M-Theory from the Superpoint

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Abstract

The "brane scan" classifies consistent Green–Schwarz strings and membranes in terms of the invariant cocycles on super-Minkowski spacetimes. The "brane bouquet" generalizes this by consecutively forming the invariant higher central extensions induced by these cocycles, which yields the complete brane content of string/M-theory, including the D-branes and the M5-brane, as well as the various duality relations between these. This raises the question whether the super-Minkowski spacetimes themselves arise as maximal invariant central extensions. Here we prove that they do. Starting from the simplest possible super-Minkowski spacetime, the superpoint, which has no Lorentz structure and no spinorial structure, we give a systematic process of consecutive maximal invariant central extensions, and show that it discovers the super-Minkowski spacetimes that contain superstrings, culminating in the 10- and 11-dimensional super-Minkowski spacetimes of string/M-theory and leading directly to the brane bouquet.

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1 Introduction

In his "vision talk" at the annual string theory conference in 2014, Greg Moore highlighted the following open question in string theory [41, section 9]:

> Perhaps we need to understand the nature of time itself better. [. . .] One natural way to approach that question would be to understand in what sense time itself is an emergent concept, and one natural way to make sense of such a notion is to understand how pseudo-Riemannian geometry can emerge from more fundamental and abstract notions such as categories of branes.

We are going to tell an origin story for spacetime, in which it emerges from the simplest kind of supermanifold: the superpoint, denoted $\mathbb{R}^{0|1}$. This is the supermanifold with no bosonic coordinates, and precisely one fermionic coordinate. From this minimal mathematical space, which has no Lorentz structure and no spin structure, we will give a systematic process to construct super-Minkowski spacetimes up to dimension 11, complete with their Lorentz structures and spinorial structures. Indeed this is the same mathematical mechanism that makes, for instance, the M2-brane and then the M5-brane emerge from 11-dimensional spacetime. It is directly analogous to the D0-brane condensation by which 11d spacetime emerges out of type IIA spacetime of dimension 10.

To make all this precise, first recall that the super p-branes of string theory and M-theory, in their incarnation as "fundamental branes"[1] or "probe branes", are mathematically embodied in terms of what are called 'κ-symmetric Green–Schwarz-type functionals'. See Sorokin [49] for review and further pointers.

Not long after Green and Schwarz [31] discovered their celebrated action functional for the superstring, Henneaux and Mezincescu observed [32] that the previously somewhat mysterious term in the Green–Schwarz action, the one which ensures its κ-symmetry, is in a fact nothing but the WZW-type functional for super-Minkowski spacetime regarded as a supergroup. This is mathematically noteworthy, because WZW-type functionals are a natural outgrowth of super Lie algebra cohomology [3, 26]. This suggests that the theory of super p-branes is to some crucial extent a topic purely in super Lie theory, hence amenable to mathematical precision and classification tools.

Indeed, Azcárraga and Townsend [4] later showed (following [1]) that it is the $\mathrm{Spin}(d-1,1)$-invariant super Lie algebra cohomology of super-Minkowski spacetime which classifies the Green–Schwarz superstring [31], the Green–Schwarz-type supermembrane [10], as well as all their double dimensional reductions [21] [28, section 2], a fact now known as the "old brane scan".[2]

For example, for minimal spacetime supersymmetry then there is, up to a constant factor, a single non-trivial invariant $(p+2)$-cocycle corresponding to a super p-brane in d dimensional spacetime, for just those pairs of (d,p) with $d \leq 11$ that are marked by an asterisk in the following table.

[1] Notice that the conformal field theories on the worldvolume of non-fundamental "solitonic branes" or "black branes" are but the perturbation theory of these Green–Schwarz-type functionals around fundamental brane configurations stretched along asymptotic boundaries of AdS spacetimes, an observation that predates the formulation of the AdS/CFT correspondence [22, 17, 18].

[2] The classification of these cocycles is also discussed by Movshev et al. [42] and Brandt [12, 13, 14]. A unified derivation of the cocycle conditions is given by Baez and Huerta [7, 8]. See also Foot and Joshi [30].

$d \backslash p$		1	2	3	4	5	6	7	8	9
11			$\star$							
10		$\star$				$\star$				
9					$\star$					
8				$\star$						
7			$\star$							
6		$\star$		$\star$						
5			$\star$							
4		$\star$	$\star$							
3		$\star$								

Table 1: The old brane scan.

Here the entry at $d = 10$ and $p = 1$ corresponds to the Green–Schwarz superstring, the entry at $d = 10$ and $p = 5$ to the NS5-brane, and the entry at $d = 11$, $p = 2$ to the M2-brane of M-theory fame [20, chapter II]. Moving down and to the left on the table corresponds to double dimensional reduction [21] [28, section 2].

This result is striking in its achievement and its failure: On the one hand it is remarkable that the existence of super p-brane species may be reduced to a mathematical classification of super Lie algebra cohomology. But then it is disconcerting that this classification misses so many p-brane species that are thought to exist: The M5-brane in $d = 11$ and all the D-branes in $d = 10$ are absent from the old brane scan, as are all their double dimensional reductions.[3]

However, it turns out that this problem is not a shortcoming of super Lie theory as such, but only of the tacit restriction to ordinary super Lie algebras, as opposed to "higher" super Lie algebras, also called 'super Lie n-algebras' or 'super L_∞-algebras' [35, 26].[4]

One way to think of super Lie n-algebras is as the answer to the following question: Since, by a classical textbook fact, 2-cocycles on a super Lie algebra classify its central extensions in the category of super Lie algebras, what do higher degree cocycles classify? The answer ([26, Prop. 3.5] based on [24, Theorem 3.1.13]) is that higher degree cocycles classify precisely *higher* central extensions, formed in the homotopy theory of super L_∞-algebras. But in fact the Chevalley–Eilenberg algebras for the canonical models of these higher extensions are well known in parts of the supergravity literature, these are just the "free differential algebras"[5] or "FDA"s of D'Auria and Fré [2].

Hence every entry in the "old brane scan", since it corresponds to a cocycle, gives a super Lie n-algebraic extension of super-Minkowski spacetime. Notably the 3-cocycles for the superstring give rise to super Lie 2-algebras and the 4-cocycles for the supermembrane give rise to super Lie 3-algebras. These are super-algebraic analogs of the *string Lie 2-algebra* [6] [25, appendix] which controls the Green–Schwarz anomaly cancellation of the heterotic string [47], and hence they are

[3] A partial completion of the old brane scan can be achieved by classifying superconformal structures that may appear in the near horizon geometry of solitonic ("black") p-branes [11, 19].

[4] Notice that these are Lie n-algebras in the sense of Stasheff [37, 38, 46] as originally found in string field theory by Zwiebach [53, section 4] not "n-Lie algebras" in the sense of Filippov. However, the two notions are not unrelated. At least the Filippov 3-algebras that appear in the Bagger-Lambert model for coincident solitonic M2-branes may naturally be understood as Stasheff-type Lie 2-algebras equipped with a metric form [43, section 2].

[5] Unfortunately, the "free differential algebras" of D'Auria and Fré are not free. In the parlance of modern mathematics, they are differential graded commutative algebras, where the underlying graded commutative algebra is free, but the differential is not. We will thus refer to them as "FDA"s, with quotes.

called the *superstring Lie 2-algebra* [35], to be denoted $\mathfrak{string}$:

$d\backslash p$	1
10	$\star$

$\leftrightarrow$

$$\begin{array}{cc} \mathfrak{string} & \begin{array}{c}\text{extended super Minkowski}\\ \text{super Lie 2-algebra}\end{array} \\ \downarrow & \\ \mathbb{R}^{9,1|\mathbf{16}} & \begin{array}{c}\text{super Minkowski}\\ \text{super Lie algebra}\end{array} \end{array}$$

and the *supermembrane Lie 3-algebra*, denoted $\mathfrak{m2brane}$:

$d\backslash p$	2
11	$\star$

$\leftrightarrow$

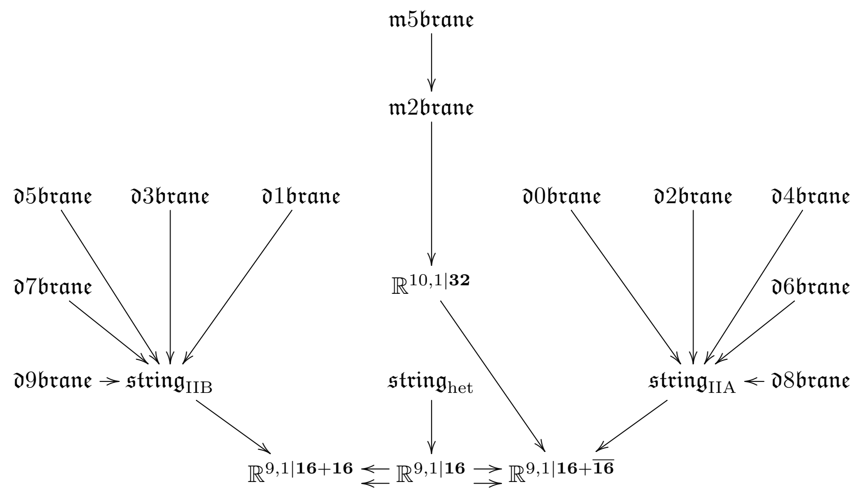
$$\begin{array}{cc} \mathfrak{m2brane} & \begin{array}{c}\text{extended super Minkowski}\\ \text{super Lie 3-algebra}\end{array} \\ \downarrow & \\ \mathbb{R}^{10,1|\mathbf{32}} & \begin{array}{c}\text{super Minkowski}\\ \text{super Lie algebra}\end{array} \end{array}$$

An exposition of these structures as objects in higher Lie theory appears in Huerta's thesis [35]. In their dual incarnation as "FDA"s these algebras are the extended super-Minkowski spacetimes considered by Chryssomalakos et al. [16]. We follow their idea, and call extensions of super-Minkowski spacetime to super Lie n-algebras such as $\mathfrak{string}$ and $\mathfrak{m2brane}$ **extended super-Minkowski spacetimes**.

