Equivariant Cohomotopy implies orientifold tadpole cancellation

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Abstract

There are fundamental open problems in the precise global nature of RR-field tadpole cancellation conditions in string theory. Moreover, the non-perturbative lift as M5/MO5-anomaly cancellation in M-theory had been based on indirect plausibility arguments, lacking a microscopic underpinning in M-brane charge quantization. We provide a framework for answering these questions, crucial not only for mathematical consistency but also for phenomenological accuracy of string theory, by formulating the M-theory C-field on flat M-orientifolds in the generalized cohomology theory called *Equivariant Cohomotopy*. This builds on our previous results for smooth but curved spacetimes, showing in that setting that charge quantization in twisted Cohomotopy rigorously implies a list of expected anomaly cancellation conditions. Here we further expand this list by proving that brane charge quantization in unstable equivariant Cohomotopy implies the anomaly cancellation conditions for M-branes and D-branes on flat orbi-orientifolds. For this we **(a)** use an unstable refinement of the equivariant Hopf-tom Dieck theorem to derive local/twisted tadpole cancellation, and in addition **(b)** the lift to super-differential cohomology to establish global/untwisted tadpole cancellation. Throughout, we use **(c)** the unstable Pontrjagin-Thom theorem to identify the brane/O-plane configurations encoded in equivariant Cohomotopy and **(d)** the Boardman homomorphism to equivariant K-theory to identify Chan-Paton representations of D-brane charge. We find that unstable equivariant Cohomotopy, but not its image in equivariant K-theory, distinguishes D-brane charge from the finite set of types of O-plane charges.

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1 Introduction

Organizing and formalizing results in the string theory literature, we start by noticing the following curious systematics, to be elaborated upon throughout the paper.

Toroidal orientifolds with ADE-singularities – A curious pattern. Consider type II superstring vacua compactified on fluxless toroidal orientifolds ([Sag88][DLP89, p. 12][Mu97][dBD$^+$02, 3]; see also [IU12, 5.3.4, 10.1.3][BLT13, 15.3]) with ADE-type singularities ([AM97][In97]; see [HSS18]), hence on orbifold quotients $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ G$ (e.g. [Ra06, 13]) of 4-tori by crystallographic point groups (22). These are finite subgroups $G \subset \mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ of the group unit quaternions acting by left multiplication on the space $\mathbb{H} \simeq_{\mathbb{R}} \mathbb{R}^4$ of all quaternions (17).

The consistency condition on such compactifications known as (Ramond-Ramond) *RR-field tadpole anomaly cancellation* ([GP96, Sec. 3][Wi12, Sec. 9.3]; see [IU12, 4.4][BLT13, 9.4]), essentially says that the joint D-brane and O-plane charge in such compact orientifolds has to vanish, albeit with some subtle fine print. Explicitly, we observe that a case-by-case analysis of the string worldsheet superconformal field theory shows (*Table 1*) that, for single wrapping number, RR-field tadpole anomaly cancellation is the following condition on the G-representation of D-brane charge and the G-set of O-planes:

(i) Local/twisted tadpole cancellation: D-brane charge is a combination of a regular representation $\mathbf{k}_{\mathrm{reg}}$ and the trivial one $\mathbf{1}_{\mathrm{triv}}$, with coefficients the number of integral and fractional branes, respectively:

$$\mathbf{c}_{\mathrm{Dbra}} = N_{\underset{\mathrm{integ}}{\mathrm{brane}}} \cdot \mathbf{k}_{\mathrm{reg}} + N_{\underset{\mathrm{frac}}{\mathrm{brane}}} \cdot \mathbf{1}_{\mathrm{triv}} \,.$$

(ii) Global/untwisted tadpole cancellation: The dimension of D-brane charge is the cardinality of the G-set of O-planes:

$$\dim(\mathbf{c}_{\mathrm{Dbra}}) = \mathrm{card}(\mathbf{c}_{\mathrm{Opla}}) \,.$$

In particular, $\mathbf{c}_{\mathrm{Dbra}}$ comes from, and $\mathbf{c}_{\mathrm{Opla}}$ gives rise to, a permutation representation, in the image of β (46) [BSS19]:

$$\begin{tikzcd}
\mathbf{c}_{\mathrm{Dbra}} \arrow[r, phantom, "\in"] & \mathrm{RO}(G) \arrow[r, phantom, "\simeq"] & \mathrm{KO}^0_G & \mathbb{S}^0_G \arrow[l, "\beta"'] \arrow[r, phantom, "\simeq"] & A(G) & G\mathrm{Set}_{/\sim} \arrow[l] \arrow[r, phantom, "\ni"] & \mathbf{c}_{\mathrm{Opla}} \\
\substack{\text{D-brane charge}\\ \text{on toroidal orientifold}} & \substack{\text{representation}\\ \text{ring}} & \substack{\text{equivariant}\\ \text{K-theory}} & \substack{\text{equivariant}\\ \text{stable Cohomotopy}} & \substack{\text{Burnside}\\ \text{ring}} & \substack{G\text{-sets}\\ (G\text{-permutations})} & \substack{\text{O-plane charge}\\ \text{on toroidal orientifold}}
\end{tikzcd} \tag{1}$$

Single D-brane species on toroidal orientifold	**Local/twisted tadpole cancellation condition**	**Global/untwisted tadpole cancellation condition**	**Comments**
Branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ G^{\mathrm{ADE}}$	$\mathbf{c}_{\mathrm{Dbra}} = N_{\underset{\mathrm{int}}{\mathrm{brane}}} \cdot \mathbf{k}_{\mathrm{reg}} + N_{\underset{\mathrm{frac}}{\mathrm{brane}}} \cdot \mathbf{1}_{\mathrm{triv}}$	$\dim(\mathbf{c}_{\mathrm{tot}}) = \mathrm{card}(\mathbf{c}_{\mathrm{Opla}})$	The general pattern of the following case-by-case results
D5/D9-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{2}_{\mathrm{reg}}$ ([BST99, (19)])	$\mathbf{c}_{\mathrm{Dbra}} = 16 \cdot \mathbf{2}_{\mathrm{reg}}$ ([BST99, (18)])	Following [GP96] [GJ96]
D5/D9-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_4$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{4}_{\mathrm{reg}}$ ([BST99, (19)])	$\mathbf{c}_{\mathrm{Dbra}} = 8 \cdot \mathbf{4}_{\mathrm{reg}}$ ([BST99, (18)])	
D4-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_k$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{k}_{\mathrm{reg}}$ ([AFIRU00a, 4.2.1])		Re-derived via M5-branes below in §4
D4-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_3$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{3}_{\mathrm{reg}}$ [AFIRU00b, (7.2)]	$\mathbf{c}_{\mathrm{Dbra}} = 4 \cdot \mathbf{3}_{\mathrm{reg}} + 4 \cdot \mathbf{1}_{\mathrm{triv}}$ ([KS02, (14)-(17)],	The special case of $k = 3$ (review in [Ma03, 4])
D8-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_3$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{3}_{\mathrm{reg}}$	$\mathbf{c}_{\mathrm{Dbra}} = 4 \cdot \mathbf{3}_{\mathrm{reg}} + 4 \cdot \mathbf{1}_{\mathrm{triv}}$ ([Hn01, 4], [Hn02, (29)])	Equivalent by T-duality to previous case ([Hn01, p.1], [Hn02, 6])
D3-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_k$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{k}_{\mathrm{reg}}$ ([FHKU01, (25)])		
D7-branes on $\mathbb{T}^{4_{\mathbb{H}}} /\!\!/ \mathbb{Z}_k$	$\mathbf{c}_{\mathrm{Dbra}} = N \cdot \mathbf{k}_{\mathrm{reg}}$ ([FHKU01, (5), (6)])		
D6-branes on $\mathbb{T}^6 /\!\!/ \mathbb{Z}_4$		$\mathbf{c}_{\mathrm{Dbra}} = 8 \cdot \mathbf{k}_{\mathrm{reg}}$ ([IKS99, (25)])	

Table 1 – Tadpole cancellation conditions between D-branes and O-planes on toroidal ADE-orientifolds as derived from case-by-case analysis in perturbative string theory. The geometric content is shown in *Figure A*. The re-derivation from *Hypothesis H* is in §4.2.

The D-brane species in *Table 1* with the most direct lift to M-theory are the D4-branes, lifting to M5-branes under double dimensional reduction ([APPS97a, 6][APPS97a, 6][LPSS11]); see *Table 7*. With an actual formulation of M-theory lacking, indirect plausibility arguments have been advanced [DM95][Wi95b, 3.3][Ho98, 2.1] that for M5-branes on M-theoretic orientifolds of the form $\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}} /\!\!/ \mathbb{Z}_2$, anomaly cancellation implies *Table 2*:

Single M-brane species on toroidal orientifold	**Local/twisted tadpole cancellation condition**	**Global/untwisted tadpole cancellation condition**	**Comments**
M5-branes on $\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}} /\!\!/ \mathbb{Z}_2$	$\mathbf{c}_{\mathrm{Mbra}} = N \cdot \mathbf{2}_{\mathrm{reg}}$ ([DM95] [Wi95b, 3.3] [Ho98, 2.1])	$\mathbf{c}_{\mathrm{Mbra}} = 16 \cdot \mathbf{2}_{\mathrm{reg}}$	plausibility arguments

Table 2 – M5/MO5 anomaly cancellation in M-theory according to Folklore 4.1. While it has remained open in which cohomology theory the charge $\mathbf{c}_{\mathrm{Mbra}}$ is quantized, the geometric picture is again that illustrated in *Figure A*.

We highlight in *Figure A* the geometric interpretation of these tadpole cancellation conditions from *Table 1* and *Table 2*. The left side of *Figure A* shows a 2-dimensional slice through the toroidal orbifold $\mathbb{T}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_4 = (\mathbb{R}^4/\mathbb{Z}^4) /\!\!/ \mathbb{Z}_4$ with transversal branes/O-plane charges appearing as points. The O-plane charges (shown as open circles) are stuck one-to-one to the fixed points of the point reflection subgroup $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ (see also *Table RT*) and, in the example shown, are permuted by the full orbifold group action of $\mathbb{Z}_4$ according to the permutation representation $2 \cdot \mathbf{1}_{\mathrm{triv}} + 1 \cdot \mathbf{2}_{\mathrm{perm}}$. The local/twisted tadpole cancellation condition says that the branes (shown as filled circles) appear in the vicinity of the O-planes with all their distinct mirror images under the full group action, thus contributing Chan-Paton fields in the regular representation $\mathbf{4}_{\mathrm{reg}}$. The global/untwisted tadpole cancellation condition says that the total charge of branes minus O-planes, hence the net charge if all branes/O-planes could freely move and pairwise annihilate, vanishes:

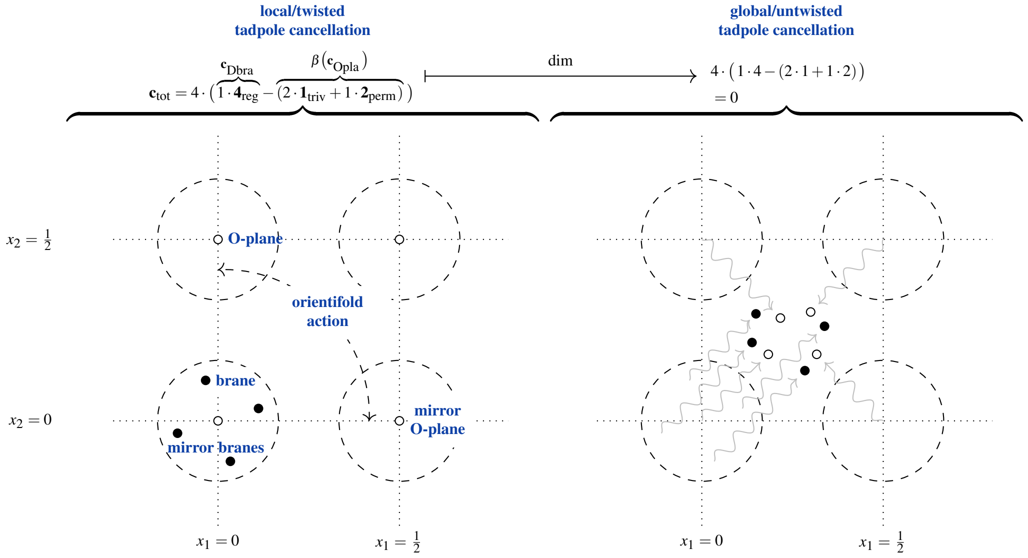

Figure A – Illustration of the geometric situation of tadpole cancellation on toroidal ADE-orientifolds according to *Table 1*, shown for the case $G^{\mathrm{ADE}} = \mathbb{Z}_4$. This is for single wrapping number of the branes along any further compact dimensions; but the general statement is just the tensor product of this situation with the cohomology of these further compact spaces.

In view of the evident pattern evidenced by *Table 1* and *Table 2*, here we ask the following question:

Is there a generalized cohomological brane charge quantization which enforces tadpole anomaly cancellation?

We show in this paper that (see *Figure U*), for fluxless toroidal ADE-orientifolds, the answer to this question is *unstable equivariant Cohomotopy* theory; see (3) below. Before explaining this, we put the open problem in perspective:

The open problem – Systematic understanding of tadpole cancellation by charge quantization. While the RR-field tadpole cancellation conditions are thought to be crucial not just for mathematical consistency, but also for phenomenological accuracy of string model building [IU12, Sec. 4.4], a real understanding of the conditions in full detail and generality has remained an open problem; see [BDS06, p. 2][Mo14, 4.6.1][HMSV19, p. 2] for critical discussion. In particular, most of the existing literature on tadpole cancellation simply regards D-brane charge as being in ordinary cohomology, while widely accepted arguments say that D-brane charge instead must be regarded in (a twisted differential enhancement of) K-theory; in this context, see [BSS19] for review, and see [GS17][GS19a][GS19b] for detailed constructions and accounts of the twisted differential case. D-brane charge in K-cohomology may be understood as a generalized *charge quantization* rule, in analogy to how Dirac's classical argument for charge quantization [Dir31] (see [Fra97, 16.4e]) expresses the electromagnetic field as a cocycle in (the differential refinement of) ordinary cohomology; see [Fre00]. Notice that cohomological charge quantization concerns the full non-perturbative structure of a physical theory, including its instanton/soliton charge content.

Accordingly, in [Ur00, 5] it was suggested that RR-tadpole cancellation must be a consistency condition expressed in K-theory. Specifically, for orientifolds this could be Atiyah's *R*eal K-theory [At66], i.e., KR-theory restricting on O-planes to KO-theory, which has been argued to capture D-brane charges on orientifolds in [Wi98, 5][Gk99][BGS01, §3]; explicit constructions are given in [DMDR14][DMDR15][HMSV15][HMSV19][GS19b]. In more detail, D-brane charge on orbifolds is traditionally expected [Wi98, 5.1][dBD$^+$02, 4.5.2][GC99] to be in equivariant K-theory (see [Gr05]). Hence orientifolds are expected to have charge quantization in a combination of these aspects in some Real equivariant K-theory [Mou11][Mou12][FM12][Go17].

However, before even formulating tadpole cancellation in Real equivariant K-theory, the full formulation of O-plane charge has remained open:

Open issue 1: Single O-plane charge. While O-plane charge is not supposed to vary over all integers, perturbative string theory predicts it to vary in the set $\{0, \pm 1\}$ (e.g. [HIS00, p. 2]), illustrated in *Figure B*.

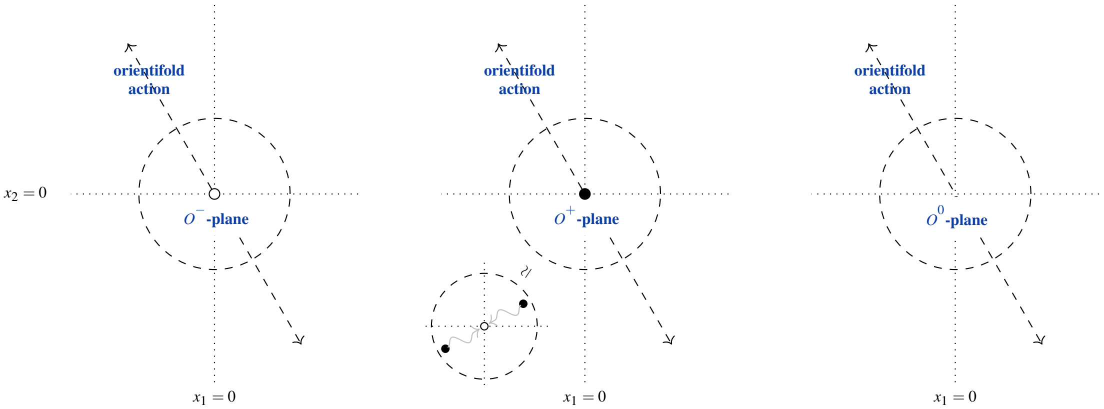

Figure B – The charge carried by a single O-plane takes values in the set $\{0, \pm 1\}$ (in units of corresponding integral D-brane charge), visualized here following the geometric illustration of *Figure A*. For O4-planes this situation lifts to MO5-planes in M-theory [Ho98][Gi98] [AKY98, II.B][HK00, 3.1]. (The notation for O^0 originates with [Ho98, p. 29][Gi98, p. 4]; see *Figure T* for more.)

