

Equivariant 2-Cohomotopy of 2-Tori

– incomplete draft –

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We compute covariantized equivariant 2-Cohomotopy of the torus, in degrees 0 and -1, hence the homotopy in degrees 0 and 1 of equivariant mapping spaces $\text{Map}(T^2, S^2)^G$ and of their homotopy-quotients by the equivariant diffeomorphism group $\text{Diff}(T^2)^G$, for $G \subset T^2$ ranging over point group actions of 2D space groups.

In application to solid state physics, the results may be understood [1] as classifying fragile topological phases and topological order of 2D 2-band crystalline (fractional) Chern insulators.

CONTENTS

| | | | |
|---|--|----|--|
| | | | |
| I. Introduction and Results | | 1 | |
| II. Equivariant 2-Tori | | 3 | |
| III. Equivariant 2-Spheres | | 12 | |
| IV. Equivariant 2-Cohomotopy of 2-Tori | | 12 | |
| A. Any G -Tori and Type 0 Coefficients | | 12 | |
| 1. p1 | | 12 | |
| 2. Other | | 13 | |
| B. Symmorphic Tori and Type I Coefficients | | 13 | |
| 1. pm | | 15 | |
| 2. cm | | 15 | |
| 3. p2 | | 15 | |
| 4. pmm | | 16 | |
| 5. cmm | | 16 | |
| 6. p3 | | 16 | |
| 7. p31m | | 17 | |
| 8. p3m1 | | 17 | |
| 9. p4 | | 18 | |
| 10. p4m | | 18 | |
| 11. p6 | | 18 | |
| 12. p6m | | 18 | |
| C. Symmorphic Tori and Type II Coefficients | | 18 | |
| D. Diamorphic Tori | | 18 | |
| V. The Equivariant Modular Groups | | 19 | |
| A. Generalities | | 19 | |
| B. Examples | | 20 | |
| 1. p2 | | 20 | |
| 2. p3 | | 20 | |
| VI. Conclusion and Outlook | | 22 | |
| A. Mathematical background | | 23 | |
| 1. Basic homotopy theory | | 23 | |
| 2. Equivariant homotopy theory | | 25 | |
| References | | 27 | |

I. INTRODUCTION AND RESULTS

Unstable n -cohomotopy [2, §VII] is the unstable/nonabelian cohomology theory [3, §2] whose classifying space is the n -sphere S^n , hence whose cohomol-

ogy sets $\pi^n(X)$ assigned to a space (CW-complex) X are the connected components π_0 of the mapping space (A8) from X to S^n :

$$\pi^n(X) := \pi_0 \text{Map}(X, S^n). \quad (1)$$

(Accordingly, the higher homotopy groups π_k of this mapping space may be thought of as the n -cohomotopy of X in degree $-k$.)

While dating back all the way to the 1930s, cohomotopy is only recently being revived with the understanding of its relevance for the description of *solitonic* topological effects in physics.

In particular, a seminal case of the celebrated phenomenon of *topological phases of quantum materials* (plausibly needed as hardware for much anticipated robust quantum computers) — namely that of 2-dimensional 2-band *Chern insulators* such as described by the famous *Haldane model* — is well-known to have its topological phases classified by the 2-cohomotopy of the material's *Brillouin torus* T^2 of *crystal momenta*, namely by the *winding number* of the *normalized Bloch Hamiltonian*, which is an integer

$$C \in \mathbb{Z} \simeq \pi_0 \text{Map}(T^2, S^2) \equiv \pi^2(T^2)$$

that, in this case, happens to be the (first and only) *Chern class* of the vector bundle of Bloch quantum states of the material's valence electrons.

Of course, in this case the role of cohomotopy is subtle and has traditionally been missed, because the *Hopf degree theorem* entails that the n -cohomotopy of closed n -manifolds X^n is nothing but their ordinary integral cohomology:

$$\pi^n(X^n) \simeq H^n(X^n; \mathbb{Z}) \simeq \mathbb{Z}.$$

However, this coincidence of cohomotopy with ordinary cohomology is lifted as soon as more details are resolved.

Crucially, the topological phases of realistic crystalline materials are typically constrained to respect *point group* symmetries G of the crystal lattice up to unitary conjugation of their Bloch Hamiltonians. Mathematically this means that the cohomotopy classification is refined to *G -equivariant cohomotopy*, this being the connected components of the space $\text{Map}(-, -)^G$ of *G -equivariant maps*

$$\pi_G^n(X) := \pi_0 \text{Map}(X, S^n)^G. \quad (2)$$

| Group G | Action on T^2 (§II) | Action on S^2 (§III) | Connected components $\{[p] \in \pi_0 \text{Map}(T^2, S^2)^G\}$ | Fundamental group $\pi_1 \text{Map}_p(T^2, S^2)^G$ | Proof & computation |
|------------------|-----------------------------|------------------------------|--|---|------------------------|
| | | 0 | | | §IV A |
| 1 | p1 | 0 | \mathbb{Z} | $\mathbb{Z} \times_2 \mathbb{Z}^2$ | §IV A 1 |
| Dih ₁ | pm | 0 | * | \mathbb{Z} | §IV A 2 |
| Dih ₁ | cm | 0 | * | 1 | §IV A 2 |
| Dih ₁ | pg | 0 | $\mathbb{Z}/_2$ | $\mathbb{Z}/_2 \times \mathbb{Z}$ | §IV A 2 |
| $\mathbb{Z}/_2$ | p2 | 0 | $\{C \in 2\mathbb{Z}\}$ | $\mathbb{Z}/_C$ | §IV A 2 |
| $\mathbb{Z}/_2$ | pmm | 0 | | | §IV A 2 |
| $\mathbb{Z}/_2$ | cmm | 0 | | | §IV A 2 |
| $\mathbb{Z}/_3$ | p3 | 0 | $\{C \in 3\mathbb{Z}\}$ | $\mathbb{Z}/_{\frac{2}{3}C}$ | §IV A 2 |
| Dih ₃ | p31m | 0 | * | 1 | §IV A 2 |
| Dih ₃ | p3m1 | 0 | * | 1 | §IV A 2 |
| $\mathbb{Z}/_4$ | p4 | 0 | $\{C \in 4\mathbb{Z}\}$ | $\mathbb{Z}/_{\frac{1}{2}C}$ | §IV A 2 |
| Dih ₄ | p4m | 0 | | | §IV A 2 |
| $\mathbb{Z}/_6$ | p6 | 0 | $\{C \in 6\mathbb{Z}\}$ | $\mathbb{Z}/_{\frac{1}{3}C}$ | §IV A 2 |
| Dih ₆ | p6m | 0 | | | §IV A 2 |
| | symmorph | I | | | §IV B |
| Dih ₁ | pm | I | $2\mathbb{Z} \times \{n, s\}^2$ | \mathbb{Z}^2 | §IV B 1 |
| Dih ₁ | cm | I | $\mathbb{Z}/_2 \times \{n, s\}$ | $\mathbb{Z}/_2$ | §IV B 2 |
| $\mathbb{Z}/_2$ | p2 | I | $2\mathbb{Z} \times \{n, s\}^4$ | \mathbb{Z}^4 | §IV B 3 |
| Dih ₂ | pmm | I | $\{n, s\}^4$ | \mathbb{Z}^3 | §IV B 4 |
| Dih ₂ | cmm | I | $\{n, s\}^3$ | \mathbb{Z}^3 | §IV B 5 |
| $\mathbb{Z}/_3$ | p3 | I | $3\mathbb{Z} \times \{n, s\}^3$ | \mathbb{Z}^3 | §IV B 6 |
| Dih ₃ | p31m | I | $\{n, s\}^3$ | \mathbb{Z}^2 | §IV B 7 |
| Dih ₃ | p3m1 | I | $\mathbb{Z}^2 \times \{n, s\}^2$ | \mathbb{Z}^3 | §IV B 8 |
| $\mathbb{Z}/_4$ | p4 | I | $\{n, s\}^2$ | \mathbb{Z}^3 | §IV B 9 |
| Dih ₄ | p4m | I | | | §IV B 10 |
| Dih ₆ | p6 | I | $6\mathbb{Z} \times \{n, s\}^3$ | \mathbb{Z}^3 | §IV B 11 |
| Dih ₆ | p6m | I | | | §IV B 12 |
| | diamorph | I | | | |
| | symmorph | II | \emptyset | – | §IV C |
| | diamorph | II | | | |

TABLE I. Our results (blank if no result yet). The integer labels of connected components are shown as the multiple as which they appear under the forgetful map to the non-equivariant components. The notation “ $E \times_{\bullet} A$ ” signifies an as yet unspecified group extension of A by E , while $\mathbb{Z}/_{2C} \times_2 \mathbb{Z}^2$ denotes specifically the level=2 integer Heisenberg group extension (Def. A.1) of \mathbb{Z}^2 .

While therefore in particular the equivariant 2-cohomotopy of (Brillouin) 2-tori is bound to be central to the understanding of the “fragile” (unstable) topological phases of (fractional) Chern insulators, the computation of these classes

$$\pi_G^2(T^2) \equiv \pi_0 \text{Map}(T^2, S^2)^G \quad (3)$$

has not received attention before. Our first result here

is a comprehensive tabulation of these topological indicators, for all possible G -actions.

But we go further and also investigate the equivariant 2-cohomotopy of tori in degree -1, hence the fundamental groups π_1 of these mapping spaces. This is motivated by the observation that in the non-equivariant case these fundamental groups are surprisingly rich and non-abelian (recalled in §IV A 1). This has dramatic implication for

the application to topological phases of what are called *fractional Chern insulators*, and a key motivation for our computations here is to see how this situation generalizes to the case of crystalline insulators and hence to equivariant 2-cohomotopy.

II. EQUIVARIANT 2-TORI

a. Background. A 2D space group or wallpaper group (cf. [4, §26][5, §2]) is a group extension S of a finite point group $G \subset O(2)$ by a lattice $\mathbb{Z}^2 \xrightarrow{\ell} \mathbb{R}^2$ fitting into a diagram of exact sequences of groups (cf. [6, Thm. 2.3][7, (22)]) like this:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \rightarrow & \mathbb{Z}^2 & \xrightarrow{\ell} & \mathbb{R}^2 & \xrightarrow{\text{mod } \ell} & \mathbb{T}^2 \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \\
 & & S & \hookrightarrow & \text{Iso}(\mathbb{R}^2) & & \\
 & & \downarrow & & \downarrow & & \\
 & & G & \hookrightarrow & O(\mathbb{R}^2) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1, & &
 \end{array} \tag{4}$$

whence the resulting torus inherits a continuous action by the point group, making it a G -space (A37):

$$G \curvearrowright \mathbb{T}^2 := G \curvearrowright \mathbb{R}^2 / \ell \mathbb{Z}^2. \tag{5}$$

Here S is called *symmorphic* if the extension is semidirect, otherwise we will call it *diamorphic* (which is not a standard term, but more succinct than *non-symmorphic*).

It is a classical fact that there are, up to isomorphism, exactly 17 such groups, cf. Table II, of which 13 are symmorphic.

We write $\mathbb{Z}/n := \mathbb{Z}/n$ for the cyclic group of order n and $\text{Dih}_n \simeq \mathbb{Z}/n \rtimes \mathbb{Z}/2$ for the corresponding dihedral group of order $2n$ (obtained by adjoining a reflection generator σ subject to $\sigma^2 = e$ and $\sigma \cdot [n] \cdot \sigma = [-n]$, cf. [4, §4]).

b. Result. We determine (Table II) minimal equivariant cell decompositions for finite group G actions (minimal G -CW complex structures, cf. [8, §I.1-2][9, Ex. 1.3.6] and §A 2 below) on the G -torus $G \curvearrowright \mathbb{T}^2$ (5) according to the 13 symmorphic wallpaper groups (4).

c. Proof. The proof of the following cell decompositions (referenced in Table II) is by immediate inspection: It may take work to find a cell decomposition but it is easy to recognize one when found. The rule is that cells appear in G -orbits (coset spaces G/H for subgroups $H \subset G$, as shown on the right of the following figures) subject to no further constraint except that their boundaries are equivariantly glued to lower-dimensional cells (as shown on the left of the following figures).