Now that each entry in the old brane scan is identified with a higher super Lie algebra in this way, something remarkable happens: there appear *new* cocycles con these extended super-Minkowski spacetimes, cocycles which do not show up on plain super-Minkowski spacetime itself. (In homotopy theory, this is a familiar phenomenon: it is the hallmark of the construction of the 'Whitehead tower' of a topological space.)

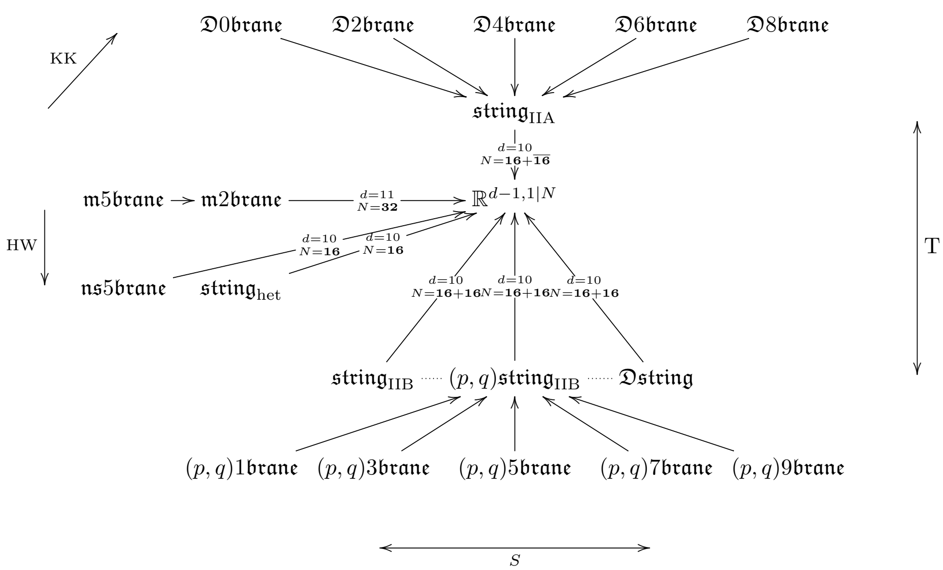
And indeed, in turns out that the new invariant cocycles thus found do correspond to the branes that were missing from the old brane scan [26]: On the super Lie 3-algebra $\mathfrak{m2brane}$ there appears an invariant 7-cocycle, which corresponds to the M5-brane, on the super Lie 2-algebra $\mathfrak{string}_{\mathrm{IIA}}$ there appears a sequence of $(p+2)$-cocycles for $p \in \{0, 2, 4, 6, 8, 10\}$, corresponding to the type IIA D-branes, and on the superstring Lie 2-algebra $\mathfrak{string}_{\mathrm{IIB}}$ there appears a sequence of $(p+2)$-cocycles for $p \in \{1, 3, 5, 7, 9\}$, corresponding to the type IIB D-branes. Under the identification of super Lie n-algebras with formal duals of "FDA"s, the algebra behind this statement is in fact an old result: For the M5-brane and the type IIA D-branes this is due to Chryssomalakos et al. [16], while for the type IIB D-branes this is due to Sakaguchi [45, section 2]. In fact, the 7-cocycle on the supermembrane Lie 3-algebra that corresponds to the M5-brane [9] was already discovered in the 1982 paper by D'Auria and Fré [2, equations (3.27) and (3.28)].

If we again denote each of these further cocycles by the super Lie n-algebra extension which it classifies [26, Prop. 3.5] [24, Theorem 3.1.13], and name these extensions by the super p-brane species whose WZW-term is given by the cocycle, then we obtain the following diagram in the category of super L_∞-algebras:



Hence in the context of higher super Lie algebra, the "old brane scan" is completed to a tree of consecutive higher central extensions emanating out of the super-Minkowski spacetimes, with one leaf for each brane species in string/M-theory and with one edge whenever one brane species may end on another [26, section 3]. This is the **brane bouquet** [26, Def. 3.9 and Section 4.5].

Interestingly, a fair bit of the story of string/M-theory is encoded in this purely super Lie-n-algebraic mathematical structure. This includes in particular the pertinent dualities: the KK-reduction between M-theory and type IIA theory, the HW-reduction between M-theory and heterotic string theory, the T-duality between type IIA and type IIB, the S-duality of type IIB, and the relation between type IIB and F-theory. All of these are reflected as equivalences of super Lie n-algebras obtained from the brane bouquet [27, 28, 36]. The diagram of super L_∞-algebras that reflects these L_∞-equivalences looks like a candidate to fill Polchinski's famous cartoon picture of M-theory [44, figure 1] [52, figure 4] with mathematical life:



Now note that not all of the super p-brane cocycles are of higher degree. One of them, the cocycle for the D0-brane, is an ordinary 2-cocycle. Accordingly, the extension that it classifies is an ordinary super Lie algebra extension. In fact one finds that the D0-cocycle classifies the central extension of 10-dimensional type IIA super-Minkowski spacetime to the 11-dimensional spacetime of M-theory. We can express these relationships by noting the following diagram of super Lie n-algebras is, in the sense of homotopy theory, a 'homotopy pullback':

$$\begin{tikzcd}
 & \mathfrak{d0brane} \arrow[dl] \arrow[dr, "\substack{\text{super } L_\infty\text{-extension} \\ \text{classified by} \\ \text{D0-cocycle}}"] & \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\substack{\text{M-theory} \\ \text{spacetime} \\ \text{extension}}"'] \arrow[rr, phantom, "(\text{pb})"] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl, "\substack{\text{super } L_\infty\text{-extension} \\ \text{classified by} \\ \text{type IIA string cocycle}}"] \\
 & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} &
\end{tikzcd}$$

This is the precise way to say that the D0-brane cocycle on $\mathfrak{string}_{\mathrm{IIA}}$ comes from pulling back an ordinary 2-cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, which in turn is extended to $\mathbb{R}^{10,1|\mathbf{32}}$ by the same 2-cocycle. Following [26, Remark 4.6] we may think of this as a super L_∞-theoretic incarnation of the observation that D0-brane condensation in type IIA string theory leads to the growth of the 11th dimension of M-theory, as explained by Polchinski [44, section 6].

This raises an evident question: Might there be a precise sense in which *all* dimensions of spacetime originate from the condensation of some kind of 0-branes in this way? Is the brane bouquet possibly rooted in the superpoint? Such that the ordinary super-Minkowski spacetimes, not just extended super-Minkowski spacetimes such as $\mathfrak{string}$ and $\mathfrak{m2brane}$, arise from a process of 0-brane condensation "from nothing"?

Since the brane bouquet proceeds at each stage by forming *maximal invariant* extensions, the mathematical version of this question is: Is there a sequence of *maximal invariant* central extensions that start at the super-point and produce the super-Minkowski spacetimes in which superstrings and supermembranes exist?

To appreciate the substance of this question, notice that it is clear that every super-Minkowski spacetime is *some* central extension of a superpoint [16, section 2.1]: the super-2-cocycle classifying this extension is just the super-bracket that turns two supercharges into a translation generator. But there are many central extensions of superpoints that are nothing like super-Minkowski spacetimes. The question hence is whether the simple principle of consecutively forming *maximal invariant* central extensions of super-Lie algebras (as opposed to more general central extensions) discovers spacetime.

We shall prove this is the case: this is our main result, Theorem 12. It says that in the following diagram of super-Minkowski super Lie algebras, each diagonal morphism is singled out as being the maximal invariant central extension of the super Lie algebra that it points to:[6]

$$
\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

Note that we do *not* specify by hand the groups under which these extensions are to be invariant. Instead these groups are being discovered stagewise, along with the spacetimes. Namely we say (Definition 6) that an extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is *invariant* if it is invariant with respect to the semi-simple factor $\mathrm{Ext}_{\mathrm{simp}}(\mathfrak{g})$ (in the sense of Lie theory) inside the external automorphism group of $\mathfrak{g}$ (Definition 1). This is a completely intrinsic concept of invariance.