But in plain KR-theory all O-planes are O$^-$-planes. To capture at least the presence of O$^+$-planes requires adding to KR-theory an extra sign choice [DMDR14]. In some cases this may be regarded as part of a twisting of KR-theory [HMSV19], but the situation remains inconclusive [HMSV19, p. 2]. [1]

Open issue 2: Total O-plane charge. As highlighted in [BGS01, p. 4, p. 25], it remains open whether a putative formalization of tadpole cancellation via Real K-theory reflects the *absolute total* charge to be carried by O-planes. This is a glaring open problem, since the absolute total charge -32 of Op-planes in toroidal orientifolds (see *Table 3*) fixes the gauge algebra $\mathfrak{so}(32)$ of type I string theory required for duality with heterotic string theory (see, e.g., [BLT13, p. 250] [APT97]) with Green-Schwarz anomaly cancellation. This core result of string theory, is the basis of the "first superstring revolution" [Sw11, p. 21], and a successful formalization of tadpole cancellation ought to reproduce it.

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[1] Note that [HMSV19, footnote 1] claims a problem with the sign choice in [DMDR14], and hence also in [Mou12]. These continuing issues with orbifold K-theory for D-brane charge may motivate but do not affect the discussion here, where instead we propose equivariant Cohomotopy theory for M-brane charge as an alternative.

A proposal for capturing absolute background charge of O-planes by equipping K-theory with a quadratic pairing has been briefly sketched in [DFM11], but the implications remain somewhat inconclusive [Mo14, p. 22]. We notice that the implications on M-brane charge quantization of analogous quadratic functions in M-theory [HS05] are reproduced by charge quantization in twisted Cohomotopy theory [FSS19b]. Here we further check this alternative proposal: That brane charge quantization is in *Cohomotopy* cohomology theory, which lifts K-theory through the Boardman homomorphism; see (4) below.

O-plane species	**Charge** $q_{\mathrm{O}p^-}/q_{\mathrm{D}p}$	**Transverse orientifold**	**Number of singularities**
O9$^-$	-32	$\mathbb{T}^0 /\!\!/ \mathbb{Z}_2$	1
O8$^-$	-16	$\mathbb{T}^1 /\!\!/ \mathbb{Z}_2$	2
O7$^-$	-8	$\mathbb{T}^2 /\!\!/ \mathbb{Z}_2$	4
O6$^-$	-4	$\mathbb{T}^3 /\!\!/ \mathbb{Z}_2$	8
O5$^-$	-2	$\mathbb{T}^4 /\!\!/ \mathbb{Z}_2$	16
O4$^-$	-1	$\mathbb{T}^5 /\!\!/ \mathbb{Z}_2$	32

Table 3 – Absolute Op-plane charge [IU12, (5.52)][BLT13, 10.212] -32 is not implied by K-theory [BGS01], but is implied by Cohomotopy.

The proposal – Charge quantization on orientifolds in Equivariant Cohomotopy theory. When educated guesswork gets stuck, it is desirable to identify principles from which to systematically *derive* charge quantization in M-theory, if possible, and seek the proper generalized cohomology theory to describe the M-theory fields, as was advocated and initiated in [Sa05a][Sa05b][Sa06][Sa10]. A first-principles analysis of super p-brane sigma-models in rational homotopy theory shows [Sa13][FSS15][FSS16a][FSS16b] that rationalized M-brane charge is quantized in rational *Cohomotopy* cohomology theory; see [FSS19a] for review. This naturally suggests the following hypothesis about charge quantization in M-theory [Sa13][FSS19b][FSS19c][BSS19][SS19] (for exposition see [Sc20]):

Hypothesis H. *The M-theory C-field is charge-quantized in Cohomotopy theory.*

Applied to toroidal orbifolds, the relevant flavor of unstable Cohomotopy theory is (see *Table 4*) *unstable equivariant Cohomotopy* ([tD79, 8.4][Cr03]), denoted $\pi^\bullet_G$ (3). This is the cohomology theory whose degrees are labeled by orthogonal linear G-representations, called the *RO-degree* (see, e.g., [Bl17, 3])

$$\underset{\text{“RO-degree”}}{\overset{\overset{\text{orthogonal linear } G\text{-representation}}{G\subset \mathrm{O}(\dim(V))}}{\overset{\circlearrowright}{V}}} \;\in\; \underset{\text{representation ring}}{\mathrm{RO}(G)} \tag{2}$$

and whose value on a topological G-space X (representing a global G-quotient orbifold $X /\!\!/ G$) with specified point at infinity $\infty \in X$ – see diagram (11) – is the *set* of G-homotopy classes (33) of pointed G-equivariant continuous functions (31) from X to the V-representation sphere S^V (21) (see §3 for details and illustration):

$$\underset{\substack{\text{equivariant Cohomotopy set}\\ \text{of the orbifold } X /\!\!/ G\\ \text{in RO-degree } V}}{\pi^V_G(X)} \;:=\; \underset{\substack{\text{set of } G\text{-homotopy classes}\\ \text{of } G\text{-equivariant continuous functions}\\ \text{from } X \text{ to } S^V}}{\left\{ \overset{\overset{G}{\circlearrowright}}{X} \xrightarrow{\;\;c\;\;} \overset{\overset{G}{\circlearrowright}}{S^V} \right\}_{/\sim}} \tag{3}$$

This is the evident enhancement to unstable G-equivariant homotopy theory (see [Bl17, 1]) of unstable plain Cohomotopy theory $\pi^\bullet$ ([Bo36][Sp49][KMT12][FSS19b, 3.1]).

Equivariant Cohomotopy is a non-abelian (i.e. “unstable”) Cohomology theory [SSS09][NSS12] that maps to equivariant K-theory via stabilization followed by the Boardman homomorphism, see §3.1.2 and [BSS19].

$$\begin{tikzcd} \underset{\substack{\text{unstable}\\ \text{equivariant}\\ \text{Cohomotopy}}}{\pi^\bullet_G} \arrow[r, "\Sigma^\infty", "\text{stablilization}"'] & \underset{\substack{\text{stable}\\ \text{equivariant}\\ \text{Cohomotopy}}}{\mathbb{S}_G} \arrow[r, "\beta", "\substack{\text{Boardman}\\ \text{homomorphism}}"'] & \underset{\substack{\text{equivariant}\\ \text{K-theory}}}{\mathrm{KO}_G} \end{tikzcd} \tag{4}$$

The solution – From Hypothesis H. In this article we explain how lifting brane charge quantization to ADE-equivariant Cohomotopy, regarded as the generalized Dirac charge quantization of the M-theory C-field (e.g. [Duf99]) on toroidal M-orientifolds ([DM95][Wi95b][Ho98][HSS18]), gives the local O-plane charges in $\{0, \pm 1\}$ from *Figure B* and enforces on D-brane charge in the underlying equivariant K-theory (4) the RR-field tadpole cancellation constraints from *Table 1* via their M-theory lift from *Table 2*.

Overall picture – M-Theory and Cohomotopy. As we further explain in [SS20], unstable equivariant Cohomotopy theory is the incarnation on flat orbifolds of *unstable twisted Cohomotopy* cohomology theory, which we showed in [FSS19b][FSS19c] implies a list of M-theory anomaly cancellation conditions on non-singular (i.e., "smooth") but topologically non-trivial spacetimes; see *Table 4*:

Spacetime	**Flat**	**Curved**
Smooth	plain Cohomotopy ([FSS15][BMSS18])	twisted Cohomotopy ([FSS19b][FSS19c])
Orbi-singular	equivariant Cohomotopy ([HSS18][BSS19] §4)	orbifold Cohomotopy ([SS20])

Table 4 – M-theory anomaly cancellation by C-field charge quantization in Cohomotopy. On smooth but curved spacetimes, Cohomotopy theory is twisted via the J-homomorphism by the tangent bundle. On flat orbi-orientifolds the spacetime curvature is all concentrated in the G-singularities, around which the tangent bundle becomes a G-representation and twisted Cohomotopy becomes equivariant Cohomotopy. In each case the respective charge quantization implies expected anomaly cancellation conditions. See also *Table 8*.

Each entry in *Table 4* supports *Hypothesis H* in different corners of the expected phase space of M-theory. This suggests that *Hypothesis H* is a correct assumption about the elusive mathematical foundation of M-theory.

The necessity of unstable = non-abelian charge quantization for O-planes. We highlight that most authors who discuss equivariant Cohomotopy consider *stable* equivariant Cohomotopy theory (e.g. [Seg71][Ca84] [Lu05]), represented by the equivariant sphere spectrum $\mathbb{S}_G$ in equivariant stable homotopy theory ([LMS86][HHR16, Appendix]); see §3.1.2 below. There are comparison homomorphisms (4) from equivariant unstable Cohomotopy to stable Cohomotopy and further to K-theory but each step forgets some information (has a non-trivial kernel) and produces spurious information (has a non-trivial cokernel); see [BSS19]. For the result presented here (just as for the previous discussion in [FSS19b][FSS19c]), it is crucial that we use the richer *unstable* version of the Cohomotopy theory, hence the *non-abelian Cohomology theory* [SSS09][NSS12], which is the one that follows from analysis of super p-brane cocycles [Sa13][FSS19a]. We find that:

(a) the difference in the behavior between the O-plane charges and the D-brane charges (in *Table 1*, *Table 2* and Figure P) and

(b) the unstable/non-abelian nature of O-plane charge itself (*Figure OP*)

are reflected in the passage from the unstable to the stable range in unstable ADE-equivariant Cohomotopy, where the O-plane charges are distinguished as being in the unstable range; see *Figure C*:

$$\begin{tikzcd}
& \text{toroidal orientifold} & \text{cocycle in equivariant Cohomotopy} & \text{equivariant cohomotopy classifying space (representation sphere)} & \\
& \mathbb{T}^{4_{\mathbb{H}}} \arrow[loop above, "\mathbb{Z}_2"] \arrow[rr, "c"] & & S^{4_{\mathbb{H}}} \arrow[loop above, "\mathbb{Z}_2"] & \\
\text{set of fixed points in orientifold (O-planes)} & \left(\mathbb{T}^{4_{\mathbb{H}}}\right)^{\mathbb{Z}_2} = \left\{0, \tfrac{1}{2}\right\}^4 \arrow[rr, "c^{(\mathbb{Z}_2)}", "\text{O-plane charge (a subset)}"'] \arrow[d, hook] & & \underbrace{\{0,\infty\}}_{S^0} = \left(S^{4_{\mathbb{H}}}\right)^{\mathbb{Z}_2} \arrow[d, hook] & \text{set of fixed points in classifying space (the 0-sphere)} \\
\text{underlying plain 4-torus} & \left(\mathbb{T}^{4_{\mathbb{H}}}\right)^1 = T^4 \arrow[rr, "(c)^1", "\text{net charge (an integer)}"'] & & S^4 = \left(S^{4_{\mathbb{H}}}\right)^1 & \text{underlying plain 4-sphere}
\end{tikzcd}$$

Figure C – A cocycle in unstable ADE-equivariant Cohomotopy on a toroidal orientifold according to (3), and its decomposition on fixed point strata into Elmendorf stages; see [Bl17, 1.3][HSS18, 3.1].

Characterizing brane/O-plane charges – Unstable (equivariant) differential topology. Since in *Figure C* the fixed locus in the classifying space is just a 0-sphere, and since the Hopf degree of maps $X^n \to S^n$ stabilizes only for $n \geq 1$ – see diagram (15) – the fixed points in the spacetime (= O-planes) carry "unstable" or "non-linear" charge: not given by a group element, but by a subset, distinguishing $O^{\pm}$-planes from O^{0}-planes as in *Figure OP*. The further distinction between O^{-}-planes and O^{+}-planes is implied by normal framing that enters in the unstable Pontrjagin-Thom theorem (discussed in §2.2). Moreover, the local/twisted tadpole cancellation condition in the vicinity of O-planes is implied by the unstable equivariant Hopf degree theorem (discussed in §3.1). Last but not least, it is the unstable Pontrjagin-Thom theorem, discussed in §2, which identifies all these charges with *sub*-manifolds, hence with actual brane/O-plane worldvolumes as shown in *Figure A*, (while the stable PT-theorem instead relates stable Cohomotopy to manifolds equipped with any maps to spacetime).

Classical theorem	Reference	**Interpretation for brane charge quantization in unstable Cohomotopy** (*Hypothesis H*)	Discussed in
Unstable Pontrjagin-Thom theorem	[Kos93, IX (5.5)]	Cohomotopy charge is sourced by submanifolds hence by worldvolumes of branes and O-planes	§2.1
Unstable Hopf degree theorem	[Kos93, IX (5.8)] [Kob16, 7.5]	Charge of flat transversal branes is integer while charge of flat transversal O-planes is in $\{0,1\}$	§2.2
Unstable equivariant Hopf degree theorem	[tD79, 8.4]	Branes appear in regular reps around O-planes = local/twisted tadpole anomaly cancellation	§3

Organization of the paper. In §2 we discuss how the classical unstable Pontrjagin-Thom isomorphism says that plain Cohomotopy classifies charge carried by brane worldvolumes. In §3 we introduce the enhancement of this situation to equivariant Cohomotopy on toroidal orbifolds, where it encodes joint D-brane and O-plane charge. We explain in §3 that now the *equivariant Hopf degree theorem* encodes the form of local/twisted tadpole cancellation conditions, and explain in §3.2 that super-differential refinement at global Elmendorf stage encodes the form of global/untwisted tadpole cancellation conditions as in *Table 1* and *Table 2*. The Pontrjagin-Thom theorem now serves to map these charges precisely to the geometric situations of the form shown in *Figure A*. Finally, in §4 we specify these general considerations to the physics of M5-branes at MO5-planes in toroidal ADE-orientifolds in M-theory, with the C-field charge-quantized in equivariant Cohomotopy theory, according to *Hypothesis H*. To set the scene, we first recall in §4.1 the situation of heterotic M-theory on ADE-orbifolds and highlight subtleties in the interpretation of MO5-planes. With this in hand, we apply in §4.2 the general discussion of equivariant Cohomotopy from §3 to ADE-singularities intersecting MO9-planes in M-theory, and find (Cor. 4.4, Cor. 4.6) that this correctly encodes the expected anomaly cancellation of M5-branes at MO5-planes, and this, upon double dimensional reduction (see *Table 7* and *Figure U*), the RR-field tadpole anomaly cancellation for D-branes on ADE-orientifolds.

2 Cohomotopy and brane charge

Before turning to equivariant/orbifold structure in §3, we first discuss basics of plain unstable Cohomotopy on plain manifolds. The key point is that the unstable *Pontrjagin-Thom theorem*, reviewed in §2.1, identifies cocycles in unstable Cohomotopy theory with cobordism classes of submanifolds carrying certain extra structure (normal framing). These submanifolds are naturally identified with the worldvolumes of branes that source the corresponding Cohomotopy charge, and the normal structure they carry corresponds to the charge carried by the branes, distinguishing branes from anti-branes. In §2.2 we highlight that coboundaries in unstable Cohomotopy accordingly correspond to brane pair creation/annihilation processes. This way the Pontrjagin-Thom theorem establishes Cohomotopy as a natural home for brane charges, as proposed in [Sa13].

