Stacks in Gauge theory

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Abstract

The development of modern physics in the first half of the 20th century was closely related to the development of differential geometry, first via Riemannian geometry in Einstein's theory of gravity and then later via Cartan geometry in Yang-Mills's theory of gauge fields. But, as highlighted by Grothendieck in the second half of the 20th century and as witnessed by a multitude of modern developments, a more natural mathematical description of many phenomena in geometry is obtained by refining from traditional geometric spaces to more refined kinds of spaces known as "stacks". In this thesis we explain in elementary terms how the possibly esoteric-seeming concepts of sheaves and stacks naturally capture fundamental aspects of modern physics, namely the gauge principle and the locality principle. We explain the universal moduli stack of the electromagnetic field configurations in a way that highlights how natural and simple this concept is from the point of view of physics. In an outlook section we indicate how this stack helps to understand non-perturbative gauge theory effects such as Dirac monopoles and charge quantization.

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1 Introduction

This thesis is about natural examples in physics of stacks on smooth manifolds, which we think of as *smooth groupoids*. Its intention is to introduce this concept to physics students and teachers alike. This is motivated by its wide application in modern research (see for instance section 6 of [6]), yet it doesn't appear in most of the standard textbooks. We shall motivate stacks as follows. In the section of smooth spaces we will discus sheaves and show how they relate to locality as we know it in physics. Locality says that from local physical data we can acquire knowledge of the global physics in our space. In the following sections stacks are introduced and it is shown how they relate to the gauge principle; which says that physical data is identifiable only up to gauge equivalence.

Our main motivation to consider sheaves and stacks is to provide a non-perturbative framework in which we can do physics. Much of gauge theory is done in perturbation theory, but in fact non-perturbative effects such as Dirac monopoles and Yang-Mills instantons play a crucial role in fundamental physics [5]. The language of stacks is the natural language for these phenomena.

The origins of current research on stacks are found in a document called "À la Poursuite des Champs" or in English "Pursuing Stacks" written in 1983 [4]. In this document Alexander Grothendieck lays out a program to motivate the use of stacks in mathematics.

In spite of all this interest from mathematicians in sheaves and stacks, they haven't received the same interest from physicists. It is often seen as an exotic and intractable branch of mathematics, with little physical application. This thesis intends to show that in fact the opposite is true. We shall explain that stacks as mathematical objects statisfy exactly the locality principle and the gauge principle, two cornerstones of modern physics:

$$\boxed{\text{stack condition} = \text{gauge principle} + \text{locality principle}}$$

By first developing the theory of sheaves on smooth manifolds in section 2, we develop a generalization of the concept of smooth manifolds. A smooth space is closely related to the spaces studied in physics in the sense that we define them by how we can map smooth manifolds into it. This process can be seen as laying out coordinate systems in our space that give us information on the way our space is structured. In a way we are *probing* the space. It gives us information on how coordinate charts map into the space and also how worldlines on which particles move are mapped into our space.

Then in section 3 we generalize from sheaves to stacks. For this we turn to spaces for which we have a concept of gauge equivalence between their points. We however are interested in what happens locally since this is the only data we can measure. We will use this to relate global space(-time)s to local space(-time)s defined by locally gauge equivalent field configurations.
Hence smooth groupoids are the natural kind of "spaces of gauge fields", also called "moduli stacks of gauge fields".

We close this thesis with an outlook on how seemingly subtle aspects of non-perturbative gauge theory are easily described using stacks: Dirac monopoles and charge quantization. This will be done in section 4.

In the appendix we recall the relevant concepts of symplectic geometry (in A) and of category theory (in B). If your not familiar with these theories, it is a good idea to briefly look at them now.

2 Smooth spaces – Sheaves

In this section we introduce a concept of smooth spaces which are more general than smooth manifolds, but which faithfully contain smooth manifolds and hence generalize them to something more flexible. Here "faithfully" means that the relationships between smooth manifolds are conserved when studying them as a sub-category of smooth spaces. We give some examples of smooth spaces that naturally occur in physics. In particular we show that there is a smooth space Ω^2_{cl} of closed differential 2-forms, and that the "slice" of the category of smooth spaces over Ω^2_{cl} faithfully contains the symplectic manifolds that appear as phase spaces in physics:

$$\mathrm{SymplecticManifolds} \hookrightarrow \mathrm{SmoothSpaces}_{/\Omega^2_{\mathrm{cl}}}$$

One may also think of Ω^2_{cl} as the space of electromagnetic Faraday tensors (electromagnetic field strengths) in which case one wants to ask for a "gauge potential" ("vector potential") for that field strength. Doing so leads further from smooth spaces to smooth groupoids below in section 3.

2.1 Smooth spaces

The motivation for looking at smooth spaces is the idea in physics that spaces (spacetimes) should be characterized by how matter propagates around in them, hence how they are probeable by coordinate charts, by worldlines (of point particles) and by higher dimensional worldvolumes (of higher dimensional branes).

The following is a way of understanding how the coordinate systems used in physics relate to mathematics.

Definition 1. The following terminology is useful:

1. We shall call a Cartesian space $\mathbb{R}^n$ an **abstract coordinate system**.
2. We shall call a smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ an **abstract coordinate transformation**.

Abstract coordinate systems (of any dimension $n \in \mathbb{N}$) with abstract coordinate transformations between them form a category (def. B.1), which we denote:

$$\mathrm{CartSp}$$

The traditional concept of a smooth manifold X is that of a space that is locally isomorphic to an abstract coordinate chart $\mathbb{R}^n$, hence such that there is an atlas by smooth maps $\mathbb{R}^n \to X$ which are diffeomorphisms onto their image. We consider now a concept of smooth space that is actually a little simpler than that, while at the same time being more general. Namely we consider a kind of space such that abstract coordinate systems may be laid out inside it, hence simply such that $\mathbb{R}^n$s may be smoothly mapped into it. This is of course also true for smooth manifolds, but we will ignore for the moment the condition that there is an atlas of coordinate charts that are diffeomorphisms onto their image, and only demand that coordinate charts may be mapped into the space:

Definition 2. A **pre-smooth space** X is an assignment

$$\mathbb{R}^n \mapsto X(\mathbb{R}^n) \in \mathrm{Set}$$

of a set $X(\mathbb{R}^n)$ to any abstract coordinate chart $\mathbb{R}^n$ (to be thought of as the set of smooth functions from $\mathbb{R}^n$ into the space X, thereby defined) together with an assignment

$$\begin{array}{ccc} \mathbb{R}^{n_1} & & X(\mathbb{R}^{n_1}) \\ \downarrow{\scriptstyle f} & \mapsto & \uparrow{\scriptstyle X(f)} \\ \mathbb{R}^{n_2} & & X(\mathbb{R}^{n_2}) \end{array}$$

of functions $X(f)$ to abstract coordinate transformations f (to be thought of as the operations of precomposing smooth maps from $\mathbb{R}^{n_2}$ to X with f); such that this assignment respects the identity functions ($X(\mathrm{id}) = \mathrm{id}$) and such that it is compatible with composition of functions:

$$\begin{array}{ccc} \mathbb{R}^{n_1} & \longmapsto & X(\mathbb{R}^{n_1}) \\ \downarrow{\scriptstyle g} & & \uparrow{\scriptstyle X(g)} \\ \mathbb{R}^{n_2} & \longmapsto & X(\mathbb{R}^{n_2}) \\ \downarrow{\scriptstyle f} & & \uparrow{\scriptstyle X(f)} \\ \mathbb{R}^{n_3} & \longmapsto & X(\mathbb{R}^{n_3}) \end{array} \qquad \mathbb{R}^{n_1} \xrightarrow{f\circ g} \mathbb{R}^{n_3}, \quad X(\mathbb{R}^{n_3}) \xrightarrow{X(f\circ g)} X(\mathbb{R}^{n_1})$$

In terms of category theory this just means that a pre-smooth space X is a presheaf (see def. 56) on CartSp. Accordingly we write

$$\mathrm{PreSmoothSpaces} := \mathrm{PSh}(\mathrm{CartSp})$$

for the category of pre-smooth spaces.

Remark 3. A pre-smooth space is defined by the sets $X(\mathbb{R}^n)$, which we have defined as the *maps* into it. We however want to relate these maps with morphisms into the pre-smooth space. These are morphisms of the form $\mathbb{R}^n \to X$ in the category PSh(CartSp).

$\mathbb{R}^n$ is a pre-smooth space by letting it work as a functor on the category CartSp. By the Yoneda lemma we have $\mathbb{R}^n(\mathbb{R}^n) \cong \mathrm{PSh}(\mathrm{CartSp})(\mathrm{Hom}_{\mathrm{CartSp}}(-, \mathbb{R}^n), \mathbb{R}^n)$ (see appendix B.2). So every $\mathbb{R}^n$ represents a (pre-)smooth space.

A pre-smooth space is a functor X on CartSp by definition. By the Yoneda lemma $X(\mathbb{R}^n) \cong \mathrm{PSh}(\mathrm{CartSp})(\mathrm{Hom}_{\mathrm{CartSp}}(-, \mathbb{R}^n), X)$. Where on the left side we have the set we defined as the *smooth maps* into X and on the right the natural transformations between $\mathrm{Hom}_{\mathrm{CartSp}}(-, \mathbb{R}^n)$ and X. So we know how maps into $\mathbb{R}^n$ correspond to maps into X, which justifies our notation.

While this definition nicely captures the simple idea that a smooth space is something that may be probed by mapping abstract coordinate systems into it, the definition does not yet know about locality.

The locality principle in physics concerns expressing physical field configurations on “large” spaces (spacetimes) by their local values. In order to precise what “locality” means here, we have to specify what it means for an abstract coordinate chart to be covered by local charts. That is what the following definition does.