We show that for $\mathfrak{g}$ a super-Minkowski spacetime, then this intrinsic group of semi-simple external automorphisms is the Spin-group cover of the Lorentz group in the corresponding dimension—this is Proposition 5. This may essentially be folklore [23, p. 95], but it seems worthwhile to pinpoint this statement. For it says that as the extension process grows out of the superpoint, not only are the super-Minkowski spacetimes being discovered as supertranslation supersymmetry groups, but also their Lorentzian metric structure is being discovered alongside.

super-Minkowski super Lie algebra	semi-simple factor of external automorphisms	induced Cartan-geometry	super-torsion freedom
$\mathbb{R}^{d-1,1\vert N}$	$\mathrm{Spin}(d-1,1)$	supergravity	in $d = 11$: Einstein equations

To amplify this, observe that with every pair (V, G) consisting of a super vector space V and a subgroup $G \subset \mathrm{GL}(V)$ of its general linear supergroup, there is associated a type of geometry, namely the corresponding *Cartan geometry*: A (V, G)-*geometry* is a supermanifold with tangent spaces isomorphic to V and equipped with a reduction of the structure group of its super frame bundle from $\mathrm{GL}(V)$ to G [40].

Now for the pairs $(\mathbb{R}^{d-1,1|N}, \mathrm{Spin}(d-1,1))$ that emerge out of the superpoint according to Proposition 5 and Theorem 12, this is what encodes a field configuration of d-dimensional N-supersymmetric supergravity: Supermanifolds locally modeled on $\mathbb{R}^{d-1,1|N}$ is precisely what underlies the *superspace* formulation of supergravity, and the reduction of its structure group to the

[6] The double arrows stand for the two different canonical inclusions of $\mathbb{R}^{d-1,1|N}$ into $\mathbb{R}^{d-1,1|N+N}$, being the identity on $\mathbb{R}^{d-1,1}$ and sending N identically either to the first or to the second copy in the direct sum $N + N$.

$\mathrm{Spin}(d-1,1)$-cover of the Lorentz group $\mathrm{SO}_0(d-1,1)$ is equivalently a choice of super-vielbein field, hence a super-pseudo-Riemannian structure, which is a field configuration of supergravity.

Observe also that the mathematically most natural condition to demand from such a super-Cartan geometry is that it be 'torsion free' [40]. In view of this it is worthwhile to recall the remarkable theorem of Howe [34], based on Candiello and Lechner [15]: For $d = 11$ the super-Einstein equations of motion of supergravity are *implied* already by the torsion freedom of the super-vielbein.

In summary, Theorem 12 shows that the brane bouquet, and with it at least a fair chunk of the structure associated with the word "M-theory", has its mathematical root in the superpoint, and proposition 5 adds that as the superspacetimes grow out of the superpoint, they consecutively discover their relevant super-Lorentzian metric structure, and finally their supergravity equations of motion.

$$
\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] \arrow[drr, phantom, "\scriptscriptstyle (\mathrm{pb})"] & & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{het}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & &
\end{tikzcd}
$$

2 Automorphisms of super-Minkowski spacetimes

For our main result, Theorem 12, we need to know the automorphisms (Definition 16) of the super Minkowski super Lie algebras $\mathbb{R}^{d-1,1|N}$ (Definition 20). The key idea is that we can extract the Lorentz symmetries of $\mathbb{R}^{d-1,1|N}$ merely from its structure as a super Lie algebra, by looking at a particular piece of the automorphisms we call the 'simple external symmetries'.

The result in this section may be folklore (see Evans [23, p. 95]), but since we did not find a full account in the literature, we provide a proof here. After some simple lemmas, the result is Proposition 5. To begin, we define the 'simple external symmetries' of the automorphisms of a super Lie algebra.

Definition 1 (external and internal symmetries)**.** Let $\mathfrak{g}$ be a super-Lie algebra (Definition 13). Its Lie algebra of *infinitesimal internal symmetries* is the stabilizer of $\mathfrak{g}_{\mathrm{even}}$ inside the automorphism Lie algebra (from Proposition 17)

$$\mathfrak{int}(\mathfrak{g}) := \mathrm{Stab}_{\mathfrak{aut}(\mathfrak{g})_{\mathrm{even}}}(\mathfrak{g}_{\mathrm{even}})\,,$$

hence is the sub-Lie algebra of derivations Δ on those which vanish on $\mathfrak{g}_{\mathrm{even}} \hookrightarrow \mathfrak{g}$. This is clearly a normal sub-Lie algebra, so that the quotient

$$\mathfrak{ext}(\mathfrak{g}) := \mathrm{aut}(\mathfrak{g})_{\mathrm{even}}/\mathfrak{int}(\mathfrak{g})$$

of all automorphisms by internal ones is again a Lie algebra, the Lie algebra of *external symmetries* of $\mathfrak{g}$, sitting in a short exact sequence

$$0 \to \mathfrak{int}(\mathfrak{g}) \hookrightarrow \mathfrak{aut}(\mathfrak{g})_{\mathrm{even}} \to \mathfrak{ext}(\mathfrak{g}) \to 0\,.$$

Finally, the Lie algebra of *simple external automorphisms*

$$\mathfrak{ext}_{\mathrm{simp}}(\mathfrak{g}) \hookrightarrow \mathfrak{ext}(\mathfrak{g}) \hookrightarrow \mathfrak{aut}(\mathfrak{g})$$

is the maximal semi-simple sub-Lie algebra of the external automorphism Lie algebra.

Example 2. The internal automorphisms (Definition 1) of the super-Minkowski Lie algebra $\mathbb{R}^{d-1,1|N}$ are the 'R-symmetries' from the physics literature [29, p. 56].

Lemma 3. *The underlying vector space of the bosonic automorphism Lie algebras* $\mathfrak{aut}(\mathbb{R}^{d-1,1|N})$ *(Proposition 17) of the super-Minkowski Lie algebras* $\mathbb{R}^{d-1,1|N}$ *(Definition 20) in any dimension* d *and for any real* $\mathrm{Spin}(d-1,1)$*-representation* N *is the graph of a surjective and* $\mathrm{Spin}(d-1,1)$*-equivariant (with respect to the adjoint action) Lie algebra homomorphism*

$$K : \mathfrak{g}_s \longrightarrow \mathfrak{g}_v\,,$$

where

$$\mathfrak{g}_s \oplus \mathfrak{g}_v \;:=\; \mathrm{im}\left(\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \longrightarrow \mathfrak{gl}(N) \oplus \mathfrak{gl}(\mathbb{R}^d)\right)$$

is the image of the canonical inclusion induced by forgetting the super Lie bracket on $\mathbb{R}^{d-1,1|N}$.

In particular the kernel of K *is the internal symmetries (Definition 1) hence the R-symmetries (Example 2):*

$$\ker(K) \simeq \mathrm{int}(\mathbb{R}^{d-1,1|N})\,.$$

Proof. Consider the corresponding inclusion at the level of groups

$$\mathrm{Aut}(\mathbb{R}^{d-1,1|N}) \longrightarrow \mathrm{GL}(N) \times \mathrm{GL}(\mathbb{R}^d)$$

with image $G_s \times G_v$. Observe that the spinor bilinear pairing

$$[-,-] \;:\; N \otimes N \longrightarrow \mathbb{R}^d$$

is surjective, because it is a homomorphism of $\mathrm{Spin}(d-1,1)$-representations, and because $\mathbb{R}^d$ is irreducible as a $\mathrm{Spin}(d-1,1)$-representation. Hence for every vector $v \in \mathbb{R}^d$ there exist $\psi, \phi \in N$ such that

$$v = [\psi, \phi]$$

and so for $(f,g) \in \mathrm{Aut}(\mathbb{R}^{d-1,1|N}) \hookrightarrow G_v \times G_s$ then f is uniquely fixed by g via

$$f(v) = [g(\psi), g(\phi)]\,.$$

The function $K\colon g \mapsto f$ thus determined is surjective by construction of G_v, is a group homomorphism because $\mathrm{Aut}(\mathbb{R}^{d-1,1|N})$ is a group, and is $\mathrm{Spin}(d-1,1)$-equivariant (with respect to the adjoint action) by the $\mathrm{Spin}(d-1,1)$-equivariance of the spinor bilinear pairing. □

Lemma 4. *Let N be a real $\mathrm{Spin}(d-1,1)$-representation in some dimension d. Then the Lie algebra $\mathfrak{g}_v$ from Lemma 3 decomposes as a $\mathrm{Spin}(d-1,1)$-representations into the direct sum of the adjoint representation with the trivial representation*

$$\mathfrak{g}_v \;\simeq\; \mathfrak{so}(d-1,1) \oplus \mathbb{R}$$

while the Lie algebra $\mathfrak{g}_s$ from Lemma 3 decomposes as a direct sum of some exterior powers of the vector representation $\mathbb{R}^d$:

$$\mathfrak{g}_s \simeq \bigoplus_i \wedge^{n_i} \mathbb{R}^d\,.$$

Proof. First assume that N is a Majorana or symplectic Majorana Spin-representation according to Example 23.