2.1 Pontrjagin-Thom theorem and brane worldvolumes

Cohomotopy cohomology theory. The special case of unstable G-equivariant Cohomotopy (3) with $G = 1$ the trivial group is unstable plain Cohomotopy theory ([Bo36][Sp49][KMT12][FSS19b, 3.1]), denoted $\pi^{\bullet} := \pi_1^{\bullet}$. This is the unstable/non-abelian cohomology theory whose degrees are natural numbers $n \in \mathbb{N}$ and which assigns to an un-pointed topological space X the *Cohomotopy set* of free homotopy classes of continuous maps into the n-sphere:

$$\underset{\substack{\text{Cohomotopy set}\\ \text{of the space } X\\ \text{in degree } n}}{\pi^n(X)} \quad := \quad \underset{\substack{\text{set of homotopy classes}\\ \text{of continuous functions}\\ \text{from } X \text{ to the } n\text{-sphere } S^n}}{\left\{ X \xrightarrow{\;\;c\;\;} S^n \right\}_{/\sim}} \tag{5}$$

The contravariant assignment $X \mapsto \pi^n(X)$ is analogous to the assignment $X \mapsto H^n(X,\mathbb{Z})$ of integral cohomology groups, or of the assignment $X \mapsto K^n(X)$ of K-theory groups, and as such may be regarded as a generalized but *non-abelian* cohomology theory [SSS09][NSS12]: For $n \geq 1$ we have (as for any connected topological space) a weak homotopy equivalence between the n-sphere and the classifying space of its loop group, $S^n \simeq_{\text{whe}} B\big(\Omega S^n\big)$, which means that the Cohomotopy sets (5)

$$\underset{n\text{-Cohomotopy set}}{\pi^n(X)} \quad \simeq \quad \underset{\substack{\text{non-abelian cohomology set}\\ \text{with coefficients in}\\ \text{loop group of } n\text{-sphere}}}{H^1(X,\Omega S^n)}$$

are equivalently the non-abelian cohomology sets with coefficient in the loop-group of the n-sphere, in direct generalization of the familiar case of non-abelian cohomology $H^1(X,G) \simeq G\mathrm{Bund}(X)_{/\sim}$ with coefficients in a compact Lie group G.

In this way we may think of (5) as defining a generalized cohomology theory, different from but akin to, say, K-theory, and as such we call it *Cohomotopy cohomology theory*, or *Cohomotopy theory* or just *Cohomotopy*, for short. The capitalization indicates that this term is the proper name of a specific cohomology theory (we might abbreviate further to *C-theory* to bring out the analogy with *K*-theory yet more) and *not* on par with *homotopy theory*, which instead is the name of the general mathematical framework within which we are speaking. In particular, *Cohomotopy cohomology theory* is *not* the dual concept of *homotopy theory*, but is the dual concept of the unstable/non-abelian generalized homology theory which assigns homotopy groups $X \mapsto \pi_n(X)$ to pointed topological spaces X (hence: *Homotopy homology theory*, mostly familiar in its stable form).

Unstable Pontrjagin-Thom theorem. Thinking of X here as spacetime, we are interested in the case that $X = X^D$ admits the structure of closed smooth manifold of dimension $D \in \mathbb{N}$. In this case, the unstable Pontrjagin-Thom theorem (7) identifies (see e.g. [Kos93, IX.5]) the degree-n Cohomotopy set of X^D (5) with the set of cobordism classes of normally framed codimension-n closed submanifolds of X^D (see e.g.[Kos93, IX.2]), hence of closed submanifolds $\Sigma^d \overset{i}{\hookrightarrow} X^D$ which are of dimension $d = D - n$ and equipped with a choice of trivialization

$$\underset{\substack{\text{normal bundle}\\ \text{of codimension } n \text{ submanifold } \Sigma\\ \text{inside ambient manifold } X}}{N_i\Sigma} \xrightarrow[\simeq]{\text{normal framing}} \underset{\substack{\text{trivial vector bundle}\\ \text{of rank } n}}{\Sigma \times \mathbb{R}^n} \tag{6}$$

of their normal vector bundle:

$$\textbf{Unstable Pontrjagin-Thom theorem} \qquad \underset{\substack{\text{Cohomotopy set in degree } n\\ \text{of closed } D\text{-dim. manifold } X}}{\pi^n\big(X^D\big)} \;\; \underset{\substack{\text{“PT collapse”}\\ \text{assign directed asymptotic distance}}}{\overset{\substack{\text{take pre-image at 0 of regular representative}\\ \mathrm{fib}_0 \circ \mathrm{reg}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \;\; \left\{ \begin{array}{c} \text{Closed submanifolds } \Sigma^d \overset{i}{\hookrightarrow} X^D \\ \text{of dimension } d = D - n \\ \text{and equipped with normal framing} \end{array} \right\}_{/\text{cobordism}} \tag{7}$$

The construction which exhibits this bijection is traditionally called the Pontrjagin-Thom *collapse*, but a more suggestive description, certainly for our application to brane charges, is this: *The Cohomotopy class corresponding to a submanifold/brane is represented by the function which assigns* directed asymptotic distance *from the submanifold/brane, as measured with respect to the given normal framing* (6) *upon identifying the normal bundle with a tubular neighborhood and regarding all points outside the tubular neighborhood as being at infinite distance.* See *Figure D*:

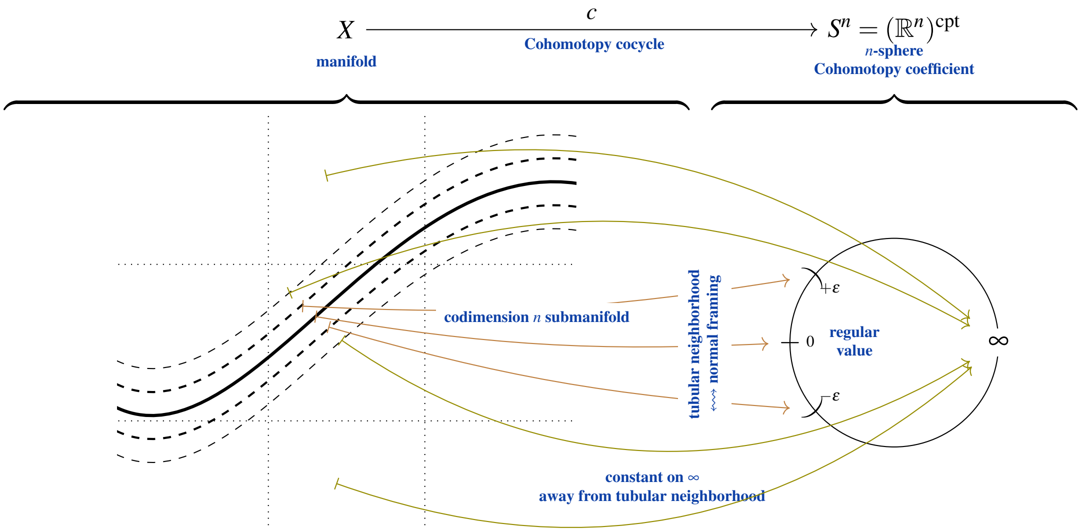


Figure D – The Pontrjagin-Thom construction which establishes the unstable Pontrjagin-Thom theorem (7). The cocycle c in Cohomotopy eq. (5) is the continuous function which sends each point to its directed asymptotic distance from the given submanifold.

One-point compactifications by adjoining the point at infinity. Here and in all of the following, we are making crucial use of the fact that the n-sphere is the one-point compactification $(-)^{\mathrm{cpt}}$ of the Cartesian space $\mathbb{R}^n$,

$$S^n \;\simeq_{\mathrm{homeo}}\; \big(\mathbb{R}^n\big)^{\mathrm{cpt}} \;:=\; \big(\{x \in \mathbb{R}^n \text{ or } x = \infty\}, \tau_{\mathrm{cpt}}\big) \qquad \text{for all } n \in \mathbb{N}, \tag{8}$$

as indicated on the right of *Figure D*. Here the one-point compactification X^{cpt} of a topological space X is defined (e.g. [Ke55, p. 150]) by adjoining one point to the underlying set of X – denoted "∞" as it becomes literally the *point at infinity* – and by declaring on the resulting set a topology τ_{cpt} whose open subsets are those of X, not containing ∞, and those containing ∞ but whose complement in X is compact. Notice that this construction also applies to topological spaces that already are compact, in which case the point at infinity appears disconnected

$$X \text{ already compact} \;\;\Rightarrow\;\; X^{\mathrm{cpt}} \;=\; X_{+} \;:=\; X \sqcup \{\infty\}\,. \tag{9}$$

This means that (8) indeed holds also in the "unstable range" of $n = 0$:

$$\big(\mathbb{R}^0\big)^{\mathrm{cpt}} \;=\; \big(\{0\}\big)^{\mathrm{cpt}} \;=\; \{0\} \sqcup \{\infty\} \;=\; S^0\,. \tag{10}$$

Cohomotopy charge vanishing at infinity. In view of the Pontrjagin-Thom theorem (7), it makes sense to say that a cocycle in Cohomotopy *vanishes* wherever it takes as value the point at infinity $\infty \in \big(\mathbb{R}^n\big)^{\mathrm{cpt}} \simeq S^n$ in the coefficient sphere, identified under (8). This means to regard the coefficient sphere as a pointed topological space, with basepoint $\infty \in S^n$. Given then a non-compact (spacetime) manifold X (such as $X = \mathbb{R}^n$), a Cohomotopy cocycle $X \longrightarrow S^n$ *vanishes at infinity* if it extends to the one-point compactification X^{cpt} (8) such as to send the actual point at infinity $\infty \in X^{\mathrm{cpt}}$ to the point at infinity in the coefficient sphere.

A Cohomotopy cocycle on a non-compact space X which *vanishes at infinity* is a Cohomotopy cocycle on the one-point compactification X^{cpt} that sends the point at infinity in the domain to that in the coefficient n-sphere.

$$\begin{tikzcd}
X^{\mathrm{cpt}} \arrow[r, "c"] & \big(\mathbb{R}^n\big) \simeq S^n \\
\{\infty\} \arrow[u, hook] \arrow[r, "c_{\vert\{\infty\}}"'] & \{\infty\} \arrow[u, hook]
\end{tikzcd} \tag{11}$$

Example 2.1 (*Figure E*)**.** For $X = \mathbb{R}^n$, we have that Cohomotopy n-cocycles on X vanishing at infinity are equivalently maps from an n-sphere to itself:

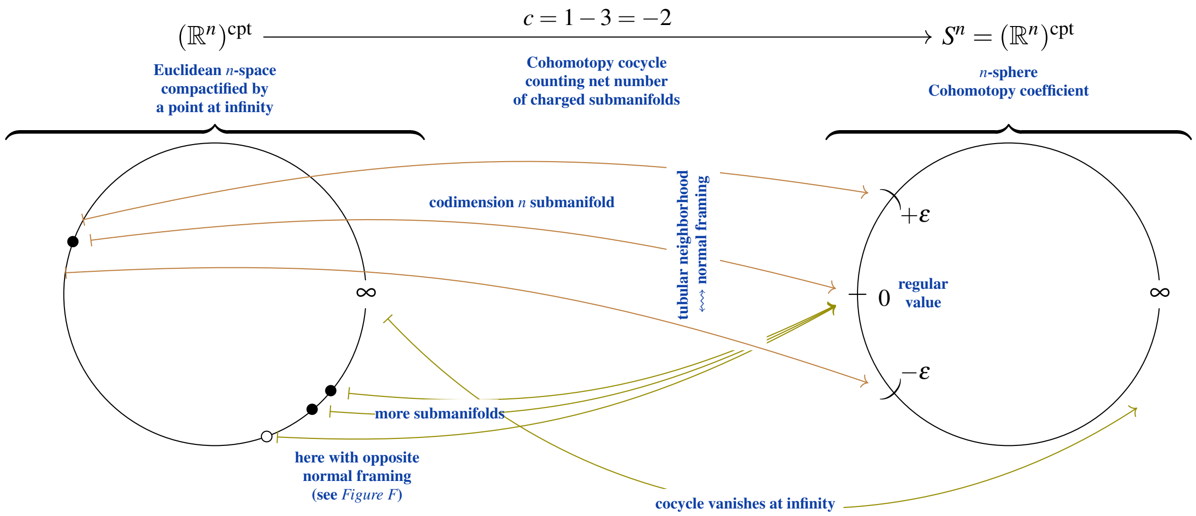


Figure E – Cohomotopy in degree n of Euclidean n-space vanishing at infinity is given by Cohomotopy cocycles (5) on the one-point compactification $(\mathbb{R}^n) \simeq S^n$ (8) that send ∞ to ∞ (11).

Of course, this is just the cohomotopical version of *instantons* in ordinary gauge theory:

Instantons and solitons. If G is a compact Lie group with classifying space BG equipped with the canonical point $* \simeq B\{e\} \longrightarrow BG$, then a *$G$-instanton sector* on Euclidean space $X = \mathbb{R}^n$ is the homotopy class of a continuous function from the one-point compactification of X to BG, which takes the base points to each other [2]

A *G instanton sector* is a cocycle in degree-1 G-cohomology which *vanishes at infinity* in that it is a cocycle on the one-point compactification X^{cpt} (8) which sends the point at infinity in the domain to the base point in the classifying space BG.

$$\begin{tikzcd}
\left(\mathbb{R}^n\right)^{\mathrm{cpt}} \arrow[r, "c"] & BG \\
\{\infty\} \arrow[u, hook] \arrow[r, "c_{|\{\infty\}}"'] & B\{e\} \arrow[u, hook]
\end{tikzcd} \tag{12}$$

Cohomotopy and $\mathrm{SU}(N)$-instanton sectors. Specifically for $n = 4$ and $G = \mathrm{SU}(N)$ any map $S^4 \xrightarrow{\varepsilon} B\mathrm{SU}(N)$ representing a generator $1 \in \mathbb{Z} \simeq \pi_4\big(B\mathrm{SU}(N)\big)$ of the 4th homotopy group of the classifying space exhibits a bijection between the 4-Cohomotopy of $\mathbb{R}^4$ vanishing at infinity (11), and the set of $\mathrm{SU}(N)$ instanton sectors

$$\pi^4\big((\mathbb{R}^n)^{\mathrm{cpt}}\big) \;=\; \big\{\, (\mathbb{R}^4)^{\mathrm{cpt}} \longrightarrow S^4 \,\big\}_{/\sim} \;\overset{\varepsilon_*}{\simeq}\; \big\{\, (\mathbb{R}^4)^{\mathrm{cpt}} \longrightarrow B\mathrm{SU}(N) \,\big\}_{/\sim} \;\simeq\; \left\{ \begin{array}{c} \mathrm{SU}(n)\text{-instanton sectors} \\ \text{on } \mathbb{R}^4 \end{array} \right\}.$$

Under this identification of $\mathrm{SU}(N)$-instanton sectors with Cohomotopy vanishing at infinity, the Pontrjagin-Thom construction (7) produces precisely the distribution of *instanton center points*, again illustrated by the left hand side in *Figure E*. To see all this in more detail, we next turn to further discussion of the charge structure encoded by Cohomotopy.

2.2 Hopf degree theorem and brane-antibrane annihilation

The classical *Hopf degree theorem* describes the n-Cohomotopy (5) of orientable closed D-manifolds X (7) in the special case where $n = D$. It says that, in the "stable range" $n \geq 1$, the Cohomotopy set is in bijection with the set of integers, where the bijection is induced by sending the continuous function representing a Cohomotopy cycle to its mapping degree (see, e.g., [Kob16, 7.5]):

Hopf degree theorem
in stable range $n \geq 1$

$$\begin{tikzcd}[column sep=large]
\underset{\text{of } n\text{-manifold}}{\overset{n\text{-Cohomotopy}}{\pi^n\big(X\big)}} \arrow[r, "S^n \xrightarrow{\varepsilon_*} K(\mathbb{Z}{,}n)", "\simeq"'] & H^n\big(X, \mathbb{Z}\big) & \simeq & \mathbb{Z} \\
\big[X^n \xrightarrow{c} S^n\big] \arrow[r, mapsto] & \big[X^n \xrightarrow{c} S^n \xrightarrow{\varepsilon} K(\mathbb{Z},n)\big] & =: & \deg(c)
\end{tikzcd} \tag{13}$$

---

[2] An actual instanton in this instanton sector is a G-principal connection on X^{cpt} whose underlying G-principal bundle has this classifying map. Ultimately we are interested in such enhancement to *differential cohomology*, but this is beyond the scope of the present article.

