\ \text{values in Jacobi diagrams}}} \\
&\simeq\; \underbrace{\mathrm{Hom}\big(\mathscr{W}^\bullet, H^\bullet(\mathscr{G})\big)}_{\substack{\text{Graded linear maps from}\\ \text{weight systems on chord diagrams}\\ \text{to graph cohomology}}}
\end{aligned}
\tag{59}
$$

which, dually, is a graded linear map, as shown on the right. Since (59) is a *universal Vassiliev invariant* [AF96, Thm. 1] (following [Kon93, Thm. 2.3][Bar95b, 4.4.2], reviewed in [CDM11, 8.8][JM19, 18]) this map in fact identifies weight systems on round chord diagrams (47) with the space of Vassiliev knot invariants [Va92] via identification (see [CCRL02, Prop. 7.6 using Thm. 7.3]) with the graph cohomology spanned by trivalent graphs

$$
\mathscr{W}^\bullet \xrightarrow{\simeq} H^\bullet(\mathscr{G})_{\curlywedge} \subset H^\bullet(\mathscr{G}) .
$$

Dual Chern-Simons theory appears. But the graph complex is what organizes Feynman diagrams for perturbative Chern-Simons theory in the presence of a framed Wilson loop [Bar91][Bar95a]. The construction [Kon92, p. 11-12][AS93][AF96, 3] of Feynman amplitudes as configuration space fiber integrals of wedge products of Chern-Simons propagators, regarded as differential forms on the configuration space of points, (see Def. 2.1) sends graph cocycles to the higher observables of Chern-Simons theory with a Wilson loop (reviewed in [Vo13, 3-4]). This is given by evaluating the corresponding Lie algebra weight system (53) (restricted to round chord diagrams via (47))

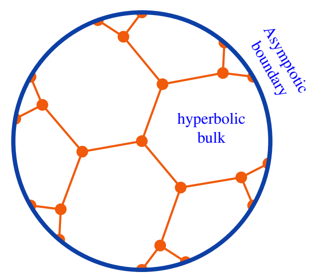
$$
\mathscr{L} \;:=\; \overbrace{\mathrm{Span}\big(\mathrm{MetLieMod}_{/\sim}\big)}^{\text{Metric Lie representations}} \xrightarrow{\text{Lie algebra weight systems}} \overbrace{\prod_{n\in\mathbb{N}} \underbrace{\big(\mathscr{W}^n\langle \hbar^n\rangle\big)}_{\substack{\text{Round}\\ \text{degree } n\\ \text{weight systems}}}}^{\substack{\text{Chordal}\\ \text{Vassiliev invariants}}}
\tag{60}
$$

on the universal Wilson loop observable (59), thus multiplying the bare Chern-Simons amplitude with the Lie algebraic contraction and tracing of the gauge indices. Here on the right of (60) we recognize the space of generating functions of round weight systems (43) as that of Vassiliev knot invariants, via the *fundamental theorem of Vassiliev invariants* [Kon93, Thm. 2.3][Bar95b, Thm. 1 (3)] (reviewed in [CDM11, 8.8][JM19, 18]).

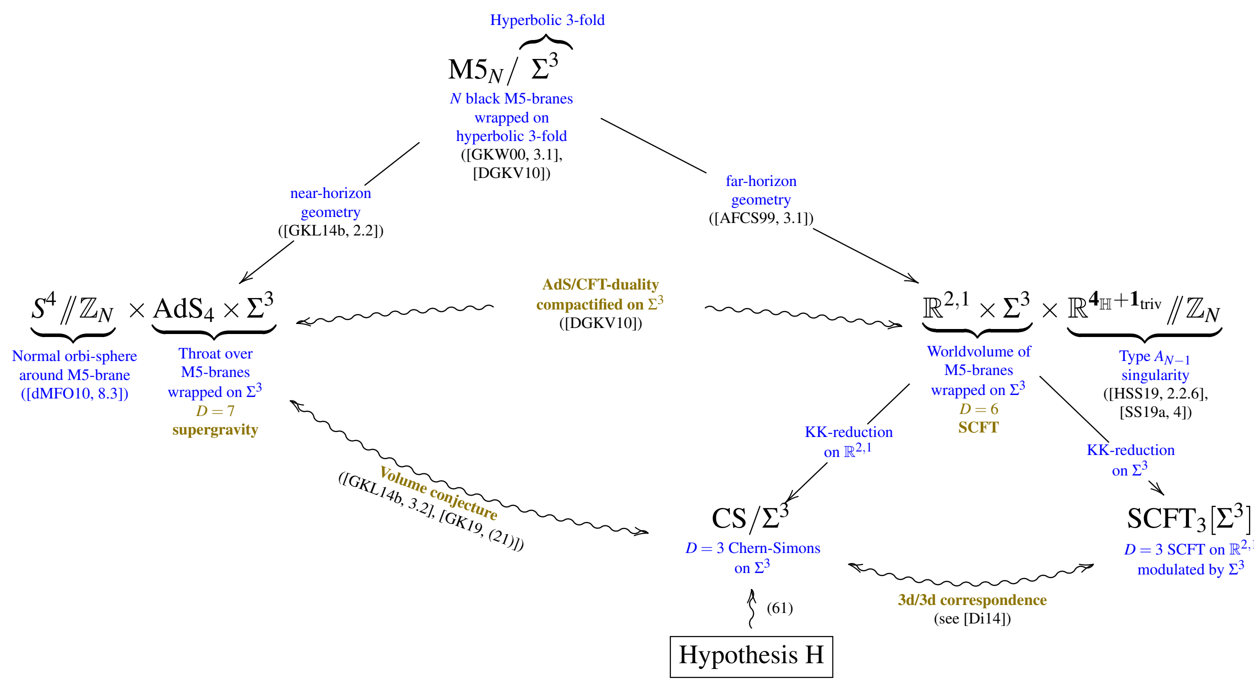
In summary, we thus find that, via the Chern-Simons Wilson loop observable, the generating functions of weight systems on round chord diagrams are equivalently the perturbative quantum observables of Chern-Simons theory with a Wilson loop knot $\mathscr{K}$:

$$
\begin{tikzcd}[column sep=large]
\underset{\text{Metric Lie representations}}{\mathscr{L}} \arrow[r] \arrow[d, phantom, "\text{Lie algebra weight systems}"] & \overset{\text{Chordal Vassiliev invariants}}{\prod_n \Big(\underbrace{\mathscr{W}^n}_{\substack{\text{Weight}\\ \text{systems}}} \langle \hbar^n \rangle \Big)} \arrow[r, "\simeq", "{\big[(w,\mathscr{K}) \mapsto \langle \mathrm{Tr}_w \mathrm{Pexp}(\int_{\mathscr{K}} A) \rangle\big]}"'] & \overset{\text{Graphical Vassiliev invariants}}{\prod_n \Big(\underbrace{H^n(\mathscr{G})}_{\substack{\text{Graph}\\ \text{cohomology}}} \langle \hbar^n \rangle \Big)} \arrow[r, hook, "\substack{\text{Vassiliev}\\ \text{knot invariants}}", "\substack{\text{Feynman rules for}\\ \text{Chern-Simons theory}}"'] & \overset{\text{Knot invariants}}{H^0\big(\mathrm{Emb}(S^1, \mathbb{R}^3)\big)[[\hbar]]} \\
(\mathfrak{g} \otimes V \xrightarrow{\rho} V) \arrow[r, mapsto] & w_V \arrow[rr, mapsto, "\text{Universal Wilson loop observable}"] & & \overset{\substack{V\text{-Wilson loop observable}\\ \text{in } \mathfrak{g}\text{-Chern-Simons theory}\\ \text{evaluated on knot } \mathscr{K}}}{\Big(\mathscr{K} \mapsto \big\langle \mathrm{Tr}_V \mathrm{Pexp}\big(\textstyle\int_{\mathscr{K}} A\big) \big\rangle_{\mathrm{CS}} \Big)}
\end{tikzcd}
\tag{61}
$$

BTZ black holes appear. In the case when $\mathscr{K} = \bigcirc$ in (61) is the unknot, its knot complement – regarded as as a hyperbolic space of *infinite* volume – is (see [Gu05, Appendix A]) the Euclidean BTZ black hole in AdS_3 [Kr00][Kr01]. We thus find the Chern-Simons sector inside $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality [GMMS04][Je10][Ke14][KL06][Kr06][GKL14a] where the Wilson loop observables (59) measure black hole entropy [ACI13][BBGR14][BR15] [HMS19][DHK19][MR19]. For the SYK-model, such chord diagram holography had been tentatively envisioned in [BINT18, p.5][Na19, p.23]. We highlight that the assignment (59) thus leads to the emergence of holography by *Hypothesis H*. It is then natural to propose our setting for studying holography.