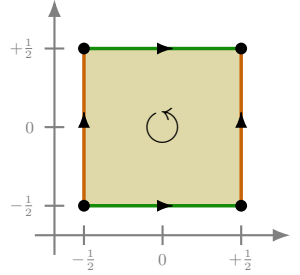
| Space group S | Point group G | Minimal cell structure on $G \curvearrowright \mathbb{R}^2 / \mathbb{Z}^2$ |
|-----------------|-----------------|--|
| p1 | $\mathbb{Z}/1$ | Fig. 1 |
| pm | Dih_1 | Fig. 2 |
| cm | Dih_1 | Fig. 3 |
| pg | Dih_1 | Fig. 4 |
| p2 | $\mathbb{Z}/2$ | Fig. 5 |
| pmm | Dih_2 | Fig. 6 |
| cmm | Dih_2 | Fig. 7 |
| pmg | Dih_2 | Fig. 8 |
| pgg | Dih_2 | Fig. 9 |
| p3 | $\mathbb{Z}/3$ | Fig. 10 |
| p31m | Dih_3 | Fig. 11 |
| p3m1 | Dih_3 | Fig. 12 |
| p4 | $\mathbb{Z}/4$ | Fig. 13 |
| p4m | Dih_4 | Fig. 14 |
| p4g | Dih_4 | Fig. 15 |
| p6 | $\mathbb{Z}/6$ | Fig. 16 |
| p6m | Dih_6 | Fig. 17 |

TABLE II. The names of the 2D space groups, arranged according to their point groups G and referenced to the minimal equivariant cell decomposition of the corresponding G -torus.

d. Status. The concept of G -equivariant cell complexes is fundamental in equivariant homotopy theory, and equivariant cell decomposition of a domain G -space is pivotal for general computation of its equivariant (generalized) cohomology. Moreover, cell decompositions of G -tori are among the simplest non-trivial examples of the general concept and of key importance for computations of crystalline topological phases (where they appear as Brillouin tori of crystal momenta subject to crystal symmetries). And yet, we are not aware that the following cell decompositions (Tbl. II) have been spelled out before.

FIG. 1. Minimal cell structure on the torus without symmetry, corresponding to the space group $p1$.

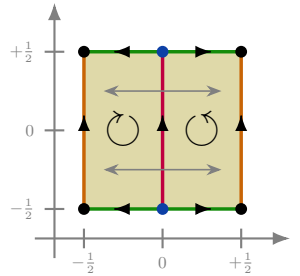
$$\begin{aligned} \mathbb{Z}_{/1} \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{p1} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [x, y] \end{aligned}$$



| Cell decomposition | |
|--|---|
| $D^0 \times \mathbb{Z}_{/1}/\mathbb{Z}_{/1}$ | • |
| $D^1 \times \mathbb{Z}_{/1}/\mathbb{Z}_{/1}$ | ↑ |
| $D^1 \times \mathbb{Z}_{/1}/\mathbb{Z}_{/1}$ | → |
| $D^2 \times \mathbb{Z}_{/1}/\mathbb{Z}_{/1}$ | ⊙ |

FIG. 2. Minimal equivariant cell structure on the torus with mirror symmetry corresponding to the space group pm .

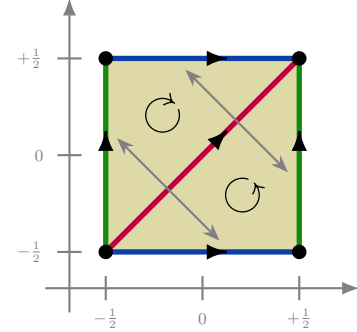
$$\begin{aligned} \mathbb{Z}_{/2} \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{pm} (\mathbb{R}^2/\mathbb{Z}^2) \\ (\sigma, [x, y]) &\mapsto [-x, y] \end{aligned}$$



| Cell decomposition | |
|--|---|
| $D^0 \times \mathbb{Z}_{/2}/\mathbb{Z}_{/2}$ | • |
| $D^0 \times \mathbb{Z}_{/2}/\mathbb{Z}_{/2}$ | • |
| $D^1 \times \mathbb{Z}_{/2}/\mathbb{Z}_{/2}$ | ↑ |
| $D^1 \times \mathbb{Z}_{/2}/\mathbb{Z}_{/2}$ | ↕ |
| $D^1 \times \mathbb{Z}_{/2}/1$ | ↔ |
| $D^2 \times \mathbb{Z}_{/2}/1$ | ⊙ |

FIG. 3. Minimal equivariant cell structure on the torus with mirror symmetry corresponding to the space group cm .

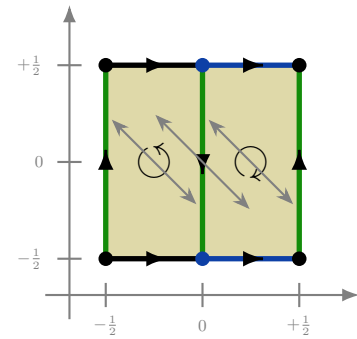
$$\begin{aligned} \text{Dih}_1 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{cm} (\mathbb{R}^2/\mathbb{Z}^2) \\ (\sigma, [x, y]) &\mapsto [y, x] \end{aligned}$$



| Cell decomposition | |
|--|---|
| $D^0 \times \text{Dih}_1/\text{Dih}_1$ | • |
| $D^1 \times \text{Dih}_1/\text{Dih}_1$ | ↗ |
| $D^1 \times \text{Dih}_1/1$ | ↑ |
| $D^1 \times \text{Dih}_1/1$ | → |
| $D^2 \times \text{Dih}_1/1$ | ⊙ |

FIG. 4. Equivariant cell structure on torus of Dih_1 -action according to the space group pg .

$$\begin{aligned} \text{Dih}_1 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{pg} (\mathbb{R}^2/\mathbb{Z}^2) \\ (\sigma, [x, y]) &\mapsto [x + \frac{1}{2}, -y] \end{aligned}$$



| Cell decomposition | |
|-----------------------------|-------|
| $D^0 \times \text{Dih}_1/1$ | • ↔ • |
| $D^1 \times \text{Dih}_1/1$ | ↔ |
| $D^1 \times \text{Dih}_1/1$ | ↕ |
| $D^2 \times \text{Dih}_1/1$ | ⊙ |

FIG. 5. Minimal equivariant cell structure on torus with 2-fold rotation symmetry corresponding to the space group $p2$.

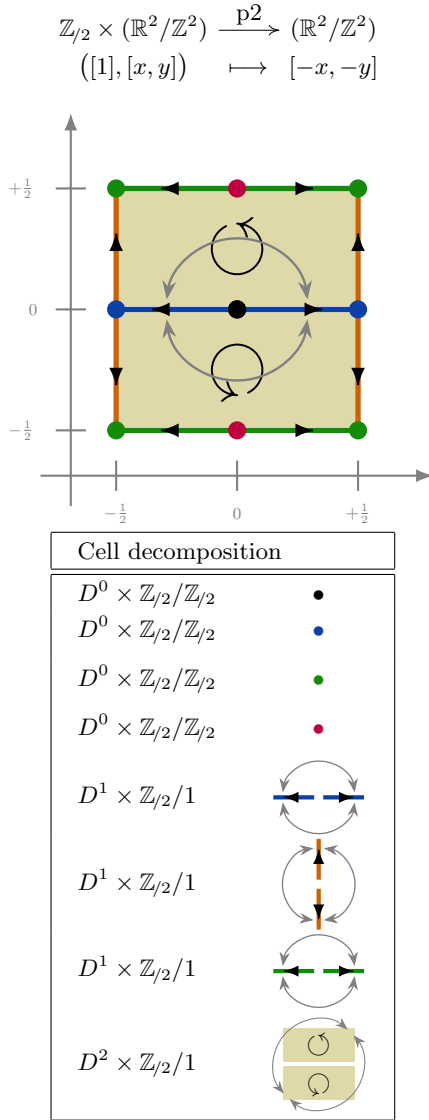


FIG. 6. Minimal equivariant cell structure on torus with 2-fold dihedral symmetry corresponding to the space group pmm .

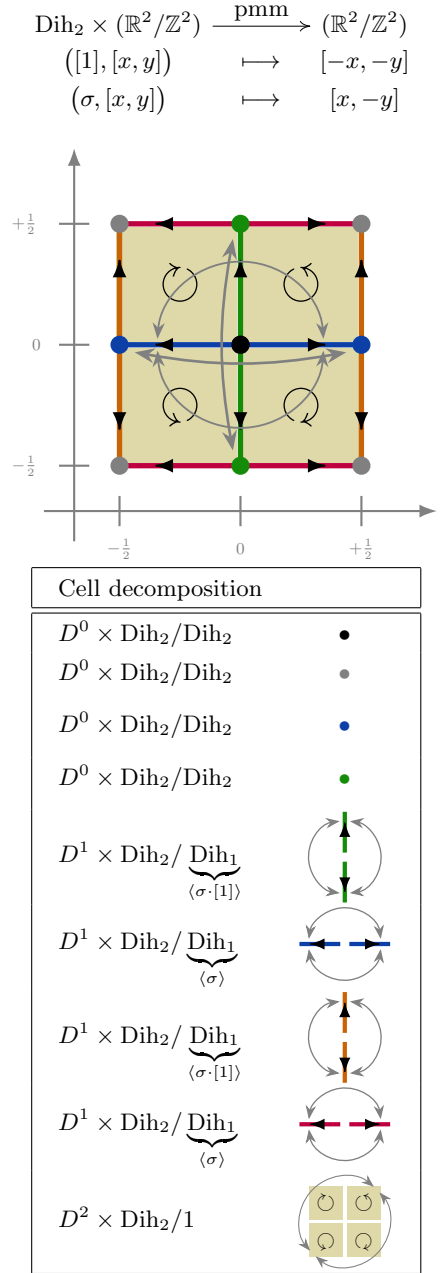


FIG. 7. Minimal equivariant cell structure on torus with 2-fold dihedral symmetry corresponding to the space group cmm .

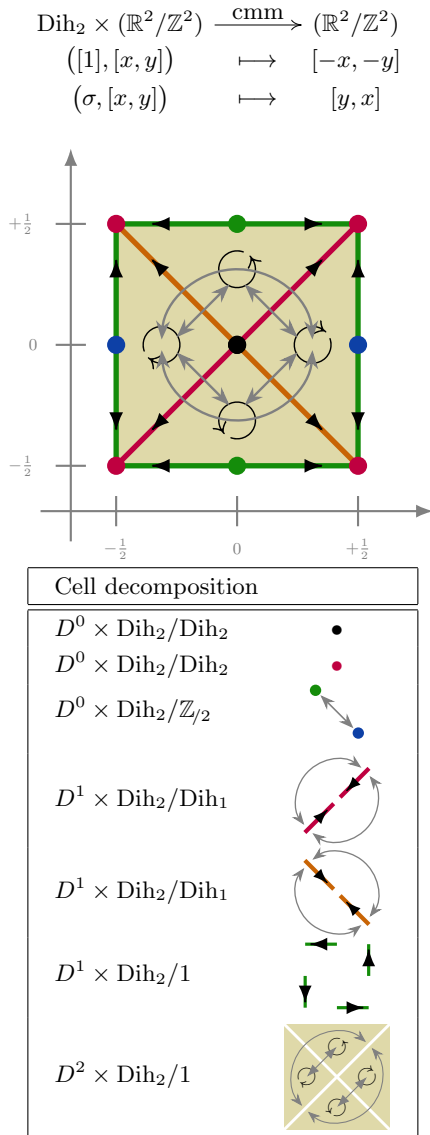


FIG. 8. Minimal equivariant cell structure on torus with symmetry corresponding to the space group pmg .

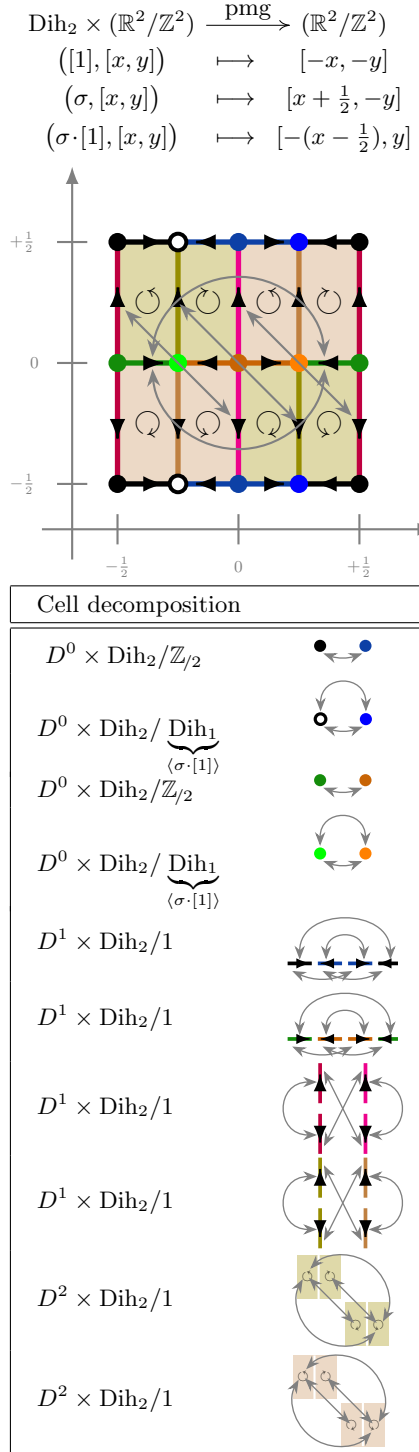


FIG. 9. Minimal equivariant cell structure on torus with rotation and glide reflection symmetry corresponding to the space group pgg.