Definition 4. For $\mathbb{R}^n$ an abstract coordinate chart according to def. 1 we call a set $\{U_i \to \mathbb{R}^n\}_{i\in I}$ of abstract coordinate transformations into it a **differentially good open cover** of $\mathbb{R}^n$ if it satisfies the following three conditions:

1. $\cup_{i\in I} U_i = \mathbb{R}^n$;
2. $U_i \to \mathbb{R}^n$ is an open embedding for every i;
3. every non-empty finite intersection of the U_i is diffeomorphic to an open ball; hence for a non-empty finite intersection $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k} \cong B^n \cong \mathbb{R}^n$.

Remark 5. In general we can also take covers $\{U_i \hookrightarrow X\}$ of smooth manifolds X. In this text we will however use covers of $\mathbb{R}^n$, these can be generalized to covers of smooth manifolds.

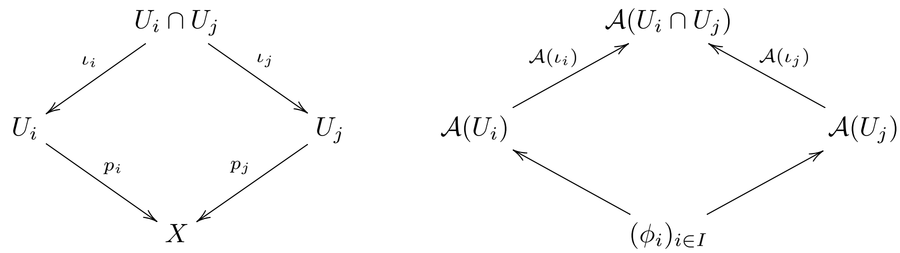
Definition 6. Consider a presheaf $\mathcal{A} \in \mathrm{PSh}(\mathrm{CartSp})$. Then for any Cartesian space $\mathbb{R}^n$ and cover $\{U_i \to \mathbb{R}^n\}_{i\in I}$ by coordinate systems, a **matching family** of local elements of $\mathcal{A}$ on $\mathbb{R}^n$ with respect to this cover is an I-tuple of elements

$$\phi_i \in \mathcal{A}(U_i)$$

such that for all pairs of patches (U_i, U_j) the restrictions of these elements to the intersections $U_i \cap U_j$ agree:

$$\mathcal{A}(U_i \cap U_j \xrightarrow{\iota_i} U_i)(\phi_i) =: \phi_i|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j} := \mathcal{A}(U_i \cap U_j \xrightarrow{\iota_j} U_j)(\phi_j)$$

This means a set $(\phi_i)_{\in iI}$ of elements such that the right diagram below commutes:



We write $Match(\mathcal{A}, \{U_i \hookrightarrow \mathbb{R}^n\})$ for the set of all matching families for the given presheaf on the given coordinate chart with respect to the given cover.

Here we can think of $\mathcal{A}(X)$ as the global field configurations and $Match(\mathcal{A}, \{U_i \hookrightarrow X\})$ as the local field configurations. For a physical space we want each local field configuration to match to a global field configuration. This is called the sheaf condition as defined now:

Definition 7. We call a presheaf $\mathcal{A} \in \mathrm{PSh}(\mathrm{CartSp})$ a **sheaf** if for every $n \in \mathbb{N}$ we have for all covers $\{U_i \hookrightarrow X\}$ that the canonical map

$$\begin{aligned} h : \mathcal{A}(X) &\to Match(\mathcal{A}, \{U_i \hookrightarrow X\}) \\ f &\mapsto \{f|_{U_i}\} \end{aligned}$$

is a bijection between the set assigned by the presheaf to $\mathbb{R}^n$ and the set of its matching families on $\mathbb{R}^n$ with respect to the given cover.

Definition 8. A **smooth space** is a pre-smooth space (def. 2) that satisfies locality in that it satisfies the sheaf condition (def. 7). We write

$$\mathrm{SmoothSpaces} \hookrightarrow \mathrm{PreSmoothSpaces}$$

for the full subcategory of smooth spaces inside pre-smooth spaces.

We want smooth spaces to generalize the concept of smooth manifolds. For this we need to (at least) check that the smooth manifolds satisty the axioms of a smooth space. Let us first recall what the definition of a smooth manifold is.

Definition 9. A **smooth n-manifold** is a topological space with the following properties:

1. second countable Hausdorff
2. locally homeomorphic to $\mathbb{R}^n$
3. has a maximal atlas: a set of charts $\{\phi_\alpha : U_i \to \mathbb{R}^n\}$ on which the transition functions $\{\psi_{ij} : U_i \to U_j | \phi_\alpha(U_i) \cap \phi_\beta(U_j) \neq \emptyset\}$ are smooth functions.

Every manifold X corresponds to a Hom-Functor $\mathrm{Hom}_{\mathrm{SmoothMfld}}(-, X)$, encoding the ways of laying out Cartesian spaces in X.

Given a smooth m-manifold X, let's look at the assignment: $X : n \mapsto C^\infty(\mathbb{R}^n, X)$, the smooth functions into X. Then the assignment $\mathbb{R}^0 \mapsto X(\mathbb{R}^0)$ gives us smooth maps to the points in X. Likewise the assignment $\mathbb{R}^1 \mapsto X(\mathbb{R}^1)$ gives us smooth maps to the lines in X (assuming $n \geq 1$).

Lemma 10. Every smooth manifold X is a pre-smooth space as defined in definition 2.

Proof. We have

1. The set $X(\mathbb{R}^n) = \mathrm{Hom}_{\mathrm{SmoothMfld}}(\mathbb{R}^n, X)$, which are the smooth functions of $\mathbb{R}^n$ into X.

2. Given a smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have the pull-back $X(f) : C^\infty(\mathbb{R}^{n_2}, X) \to C^\infty(\mathbb{R}^{n_1}, X)$. This is precomposition of f, so we have $X(f) := (-) \circ f$. Let $g \in X(\mathbb{R}^{n_2})$, then $X(f)(g) = g(f) : \mathbb{R}^{n_1} \xrightarrow{f} \mathbb{R}^{n_1} \xrightarrow{g} X$.

We now check that the pull-back satisfies the two properties required for X to be a smooth space.

1. For $g \in X(\mathbb{R}^n)$, $X(id_{\mathbb{R}^n})(g) = g(id_{\mathbb{R}^n}) = g$

2. Given two smooth functions $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $g : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$. We have $X(f) \circ X(g) = X(g) \circ f = ((-) \circ g) \circ f = (-) \circ (f \circ g) = X(f \circ g)$ □

Lemma 11. The pre-smooth space represented by a manifold X is a smooth space as in definition 8.

Proof. We need to show that we have a bijection: $h : X(\mathbb{R}^n) \to Match(X, \{U_i \hookrightarrow \mathbb{R}^n\})$. Where h sends every smooth function in $X(\mathbb{R}^n)$ to a matching family; this is the mapping $f \mapsto \{f|_{U_i}\}_{i \in I}$.

Given a matching family $\{f|_{U_i}\}$, we know that $f|_{U_i}(x) = f|_{U_j}(x)$ for $x \in U_i \cap U_j$. This implies that $f|_{\cup U_i}$ is a smooth function of $\cup U_i = \mathbb{R}^n$ to X. Thus h is surjective.

Now given a matching family $\{f|_{U_i}\}_{i \in I}$. Assume we have a f' such that, $h(f') = h(f)$ and $f \neq f''$. Then $f|_{U_i} = f'|_{U_i}$ for all $i \in I$. The patches agree on the overlap, so we have $f|_{\mathbb{R}^n} = f|_{\cup U_i} = f'|_{\cup U_i} = f'|_{\mathbb{R}^n}$, which is in contradiction with our assumption that $f \neq f'$. Thus h is injective. □

Corrollary 12. A direct corrollary is that Cartesian spaces are smooth spaces and in fact that the category of smooth manifolds is a full subcategory of that of smooth spaces:

$$\mathrm{SmoothManifolds} \hookrightarrow \mathrm{SmoothSpaces}.$$

The following is another simple example of a smooth space, which is however far from being a smooth manifold and hence is our first example that highlights the usefulness of smooth spaces.

Lemma 13. For every $m \in \mathbb{N}$ the space of smooth differential m-forms is a smooth space under the assignment $\Omega^m : \mathbb{R}^n \mapsto \Omega^m(\mathbb{R}^n)$.

Proof. We have

1. $\Omega^m(\mathbb{R}^n)$, the set of smooth differential m-forms on $\mathbb{R}^n$;
2. we define $\Omega^n(f) := f^*$ for a smooth f, where f^* is the pull-back as defined in definition 42.

We now check that this assignment satisfies the demands for a smooth space:

1. If we take the pull-back allong the identity function $id_{\mathbb{R}^n}$, we have $\Omega^n(id_{\mathbb{R}^n})$ $\omega_p(v_1, ...v_n) = (id^*_{\mathbb{R}^n}\omega)_p(v_1, ...v_n) = \omega_{id_{\mathbb{R}^n}(p)}(id_{\mathbb{R}^n *}v_1, ..., id_{\mathbb{R}^n *}v_n) = \omega_p(v_1,$ $..., v_n) = id_{\Omega^n(\mathbb{R}^n)}\omega_p(v_1, ..., v_n)$, since $id_{\mathbb{R}^n *}(p) = id_{\mathbb{R}^n}(p) = p$. So the pull-back along the identity function on $\mathbb{R}^n$ gives the identity on the pre-smooth space $\Omega^m(\mathbb{R}^n)$.
2. Now given $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $g : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$, $\Omega^m(g \circ f)(\omega_p(v_1, ..., v_n))$ $= (g \circ f)^*\omega_p(v_1, ..., v_n) = \omega_{g\circ f(p)}((g \circ f)_*v_1, ..., (g \circ f)_*v_n) = \omega_{g\circ f(p)}(g_* \circ$ $f_*v_1, ..., g_*\circ f_*v_n) = f^*\omega_{f(p)}(g_*v_1, ..., g_*v_n) = f^*\circ g^*\omega_p(v_1, ..., v_n) = \Omega^m(f)$ $\circ\Omega^m(g)(\omega_p(v_1, ..., v_n))$.
3. Now we check that Ω^m satisfies the sheaf condition. Define:

$$h : \Omega^m(\mathbb{R}^n) \to Match(\Omega^m, \{U_i \hookrightarrow \mathbb{R}^n\})$$
$$\omega \mapsto \{\omega|_{U_i}\}$$

Given two $\omega, \omega' \in \Omega^m(\mathbb{R}^n)$ that agree on a matching family, they agree on every point $p \in \mathbb{R}^n$ so they are equal. So h is injective.