Under the Pontrjagin-Thom theorem (7) the Hopf degree theorem (13) translates into the following geometric situation for signed (charged) points in X^n (see [Kos93, IX.4]): A codimension-n submanifold in an n-manifold X^n is a set of points in X^n, and a choice of normal framing (6) is, up to normally framed cobordism, the same as choice of sign (charge) in $\{\pm 1\}$ for each point, as shown in *Figure F*:

$$\underset{\text{manifold}}{X^n} \xrightarrow[\text{Cohomotopy cocycle}]{c} \underset{\substack{\text{sphere}\\ \text{Cohomotopy coefficient}}}{S^n = D(\mathbb{R}^n)/S(\mathbb{R}^n)}$$

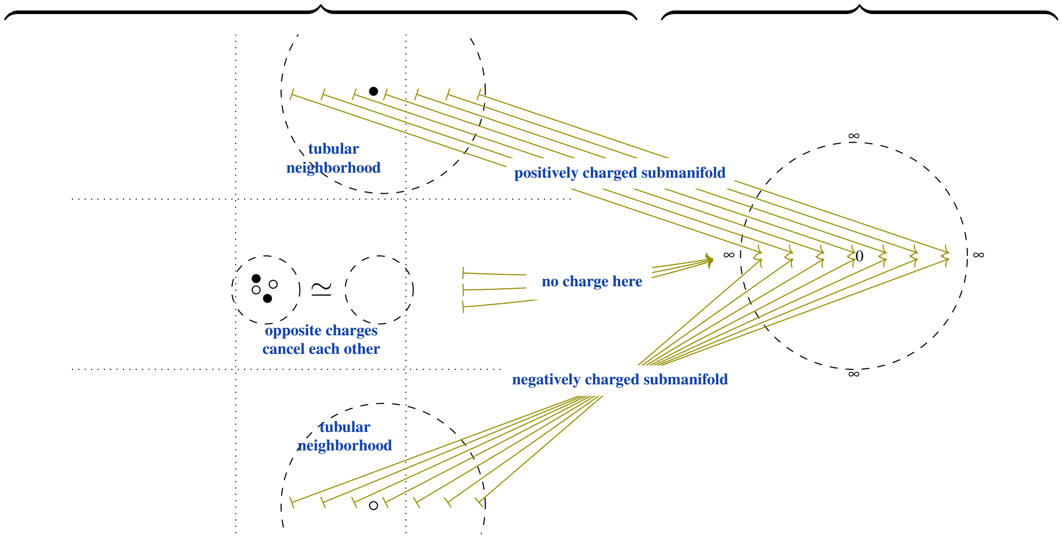


Figure F – Charge in Cohomotopy carried by submanifolds, under the PT-isomorphism (7) is encoded in their normal framing (6). In full codimension the normal framing is a normal orientation and hence a choice in $\{\pm 1\}$, which we indicate graphically by $\begin{array}{ccc} \bullet & \leftrightarrow & -1 \\ \circ & \leftrightarrow & +1 \end{array}$

Under this geometric translation, we have the correspondence

$$\begin{array}{c}\text{Hopf degree}\\ \text{of Cohomotopy cocycle on } X\end{array} \overset{\text{PT}}{\longleftrightarrow} \begin{array}{c}\text{Net number of } \pm\text{-charges}\\ \text{carried by points in } X\end{array}$$

The mechanism which implements this on the geometric right hand side is that points of opposite sign/normal framing are cobordant to the empty collection of points, hence mutually annihilate each other via coboundaries in Cohomotopy, as shown in *Figure G*:

$$\underset{\substack{\text{product space}\\ \text{of interval with manifold}}}{[0,1]\times X} \xrightarrow[\text{Cohomotopy coboundary}]{0 \simeq (-1)+(+1)} \underset{\substack{n\text{-sphere}\\ \text{Cohomotopy coefficient}}}{S^n = (\mathbb{R}^n)^{\mathrm{cpt}}}$$

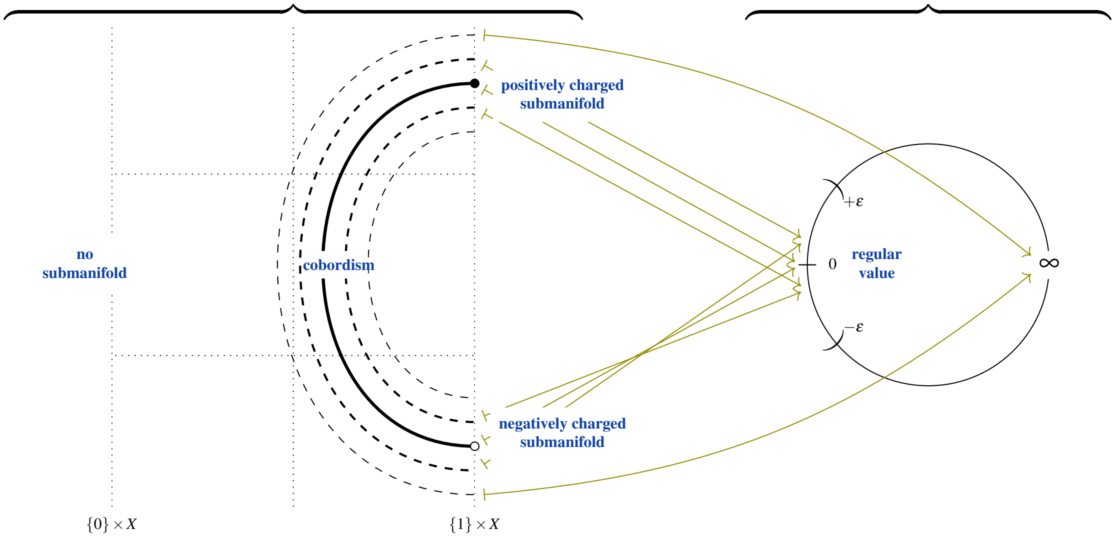


Figure G – Cobordisms between submanifolds of opposite normal framing as in *Figure F* exhibit their pair creation/annihilation. This is the geometric mechanism which underlies the Hopf degree theorem (13) when translating via the Pontrjagin-Thom theorem (7) between Cohomotopy charge and the submanifolds sourcing it, as in *Figure D*.

Hopf degree in unstable range. The classical Hopf degree theorem (13) is stated only in the stable range $n \geq 1$, but it is immediate to extend it to the unstable range. While this is a simple statement in itself, it is necessary to conceptually complete the discussion of the equivariant Hopf degree theorem in §3.1 below, where the ordinary Hopf degree appears jointly in stable and unstable range, with the distinction being responsible for the difference in nature between O-plane charge (unstable range) and D-brane charge (stable range): For $X = X^0$ a compact 0-manifold, hence a finite set, and $X^{\mathrm{cpt}} = X_+ = X \sqcup \{\infty\}$ the same set with a "point at infinity" adjoined (9), its unstable Cohomotopy classes (5) in degree 0, being functions to the 0-sphere hence to the 2-element set $S^0 = \{0, \infty\}$ that take $\infty \mapsto \infty$

$$\pi^0\big(X^{\mathrm{cpt}}\big) \;=\; \big\{X \xrightarrow{\;c\;} S^0\big\},$$

are in bijection to the subsets $S \subset X$ of X, by the assignment that sends c to the pre-image $c^{-1}\big(\{0\}\big)$ of $0 \in S^0$ under c. We may think of these subsets as elements of the power set $\{0,1\}^X$ and as such call them the sets $\deg(c)$ of Hopf degrees in $\{0,1\}$ for $n = 0$:

$$\begin{tikzcd}
 & \text{0-Cohomotopy of 0-manifold} & & & \text{sets of unstable Hopf degrees} \\
\textbf{Hopf degree theorem} & \pi^0\big(X^{\mathrm{cpt}}\big) \arrow[r, "S^0=\{0,\infty\}", "\simeq"'] & \mathrm{Subsets}(X) & \simeq & \{0,1\}^X \\
\text{in unstable range } n=0 & \big[X^0 \xrightarrow{c} S^0\big] \arrow[r, mapsto] & \big[c^{-1}\big(\{0\}\big) \subset X\big] & =: & \deg(c)
\end{tikzcd} \tag{14}$$

Example 2.2. For $X = \{0\}$ the single point so that, with (10), X^{cpt} is the 0-sphere, we have $\pi^0\big(\{0\}^{\mathrm{cpt}}\big) \simeq \{0,1\}$, as illustrated in the following figure:

Figure H – Hopf degree in the unstable range takes values in the set $\{0,1\}$ (14), corresponding to the binary choice of there being or not being a unit charge at the single point.

The point of unstable Hopf degree in $\{0,1\}$ is that it exhibits *homogeneous behavior under suspension* Σ^1 (41) across the unstable and stable range of Hopf degrees, with the unstable Hopf degrees in $\{0,1\}$ injecting into the full set of integers in the stable range:

$$\begin{tikzcd}
\text{unstable Hopf degrees} & \text{stable Hopf degrees} & \text{stable Hopf degrees} & \text{stable Hopf degrees} & \\
\{0,1\} \arrow[r, hook, "\text{injection}"] & \mathbb{Z} \arrow[r, "="] & \mathbb{Z} \arrow[r, "="] & \mathbb{Z} \arrow[r] & \cdots \\
\pi^0\big(S^0\big) \arrow[u, "\wr", "(14)"'] \arrow[r, "\text{suspension}", "\Sigma^1"'] & \pi^1\big(S^1\big) \arrow[u, "\wr", "(13)"'] \arrow[r, "\Sigma^1"'] & \pi^2\big(S^2\big) \arrow[u, "\wr", "(13)"'] \arrow[r, "\Sigma^1"'] & \pi^3\big(S^3\big) \arrow[u, "\wr", "(13)"'] \arrow[r] & \cdots \arrow[u, phantom, "\vdots"]
\end{tikzcd} \tag{15}$$

As we next turn from plain to equivariant Cohomotopy in §3, we find that unstable and stable Hopf degrees unify in the equivariant Hopf degree theorems, and that the *D-brane charge is what appears in the stable range*, while the *O-plane charge is what appears in the unstable range* (in particular, via the proof of Theorem 3.13 below).

3 Equivariant Cohomotopy and tadpole cancellation

We now turn to the equivariant enhancement (3) of Cohomotopy theory. We discuss in §3.1 and in §3.2, respectively, how this captures the form of the local/twisted (see Diagram (58) in §3.2) and of the global/untwisted tadpole cancellation conditions (see §4.1) according to *Table 1* and *Table 2*, by appeal to the equivariant enhancement of the Hopf degree theorem applied to representation spheres, which we state as Theorem 3.10 and Theorem 3.13.

Basic concepts of unstable equivariant homotopy theory. To set up notation, we start with reviewing a minimum of underlying concepts from unstable equivariant homotopy theory (see [Bl17, 1][HSS18, 3.1] for more).

Topological G-spaces. For G a finite group, a *topological G-space* $\overset{\overset{G}{\curvearrowright}}{X}$ (or just *G-space*, for short) is a topological space X equipped with a continuous G-action, hence with a continuous function $G \times X \overset{\cdot}{\longrightarrow} X$ such that for all $g_i \in G$ and $x \in X$ we have $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $e \cdot x = x$ (where $e \in G$ is the neutral element).

Here we are concerned with the **classes of examples of G-spaces** shown in *Table 5*:[3]

G-representation	G-space	G-orbifold	Terminology	
G **finite group**	$\overset{\overset{G}{\curvearrowright}}{\mathbb{R}^V}$ **Euclidean G-space** (19)	$\mathbb{R}^V /\!\!/ G$ **Euclidean orbifold**	**singularity**	**ADE-singularities** $G \subset \mathrm{SU}(2)$ **finite subgroup of** $\mathrm{SU}(2)$ (16) $V = \mathbf{4}_{\mathbb{H}}$ **quaternionic representation** (17)
$V \in \mathrm{RO}(G)$ **orthogonal linear G-representation**	$\overset{\overset{G}{\curvearrowright}}{S^V}$ **G-representation sphere** (21)	$S^V /\!\!/ G = \big(\mathbb{R}^V /\!\!/ G\big)^{\mathrm{cpt}}$ **Euclidean orbifold including point at infinity** (11)	**vicinity of singularity**	
$G \rtimes \mathbb{Z}^{\dim(V)} \subset \mathrm{Iso}\big(\mathbb{R}^V\big)$ **crystallographic group** (22))	$\overset{\overset{G}{\curvearrowright}}{\mathbb{T}^V}$ **G-representation torus** (23)	$\mathbb{T}^V /\!\!/ G = \big(\mathbb{R}^V /\!\!/ G\big)/\mathbb{Z}^{\dim(V)}$ **toroidal orbifold**	**flat, compact singular space**	

Table 5 – Flat G-orbifolds and the G-spaces covering them. Examples arising in application to M-theory are discussed in §4.1.

Orbifold terminology. As common in string theory, we will be thinking of G-spaces X as stand-ins for their homotopy quotients $X /\!\!/ G$, which are the actual orbifolds. This is mathematically fully justified by the fact that the proper notion of generalized cohomology of such global quotient orbifolds $X /\!\!/ G$ is equivalently the G-equivariant generalized cohomology of the space X. We relegate a comprehensive discussion of this technical point to [SS20], but this is mathematical folklore: see [PrSc10, §1][Schw17, p. 1][Schw18, p. ix-x]. Moreover, in the specific application to M-theory, below in §4, the relevant orbifolds are always part of *orbi-orientifolds*, in that a subgroup $\mathbb{Z}_2^{\mathrm{refl}}$ of the orbifold quotient group $G = G^{\mathrm{ADE}}$ combines with a Hořava-Witten-involution $\mathbb{Z}_2^{\mathrm{HW}}$ to an orientation-changing involution $\mathbb{Z}_2^{\mathrm{HW+refl}}$ which fixes an "MO5-plane". This is made precise in §4.1 below, see (67) there. Since, with passage to the $\mathbb{Z}_2^{\mathrm{HW}}$-fixed locus (the "MO9-pane") understood (80), the further localization to the MO5-plane coincides with the orbifold singularity, we will often refer here to orbifold fixed points as orientifold fixed points, wherever this serves the preparation of the application in §4.

Accordingly, the orbifold singularities in the applications below in §4 are always inside an O-plane, so that the relevant flavor of equivariant K-theory considered below in Prop. 3.14 and in *Figure P*, *Figure M* is KO.

Linear G-representations. The G-spaces of interest for the discussion of toroidal orbifolds all come from *orthogonal linear G-representations V*: finite-dimensional Euclidean vector spaces equipped with a linear action by G factoring through the canonical action of the orthogonal group. We will denote concrete examples of such V of dimension $n \in \mathbb{N}$ and characterized by some label "l" in the form $V = \mathbf{n}_{\mathrm{l}}$, and also refer to them as an *RO-degree* (2).

The *key class of examples* of interest here are finite subgroups (see, e.g., [BSS19, A.1])

$$G^{\mathrm{ADE}} \;\subset\; \mathrm{SU}(2) \;\simeq\; \mathrm{Sp}(1) \;\simeq\; U(1,\mathbb{H}) \;\simeq\; S(\mathbb{H}) \tag{16}$$

of the multiplicative group of unit norm elements $q \in S(\mathbb{H})$ in the vector space $\mathbb{H} \simeq_{\mathbb{R}} \mathbb{R}^4$ of quaternions, and their defining 4-dimensional linear representation on this space (by left quaternion multiplication), which we denote by

$$\mathbf{4}_{\mathbb{H}} \;\in\; \mathrm{RO}\big(G^{\mathrm{ADE}}\big)\,. \tag{17}$$

---

[3] For our purposes here, the covering G-space X is all we need to speak about the corresponding orbifold $X /\!\!/ G$. For a dedicated discussion of geometric orbifolds we refer to [Ra06, 13][SS20]. Note that [Ra06, 13] says "Euclidean orbifold" for any flat orbifold.

All of these, except the cyclic groups of odd order, contain the subgroup

$$\mathbb{Z}_2^{\mathrm{refl}} := \big\langle -1 \in S(\mathbb{H}) \big\rangle \subset G^{\mathrm{A_{ev}DE}} \tag{18}$$

generated by the quaternion $-1 \in \mathbb{H}$. This acts on the 4-dimensional quaternionic representation (17) by point reflection at the origin, hence as the 4-dimensional sign representation

$$\overset{\mathbb{Z}_2^{\mathrm{refl}}}{\curvearrowright}\!\!\!\!\!\! \quad \mathbb{R}^{\mathbf{4}_{\mathbb{H}}} \;\simeq\; \overset{\mathbb{Z}_2}{\curvearrowright}\!\!\!\! \quad \mathbb{R}^{\mathbf{4}_{\mathrm{sgn}}}\,,$$

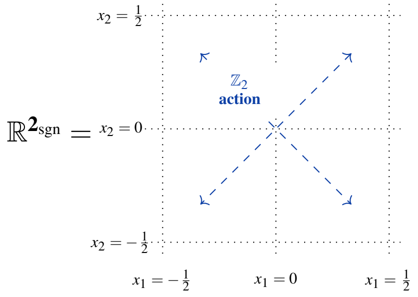
as illustrated for 2 of 4 dimensions in *Figure I*.