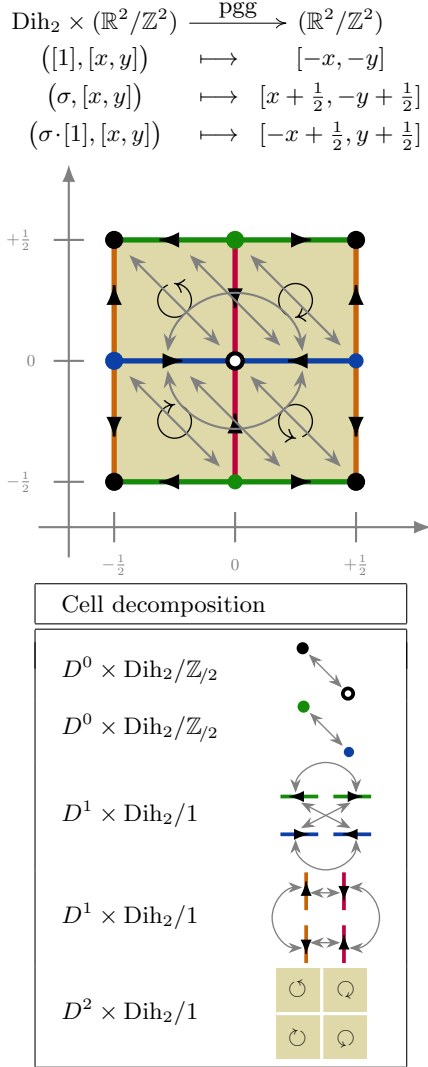


FIG. 10. Minimal equivariant cell structure on torus with 3-fold rotation symmetry corresponding to the space group p3.

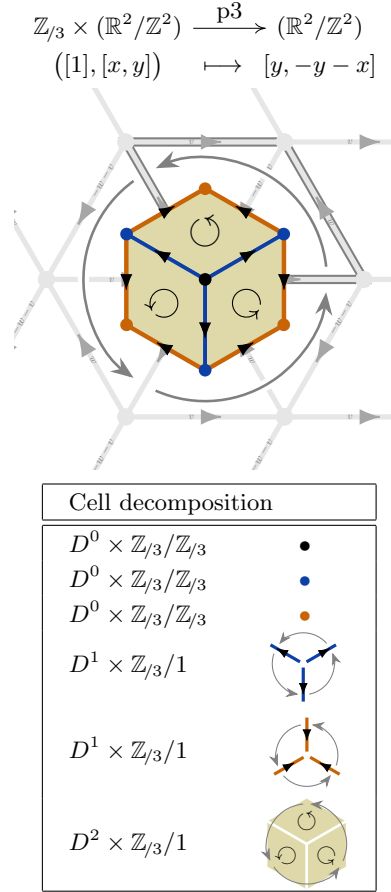
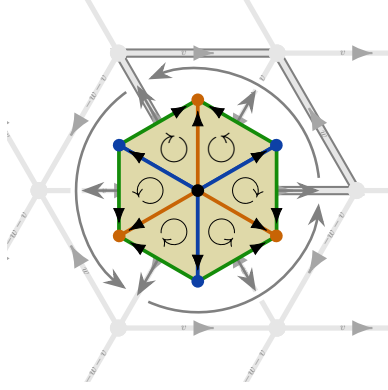


FIG. 11. Minimal equivariant cell structure on torus with 3-fold dihedral symmetry corresponding to the space group $p31m$.

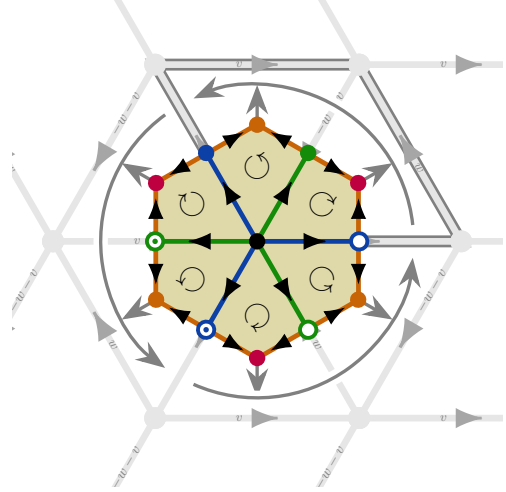
$$\begin{aligned} \text{Dih}_3 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{p31m} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [y, -y-x] \\ (\sigma, [x, y]) &\mapsto [-y, -x] \end{aligned}$$



| Cell decomposition | |
|--|---|
| $D^0 \times \text{Dih}_3/\text{Dih}_3$ | • |
| $D^0 \times \text{Dih}_3/\text{Dih}_3$ | • |
| $D^0 \times \text{Dih}_3/\text{Dih}_3$ | • |
| $D^1 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^1 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \cdot [1] \rangle}$ | |
| $D^1 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^2 \times \text{Dih}_3/1$ | |

FIG. 12. Minimal equivariant cell structure on torus with 3-fold dihedral symmetry corresponding to the space group $p3m1$.

$$\begin{aligned} \text{Dih}_3 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{p3m1} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [y, -y-x] \\ (\sigma, [x, y]) &\mapsto [x, -y-x] \end{aligned}$$



| Cell decomposition | |
|--|---|
| $D^0 \times \text{Dih}_3/\text{Dih}_3$ | • |
| $D^0 \times \text{Dih}_3/\mathbb{Z}_3$ | |
| $D^0 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^0 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^1 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^1 \times \text{Dih}_3/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^1 \times \text{Dih}_3/1$ | |
| $D^2 \times \text{Dih}_3/1$ | |

FIG. 13. Minimal equivariant cell structure on torus with 4-fold rotation symmetry corresponding to the space group $p4$.

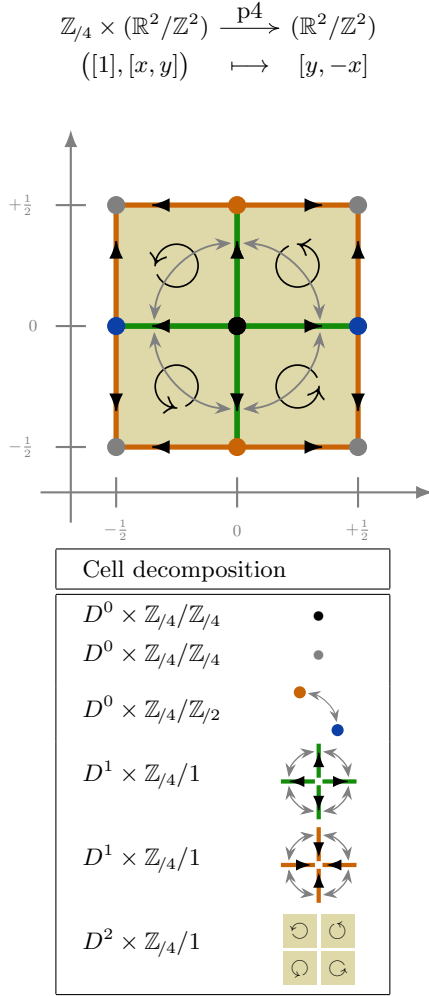


FIG. 14. Minimal equivariant cell structure on torus with 4-fold dihedral symmetry corresponding to the space group $p4m$.

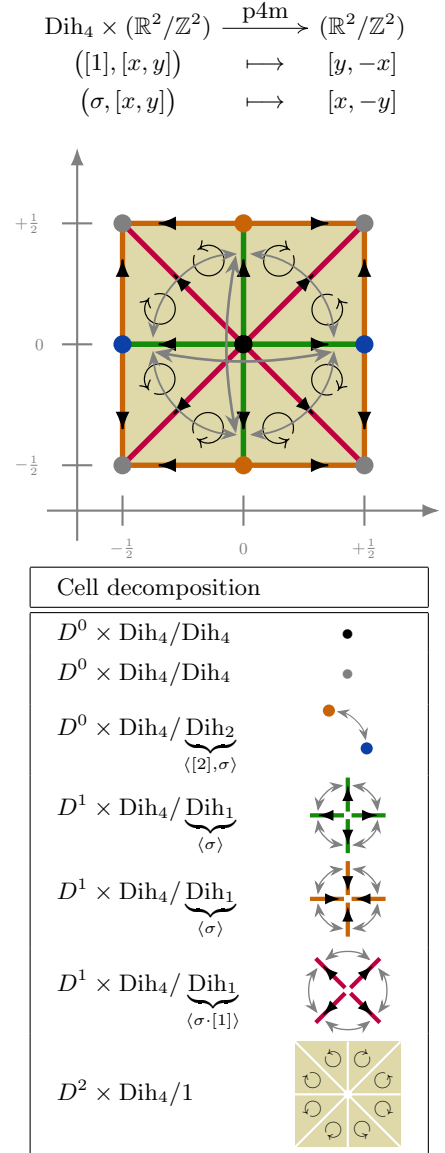
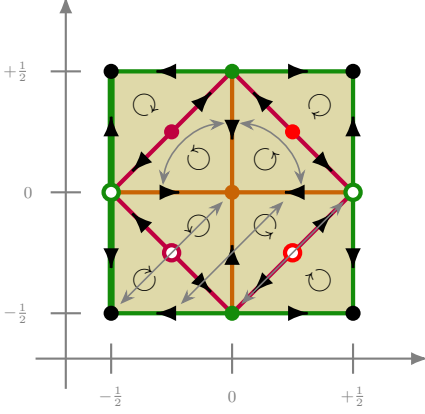


FIG. 15. Minimal equivariant cell structure on torus with rotation and glide reflection symmetry corresponding to the space group p4g.

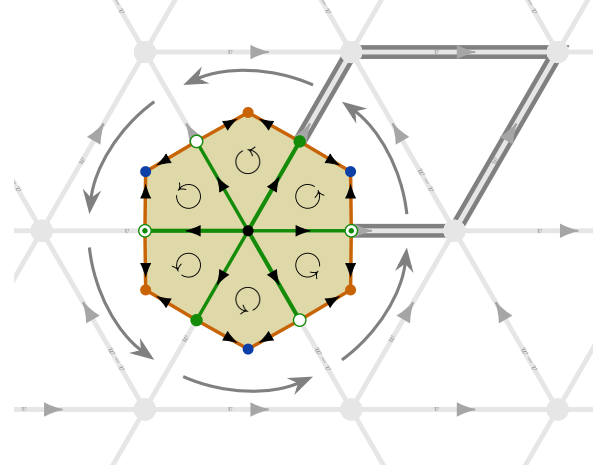
$$\begin{aligned} \text{Dih}_4 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p4g}} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [y, -x] \\ (\sigma, [x, y]) &\mapsto [x + \frac{1}{2}, -y + \frac{1}{2}] \end{aligned}$$



| Cell decomposition | |
|---|--|
| $D^0 \times \text{Dih}_4/\mathbb{Z}_4$ | |
| $D^0 \times \text{Dih}_4/\underbrace{\text{Dih}_2}_{\langle [2], \sigma \cdot [1] \rangle}$ | |
| $D^0 \times \text{Dih}_4/\underbrace{\text{Dih}_1}_{\langle \sigma \cdot [3] \rangle}$ | |
| $D^1 \times \text{Dih}_4/1$ | |
| $D^1 \times \text{Dih}_4/1$ | |
| $D^2 \times \text{Dih}_4/1$ | |

FIG. 16. Minimal equivariant cell structure on torus with 6-fold rotation symmetry corresponding to the space group p6.

