Gauge principle and groupoids

Pursuing stacks

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Outline

Physics meets mathematics

Physics		Mathematics
- Locality	Stacks	- Differential geometry
- Gauge principle		- Homotopy theory

Locality principle

What does it mean physically

Global field configurations have to be obtained by summing up local data. That means by gluing local field configurations together.

What does it mean mathematically

This is captured by the theory of *sheaves*.

Gauge principle

What does it mean physically

Given two electromagnetic field configurations it only makes sense to ask if they are equal up to gauge equivalence.

What does it mean mathematically

This motivates the study of *groupoids*.

Idea of a smooth space

1. We want to make it smooth, because physics happens on smooth spaces.
2. We want our space to be probable by test functions.
3. We want it to be satisfy the locality principle.

Definition of a smooth space

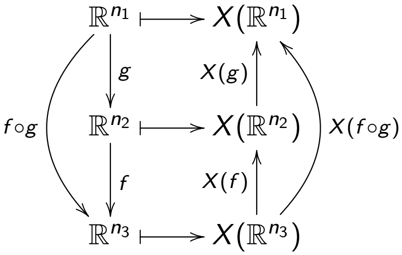
Definition

1. $X(\mathbb{R}^n)$ a set of functions from $\mathbb{R}^n$ to X.
2. We have a pull-back $X(f) : X(\mathbb{R}^{n_2}) \to X(\mathbb{R}^{n_1})$ for every smooth $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$

With the following demands:

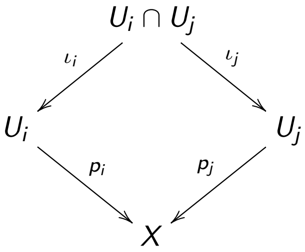
1. $X(Id_{\mathbb{R}^n}) = Id_{X(\mathbb{R}^n)}$
2. We have $X(g) \circ X(f) = X(f \circ g)$ for smooth f and g.
3. It satisfies the sheaf condition.

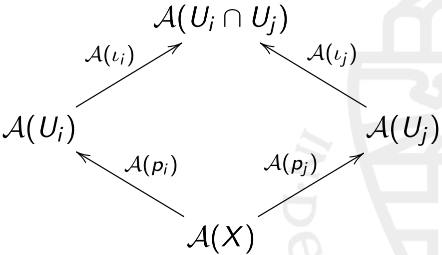
Assignment of a smooth space



Note: pullback of functions may be composed consistently, meaning that it is functorial

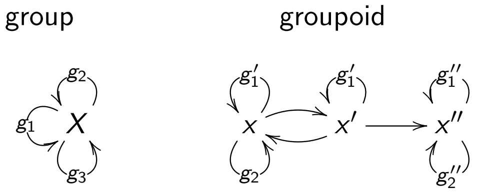
The sheaf condition





Given a cover by two patches as on the left, the sheaf condition says that $\mathcal{A}(X)$ is the universal solution (the fiber product) of completing this square.

Groups vs groupoids



Groupoids

Definition

A groupoid G consists of:

- A set of objects G_0
- A set of morphisms G_1

We demand for morphisms:

- Composition: $f_2 \circ f_1 = x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \in \mathcal{G}_1$
- Associativity: $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$
- We have an identity morphism: $x \xrightarrow{Id_x} x \in \mathcal{G}_1 \ \forall x \in G_0$
- Given $f \in \mathcal{G}_1$ we have an inverse f^{-1} such that $f^{-1} \circ f = Id_x \in \mathcal{G}_1$

Electromagnetic field configurations

The electromagnetic field configurations form a groupoid

- The set of objects given by $\Omega^1(\mathbb{R}^n) = \{A |\text{ the vector potentials}\}$
- The morphisms are given by $\Omega^1(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$. *Motivation:* $\mathbf{d}(A + \mathbf{d}f) = \mathbf{d}A$

$$A \xrightarrow{f} (A + \mathbf{d}f) \xrightarrow{g} (A + \mathbf{d}(f + g))$$

Pre smooth groupoids

Idea: characterize general smooth groupoids by how to smoothly map charts into them

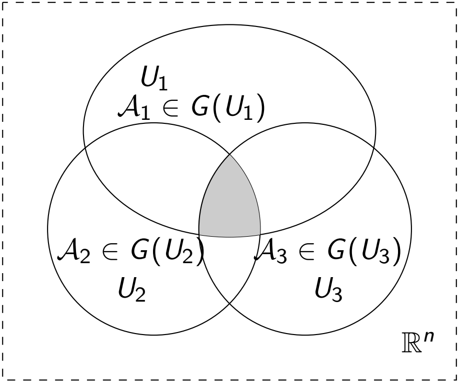
Definition

A pre smooth groupoid $G_\bullet$:

- Is an assignment: $n \mapsto G_\bullet(\mathbb{R}^n)$ which assigns to every $n \in \mathbb{N}$ to a groupoid
- For every $\phi : \mathbb{R}^{n_1} \mapsto \mathbb{R}^{n_2}$ a pull-back $G_\bullet(\phi) : G_\bullet(\mathbb{R}^{n_2}) \mapsto G_\bullet(\mathbb{R}^{n_1})$ such that it has associative and unital composition.

Stack condition I

Given a cover $\{U_i \hookrightarrow \mathbb{R}^n\}$ and a pre smooth groupoid $G_\bullet$...



Stack condition II

On the triple intersections we have a commuting diagram:

$$\begin{array}{ccc} \mathcal{A}_1|_{U_1\cap U_2\cap U_3} & \xrightarrow[g_{12}]{\simeq} & \mathcal{A}_2|_{U_1\cap U_2\cap U_3} \\ & {\scriptstyle g_{13}}\searrow^{\simeq} \quad {}^{\simeq}\swarrow{\scriptstyle g_{23}} & \\ & \mathcal{A}_3|_{U_1\cap U_2\cap U_3} & \end{array}$$

Stack condition

The globally assigned groupoid is the local assignments glued by equivalences.

Electromagnetic field configurations

Proposition

The electromagnetic field configurations form a smooth groupoid (hence a stack!).

- Data measured by observers (locally) will only agree up to gauge equivalence.
- We know that matching local data corresponds to the same global electromagnetic field configuration.

In terms of the previous definitions this is simply the assignment that to each spacetime $\mathbb{R}^n$ assigns the groupoid of gauge field configurations on it.

Questions?