Euclidean G-Spaces. The underlying Euclidean space of a linear G-representation V is of course a G-space, hence a *Euclidean G-space*, which we suggestively denote by $\mathbb{R}^V$:

$$\underset{\text{linear } G\text{-representation}}{} \quad V \in \mathrm{RO}(G) \quad \Rightarrow \quad \overset{G}{\curvearrowright}\!\!\!\! \quad \mathbb{R}^V \quad \underset{\text{Euclidean } G\text{-space}}{} \tag{19}$$

Example 3.1 (*Figure I*)**.** With $G = \mathbb{Z}_2$ and $V = \mathbf{2}_{\mathrm{sgn}}$ its 2-dimensional sign representation, the Euclidean G-spaces $\mathbb{R}^{\mathbf{2}_{\mathrm{sgn}}}$ is the Cartesian plane equipped with the action of point reflection at the origin:

Figure I – The Euclidean $\mathbb{Z}_2$-space (19) of the 2-dimensional sign representation $\mathbf{2}_{\mathrm{sgn}}$. The underlying topological space is the Euclidean plane $\mathbb{R}^2$, with group action by point reflection at the origin.



Notice that for $V, W \in \mathrm{RO}(G)$ two orthogonal linear G-representations, with $V \oplus W \in \mathrm{RO}(G)$ their direct sum representation, the Cartesian product of their Euclidean G-spaces (19) is the Euclidean G-space of their direct sum:

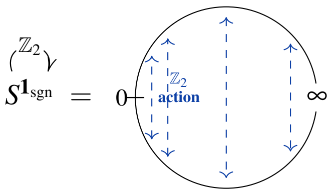
$$\mathbb{R}^V \times \mathbb{R}^W \simeq \mathbb{R}^{V \oplus W}\,. \tag{20}$$

G-Representation spheres. The one-point compactification (8) of a Euclidean space $\mathbb{R}^V$ (19) becomes itself a G-space, with the point at infinity declared to be fixed by all group elements; this is called the *representation sphere* of V (see, e.g., [Bl17, 1.1.5]):

$$\underset{\substack{\text{representation}\\ \text{sphere}}}{S^V} \;:=\; \overset{\substack{\text{one-point compactification}\\ \text{of Euclidean space } V}}{\big(\mathbb{R}^V\big)^{\mathrm{cpt}}} \;\simeq\; \underset{\substack{\text{unit disk in } V\\ \text{with boundary collapsed}\\ \text{to the point at infinity}}}{D\big(\mathbb{R}^V\big)/S\big(\mathbb{R}^V\big)} \;\simeq\; \overset{\substack{\text{unit sphere}\\ \text{in product of } V\\ \text{with the 1d trivial representation}}}{S\big(\mathbb{R}^{\mathbf{1}_{\mathrm{triv}} \oplus V}\big)}\,. \tag{21}$$

Example 3.2 (*Figure J*)**.** With $G := \mathbb{Z}_2$ the group of order 2 and $\mathbf{1}_{\mathrm{sgn}}$ its 1-dimensional sign-representation, the corresponding representation sphere (21) is the circle equipped with the $\mathbb{Z}_2$-action that reflects across an equator:

Figure J – The $\mathbb{Z}_2$-representation sphere of the 1-dimensional sign representation $\mathbf{1}_{\mathrm{sgn}}$ is the $\mathbb{Z}_2$-space whose underlying topological space is the circle, and equipped with the $\mathbb{Z}_2$-action that reflects points across the equator through 0 and the point at infinity.



G-Representation tori. Similarly, consider the linear G-representation V such that $G \subset \mathrm{Iso}\big(\mathbb{R}^{\dim(V)}\big)$ is the point group of a crystallographic group C (see, e.g., [Far81]) of the underlying Euclidean space $\mathbb{R}^{\dim(V)}$ with corresponding translational sub-lattice $\mathbb{Z}^n \subset \mathrm{Iso}(n)$ inside the Euclidean group in $n = \dim(V)$ dimensions. This means we have an exact sequence of this form:

$$\begin{tikzcd}
& \substack{\text{lattice of translations}\\\text{normal subgroup}} & \substack{\text{crystallographic}\\\text{group}} & \substack{\text{point group}\\\simeq C/\mathbb{Z}^n} & \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r, hook] \arrow[d, hook] & C \arrow[r, two heads] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathbb{R}^n \arrow[r, hook] & \mathrm{Iso}(n) \arrow[r, two heads] & \mathrm{O}(n) \arrow[r] & 1 \\
& \text{translation group} & \substack{\text{Euclidean group}\\\text{(isometries of } \mathbb{R}^n\text{)}} & \text{orthogonal group} &
\end{tikzcd} \tag{22}$$

Then the corresponding torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ inherits a G-action from $\mathbb{R}^V$. We may call the resulting G-space the *representation torus* of V. This is the type of G-space whose global quotients are *toroidal orbifolds*:

$$\begin{array}{ccccccc}
V \in \mathrm{RO}(G) & \Rightarrow & \overset{\overset{G}{\curvearrowright}}{\mathbb{R}^V} & \Rightarrow & \overset{\overset{G}{\curvearrowright}}{\mathbb{T}^V} := \overset{\overset{G}{\curvearrowright}}{\mathbb{R}^V} /\mathbb{Z}^n & \Rightarrow & \mathbb{T}^V /\!\!/ G\,. \\
\substack{\text{orthogonal linear}\\G\text{-representation}} & & \substack{\text{Euclidean}\\G\text{-space}} & \substack{\text{if } G \text{ is point group}\\\text{of crystallographic group}} & \text{representation torus} & & \substack{\text{toroidal}\\\text{orbifold}}
\end{array} \tag{23}$$

Example 3.3 (*Figure K*)**.** For $G = \mathbb{Z}_4$ the cyclic group of order 4 and $\mathbf{2}_{\mathrm{rot}}$ its 2-dimensional linear representation given by rotations around the origin by integer multiples of $\pi/2$, this action descends to the 2-torus quotient to give the representation torus $\mathbb{T}^{\mathbf{2}_{\mathrm{rot}}}$:

Figure K – The $\mathbb{Z}_4$-representation torus (23) of the 2-dimensional rotational representation $\mathbf{2}_{\mathrm{rot}}$. The underlying topological space is the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, of which we show the canonical covering $\mathbb{R}^2$-coordinate chart. Due to the coordinate identifications

$$\big([x_1],[x_2]\big) = \big([x_1+n],[x_2+m]\big) \;\in\; \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

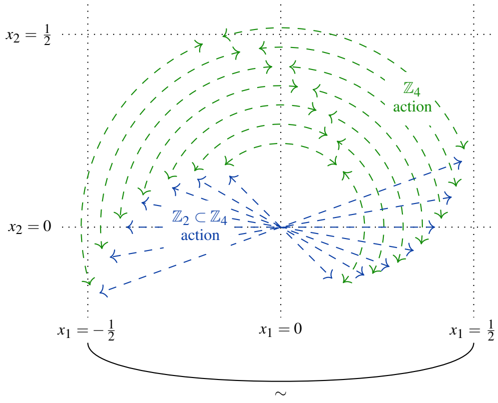
the fixed point set (24) of the $\mathbb{Z}_2$-subgroup has four points is

$$\big(\mathbb{T}^{\mathbf{2}_{\mathrm{rot}}}\big)^{\mathbb{Z}_2} \;=\; \Big\{\big([0],[0]\big),\big([\tfrac{1}{2}],[\tfrac{1}{2}]\big),\big([0],[\tfrac{1}{2}]\big),\big([\tfrac{1}{2}],[0]\big)\Big\} \subset \mathbb{T}^2\,.$$

while that of the full group has two points

$$\big(\mathbb{T}^{\mathbf{2}_{\mathrm{rot}}}\big)^{\mathbb{Z}_4} \;=\; \Big\{\big([0],[0]\big),\big([\tfrac{1}{2}],[\tfrac{1}{2}]\big)\Big\} \subset \mathbb{T}^2\,.$$

$$\overset{\overset{\mathbb{Z}_4}{\curvearrowright}}{\mathbb{T}^{\mathbf{2}_{\mathrm{rot}}}} \;=$$



H**-Fixed subspaces and isotropy groups.** For $\overset{\overset{G}{\curvearrowright}}{X}$ a G-space and $H \subset G$ any subgroup, the H*-fixed subspace*

$$X^H \;:=\; \big\{x \in X \,\big|\, h\cdot x = x \text{ for all } h \in H\big\} \;\subset\; X \tag{24}$$

is the topological subspace of X on those points which are fixed by the action of H. In particular, for $1 \subset G$ the trivial group we have $X^1 = X$. We also write

$$\mathrm{Isotr}_X(G) \;:=\; \big\{\mathrm{Stab}_G(x) \subset G \,\big|\, x \in X\big\} \tag{25}$$

for the set of *isotropy subgroups* of G, hence those that appear as stabilizer groups of some point, namely as maximal subgroups fixing a point: $\mathrm{Stab}_G(x) \;:=\; \big\{g \in G \big| g\cdot x = x\big\} \;\subset\; G$. It is the isotropy subgroups (25), but not necessarily the generic subgroups, which serve to filter a G-space in a non-degenerate way, since if one isotropy subgroup is strictly larger than another, then its fixed subspace (24) is strictly smaller

$$H_1 \subsetneq H_2 \;\in\; \mathrm{Isotr}_X(G) \qquad \Rightarrow \qquad X^{H_2} \subsetneq X^{H_1}\,.$$

Example 3.4 (fixed subspaces of ADE-singularities)**.** The non-trivial fixed subspaces of the Euclidean G-space (19) of the quaternionic representation $\mathbf{4}_{\mathbb{H}}$ (17) are all the singleton sets consisting of the origin:

$$\big(\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}\big)^{H} \;=\; \begin{cases} \mathbb{R}^4 & \text{if } H = 1 \\ \{0\} & \text{otherwise.} \end{cases} \tag{26}$$

Example 3.5 (*Figure K*)**.** For $G = \mathbb{Z}_2$ and $\mathbf{n}_{\mathrm{sgn}}$ the n-dimensional sign representation, the corresponding representation torus (23) has as $\mathbb{Z}_2$-fixed space (24) the 0-dimensional space which is the set of points whose canonical coordinates are all either 0 mod $\mathbb{Z}$ or $\frac{1}{2}$ mod $\mathbb{Z}$:

$$\begin{tikzcd} \mathbb{T}^{\mathbf{n}_{\mathrm{sgn}}} \;:=\; (\mathbb{R}^n/\mathbb{Z}^n) \arrow[loop above, "{[x]\mapsto[-x]}"] \end{tikzcd} \qquad \Longrightarrow \qquad \big(\mathbb{T}^{\mathbf{n}_{\mathrm{sgn}}}\big)^{\mathbb{Z}_2} \;=\; \big\{[0],[\tfrac{1}{2}]\big\}^n \;\subset\; \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n. \tag{27}$$

Example 3.6 (Kummer surface). The reflection ADE-action (18) $\begin{tikzcd} \mathbb{R}^{\mathbf{4}_{\mathbb{H}}} \arrow[loop above, "\mathbb{Z}_2^{\mathrm{refl}}"] \end{tikzcd}$ is clearly crystallographic (22). The orbifold $\mathbb{T}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}} \simeq \mathbb{T}^{\mathbf{4}_{\mathrm{sgn}}} /\!\!/ \mathbb{Z}_2$ presented by the corresponding representation torus (23) (when equivalently thought of as an orbifold of the complex 2-dimensional torus) known as the *Kummer surface* (e.g. [BDP17, 5.5]). The cardinality of its fixed point set (24) is (by Example 3.5)

$$\Big|\big(\mathbb{T}^{\mathbf{4}_{\mathbb{Z}}}\big)^{\mathbb{Z}_2^{\mathrm{refl}}}\Big| \;=\; \big|\{[0],[\tfrac{1}{2}]\}^4\big| \;=\; 16.$$

Residual action on fixed spaces. There is a residual group action on any H-fixed subspace X^H (24) inherited from the G-action on all of X, with the residual group being the "Weyl group" [May96, p. 13]

$$W_G(H) \;:=\; N_G(H)/H \tag{28}$$

which is the quotient group of the maximal subgroup $N_G(H) \subset G$ for which H is a normal subgroup (the normalizer of H in G) by H itself. Thereby any H-fixed subspace becomes itself a $W_G(H)$-space:

$$\begin{tikzcd} X \arrow[loop above, "G"] & : & (H \subset G) \arrow[r, mapsto] & X^H \arrow[loop above, "W_G(H)"] \; . \\ \text{A G-space induces} & & \text{for each subgroup H} & \text{the H-fixed space with residual $W_G(H)$-action} \end{tikzcd} \tag{29}$$

Notice the two extreme cases of the Weyl group (28):

$$W_G(1) = G \qquad \text{and} \qquad W_G(G) = 1\,. \tag{30}$$

Maps between G-spaces and their Elmendorf stages. The relevant *morphisms between G-spaces* are continuous functions between the underlying spaces that are G-equivariant:

$$\begin{tikzcd} X \arrow[loop above, "G"] \arrow[r, "f"] & Y \arrow[loop above, "G"] \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} X \arrow[r, "f"] & Y \end{tikzcd} \quad \begin{array}{l} \text{such that } f(g\cdot x) = g\cdot f(x) \\ \text{for all } g \in G \text{ and all } x \in X. \end{array} \tag{31}$$

This G-equivariance implies that H-fixed points are sent to H-fixed points, for every subgroup $H \subset G$, hence that every G-equivariant continuous function (31) induces a system of plain continous functions $f^H := f_{|X^H}$ between H-fixed point spaces (24), which are each equivariant with respect to the residual $W_G(H)$-action (28) and compatible with each other with respect to inclusions $H_i \subset H_j$ of subgroups:

$$\begin{tikzcd}
 & & G \curvearrowright X \arrow[r, "f"] & Y \curvearrowleft G & & 1 \arrow[d, hook] \\
 & & W_G(H_i) \curvearrowright X^{H_i} \arrow[u, hook] \arrow[r, "f^{H_i}"] & Y^{H_i} \curvearrowleft W_G(H_i) \arrow[u, hook] & & H_i \arrow[d, hook] \\
\overset{\overset{G}{\circlearrowright}}{X} \arrow[r, "f"] & \overset{\overset{G}{\circlearrowright}}{Y} \arrow[r, phantom, "\Rightarrow"] & W_G(H_j) \curvearrowright X^{H_j} \arrow[u, hook] \arrow[r, "f^{H_j}"] & Y^{H_j} \curvearrowleft W_G(H_j) \arrow[u, hook] & \text{for all} & H_j \arrow[d, hook] \\
 & & X^G \arrow[u, hook] \arrow[r, "f^G"] & Y^G \arrow[u, hook] & & G
\end{tikzcd} \tag{32}$$

G-equivariant function between *G*-spaces induces system of $W_G(H)$-equivariant functions between *H*-fixed subspaces

We will refer to the component f^H here as the *Elmendorf stage* labeled by H [Bl17, 1.3][HSS18, 3.1].

Finally, a *G-homotopy* between two G-equivariant functions f_1, f_2 (31)

$$\begin{tikzcd}
G \curvearrowright X \arrow[r, bend left, "f_0"{name=U}] \arrow[r, bend right, "f_1"'{name=D}] & Y \curvearrowleft G
\arrow[from=U, to=D, Rightarrow, "\eta"]
\end{tikzcd} \tag{33}$$

is a homotopy $[0,1] \times X \xrightarrow{\eta} X$ between the underlying continuous functions, hence such that $f_i = \eta(i,-)$, which is equivariant as a function on the product G-space $X \times [0,1]$, where the G-action on the interval $[0,1]$ is taken to be trivial.

3.1 Equivariant Hopf degree on spheres and Local tadpole cancellation

We discuss the unstable (Theorem 3.10) and the stabilized (Theorem 3.13) equivariant Hopf degree theorem for representation spheres, which characterizes equivariant Cohomotopy in compatible RO-degree (Def. 3.7 below), on Euclidean G-spaces and vanishing at infinity, hence of the vicinity of G-singularities inside flat Euclidean space (Def. 3.9 below). Using this we show (Prop. 3.14) that equivariant Cohomotopy implies the form of the local/twisted tadpole cancellation conditions from *Table 1*, *Table 2*.

3.1.1 Unstable equivariant Hopf degree

For stating the equivariant Hopf degree theorem, we need the following concept of *compatible RO-degree* for equivariant Cohomotopy. This condition is really a reflection of the structure of *J-twisted* Cohomotopy (as in [FSS19b][FSS19c]) in its version on flat orbifolds, and as such is further developed in [SS20].