Consider the statement for $\mathfrak{g}_s$. Since the (symplectic) Majorana representation N is a real sub-representation of a complex Dirac representation $\mathbb{C}^{\dim_{\mathbb{R}}(N)}$ there is a canonical $\mathbb{R}$-linear inclusion

$$\mathrm{End}_{\mathbb{R}}(N) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\dim_{\mathbb{R}}(N)})\,.$$

Therefore it is sufficient to observe that the space of endomorphisms of the Dirac representation over the complex numbers decomposes into a direct sum of exterior powers of the vector representation. This is the case due to the inclusion

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\dim_{\mathbb{R}}(N)}) \hookrightarrow \mathrm{Cl}(\mathbb{R}^{d-1,1}) \otimes \mathbb{C}\,.$$

Explicitly, in terms of the Dirac Clifford basis of Example 23, the decomposition is given by the usual component formula

$$\psi\overline{\phi} \;=\; \left(\overline{\phi}\psi\right) + \left(\overline{\phi}\Gamma_a\psi\right)\Gamma^a + \tfrac{1}{2}\left(\overline{\phi}\Gamma_{ab}\psi\right)\Gamma^{ab} + \tfrac{1}{3!}\left(\overline{\phi}\Gamma_{a_1a_2a_3}\psi\right)\Gamma^{a_1a_2a_3} + \cdots\,.$$

Now consider $\mathfrak{g}_v$. Recall that, by definition, the bosonic automorphism group of $\mathbb{R}^{d-1,1|N}$ is

$$\mathrm{Aut}(\mathbb{R}^{d-1,1|N}) := \left\{ f \in \mathrm{GL}(\mathbb{R}^d)\,, g \in \mathrm{GL}(N) \mid \underset{\psi,\phi \in N}{\forall} \;:\; f([\psi,\phi]) = [g(\psi), g(\phi)] \right\}$$

and its Lie algebra is

$$\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) = \left\{ (X,Y) \in \mathfrak{gl}(\mathbb{R}^d) \oplus \mathfrak{gl}(N) \mid \underset{\psi,\phi \in N}{\forall} \;:\; X([\psi,\phi]) = [Y(\psi),\phi] + [\psi, Y(\phi)] \right\}\,.$$

Of course $\mathrm{Aut}(\mathbb{R}^{d-1,1|N})$ always contains $\mathrm{Spin}(d-1,1))$, acting canonically, since the spinor bilinear pairing is $\mathrm{Spin}(d,1-1)$-equivariant. Another subgroup of automorphisms that exists generally is a copy of the multiplicative group of real numbers $\mathbb{R}^\times$ where $t \in \mathbb{R}^\times$ acts on spinors ψ as rescaling by t and on vectors v as rescaling by t^2:

$$\begin{aligned} v &\mapsto t^2 v \\ \psi &\mapsto t\psi \end{aligned} .$$

The Lie algebra of this scaling action is the scaling derivations of Example 18. Hence for all d and N we have the obvious Lie algebra inclusion

$$\mathfrak{so}(d-1,1) \oplus \mathbb{R} \hookrightarrow \mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \,.$$

This shows that there is an inclusion

$$\mathfrak{so}(d-1,1) \oplus \mathbb{R} \hookrightarrow \mathfrak{g}_v \hookrightarrow \mathfrak{gl}(\mathbb{R}^d) \,.$$

Hence it now only remains to see that there is no further summand in $\mathfrak{g}_v$. But we know that there is at most one further summand in $\mathfrak{gl}(\mathbb{R}^d)$, since this decomposes in the form

$$\mathfrak{gl}(\mathbb{R}^d) \simeq \mathfrak{so}(d-1,1) \oplus \mathbb{R} \oplus \left(\mathbb{R}^d \otimes \mathbb{R}^d\right)_{\substack{\text{symmetric} \\ \text{traceless}}}$$

It follows that the only further summand that could appear in $\mathfrak{g}_v$ is the symmetric traceless $d \times d$-matrices. Now by Lemma 3, the homomorphism $K : \mathfrak{g}_s \to \mathfrak{g}_v$ is surjective, hence if the symmetric traceless matrices were a summand in $\mathfrak{g}_s$, they would have to be in the image of K. But since the symmetric traceless matrices as well as the exterior powers $\wedge^\bullet \mathbb{R}^d$ all are irreducible representations, they do not map to each other under the Spin-equivariant function K. Therefore the nature of $\mathfrak{g}_s$ established above implies that the symmetric tracless matrices do not appear in $\mathfrak{g}_v$.

This concludes the proof for the case that N is a (symplectic) Majorana representation. Now a general real spin representation is a direct multiple of N or a direct sum of the two Weyl sub-representations of $N \simeq N_- \oplus N_+$. We generalize to these cases, as follows.

First, we consider nN, a direct multiple of N, for n some nonnegative integer. Since $\mathrm{End}_{\mathbb{R}}(nN) \simeq n^2 \mathrm{End}_{\mathbb{R}}(N)$, the left hand side is indeed a sum of exterior powers.

Finally, if N decomposes as $N_- \oplus N_+$, a general spin representation is of the form $n_- N_- \oplus n_+ N_+$, for n_- and n_+ nonnegative integers. We wish to show that

$$\mathrm{End}_{\mathbb{R}}(n_- N_- \oplus n_+ N_+) \simeq n_-^2 \mathrm{End}_{\mathbb{R}}(N_-) \oplus n_- n_+ \mathrm{Hom}_{\mathbb{R}}(N_-, N_+) \oplus n_+ n_- \mathrm{Hom}_{\mathbb{R}}(N_+, N_-) \oplus n_+^2 \mathrm{End}_{\mathbb{R}}(N_+)$$

is a sum of exterior powers. Yet we have already shown that

$$\mathrm{End}_{\mathbb{R}}(N_- \oplus N_+) \simeq \mathrm{End}_{\mathbb{R}}(N_-) \,\oplus\, \mathrm{Hom}_{\mathbb{R}}(N_-, N_+) \,\oplus\, \mathrm{Hom}_{\mathbb{R}}(N_+, N_-) \,\oplus\, \mathrm{End}_{\mathbb{R}}(N_+)$$

is a sum of exterior powers. Thus, every summand on the right hand side is a sum of exterior powers, and it follows that $\mathrm{End}_{\mathbb{R}}(n_- N_- \oplus n_+ N_+)$ is also.

□

Proposition 5. *For all $d \in \mathbb{N}$ and $N \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(d-1,1))$, then the Lie algebra of external automorphism (Definition 1) of the super-Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ (Definition 20) is the direct sum*

$$\mathfrak{ext}(\mathbb{R}^{d-1,1|N}) \;\simeq\; \mathfrak{so}(d-1,1) \oplus \mathbb{R}$$

of the canonical Spin-Lorentz action (from Definition 20) and the scaling action from example 18.

Proof. By Lemma 3 we have $\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \simeq \mathfrak{g}_s$ and by Lemma 4 we have a decomposition as *Spin*-representations

$$\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \simeq (\mathfrak{so}(d-1,1) \oplus \mathbb{R}) \oplus \underbrace{\ker(K)}_{=\mathfrak{int}(\mathbb{R}^{d-1,1|N})},$$

where the last summand is the internal automorphisms (Definition 1), hence the internal R-symmetries (Example 2). Therefore the claim follows by Definition 1. $\square$

3 The consecutive maximal invariant central extensions of the superpoint

We now compute consecutive maximal invariant central extensions of the superpoint. This is our main result Theorem 12. First we state the precise definition of the extension process:

Definition 6 (maximal invariant central extensions)**.** Let $\mathfrak{g}$ be a super Lie algebra (Definition 13), let $\mathfrak{h} \hookrightarrow \mathfrak{aut}(\mathfrak{g})_{\mathrm{even}}$ be a sub-Lie algebra of its automorphism Lie algebra (Proposition 17) and let

$$\begin{tikzcd} V \arrow[r, hook] & \widehat{\mathfrak{g}} \arrow[d] \\ & \mathfrak{g} \end{tikzcd}$$

be a central extension of $\mathfrak{g}$ by a vector space V in even degree. Then we say that: $\widehat{\mathfrak{g}}$ is

1. an $\mathfrak{h}$*-invariant central extension* if the 2-cocycles that classify the extension, according to Example 15, are $\mathfrak{h}$-invariant 2-cocycles according to Definition 19;
2. an *invariant central extension* if it is $\mathfrak{h}$-invariant and $\mathfrak{h} = \mathfrak{ext}_{\mathrm{simp}}(\mathfrak{g})$ is the semi-simple part of its external automorphism Lie algebra (Definition 1);
3. a *maximal* $\mathfrak{h}$*-invariant central extension* if it is an $\mathfrak{h}$-invariant central extension such that the n-tuple of $\mathfrak{h}$-invariant 2-cocycles that classifies it (according to Example 15) is a linear basis for the $\mathfrak{h}$-invariant cohomology $H^2(\mathfrak{g},\mathbb{R})^{\mathfrak{h}}$ (Definition 19).

The *maximal invariant central extensions* (i.e. the maximal $\mathfrak{h}$-invariant central extensions for $\mathfrak{h} = \mathfrak{ext}_{\mathrm{simp}}(\mathfrak{g})$) we indicate by the following symbols:

$$\begin{tikzcd} V \arrow[r, hook] & \widehat{\mathfrak{g}} \arrow[d, "\star"] \\ & \mathfrak{g} \end{tikzcd}$$

Proposition 7. *The maximal invariant central extension (Definition 6) of the superpoint $\mathbb{R}^{0|2}$ (Example 22) is the 3-dimensional super Minkowski super Lie algebra $\mathbb{R}^{2,1|N=\mathbf{2}}$ according to Definition 20, Example 27,*

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, hook] & \mathbb{R}^{2,1|\mathbf{2}} \arrow[d, "\star"] \\ & \mathbb{R}^{0|2} \end{tikzcd}$$

with $N = \mathbf{2}$ the irreducible real representation of $\mathrm{Spin}(2,1)$ *from Proposition 27.*

Proof. Since $(\mathbb{R}^{0|2})_{\mathrm{even}} = 0$, here $\mathfrak{ext}(\mathbb{R}^{0|2}) = 0$ and so for the case of the superpoint every central extension is invariant in the sense of Definition 6.