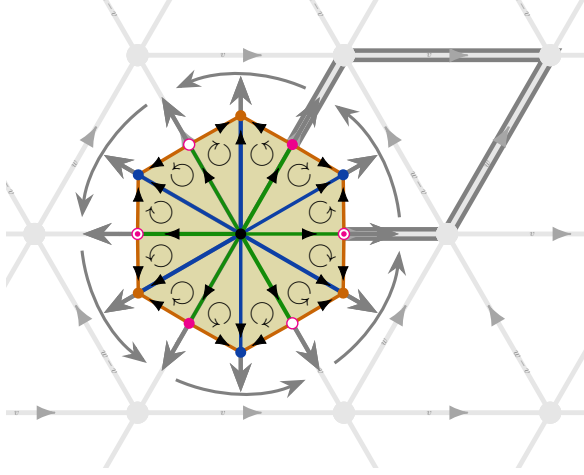
$$\begin{aligned} \mathbb{Z}_6 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p6}} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [y, y - x] \end{aligned}$$



| Cell decomposition | |
|--|--|
| $D^0 \times \mathbb{Z}_6/\mathbb{Z}_6$ | |
| $D^0 \times \mathbb{Z}_6/\mathbb{Z}_3$ | |
| $D^0 \times \mathbb{Z}_6/\mathbb{Z}_2$ | |
| $D^1 \times \mathbb{Z}_6/1$ | |
| $D^1 \times \mathbb{Z}_6/1$ | |
| $D^2 \times \mathbb{Z}_6/1$ | |

FIG. 17. Minimal equivariant cell structure on torus with 6-fold dihedral symmetry corresponding to the space group p6m.

$$\begin{aligned} \text{Dih}_6 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p6m}} (\mathbb{R}^2/\mathbb{Z}^2) \\ ([1], [x, y]) &\mapsto [y, y-x] \\ (\sigma, [x, y]) &\mapsto [x, x-y] \end{aligned}$$



| Cell decomposition | |
|---|--|
| $D^0 \times \text{Dih}_6/\text{Dih}_6$ | |
| $D^0 \times \text{Dih}_6/\underbrace{\text{Dih}_3}_{\langle [2], \sigma \cdot [1] \rangle}$ | |
| $D^0 \times \text{Dih}_6/\underbrace{\text{Dih}_2}_{\langle [3], \sigma \cdot [1] \rangle}$ | |
| $D^1 \times \text{Dih}_6/\underbrace{\text{Dih}_1}_{\langle \sigma \cdot [1] \rangle}$ | |
| $D^1 \times \mathbb{Z}_6/\underbrace{\text{Dih}_1}_{\langle \sigma \rangle}$ | |
| $D^1 \times \text{Dih}_6/\underbrace{\text{Dih}_1}_{\langle \sigma \cdot [1] \rangle}$ | |
| $D^2 \times \text{Dih}_6/1$ | |

TABLE III. Quotients of the torus by actions of the symmorphic point groups (where “ \sim ” denotes identification up to homotopy equivalence, otherwise up to homeomorphism). These may be read off from the equivariant cell decompositions shown above,

| Space group | T^2/G |
|-------------|------------|
| p1 | T^2 |
| pm | $\sim S^1$ |
| cm | $\sim *$ |
| p2 | S^2 |
| pmm | |
| cmm | |
| p3 | S^2 |
| p31m | $\sim *$ |
| p3m1 | $\sim *$ |
| p4 | S^2 |
| p4m | |
| p6 | S^2 |
| p6m | |
| pg | K^2 |

III. EQUIVARIANT 2-SPHERES

We consider the 2-sphere $S^2 \simeq S(\mathbb{R}^3)$ as equipped with G -actions of finite subgroups $G \subset \text{SO}(3)$, under the defining action of $\text{SO}(3)$. These subgroups famously fall into an ADE-classification, cf. Table. IV.

TABLE IV. The finite subgroups of $\text{SO}(3)$.

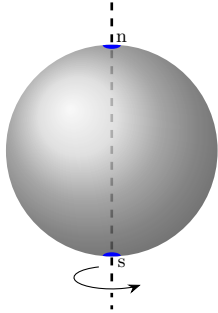
| Type | Name | Symbol | Range |
|-------|-------------|---------------------|-----------------------------|
| A_n | cyclic | $\mathbb{Z}_{/n+1}$ | $n \in \mathbb{N}$ |
| D_n | dihedral | Dih_{n-2} | $n \in \mathbb{N}_{\geq 2}$ |
| E_6 | tetrahedral | T | |
| E_7 | octahedral | O | |
| E_8 | icosahedral | I | |

But what matters in the following are only the fixed point sets of these actions, which fall into three classes according to Table V. The case I of exactly 2 fixed points is illustrated in Fig. 18.

TABLE V. The fixed point structure of the actions of finite subgroups of $\text{SO}(3)$ (Table IV) on the 2-sphere.

| Case | Points fixed | Groups |
|------|--------------|-----------------------------|
| 0 | all | $\mathbb{Z}_{/1}$ |
| I | two | $\mathbb{Z}_{/n}, n \geq 2$ |
| II | none | all other |

FIG. 18. The action on the 2-sphere of the non-trivial cyclic groups $\mathbb{Z}_{/n}, n \geq 1$, is by rotation around a fixed axis and hence fixes precisely an antipodal pair of points, to be denoted n and s .



Here:

- The case (0) of the trivial action on the coefficients 2-sphere uniformly reduces G -equivariant 2-Cohomotopy to the ordinary 2-Cohomotopy of the domain's quotient space T^2/G . This situation we discuss in §IV A.
- The case (II) of no fixed points on the coefficient 2-sphere makes the equivariant 2-cohomotopy be

empty as soon as there is a fixed point in the domain, which is the case for all symmmorphic actions (cf. Table II).

- Therefore case (I) of exactly 2 fixed points on the coefficient sphere is the only substantially relevant one for symmmorphic 2-tori. This situation we discuss in §IV B.

IV. EQUIVARIANT 2-COHOMOTOPY OF 2-TORI

A. Any G -Tori and Type 0 Coefficients

When the G -action on the 2-sphere coefficient is trivial (“type 0” according to Table V), G -equivariant cohomotopy reduces to the ordinary cohomotopy of the quotient domain:

$$G \zeta S^2 \text{ trivial} \Rightarrow \text{Map}(T^2, S^2)^G \simeq \text{Map}(T^2/G, S^2).$$

The homotopy types of quotients that appear this way are those of the point, the circle and the 2-sphere (cf. Table III).

1. $p1$

The case of trivial G action both on the domain 2-torus and on the coefficient 2-sphere is the case of non-equivariant 2-cohomotopy of the 2-torus. For completeness, we recall the remarkable fact (Prop. A.3, due to Larmore & Thomas 1980 [10], going back to Hansen 1974[11], reproved in [12]) how that yields (besides the usual winding number in degree 0) a non-abelian fundamental group, namely an *integer Heisenberg group* (re-called as Def. A.1 below):

The 1-skeleton in Fig. 1 is the wedge of two circles

$$\text{sk}_1(T^2) \simeq S^1 \vee S^1 = \left\{ \begin{array}{c} \text{---} \rightarrow \text{---} \\ \uparrow \quad \downarrow \\ \text{---} \leftarrow \text{---} \end{array} \right\}$$

and the 2-cell attachment is

$$\begin{array}{ccc} S^1 & \xrightarrow{\circlearrowright} & \text{sk}_1(T^2) \\ \downarrow & & \downarrow \\ D^2 & \xrightarrow{\quad (p_0) \quad} & T^2, \end{array} \tag{6}$$

where the top map picks the boundary of the 2-cell as shown in Fig. 1.

Definition A.1. For $k \in \mathbb{Z}$, the level=2 $\mathbb{Z}_{/n}$ -extended *integer Heisenberg group* has underlying set $\mathbb{Z}_{/k} \times \mathbb{Z}^2$ with

group operation

$$\begin{aligned} ((\mathbb{Z}/k \times_2 \mathbb{Z}^2) \times (\mathbb{Z}/k \times_2 \mathbb{Z}^2)) &\longrightarrow \mathbb{Z}/k \times_2 \mathbb{Z}^2 \\ ((a, b, [n]), (a', b', [n'])) &\mapsto \left(a + a', \begin{bmatrix} n + n' + \\ ab' - a'b \end{bmatrix} \right) \end{aligned} \quad (7)$$

Proposition A.2. *The pointed mapping space $\text{Map}^*(T^2, S^2)$ has connected components*

$$\pi_0 \text{Map}^*(T^2, S^2) \simeq \mathbb{Z} \quad (8)$$

and each component $C \in \mathbb{Z}$ has fundamental group the level=2 \mathbb{Z} -extended integer Heisenberg group (A.1)

$$\pi_1 \text{Map}_C^*(T^2, S^2) \simeq \mathbb{Z}/0 \times_2 \mathbb{Z}^2. \quad (9)$$

Proof. Regarding the 2-cell attachment (6) as a pushout of pointed spaces and using that $\text{Map}^*(D^2, T^2) \sim *$, we have the following homotopy fiber sequence and its induced homotopy long exact sequence:

$$\begin{array}{ccc} \text{Map}^*(T^2, S^2) & \longrightarrow & (\Omega S^2)^2 \xrightarrow{\circ^*} \Omega S^2 \\ & & \left. \begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{0} & \mathbb{Z} \\ \pi_1 \text{Map}^*(T^2, S^2) & \longrightarrow & \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \\ \pi_0 \text{Map}^*(T^2, S^2) & \longrightarrow & * \longrightarrow * \end{array} \right\} \end{array} \quad (10)$$

Here $\pi_n(\circ^*) = 0$, for $n > 0$, because its image is manifestly in group commutators inside abelian groups (cf. [11, Prop. 2]).

This already implies the first statement (8) and the second statement (9) up to the determination of the group extension. That the latter is indeed the central extension at level=2 is due to Kallel 2001 [13, Cor. 6.14]. \square

Proposition A.3. (i) *The plain mapping space $\text{Map}(T^2, S^2)$ has connected components*

$$\pi_0 \text{Map}(T^2, S^2) \simeq \mathbb{Z} \quad (11)$$

and the component $C \in \mathbb{Z}$ has fundamental group the level=2 $\mathbb{Z}/2C$ -extended integer Heisenberg group (7)

$$\pi_1 \text{Map}_C(T^2, S^2) \simeq \mathbb{Z}/2C \times_2 \mathbb{Z}^2. \quad (12)$$

(ii) *Moreover, the canonical map from the pointed mapping space yields the canonical surjections on these homotopy groups (cf. Prop. A.2).*

Proof. The statement (12) is due to Larmore & Thomas 1980 [10, Thm. 1] following [11]; a transparent new proof is given in [12, Thm. 3.3]. \square

2. Other

For the nontrivial space groups in Table III we have:

$$\begin{aligned} \pi_0 \text{Map}(*, S^2) &\simeq * \\ \pi_1 \text{Map}(*, S^2) &\simeq 1 \end{aligned} \quad (13)$$

$$\begin{aligned} \pi_0 \text{Map}(S^1, S^2) &\simeq * \\ \pi_1 \text{Map}(S^1, S^2) &\simeq \mathbb{Z} \end{aligned} \quad (14)$$

$$\begin{aligned} \pi_0 \text{Map}(S^2, S^2) &\simeq \mathbb{Z} \\ \pi_1 \text{Map}_C(S^2, S^2) &\simeq \mathbb{Z}/2C, \end{aligned} \quad (15)$$

$$\begin{aligned} \pi_0 \text{Map}(K^2, S^2) &\simeq \mathbb{Z}/2 \\ \pi_1 \text{Map}_C(K^2, S^2) &\simeq \mathbb{Z}/2 \times \mathbb{Z}, \end{aligned} \quad (16)$$

where (14) follows by (A27), (15) follows by [14, Thm. 5.3(1)][15, Lem. 3.1] and (16) is [11, Thm. 2].

B. Symmorphic Tori and Type I Coefficients

We compute here the equivariant 2-cohomotopy of 2-tori equipped with symmorphic G -action according to Table II. Since these G -actions all have fixed points, their equivariant 2-cohomotopy for coefficient 2-sphere with actions of Type II (Table V) is empty. Therefore we may focus on 2-spheres with type I actions, hence with non-trivial finite rotation actions of \mathbb{Z}/n (Fig. 18) which we regard as G -actions via any non-trivial homomorphism $G \rightarrow \mathbb{Z}/n$ (which clearly exists for each symmorphic G).

In this situation, the G -action on S^2 has precisely two fixed points (Fig. 18) and the following results depend otherwise only on the given G -action on T^2 .

In this situation, the following computation strategy works uniformly for essentially all cases (the few exceptions being discussed separately below); the reader may want to compare the following general prescription to, say, the first example in §IV B 1 below:

- (i) Observe that equivariance forces each of k_0 0-cells of the 2-torus to be mapped to a fixed point, hence to one of the two fixed poles $n, s \in S^2$ (Fig. 18). This decomposes the equivariant mapping space into a set $\{n, s\}^{k_0}$ of disjoint components.
- (ii) Observe that the images of orbits of the k_1 1-cells of the 2-torus are determined by the image of any one of their members, which is free except that its endpoints are stuck to the fixed poles as per the previous choice of images of the 0-cells. This makes the equivariant mapping space out of the equivariant 1-skeleton $\text{sk}_1^G(T^2)$ (A5) be the k_0 -fold disjoint union of k_1 -fold products of bipointed mapping spaces,

the latter each equivalent (A34) to the based loop space:

$$\begin{aligned} & \text{Map}(\text{sk}_1^G(T^2), S^2)^G \\ & \simeq \prod_{\{n,s\}^{k_0}} \left(\begin{array}{c} \text{Map}^{*1,*2}(D^1, S^2) \\ \times \text{Map}^{*3,*4}(D^1, S^2) \\ \vdots \end{array} \right) \\ & \sim \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \end{aligned} \quad (17)$$

(where for the disk $*_{2i} = 0 \in D^1$ and $*_{2i+1} = 1 \in D^1$, while for the sphere $*_i \in \{n, s\}$ as per the given disjoint component).