Definition 3.7 (Compatible RO-degree). Given a G-space $\overset{\overset{G}{\circlearrowright}}{X}$ such that each H-fixed subspace X^H (24) for isotropy groups $H \in \mathrm{Isotr}_X(G)$ (25) admits the structure of an orientable manifold, we say that an orthogonal linear G-representation V is a *compatible RO-degree for equivariant Cohomotopy of* X if for each isotropy subgroup $H \in \mathrm{Isotr}_X(G)$ (25) the following two conditions hold: [4]

(i) Compatible fixed space dimensions: the dimension of the H-fixed subspace of V equals that of the H-fixed subspace of X:

$$\dim\left(X^H\right) \;=\; \dim\left(V^H\right). \tag{34}$$

---

[4] These conditions are a specializations of the conditions stated in [tD79, p. 212-213], streamlined here for our purpose.

(ii) **Compatible orientation behavior:** the action (29) of an element $[g] \in W_G(H)$ (28) on V^H is orientation preserving or reversing, respectively, precisely if it is so on X^H

$$\mathrm{orient}\left(\begin{array}{c} {\scriptstyle [g]\in W_H(H)} \\ \curvearrowright \\ X^H \end{array} \right) \;=\; \mathrm{orient}\left(\begin{array}{c} {\scriptstyle [g]\in W_H(H)} \\ \curvearrowright \\ (S^V)^H \end{array} \right). \tag{35}$$

Example 3.8 (**Compatible RO-degree for representation-spheres and -tori).** We observe that every real linear G-representation V is a compatible RO-degree (Def. 3.7)

(i) for the corresponding representation sphere S^V (21);

(ii) and for the corresponding representation torus $\mathbb{T}^V$ (23)

If the latter exists, hence if G is the point group of a crystallographic group on $\mathbb{R}^V$ (22).

For brevity, we introduce the following terminology, following *Table 5*, for the situation in which we will now consider equivariant Cohomotopy in compatible RO-degree:

Definition 3.9 (Cohomotopy of vicinity of the singularity)**.** Given a finite group G and an orthogonal linear G-representation $V \in \mathrm{RO}(G)$, we say that the *Cohomotopy of the vicinity of the singularity* is the unstable G-equivariant Cohomotopy (3)

$$\pi_G^V\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \;=\; \pi_G^V\big(S^V\big)$$

in compatible RO-degree V (Def. 3.7, Example 3.8) of the Euclidean G-space $\mathbb{R}^V$ (19) and vanishing at infinity (11), hence of the representation sphere S^V (21) and preserving the point at infinity.

The key implication of the first clause (34) on compatible RO-degrees is that each Elmendorf stage c^H (32) of a G-equivariant Cohomotopy cocycle c is a cocycle in ordinary Cohomotopy (5) to which the ordinary Hopf degree theorem applies, either in its stable range (13) or in the unstable range (14):

$$\begin{tikzcd}[column sep=small]
 & & & X \arrow[rr, "c"] & & S^{\dim(X)>0} & & \deg(f)\in\mathbb{Z} \\
 & & & \vdots \arrow[u, hook] & & \vdots \arrow[u, hook] & & \\
\overset{G}{\circlearrowright} X \arrow[r, "c"] & \overset{G}{\circlearrowright} S^V & \Rightarrow & X^H \arrow[u, hook] \arrow[rr, "c^H"] & & S^{\dim(X^H)>0} \arrow[u, hook] & & \deg(c^H)\in\mathbb{Z} \\
\text{equivariant Cohomotopy cocycle (3)} & & & & & & & \text{ordinary stable Hopf degree (13)} \\
\text{in compatible RO-degree } V \text{ (34)} & & & \vdots \arrow[uu, hook] & \ddots \arrow[uul, hook] \arrow[rr] & \vdots \arrow[uu, hook] & \ddots \arrow[uul, hook] & \\
 & & \text{Elmendorf stages (32)} & X^K \arrow[u, hook] \arrow[rr, "c^K"] & \vdots \arrow[u, hook] & S^{\dim(V^K)>0} \arrow[u, hook] & \vdots \arrow[u, hook] & \deg(c^K)\in\mathbb{Z} \\
 & & & \vdots \arrow[u, hook] & \ddots \arrow[u, hook] \arrow[ul, hook] \arrow[rr] & \vdots \arrow[u, hook] & \ddots \arrow[u, hook] \arrow[ul, hook] & \\
 & & & X^J \arrow[u, hook] \arrow[rr, "c^J"] & \vdots \arrow[u, dotted] & S^{\dim(V^J)=0} \arrow[u, hook] & \vdots \arrow[u, dotted] & \deg(c^J)\in\{0,1\}^{(X^J)} \\
 & & & \vdots \arrow[u, dotted] & & \vdots \arrow[u, dotted] & & \text{ordinary unstable Hopf degree (14)}
\end{tikzcd} \tag{36}$$

Theorem 3.10 (Unstable equivariant Hopf degree theorem for representation spheres). *The unstable Cohomotopy of the vicinity of a G-singularity $\mathbb{R}^V$ (Def. 3.9) is in bijection to the product set of one copy of the integers for each isotropy group* (25) *with positive-dimensional fixed subspace* $\mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)$ (24)*, and one copy of* $\{0,1\}$ *if there is an isotropy group with 0-dimensional fixed subspace* $\mathrm{Isotr}_X^{d_{\mathrm{fix}}=0}(G)$ *(which is then necessarily unique and, in fact, the group G itself):*

$$\pi_G^V\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \xrightarrow[\simeq]{c \mapsto (H \mapsto \mathbf{N_H}(c))} \mathbb{Z}^{\mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)} \times \{0,1\}^{\mathrm{Isotr}_X^{d_{\mathrm{fix}}=0}(G)} \,, \tag{37}$$

where, for $H \in \mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)$*, the ordinary Hopf degree at Elmendorf stage H* (36) *is of the form*

$$\underset{\substack{\textbf{The ordinary Hopf degree } (13) \\ \textbf{at Elmendorf stage } K \ (32)}}{\deg\big(c^H\big)} \quad = \quad \underset{\substack{\textbf{offset, being a function } \phi_H \textbf{ of} \\ \textbf{the Hopf degrees at all lower stages.}}}{\phi_H\Big(\big\{\deg\big(c^K\big)\,\big|\, K \supsetneq H \in \mathrm{Isotr}_X(G)\big\}\Big)} \quad - \quad \underset{\substack{\textbf{an integer multiple of} \\ \textbf{the order of the Weyl group } (28)}}{\mathbf{N_H}(c) \cdot \big|\big(W_G(H)\big)\big|} \quad \in \mathbb{Z}. \tag{38}$$

The isomorphism (37) *is exhibited by sending an equivariant Cohomotopy cocycle c to the sequence of the integers* $\mathbf{N_H}(c)$ *from* (38) *in positive fixed subspace dimensions, together with possibly the choice of an element of* $\{0,1\}$*, which is the unstable Hopf degree in dimension 0* (14)*, at Elmendorf stage G (if* $\dim(V^G)=0$*).*

Proof. In the special case that no subgroup $H \subset G$ has a fixed subspace of vanishing dimension, this is [tD79, Theorem 8.4.1] (the assumption of positive dimension is made "for simplicity" in [tD79, middle of p. 212]). Hence we just need to convince ourselves that the proof given there generalizes: in the present case of representation spheres, the only possible 0-dimensional fixed subspace is the 0-sphere. Hence we need to consider the case that $(S^V)^G = S^0$.

To generalize the inductive argument in [tD79, p. 214] to this case, we just need to see that every function $(S^V)^G \to (S^V)^G$ extends to a $W_G(H)$-equivariant function $(S^V)^H \to (S^V)^H$ on a next higher Elmendorf stage H. But this holds in the present case: every function from $S^0 = \{0,\infty\}$ to itself (as in *Figure H*) readily extends even to a G-equivariant function $S^V \to S^V$, and by assumption of vanishing at infinity (11) one of exactly two extensions will work, namely either the identity function or the function constant on $\infty \in S^V$:

$$\begin{tikzcd}
 & \{0,1\} & & \pi^V\big(S^V\big) \arrow[ll, "\deg\left((-)^G\right)"'] & \\
\substack{\textbf{configuration of} \\ \textbf{a single point in } S^0 \\ \textbf{sitting at } 0 \in S^0} & \Big[S^0 \xrightarrow{\mathrm{id}_{S^0}} S^0\Big] & \longleftarrow\!\shortmid & \Big[S^V \xrightarrow{c=\mathrm{id}_{S^V}} S^V\Big] & \substack{\textbf{configuration of} \\ \textbf{a single charged point in } S^V \\ \textbf{which is sitting at } 0 \in S^V} \\
\substack{\textbf{configuration of} \\ \textbf{no point in } S^0} & \Big[S^0 \xrightarrow{\mathrm{const}_\infty} S^0\Big] & \longleftarrow\!\shortmid & \Big[S^V \xrightarrow{c=\mathrm{const}_\infty} S^V\Big] & \substack{\textbf{configuration of} \\ \textbf{no point in } S^V}
\end{tikzcd} \tag{39}$$

From this induction forward, the proof of [tD79, 8.4.1] applies verbatim and shows that on top of this initial Hopf degree number of -1 (a charge at $0 \in S^0$) or 0 (no charge at $0 \in S^0$) there may now be further $N_H \cdot |W_G(H)|$-worth of Hopf degree at the next higher Elmendorf stage H, and so on. ☐

Example 3.11 ($\mathbb{Z}_2$-equivariant Cohomotopy)**.** Consider

$$c \;\in\; \pi_{\mathbb{Z}_2}^{\mathbf{n}_{\mathrm{sgn}}}\big((\mathbb{R}^{\mathbf{n}_{\mathrm{sgn}}})^{\mathrm{cpt}}\big)$$

(i.e., a cocycle in $\mathbb{Z}_2$-equivariant Cohomotopy vanishing at infinity (11) of the n-dimensional Euclidean orientifold $\mathbb{R}^{\mathbf{n}_{\mathrm{sgn}}}$ (19) underlying the n-dimensional sign representation $\mathbf{n}_{\mathrm{sgn}}$, as in *Figure I*, hence the equivariant Cohomotopy of the representation sphere $S^{\mathbf{n}_{\mathrm{sgn}}}$ (21), as in *Figure J*, in compatible RO-degree $\mathbf{n}_{\mathrm{sgn}}$, by Example 3.8). Then the unstable equivariant Hopf degree theorem 3.10 says, when translated to a geometric situation via the unstable Pontrjagin-Thom theorem (7), that:

- **(i)** there either is, or is not, a single charge sitting at the finite fixed point $0 \in S^{\mathbf{n}_{\mathrm{sgn}}}$, corresponding, with (39), to an offset of -1 or 0, respectively, in (38);
- **(ii)** in addition, there is any integer number (the $N_1 \in \mathbb{N}$ in (38)) of orientifold mirror pairs (since $|W_{\mathbb{Z}_2}(1)| = |\mathbb{Z}_2| = 2$, by (30)) of charges floating in the vicinity.

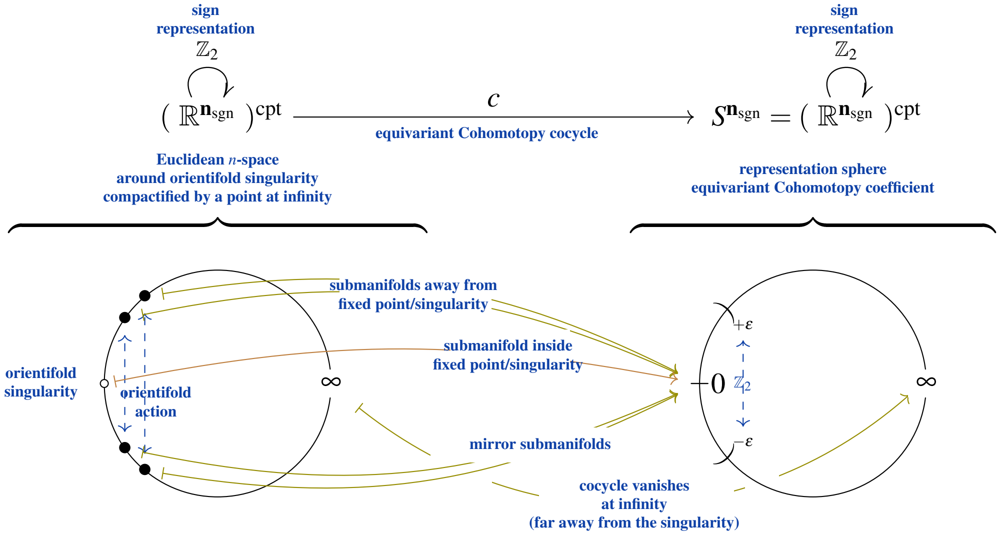


Figure L – The $\mathbb{Z}_2$-Equivariant Cohomotopy of Euclidean n-orientifolds vanishing at infinity according to the unstable equivariant Hopf degree theorem 3.10 applied to sign-representation spheres (*Figure J*) and visuallized by the corresponding configurations of charged points via the unstable Pontrjagin-Thom construction (7), in equivariant enhancement of the situation show in *Figure E*. The same situation, just crossed with an interval, appears in the application to M5/MO5 charge in *Figure V*.

It is possible and instructive to make this fully explicit in the simple special case of the 1-dimensional sign representation, where the statement of the equivariant Hopf degree theorem 3.10 may be found in elementary terms: It is readily checked that all the continuous functions $c^1 : S^1 \to S^1$ which take 0 to either of $0, \infty \in S^1$ and wind around at constant parameter speed are $\mathbb{Z}_2$-equivariant, hence are Elmendorf stages (32) of $\mathbb{Z}_2$-equivariant cocycles c:

$$\begin{tikzcd}
\left(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}}\right)^{\mathrm{cpt}} \simeq S^{\mathbf{1}_{\mathrm{sgn}}} \arrow[r, "c"] & S^{\mathbf{1}_{\mathrm{sgn}}} \; . \\
S^1 \arrow[r, "c^1"] & S^1 \\
S^0 \arrow[u, hook] \arrow[r, "c^{\mathbb{Z}_2}"] & S^0 \arrow[u, hook]
\end{tikzcd} \tag{40}$$

If such a function vanishes at infinity (11), in that it takes $\infty \mapsto \infty$ as shown in *Figure L*, then we have one of two cases:

(i) either c^1 *winds an odd number* of times, so that (38) reads:

$$\deg(c^1) = \overbrace{1}^{\text{offset}} - N_1 \cdot 2\,,$$

in which case it satisfies $c^1(0) = 0$, so that under the PT-theorem (7) there is precisely one charge at the singular fixed point, together with the even integer number $2 \cdot N_1 \in \mathbb{Z}$ of net charges in its "vicinity" (namely: away from infinity) which are arranged in $\mathbb{Z}_2$-mirror pairs, due to the $\mathbb{Z}_2$-equivariance of c; this is what is shown on the left of *Figure L*;

(ii) or c^1 *winds an even number* of times so that (38) reads:

$$\deg(c^1) = \overbrace{0}^{\text{offset}} - N_1 \cdot 2\,,$$

in which case it satisfies $c^1(0) = \infty$, so that under the PT-theorem (7) there is no charge at the singular fixed point, but a net even integer number $2 \cdot N_1 \in \mathbb{Z}$ of charges in its vicinity, as before.

Remark 3.12 (Number of branes and offset)**.** Notice that:

(i) For $N_1 = 0$ (no branes) this is the situation of (39): either there is a non-vanishing charge associated with the singular fixed point (O-plane charge), or not.

(ii) Furthermore, if there is, it is either +1 or -1, so that in general the charge associated with the singular fixed point is in $\{0, \pm 1\}$, as befits O-plane charge according to *Figure OP*.

(iii) The offset is relevant only modulo 2, so that we could have chosen an offset of $+1$ instead of as -1 in the first case. This choice just fixes the sign convention for D-brane/O-plane charge.

Characterizing the brane content around a singularity. In the above example in RO-degree $\mathbf{1}_{\mathrm{sgn}}$ (40), it is clear that the configurations of branes implied by the unstable equvariant Hopf degree theorem (Theorem 3.10) appear in multiples of the regular G-set around a fixed O-plane charge stuck in the singularity, as illustrated in *Figure L* and as demanded by the local/twisted tadpole cancellation conditions according to *Table 1*. In order to prove that this is the case generally, we now turn to the stabilized equivariant Hopf degree theorem (Theorem 3.13 below), which concretely characterizes the (virtual) G-sets of branes that may appear classified by equivariant Cohomotopy.

3.1.2 Stable equivariant Hopf degree

In a homotopy-theoretic incarnation of perturbation theory, we may approximate unstable equivariant Cohomotopy (Theorem 3.13) by its homotopically linearized, namely stabilized (see [BMSS18]) version. We briefly recall the basics of stable equivariant Cohomotopy in RO-degree 0 ([Seg71], see [tD79, 7.6 & 8.5][Lu05]) before applying this in Theorem 3.13 and Prop. 3.14 below.