Holographic wrapped 5-branes appear. Now consider instead the case that the knot $\mathscr{K}$ in (61) is a hyperbolic knot [FKO17], hence such that its complement $S^3 \setminus \mathscr{K}$ carries the structure of a hyperbolic space with *finite* volume, then necessarily unique, by Mostow rigidity [Mo68] (reviewed in [Bo18]). In this case the *volume conjecture* asserts [Ka96][MM01] (reviewed in [Mu10]) that the Wilson loop observables (61) for the N-dimensional complex representation of $\mathfrak{su}(2)$ tends in the large N limit, $N \to \infty$, to that finite volume. Moreover, the *3d-3d correspondence* (see the review [Di14]) asserts that the Wilson loop observables (61) are dually observables on the worldvolume theory of M5-branes wrapped on $\Sigma^3 := S^3 \setminus \mathscr{K}$. Furthermore, with this identification the statement of the volume conjecture is part of the statement of holographic AdS/CFT duality for such configurations [GKL14b, 3.2] (see also [BGL16]). We thus have the following web of relations connecting to Hypothesis H:



4.5 Round weight systems contain supersymetric indices

We observe here that round weight systems encode the Witten indices of $D = 3, \mathcal{N} = 4$ super Yang-Mills theories, computing the $\widehat{A}$-genus of Coulomb branches of intersecing branes given by Atiyah-Hitchin moduli space of Yang-Mills monopoles.

Coulomb branches of $D = 3, \mathcal{N} = 3$ SYM and monopole moduli. The worldvolume gauge theory of D$p \perp$ D$(p+2)$-brane intersections is thought to be $D = 3, \mathcal{N} = 4$ super Yang-Mills theory, at least for $p = 3$ [HW97]. The moduli spaces of vacua of $D = 3$ $\mathcal{N} = 4$ super Yang-Mills theory, both the Coulomb branches and the Higgs branches, are hyperkähler manifolds $\mathscr{M}^{4n}$ [SW96] (see, e.g., [dBHOO97]), which are either

(1) asymptotically flat (ALE-spaces) and dual to branes transversal to ADE-singularities;

(2) or compact and dual to branes transversal to a K3 surfaces or to a 4-torus $\mathbb{T}^4$.

Specifically, the (classical) Coulomb branches of these theories are the Atiyah-Hitchin moduli spaces of Yang-Mills monopoles [AH16] on the transversal space [DKMTV97][To99][BDG15], which are often identified with Hilbert schemes of points [dBHOO97][dBHOO97][CHZ14, (4.4)].

In particular, if the transversal space is a K3 surface Σ^4_{K3}, then the corresponding moduli space is the Hilbert scheme of points $\mathscr{M}^{4n} = (\Sigma^4_{\mathrm{K3}})^{[n]}$ [VW94][Va96], which is an example of a *compact* hyperkähler manifold. In fact, all known examples of compact hyperkähler manifolds are Hilbert schemes either of K3 surfaces or of the 4-torus [Be83], with two exceptional variants found in [O'G98][O'G00] (reviewed in [Sa04, 5.3]). These compact Coulomb branches come from $D = 3, \mathcal{N} = 4$ SYM theories that are obtained by KK-compactification of little string theories [In99].

Rozansky-Witten theory. The topological C-twist of $D = 3$ $\mathcal{N} = 4$ SYM is Rozansky-Witten theory [RzW97], which, after gauge fixing and suitable field identifications, turns out to have same Feynman rules as 3d Chern-Simons theory. This is in the sense that the only relevant propagator is the Chern-Simons propagator, and the only relevant Feynman diagrams are trivalent, the only difference being that the Lie algebra weights of Chern-Simons theory are replaced by *Rozansky-Witten weights* [RzW97, 3][RW06] which depend (only) on the hyperkähler moduli space $\mathscr{M}^{4n}$, and in fact only on its underlying holomorphic symplectic manifold [Ka99].

Hence the assignment of Rozansky-Witten weights is a linear map from the linear span of the set of isomorphism classes of such gauge theories

$$\mathscr{G} := \overset{D=3,\,\mathcal{N}=4\text{ gauge theories}}{\mathrm{Span}\big(\mathrm{SYM}^{D=3,\mathcal{N}=4}_{/\sim}\big)} \xrightarrow{\text{Rozansky-Witten weight systems}} \overset{\substack{\text{Chordal}\\ \text{Vassiliev invariants}}}{\underset{\substack{\text{Round}\\ \text{degree } n\\ \text{weight systems}}}{\prod_{n \in \mathbb{N}} \big(\mathscr{W}^n \langle \hbar^n \rangle\big)}} \tag{62}$$

directly analogous to the assignment of Lie algebra weight systems (60). Furthermore, the Wilson loop observables of Rozansky-Witten C-twisted $D = 3, \mathcal{N} = 4$ super Yang-Mills theory are obtained by evaluating these weights on the universal Vassiliev Wilson loop observable, in direct analogy to the Wilson loop observables (61) of Chern-Simons theory:

$$\begin{array}{ccccccc}
\overset{\substack{D=3,\,\mathcal{N}=4\\ \text{gauge theories}}}{\mathscr{G}} & \xrightarrow{} & \overset{\substack{\text{Chordal}\\ \text{Vassiliev invariants}}}{\prod_n \Big(\underset{\substack{\text{Weight}\\ \text{systems}}}{\mathscr{W}^n} \langle \hbar^n \rangle \Big)} & \underset{\text{Universal Wilson loop observable}}{\xrightarrow[{[(w,\mathscr{K}) \mapsto \langle \mathrm{Tr}_w \mathrm{Pexp}(\int_{\mathscr{K}} A)\rangle]}]{\simeq}} & \overset{\substack{\text{Graphical}\\ \text{Vassiliev invariants}}}{\prod_n \Big(\underset{\substack{\text{Graph}\\ \text{cohomology}}}{H^n(\mathscr{G})} \langle \hbar^n \rangle \Big)} & \underset{\substack{\text{Feynman rules for}\\ \text{RW-twisted SYM theory}}}{\overset{\substack{\text{Vassiliev}\\ \text{knot invariants}}}{\hookrightarrow}} & \overset{\text{Knot invariants}}{H^0\big(\mathrm{Emb}(S^1, \mathbb{R}^3)\big)[[\hbar]]} \\
\text{Rozansky-Witten weight systems} & & & & & & \\
T & \longmapsto & w_T & & \longmapsto & & \underset{\substack{\text{Wilson loop observable}\\ \text{in RW-twisted SYM theory } T\\ \text{evaluated on knot } \mathscr{K}}}{\Big(\mathscr{K} \mapsto \big\langle \mathrm{TrPexp}\big(\int_{\mathscr{K}} (\Gamma + \Omega)\big) \big\rangle_T \Big)}.
\end{array} \tag{63}$$

The index of $D=3$, $\mathscr{N}=4$ SYM. In the case that the knot $\mathscr{K}=\bigcirc$ in (63) is the unknot, the Rozansky-Witten Wilson loop observable (63) computes the square root of the $\widehat{A}$-genus of the moduli space $\mathscr{M}^4 n_T$ of the given C-twisted $D=3, \mathscr{N}=4$ SYM theory T ([RW06, Lem. 8.6], using the wheeling theorem [BNTT03] and Hitchin-Sawon theorem [HS99]):

$$\left\langle \mathrm{Tr}\left(\mathrm{P}\exp\left(\int_{\bigcirc}(\Gamma+\Omega)\right)\right)\right\rangle_T = \sqrt{\widehat{A}(\mathscr{M}_T^{4n})}\,.$$

This genus is part of the expression of the Witten index of the theory T [BFK18].