- (iii) Observe that the images of the k_2 2-cells are similarly fixed by that of any one of its members, which now means that equivariantly mapping out of the 2-cell attachment diagram (A43) characterizes the equivariant mapping space as a pullback (A48) of this form:

$$\begin{aligned} & \text{Map} \left(\begin{array}{ccc} \prod_{k_2} S^1 \times G/1 & \xrightarrow{(\circlearrowleft^*)_{i=1}^{k_2}} & \text{sk}_1^G(T^2) \\ \downarrow \sqcup_{\iota_2} & & \downarrow \\ \prod_{k_2} D^2 \times G/1 & \xrightarrow{\text{(po)}} & T^2 \end{array} \right), S^2 \\ & \simeq \begin{array}{ccc} (\mathcal{L}S^2)^{k_2} & \hookrightarrow & (\Omega S^2)^{k_2} \longleftarrow \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \\ \uparrow \sqcap_{\iota_n^*} & & \uparrow \text{(pb)} \\ (\text{Map}(D^2, S^2))^{k_2} & \longleftarrow & \text{Map}(T^2, S^2)^G \end{array} \end{aligned} \quad (18)$$

where for the top map we have recognized, on the left, the free loop space $\mathcal{L}S^2 \equiv \text{Map}(S^1, S^2)$ and, in the middle, that the map actually factors through the inclusion of based loops into free loops (A10), due to the previous observation that all 0-vertices must map locally constantly to a fixed pole.

- (iv) Observe that with its left map being a fibration (A20), this pullback square (18) is actually a homotopy pullback (A29), in which guise we may replace $D^2 \sim *$ and use the pasting law (A35) to factor as

$$\begin{array}{ccc} (\mathcal{L}S^2)^{k_2} & \longleftarrow & (\Omega S^2)^{k_2} \xleftarrow{(\circlearrowleft^*)_{i=1}^{k_2}} \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \\ \uparrow \sqcap_{\text{cnst}} & & \uparrow \text{(hpb)} \\ (S^2)^{k_2} & \longleftarrow & * \longleftarrow \text{Map}(T^2, S^2)^G \end{array} \quad (19)$$

where the left square follows by (A36).

- (v) In most cases the dual attachment map $(\circlearrowleft^*)_{i=0}^{k_2}$ in (19) happens to be null homotopic (homotopic to a map factoring through the point), in which case:

Observe that the right square in (19) factors further as

$$\begin{array}{ccc} & \xleftarrow{(\circlearrowleft^*)_{i=1}^{k_2}} & \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \\ (\Omega S^2)^{k_2} & \longleftarrow * & \longleftarrow \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \\ \uparrow & \text{(hpb)} \uparrow & \uparrow \text{(hpb)} \\ * & \longleftarrow (\Omega^2 S^2)^{k_2} & \longleftarrow \text{Map}(T^2, S^2)^G \end{array} \quad (20)$$

where the left square is a homotopy pullback by (A31), and then the right square is so again by the pasting law (A35). But this implies (A32) that

$$\text{Map}(T^2, S^2)^G \sim \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \times (\Omega^2 S^2)^{k_2} \quad (21)$$

and hence in particular that its first homotopy groups are, for all basepoints:

$$\begin{aligned} \pi_0 \text{Map}(T^2, S^2)^G & \simeq \mathbb{Z}^{k_2} \times \{n, s\}^{k_0} \\ \pi_1 \text{Map}(T^2, S^2)^G & \simeq \mathbb{Z}^{k_1} \times \mathbb{Z}^{k_2}. \end{aligned} \quad (22)$$

- (vi) Generally, observe that (19) exhibits the equivariant mapping space as the homotopy fiber (A30) of the dual 2-cell attachment map $(\circlearrowleft^*)_{i=1}^{k_2}$ exhibited on its free 1-cell data, and thus induces, for any choice of basepoint b , the corresponding long exact sequence of homotopy groups (A21)

$$\begin{array}{ccc} \text{Map}(T^2, S^2)^G & \longrightarrow & \prod_{\{n,s\}^{k_0}} (\Omega S^2)^{k_1} \longrightarrow (\Omega S^2)^{k_2} \\ & & \mathbb{Z}^{k_1} \xrightarrow{\phi} \mathbb{Z}^{k_2} \\ \pi_1 \text{Map}_b(T^2, S^2)^G & \longrightarrow & \mathbb{Z}^{k_1} \xrightarrow{\phi} \mathbb{Z}^{k_2} \\ \pi_0 \text{Map}(T^2, S^2)^G & \longrightarrow & \{n, s\}^{k_0} \longrightarrow * \end{array} \quad (23)$$

where the \mathbb{Z}^{k_i} entries follow (A15) from the homotopy groups of the 2-sphere (A16), and where the homomorphism ϕ is known explicitly as the matrix reflecting how the 1-cells are composed (and hence linearly added up, under looping) to boundaries of 2-cells under the attaching map.

In conclusion then, the truncation (A28) of this long exact sequence (23) yields the desired connected components and fundamental group of the equivariant mapping space (the latter up to extension), in that it gives short exact sequences

$$\begin{aligned} 1 & \rightarrow \frac{\mathbb{Z}^{k_2}}{\text{im}(\phi)} \longrightarrow \pi_1 \text{Map}_b(T^2, S^2)^G \longrightarrow \ker(\phi) \rightarrow 1 \\ 1 & \rightarrow \frac{\mathbb{Z}^{k_2}}{\text{im}(\phi)} \longrightarrow \pi_0 \text{Map}(T^2, S^2)^G \longrightarrow \{n, s\}^{k_0} \rightarrow * \end{aligned}$$

finally implying that

$$\pi_0 \text{Map}(T^2, S^2)^G \simeq \mathbb{Z}^{k_2 - \dim(\text{im}(\phi))} \times \{\mathfrak{n}, \mathfrak{s}\}^{k_0} \quad (24)$$

and for each $(C, s) \in \pi_0 \text{Map}(T^2, S^2)^G$

$$\pi_1 \text{Map}_{(C,s)}(T^2, S^2)^G \simeq \mathbb{Z}^{k_2 - \dim(\text{im}(\phi))} \times_{\bullet} \ker(\phi), \quad (25)$$

where “ $E \times_{\bullet} A$ ” denotes an unspecified group extension of A by E .

We next spell out this computation strategy for each of the symmorphic actions from Table II (except for the case p1, since this has only type 0 coefficients and as such is discussed in §IV A 1).

1. pm

Specializing the mapping out of the 2-cell attachment (18) to the situation for space group pm shown in Fig. 2, and then factoring the dual attaching map through based loops (19) gives

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & \prod_{\{\mathfrak{n}, \mathfrak{s}\}^2} \text{Map}^{*,*}(D^1, S^2) \\ \uparrow & & \uparrow & & \uparrow \\ S^2 & \longleftarrow & * & \longleftarrow & \text{Map}(T^2, S^2)^G, \end{array}$$

(hpb) (hpb)

where the top map assigns

$$\circlearrowleft^*: \gamma \mapsto \text{cnc}(\gamma, \bar{\gamma}).$$

Since this is null homotopic, we find with (21) that:

$$\pi_0 \text{Map}(T^2, S^2)^G \simeq \mathbb{Z} \times \{\mathfrak{n}, \mathfrak{s}\}^2, \quad (26)$$

$$\pi_1 \text{Map}_{(C,s)}(T^2, S^2)^G \simeq \mathbb{Z}^2. \quad (27)$$

Finally, the map forgetting the G -action on 1-skeleta is homotopic to multiplication by 2 on the attachment

$$\begin{array}{ccc} \gamma_1 & \mapsto & \begin{array}{c} \text{cnc} \\ \uparrow \\ \text{cnc}(\sigma \cdot \gamma_1, \gamma_1) \end{array} \\ \prod_{\{\mathfrak{n}, \mathfrak{s}\}^2} \text{Map}^{*,*}(D^1, S^2) & \longrightarrow & \text{Map}(\text{sk}_1(T^1), S^2) \\ \downarrow \circlearrowleft^* & & \downarrow \circlearrowleft^* \\ \Omega S^2 & \xrightarrow{2} & \Omega S^2 \end{array}$$

which implies that so is the comparison map on the \mathbb{Z} -factor in (26):

$$\begin{array}{ccc} \text{Map}(T^2, S^2)^G & \longrightarrow & \text{Map}(T^2, S^2) \\ \parallel & & \parallel \\ \mathbb{Z} \times \{\mathfrak{n}, \mathfrak{s}\}^2 & \xrightarrow{(n,s) \mapsto 2n} & \mathbb{Z} \end{array}$$

2. cm

Mapping out of the 2-cell attachment (Fig. 3) and factoring the dual attaching map through based loops gives

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & \prod_{\{\mathfrak{n}, \mathfrak{s}\}} \text{Map}^{*,*}(D^1, S^2) \\ \uparrow & & \uparrow & & \uparrow \\ S^2 & \longleftarrow & * & \longleftarrow & \text{Map}(T^2, S^2)^G, \end{array}$$

(hpb) (hpb)

where the top map assigns, up to homotopy:

$$\circlearrowleft^*: \gamma \mapsto \text{cnc}(\gamma, \gamma).$$

In the induced homotopy LES, for any basepoint,

$$\text{Map}(T^2, S^2)^G \longrightarrow \Omega S^2 \longrightarrow \Omega S^2$$

$$\begin{array}{ccc} & \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} & \\ \downarrow & \searrow & \\ \pi_1 \text{Map}(T^2, S^2)^G & \longrightarrow & \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \\ \downarrow & \searrow & \\ \pi_1 \text{Map}(T^2, S^2)^{\mathbb{Z}/2} & \longrightarrow & \{\mathfrak{n}, \mathfrak{s}\}^4 \longrightarrow * \end{array}$$

we therefore have, as a 1×1 matrix,

$$\phi = (2).$$

This implies that

$$\mathbb{Z}/\text{im}(\phi) \simeq \mathbb{Z}/2 \quad \text{and} \quad \ker(\phi) \simeq 1$$

so that the homotopy LES truncates to

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathbb{Z}/2 & \rightarrow & \pi_1 \text{Map}(T^2, S^2)^G & \rightarrow & 1 \rightarrow 1, \\ 1 & \rightarrow & \mathbb{Z}/2 & \rightarrow & \pi_0 \text{Map}(T^2, S^2)^G & \rightarrow & * \rightarrow \{\mathfrak{n}, \mathfrak{s}\}. \end{array}$$

3. p2

Mapping out of the 2-cell attachment (Fig. 5) and factoring the dual attaching map through based loops gives

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & \prod_{\{\mathfrak{n}, \mathfrak{s}\}^4} \left(\begin{array}{c} \text{Map}^{*,*}(D^1, S^2) \\ \times \text{Map}^{*,*}(D^1, S^2) \\ \times \text{Map}^{*,*}(D^1, S^2) \end{array} \right) \\ \uparrow & & \uparrow & & \uparrow \\ S^2 & \longleftarrow & * & \longleftarrow & \text{Map}(T^2, S^2)^G, \end{array}$$

(hpb) (hpb)

Therefore, the equivariant mapping space is equivalent to

$$\text{Map}(T^2, S^2)^G \sim \prod_{\{n,s\}^2} \Omega S^2 \times (\Omega^2 S^2)^2$$

and hence

$$\begin{aligned} \pi_0 \text{Map}(T^2, S^2)^G &\simeq \mathbb{Z}^2 \times \{n, s\}^2 \\ \pi_1 \text{Map}(T^2, S^2)^G &\simeq \mathbb{Z}^3. \end{aligned}$$

9. p4

(Assuming that $[2] \in \mathbb{Z}/4$ does not act trivially on the 2-sphere.)

Mapping out of the 2-cell attachment (Fig. 13) and factoring the dual attaching map through based loops gives

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \downarrow \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & \prod_{\{n,s\}^2} \left(\begin{array}{c} \text{Map}^{*,*}(D^1, S^2) \\ \times \text{Map}^{*,*}(D^1, S^2) \end{array} \right) \\ \uparrow & & \uparrow & & \uparrow \\ S^2 & \longleftarrow & * & \longleftarrow & \text{Map}(T^2, S^2)^G, \end{array}$$

(hpb) (hpb)

where the top map assigns

$$\circlearrowleft^*: (\gamma_1, \gamma_2,) \mapsto \text{cnc}(\gamma_1, \gamma_2, \overline{\gamma_2}, \overline{\gamma_1}).$$

This is null homotopic, so that we have the further factorization

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \downarrow \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & * \longleftarrow \prod_{\{n,s\}^2} \left(\begin{array}{c} \text{Map}^{*,*}(D^1, S^2) \\ \times \text{Map}^{*,*}(D^1, S^2) \end{array} \right) \\ \uparrow & & \uparrow & & \uparrow \\ S^2 & \longleftarrow & * & \longleftarrow & \Omega^2 S^2 \longleftarrow \text{Map}(T^2, S^2)^G \end{array}$$

(hpb) (hpb) (hpb)

which exhibits

$$\text{Map}(T^2, S^2)^G \sim \prod_{\{n,s\}^2} (\Omega S^2)^2 \times \Omega^2 S^2.$$

10. p4m

(...)

Fig. 14

(...)

11. p6

(Assuming that $[2], [3] \in \mathbb{Z}/6$ do not act trivially on the 2-sphere.)