Equivariant suspension. For $V, W \in \mathrm{RO}(G)$ two orthogonal linear G-representations, and for

$$\big[S^V \simeq (\mathbb{R}^V)^{\mathrm{cpt}} \xrightarrow{\;c\;} (\mathbb{R}^V)^{\mathrm{cpt}} \simeq S^V \big] \;\in\; \pi_G^V\big((\mathbb{R}^V)^{\mathrm{cpt}} \big)$$

the class of a cocycle in the equivariant Cohomotopy (3) of the Euclidean G-space $\mathbb{R}^V$ (19) in compatible RO-degree V (Example 3.8) and vanishing at infinity (11), we obtain the class of a cocycle vanishing at infinity on the product G-space $\mathbb{R}^{V\oplus W}$ (20) in compatible degree $V \oplus W$, simply by forming the Cartesian product of c with the identity on $\mathbb{R}^W$. This is the *equivariant suspension* of c by RO-degree W:

$$\underset{\substack{\text{equivariant suspension}\\ \text{by RO-degree } W\\ \text{of equivariant Cohomotopy cocycle}}}{\Sigma^W c} \;:=\; \Big[\underset{\substack{\text{compactified Cartesian product}\\ \text{of cocycle with identity on } \mathbb{R}^W}}{(\mathbb{R}^V \times \mathbb{R}^W)^{\mathrm{cpt}} \xrightarrow{\; c \times \mathrm{id}_{\mathbb{R}^W} \;} (\mathbb{R}^V \times \mathbb{R}^W)^{\mathrm{cpt}}} \Big] \;\in\; \pi_G^{V\oplus W}\big((\mathbb{R}^{V\oplus W})^{\mathrm{cpt}} \big) . \tag{41}$$

Note that this reduces to the ordinary suspension operation eq. (15) for $G = 1$ the trivial group, hence for RO-degrees $\mathbf{n}_{\mathbf{triv}} = n$. These equivariant suspension operations form a directed system on the collection of equivariant Cohomotopy sets (3), indexed by inclusions of orthogonal linear representations:

$$\big(V \hookrightarrow V \oplus W \big) \;\longmapsto\; \Big(\pi_G^V\big((\mathbb{R}^V)^{\mathrm{cpt}} \big) \xrightarrow{\;\Sigma^W\;} \pi_G^{V\oplus W}\big((\mathbb{R}^{V\oplus W})^{\mathrm{cpt}} \big) \Big) . \tag{42}$$

Stable equivariant Cohomotopy. As a consequence of the above, one may consider the union of all unstable equivariant Cohomotopy sets of representation spheres in all compatible degrees, with respect to the identifications along the equivariant suspension maps (42) (the colimit of this system):

$$\underset{\substack{\text{unstable equivariant Cohomotopy set}\\ \text{in compatible RO-degree } V}}{\pi_G^V\big((\mathbb{R}^V)^{\mathrm{cpt}} \big)} \quad \underset{\substack{\text{stabilization map}\\ \text{(coprojection into colimit)}}}{\xrightarrow{\qquad \Sigma^\infty \qquad}} \quad \underset{\substack{\text{union (colimit) of}\\ \text{all unstable Cohomotopy sets}\\ \text{in compatible RO-degrees}\\ \text{identified along equivariant suspensions}}}{\underset{\underset{W}{\longrightarrow}}{\lim}\, \pi_G^W\big((\mathbb{R}^W)^{\mathrm{cpt}} \big)} \quad = \quad \underset{\substack{\text{stable}\\ \text{equivariant Cohomotopy ring}\\ \text{in RO-degree 0}}}{\mathbb{S}_G^0} . \tag{43}$$

Since the resulting union/colimit is, by construction, stable under taking further such suspensions, this is called the *stable equivariant Cohomotopy in degree 0* ([Seg71, p. 1], see [Lu05, p. 9-10]) also called the *0th stable G-equivariant homotopy group of spheres* or the *G-equivariant stable 0-stem* or similar (see [May96, IX.2][Schw, 3]). Notice that here the stable RO-degree is the formal difference of the unstable RO-degree by the RO-degree of the singularity, so that vanishing stable RO-degree is another expression of compatibility of unstable degree, in the sense of Example 3.8:

$$-\underbrace{V}_{\substack{\text{RO-degree of}\\ \text{singularity}}} + \underbrace{V}_{\substack{\text{compatible RO-degree of}\\ \text{unstable Cohomotopy}}} = \underbrace{0}_{\substack{\text{RO-degree of}\\ \text{stable Cohomotopy}}}$$

via

$$\pi_V(S^W) = [S^V, S^W] = \pi^W(S^V) \xrightarrow{\Sigma^\infty} S^{W-V}.$$

Extensive computation of stable $\mathbb{Z}_2$-equivariant Cohomotopy of representation spheres in non-vanishing RO-degrees, i.e., computation of the abelian groups $\mathbb{S}_{\mathbb{Z}_2}^{\mathbf{n}_{\mathrm{sgn}}+\mathbf{m}_{\mathrm{triv}}}$, is due to [AI82][Ir82]; see also [DI16, 5][Du08, p. 10-15]. Under *Hypothesis H*, these groups are relevant for tadpole cancellation with branes wrapping orientifold singularities non-transversally. This is of interest to us but goes beyond the scope of this article.

Equivalence to the Burnside ring. Due to the stabilization, the stable equivariant Cohomotopy set (43) has the structure of an abelian group, in fact the structure of a ring. As such, it is isomorphic to the *Burnside ring* $A(G)$ of virtual G-sets ([Bur01][So67][tD79, 1], for exposition in our context see [BSS19, 2]):

$$\overset{\substack{\text{stable}\\ \text{equivariant Cohomotopy}}}{\mathbb{S}^0_G} \simeq \overset{\text{Burnside ring}}{A(G)} = \big\{\text{Virtual } G\text{-sets}\big\}. \tag{44}$$

This result is due to [Seg71, p. 2]; see [tD79, 7.6.7 & 8.5.1][Lu05, 1.13], we highlight its geometric meaning below; see *Figure M*. This is a non-linear analog (more precisely, the analog over the absolute base “field” $\mathbb{F}_1$ [Co04, p. 3][Dur07, 2.5.6]) of the fact that the equivariant K-theory in degree 0 is the representation ring of virtual linear G-representations over the field of real numbers (see, e.g., [Gr05, 3]):

$$\overset{\text{equivariant K-theory}}{\mathrm{KO}^0_G} \simeq \overset{\text{representation ring}}{\mathrm{RO}(G)} = \big\{\text{Virtual } G\text{-representations}\big\}. \tag{45}$$

In fact, the operation $S \mapsto \mathbb{R}[S]$ that sends a (virtual) G-set $S \in \mathrm{A}(G)$ to its linearization, hence to its linear span $\mathbb{R}[S]$, hence to the (virtual) permutation representation that it induces (see [tD79, 4][BSS19, 2]), is a ring homomorphism from the Burnside ring to the representation ring. Furthermore, it exhibits the value on the point of unique multiplicative morphism from equivariant stable Cohomotopy theory to equvariant K-theory, called the *Boardman homomorphism* [Ada74, II.6], which is the Hurewicz homomorphism generalized from ordinary cohomology to generalized cohomology theories:

$$\begin{tikzcd}
\text{equivariant stable Cohomotopy} & \mathbb{S}^0_G \arrow[r, "\overset{\text{Boardman homomorphism}}{\beta}"] \arrow[d, phantom, "\wr|"] & \mathrm{KO}^0_G \arrow[d, phantom, "\wr|"] & \text{equivariant K-theory} \\
\text{Burnside ring} & \mathrm{A}(G) \arrow[r, "\substack{S \mapsto \mathbb{R}[S]\\ \text{linearization}\\ \text{sending } G\text{-sets } S \text{ to}\\ \text{linear } G\text{-representations } \mathbb{R}[S]}"'] & \mathrm{RO}(G) & \text{representation ring}
\end{tikzcd} \tag{46}$$

In summary, the composite of the stabilization morphism (43) with the isomorphism (44) to the Burnside ring explicitly extracts from any cocycle c in unstable equivariant Cohomotopy a virtual G-set $\{\text{branes}\}$, hence a virtual G-permutation representation $\mathbb{R}[\{\text{branes}\}]$. The following theorem explicitly identifies this G-set $\{branes\}$ in terms of the Elmendorf stage-wise Hopf degrees of the cocycle c; see *Figure M* below for illustration.

Theorem 3.13 (Stabilized equivariant Hopf degree theorem for representation spheres). *Consider a cocycle c in unstable Cohomotopy of the vicinity of a G-singularity $\mathbb{R}^V$ (Def. 3.9). Its image under stabilization in equivariant stable Cohomotopy* (43) *is, under the identification* (44) *with the Burnside ring, precisely that virtual G-set $\{\text{branes}\} \in A(G)$ whose net number of H-fixed points (“Burnside marks”, see [BSS19, 2]), equals the Hopf degree of c at any Elmendorf stage $H \in \mathrm{Isotr}_X(G)$* (36)*. Hence if $H = \langle g \rangle$ is a cyclic group generated by an element $g \in G$, this number also equals the character value at g (i.e., the trace of the linear action of g) on the linear representation $\mathbb{R}\big[\{\text{branes}\}\big]$:*

$$\begin{tikzcd}[column sep=large]
& \overset{\substack{\text{cocycle in}\\ \text{unstable equivariant Cohomotopy}}}{c} \arrow[r, mapsto] & \overset{\substack{\text{cocycle in stable equivariant Cohomotopy}\\ \simeq \text{ virtual } G\text{-set of branes}}}{\{\text{branes}\}} \arrow[r, mapsto] & \overset{\substack{\text{virtual linear } G\text{-representation}\\ \text{spanned by virtual } G\text{-set of branes}}}{\mathbb{R}[\{\text{branes}\}]} \\
& \pi^V_G\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \arrow[u, phantom, sloped, "\in"] \arrow[r, "\substack{\text{stabilization}\\ \Sigma^\infty}"] \arrow[d, "\wr", "(37)"'] & \mathbb{S}^0_G \arrow[u, phantom, sloped, "\in"] \arrow[r, "\beta"] \arrow[d, "\wr", "(44)"'] & \mathrm{KO}^0_G \arrow[u, phantom, sloped, "\in"] \arrow[d, "\wr", "(45)"'] \\
& \mathbb{Z}^{\mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)} \times \{0,1\}^{\mathrm{Isotr}_X^{d_{\mathrm{fix}}=0}(G)} \arrow[r] & A(G) \arrow[r, "\substack{\mathbb{R}[-]\\ \text{linearization}}"'] & \mathrm{RO}(G)
\end{tikzcd} \tag{47}$$

$$\begin{tikzcd}[column sep=large]
\underset{\substack{\text{Elmendorf}\\ \text{stage (32)}}}{G \supset H :} & \underset{\substack{\text{Hopf degree at stage } H \text{ (36)}\\ \text{of Cohomotopy cocycle}}}{\deg\big(c^H\big)} \arrow[r, equal] & \underset{\substack{\text{net number of } H\text{-fixed elements}\\ = \text{Burnside marks at stage } H\\ \text{in virtual set of branes}}}{\big|\{\text{branes}\}^H\big|} & \\
\underset{\substack{\text{Elmendorf stage}\\ \text{at cyclic subgroup}\\ \text{generated by } g \in G}}{H = \langle g \rangle :} & \underset{\substack{\text{Hopf degree at stage } H = \langle g \rangle \text{ (36)}\\ \text{of Cohomotopy cocycle}}}{\deg\big(c^{\langle g \rangle}\big)} \arrow[r, equal] & \underset{\substack{\text{virtual number of fixed points}\\ \text{under action of } g \in G\\ \text{on virtual set of branes}}}{\big|\{\text{branes}\}^{\langle g \rangle}\big|} \arrow[r, equal] & \underset{\substack{\text{character value at } g\\ \text{of virtual permutation representation}\\ \text{spanned by branes}}}{\mathrm{Tr}_g\big(\mathbb{R}[\{\text{branes}\}]\big)}
\end{tikzcd}$$

Proof. For the case that all fixed subspace dimensions are positive, this is essentially the statement of [tD79, 8.5.1], after unwinding the definitions there (see [tD79, p. 190]). We just need to see that the statement generalizes as claimed to the case where the full fixed subspace $\big(S^V\big)^G = S^0$ is the 0-sphere. But, under stabilization map Σ^∞ (43), the image of a Cohomotopy cocycle $S^V \xrightarrow{c} S^V$ and its equivariant suspension (41) $S^{\mathbf{1}_{\mathrm{triv}} \oplus V} \xrightarrow{\Sigma^{\mathbf{1}_{\mathrm{triv}}} c} S^{\mathbf{1}_{\mathrm{triv}} \oplus V}$ by, in particular, the trivial 1-dimensional representation, have the same image $\Sigma^\infty(c) \simeq \Sigma^\infty\big(\Sigma^{\mathbf{1}_{\mathrm{triv}}} c\big)$. Now to the suspended cocycle $\Sigma^{\mathbf{1}_{\mathrm{triv}}} c$ the theorem [tD79, 8.5.1] applies, and hence the claim follows from the fact (15) that the unstable Hopf degree in $\{0,1\}$ injects under suspension into the stable Hopf degrees:

$$\Big[S^0 = \big(S^V\big)^G \xrightarrow{c^G} \big(S^V\big)^G = S^0\Big] \;\in\; \{0,1\} \hookrightarrow \mathbb{Z}\,. \qquad \square$$

For ADE-singularities, this implies the following (see *Figure M*):

Proposition 3.14 (Classification of Cohomotopy charge in the vicinity of ADE-singularities). *Consider $G = G^{\mathrm{ADE}} \subset \mathrm{SU}(2)$ a finite ADE-group* (16) *and $\mathbf{4}_{\mathbb{H}}$ its canonical quaternionic representation* (17)*. Then the homomorphism* (47) *from Theorem 3.13 identifies the unstable Cohomotopy of the vicinity of the G^{ADE}-singularity $\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}$ (Def. 3.9) with its image in the representation ring*

$$\pi^{\mathbf{4}_{\mathbb{H}}}_{G^{\mathrm{ADE}}}\Big(\big(\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}\big)^{\mathrm{cpt}}\Big) \xhookrightarrow{\;\beta \circ \Sigma^\infty\;} \mathrm{KO}^0_G \simeq \mathrm{RO}(G)$$

which consists of all the virtual representations of the form

$$\mathbb{R}[\{\text{branes}\}] \;=\; N_{\mathrm{Opla}} \cdot \mathbf{1}_{\mathrm{triv}} - N_{\substack{\mathrm{brane}\\ \mathrm{int}}} \cdot \mathbf{k}_{\mathrm{reg}} \qquad \text{for} \qquad \begin{array}{rcl} N_{\substack{\mathrm{brane}\\ \mathrm{int}}} & \in & \mathbb{N}\,, \\ N_{\mathrm{Opla}} & \in & \{0,1\} \end{array} \tag{48}$$

hence of the form of the local/twisted tadpole cancellation conditions in Table 1 and Table 2.