Given a matching family $\{\omega|_{U_i}\}_i$ then we can define $\omega := \omega|_{U_i}$ on U_i. Given a $p \in \mathbb{R}^n$ then ω_p is a m-form in p. So we can conclude ω is a m-form. So h is surjective.

So h is a bijection.

We can now conclude that Ω^m is a smooth space – the smooth space of all smooth m-forms! □

Using this smooth space of differential forms, we may now in section 2.2 give a useful reformulation of symplectic geometry as it appears in the physics of phase space.

2.2 Symplectic smooth spaces

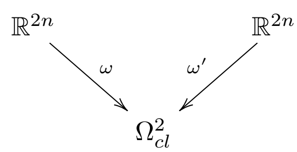
In the previous section we defined smooth spaces, which are a generalization of smooth manifolds. We will now show how the symplectic manifolds relate to these smooth spaces.

Remark 14. By the Yoneda lemma (see appendix B.2) we have:

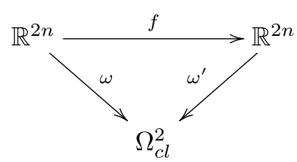
$$\mathrm{Hom}_{\mathrm{SmoothSpaces}}(\mathbb{R}^{2n}, \Omega^2_{cl}) \cong \Omega^2_{cl}(\mathbb{R}^{2n})$$

This says that 2-forms in $\Omega^2_{cl}(\mathbb{R}^{2n})$ are naturally identified with maps of smooth spaces of the form $\omega : \Omega^2_{cl} \mapsto \mathbb{R}^{2n}$.

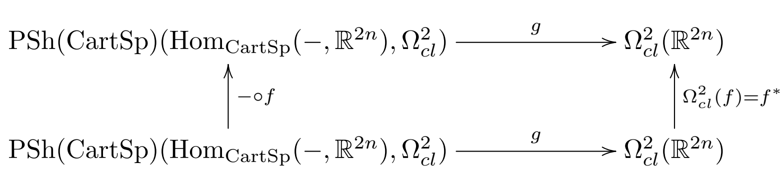
Theorem 15. Given two symplectic manifolds of the form $(\mathbb{R}^{2n}, \omega)$ and $(\mathbb{R}^{2n}, \omega')$, thought of by remark 14 as maps of smooth spaces



then symplectomorphisms $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ between them (def. 49) are equivalently diagrams of smooth spaces of the form:



Proof. $\Longrightarrow$ Given two symplectic structures $\omega, \omega' \in \mathrm{PSh}(\mathrm{CartSp})(\mathbb{R}^{2n}, \Omega^2_{cl})$ and a symplectomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Since f is a symplectomorphism it is a bijective smooth function from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ for which we have a pull-back f^*. By the Yoneda lemma (see appendix B.2) we have a natural isomorphism g : $\mathrm{PSh}(\mathrm{CartSp})(\mathbb{R}^{2n}, \Omega^2_{cl}) = \mathrm{PSh}(\mathrm{CartSp})(\mathrm{Hom}_{\mathrm{CartSp}}(-, \mathbb{R}^{2n}), \Omega^2_{cl}) \xrightarrow{\cong} \Omega^2_{cl}(\mathbb{R}^{2n})$. This gives a diagram:



Tracing an element ω' from the bottom left of the diagram to the top right we get: $g(\omega' \circ f)$ clockwise and $f^*(g(\omega')) = g \circ \omega'(f) = g(\omega' \circ f)$ counter-clockwise, which implies that this diagram commutes.

$\Longleftarrow$ Given a commuting diagram:

$$\begin{array}{ccc} \mathbb{R}^{2n} & \xrightarrow{f} & \mathbb{R}^{2n} \\ & {\scriptstyle \omega}\searrow \quad \swarrow{\scriptstyle \omega'} & \\ & \Omega^2_{cl} & \end{array}$$

ω and ω' correspond to representable presheaves, as defined in definition 57. These representable presheaves are $\mathbb{R}^{2n} \xrightarrow{\omega} \Omega^2_{cl}$ and $\mathbb{R}^{2n} \xrightarrow{\omega'} \Omega^2_{cl}$. Combining these with the given $f \in \mathrm{Hom}(\mathrm{CartSp})$, this results in the diagram below:

$$\begin{array}{ccc} \mathbb{R}^{2n} & \xrightarrow{\omega} & \Omega^2_{cl}(\mathbb{R}^{2n}) \\ \downarrow{\scriptstyle f} & & \uparrow{\scriptstyle f^*} \\ \mathbb{R}^{2n} & \xrightarrow{\omega'} & \Omega^2_{cl}(\mathbb{R}^{2n}) \end{array}$$

Here we have $f^* = \Omega^2_{cl}(f)$, since Ω^2_{cl} is a smooth space, which implies we have a pull-back. So we have that f is smooth since it is a element in CartSp and we have a pull-back along symplectic structures, which implies that f is a symplectomorphism. $\square$

Remark 16. In terms of the concept of slice categories (def. 58) one may formulate theorem 15 as follows: when a symplectic smooth space (X, ω) is regarded as an map $\omega : X \to \Omega^2_{\mathrm{cl}}$ via remark 14, hence as an object in the slice category $\mathrm{SmoothSpaces}_{/\Omega^2_{\mathrm{cl}}}$, then morphisms in the slice category between such objects are equivalently symplectomorphisms. In particular this means that there is a fully faithful embedding:

$$\mathrm{SymplecticSmoothManifolds} \hookrightarrow \mathrm{SmoothSpaces}_{/\Omega^2_{\mathrm{cl}}}$$

3 Smooth groupoids – Stacks

In this section we will generalize the notion of a smooth space even further. We will motivate this generalization by an example in physics. First we take a look at the original formulation of electromagnetics according to Maxwell. Take two observers who are measuring the electromagnetic field strength F, a 2-form. They will stricly agree on this, minus of course measurement errors. In example 46 in appendix A it is shown how Maxwell's equations can be expressed in 2-forms.

An electromagnetic field strength F corresponds locally to a 1-form $\mathcal{A}$ in the sense that $F = \mathbf{d}\mathcal{A}$. We call this $\mathcal{A}$ the electromagnetic vector potential. This more general notion is according to Dirac. When measuring $\mathcal{A}$, two observers will only agree on their data up to gauge transformation, which means that multiple vector potentials will correspond to the same field strength. This is an example of a groupoid.

The electromagnetic field strength varies smoothly in a space. The same goes for vector potentials relating to that field strength. To incorporate this into our theory we will generalize smooth spaces to smooth groupoids.

We will now turn to the definition of a groupoid.

3.1 Groupoids

In this section we study the mathematics that allow us to deal with fields for which we have a gauge equivalence. The structures we study for this are the groupoids.

Groupoids are a generalization of groups. A groupoid consists of a set of objects and a set of morphisms f. The demands on the morphisms are equivalent to that of a group. So for these we have:

1. Composition of morphisms: if $x_1 \xrightarrow{f_1} x_2, x_2 \xrightarrow{f_2} x_3 \in \mathcal{G}_1$, then $f_2 \circ f_1 = x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \in \mathcal{G}_1$
2. For each $x \in \mathcal{G}_0$ we have an identity morphism: $x \xrightarrow{id_x} x \in \mathcal{G}_1$
3. For every $f \in \mathcal{G}_1$ we have an inverse $f^{-1} : x \xrightarrow{f^{-1}} x$ such that $f^{-1} \circ f = id_x \in \mathcal{G}_1$

As can be seen, a groupoid whose set of objects is a singleton, is a group. Now that we know what a groupoid is made of, we turn to the defintion.

Definition 17. A **small groupoid** is a made up of three sets:

1. X_0, the set of objects.
2. X_1, the set of morphisms.
3. $X_1 \circ_{X_0} X_1$, the set of composable morphisms. We shall call this the fiber product of X_1 over X_0.

and the following four assignments:

$s : X_1 \to X_0$ assigns to each morphism $g \in X_1$ an element $x \in X_0$, which we shall call the source of g.

$t : X_1 \to X_0$ assigns to each morphism $g \in X_1$ an element $x \in X_0$, which we shall call the target of g.

$i : X_0 \to X_1$ assigns to each $x \in X_0$ an identity element $g \in X_1$, such that $t(g) = s(g) = x$.

$\circ : X_1 \circ_{X_0} X_1 \to X_1$ assigns to each two elements $g, h \in X_1$ with $t(g) = s(h)$ a composition. A function $f \in X_1$ such that $g \circ h = f$.

and lastly we demand:

i. $\circ$ is associative. This means for composable $f, g, h \in X_1$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

ii. i is a unit for $\circ$. This means that for $f \in X_1$, we have $f = f \circ i = i \circ f$.

iii. Every morphism has an inverse under $\circ$.

We can summarize the above in the following diagram:

$$X_0 \; \begin{array}{c} \xrightarrow{\;s\;} \\ \xleftarrow{\;i\;} \\ \xrightarrow{\;t\;} \end{array} \; X_1 \; \xleftarrow{\;\circ\;} \; X_1 \circ_{X_0} X_1$$

Remark 18. We shall also call the objects *configurations* and the morphisms *transformations*. These terms are motivated by field configurations and gauge transformations in physics.