Mapping out of the 2-cell attachment (Fig. 16) and factoring the dual attaching map through based loops gives

$$\begin{array}{ccccc} & & \circlearrowleft^* & & \\ & & \downarrow & & \downarrow \\ \mathcal{L}S^2 & \longleftarrow & \Omega S^2 & \longleftarrow & \prod_{\{n,s\}^3} \left(\begin{array}{c} \text{Map}^{*,*}(D^1, S^2) \\ \times \text{Map}^{*,*}(D^1, S^2) \end{array} \right) \\ \uparrow & & \uparrow & & \uparrow \\ (S^2) & \longleftarrow & * & \longleftarrow & \text{Map}(T^2, S^2)^G, \end{array}$$

(hpb) (hpb)

where the top map assigns, up to homotopy,

$$\circlearrowleft^*: (\gamma_1, \gamma_2) \mapsto \text{cnc}(\gamma_1, \gamma_2, \overline{\gamma_2}, \overline{\gamma_1}), .$$

Since this is null-homotopic, (21) gives

$$\begin{aligned} \pi_1 \text{Map}(T^2, S^2)^{\mathbb{Z}/6} &\simeq \mathbb{Z} \times \mathbb{Z}^2 \\ \pi_1 \text{Map}(T^2, S^2)^{\mathbb{Z}/6} &\simeq \mathbb{Z} \times \{n, s\}^3 \end{aligned}$$

12. p6m

(...)

C. Symmorphic Tori and Type II Coefficients

Since all symmorphic actions have fixed points, while the Type II action on the 2-sphere by definition has no fixed point, there is *no* equivariant map from a symmorphic G -torus to such a 2-sphere coefficient.

D. Diamorphic Tori

(...)

V. THE EQUIVARIANT MODULAR GROUPS

Here we analyze the equivariant mapping class groups of the torus (the *equivariant modular groups*), for various point group symmetries.

Ultimately we are interested in the homotopy quotients of equivariant mapping spaces to the sphere by these equivariant mapping class groups.

A. Generalities

Definition A.1 (Equivariant mapping class group). Given a smooth action $G \curvearrowright \Sigma$ of a finite group G on a (connected, orientable) surface Σ^2 , say that the G -equivariant mapping class group is the group of connected components of the equivariant orientation-preserving diffeomorphism group of Σ^2 :

$$\text{MCG}^G(\Sigma^2) := \pi_0(\text{Diff}^+(\Sigma^2)^G). \quad (28)$$

For $G = 1$ this reduces to the plain mapping class group $\text{MCG}(\Sigma^2)$, and for $\Sigma^2 \equiv T^2$ this is also called the equivariant *modular group*.

Since equivariant maps send fixed points to fixed points, evaluation of equivariant mapping classes (28) constitutes a group homomorphism

$$\text{MCG}^G(\Sigma^2) \longrightarrow \text{Sym}((\Sigma^2)^G). \quad (29)$$

to the symmetric group of the set $(\Sigma^2)^G$ of fixed points.

Example A.2. With the G -action on $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ induced from a linear action of $L \subset \text{O}(2)$, rigid translation

$$\begin{aligned} T^2 &\longrightarrow T^2 \\ [x] &\mapsto [x + p] \end{aligned}$$

by the vector representative $p \in \mathbb{R}^2$ of any fixed point

$$G \cdot [p] = \{[p]\}$$

is an equivariant diffeomorphism, since

$$\forall_{[x] \in T^2} \quad \begin{aligned} L \cdot [x + p] &= [L \cdot x + L \cdot p] \\ &= [L \cdot x + p]. \end{aligned}$$

For some of the following computations we need to compare G -equivariant maps to more general maps that are only required to preserve the fibers of the G -quotient coprojection:

Definition A.3 (Coprojection fiber-preserving maps). With q denoting the quotient coprojection of the G -action on Σ^2 , a *fiber-preserving* homeomorphism $\hat{\phi}$ of Σ^2

is a homeomorphism of Σ^2 which fits into a commuting diagram of this form:

$$\begin{array}{ccc} \Sigma^2 & \xrightarrow[\sim]{\hat{\phi}} & \Sigma^2 \\ \downarrow q & & \downarrow q \\ \Sigma^2/G & \xrightarrow[\sim]{\phi} & \Sigma^2/G. \end{array} \quad (30)$$

for ϕ a homeomorphism of the quotient space. Write

$$\text{Diff}_q(\Sigma^2) \subset \text{Diff}(\Sigma^2) \quad (31)$$

for the subgroup of fiber-preserving diffeomorphisms and

$$\text{MCG}_q(\Sigma^2) := \pi_0(\text{Diff}_q^+(\Sigma^2)) \quad (32)$$

for the *fiber-preserving mapping class group*.

Remark A.4. Beware that the influential work [16] considers instead fiberwise diffeomorphisms modulo continuous paths of *general* diffeomorphisms (cf. p. 432 there). That definition however makes the main Theorem 5 there fail in general (cf. Rem. B.5 below) while with our definition (32) the relevant cases are salvaged (cf. Lem. B.6).

Example A.5. Equivariant diffeomorphisms are coprojection fiber-preserving, hence there are group homomorphisms:

$$\begin{array}{ccc} \text{Diff}^+(\Sigma^2)^G & \longrightarrow & \text{Diff}_q^+(\Sigma^2) \\ \downarrow & & \downarrow \\ \text{MCG}^G(\Sigma^2) & \longrightarrow & \text{MCG}_q(\Sigma^2) \end{array} \quad (33)$$

Lemma A.6 (Covering maps). *If the action of G is effective and orientation-preserving, then the fiber-preserving homeomorphisms (30) which cover the identity map (the covering maps) are precisely those given by the action of an element $g \in G$:*

$$G \simeq \left\{ \begin{array}{ccc} \Sigma^2 & \xrightarrow{\sim} & \Sigma^2 \\ q \searrow & & \swarrow q \\ & \Sigma^2/G & \end{array} \right\} \subset \text{Diff}_q^+(\Sigma^2). \quad (34)$$

Proof. It is clear that every element of G acts as a covering map. Conversely, given a covering map $\hat{\phi}$ we are to find the corresponding unique group element:

Consider the subset $\Sigma_{\text{reg}}^2 \subset \Sigma^2$ of *regular points* of the G -action. Since the action is effective, this is a dense subset (by Newman's theorem, cf. [17]), and since the action is orientation-preserving, its non-regular points are isolated (by Kerékjártó's theorem, cf. [18]) so that with Σ^2 also Σ_{reg}^2 is connected.

Now for $s \in \Sigma_{\text{reg}}^2$ any such a regular point, regularity of the action on Σ_{reg}^2 implies that there is a unique $g \in G$ such that $\hat{\phi}(s) = g \cdot s$. But since Σ_{reg}^2 is connected, the *unique lifting property* of covering spaces ([19, Prop. 1.34]) implies that $\hat{\phi}(-) = g \cdot (-)$ everywhere on Σ_{reg}^2 . Finally, by continuity, this also holds away from the regular points on all of Σ^2 . \square

Lemma A.7. *If the action of G is effective and orientation-preserving, then the covering maps (34) are normal as a subgroup of the fiber-preserving diffeomorphisms $\widehat{\phi}$ (31):*

$$\widehat{\phi} \circ (-) \circ \widehat{\phi}^{-1} : G \xrightarrow{\sim} G. \quad (35)$$

Proof. By Lem. A.6 it is sufficient to check, for $g \in G$, that:

$$q \circ (\widehat{\phi} \circ g \circ \widehat{\phi}^{-1}) = q.$$

But this follows by repeated use of the fiber-preserving property (30) of $\widehat{\phi}$ and of the covering-map property (34) of g :

$$\begin{aligned} q \circ \widehat{\phi} \circ g \circ \widehat{\phi}^{-1} &= \phi \circ q \circ g \circ \widehat{\phi}^{-1} \\ &= \phi \circ q \circ \widehat{\phi}^{-1} \\ &= \phi \circ \phi^{-1} \circ q \\ &= q. \quad \square \end{aligned}$$

Lemma A.8. *A continuous path of fiber-preserving diffeos that starts at an equivariant map (33) consists entirely of equivariant maps.*

Proof. Observing that a fiber-preserving diffeo $\widehat{\phi}$ is equivariant iff the induced isomorphism (35) is the identity, this follows by continuity, since the automorphism group of our discrete group G is itself discrete. \square

B. Examples

Here we work out examples of equivariant modular groups, hence of equivariant mapping class groups (Def. A.1) of G -tori.

1. $p2$

We discuss aspects of the equivariant modular group for $p2$ -symmetry (cf. Fig. 5).

An easy but consequential observation gives:

Proposition B.1. *We have a subgroup inclusion*

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}/2)^2 \subset \mathrm{MCG}^{\mathbb{Z}/2}(T^2).$$

Proof. Since $\mathbb{Z}/2$ -symmetry is implemented by the action of the diagonal matrix $-\mathrm{id}_2$, it is immediate that all elements of $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{MCG}(T^2)$ are $\mathbb{Z}/2$ -equivariant. The remaining normal subgroup in the claim is that by translation along fixed point vectors according to Ex. A.2. \square

2. $p3$

For the case of $p3$ equivariance (Fig. 10) with $G \equiv \mathbb{Z}/3$ we analyze the equivariant modular group. First we explicitly construct some elements (in Prop. B.2) and then we use Birman-Hilden theory to show that these generate the full group (Thm. B.10).

Proposition B.2. *For the $p3$ -action of $\mathbb{Z}/3$ on T^2 (Fig. 10), the equivariant modular group (Def. 28) contains, besides $\mathbb{Z}/3$ itself, the symmetric group on the three G -fixed points via a section of the evaluation map (29):*

$$\begin{array}{ccc} & & \mathrm{MCG}^{\mathbb{Z}/3}(T^2) \\ & \nearrow \widehat{(-)} & \downarrow \\ \mathrm{Sym}_3 \times \mathbb{Z}/3 & \xrightarrow{\mathrm{Pr}_1} & \mathrm{Aut}((T^2)^{\mathbb{Z}/3}). \end{array} \quad (36)$$

Proof. A matrix representative of the generator $[1] \in \mathbb{Z}/3$ acting on $\mathbb{R}^2/\mathbb{Z}^2$ is (using the ‘‘hexagonal’’ lattice from Fig. 10):

$$L_{[1]} := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad (37)$$

whose fixed points $[p]$ have coordinate representatives

$$p_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p_1 := \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \quad p_2 := \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Consider now the equivariant diffeomorphisms given by translation along these fixed points (per Ex. A.2) and the further one given by π -rotation: $[x] \mapsto [-x]$. Inspection shows that their action on the set of fixed points is:

(i) for translation by p_1 : a cyclic permutation

$$[p_0] \mapsto [p_1] \mapsto [p_2] \mapsto [p_0],$$

(ii) for translation by p_2 : the inverse cyclic permutation

$$[p_0] \mapsto [p_2] \mapsto [p_1] \mapsto [p_0],$$

(iii) for π -rotation: a transposition

$$[p_0] \mapsto [p_0], \quad [p_1] \mapsto [p_2] \mapsto [p_1].$$

But a cyclic permutation and a transposition generate Sym_3 , so that the evaluation homomorphism (29) is surjective. To see that it has a section $\widehat{(-)}$ (36), it is sufficient to observe that setting

$$\widehat{(123)} : [x] \mapsto [x + p_1]$$

and

$$\widehat{(1)(23)} : [x] \mapsto [-x]$$

constitutes a group homomorphism, because the relations characterizing the symmetric group are respected:

$$\begin{aligned} \widehat{(123)}^3 &= e, \\ \widehat{(1)(23)}^2 &= e, \\ \widehat{(123)} \circ \widehat{(1)(23)} &= \widehat{(1)(23)} \circ \widehat{(123)}^{-1} \end{aligned}$$

But this is immediate (note that the last one translates to: $[-x + p_1] = [-(x - p_1)]$). This establishes the Sym_3 subgroup.

Finally, it is clear that $L_{[1]}$ and $L_{[1]}^2$ are equivariant. Since their representing matrices (37) are not the identity matrix, the corresponding diffeomorphisms are not homotopic to the identity matrix even non-equivariantly. Since by definition these elements fix the fixed points, this yields in total the claimed product group $\text{Sym}_3 \times \mathbb{Z}/3$. \square

Proposition B.3. *The inclusion (36) is retractive, in that there is a surjective homomorphism which postcomposes it to the identity:*

$$\mathbb{Z}/3 \times \text{Sym}_3 \hookrightarrow \text{MCG}^{\mathbb{Z}/3}(T^2) \twoheadrightarrow \mathbb{Z}/3 \times \text{Sym}_3. \quad (38)$$

$\underbrace{\hspace{10em}}_{\text{id}}$

Proof. We appeal to aspects of *Birman-Hilden theory* (cf. [20]). We will write $S_{g,n}^2$ for the oriented surface of genus g with n punctures.