Observation 4.2 (**Dualities).** From the point of view of *Hypothesis H*, the genuine observables on the brane configurations are the abstract weight systems in $\prod\limits_{n\in\mathbb{N}}\big(\mathscr{W}^n\langle\hbar^n\rangle\big)$, by Prop. 2.16. One may then ask which physics is compatible with these observables, much like one asks which target space geometry emerges from a given worldsheet CFT. We saw in §4.4 and §4.5 that a range of quantum field theories has these weight systems as their observables, including Chern-Simons theories and Rozansky-Witten C-twisted 3d super Yang-Mills theories. These are reflected by canonical maps (60) and (62) from the spaces of these theories into the space of observables:

$$\begin{tikzcd}[column sep=huge]
\underset{\text{field theories}}{\overset{\text{Chern-Simons \& Rozansky-Witten}}{\mathscr{L}\oplus\mathscr{G}}} \arrow[r, "\text{Extract observables}", "\underset{(60)}{\text{Lie algebra weight systems}}\ \oplus\ \underset{(62)}{\text{Rozansky-Witten weight systems}}"'] & \overset{\text{Chordal Vassiliev invariants}}{\prod\limits_{n\in\mathbb{N}}\big(\mathscr{W}^n\langle\hbar^n\rangle\big)} \arrow[r, hook, "\underset{(47)}{\mathrm{comp}^*}"'] & \overset{\text{Higher observables on D6-D8-brane intersections by } \textit{Hypothesis H}}{\prod\limits_{n\in\mathbb{N}}\Big(\big(\mathscr{W}^{\mathrm{pb}}\big)^n\langle\hbar^n\rangle\Big)}.
\end{tikzcd}$$

But this operation of extracting observables from field theories has a large kernel, equivalently a non-trivial fiber product

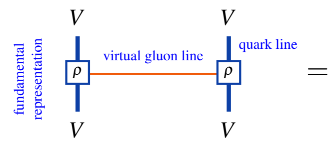
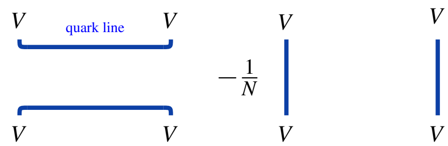
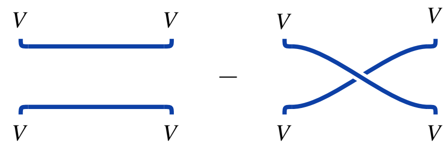
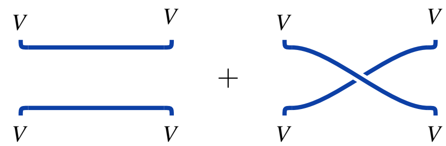
$$\begin{tikzcd}
& \boxed{\text{Dual pairs}} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{(pb)}"] & \\
\text{Field theories}\ \mathscr{L}\oplus\mathscr{G} \arrow[dr, "\text{extract observables}"'] & & \mathscr{L}\oplus\mathscr{G}\ \text{Field theories} \arrow[dl, "\text{extract observables}"] \\
& \underset{\text{Space of higher observables by } \textit{Hypothesis H}}{\prod\limits_{n\in\mathbb{N}}\big(\mathscr{W}^n\langle\hbar^n\rangle\big)} &
\end{tikzcd} \tag{64}$$

corresponding to different gauge theories which have *indistinguishable observables*, hence which are physical duals. We thus see that *Hypothesis H*, not only sees the genuine observables on the brane configurations as the abstract weight systems but also encodes duality in the corresponding field theories in a compatible manner.

4.6 Round weight systems encode 't Hooft string amplitudes

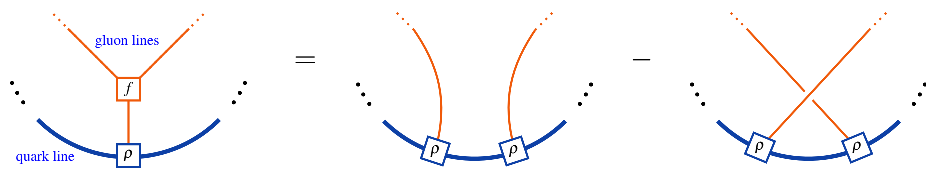
We have seen in (46) that round chord diagrams modulo 4T relations are equivalently Jacobi diagrams (40) modulo STU-relations, and that weight systems (47) exhibit the latter as the Feynman diagrams of Chern-Simons theory (§4.4) and of Rozansky-Witten theory (§4.5). We now observe that Lie algebra weight systems (§3.3) also know about the 't Hooft double line reformulation ['tH74] of these Feynman diagrams as well as about the resulting identification of Chern-Simons amplitudes with topological open string amplitudes [Wi92] (reviewed in [Mar04]).

't Hooft double line notation. One observes that Lie algebra weight systems (60) for $\mathfrak{g}$ a semisimple Lie algebra and V its fundamental representation, evaluate a single chord (58) to a linear combination of a *double line* of strands, in terms of the Penrose notation from §3.3, as shown in the following table:

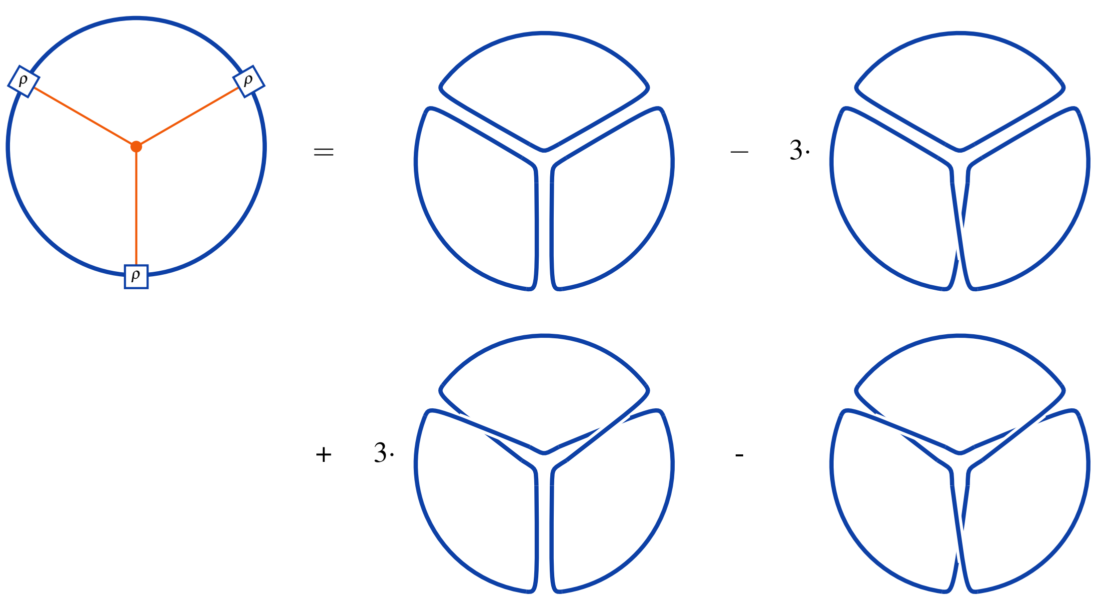
Metric Lie algebra $\mathfrak{g}$	**Metric contraction of fundamental action tensors**
	
$\mathfrak{su}(N)$	
$\mathfrak{so}(N)$	
$\mathfrak{sp}(N)$	

(65)

Applying this iteratively on the right hand side of the Jacobi identity/Lie action property (3.3)



identifies the corresponding Lie algebra weight of any Jacobi/Feynman diagram with that of a linear combiantion of purely *double line diagrams* where, in Feynman diagram language, all virtual gluon lines have turned into double quark lines. For example:



(66)