According to Example 15 the maximal central extension is that induced by the maximal space of super Lie algebra 2-cocycles on $\mathbb{R}^{0|2}$ according to Definition 14. Since $\mathbb{R}^{0|2}$ is concentrated in odd degree and has trivial super Lie bracket, a 2-cocycle here is simply a symmetric bilinear form on $(\mathbb{R}^{0|2})_{\mathrm{odd}} = \mathbb{R}^2$. There is a 3-dimensional real vector space of such. This shows that the underlying super-vector space of the maximal central extension is $\mathbb{R}^{3|2}$. It remains to see that the induced super Lie bracket is that of 3d super-Minkowski.

If we let $\{d\theta_1, d\theta_2\}$ denote the canonical basis of $\wedge^1(\mathbb{R}^{0|2})^*$ then the space of 2-cocycles is spanned by these three elements:

$$\begin{array}{cc} d\theta^1 \wedge d\theta^1 & d\theta^1 \wedge d\theta^2 \\ & d\theta^2 \wedge d\theta^2 \end{array} .$$

By the formula for the induced central extension from Example 15, this means that the super Lie bracket is given on 2-component spinors $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ by

$$[\psi, \phi] = \begin{pmatrix} \psi_1\phi_1 & \frac{1}{2}(\psi_1\phi_2 + \phi_1\psi_2) \\ & \psi_2\phi_2 \end{pmatrix} \quad = \quad \tfrac{1}{2}\left(\psi\phi^\dagger + \phi^\dagger\psi\right) \quad .$$

As shown on the far right, by Example 27 this is indeed the spinor-to-vector pairing on the real representation $\mathbf{2}$ of $\mathrm{Spin}(2,1)$. $\square$

To deduce the following consecutive maximal invariant central extensions of $\mathbb{R}^{2,1|2}$, we first establish a few lemmas:

Lemma 8. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ from Example 25, and let $\mathbb{K} \hookrightarrow \mathbb{K}_{\mathrm{dbl}}$ be its Cayley-Dickson double according to Definition 24. Consider the induced real spin representations via Proposition 27. Then under the induced inclusion of Spin-groups*

$$\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1) \hookrightarrow \mathrm{Spin}(2\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1)$$

the irreducible real $\mathrm{Spin}(2\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1)$-representation N_{dbl} (either of the two) branches into the direct sum of the two irreducible real $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1)$-representations N_+, N_-:

$$N_{\mathrm{dbl}} \simeq N_+ \oplus N_- \,.$$

Proof. By Proposition 27 the spin representation $N_{\mathrm{dbl},+}$ is on the real vector space $\mathbb{K}^2_{\mathrm{dbl}}$. By Cayley–Dickson doubling (Definition 27), this is is given in terms of $\mathbb{K}$ as the direct sum

$$\mathbb{K}^2_{\mathrm{dbl}} \simeq \mathbb{K}^2 \oplus \mathbb{K}^2\ell \,.$$

This makes it immediate that the first summand $\mathbb{K}^2$ is N_+, as a restricted representation. We need to show that the second summand is isomorphic to N_- as we restrict the Spin action on $N_{\mathrm{dbl},+}$.

To that end observe, by the relations in the Cayley–Dickson construction (Defintion 24), that for $\psi \in \mathbb{K}^2$ and $B \in \mathrm{Mat}^{\mathrm{herm}}_{2\times 2}(\mathbb{K})$ we have the following identity:

$$\begin{aligned} B(\psi\ell) &= B(\ell\overline{\psi}) \\ &= \ell(\overline{B\psi}) \\ &= \ell\overline{(\psi B)} \\ &= (\psi B)\ell \\ &= (B_R\psi)\ell \,, \end{aligned}$$

where B_R denotes matrix multiplication with componentwise action from the right. It follows that the Clifford action of $\mathrm{Mat}^{\mathrm{her}}_{2\times 2}(\mathbb{K}) \hookrightarrow \mathrm{Mat}^{\mathrm{her}}_{2\times 2}(\mathbb{K}_{\mathrm{dbl}})$ on the summand $\mathbb{K}^2\ell$ is actually by right linear action on the factor $\mathbb{K}^2$:

$$\begin{aligned} \tilde{A}_L \circ B_L(\psi\ell) &= \tilde{A}(B(\psi\ell)) \\ &= \left(\tilde{A}_R \circ B_R(\psi)\right)\ell \end{aligned}$$

Therefore we are now reduced to showing that this right linear action

$$\psi \mapsto \tilde{A}_R \circ \tilde{B}_R(\psi)$$

is isomorphic to the left linear action with the position of the trace reversal (def. 26) exchanged:

$$\psi \mapsto A_L \circ \tilde{B}_L(\psi)\,.$$

We claim that such an isomorphism is established by

$$F : \psi \mapsto J\overline{\psi}\,,$$

with

$$J := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right).$$

To see this, use the relation

$$J\overline{A} = -\tilde{A}J$$

(which is directly checked) to deduce that

$$\begin{aligned} F(\tilde{A}_R \circ B_R(\psi)) &= J\,\overline{\tilde{A}_R \circ B_R(\psi)} \\ &= J\,\overline{\tilde{A}}\,(\overline{B}\,\overline{\psi}) \\ &= A(\tilde{B}J\overline{\psi}) \\ &= A_L \circ \tilde{B}_L(F(\psi)) \end{aligned}$$

$\square$

Lemma 9. *For* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ *(Example 25) and for* $N_\pm$ *the real* $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K})+1, 1)$*-representations from Proposition 27, then*

1. $(\mathrm{End}(N_\pm))^{\mathrm{Spin}} \simeq \left\{\begin{array}{ll} \mathbb{K} & |\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \\ \mathbb{R} & |\mathbb{K} = \mathbb{O} \end{array}\right.$

2. $\left(\mathrm{Sym}^2(N_\pm)\right)^{\mathrm{Spin}} \simeq 0$

Proof. For part 1, the algebra of $\mathrm{Spin}(k+1,1)$-equivariant real linear endomorphisms of $N_\pm$:

$$\mathrm{End}_{\mathrm{Spin}(k+1,1)}(N_\pm) = (\mathrm{End}(N_\pm))^{\mathrm{Spin}}$$

is called the *commutant* of $N_\pm$. For an irreducible representation such as $N_\pm$, Schur's lemma tells us the commutant must be a division algebra. By the Frobenius theorem, the only associative real division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$. We must now determine which case occurs, but this is done by Varadarajan [51, theorem 6.4.2].

For part 2, recall from Proposition 27 that we have an invariant pairing

$$\langle -, - \rangle \colon N_+ \otimes N_- \to \mathbb{R}.$$

Thus $N_\pm \simeq N_\mp^*$, and in particular, $\mathrm{Sym}^2 N_\pm \simeq \mathrm{Sym}^2 N_\mp^*$. But the latter is the space of symmetric pairings:

$$\mathrm{Sym}^2 N_\mp \to \mathbb{R},$$

which is a subspace of the space of all pairings on $N_\mp$. The invariant elements of the space of all pairings are tabulated according to dimension and signature mod 8 by Varadarajan [51, theorem 6.5.10]. In particular, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ where $N_\mp = \mathbb{K}^2$ are the spinors in signature $(2,1)$ and $(3,1)$ respectively, the nonzero invariant pairings are antisymmetric, so $\big(\mathrm{Sym}^2(N_\pm)\big)^{\mathrm{Spin}} = 0$. For $\mathbb{K} = \mathbb{H}, \mathbb{O}$, where $N_\mp = \mathbb{K}^2$ is the space of spinors in signature $(5,1)$ and $(9,1)$ respectively, $N_\mp$ is not self-dual, so there are no nonzero invariant pairings, and again we conclude $\big(\mathrm{Sym}^2(N_\pm)\big)^{\mathrm{Spin}} = 0$. □

Proposition 10. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then the maximal invariant central extension of $\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1|N_+ \oplus N_-}$ with $N_\pm$ the irreducible real spin representations from proposition 27, is $\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K}_{\mathrm{dbl}})+1,1|N_{\mathrm{dbl}}}$, for $\mathbb{K}_{\mathrm{dbl}}$ the Cayley–Dickson double of $\mathbb{K}$ (Definition 24):*

$$\begin{tikzcd}
\mathbb{K} \arrow[r, hook] & \mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K}_{\mathrm{dbl}})+1,1|N_{\mathrm{dbl}}} \arrow[d, "\star"] \\
& \mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1|N_+ \oplus N_-}
\end{tikzcd}$$

Proof. By Proposition 5 we need to compute the $\mathfrak{so}(\dim_{\mathbb{R}}(\mathbb{K})+1,1)$-invariant cohomology of $\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1}$ in degree 2. It is clear that such Spin-Lorentz invariant 2-cocycles need to pair two fermions. Due to the simple nature of the Lie bracket on super Minkowski spacetime, this means that we need to compute the space of $\mathfrak{so}(\dim_{\mathbb{R}}(\mathbb{K})+1,1)$-invariant symmetric bilinear forms on $N_+ \oplus N_-$. We now observe that Lemma 8 produces examples of these, and then we check that these examples already exhaust the space of possibilities.