Remark 19. We add *small* to the definition of a groupoid in definition 17 because the configurations form a set, not a proper class.

Remark 20. We could have equivalently defined a small groupoid as a category in which the objects form a set and all morphisms are isomorphisms.

Since we are concerned with the structure on groupoids, we now turn to functors. The following definition is relevant for equivalences between groupoids.

Definition 21. A **functor** $F_\bullet$ between groupoids X and Y is a function $F_0 : X_0 \to Y_0$ and a function $F_1 : X_1 \to Y_1$ such that the diagram below commutes.

$$\begin{array}{ccc} X_1 \circ_{X_0} X_1 & \xrightarrow{(F_1, F_1)} & Y_1 \circ_{Y_0} Y_1 \\ \circ \downarrow & & \downarrow \circ \\ X_1 & \xrightarrow{\;F_1\;} & Y_1 \\ s \downarrow \; i \uparrow \; t \downarrow & & s \downarrow \; i \uparrow \; t \downarrow \\ X_0 & \xrightarrow{\;F_0\;} & Y_0 \end{array}$$

In the above diagram commutation implies that if we trace an element two ways through the diagram, that both paths agree. Let us chase a $x \in X_1$ for instance to Y_0, so we have $t(F_1(x)) = F_1(t(x))$ and $s(F_1(x)) = F_1(s(x))$. It captures that the source and target of a morphism f are mapped to the source and target of $F_1(f)$ and also that this mapping respects the identity and composition. Commutativity here does not mean that $s(F_1(x))$ and $F_1(t(x))$ agree!

Definition 22. We call a functor between two groupoids an **equivalence** if it is a equivalence of categories as defined in appendix B definition 54.

We now have the category of groupoids. The objects are groupoids and the morphisms are the functors between them. The isomorphisms are given exactly by the equivalences as defined above. We denote this category:

$$\mathbf{Grpd}$$

Remark 23. Let us check that the above definition is equivalent to definition 51 given in appendix B. Given a diagram of the form $x_1 \xrightarrow{f} x_2$ in $X_\bullet$, then we have for $F_1(f)$ in Y_1: $F_1(x_1) = y_1$ and $F_1(x_2) = y_2$ for certain $y_1, y_2 \in Y_0$. Furthermore writing this out gives us: $s(F_1(g)) = y_1 = F_0(x_1) = F_0(s(g))$ and $t(F_1(g)) = y_2 = F_0(x_2) = F_0(t(g))$, since the diagram commutes.

This tells us that for every morphism in $X_\bullet$, we have a morphism in $Y_\bullet$. Also source and target are preserved. So given two morphisms in $X_\bullet$, $x_1 \xrightarrow{f} x_2$ and $x_2 \xrightarrow{g} x_3$. We have $F_0(x_1) \xrightarrow{F_1(f)} F_1(x_2) = y_1 \xrightarrow{f'} y_2$ and $F_0(x_2) \xrightarrow{F_1(g)} F_1(x_3) = y_2 \xrightarrow{g'} y_3$. For the composition we have $F_0(x_1) \xrightarrow{F_1(f\circ g)} F_0(x_3) = y_1 \xrightarrow{(f\circ g)'} y_3$. So the above definition respects composition.

For the functor working on identity morphisms we have: $F_0(x) \xrightarrow{F_1(id_x)} F_0(x) = F_0(x) \xrightarrow{id_{F_0(x)}} F_0(x)$.

This verifies that our above definition of functors agrees with the definition given in appendix B.

3.2 The electromagnetic field configurations as a groupoid

In physics we are used to dealing with global field configurations on $\mathbb{R}^n$. This is justified by the Poincaré lemma, which says that closed forms are exact on contractible manifolds. In terms of differential forms the electromagnetic field strength is a closed 2-form as explained in appendix A. So the Poincaré lemma says that there is a 1-form $\mathcal{A}$ such that $\mathbf{d}\mathcal{A} = F$. On $\mathbb{R}^n$ it is therefore logical to define the groupoid of gauge fields as the smooth space that assigns 1-forms to every $\mathbb{R}^n$. The gauge transformations are the 0-forms. We shall call this groupoid $\mathbb{B}U(1)_{conn}$.

We will now turn to the following assignment. It sends a spacetime $\mathbb{R}^4$ to the electromagnetic field configurations $\mathbb{B}U(1)_{\text{conn}}(\mathbb{R}^4)$. This is the groupoid whose objects are the 1-forms on $\mathbb{R}^4$ (see example 46) and whose morphisms are pairs of 0-forms and 1-forms on $\mathbb{R}^4$, where the 1-form serves to remember what the source of the morphism is:

$$\mathbb{R}^4 \mapsto \mathbb{B}U(1)_{\text{conn}}(\mathbb{R}^4) \in \mathbf{Grpd}$$

This forms a groupoid as we shall prove in lemma 25. For clarification we give the internal definition of this groupoid (def. 17). Notice here that what we know as *gauge transformations* in physics are the morphisms of this groupoid:

$$
\mathbb{B}U(1)_{\text{conn}}(\mathbb{R}^4) \quad = \quad
\begin{array}{ccl}
\Omega^1(\mathbb{R}^4)\times C^\infty(\mathbb{R}^4,U(1)) = & \{\mathcal{A} \xrightarrow{g} \mathcal{A}^g = \\
& \mathcal{A}+\mathbf{d}g = t(\mathcal{A},g)\} \\
s\Big\downarrow \Big\uparrow i \Big\downarrow t & \\
\Omega^1(\mathbb{R}^4) & =[\mathcal{A}] = \{\mathcal{A}_0dx^0+\mathcal{A}_1dx^1 \\
& \quad +\mathcal{A}_2dx^2+\mathcal{A}_3dx^3\}
\end{array}
$$

Here it is made explict that the objects of $\mathbb{B}U(1)_{\text{conn}}$ are given by electromagnetic field configurations $\mathcal{A}$ and the morphisms are given by pairs $(\mathcal{A}, g)$ and $g \in C^\infty(\mathcal{A}, U(1))$.

We now turn to the definition of the electromagnetic field configurations as a groupoid on given $\mathbb{R}^n$.

Definition 24. We define a groupoid $\mathbb{B}U(1)_{conn}(\mathbb{R}^n)$ for a spacetime $\mathbb{R}^n$ by:

1. the objects are $\mathbb{B}U(1)_0 := \Omega^1(\mathbb{R}^n)$, the 1-forms on $\mathbb{R}^n$.
2. the morphisms are $\mathbb{B}U(1)_1 := \Omega^1(\mathbb{R}^n)\times\Omega^0(\mathbb{R}^n,\mathbb{R}/\mathbb{Z}) = \Omega^1(\mathbb{R}^n)\times C^\infty(\mathbb{R}^n,\mathbb{R}/\mathbb{Z})$, which are pairs $(\mathcal{A}, f)$ where f is a smooth function into $\mathbb{R}/\mathbb{Z}$.

We have now assigned to each morphism a source, target and identity. To two morphisms we assign a composition:

1. define the source map $s : \mathbb{B}U(1)_1 \to \mathbb{B}U(1)_0$ for which $s((\mathcal{A}, f)) := \mathcal{A}$.
2. define the target map $t : \mathbb{B}U(1)_1 \to \mathbb{B}U(1)_0$ for which $t((\mathcal{A}, f)) := \mathcal{A}+\mathbf{d}f$.
3. define the identity map $i : \mathbb{B}U(1)_0 \to \mathbb{B}U(1)_1$ for which $i(\mathcal{A}) := (\mathcal{A}, 0)$.
4. given two morphisms such that $t((\mathcal{A}, f)) = s((A', g))$, we define the composition: $\circ((\mathcal{A}, f),(\mathcal{A}', g)) := (\mathcal{A}, f + g)$.

Lemma 25. $\mathbb{B}U(1)_{conn}(\mathbb{R}^n)$ is a groupoid.

Proof. We will now check that the defined groupoid satisfies the demands given in definition 17:

1. Given two composable morphisms $((\mathcal{A}, f),(\mathcal{A}', g)),((\mathcal{A}', g),(\mathcal{A}'', h)) \in \mathbb{B}U(1)_1 \times_{\mathbb{B}U(1)_0} \mathbb{B}U(1)_1$, we have $((\mathcal{A}, f)\circ(\mathcal{A}', g))\circ(\mathcal{A}'', h) = (\mathcal{A}, f+g)\circ(\mathcal{A}'', h) = (\mathcal{A},(f+g)+h) = (\mathcal{A}, f+(g+h)) = (\mathcal{A}, f)\circ(\mathcal{A}', g+h) = (\mathcal{A}, f)\circ((\mathcal{A}', g)\circ(\mathcal{A}'', h))$. Proving that $\circ$ is associative.

2. Now we check that the identity morphism satisfies $i \circ (\mathcal{A}, f) = (\mathcal{A}, f) \circ i$ for every $(\mathcal{A}, f) \in \mathbb{B}U(1)_1$. We have $i(\mathcal{A}+\mathbf{d}f)\circ(\mathcal{A}, f) = (\mathcal{A}+\mathbf{d}f, 0)\circ(\mathcal{A}, f) = (\mathcal{A}, f+0) = (\mathcal{A}, 0+f) = (A, f)\circ(\mathcal{A}, 0) = (\mathcal{A}, f)\circ i(\mathcal{A})$, which verifies that i acts as the unit on morphisms.