In the context of gauge theory this was famously observed in ['tH74] for $\mathfrak{g} = \mathfrak{u}(N)$, in which case only the first summands in (65) and (66) appear; the generalization to arbitrary semisimple Lie algebras was observed in [Cv76,

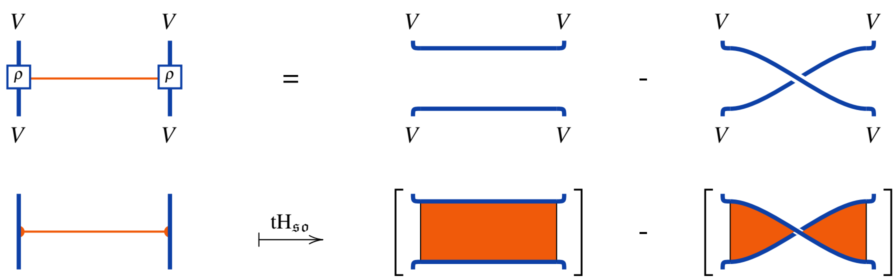
Figure 14] and partially again in [Ci82]. Later [Bar95b, 6.3] reconsidered this, apparently independently, in the general context of Lie algebra weight systems, which is reviewed in [CDM11, 6.2.6]. The case $\mathfrak{g} = \mathrm{sgl}(N)$ was considered in [BN96, 2.2][CDM11, 6.2.5] and the case $\mathfrak{g} = \mathrm{sl}(N)$ in [CDM11, 6.2.3][JM19, 14.4].

We focus on the case $\mathfrak{g} = \mathfrak{so}(N)$.

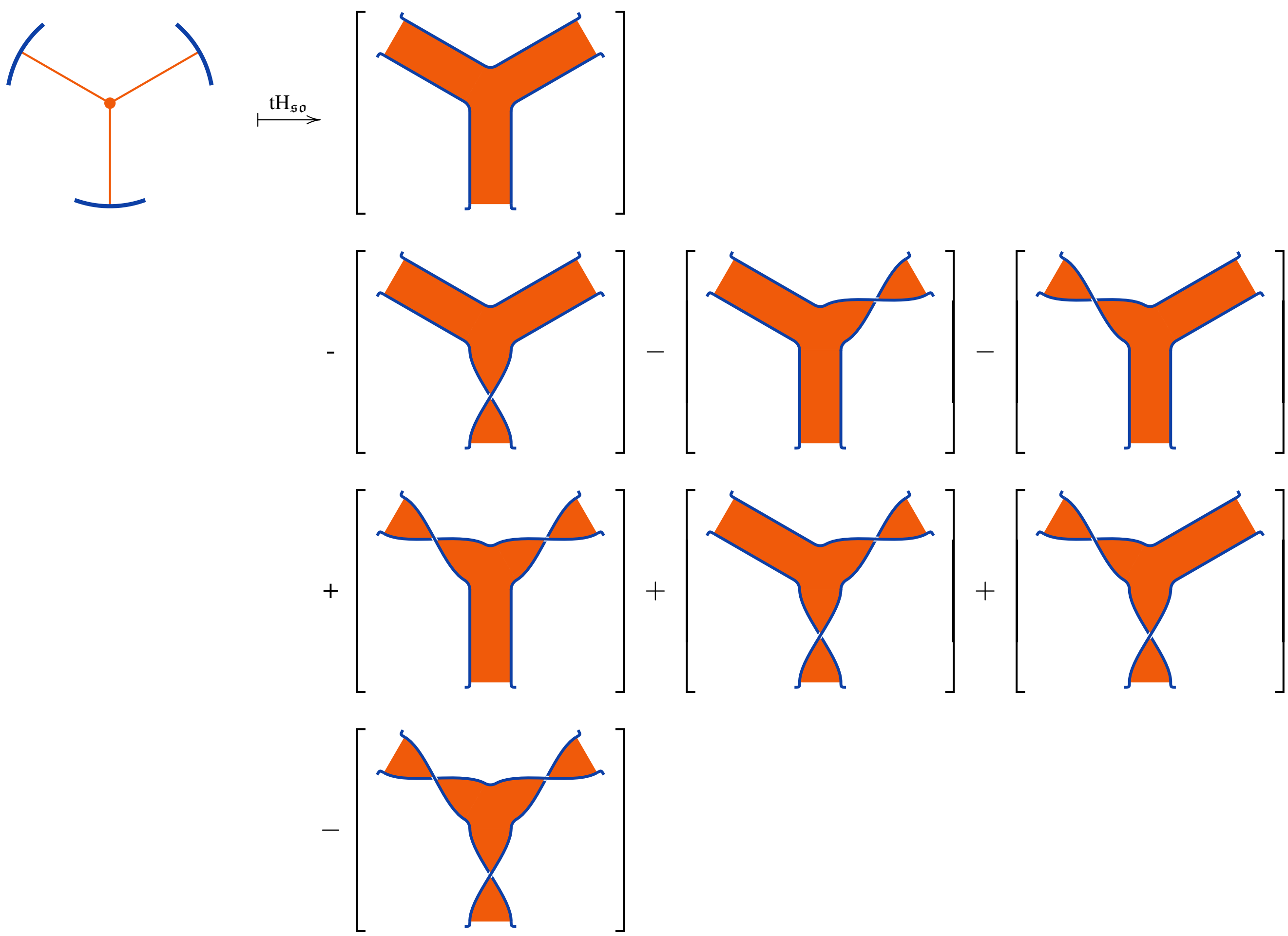
Emerging string worldsheets. The 't Hooft double line construction (66) exhibits each Jacobi/Feynman diagram as a linear combination of ribbon graphs ("fatgraphs"), underlying which are isomorphism classes of surfaces with marked boundaries (see [Bar95b, Def. 1.12]). This defines a linear function

$$\overset{\substack{\text{Linear combinations of}\\ \text{CS/RZ Feynman diagrams}\\ \text{(Jacobi diagrams)}}}{\mathrm{Span}\big(\mathscr{D}^{\mathrm{t}}\big)} \xrightarrow{\overset{\text{'t Hooft construction}}{\mathrm{tH}_{\mathfrak{g}}}} \overset{\substack{\text{Linear combinations of}\\ \text{topological open string worldsheets}\\ \text{(marked surfaces)}}}{\mathrm{Span}\big(\mathrm{MarkedSurfaces}_{/\sim}\big)} \tag{67}$$

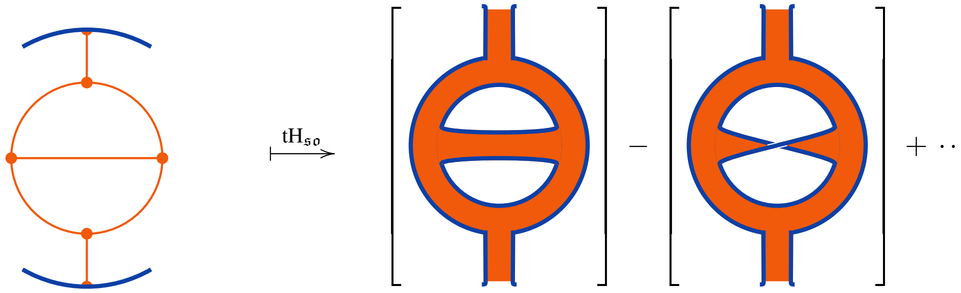
from the linear span of the set (40) of Jacobi diagrams to the linear span of the set of isomorphism classes of marked surfaces. Specifically for $\mathfrak{g} = \mathfrak{so}(N)$ this function is given on single chords (58) by



and on single internal vertices by



For example:



(68)

This is the generalization to unoriented open string worldsheets of the ’t Hooft construction for Chern-Simons theory as an open topological string theory [Wi92, Figures 1 & 2].[5]

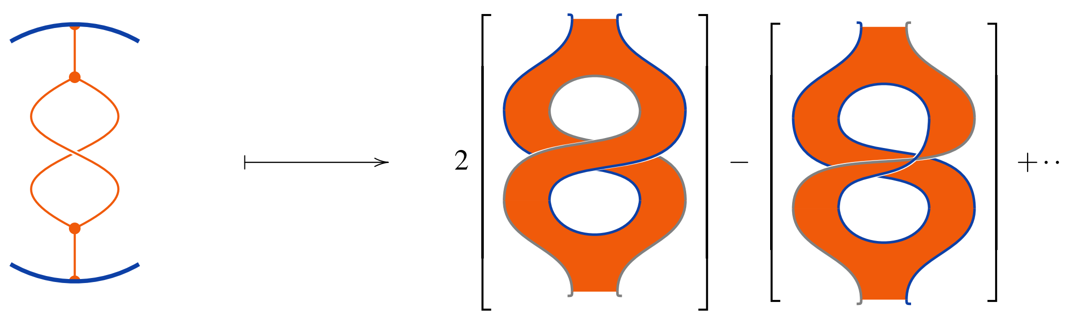
Chern-Simons observables as topological string amplitudes. We now observe that the open topological string worldsheets as in (68), given by the ’t Hooft construction (67), are reflected by the higher observables (Prop. 2.18) in that the image of *stringy weight systems*, assigning weight amplitudes to open string worldsheets, embed into the space of weight systems on chord diagrams.