Namely let $v \in \mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K}_{\mathrm{dbl}})+1,1}$ be any vector in the complement of $\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1}$. Then the symmetric pairing

$$\begin{aligned} N_{\mathrm{dbl},+} \otimes N_{\mathrm{dbl},+} &\longrightarrow \mathbb{R} \\ \psi \otimes \phi &\mapsto \eta(v, [\psi, \phi]) \end{aligned}$$

is clearly $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K})+1,1)$-invariant, by the Spin-equivariance of the spinor pairing (Proposition 27) and the assumption on v. But by Lemma 8 $N_{\mathrm{dbl},+}$ is $N_+ \oplus N_-$ as a $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K})+1,1)$-representation. Therefore this construction yields a $\dim_{\mathbb{R}}(\mathbb{K})$-dimensional space of Spin-invariant symmetric bilinear pairings on $N_+ \oplus N_-$. Moreover, by the very definition of the pairing above, it follows that the central extension classified by these pairings, regarded as 2-cocycles, is $\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1|\mathrm{N}_{\mathrm{dbl}}}$.

Hence to conclude the proof we are now reduced to showing that this invariant extension is in fact maximal, hence that $\dim_{\mathbb{R}}(\mathbb{K})$ already equals the space of all Spin-invariant symmetric pairings on $N_+ \oplus N_-$. The space of all symmetric pairings, invariant or not, is:

$$\mathrm{Spin}^2(N_+ \oplus N_-) \;\simeq\; \mathrm{Sym}^2(N_+) \;\oplus\; N_+ \otimes N_- \;\oplus\; \mathrm{Sym}^2(N_-)\,.$$

So, we seek the *Spin*-invariant elements of the latter space. By Lemma 9 the Spin-invariant subspaces of $\mathrm{Sym}^2(N_\pm)$ vanishes. Therefore the space of 2-cocycles is that of Spin-invariant elements in $N_+ \otimes N_-$. By the spinor-to-scalar pairing from Prop. 27 the two spaces N_+ and N_- are linear dual to each other, as Spin-representations. Therefore the Spin-invariant elements in $N_+ \otimes N_-$ are equivalently the Spin-equivariant linear endomorphisms of the form

$$N_+ \to N_+\,.$$

By Lemma 9 this space of invariant endomorphisms is identified with $\mathbb{K}$

$$(\mathrm{End}(N_+))^{\mathrm{Spin}} \simeq \mathbb{K}\,.$$

Hence the dimension of this space is $\dim_{\mathbb{R}}(\mathbb{K})$, which concludes the proof. □

Proposition 11. *The maximal invariant central extension (Definition 6) of the type IIA super-Minkowski spacetime $\mathbb{R}^{9,1|\mathbf{16}\oplus\overline{\mathbf{16}}}$ is $\mathbb{R}^{d-1,1|\mathbf{32}}$:*

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & \mathbb{R}^{10,1|\mathbf{32}} \arrow[d, "\star"] \\
& \mathbb{R}^{9,1|\mathbf{16}\oplus\overline{\mathbf{16}}}
\end{tikzcd}$$

Proof. By Proposition 4 we need to consider $\mathrm{Spin}(9,1)$-invariance. Since the extension in question is clearly $\mathrm{Spin}(9,1)$-invariant, it is now sufficient to show that the space of all $\mathrm{Spin}(9,1)$-invariant 2-cocycles on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ is 1-dimensional. As in the proof of Proposition 10, that space is equivalently the space of $\mathrm{Spin}(9,1)$-invariant elements in

$$\mathrm{Sym}^2(N_+ \oplus N_-) \simeq \mathrm{Sym}^2(N_+) \ \oplus\ N_+ \oplus N_- \ \oplus\ \mathrm{Sym}^2(N_-)\,.$$

By Lemma 9 all invariants in $\mathrm{Sym}^2(N_\pm)$ are trivial and the space of invariants in $N_+ \otimes N_1$ is one-dimensional. ☐

In summary, this proves the main theorem:

Theorem 12. *The process that starts with the superpoint $\mathbb{R}^{0|1}$ and then consecutively doubles the supersymmetries and forms the maximal invariant central extension according to Definition 6 discovers the super-Minkowski super Lie algebras $\mathbb{R}^{d-1,1|N}$ from Definition 20 in dimensions $d \in \{2,3,6,10,11\}$ and for $N = 1$ and $N = 2$ supersymmetry: there is a diagram of super Lie algebras of the following form*

$$\begin{tikzcd}
& \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\star"] & \\
& \mathbb{R}^{9,1|\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, "\star"] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
& \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl, "\star"'] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl, "\star"'] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl, "\star"'] & \\
\mathbb{R}^{0|1+1} & \mathbb{R}^{0|1} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

where each single arrow $\xrightarrow{\ \star\ }$ denotes a maximal invariant central extension according to Definition 6 and where each double arrow denotes the two evident injections (Remark 21).

Proof. This is the joint statement of Proposition 7, Proposition 10 and Proposition 11. Here we use in Proposition 10 that for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the two representations $N_\pm$ from Proposition 27 are in fact isomorphic. ☐

4 Outlook

In view of the brane bouquet [26], Theorem 12 is suggestive of phenomena still to be uncovered. Further corners of M-theory, currently less well understood, might be found by following the process of maximal invariant central extensions in other directions. Because, notice that Theorem 12 only states that the central extensions that it does show are maximal invariant central extensions, but it does not claim that there are not further maximal central extensions in other directions.

For example, the $N = 1$ superpoint $\mathbb{R}^{0|1}$ also has a maximal central extension, namely the super-line $\mathbb{R}^{1|1} = \mathbb{R}^{1,0|1}$

$$\begin{tikzcd} \mathbb{R}^{1|1} \arrow[d, "\star"] \\ \mathbb{R}^{0|1} \end{tikzcd}$$

This follows immediately with the same argument as in Proposition 7.

Therefore we next ought to ask: What is the bouquet of maximal central extensions emerging out of $\mathbb{R}^{0|3}$? It is clear that the first step yields $\mathbb{R}^{6|3}$, with the underlying bosonic 6-dimensional vector space canonically identified with the 3×3 symmetric matrices with entries in the real numbers. Now if an analogue of Proposition 10 would still be true in this case (which needs to be checked), then the further consecutive maximal invariant extensions might involve the 3×3 hermitian matrices with coefficients in $\mathbb{C}$, $\mathbb{H}$ and then $\mathbb{O}$. The last of these, $\mathrm{Mat}^{\mathrm{herm}}_{3\times3}(\mathbb{O})$ is the famous exceptional Jordan algebra or *Albert algebra* of dimension 27. Just as $\mathrm{Mat}^{\mathrm{herm}}_{2\times2}(\mathbb{O})$, equipped with the determinant function, is isomorphic to Minkowski spacetime $\mathbb{R}^{9,1}$, so $\mathrm{Mat}^{\mathrm{herm}}_{3\times3}(\mathbb{O})$ is isomorphic to the 27-dimensional direct sum $\mathbb{R}^{9,1} \oplus \mathbf{16} \oplus \mathbb{R}$ consisting of 10d-spacetime, one copy of the real 10d spinors and a scalar, see [5, section 3.4]. This kind of data is naturally associated with heterotic M-theory, and grouping its spinors together with the vectors and the scalar to a 27-dimensional bosonic space is reminiscent of the speculations about bosonic M-theory in [33]. Therefore, should the bouquet of maximal invariant extensions indeed arrive at $\mathrm{Mat}^{\mathrm{herm}}_{3\times3}(\mathbb{O})$, this might help to better understand the nature of the bosonic and/or heterotic corner of M-theory.

In a similar vein, we ought to ask how the tower of steps in Theorem 12 continues beyond dimension 11, and what the resulting structures mean.

A Background

For reference, we briefly recall here some definitions and facts that we use in the main text.

A.1 Super Lie algebra cohomology

We recall the definition of super Lie algebras and their (invariant) cohomology. All our vector spaces and algebras are over $\mathbb{R}$, and they are all of finite dimension.

Definition 13. The *tensor category of super vector spaces* is that of $\mathbb{Z}/2$-graded vector spaces, but equipped with the unique non-trivial braiding, the one which on elements v_i of homogeneous degree $\sigma_i \in \mathbb{Z}/2$ is given by

$$\tau^{\mathrm{super}} : v_1 \otimes v_2 \mapsto (-1)^{\sigma_1\sigma_2}\, v_2 \otimes v_1$$

A *super Lie algebra* is a Lie algebra internal to super vector spaces, hence is a super vector space

$$\mathfrak{g} = \mathfrak{g}_{\mathrm{even}} \oplus \mathfrak{g}_{\mathrm{odd}}$$

equipped with a bilinear map (the *super Lie bracket*)

$$[-,-] \,:\, \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

which is graded skew symmetric in that

$$[v_1, v_2] = (-1)^{\sigma_1 \sigma_2 + 1}[v_2, v_1]$$

and which satisfies the *super Jacobi identity*:

$$[v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{\sigma_1 \sigma_2}[v_2, [v_1, v_3]] \,.$$

A homomorphism of super Lie algebras $\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a linear function on the underlying vector space, such that this preserves the $\mathbb{Z}/2$-grading and respects the bracket.