3. Given a morphism $(\mathcal{A}, f) \in \mathbb{B}U(1)_1$ and given $f \in C^\infty(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$ then we have a $-f \in C^\infty(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$; an inverse given by $(\mathcal{A} + \mathbf{d}f, -f)$, since $(\mathcal{A} + \mathbf{d}f, -f) \circ (\mathcal{A}, f) = (\mathcal{A}, f + -f) = (A, 0) = i(\mathcal{A})$ as demanded. □

Corollary 26. Thus far we have only worked on electromagnetic gauge fields on Cartesian spaces. For electromagnetic field configurations $\mathcal{A}$ on a spacetime $\mathbb{R}^4$ the Yoneda lemma gives a natural correspondence: $\mathrm{Hom}_{\mathrm{Grpd}}(\mathbb{R}^n, \mathbb{B}U(1)_{conn}) \simeq \mathbb{B}U(1)_{conn}(\mathbb{R}^n)$. This motivates us to generalize this definition to smooth manifolds. We are interested in what happens when we look at the morphism $X \to \mathbb{B}U(1)_{conn}$ mapping a manifold to $\mathbb{B}U(1)_{conn}$. If we could apply our current version of the Yoneda lemma B.2 on the groupoid of morphisms from a manifold X into $\mathbb{B}U(1)_{conn}$ this would give the groupoid of electromagnetic field configurations on the manifold X. With our currect tools we can't define this yet on a smooth manifold X. We will turn to $S^2 \to \mathbb{B}U(1)_{conn}$ later on, which is the Dirac monopole.

Definition 27. Given the groupoid $\mathbb{B}U(1)_{conn}(\mathbb{R}^4)$ and the smooth space Ω^2_{cl}, we have a functor $F(-)$, that sends gauge equivalent elements in $\mathbb{B}U(1)_{conn}$ to the corresponding field strength and the gauge transformations between these to the identity function. This is shown by the following diagram:

$$\mathbb{B}U(1)_{conn}(\mathbb{R}^4) \xmapsto{F_{(-)}} \Omega^2_{cl}(\mathbb{R}^4)$$

$$\begin{array}{ccc} \mathcal{A} & \xmapsto{\quad g \quad} & \mathbf{d}\mathcal{A} = F \\ \downarrow & & \downarrow{\scriptstyle id} \\ \mathcal{A}^g & \longmapsto & \mathbf{d}\mathcal{A}^g = F \end{array}$$

where $\mathbf{d}\mathcal{A}^g = F$ follows because $\mathbf{d}\mathcal{A}^g = \mathbf{d}(\mathcal{A} + \mathbf{d}g) = \mathbf{d}\mathcal{A} = F$.

Remark 28. Here $F = \mathbf{d}A$ is the Faraday tensor, encoding the electromagnetic field strength.

3.3 Smooth groupoids and the locality principle

We are now going to generalize smooth spaces to smooth groupoids. We have just discussed what a groupoid is. The question remains: what does it mean for a groupoid to be smooth? We will now look at matching families of groupoids. The main difference with the preceding discussion on smooth spaces is that now the matching family is a groupoid itself. This means we shall look for a way to define an equivalence of groupoids.

We shall also limit ourselves to good open covers of $\mathbb{R}^n$. We can do this without losing generality of our spaces.

Definition 29. A **pre-smooth groupoid** is a groupoid-valued presheaf, hence a functor $\mathbb{X}_\bullet : \mathrm{Cart}^{op} \to \mathbf{Grpd}$ such that:

1. $\mathbb{X}_\bullet$ assigns to every $n \in \mathbb{N}$ a groupoid $\mathbb{X}_\bullet(\mathbb{R}^n)$.

2. For every smooth $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have a functor $\mathbb{X}_\bullet(\phi) : \mathbb{X}_\bullet(\mathbb{R}^{n_2}) \to \mathbb{X}_\bullet(\mathbb{R}^{n_1})$ that respects composition: $\mathbb{X}_\bullet(\phi) \circ \mathbb{X}_\bullet(\rho) = \mathbb{X}_\bullet(\phi \circ \rho)$ for $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $\rho : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$. *We call this functor the pull-back.*

Definition 30. We generalize $\mathbb{B}U(1)_{conn}$ (def. 24) with the following assignment:

$$\mathbb{B}U(1)_{conn} : \qquad \begin{array}{ccc} \mathbb{R}^{n_1} & & \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_1}) \\ \downarrow \phi & \longmapsto & \uparrow \phi^* \\ \mathbb{R}^{n_2} & & \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_2}) \end{array}$$

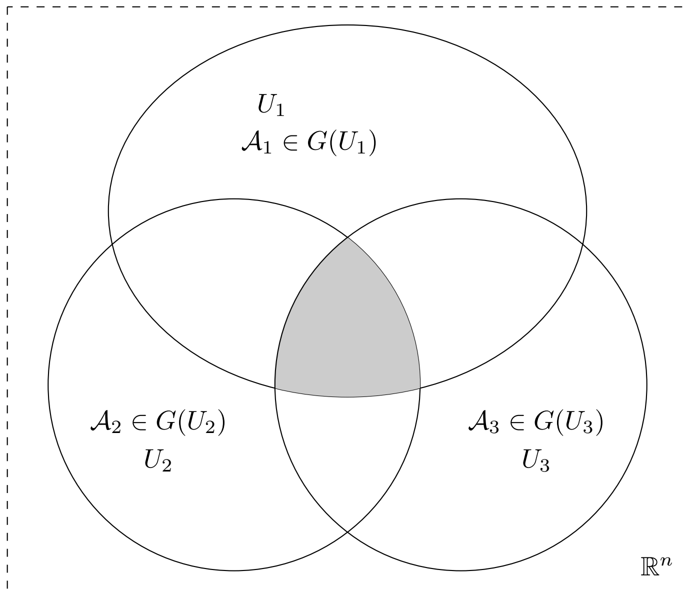
where for a smooth function $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have the pull-back $\phi^* := (-) \circ \phi : \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_2}) \to \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_1})$.

Lemma 31. $\mathbb{B}U(1)_{conn}$ is a pre-smooth groupoid.

Proof. $\mathbb{B}U(1)_{conn}$ assigns every $\mathbb{R}^n$ to a groupoid $\mathbb{B}U(1)_{conn}(\mathbb{R}^n)$ by lemma 25, so it satisfies the first demand of a smooth groupoid.

In example 13 we have shown that Ω^n_{cl} is a smooth space. So precomposition acts as a pull-back on both the objects and morphisms in $\mathbb{B}U(1)_{conn}(\mathbb{R}^n)$, from which follows that the pull-back behaves as we demanded. □

Now that we have pre-smooth groupoids, we want to pass on to smooth groupoids. In section 2 we have the sheaf condition on smooth spaces. Here we demanded that given differentially good open cover $\{U_i \to \mathbb{R}^n\}$ we have a natural bijection $X(\mathbb{R}^n) \to Match(X, \{U_i \hookrightarrow \mathbb{R}^n\})$. We will impose similar restrictions on our pre-smooth groupoids. This we shall call the stack condition.



We turn our attention to the gray part in the above diagram: the triple intersection $U_1 \cap U_2 \cap U_3$. Here we will demand that the elements $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ agree. This means we demand that we have isomorphisms $g_{ij} : \mathcal{A}_i|_{U_i \cap U_j} \to \mathcal{A}_j|_{U_i \cap U_j}$ for $i, j \in \{1, 2, 3\}$ on the double intersections. On the triple intersection we demand that: $g_{12} \circ g_{23}|_{U_1 \cap U_2 \cap U_3} \cong g_{13}|_{U_1 \cap U_2 \cap U_3}$. Equivalently we ask for the following diagram to commute:

$$\begin{array}{ccc} \mathcal{A}_1|_{U_1\cap U_2\cap U_3} & \xrightarrow[g_{12}]{\simeq} & \mathcal{A}_2|_{U_1\cap U_2\cap U_3} \\ & {\scriptstyle g_{13}}\searrow^{\simeq} \quad {}^{\simeq}\swarrow{\scriptstyle g_{23}} & \\ & \mathcal{A}_3|_{U_1\cap U_2\cap U_3} & \end{array}$$

The above theory works well when we are dealing with a single groupoid. In physics we are however dealing with different groupoids that carry the same physical information. Therefore we also need to define what makes two groupoids equivalent.

Remark 32. We can think of $\{\mathcal{A}_i\}$ and $\{\mathcal{A}'_i\}$ as vector potentials. We have shown in example 25 that these form a groupoid. So we can have mutiple vector potentials corresponding to the same field. We however want these vector potentials to be considered the "same". Now if one measurement yields $\{\mathcal{A}_i\}$ and another measurement of the same field yields $\{\mathcal{A}'_i\}$, we are able to relate these with $\{h_i\}$.

Definition 33. Given a pre-smooth groupoid $\mathbb{X}_\bullet$ and a good cover $\{U_i \to \mathbb{R}^n\}$, then the groupoid $Match(\mathbb{X}_\bullet, \{U_i \to \mathbb{R}^n\})$ of matching families is the following:

1. its objects are pairs consisting of:

   a) tuples $\{\mathcal{A}_i\}$ in $\mathbb{X}_0(U_i)$.

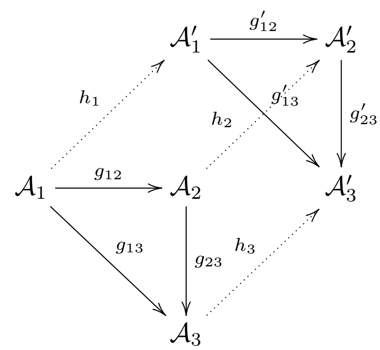
   b) tuples g_{ij} in $\mathbb{X}_1(U_i \cap U_j)$.

   such that

   a) on double intersections: $\mathcal{A}_j|_{U_i \cap U_j} = g_{ij}(\mathcal{A}_i|_{U_i \cap U_j})$

   b) on triple intersections: $g_{12} \circ g_{23}|_{U_1 \cap U_2 \cap U_3} = g_{13}|_{U_1 \cap U_2 \cap U_3}$.