To fully account for the quark/Wilson loop, consider the function

$$\mathrm{Span}\big(\mathscr{D}^{\mathrm{t}}\big) \xrightarrow{\ \mathrm{perm}\ } \mathrm{Span}\big(\mathscr{D}^{\mathrm{t}}\big) \tag{69}$$

which sends a Jacobi diagram with n external vertices to the linear combination of the $n!$ ways of permuting them along the Wilson loop circle. Then the composition of perm (69) with the ’t Hooft double line construction $\mathrm{tH}_{\mathfrak{so}}$ (67) respects the STU-relations (44) ([Bar95b, Thm. 10 with Thm. 8]) and thus descends to linear map on $\mathscr{A}_\bullet := \mathscr{A}^{\mathrm{t}}$ (46)

$$\underset{\substack{\text{CS/RW Feynman diagrams}\\ \text{(Jacobi diagrams) modulo STU}}}{\bigoplus_{n\in\mathbb{N}} \mathscr{A}_n} \xrightarrow[\qquad\qquad\qquad]{\overset{\text{'t Hooft construction}}{\mathrm{tH}\circ\mathrm{perm}}} \underset{\text{String worldsheets}}{\mathrm{Span}\big(\mathrm{MarkedSurfaces}_{/\sim}\big)}. \tag{70}$$



Given that weight systems (47) on Jacobi diagrams reflect assignments of Feynman amplitudes to Feynman diagrams (for Chern-Simons theory §4.4 or Rozansky-Witten theory §4.5) we are to regard *stringy weight systems*

$$\underset{\text{Stringy weight systems}}{\mathscr{S} := \Big(\mathrm{Span}\big(\mathrm{MarkedSurfaces}_{/\sim}\big)\Big)^*} \xrightarrow{\overset{\text{point-particle limit}}{(\mathrm{tH}\circ\mathrm{perm})^*}} \underset{\text{Ordinary weight systems}}{\prod_{n\in\mathbb{N}} \mathscr{W}^n\langle\hbar^n\rangle} \tag{71}$$

as encoding open string scattering amplitudes.

[5]Beware that (only) for *closed* string gravity duals of Chern-Simons theory [GB99] are these open worldsheets turned into closed string worldsheets by gluing disks onto all their free boundaries, see [GR03, 1.1][Mar04, III, p. 14].

One finds ([Bar95b, Thm. 11]) that stringy weight systems span those Lie algebra weight systems (60) that come from metric Lie representations of $\mathfrak{gl}(N)$ and $\mathfrak{so}(N)$, and contains those coming from $\mathfrak{sp}(N)$, with N ranging over the natural numbers :

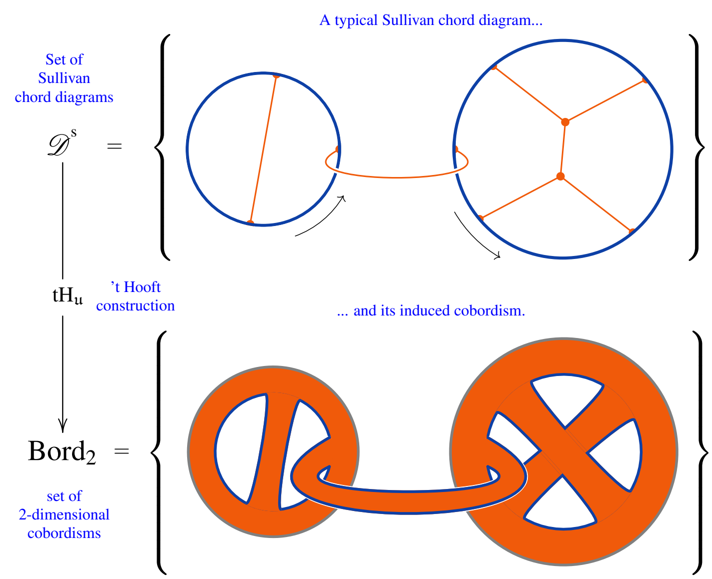
$$
\begin{tikzcd}
\underset{\text{Stringy weight systems}}{\mathrm{Im}\big(\mathscr{S}\big)} \arrow[dr, hook, "(71)"'] & \simeq \; \underset{\text{Classical Lie algebra weight systems}}{\mathrm{Im}\big(\mathscr{L}_{\mathfrak{gl}} \oplus \mathscr{L}_{\mathfrak{so}}\big)} \arrow[d, hook, "(60)"] & \supset \; \mathrm{Im}\big(\mathscr{L}_{\mathfrak{sp}}\big). \arrow[dl, hook', "(60)"] \\
& \underset{\text{All weight systems}}{\prod\limits_{n\in\mathbb{N}} \big(\mathscr{W}^n\langle \hbar^n\rangle\big)} &
\end{tikzcd}
$$

It follows from this weight-theoretic result that the perturbative Wilson loop observables of Chern-Simons theory (61), for $\mathfrak{g} = \mathfrak{so}(N)$ and with the Wilson loop in the fundamental representation, are equivalent, under the 't Hooft construction (70), to observables of a unoriented open topological string theory, as in the identification of Chern-Simons theory as a topological string theory in [Wi92] (reviewed in [Mar04]).

4.7 Horizontal weight systems observe string topology operations

A *Sullivan chord diagram* ([CG04, Def. 1], following [CS02]) is a finite undirected graph equipped with cyclic orderings of the edges around each of its vertices, which arises from attaching the external vertices of trees to a number of oriented and disjointly embedded circles, which give boundary components of the corresponding ribbon graph. We write $\mathscr{D}^{\mathrm{s}}$ for the set of Sullian chord diagrams.

Applying the 't Hooft surface construction as in §4.6 for $\mathfrak{g} = \mathfrak{u}$, now including the boundary circles (shown in grey in the example on the right), turns a Sullivan diagran into a cobordism to these boundary circles from the remaining boundary components (shown in blue).



(72)

For example, every round chord diagram (39) is a Sullivan chord diagram, but a Jacobi diagram (40) is only a Sullivan chord diagram if its internal edges form a tree (so the Jacobi diagram shown in (70) is *not* a Sullivan chord diagram).