To set this up, notice that the complement of the three fixed points inside the quotient space (Tbl. III) is the 3-punctured sphere $S_{0,3}^2$:

$$(S_{1,0}^2 - (S_{1,0}^2)^{\mathbb{Z}/3})/\mathbb{Z}/3 \simeq S_{0,3}^2,$$

so that equivariant homeomorphisms of the torus descend to homeomorphisms of the 3-punctured sphere, via a topological group homomorphism:

$$\text{Diff}(T^2)^{\mathbb{Z}/3} \twoheadrightarrow \text{Diff}(\Sigma_{0,3}^2). \quad (39)$$

Now by Lem. B.7 we have an isomorphism

$$\text{MCG}_q(S_{1,0}^2) \simeq \mathbb{Z}/3 \times \text{Sym}_3$$

and by Lem. B.4 we have a surjective homomorphism

$$\text{MCG}^{\mathbb{Z}/3}(S_{1,0}^2) \twoheadrightarrow \text{MCG}_q(S_{1,0}^2) \simeq \mathbb{Z}/3 \times \text{Sym}_3.$$

which is the claimed retraction. \square

Lemma B.4. *The $p3$ -equivariant modular group (28) homomorphically surjects onto the fiber-preserving mapping class group (32):*

$$\text{MCG}^{\mathbb{Z}/3}(S_{1,0}^2) \twoheadrightarrow \text{MCG}_q(S_{1,0}^2). \quad (40)$$

Proof. Given a fiber-preserving diffeo $\widehat{\phi}$ and writing $\widehat{\tau}$ for a generator of $\mathbb{Z}/3 \subset \text{Diff}_q(S_{1,0}^2)$ (34), Lem. A.7 implies that there exists $[k] \in \mathbb{Z}/3$ such that

$$\widehat{\phi} \circ \widehat{\tau} \circ \widehat{\phi}^{-1} = \widehat{\tau}^k.$$

Hence the claim is equivalent to the statement that $[k] = [1]$ for all orientation-preserving $\widehat{\phi}$.

To that end, consider the differentials $d\phi_p$ and $d\tau_p$ at a fixed point $[p]$. Applying the chain rule to the above conjugation condition implies

$$d\phi_p \circ d\tau_p = d(\tau^k)_{\phi(p)} \circ d\phi_p.$$

Here $d\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by the rotation matrix $R_{2\pi/3}$, hence $d(\tau^k)$ by $R_{2\pi k/3}$, and $d\phi_p$ may be parameterized by a real matrix

$$d\phi_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $\det(d\phi_p) \equiv ad - bc > 0$.

Now observe that we only need to rule out $[k] = [2]$, since $k = [0]$ cannot happen for a homeomorphism (whose conjugation action must take generators to generators).

So it remains now to derive a contradiction from the above chain rule equation for $[k] = 2$, which is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for $\theta = 2\pi/3$. But in its left column this matrix equation implies $b = c$ and $d = -a$, and hence implies:

$$d\phi_p = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

But invertible real matrices of this form manifestly have negative determinant, which contradicts the assumption that ϕ is orientation-preserving.

Therefore $[k] = [2]$ is seen to be inconsistent and hence only the claimed case $[k] = [1]$ remains. \square

Remark B.5. The following Lemma B.6 is a corrected version of [16, Thm. 5] for our case, using our Definition A.3 of fiberwise mapping classes (cf. Rem. A.4).

Beacuse, as stated, [16, Thm. 5] actually fails in our case: This was noticed in [21] and acknowledged in the erratum [22]. (Our $p3$ -action is just the case $(n = 3, k = 3)$ excluded in [21, Thm. 1.3].)

In fact, this failure is also seen immediately from our analysis: The subgroup $\text{Sym}_3 \subset \text{Diff}_q(T^2)$ (36) permuting the fixed points is generated by *translations* along the fixed points (Ex. A.2), and translations clearly are homotopic to the identity if we do not require the homotopy itself to be fiber preserving. Therefore fiber-preserving maps modulo unconstrained homotopy cannot surject onto the mapping classes of $S_{0,3}^2$.

Lemma B.6. *We have an isomorphism:*

$$\mathrm{MCG}_q(S_{1,0}^2)/\mathbb{Z}_3 \xrightarrow{\sim} \mathrm{MCG}(S_{0,3}^2).$$

Proof. It is sufficient to show that the comparison homomorphism is both surjective and injective.

The claim that this kind of map is surjective goes back to [16]; and despite a missing assumption there (cf. Rem. B.5), this does remain true in our case, by [21, Thm. 1.1]: In the notation there, we are in the case $n = 3$, $k = 3$, $A = \mathbb{Z}_3$, and $(a_1, a_2, a_3) = ([1], [1], [1])$.

For injectivity it is sufficient to show that for $\widehat{\phi}$ a fiber-preserving diffeo such that the restriction ϕ_0 of its base map ϕ to $S_{0,3}^2 \subset S_{0,0}^2 \simeq S_{1,0}^2/\mathbb{Z}_3$ has a homotopy ϕ_t to $\phi_0 := \mathrm{id}_{S_{0,3}^2}$, then there exists a fiber-preserving homotopy $\widehat{\phi}_t$ connecting it to a covering map. But over $S_{1,3}^2$ this follows by the *homotopy lifting property* (cf. [19, Prop. .30]) of the covering space $S_{1,3}^2 \twoheadrightarrow S_{0,3}^2$ and from there it extends uniquely to $S_{1,0}^2$. \square

Lemma B.7. *The fiber-preserving mapping class group (32) is:*

$$\mathrm{MCG}_q(T^2) \simeq \mathbb{Z}_3 \times \mathrm{Sym}_3,$$

where the second factor Sym_3 records the permutation action on the fixed/branch points.

Proof. By Prop. B.2 and via Lem. B.4, the right hand side is indeed a subgroup

$$\underbrace{\mathbb{Z}_3 \times \mathrm{Sym}_3 \hookrightarrow \mathrm{MCG}^{\mathbb{Z}_3}(T^2) \twoheadrightarrow \mathrm{MCG}_q(T^2)}_{\uparrow}$$

(The composite is still an injection since the representing integer matrices of distinct group elements on the left are clearly distinct.)

But since (by Lem. B.6 and [23, Prop. 2.3]) we also have

$$\mathrm{MCG}_q(T^2)/\mathbb{Z}_3 \simeq \mathrm{MCG}(S_{0,3}^2) \simeq \mathrm{Sym}_3,$$

this subgroup must already be the full group. \square

Lemma B.8. *The $p3$ -equivariant modular group is a semidirect product with normal subgroup $\mathbb{Z}_3 \times \mathrm{Sym}_3$:*

$$\mathrm{MCG}^{\mathbb{Z}_3}(T^2) \simeq K_{p3} \ltimes (\mathbb{Z}_3 \times \mathrm{Sym}_3),$$

where K_{p3} is the kernel of (40).

Proof. This follows by Prop. B.3 from standard facts (cf. [24, Lem. 7.20]). \square

Lemma B.9. *The kernel of (40) is trivial:*

$$K_{p3} \simeq 1.$$

Proof. Lem. B.7 implies that every element in the kernel

$$K_{p3} \hookrightarrow \mathrm{MCG}^{\mathbb{Z}_3}(S_{1,0}^2) \twoheadrightarrow \underbrace{\mathrm{MCG}_q(S_{1,0}^2)}_{\mathbb{Z}_3 \times \mathrm{Sym}_3}$$

preserves the three fixed/branch points, individually. Therefore if $S_{1,1}^2$ denotes the torus with one of the fixed/branch points removed, the alternative kernel

$$K'_{p3} \hookrightarrow \mathrm{MCG}^{\mathbb{Z}_3}(S_{1,1}^2) \twoheadrightarrow \mathrm{MCG}_q(S_{1,1}^2)$$

surjects onto the kernel in question

$$K'_{p3} \twoheadrightarrow K_{p3}.$$

Therefore it now suffices to show that K'_{p3} is trivial. But in this case of the punctured torus $S_{1,1}^2$, [16, Thm. 1] applies and implies that any representative ϕ of an element $[\phi] \in K'_{p3}$ is connected to the identity by a path of fiber-preserving maps. By Lem. A.8 this must be a path of \mathbb{Z}_3 -equivariant maps, and hence $[\phi]$ is the trivial element. \square

In conclusion:

Theorem B.10. *The $p3$ -equivariant modular group is*

$$\mathrm{MCG}^{\mathbb{Z}_3}(T^2) \simeq \mathbb{Z}_3 \times \mathrm{Sym}_3.$$

Proof. By Lem. B.9 used in Lem. B.8. \square

VI. CONCLUSION AND OUTLOOK

The above computations of low degree (covariantized) equivariant 2-Cohomotopy of G -tori appear to be largely novel, despite these being among the first nontrivial examples that suggest themselves mathematically, and despite the considerable interest that these have as classes of fragile crystalline phases of matter [1].

In particular, we found (cf. Tbl. II) that:

- (i) in degree=0 the usual Chern classification known in the condensed matter literature splits into finite sets of further topological sectors labeled by the assignment of the *high symmetry points* (fixed points) of the domain torus to those of the coefficient 2-sphere;
- (ii) in degree=-1 that the remarkable anyonic signature (the nonabelian integral Heisenberg group) of the non-equivariant 2-Cohomotopy of the torus (cf. §IV A 1) gets replaced by abelian groups as soon as there is nontrivial symmetry;

but that

- (iii) interesting nonabelian groups appear after ‘‘covariantization’’, namely in the equivariant modular groups (§V) acting on the equivariant 2-Cohomotopy.

(...)

The *fundamental group* of a pointed space A is the connected components of its based loop space (A10):

$$\pi_1(A) \equiv \pi_0(\Omega A), \quad (\text{A13})$$

and iteratively so for the higher *homotopy groups*:

$$\pi_n(A) \equiv \pi_0(\Omega^n A) \equiv \pi_0(\underbrace{\Omega \cdots \Omega A}_n), \quad (\text{A14})$$

which implies in particular that

$$\pi_n(\Omega^k A) = \pi_{n+k}(A). \quad (\text{A15})$$

For example, the non-torsion homotopy groups of the 2-sphere are

$$\pi_2(S^2) \simeq \mathbb{Z}, \quad \pi_3(S^2) \simeq \mathbb{Z}. \quad (\text{A16})$$

A map $f : X \rightarrow Y$ induces a map of connected components π_0 as well as homomorphisms between all homotopy groups π_n (A14) for all basepoints $x \in X$ and induced basepoints $f(x) \in Y$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y) \\ \pi_n(X, x) & \xrightarrow{\pi_n(f)} & \pi_n(Y, f(x)). \end{array} \quad (\text{A17})$$

If $\pi_0(f)$ is a bijection and $\pi_n(f, x)$ is an isomorphism for all n and x , then f is called a *weak homotopy equivalence*, to be denoted “ $\xrightarrow{\sim}$ ” and X is said to be (weak homotopy) *equivalent* to Y , denoted

$$\exists (f : X \xrightarrow{\sim} Y) \quad \Leftrightarrow \quad X \sim Y. \quad (\text{A18})$$

A map $f : X \rightarrow Y$ is a (Serre-) *fibration*, to be denoted “ \twoheadrightarrow ”, if all families of paths continuously parameterized by higher dimensional disks D^n may be lifted through f for arbitrary lifts of their starting points:

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{\forall} & X \\ \downarrow & \dashrightarrow \exists & \downarrow \\ D^n \times [0, 1] & \xrightarrow{\forall} & Y. \end{array} \quad (\text{A19})$$

For instance, the restriction of maps (A8) along cell boundary inclusion (A4) is a fibration:

$$\text{Map}(D^n, X) \xrightarrow{\iota_n^*} \text{Map}(S^{n-1}, X). \quad (\text{A20})$$

For $A \xrightarrow{\mathcal{L}} B$ a fibration (A19) of pointed spaces with fiber F (A2) — hence: homotopy fiber (A30) —, there is induced a long exact sequence of homotopy groups (A14)

of the following form (ending in an exact sequence of pointed sets of connected components):

$$\begin{array}{ccccccc} F & \xrightarrow{\text{fib}_p} & A & \xrightarrow{p} & B & & \\ \Rightarrow & \cdots & \xrightarrow{\pi_3} & \pi_3(S) & \xrightarrow{\pi_3 p} & \pi_3(B) & \\ & \searrow & \delta_2 & \xrightarrow{\pi_2} & \pi_2 F & \xrightarrow{\pi_2 \text{fib}_p} & \pi_2 A & \xrightarrow{\pi_2 p} & \pi_2 B & \\ & \searrow & \delta_1 & \xrightarrow{\pi_1} & \pi_1 F & \xrightarrow{\pi_1 \text{fib}_p} & \pi_1 A & \xrightarrow{\pi_1 p} & \pi_1 B & \\ & \searrow & \delta_0 & \xrightarrow{\pi_0} & \pi_0 F & \xrightarrow{\pi_0 \text{fib}_p} & \pi_0 A & \xrightarrow{\pi_0 p} & \pi_0 B, \end{array} \quad (\text{A21})$$

where δ_0 factors through the center $Z(-)$ [29]:

$$\pi_1 B \dashrightarrow Z(\pi_1 F) \hookrightarrow \pi_1(F). \quad (\text{A22})$$