Definition 14 (super Lie algebra cohomology)**.** Let V be a super $\mathbb{R}$-vector space of finite dimension. Then the *super-Grassmann algebra* $\wedge^\bullet V^*$ is the $\mathbb{Z} \times (\mathbb{Z}/2)$-bigraded-commutative associative $\mathbb{R}$-algebra, generated from the elements in $\mathfrak{g}^*_{\mathrm{even}}$ regarded as being in bidegree $(1, \mathrm{even})$, and from the element in $\mathfrak{g}^*_{\mathrm{odd}}$ regarded as being in bidegree $(1, \mathrm{odd})$, subject to the relation that for α_i two elements of homogeneous bidegree (n_i, σ_i), then

$$\alpha_1 \wedge \alpha_2 \;=\; (-1)^{n_1 n_2}(-1)^{\sigma_1 \sigma_2}\, \alpha_2 \wedge \alpha_1 \,.$$

Let $(\mathfrak{g}, [-,-])$ be a super Lie algebra of finite dimension. Then its *Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ is the dg-algebra whose underlying graded algebras is the super-Grassmann algebra $\wedge^\bullet \mathfrak{g}^*$, equipped with the differential which on $\wedge^1 \mathfrak{g}^*$ is given by the linear dual of the super Lie bracket

$$d_{\mathrm{CE}} : \wedge^1 \mathfrak{g}^* \xrightarrow{[-,-]^*} \wedge^2 \mathfrak{g}^* \hookrightarrow \wedge^\bullet \mathfrak{g}^*$$

and which is extended from there to all of $\wedge^\bullet \mathfrak{g}^*$ as a bi-graded derivation of bi-degree $(1, \mathrm{even})$.

For $p \in \mathbb{N}$ we say that a $(p+2)$*-cocycle* on $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is a d_{CE}-closed element in $\wedge^{p+2}\mathfrak{g}^*$. The *super Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is the cochain cohomology of the super-Chevalley-Eilenberg algebra:

$$H^\bullet(\mathfrak{g}, \mathbb{R}) := H^\bullet(\mathrm{CE}(\mathfrak{g})) \,.$$

Example 15. For $\mathfrak{g}$ a finite dimensional super Lie algebra, and for $\omega \in \wedge^2 \mathfrak{g}^*$ a 2-cocycle according to Definition 14, then there is a new super Lie algebra $\widehat{\mathfrak{g}}$ whose underlying super vector space is

$$\widehat{\mathfrak{g}} := \underbrace{\mathfrak{g}_{\mathrm{even}} \oplus \mathbb{R}}_{\mathrm{even}} \oplus \underbrace{\mathfrak{g}_{\mathrm{odd}}}_{\mathrm{odd}}$$

and with super Lie bracket given by

$$[(x_1, c_1), (x_2, c_2)] \;=\; ([x_1, x_2], \omega(x_1, x_2)) \,.$$

The evident forgetful morphism exhibits $\widehat{\mathfrak{g}}$ as a central extension of super Lie algebras

$$\begin{tikzcd} \mathbb{R} \arrow[r, hook] & \widehat{\mathfrak{g}} \arrow[d] \\ & \mathfrak{g} \end{tikzcd} .$$

Just as for ordinary Lie algebras, this construction establishes a natural equivalence between central extensions of $\mathfrak{g}$ by $\mathbb{R}$ (in even degree) and super Lie algebra 2-cocycles on $\mathfrak{g}$.

More generally, an central extension in even degree is by some vector space $V \simeq \mathbb{R}^n$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, hook] & \widehat{\mathfrak{g}} \arrow[d] \\ & \mathfrak{g} \end{tikzcd}$$

which is equivalently the result of extending by n 2-cocycles consecutively.

We will be interested not in the full super Lie algebra cohomology, but in the *invariant* cohomology with respect to some action:

Definition 16. For $\mathfrak{g}$ a super Lie algebra (Definition 13), then its *bosonic automorphism group* is the Lie subgroup

$$\mathrm{Aut}(\mathfrak{g})_{\mathrm{even}} \hookrightarrow \mathrm{GL}(\mathfrak{g}_{\mathrm{even}}) \times \mathrm{GL}(\mathfrak{g}_{\mathrm{odd}})$$

of the group of general degree-preserving linear maps on the underlying vector space, on those elements which are Lie algebra homomorphisms, hence which preserve the super Lie bracket.

Proposition 17. *For $\mathfrak{g}$ a super Lie algebra, then the Lie algebra of its automorphism Lie group (Definition 16)*

$$\mathfrak{aut}(\mathfrak{g})_{\mathrm{even}}$$

called the the automorphism Lie algebra *of $\mathfrak{g}$ (or* derivation Lie algebra*), is the Lie algebra whose underlying vector space is that of those linear maps $\Delta : \mathfrak{g} \to \mathfrak{g}$ which preserve the degree and satisfy the derivation property:*

$$\Delta([x,y]) = [\Delta(x), y] + [x, \Delta(y)]$$

for all $x, y \in \mathfrak{g}$. The Lie bracket on $\mathfrak{aut}(\mathfrak{g})_{\mathrm{even}}$ is the commutator operation:

$$[\Delta_1, \Delta_2] := \Delta_1 \circ \Delta_2 - \Delta_2 \circ \Delta_1 \,.$$

Example 18. The super Minkowski super Lie algebras $\mathbb{R}^{d-1,1|N}$ from Definition 20 all carry an automorphism action of the abelian Lie algebra $\mathbb{R}$ which is spanned by the *scaling* derivation that acts on vectors $v \in \mathbb{R}^d$ by

$$v \mapsto v$$

and on spinors $\psi \in N$ by

$$\psi \mapsto \tfrac{1}{2}\psi \,.$$

Definition 19. Let $\mathfrak{g}$ be a super Lie algebra (Definition 13) and let

$$\mathfrak{h} \hookrightarrow \mathfrak{aut}(\mathfrak{g})_{\mathrm{even}}$$

be a sub-Lie algebra of its automorphism Lie algebra (Proposition 17).

Write

$$(-)_{\mathrm{CE}} : \mathfrak{h} \longrightarrow \mathrm{Der}(\mathrm{CE}(\mathfrak{g}))_0$$

for the canonical Lie algebra homomorphism to the Lie algebra of derivations on the Chevalley-Eilenberg dg-algebra of $\mathfrak{g}$ (Definition 14), which on $\wedge^1 \mathfrak{g}^*$ acts as the linear dual Δ^* of Δ

$$\Delta_{\mathrm{CE}} : \wedge^1 \mathfrak{g}^* \xrightarrow{\Delta^*} \wedge^1 \mathfrak{g}^*$$

and which is extended from there to all of $\wedge^\bullet \mathfrak{g}^*$ as a derivation of vanishing bi-degree. That this commutes with the Chevalley-Eilenberg differential

$$d_{\mathrm{CE}} \circ \Delta_{\mathrm{CE}} - \Delta_{\mathrm{CE}} \circ d_{\mathrm{CE}} = 0$$

is equivalent to the Lie derivation property of Δ.

Hence those elements of $\mathrm{CE}(\mathfrak{g})$ which are annihilated by Δ_{CE} for all $\Delta \in \mathfrak{h}$ form a sub-dg-algebra

$$\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}} \hookrightarrow \mathrm{CE}(\mathfrak{g})$$

Then an $\mathfrak{h}$-*invariant* $(p+2)$-*cocycle* on $\mathfrak{g}$ is an element in $\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}}$ which is d_{CE}-closed and the $\mathfrak{h}$-*invariant cohomology* of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is the cochain cohomology of this subcomplex:

$$H^\bullet(\mathfrak{g}, \mathbb{R})^{\mathfrak{h}} := H^\bullet(\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}}) \,.$$

A.2 Super Minkowski spacetimes

We recall the definition of super-Minkowski super Lie algebras (the super-translation supersymmetry algebras, Definition 20 below) as well as their construction, on the one hand via (symplectic) Majorana spinors (Example 23 below), on the other hand via the four normed division algebras (Proposition 27 below). We freely use basic facts about spinors, as may be found for instance in [39].

Definition 20 (super Minkowski Lie algebras)**.** Let $d \in \mathbb{N}$ (spacetime dimension) and let $N \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(d-1,1))$ be a real representation of the Spin-cover of the Lorentz group in this dimension. Then *d-dimensional N-supersymmetric super-Minkowski spacetime* $\mathbb{R}^{d-1,1|N}$ is the super Lie algebra (Definition 13) whose underlying super-vector space is

$$\mathbb{R}^{d-1,1|N} := \underbrace{\mathbb{R}^d}_{\text{even}} \oplus \underbrace{N}_{\text{odd}}$$

and whose only non-trivial component of the super Lie bracket is the odd-odd-component which is given by the symmetric bilinear $\mathrm{Spin}(d-1,1)$-equivariant spinor-to-vector pairing

$$[-,-] \,:\, N \otimes_{\mathbb{R}} N \longrightarrow \mathbb{R}^d$$

which comes with any real spin representation.