String topology TQFT. The tree-condition ensures [CG04, 2] that for $D \in \mathscr{D}^{\mathrm{s}}$ a Sullivan chord diagram with $n_{\mathrm{in}}, n_{\mathrm{out}}$ in/out-going boundary components, the pull-push operation in homology through the mapping space out of $\mathrm{tH}_{\mathfrak{u}}(D)$ exists [CG04, Theorem 4], for X an oriented target manifold, with free loop space $\mathscr{L}X := \mathrm{Maps}(S^1, X)$:

$$
\begin{tikzcd}
& \underset{\text{space of maps from worldsheet } \mathrm{tH}_{\mathfrak{u}}(D) \text{ to spacetime } X}{\mathrm{Maps}\big(\mathrm{tH}_{\mathfrak{u}}(D), X\big)} \arrow[dl, "p_{\mathrm{in}}"'] \arrow[dr, "p_{\mathrm{out}}"] & \\
\underset{\text{space of incoming string configurations}}{(\mathscr{L}X)^{\times n_{\mathrm{in}}} \simeq \mathrm{Maps}\big(\partial_{\mathrm{in}}\mathrm{tH}_{\mathfrak{u}}(D), X\big)} & & \underset{\text{space of outgoing string configurations}}{\mathrm{Maps}\big(\partial_{\mathrm{out}}\mathrm{tH}_{\mathfrak{u}}(D), X\big) \simeq (\mathscr{L}X)^{\times n_{\mathrm{out}}}} \\
\underset{\text{space of incoming string states}}{\big(H_\bullet(\mathscr{L}X)\big)^{\otimes n_{\mathrm{in}}}} \arrow[rr, "(p_{\mathrm{out}})_* \circ (p_{\mathrm{in}})^*", "\text{homological path integral}"'] & & \underset{\text{space of outgoing string states}}{\big(H_\bullet(\mathscr{L}X)\big)^{\otimes n_{\mathrm{out}}}}
\end{tikzcd}
\tag{73}
$$

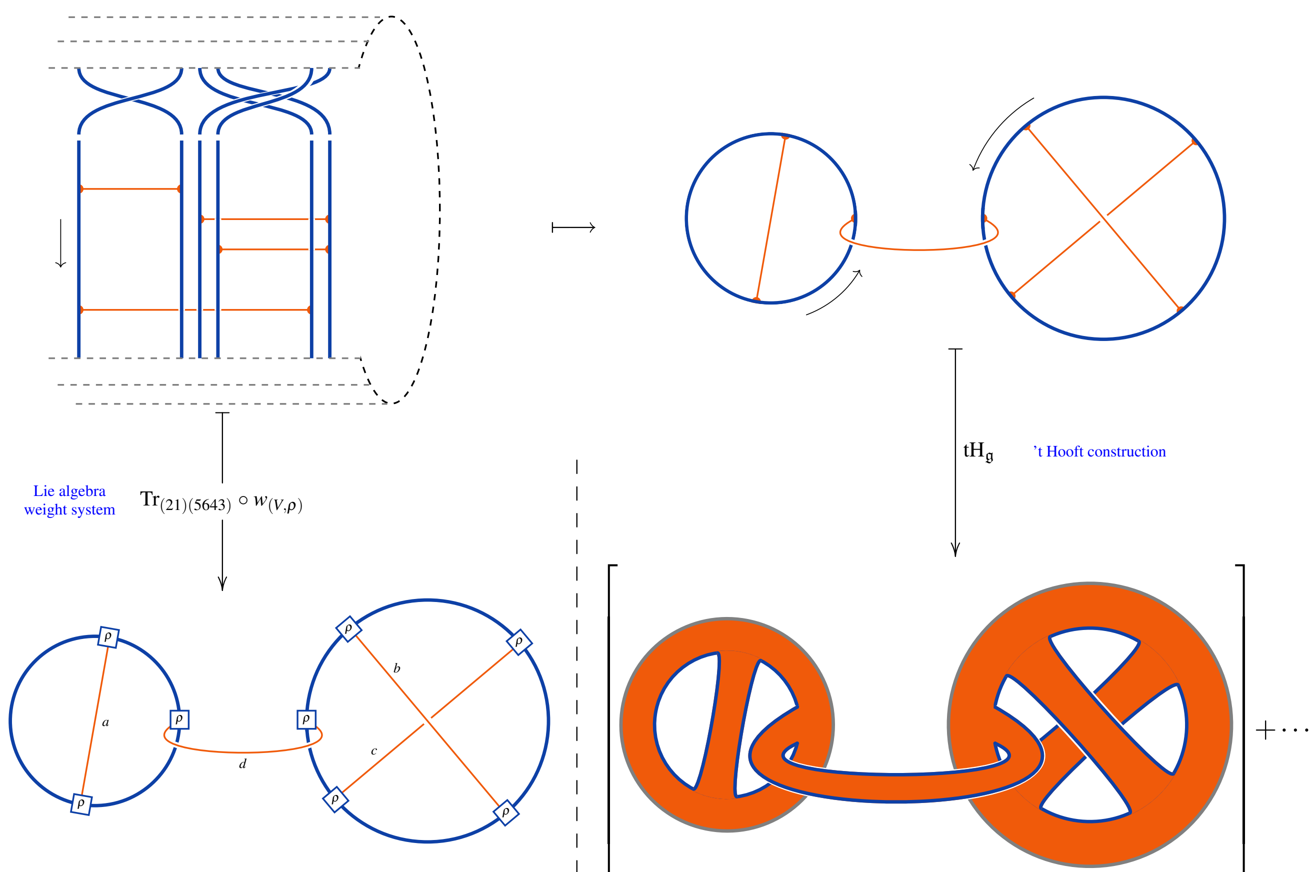
There is a precise sense [Sc14, Ex. 7.1] in which this pull-push operation (73) is the *cohomological path-integral* of a topological closed string theory with target space X and worldsheet geometry $\mathrm{tH}_{\mathfrak{u}}(D)$. Indeed, as the Sullivan chord diagram D and hence the worldsheet topology $\mathrm{tH}_{\mathfrak{u}}(D)$ varies, the operations (73) organize into the propagators of a 2d topological field theory (see [CV05, 3]).

String topology operations from horizontal chord diagrams. We observe that Sullivan chord diagrams (72) without any internal vertices, and hence the corresponding string topology operations (73), arise precisely as the closures of horizontal chord diagrams (30) with respect to general monodromy permutations σ as in (53). If $\sigma = (N_{\mathrm{f}}12\cdots)$ has only a single cycle (single orbit) the result is a round chord diagram as in (38):

$$\overset{\text{Horizontal chord diagrams}}{\mathscr{D}^{\mathrm{pb}}_{N_{\mathrm{f}}}} \xrightarrow{\ \mathrm{close}_{(N_{\mathrm{f}}12\cdots)}\ } \overset{\text{Round chord diagrams}}{\mathscr{D}^{\mathrm{c}}} \hookrightarrow \overset{\text{Sullivan chord diagrams}}{\mathscr{D}^{\mathrm{s}}} .$$

But for general permutations σ with n cycles (n orbits) as in (53) the result is a Sullivan chord diagram whose corresponding cobordism has n outgoing boundary components. For example:

$$\overset{\text{Horizontal chord diagrams}}{\mathscr{D}^{\mathrm{pb}}_{N_{\mathrm{f}}=6}} \xrightarrow[\text{}]{\overset{\text{close up strands after permutation}}{\mathrm{close}_{(21)(5643)}}} \overset{\text{Sullivan chord diagrams}}{\mathscr{D}^{\mathrm{s}}} \qquad (74)$$



$$= \mathrm{Tr}_V\left(\rho_a \cdot \rho_d \cdot \rho^a\right) \mathrm{Tr}_V\left(\rho_b \cdot \rho_c \cdot \rho^d \cdot \rho^b \cdot \rho^c\right)$$

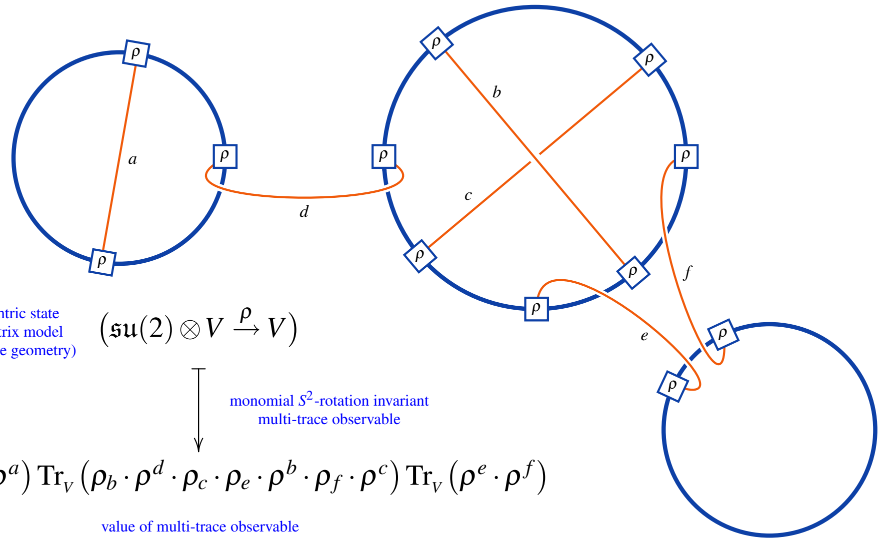
multi-trace observable

On the bottom left of (74) we are showing the form of the associated Lie algebra weights (53), which now are BMN multi-trace observables, see §4.8.