2. given two tuples $\{\mathcal{A}_i\}$ and $\{\mathcal{A}'_j\}$, the morphisms of $Match(\mathbb{X}_\bullet, \{U_i \to \mathbb{R}^n\})$ are tuples $\{h_i\}$ such that the diagram below commutes:



Definition 34. Given a pre-smooth groupoid $\mathbb{X}_\bullet$ with objects $\mathcal{A} \in \mathbb{X}_0$ and morphisms $g \in \mathbb{X}_1$, then we have a groupoid $Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n, \})$. We say that $\mathbb{X}_\bullet$ is a **smooth groupoid** if we have for every $\mathbb{R}^n$ an equivalence of groupoids given by the functor:

$$\begin{aligned} F_\bullet : \mathbb{X}_\bullet(\mathbb{R}^n) &\to Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n\}) \\ \mathcal{A} &\mapsto \{\mathcal{A}_i := \mathcal{A}|_{U_i}, g_{ij} := id\} \\ g &\mapsto \{h_i := g|_{U_i}\} \end{aligned}$$

This means $F_\bullet$ is fully faithful and essentially surjective.

We shall also denote this $\mathbb{X}_\bullet \xrightarrow{\simeq} Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n, \})$ or say equivalently $\mathbb{X}_\bullet \in$ Sheaf(Grpd). The stack condition is the condition that this functor $F_\bullet$ just defined is an equivalence of groupoids for all $\mathbb{R}^n$ and all covers $\{U_i \hookrightarrow \mathbb{R}^n\}$.

Now we will show how the smooth groupoids relate to the smooth spaces. A smooth space naturally gives rise to a smooth groupoid. We show this now.

Lemma 35. Every smooth space $\mathbb{X}$ is also a smooth groupoid $\mathbb{X}_\bullet$, under the following assignments:

1. $\mathbb{X}_0(\mathbb{R}^n) := \mathbb{X}(\mathbb{R}^n)$

2. $\mathbb{X}_1(\mathbb{R}^n) := \{f \xrightarrow{id} f | f \in \mathbb{X}(\mathbb{R}^n)\}$

Furthermore the pull-back of a function works on the set of objects like it would on the smooth space and it sends identity functions to identity functions.

Proof. We have a pull-back and thus a pre-smooth groupoid. What is left to show is that we have an equivalence as defined in definition 34:

$$
\begin{aligned}
F_\bullet : \mathbb{X}_\bullet &\to Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n\}) \\
f &\mapsto \{f|_{U_i}, g_{ij} = id\} \\
id &\mapsto \{h_i = id|_{U_i}\}
\end{aligned}
$$

Since $\mathbb{X}_0$ is a smooth space, each matching tuple $\{f|_{U_i}\}$ corresponds uniquely to a $f \in \mathbb{X}_0(\mathbb{R}^n)$. This implies that the first assignment is a bijection, since each f corresponds to a unique identity function. The second assignment is also a bijection. Thus $F_\bullet$ is a bijection and an equivalence of groupoids. □

Remark 36. From this follows that we have an inclusion:

$$\text{SmoothSpaces} \hookrightarrow \text{SmoothGroupoids}$$

With a similar proof the inclusion below follows. This is however less relevant for our purposes so we omit the proof here.

$$\text{PreSmoothSpaces} \hookrightarrow \text{PreSmoothGroupoids}$$

Theorem 37. $\mathbb{B}U(1)_{conn}$ is a smooth groupoid.

Proof. We have proven in lemma 31 that electromagnetic field configurations $\mathbb{B}U(1)_{conn}$ form a pre-smooth groupoid under the assignment $\mathbb{B}U(1)_{conn} : n \mapsto \mathbb{B}U(1)_{conn}(\mathbb{R}^n)$. Composition of two smooth functions is again smooth. So given a smooth $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have a pullback given by precomposition $(-) \circ f : \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_1}) \to \mathbb{B}U(1)_{conn}(\mathbb{R}^{n_2})$.

Given a good cover $\{U_i \to \mathbb{R}^n\}$, then we define the groupoid $Match(\mathbb{B}U(1)_{conn}, \{U_i \to \mathbb{R}^n\})$ to be the following:

1. its objects are pairs consisting of

   a) tuples $\{A_i\}$ in $A_0(U_i)$ of 1-forms in each patch.

   b) tuples g_{ij} in $A_1(U_i \cap U_j)$ of 0-forms in each double intersection.

   such that

   a) on double intersections: $A_j|_{U_i \cap U_j} = A_i|_{U_i \cap U_j} + dg_{ij}|_{Ui \cap U_j}$.

   b) on triple intersections: $g_{12} \circ g_{23}|_{U_1 \cap U_2 \cap U_3} = g_{13}|_{U_1 \cap U_2 \cap U_3}$.

2. Given two tuples $\{\mathcal{A}_i\}$ and $\{\mathcal{A}'_j\}$, the morphisms of $Match(\mathbb{R}_\bullet, \{U_i \to \mathbb{R}^n\})$ are equivalences $\{h_i\}$ as defined in definition 33.

We need to show that we have an equivalence as in definition 34:

$$\begin{aligned} F_\bullet : \mathbb{X}_\bullet &\to Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow X\}) \\ \mathcal{A} &\mapsto \{\mathcal{A}_i := \mathcal{A}|_{U_i}, g_{ij} := id\} \\ g &\mapsto \{h_i := g|_{U_i}\} \end{aligned}$$

This functor is well defined since given $\mathcal{A}, \mathcal{A}' \in \mathbb{B}U(1)_0$ and a global equivalence $(\mathcal{A}, g)$ such that: $\mathcal{A} \xrightarrow{g} \mathcal{A}'$. Then we have locally on a patch U_i: $\mathcal{A}|_{U_i} \xrightarrow{g|_{U_i}} \mathcal{A}'|_{U_i}$.

We will now show that $F_\bullet$ is bijective on hom-sets. Given $\mathcal{A}, \mathcal{A}' \in \mathbb{X}_\bullet(\mathbb{R}^n)$ we have $F_\bullet(\mathcal{A}) = \{\mathcal{A}, g_{ij} = id\}$ and $F_\bullet(\mathcal{A}') = \{\mathcal{A}', g'_{ij} = id\}$.

If $\{h_i = g|_{U_i}\}, \{h_i = g'|_{U_i}\} \in \mathrm{Hom}_{Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n\})}(\{\mathcal{A}_i, g_{ij}\}, \{\mathcal{A}'_i, g'_{ij}\})$ such that $\{h_i = g|_{U_i}\} = \{h_i = g'|_{U_i}\}$, equivalently $F_\bullet((g, \mathcal{A})) = F_\bullet((g', \mathcal{A}))$. So g and g' agree on all patches U_i on $\mathbb{R}^n$. Thus $g = g'$. This implies that $F_\bullet$ is injective.

Given $\{h_i = g|_{U_i}\} \in \mathrm{Hom}_{Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n\})}(\{\mathcal{A}_i, g_{ij}\}, \{\mathcal{A}'_i, g'_{ij}\})$, we define $g := g|_{U_i}$ on U_i. We have g_{ij} between all patches $\mathbb{A}|_{U_i}$ which match on the overlaps. This implies $g : \mathcal{A} \to \mathcal{A}'$. So $F_\bullet$ is surjective on hom-sets.

We will now show that $F_\bullet$ is essentially surjective. Given $\{\mathcal{A}_i, g_{ij}\} \in Match(\mathbb{X}_\bullet, \{U_i \hookrightarrow \mathbb{R}^n\})$, on every overlap of two $\mathcal{A}_i, \mathcal{A}_j$ we have gauge transformations g_{ij} such that $\mathcal{A}_i = \mathcal{A}_j - \mathbf{d}g_{ij}$. We can do this for every overlap inductively, which gives a set of the form $\{\mathcal{A}_i, g_{ij} = id\}$, an element on which $F_\bullet$ maps. □

Remark 38. Theorem 37 says that the electromagnetic vector potentials form a smooth groupoid. This means that it satisfies the stack condition as defined in definition 34. Physically this means that when we measure the electromagnetic field configurations in a given space we can find a global electromagnetic field configuration that is locally gauge equivalent to the measured values. Equivalently this means that if we measure the electromagnetic patches through a space and we glue these patches together, where the gluing is done by means of gauge equivalences such that all the patches agree, then this gives rise to a global field configuration.

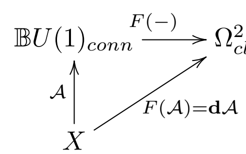
4 Outlook: examples and applications

In this section we will look at the applications of stacks in physics. We shall start by showing the concept of prequantization. After that we shall turn our attention to the Dirac monopole. Here we use $\mathbb{B}U(1)_{conn}$ to show that flux is quantized.

4.1 Prequantization

We will now discuss how the Faraday tensor F, and the vector potential $\mathcal{A}$ relate. To do this, we show in a diagram how the field strengths relate to the more general vector potentials.

Example 39. Given $\mathbb{B}U(1)_{conn}$,Ω^2_{cl} and a smooth space X, then we have a commuting diagram of the following form:



Here $\mathbb{B}U(1)_{conn}$ is a smooth groupoid by theorem 37. Now given a functor $F(-)$ that sends every $\mathcal{A} \in \mathbb{B}U(1)_{conn}(X)$ to its field strength $F \in \Omega^2_{cl}$, we can again relate the field strength to $\mathbf{d}\mathcal{A}$. Now we have a concept for field strength in more general spaces.