There is a canonical action of $\mathrm{Spin}(d-1,1)$ on $\mathbb{R}^{d-1,1|N}$ by Lie algebra automorphisms, and the corresponding semidirect product Lie algebra is the *super Poincarée super Lie algebra*

$$\mathfrak{iso}(\mathbb{R}^{d-1,1|N}) = \mathbb{R}^{d-1,1|N} \rtimes \mathfrak{so}(d-1,1)\,,$$

often called just the *supersymmetry algebra.* In this terminology then

$$\mathbb{R}^{d-1,1|N} \hookrightarrow \mathfrak{iso}(\mathbb{R}^{d-1,1|N})$$

is the super-translation subalgebra of the supersymmetry algebra.

Remark 21 (number of super-symmetries)**.** In the physics literature the choice of real spin representation in Definition 20 is often referred to as the "number of supersymmetries". While this is imprecise, it fits well with the convention of labelling irreducible representations by their linear dimension in boldset. For example for $d = 10$ then there are two irreduible real spin representations, both of real dimension 16, but of opposite chirality, and hence traditionally denoted $\mathbf{16}$ and $\overline{\mathbf{16}}$. Hence we may speak of $N = \mathbf{16}$ supersymmetry (also called $N = 1$ or *heterotic*) and $N = \mathbf{16} \oplus \overline{\mathbf{16}}$ supersymmetry (also called $N = (1,1)$ of *type IIA*) and $N = \mathbf{16} \oplus \mathbf{16}$ supersymmetry (also called $N = (2,0)$ or *type IIB*).

In Section 3 the generalization of the last of these cases plays a central role, where for any given real spin representation N we pass to the doubled supersymmetry $N \oplus N$. Observe that the two canonical linear injections $N \to N \oplus N$ into the direct sum induce two super Lie algebra homomorphisms

$$\begin{tikzcd} \mathbb{R}^{d-1,1|N} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{R}^{d-1,1|N\oplus N} \end{tikzcd}\,.$$

The following degenerate example will play a key role:

Example 22 (superpoint)**.** For $d = 0$ then Definition 20 reduces to that of the *superpoint*: the super Lie algebra

$$\mathbb{R}^{0|\mathbf{N}}$$

which has zero super Lie bracket, and whose underlying super vector space is all in odd degree, where it is given by some vector space $\mathbf{N}$.

We will use two different ways of constructing real spin representations, and hence super-Minkowski spacetimes: via (symplectic) Majorana spinors (Example 23 below) and via real normed division algebras (Proposition 27 below).

Example 23 (Majorana representations)**.** For $d = 2\nu$ or $2\nu + 1$ then there exists a complex representation of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{d-1,1}) \otimes \mathbb{C}$, hence of the spin group $\mathrm{Spin}(d-1,1)$ on $\mathbb{C}^{2^\nu}$ such that

1. all skew-symmetrized products of $p \geq 1$ Clifford elements $\Gamma_{a_1 \cdots a_p}$ are traceless;
2. $\Gamma^\dagger_{\mathrm{temporal}} = \Gamma_{\mathrm{temporal}}$ and $\Gamma^\dagger_{\mathrm{spatial}} = -\Gamma_{\mathrm{spatial}}$.

This is the *Dirac representation* in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{Spin}(d-1,1))$. For $d = 2\nu$ this is the direct sum of two subrepresentations on $\mathbb{C}^{2^\nu - 1}$, the *Weyl representations*.

For $d \in \{3, 4, 8, 9, 10, 11\}$, there exists a real structure J on the complex Dirac representation, restricting to the Weyl representations for $d = 10$, hence a $\mathrm{Spin}(d-1,1)$-equivariant linear endomorphism $J : S \to S$ which squares to unity, $J^2 = +1$. This carves out a real representation $N := \mathrm{Eig}(J, +1)$, being the eigenspace of J of eigenvalue $+1$, whose elements are called the *Majorana spinors* inside the Dirac/Weyl representation. In this case the *Dirac conjugate* $\psi \mapsto \psi^\dagger \Gamma_0$ on elements $\psi \in \mathbb{C}^{[d/2]}$ restricts to N and is called there the *Majorana conjugation* operation denoted $\overline{(-)}$. In terms of this matrix representation then the spinor bilinear pairing that appears in Definition 20 is given by the following matrix product expression:

$$[\psi, \phi] \;=\; \left(\overline{\psi}\Gamma^a \phi\right)_{a=0}^{d-1} .$$

Similarly, for $d \in \{5, 6, 7\}$ then there exists a quaternionic structure on the Dirac representation, hence an endomorphism $\tilde{J}$ as above which however squares to minus the identity, $\tilde{J}^2 = -1$. It follows that

$$J := \begin{pmatrix} 0 & -\tilde{J} \\ \tilde{J} & 0 \end{pmatrix}$$

is a real structure on the direct sum of the Dirac representation with itself. Hence as before $N := \mathrm{Eig}(J, +1)$ is a real sub-representation, called the representation of *symplectic Majorana spinors*. For these the spinor-to-vector bilinear pairing is again of the above form.

Definition 24 (Cayley–Dickson double, see e.g. [5, section 2.2])**.** Let $\mathbb{K}$ be a real star-algebra, i.e. a real algebra (not necessarily associative) equipped with an algebra anti-automorphism $\overline{(-)} : \mathbb{K} \to \mathbb{K}$. Then the *Cayley–Dickson double* $\mathbb{K}'$ of $\mathbb{K}$ is the real star-algebra obtained from $\mathbb{K}$ by adjoining one element ℓ such that $\ell^2 = -1$ and such that the following relations hold, for all $a, b \in \mathbb{K}$:

$$a(\ell b) = \ell(\overline{a} b)\,, \quad (a\ell)b = (a\overline{b})\ell\,, \quad (\ell a)(b\ell) = -\,\overline{(ab)}\,.$$

Finally, the algebra-anti-automorphism $\overline{(-)}$ on $\mathbb{K}_{\mathrm{dbl}}$ is given by that of $\mathbb{K}$ on the elements coming from there and by $\overline{\ell} = -\ell$.

Example 25. For $\mathbb{R}$ the real numbers regarded as a star algebra with trivial involution, then its Cayley–Dickson double (Definition 24) is the complex numbers $\mathbb{C}$, the Cayley–Dickson double of these is the quaternions $\mathbb{H}$ and the Cayley–Dickson double of those is the octonions $\mathbb{O}$.

By a classical result of Hurwitz, these four are exactly the normed division algebras over the real numbers, see Baez [5] for review.

The following definition is elementary but important for the characterization of real spin representations via the real normed division algebras below in prop. 27.

Definition 26 ([48]). For $A \in \mathrm{Mat}^{\mathrm{herm}}_{2\times 2}(\mathbb{K})$ a hermitian matrix with coefficients in one of the four ral normed division algebras from example 25. Then its *trace reversal* is

$$\widetilde{A} := A - \mathrm{tr}(A) \cdot \mathbf{1} .$$

Proposition 27 ([7]). *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be one of the real algebras from Example 25. Write $\mathrm{Mat}^{\mathrm{herm}}_{2\times 2}(\mathbb{K})$ for the $\mathbb{R}$-vector space of 2×2 hermitian matrices with coefficients in $\mathbb{K}$.*

Then:

1. *There is an isomorphism of inner product spaces ("forming Pauli matrices over $\mathbb{K}$")*

$$(\mathbb{R}^{\dim_{\mathbb{R}}(\mathbb{K})+1,1}, \eta) \xrightarrow{\simeq} \left(\mathrm{Mat}^{\mathrm{herm}}_{2\times 2}(\mathbb{K}), -\det\right)$$

identifying $\mathbb{R}^d$ equipped and its Minkowski inner product

$$\eta := \mathrm{diag}(-, +, +, \cdots, +)$$

with the space of hermitian matrices equipped with the negative of the determinant operation.

2. *There are irreducible real $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1)$-representations $N_\pm$, whose underlying vector space is, in both cases,*

$$N_\pm = \mathbb{K}^2$$

and the Clifford action on which is given by

$$\Gamma_+(A)\Gamma_+(B) = \tilde{A}_L \circ B_L$$

$$\Gamma_-(A)\Gamma_-(B) = A_L \circ \tilde{B}_L ,$$

where we are identifying spacetime vectors A, B with 2×2 matrices by the above, where $\widetilde{(-)}$ is the trace reversal operation from def. 26, and where $(-)_L$ denotes the linear map given by left matrix multiplication.

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ then these two representations are in fact isomorphic and are the Majorana representation *of $\mathrm{Spin}(\dim_{\mathbb{R}}(\mathbb{K}) + 1, 1)$, while for $\mathbb{K} \in \{\mathbb{H}, \mathbb{O}\}$ they are the two non-isomorphic Majorana-Weyl representations of $\mathrm{Spin}(5, 1)$ and $\mathrm{Spin}(9, 1)$, respectively.*

3. *Under the above identifications the symmetic bilinear Spin-equivariant spinor-to-vector pairing is given by*

$$N_\pm \otimes N_\pm \longrightarrow \mathrm{Mat}^{\mathrm{herm}}_{2\times 2}(\mathbb{K})$$

$$\psi, \phi \mapsto \tfrac{1}{2}\left(\psi\phi^\dagger + \phi\psi^\dagger\right) .$$

4. *There is in addition a bilinear symmetric, non-degenerate and* Spin*-invariant spinor-to-scalar pairing give by*

$$N_\pm \otimes N_\mp \longrightarrow \mathbb{R}$$

$$\psi, \phi \mapsto \mathrm{Re}(\psi^\dagger \cdot \phi)$$

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