4.8 Horizontal chord diagrams are BMN model multi-trace observables

Where the single-trace gauge theory observables from §4.1 correspond to single-string states under the AdS/CFT correspondence, general multi-string states correspond [CS99] to *multi-trace observables* [BDHM98] hence to polynomials in single-trace observables [Wi01, p. 1].

Invariant multi-trace observables in the BMN matrix model. Hence, in generalization of the discussion in §4.1, a supersymmetric and S^2-rotation invariant multi-trace observable in the BMN matrix model sends a supersymmetric state given by a complex Lie algebra representation $\mathfrak{su}(2) \otimes V \xrightarrow{\rho} V$ to expressions like the following:



supersymemtric state of BMN matrix model (fuzzy 2-sphere geometry)

$$\big(\mathfrak{su}(2) \otimes V \xrightarrow{\rho} V\big)$$

$\big\downarrow$ monomial S^2-rotation invariant multi-trace observable

$$\mathrm{Tr}_V\big(\rho_a \cdot \rho_d \cdot \rho^a\big)\,\mathrm{Tr}_V\big(\rho_b \cdot \rho^d \cdot \rho_c \cdot \rho_e \cdot \rho^b \cdot \rho_f \cdot \rho^c\big)\,\mathrm{Tr}_V\big(\rho^e \cdot \rho^f\big) \tag{75}$$

value of multi-trace observable

Horizontal chord diagrams are BMN matrix model multi-trace observables. The multi-trace expressions (75) are manifestly the values (53) of the Lie algebra weight system w_V corresponding to the given BMN model state on horizontal chord diagrams encoding the multi-trace observable, as in (74).

But the *fundamental theorem of horizontal weight systems* Prop. 3.4 says that *every* horizontal weight system arises this way (54), hence that:

Weight systems on horizontal chord diagrams are equivalently the supersymmetric BMN model states as seen by the colleciton of S^2-invariant multi-trace observables, which in turn are encoded by chord diagrams.

In summary, this means we have found the following identifications (see Figure 2):

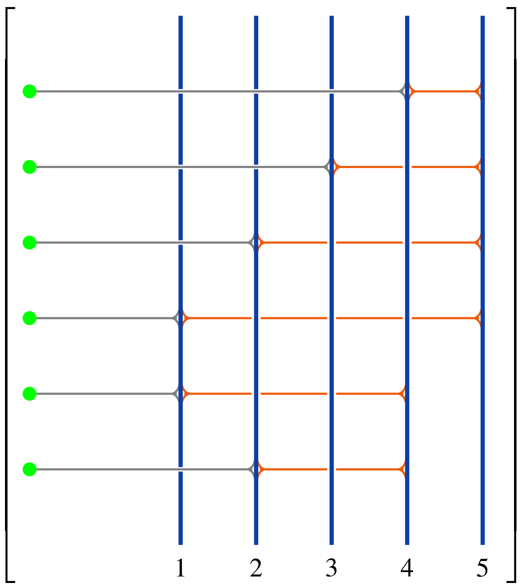
$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{higher observables on}\\ \mathrm{D}p \perp \mathrm{D}(p+2)\text{ intersections}\\ \text{by }\textit{Hypothesis H}\end{array}\right\} & & \left\{\begin{array}{c}\text{higher co-observables on}\\ \mathrm{D}p \perp \mathrm{D}(p+2)\text{ intersections}\\ \text{by }\textit{Hypothesis H}\end{array}\right\} \\
\simeq\updownarrow & \text{Prop. 2.16, Prop. 2.18} & \simeq\updownarrow \\
\{\ \text{horizontal weight systems}\ \} & & \{\ \text{horizontal chord diagrams}\ \} \\
\simeq\updownarrow & \text{Prop. 3.4} & \simeq\updownarrow \\
\left\{\begin{array}{c}\text{supersymmetric states}\\ \text{of the BMN matrix model}\\ \text{and }\mathrm{D}p \perp \mathrm{D}(p+2)\text{ fuzzy funnels}\end{array}\right\} & & \left\{\begin{array}{c}\text{invariant multi-trace observables}\\ \text{of the BMN matrix model}\\ \text{and }\mathrm{D}p \perp \mathrm{D}(p+2)\text{ fuzzy funnels}\end{array}\right\}
\end{array}$$

4.9 Horizontal chord diagrams encode Hanany-Witten states

The graded-commutative algebra of horizontal chord diagrams. Recall from (36) that $\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}}$ is the free *graded associative* algebra on generators $\big\{t_{ij} = t_{ji} | i \neq j \in \{1, \cdots N_{\mathrm{f}}\}\big\}$ in degree 1, modulo the 2T and 4T relations. By skew-symmetrizing this induces the *graded commutative* algebra obtained from the same generators and relations:

$$\mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}} \;:=\; \mathrm{GradedComm}\Big(\big\{\underset{\mathrm{deg}\,=\,1}{t_{ij=t_{ji}}} | i \neq j \in \{1, \cdots N_{\mathrm{f}}\}\big\}\Big)\big/(2\mathrm{T}, 4\mathrm{T})\,. \tag{76}$$

Horizontal chord diagrams (30) still represent generators in this graded-commutative algebra. To indicate that we think of a horizontal chord diagram as a generator in $\mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}}$, we complete each chord by a gray line to the left or to the right, as in the following example:

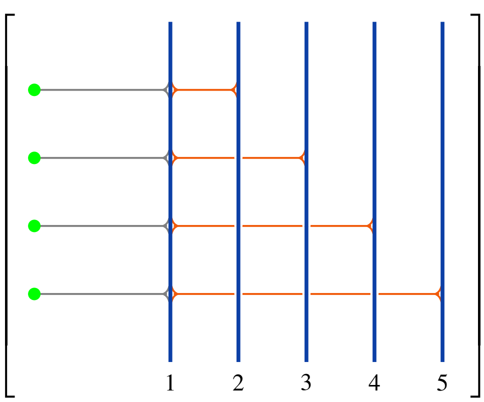


$$= \; t_{45} \wedge t_{35} \wedge t_{25} \wedge t_{15} \wedge t_{14} \wedge t_{24} \;\; \in \;\; \mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}=5}\,. \tag{77}$$

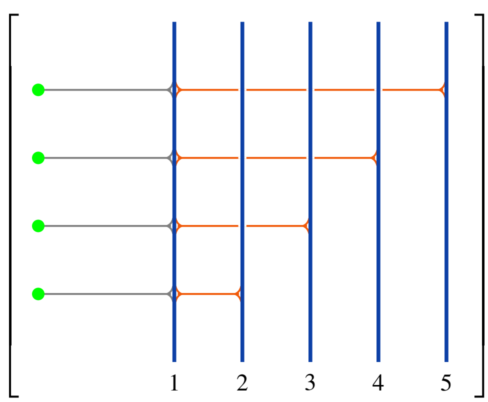
In fact, this element vanishes, because the 2T-relations (33) now say that the product of two chords vanishes if they do not connect to one common strand. In the example (77) the 2T relation gives

$$t_{15} \wedge t_{24} \;=\; 0\,.$$

Therefore, a non-vanishing homogeneous element in (76) has to look either like this:

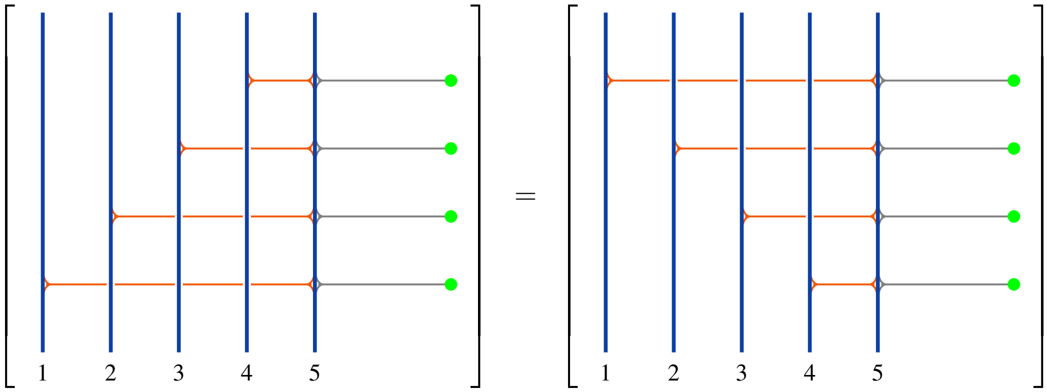


$$=$$



$$= \; t_{12} \wedge t_{13} \wedge t_{14} \wedge t_{15} \;\; \in \;\; \mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}=5}$$

or like this:

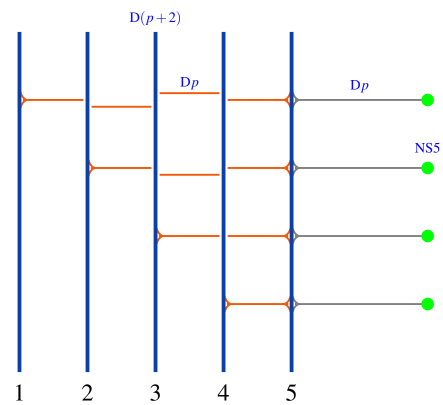


$$= t_{45} \wedge t_{35} \wedge t_{25} \wedge t_{15} \in \mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}=5}$$

up to permutation of strands.

Hanany-Witten theory. We observe that the elements of the skew-symmetrized graded-commutative algebra of $\mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}}$ (76) of horizontal chord diagrams reflect the diagrammatics of Hanany-Witten $\mathrm{D}p - \mathrm{D}(p+2)$ brane configurations according to [HW97, 6][GW08, 3] (see also [HOO98, 23][GK99, p. 83-][GKSTY01, (6.12)][Fa17, Fig. 3.13]) if we identify:

- **(i)** strands as $\mathrm{D}(p+2)$-branes;
- **(ii)** chords as $\mathrm{D}p$-branes, stretching between $\mathrm{D}(p+2)$s;
- **(iii)** green dots as NS5-branes;
- **(iv)** gray lines as $\mathrm{D}p$-branes, stretching from NS5 to $\mathrm{D}(p+2)$.

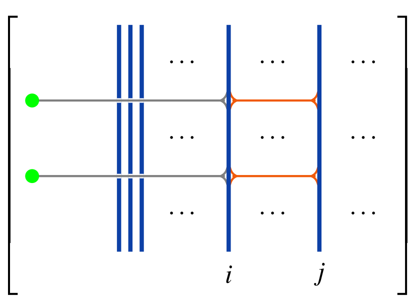


(78)

With this identification we find that the algebra of horizontal chord diagrams reflects the following rules of Hanany-Witten theory:

(1) *The s-rule.*
(2) *The breaking of* $\mathrm{D}p$*-branes on* $\mathrm{D}(p+2) -$ *branes.*
(3) *The ordering constraint.*

(1) The s-rule. A direct consequence of the graded-commutativity in (76) and the fact that the chord generators are in degree 1 is that diagrams of the following form vanish:

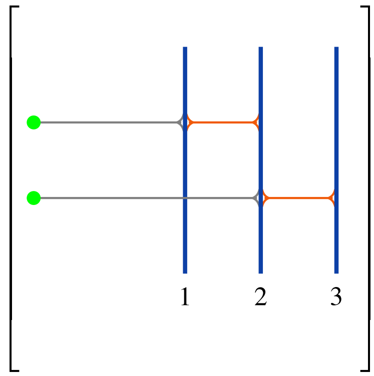


$$= \quad 0$$

Under the identification (78), these are the configurations where two $\mathrm{D}p$-branes end on the same $\mathrm{D}(p+2)$-brane. That these configurations are excluded (if supersymmetry is required) is known as the *s-rule* of Hanany-Witten theory, going back to the discussion of *s-configurations* in [HW97] and made explicit in [GK99, p. 83-].

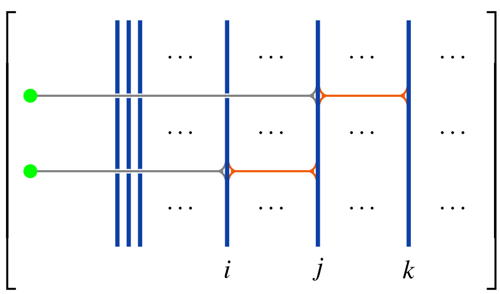
We notice that in [BGS97, 2.3][BG98] the s-rule has been argued to be nothing but the implication of the *Pauli exclusion principle* for the fermions on the intersecting branes. But of course the mathematical reflection of the Pauli exclusion principle is, at its core, precisely the graded-commutativity as in (76).

(2) Breaking of Dp-branes on D$(p+2)$-branes. A non-vanishing element of $\mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}}$ (76) may also be of the form

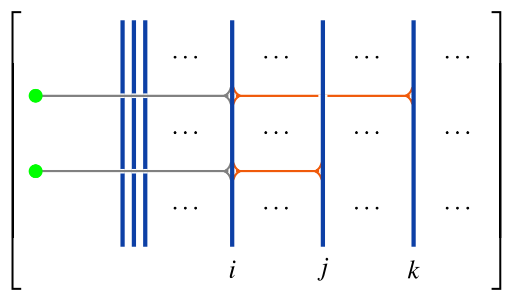


$$= \; t_{12} \wedge t_{23} \;\in\; \mathscr{A}^{\mathrm{hw}}_{N_{\mathrm{f}}=5}$$

Under the identification (78) this corresponds to a Dp-brane which crosses a D$(p+2)$-brane without having broken up into segments. But the *4T-relation* (42) in the graded commutative algebra (76) now implies that this configuration equivalently transmutes to the one on the right of the following:



$$= \; -$$



Under the identification (78), this process is the breaking up of a Dp-brane where it crosses a D$(p+2)$-brane, as expected in Hanany-Witten theory.

(3) The ordering constraint. Under the identification (78) and by the discussion in §2.5, we obtain the higher observables on Hanany-Witten D$p \perp$ D$(p+2)$-configurations by passing to weight systems evaluated on the skew-symmetrized horizontal chord diagrams in (76). By Prop. 3.4 this introduces two extra pieces of data, namely:

(i) numbers $N_{\mathrm{c},i}$ of coincident Dp-branes ending on the ith strand, and

(ii) winding monodromies σ of these strands, modulo some equivalence relations.

But from (53) it is evident that up to these equivalence relations only the conjugacy class of the winding monodromy $\sigma \in \mathrm{Sym}(N_{\mathrm{f}})$ matters, where an equivalence

$$\sigma \;\sim\; \tilde{\sigma} \circ \sigma \circ \tilde{\sigma}^{-1}$$

corresponds to reordering the strands according to any other permutation $\tilde{\sigma} \in \mathrm{Sym}(N_{\mathrm{f}})$. With the tuple $\vec{N}_{\mathrm{f}}$ of numbers of coincident Dp-branes specified, this means that we may partially gauge-fix this freedom in the winding monodromy σ by requiring that the elements of $\vec{N}_{\mathrm{f}}$ are monotonically ordered:

$$N_{\mathrm{c},1} \;\leq\; N_{\mathrm{c},2} \;\leq\; \cdots \;\leq\; N_{\mathrm{c},N_{\mathrm{f}}}\,.$$

Under the identification (78) this is the *ordering constraint* that was found in [GW08, 3.5].

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