4.2 The Dirac monopole

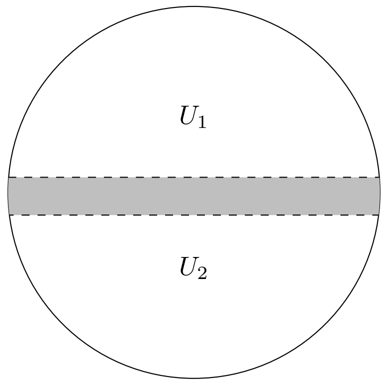
In the previous chapters we have set up all the theory to think of electromagnetic field configurations on the site of smooth manifolds. Until this point we have however only considered covers of Cartesian spaces. If we study these electromagnetic field configurations, Poincaré's lemma says that every closed 2-form on such a space is exact. With Stokes theorem we find for the flux: $\int_{D^3} F = \int_{D^3} \mathbf{d}\mathcal{A} = \int_{S^2} \mathcal{A} = 0$ (see page 156 in [3]). There is no magnetic charge in a Cartesian space. We will now show how the theory of stacks naturally gives rise to the quantization of the Dirac monopole.

The **Dirac monopole** is a spacetime $\mathbb{R}^3 \backslash \{(0,0,0)\} \times \mathbb{R}$ with a non-zero magnetic field supported by a magnetic charge at the origin. From topology we know that $\mathbb{R}^3 \backslash \{(0,0,0)\} \times \mathbb{R} \cong S^2 \times (0,\infty) \times \mathbb{R}$. Therefore the Dirac monopole is equivalently a space which is a 2-sphere.

The origin is removed from the spacetime $\mathbb{R}^4$ to ensure it is not singular. As discussed we don't have magnetic charge for the electromagnetic field configurations on Cartesian spaces. However, now something interesting happens. The Poincaré lemma doesn't hold for non-contractable manifolds. The magnetic charge of a smooth manifold does not have to be 0.

We shall now show that the flux through the Dirac monopole is quantized. Let us look at a Dirac monopole covered by two patches U_1 and U_2 that cover respectively the northern- and southern hemisphere, which overlap. On both patches we have electromagnetic field configurations such that they are gauge equivalent on the equator. We call these fields $\mathcal{A}_1$ and $\mathcal{A}_2$. This is the electromagnetic groupoid as defined in theorem 37.

The situation described can be visualized as follows:



On the overlap we have a gauge tranformation g such that $\mathbf{d}g = \mathcal{A}_2 - \mathcal{A}_1$. If we take the patch U_2 to cover all of the sphere except for the north pole. We take U_2 to have a boundery of a small size $\varepsilon > 0$. Furthermore $U_1 \cap U_2 \cong S^1$. So we get the integral of the gauge tranformation g on the overlap $\oint_{S^1} g = \oint_{S^1} \mathcal{A}_2 - \mathcal{A}_1 = \oint_{S^1} \mathcal{A}_2 - \oint_{S^1} \mathcal{A}_1 = \int_{B^2-(0,0,1)} \mathbf{d}\mathcal{A}_2 - \int_{|(1,0,0)-x|<\varepsilon} \mathbf{d}\mathcal{A}_2$, where we applied Stokes theorem and $x \in S^2$. Then $\oint_{S^1} g = \int_{B^2-(0,0,1)} \mathbf{d}\mathcal{A}_2 = \int_{B^2-(0,0,1)} F = \phi_{mag}$, the magnetic flux through the monopole.

We also have $g \in C^\infty(U_1 \cap U_2, U(1)) \cong C^\infty(S^1, S^1)$. Furthermore we can parametrize S^1 with a function $\phi(t) : [0,T] \to S^1$ which gives a unique t for every $x \in S^2$. So we have $g(x) = g(\phi(t))$ with for every x a unique $t \in [0,T]$. For g to be smooth it needs to be infinitely differentiable on its starting and end point. This gives: $g(0) = g(\phi(0)) = g(\phi(T)) = g(T)$. Now we turn to the integral $\oint_{S^1} g$. We know it to turn a fixed number of times around S^1 since it needs to be smooth. So every $g \in C^\infty(S^1, S^1)$ has to be a multiple of the integral $\oint_{S^1} id_{S^1}$. We call this number the winding number $n \in \mathbb{N}$.

Putting it all together:

$$\phi_{mag} = \oint_{S^1} g = n \oint_{S^1} id_{S^1}$$

from which we can deduce that the magnetic flux ϕ_{mag} is quantized.

A Symplectic geometry

What we are interested in in physics is information with invariant meaning. We usually express formulas in coordinates as a means to relating them to the world around us. However, coordinates in different reference frames have different meanings. This is why it is beneficial to see if we can describe the world around us by surface elements. This is done studying n-forms on spaces of different kinds.

A.1 Differential forms

In this section we will shortly review differential n-forms. See for instance page 40 of [3] and see [2].

Definition 40. Given a function $f : M \to \mathbb{R}$ where M is a manifold. We define the **differential** $f_* : M_p^n \to \mathbb{R}$ in a point $p \in M$ by $f^* := df(\mathbf{v}) = \mathbf{v}_p(f) = \sum v^j(p)\frac{\partial f}{\partial x^j}(p)$.

Definition 41. A **exterior** n**-form** is a skew symmetric covariant n-tensor $\alpha \in \bigotimes^n E^*$. So we have $\alpha : E \times E \times ... \times E \to \mathbb{R}$ with E a vector space. Skew symmetry implies that $\alpha(..., v_i, ..., v_j, ...) = -\alpha(..., v_j, ..., v_i, ...)$. We denote the space of all n-forms on a manifold M by $\Omega^n M$

Definition 42. The **pull-back** of a m-form along a function $f : M \to N$ from a M to a manifold N is defined for every $p \in M$ by: $(f^*\omega)_p(v_1, ...v_n) = \omega_{f(p)}(f_*v_1, ..., f_*v_n)$, where f_* is the differential along f.

Definition 43. Define **exterior differentiation** on n-forms by: $\mathbf{d} :\Omega^n(M) \to \Omega^{n+1}(M)$.

Let $\alpha \in \Omega^n(M)$ and $\beta \in \Omega^m(M)$, then $\mathbf{d}$ satisfies the following rules:

1. $\mathbf{d}(\alpha + \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$
2. For f a 0-form $\mathbf{d}f = \sum_{x_i} \frac{\partial f}{\partial x_i} dx_i$, the differential of f.
3. $\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^n \alpha \wedge \mathbf{d}\beta$
4. For all $n \in \mathbb{N}$ we have $\mathbf{d}^2\alpha = 0$

Example 44. The following are respectively examples of 0, 1, 2 and 3 forms.

1. A smooth function $f : M \to \mathbb{R}$ from a manifold into $\mathbb{R}$ is a 0-form.
2. The **gradient** $\mathbf{d}f = \sum_i \frac{\partial f}{\partial x_i} dx_i$ of a smooth function f is a 1-form.
3. The **curl** of a smooth 1-form $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ is given by $\mathbf{d}\omega = (\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z})dy \wedge dz + (\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x})dz \wedge dx + (\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y})dx \wedge dy$, a 2 - form.
4. The **divergence** of a smooth 2-form $\alpha = Qdx \wedge dy + Rdy \wedge dz + Sdz \wedge dx$ is given by $\mathbf{d}\alpha = (\frac{\partial Q}{\partial x} + \frac{\partial R}{\partial y} + \frac{\partial S}{\partial z})dx \wedge dy \wedge dz$, a 3 - form.

Definition 45. The **Hodge star operator** sends n-forms to their dual. Given a $\alpha \in \Omega^n(M)$ with M of dimension m and for every $p \in M$ an orthonormal basis $(e_1, ...e_n)$ with respect to a metric g, we define:

$$
\begin{aligned}
\star : &\Omega^n(M) \to \Omega^{m-n}(M) \\
&e_1 \wedge ... \wedge e_n \mapsto e_{n+1} \wedge ... \wedge e_m
\end{aligned}
$$

Example 46. Using these new operators we can define electromagnetics using forms. It turns out that we can capture electromagnetics in the following formulae.

A differential 1-form $A \in \Omega^1(\mathbb{R}^4)$ in Minkowski space has components:

$$A = A_1\mathbf{d}x^1 + A_2\mathbf{d}x^2 + A_3\mathbf{d}x^3 + \phi dx^0 \,,$$

which we may identify with the vector potential known in physics. One checks then that the exterior derivative:

$$
\begin{aligned}
F = \mathbf{d}A = &E_1\mathbf{d}x^1 \wedge \mathbf{d}x^0 + E_2\mathbf{d}x^2 \wedge \mathbf{d}x^0 + E_3\mathbf{d}x^3 \wedge \mathbf{d}x^0 \\
&+ B_1\mathbf{d}x^2 \wedge \mathbf{d}x^3 + B_2\mathbf{d}x^3 \wedge \mathbf{d}x^1 + B_3\mathbf{d}x^1 \wedge \mathbf{d}x^2
\end{aligned}
$$

as components is precisely the Faraday tensor of the corresponding electromagnetic field strength. In terms of this Maxwell's equations read:

$$
\begin{aligned}
dF &= 0 \\
\star d \star F &= J
\end{aligned}
$$

For a further discussion refer to section 3.5c of reference [3].

A.2 Symplectic manifolds

Example 47. Let $(\mathbb{R}^2, \omega)$ be the Cartesian space with $\omega := \mathbf{dp} \wedge \mathbf{dq}$. Given a smooth function $H : \mathbb{R}^2 \to \mathbb{R}$ and v a velocity vector of a particle in phase space, then we have:

$$
\begin{aligned}
dH &= \frac{\partial H}{\partial q}\mathbf{dq} + \frac{\partial H}{\partial p}\mathbf{dp} \\
i_v\omega(-,-) &:= (v_q\partial_q + v_p\partial_p)\mathbf{dq} \wedge \mathbf{dp} = v_q\mathbf{dp} - v_p\mathbf{dq} = \frac{\partial q}{\partial t}\mathbf{dp} - \frac{\partial p}{\partial t}\mathbf{dq}
\end{aligned}
$$

If we demand that $dH = i_v\omega(-,-)$, Hamiltons equations emerge:

$$\frac{\partial H}{\partial p} = \frac{\partial q}{\partial t}$$
$$\frac{\partial H}{\partial q} = -\frac{\partial p}{\partial t}$$

This example illustrates that 2-forms relate to phase space.