These long exact sequences are *natural* in that a commuting square of pointed maps induces a commuting ladder of homotopy groups:

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ & & \pi_{n+1} B' & \rightarrow & \pi_{n+1} B \\ & & \downarrow & & \downarrow \\ A' & \rightarrow & A & \Rightarrow & \pi_n F' & \rightarrow & \pi_n F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B' & \rightarrow & B & & \pi_n A' & \rightarrow & \pi_n A \\ & & & & \downarrow & & \downarrow \\ & & & & \pi_n B' & \rightarrow & \pi_n B. \end{array} \quad (\text{A23})$$

For example, the based loop space (A10) is the fiber of the point-evaluation map of its free loop space

$$\underbrace{\text{Map}^*(S^1, A)}_{\Omega A} \xrightarrow{\text{fib}_{\text{ev}}} \underbrace{\text{Map}(S^1, A)}_{\mathcal{L}A} \xrightarrow{\text{ev}} A \quad (\text{A24})$$

$\longleftarrow \text{cnst}$

(as well as the fiber of the *based path space fibration*

$$\underbrace{\text{Map}^*(S^1, A)}_{\Omega A} \xrightarrow{\text{fib}_{\text{ev}_1}} \text{Map}^*(D^1, A) \xrightarrow{\text{ev}_1} A, \quad (\text{A25})$$

where in the second case $* := 0 \in D^1$ is one endpoint of the interval, while $1 \in D_1$ is the other). Due to the section cnst (A24) the induced homotopy long exact sequence (A21) splits to yield:

$$\pi_n \mathcal{L}A \simeq \pi_n \Omega A \times \pi_n A \quad (\text{A26})$$

(where the base point of the loop spaces is the loop constant on the base points of A). For the case $A = S^2$ this

gives, with (A16) and (A15):

$$\begin{array}{ccc}
\pi_1 \Omega S^2 & \xrightarrow{\pi_1 \text{fib}_{\text{ev}}} & \pi_1 \mathcal{L} S^2 \\
\parallel & \xrightarrow{\text{id}} & \parallel \\
\mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z}, \\
\pi_2 \Omega S^2 & \xrightarrow{\pi_2 \text{fib}_{\text{ev}}} & \pi_2 \mathcal{L} S^2 \xleftarrow{\pi_2 \text{cnst}} \pi_2 S^2 \\
\parallel & \xrightarrow{\quad} & \parallel \xleftarrow{\pi_2 \text{ev}} \parallel \\
\mathbb{Z} & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z} \\
& & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & \mathbb{Z}.
\end{array} \tag{A27}$$

Note in this context that from long exact sequences of groups there are induced short exact sequences by *truncation*

$$\begin{array}{ccccccc}
A_{-2} & \xrightarrow{f_{-2}} & A_{-1} & \xrightarrow{f_{-1}} & A_0 & \xrightarrow{f_1} & A_1 \\
1 & \longrightarrow & A_{-1}/A_{-2} & \xrightarrow{f_{-1}} & A_0 & \xrightarrow{f_1} & \text{im}(f_1) \longrightarrow 1.
\end{array} \tag{A28}$$

More generally, a commuting diagram of spaces is said to be a *homotopy pullback* (hpb) if there exists a factoring via weak homotopy equivalences through an ordinary pullback of a fibration:

$$\begin{array}{ccc}
A' \longrightarrow A \\
f' \downarrow \text{(hpb)} \quad \downarrow f \\
B' \longrightarrow B
\end{array} \Leftrightarrow \exists f' \left[\begin{array}{ccc}
A & \longrightarrow & A \\
\downarrow \wr & & \downarrow \wr \\
\widehat{A}' & \longrightarrow & \widehat{A} \\
\downarrow \text{(pb)} & & \downarrow \\
B' & \longrightarrow & B
\end{array} \right] f \tag{A29}$$

For example a general *homotopy fiber* F_b of a map $F \rightarrow B$ is the homotopy pullback of that map to the given base point b :

$$\begin{array}{ccc}
F_b & \xrightarrow{\text{hfib}_p} & F \\
\downarrow \text{(hpb)} & & \downarrow p \\
* & \xrightarrow{b} & B,
\end{array} \tag{A30}$$

and a based loop space (A10) is equivalently the homotopy pullback of the base point inclusion along itself:

$$\begin{array}{ccc}
\Omega X & \longrightarrow & * \\
\downarrow \text{(hpb)} & & \downarrow \\
* & \longrightarrow & X,
\end{array} \tag{A31}$$

while a product is just a homotopy pullback from the point:

$$\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow \text{(hpb)} & & \downarrow \\
X & \longrightarrow & *.
\end{array} \tag{A32}$$

A further important class of example of homotopy fiber sequences (A30) are *homotopy quotients* $X // G$ of spaces X by the continuous actions $G \curvearrowright X$ of (well-pointed) topological groups G , modeled by the *Borel construction*, as follows [9, Lem. 3.3.23] (here $EG \twoheadrightarrow BG$ is the *universal*

principal G -bundle and $X \times_G EG$ its associated X -fiber bundle, cf. [9, §3.3.1]):

$$X \xrightarrow{\text{hfib}_p} \underbrace{X \times_G EG}_{X // G} \xrightarrow{p} BG. \tag{A33}$$

Homotopy pullbacks (A29) are invariant, up to equivalence, under homotopy (A6). For example, the space of x_0 -based loops in a connected space X is equivalent to the space of paths (maps $D^1 \rightarrow X$) whose endpoints are pinned to a pair of not necessarily coincident points $x_0, x_1 \in X$:

$$\begin{array}{ccc}
\text{Map}^{*0,*1}(D^1, X) & \longrightarrow & \text{Map}^{*0}(D^1, X) \\
\downarrow \sim & \searrow \text{(hpb)} & \downarrow \text{ev}_1 \\
\Omega_{x_0} X & \xrightarrow{\quad} & * \\
\downarrow & \searrow & \downarrow \\
* & \xrightarrow{x_1} & X \\
& \searrow x_0 & \uparrow \\
& & *
\end{array} \tag{A34}$$

Moreover, homotopy pullbacks (A29) satisfy the *past-ing law* saying that if in a commuting diagram of the form

$$\begin{array}{ccccc}
A'' & \longrightarrow & A' & \longrightarrow & A \\
\downarrow & & \downarrow \text{(hpb)} & & \downarrow \\
B'' & \longrightarrow & B' & \longrightarrow & B.
\end{array} \tag{A35}$$

the right square is a homotopy pullback, then the left square is so if and only if the total rectangle is.

For example, the left square in the following diagram is a homotopy pullback:

$$\begin{array}{ccccc}
* & \longrightarrow & \Omega X & \longrightarrow & * \\
\downarrow \text{(hpb)} & & \downarrow \text{(hpb)} & & \downarrow \\
X & \xrightarrow{\text{cnst}} & \mathcal{L} X & \xrightarrow{\text{ev}} & X, \\
\downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id}
\end{array} \tag{A36}$$

because the right square is so by (A24) and the total rectangle is so because the (homotopy) pullback of an identity is (equivalent to) an identity.

2. Equivariant homotopy theory

All of the above generalizes to spaces equipped with an *action* $G \curvearrowright X$ (G -spaces) by a finite group G (regarded as a discrete topological group), given by maps

$$\begin{array}{ccc}
G \times X & \xrightarrow{(-) \cdot (-)} & X \\
(g, x) & \longmapsto & g \cdot x
\end{array} \tag{A37}$$

$$\text{s.t. } \forall_{\substack{g_i \in G \\ x \in X}} \begin{cases} e \cdot x = x \\ (g_2 \cdot g_1) \cdot x = g_2 \cdot (g_1 \cdot x). \end{cases}$$

The *subspace of fixed points* (or *fixed locus*) of such a G -space is

$$X^G := \left\{ x \in X \mid \forall_{g \in G} g \cdot x = x \right\} \subset X. \quad (\text{A38})$$

and the *equivariant maps* between pairs of such G -spaces $G \zeta X$ and $G \zeta Y$ are the maps $X \xrightarrow{f} Y$ between the underlying spaces which respect the G -action:

$$\forall_{\substack{g \in G \\ x \in X}} f(g \cdot x) = g \cdot f(x) \Leftrightarrow \begin{array}{ccc} & \begin{array}{c} \curvearrowright \\ X \end{array} & \xrightarrow{f} & \begin{array}{c} \curvearrowright \\ Y \end{array} \\ & & & \end{array}. \quad (\text{A39})$$

For example, for a pair of G -spaces $G \zeta X$ and $G \zeta Y$, the mapping space (A7) between their underlying spaces becomes a G -space via the *conjugation action*

$$\begin{array}{ccc} G \times \text{Map}(X, Y) & \longrightarrow & \text{Map}(X, Y) \\ (g, f) & \longmapsto & (x \mapsto g^{-1} \cdot f(g \cdot x)), \end{array} \quad (\text{A40})$$

whose G -fixed points (A38) are exactly the G -equivariant maps (A39). Therefore the *equivariant mapping space* is

$$\text{Map}(X, Y)^G \subset \text{Map}(X, Y). \quad (\text{A41})$$

Another basic example of G -spaces are the (discrete) coset spaces G/H for subgroups $H \subset G$ (with the action given by left multiplication in G). These are the canonical *orbits* of points under G , in that for a point $x \in X$ with *stabilizer* or *isotropy* group

$$\text{Stab}(x) := \{ g \in G \mid g \cdot x = x \} \subset G \quad (\text{A42})$$

the G -orbit of all images of x under the action of G is G -equivariantly identified with $G/\text{Stab}(x)$.

The special case $G/G = *$ of the point equipped with the trivial G -action is important: An equivariant map (A39) out of G/G is a *fixed point*

Pullbacks (A1) and pushouts (A3) along G -equivariant maps are constructed as for the underlying plain maps and inherit a unique G -space structure (cf. [9, Lem 1.1.10]).

For example, an *equivariant n -cell attachment* to a G -space $G \zeta X$ is like a plain cell attachment (A4) but now by G -orbits of cells:

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H & \xrightarrow[\text{equiv. attaching map}]{f} & X \\ \sqcup_i \iota_{(n,H)} \downarrow & & \downarrow \\ \coprod_i D^n \times G/H & \longrightarrow & (D^n \times G/H) \cup_f X. \end{array} \quad (\text{A43})$$

Constructing a space by iterative G -cell attachments starting with $S^{-1} \times G/H \equiv \emptyset$ means to give it a G -cell complex structure or a G -cell decomposition (cf. [9, Ex. 1.3.6]).

For example, the G/G -orbits of cells in a G -space are its G -fixed points which in crystallography, with G a crystallographic point group, are known as the *high symmetry points*.

In equivariantly mapping out of an equivariant cell complex, note that, for $X \in \text{Top}^{(*)}$ a (pointed) space with trivial G -action and $G \zeta Y$ any (pointed) G -space:

- G -Equivariant maps out of a free action permuting disjoint components are equivalently ordinary maps out of one component:

$$\text{Map}(X \times G/1, Y)^G \simeq \text{Map}(X, Y). \quad (\text{A44})$$

- G -Equivariant maps out of a trivial action are equivalently maps to the G -fixed locus $(-)^G$ (A38) inside the domain space:

$$\text{Map}(X \times G/G, Y)^G \simeq \text{Map}(X, Y^G), \quad (\text{A45})$$

In a similar vein, now for Y with trivial G -action and $G \zeta X$ any G -space, we have:

- G -Equivariant maps into a trivial G -action are equivalently maps out of the quotient space X/G :

$$\text{Map}(X, Y \times G/G)^G \simeq \text{Map}(X/G, Y). \quad (\text{A46})$$

Passage to G -fixed loci (A38) preserves pullbacks (A1):

$$\begin{array}{ccc} G \zeta A' \longrightarrow G \zeta A & A'^G \longrightarrow A^G \\ \downarrow \text{(pb)} & \downarrow \Rightarrow \downarrow \text{(pb)} & \downarrow \\ G \zeta B' \longrightarrow G \zeta B & B'^G \longrightarrow B^G. \end{array} \quad (\text{A47})$$

In particular, with (A41) and (A12) this implies that forming equivariant maps out of an equivariant pushout yields a pullback:

$$\begin{array}{ccc} G \zeta A' \longrightarrow G \zeta A \\ \downarrow & \text{(po)} & \downarrow \\ G \zeta B' \longrightarrow G \zeta B \\ \downarrow & & \downarrow \\ \text{Map}^{(*)}(A', X)^G \longleftarrow \text{Map}^{(*)}(A, X)^G \\ \uparrow & & \uparrow \\ \text{Map}^{(*)}(B', X)^G \longleftarrow \text{Map}^{(*)}(B, X)^G. \end{array} \quad (\text{A48})$$

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