Definition 48. A **symplectic manifold** is a smooth manifold X with a non-degenerate closed 2-form ω.

Definition 49. A **symplectomorphism** ϕ is a diffeomorphism between two symplectic manifolds (X_1, ω_1) and (X_2, ω_2), such that $\omega_1 = \phi^*\omega_2$.

B Category theory

In this section we give a short introduction to category theory. It plays an important rôle in mathematics since it provides an ideal way to see the relationships between different theories. It studies the relationships between different objects by looking at the morphisms between them. In the most basic picture we think of objects as elements and morphisms as arrows. For a more thorough introduction to the subject see [8].

B.1 Basic definitions

Definition 50. A **category** $\mathcal{C}$ is a class of objects $\mathrm{Obj}(\mathcal{C})$ and a class $\mathrm{Hom}(\mathcal{C})$ of morphisms between objects, where a class is a collection of sets.

Morphisms have the following properties:

1. A **morphism** maps an object to another unique object. We denote the set of morphisms from a to b in $\mathcal{C}$, two objects $a, b \in \mathrm{Obj}(\mathcal{C})$ by $\mathrm{Hom}_{\mathcal{C}}(a, b)$.

2. We have composition of morphisms. Given morphisms $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$, thus $f : a \to b$ and $g : b \to c$ we have $g \circ f : a \to c$.

Morphisms satisfy the following axioms:

1. Morphisms are associative. Given $f : a \to b$, $g : b \to c$ and $h : c \to d$, $a, b, c \in \mathrm{Obj}(\mathcal{C})$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

2. For every object $a \in \mathrm{Obj}(\mathcal{C})$ we have an identity morphism $id_{\dashv} : a \to a$.

Definition 51. A **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ has the following properties:

1. $F(g \circ f) = F(g) \circ F(f)$, where $f, g \in hom(\mathcal{C})$. *Functors preserve composition.*

2. $F(id_a) = id_{F(a)}$, where $a \in \mathcal{C}$. *Functors preserve identity morphisms.*

Definition 52. Given a functor $F : \mathcal{C} \to \mathcal{D}$:

1. We call F **full** if F is surjective on every Hom-set. This means every $F : \mathrm{Hom}_C(a, b) \to \mathrm{Hom}_C(F(a), F(b))$ is surjective.

2. We call F **faithful** if F is injective on every Hom-set. This means every $F : \mathrm{Hom}_C(a, b) \to \mathrm{Hom}_C(F(a), F(b))$ is injective.

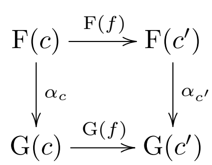
3. We call F **essentially surjective** if for every object y in $\mathcal{D}$ we have a x in $\mathcal{C}$ such that for $F(x)$ we have an isomorphism $F(x) \xrightarrow{f} y$ in $\mathcal{C}$.

If a functor is both full and faithful then we shall call it **fully faithful**.

Example 53. An example from section 2. CartSp $\hookrightarrow$ SmoothMfld $\hookrightarrow$ SmoothSpaces are all fully faithful embeddings. Here SmoothMfld is the category of smooth manifolds.

Definition 54. We shall call a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories an **equivalence** if it is fully faithful and essentially surjective.

Definition 55. Given two categories $\mathcal{C}, \mathcal{D}$ and two functors: $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, we call a transformation $\alpha : F \to G$ a **natural transformation** if the following diagram commutes for $c, c' \in \mathcal{C}$:



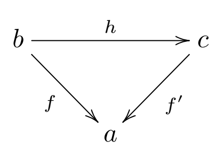
Definition 56. A **presheaf** is a functor $F : C^{\mathrm{op}} \to \mathrm{Set}$. The **category of presheaves** $\mathrm{PSh}(\mathcal{C})$ of a category $\mathcal{C}$ is defined by:

1. $\mathrm{Obj}(\mathrm{PSh}(\mathcal{C}))$ are the presheaves on $\mathcal{C}$

2. $\mathrm{Hom}(\mathrm{PSh}(\mathcal{C}))$ are the natural tranformations between presheaves on $\mathcal{C}$

Definition 57. A **representable presheaf** is a functor $F : C^{op} \to \mathbf{Set}$ that is naturally isomorphic to a functor $\mathrm{Hom}_C(-, c) \to F$ for a $c \in C$.

Definition 58. Given a category $\mathcal{C}$, the **slice category** over an object a is denoted $\mathcal{C}/a$. It is the category with $\mathrm{Obj}(\mathcal{C}/a) := \{b \xrightarrow{f} a | b \in \mathrm{Obj}(\mathcal{C})\}$, the morphisms into a. The morphisms $\mathrm{Hom}(\mathcal{C}/a)$ are given by commuting diagrams of the form:



B.2 Yoneda lemma

We recall the statement and the proof of the Yoneda lemma.

Lemma 59. Given a small category D, we have for $\mathrm{PSh}(D)$ a natural isomorphism:

$$\mathcal{A}(d) \cong \mathrm{PSh}(D)(\mathrm{Hom}_D(-, d), \mathcal{A})$$

Proof. Let F and G be objects in $\mathrm{PSh}(D)$. Thus F and G are functors from D^{op} to **Set**. Let α be a natural transformation from F to G. This gives the situation in the following diagram:

$$D^{\mathrm{op}} \underset{\mathrm{G}}{\overset{\mathrm{F}}{\underset{\Downarrow \alpha}{\rightrightarrows}}} \mathbf{Set}$$

This means that for every c', c objects in D for which we have a morphism $f : c \to c'$ in D, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{F}(c') & \xrightarrow{\mathrm{F}(f)} & \mathrm{F}(c) \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha_c} \\ \mathrm{G}(c') & \xrightarrow{\mathrm{G}(f)} & \mathrm{G}(c) \end{array}$$

Now pick $c' = d$, $\mathrm{F} = \mathrm{Hom}_D(\text{-},d)$ and denote $\mathrm{G} = \mathcal{A}$, a functor from D^{op} to **Set**:

$$\begin{array}{ccc} \mathrm{Hom}_D(d, d) & \xrightarrow{\mathrm{Hom}_D(f,d)} & \mathrm{Hom}_D(d, c) \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha_c} \\ \mathcal{A}(d) & \xrightarrow{\mathcal{A}(f)} & \mathcal{A}(c) \end{array}$$

Now we trace $id_D \in \mathrm{Hom}_D(d, d)$ through the diagram.

Clockwise this gives us: $id_D \mapsto \alpha_c(\mathrm{Hom}_D(f, d)(id_D)) = \alpha_c(f \circ id_D) = \alpha_c(f)$

Counterclockwise it gives us: $id_D \mapsto \mathcal{A}(f)(\alpha_d(id_D))$

Since the diagram commutes, we have: $\mathcal{A}(f)(\alpha_d(id_D)) = \alpha_c(f)$. If we know what element $\alpha_d(id_D)$ is mapped to, we know α_c for every $c \in D$. This implies that to every $a \in \mathcal{A}(d)$ corresponds a natural isomorphism in $\mathrm{PSh}(D)(\mathrm{Hom}_D(-, d), \mathcal{A})$.

Now we still have left to prove that the correspondence we have is a natural isomorphism. Given a morphism f from d to d', we can use precomposition to send natural isomorphisms from $\mathrm{PSh}(D)(\mathrm{Hom}_D(-, d))$ to $\mathrm{PSh}(D)(\mathrm{Hom}_D(-, d'))$:

$$
\begin{array}{ccc}
\mathrm{PSh}(D)(\mathrm{Hom}_D(-,d),\mathcal{A}) & \xrightarrow{\;g\;} & \mathcal{A}(d) \\
\uparrow{\scriptstyle -\circ f} & & \uparrow{\scriptstyle \mathcal{A}(f)} \\
\mathrm{PSh}(D)(\mathrm{Hom}_D(-,d'),\mathcal{A}) & \xrightarrow{\;g'\;} & \mathcal{A}(d')
\end{array}
$$

Let $a \in \mathrm{PSh}(D)(\mathrm{Hom}_D(-,d'),\mathcal{A})$ and g' be given bijection. We have just proved this bijection exists. For a given f the morphism $\mathcal{A}(f)$ is fixed. So if we chase a counterclockwise through the diagram, we get $(\mathcal{A}(f) \circ g')(a)$. g' is a bijection and $\mathcal{A}(f)$ is a morphism so $(\mathcal{A}(f) \circ g')$ is onto. So we can pick a bijection g that maps $a \circ f \in \mathrm{PSh}(D)(\mathrm{Hom}_D(-,d),\mathcal{A})$ to $(\mathcal{A}(f) \circ g')(a)$. This proves that the correspondence is a natural isomorphism.

□

Corrollary 60. The Yoneda embedding is given by:

$$
\begin{aligned}
\mathcal{J}(-) :& D \to \mathrm{PSh}(D) \\
& \mathrm{Hom}_D(a,b) \mapsto \mathrm{PSh}(D)(\mathrm{Hom}_D(-,a),\mathrm{Hom}(-,b))
\end{aligned}
$$

This is full and faithful. See appendix A for these definitions.

Proof. For the embedding to be fully faithful it needs to be a bijection on the Hom-sets. Thus $\mathrm{Hom}_D(a,b) \cong \mathrm{Hom}_{\mathrm{PSh}(D)}(a,b)$. The Yoneda lemma says $\mathrm{PSh}(D)(\mathrm{Hom}_D(-,a),\mathrm{Hom}(-,b)) \cong \mathrm{Hom}(a,b)$, since $\mathrm{Hom}(-,b)$ is an object in $\mathrm{PSh}(D)$. □

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