Proof. By (26), the representation $\mathbf{4}_{\mathbb{H}}$ is such that *every* non-trivial subgroup $1 \neq H \subset G$ has a 0-dimensional fixed space:

$$\dim\Big(\big(\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}\big)^H\Big) \;=\; \left\{ \begin{array}{ll} 4 & \text{if } H = 1 \\ 0 & \text{otherwise.} \end{array} \right.$$

This means that for $c \in \pi^{\mathbf{4}_{\mathbb{H}}}\Big(\big(\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}\big)^{\mathrm{cpt}}\Big)$ an equivariant Cohomotopy cocycle in the vicinity of an ADE-singularity, its only Elmendorf stage-wise Hopf degree (36) in positive dimension is, by equation (38) in Theorem 3.10, of the form

$$\deg\big(c^1\big) \;=\; \overbrace{Q_{\mathrm{Opla}}}^{\in \{0,1\}} - N_1 \cdot |G|\,,$$

where we used the fact that $W_G(1) = G$ (30). But, by Theorem 3.13, this implies that the virtual G-set $\{\text{branes}\}$ of branes corresponding to c has the following Burnside marks

$$\{\text{branes}\}^H \;=\; \left\{ \begin{array}{cc} Q_{\text{Opla}} - N_1 \cdot |G| & \text{if } H = 1 \\ Q_{\text{Opla}} & \text{otherwise}, \end{array} \right.$$

hence that the corresponding permutation representation of branes has the following characters:

$$\mathrm{Tr}_g\big(\mathbb{R}\{\text{branes}\}\big) \;=\; \left\{ \begin{array}{cc} Q_{\text{Opla}} - N_1 \cdot |G| & \text{if } g = e \\ Q_{\text{Opla}} & \text{otherwise}. \end{array} \right.$$

The unique G-set/G-representation with these Burnside marks/characters is the sum of the N_1-fold multiple of the regular G-set/G-representation and the Q_{Opla}-fold multiple of the trivial representation (see *Figure M*):

$$\mathbb{R}[\{\text{branes}\}] \;=\; Q_{\text{Opla}} \cdot \mathbf{1}_{\text{triv}} \;-\; N_1 \cdot \mathbf{k}_{\text{reg}}\,, \qquad \square$$

The situation is illustrated by *Figure M*:

$$\begin{tikzcd}
\begin{array}{c}\textbf{equivariant Cohomotopy}\\ \textbf{vanishing at infinity}\\ \textbf{of Euclidean } G\textbf{-space}\\ \textbf{in compatible RO-degree } V\end{array} & & \begin{array}{c}\textbf{stable}\\ \textbf{equivariant}\\ \textbf{Cohomotopy}\end{array} & & \begin{array}{c}\textbf{equivariant}\\ \textbf{K-theory}\end{array} \\
\pi^V_G\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \arrow[rr, "\Sigma^\infty", "\textbf{stabilization}"'{yshift=30pt}] & & \mathbb{S}^0_G \arrow[rr, "\beta", "\textbf{Boardman homomorphism}"'{yshift=30pt}] \arrow[d, phantom, "\wr"] & & \mathrm{KO}^0_G \arrow[d, phantom, "\wr"] \\
& & A_G \arrow[rr, "\mathbb{R}[-]", "\textbf{linearization}"'] & & \mathrm{RO}(G) \\
& & \begin{array}{c}\textbf{Burnside}\\ \textbf{ring}\end{array} & & \begin{array}{c}\textbf{representation}\\ \textbf{ring}\end{array}
\end{tikzcd}$$

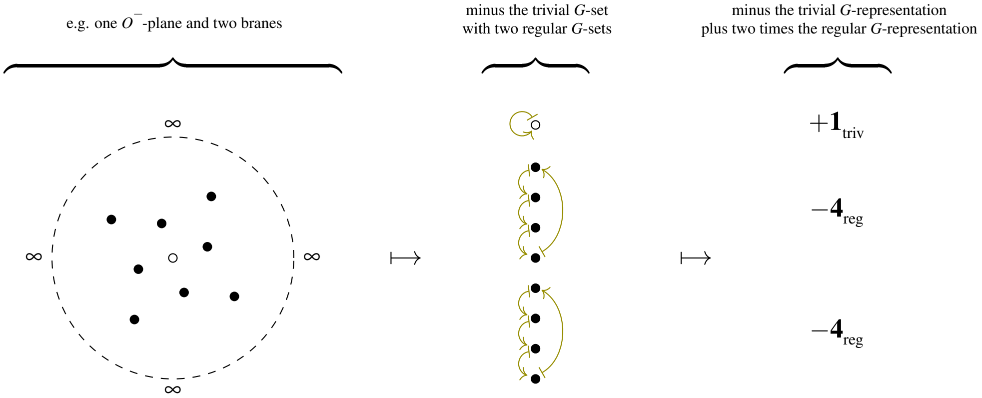


Figure M – Virtual G-representations of brane configurations classified by equivariant Cohomotopy in the vicinity of ADE-singularities (Def. 3.9), according to Prop. 3.14, following Theorem 3.10 and Theorem 3.13. The results reproduces the form of the local/twisted tadpole cancellation conditions in *Table 1*, *Table 2*. Shown is a situation for $G = \mathbb{Z}_4$ and $V = \mathbf{2}_{\text{rot}}$ as in *Figure K*.

3.2 Equivariant Hopf degree on tori and Global tadpole cancellation

We now globalize the characterization of equivariant Cohomotopy from the vicinity of singular fixed points to compact toroidal orbifolds, in Theorem 3.17 below. Prop. 3.18 below shows that the two are closely related, implying that the local/twisted tadpole cancellation carries over to toroidal orbifolds. Then we informally discuss the enhancement of unstable equivariant Cohomotopy to a super-differential cohomology theory (55) and show that its implications (58) on the underlying equivariant Cohomotopy enforce the form of the global/untwisted tadpole cancellation conditions.

Globalizing from Euclidean orientifolds to toroidal orientifolds. In §3.1 we discussed the characterization of equivariant Cohomotopy in the vicinity of singularities (according to *Table 5*). We may globalize this to compact toroidal orientifolds by applying this local construction in the vicinity of each singularity, using that the condition of “vanishing at infinity” (11) with respect to any one singularity means that the local constructions may be glued together. This *local-to-global* construction is indicated in *Figure N*:

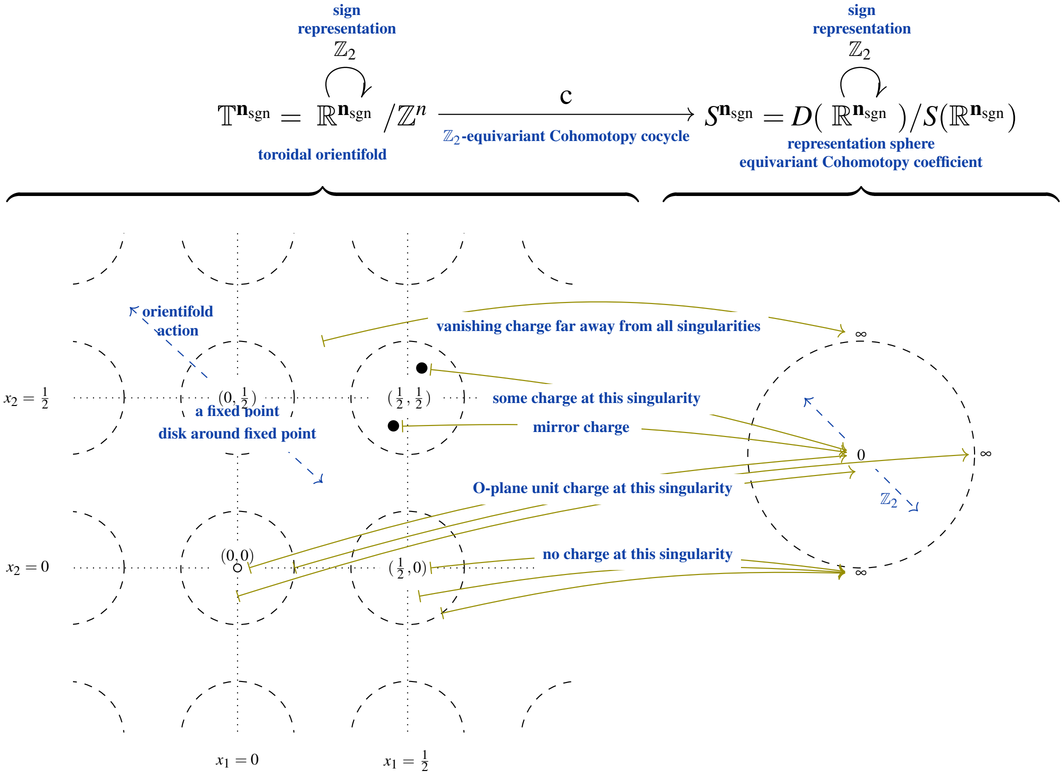


Figure N – Equivariant Cohomotopy cocycle on toroidal orbifolds glued from local cocycles in the vicinity of singularities, as formalized in the proof of Theorem 3.17. Shown is a situation for $G = \mathbb{Z}_2$, as in *Figure I* and *Figure J*.

Well-isolated singularities. In order to formalize this local-to-global construction conveniently, we make the following sufficient assumption on the G-spaces to which we apply it:

Definition 3.15. We say that a G-space $\overset{G \circlearrowright}{X}$ has *well-isolated singularities* if all the *minimal* subgroups with 0-dimensional fixed subspaces (24) are in the center of G, i.e. if the following condition holds:

$$H \subset G \text{ minimal such that } \dim\big(X^G\big) = 0 \quad \Rightarrow \quad H \subset \mathrm{Center}(G)\,. \tag{49}$$

Example 3.16. The ADE-singularites (*Table 5*) with well-isolated fixed points in the sense of (49) are all those in the $\mathbb{A}$-series, as well as the generalized quaternionic ones in the $\mathbb{D}$-series – see *Table 6*. This is because, for ADE-singularities, all non-trivial subgroups have 0-dimensional fixed space (26), so that here the condition of well-isolated singularities (49) requires that all non-trivial minimal elements in the subgroup lattice be in the center. This is trivially true for the cyclic groups in the $\mathbb{A}$-series, since they are abelian. For the generalized quaternionic groups in the $\mathbb{D}$-series there is in fact a unique minimal non-trivial subgroup, and it in fact it is always the orientifold action $H_{\min} = \mathbb{Z}_2$ which coincides with the center, as shown for the first few cases in *Table 6*.

The point of the notion of well-isolated fixed points (49) is that it is sufficient to guarantee that the action of the full group restricts to the union of the 0-dimensional fixed subspaces, since then

$$H \cdot x_{\mathrm{fixed}} \;=\; x_{\mathrm{fixed}} \quad \Rightarrow \quad H \cdot (g \cdot x_{\mathrm{fixed}}) \;=\; (H \cdot g) \cdot x_{\mathrm{fixed}} \;=\; (g \cdot H) \cdot x_{\mathrm{fixed}} \;=\; g \cdot (H \cdot x_{\mathrm{fixed}}) \;=\; g \cdot x_{\mathrm{fixed}}, \tag{50}$$

for all $g \in G$. Hence, with (49), the quotient set

$$\mathrm{IsolSingPts}_G(X) \;:=\; \Bigg(\bigcup_{H \subset G,\, \dim(X^H) = 0} X^H \Bigg) \Big/ G \tag{51}$$

exists and is the set of isolated singular points in the orbifold $X /\!\!/ G$.

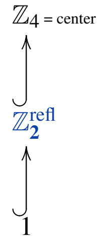
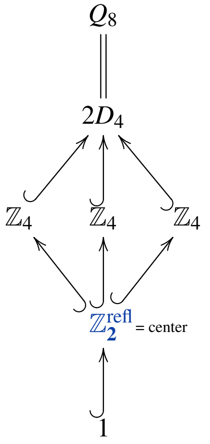
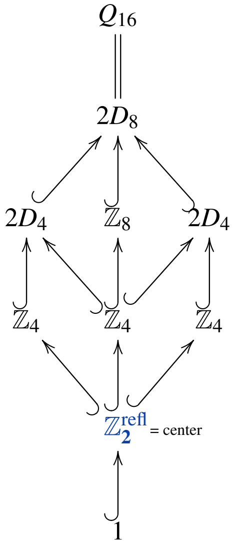
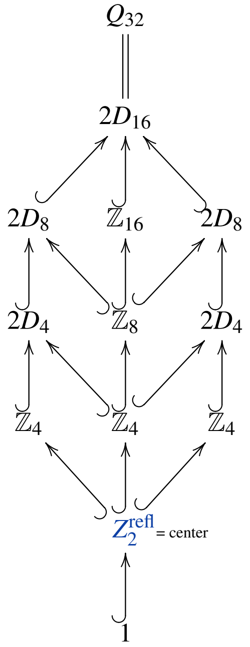
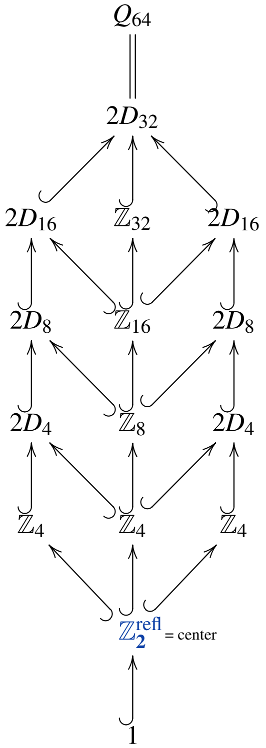
Dynkin label	$\mathbb{A}3 = \mathbb{D}3$	$\mathbb{D}4$	$\mathbb{D}6$	$\mathbb{D}10$	$\mathbb{D}18$
$G \subset \mathrm{Sp}(1)$ **subgroup lattice**					

Table 6 – The ADE-Singularities $\mathbb{R}^{\mathbf{4}_{\mathbb{H}}} \!\sslash\! G^{\mathrm{ADE}}$ **with well-isolated fixed points** according to (49) are those in the $\mathbb{A}$-series $G^{\mathrm{ADE}} = \mathbb{Z}_n$ and the quaternionic groups $Q_{2^{n+2}} = 2D_{2^n+2}$ in the $\mathbb{D}$-series. For the latter and for the even-order cyclic groups, the minimal non-trivial central subgroup is unique and given by the point reflection group $\mathbb{Z}_2^{\mathrm{refl}}$ (18).

Unstable equivariant Hopf degree of representation tori. With these preliminaries in hand, we may now state and prove the unstable equivariant Hopf degree theorem for representation tori with well-isolated singularities, Theorem 3.17 below. Its statement and proof are directly analogous to the case for representation spheres in Theorem 3.10. The difference here, besides the passage from spheres to tori, is the extra assumption on well-isolated singularities and the fact that the proof here invokes the construction of the previous proof around each one of the well-isolated singularities.

Theorem 3.17 (**Unstable equivariant Hopf degree theorem for representation tori**)**.** *The unstable equivariant Cohomotopy* (3) *of a G-representation torus* $\mathbb{T}^V$ (23) *with well-isolated singularities* (49) *and with a point at infinity adjoined* (9) *in compatible RO-degree V (Example 3.8) is in bijection to the product set of one copy of the integers for each isotropy group* (25) *with positive dimensional fixed subspace* (24)*, and one copy of* $\{0,1\}$ *for each well-isolated fixed point* (51)

$$\pi_G^V\big((\mathbb{T}^V)_+\big) \xrightarrow[\simeq]{c \,\mapsto\, (H \mapsto \mathbf{N_H}(c))} \mathbb{Z}^{\mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)} \times \{0,1\}^{\mathrm{IsolSingPts}_G(X)}\,, \tag{52}$$

where, for $H \in \mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)$*, the ordinary Hopf degree at Elmendorf stage H* (36) *is of the form*

$$\underset{\substack{\text{The ordinary Hopf degree (13)}\\ \text{at Elmendorf stage } K \text{ (32)}}}{\deg\big(c^H\big)} \;=\; \underset{\substack{\text{a fixed offset, being a function of}\\ \text{the Hopf degrees at all lower stages.}}}{\phi_H\big(\{\deg(c^K)\,\big|\,K \supsetneq H \in \mathrm{Isotr}_X(G)\}\big)} \;-\; \underset{\substack{\text{an integer multiple of}\\ \text{the order of the Weyl group (28)}}}{\mathbf{N_H}(c)\cdot\big|\big(W_G(H)\big)\big|} \;\in \mathbb{Z}. \tag{53}$$

The isomorphism (52) *is exhibited by sending an equivariant Cohomotopy cocycle to the sequence of the integers* $\mathbf{N_H}$ *from* (53) *in positive fixed subspace dimensions, together with the collection of elements in* $\{0,1\}$*, which are the unstable Hopf degrees in dimension 0* (14)*, at Elmendorf stage G at each one of the well-isolated singularities.*

Proof. In the special case when no subgroup $H \subset G$ has a fixed subspace of vanishing dimension, this is [tD79, Theorem 8.4.1], where the assumption of positive dimension is made "for simplicity" in [tD79, middle of p. 212]. Hence we just need to convince ourselves that the proof given there generalizes.

To that end, assume that $\dim\big(V^G\big) = 0$. To generalize the inductive argument in [tD79, p. 214] to this case, we just need to see that every G-invariant function on the isolated fixed (51)

$$\begin{tikzcd}
\mathrm{IsolSingPts}_G(X) \arrow[r] & S^0 \arrow[d, equal] \\
\underset{H \subset G, \dim(X^H) = 0}{\bigcup} X^H \arrow[u, "q"] \arrow[r, "(c^H)"] & S^0
\end{tikzcd} \tag{54}$$

extends to a $W_G(K)$-equivariant function $(S^V)^K \to (S^V)^K$ on the next higher Elmendorf stage $K \in \mathrm{Isotr}_X^{d_{\mathrm{fix}}>0}(G)$. For this, consider a G-equivariant tubular neighborhood around the well-isolated fixed points. This is guaranteed to exist on general grounds by the equivariant tubular neighborhood theorem, since, by assumption (49), the set of points (in the bottom left of (54)) is an equivariant (and of course closed) subspace, by (50). In fact, in the present specific situation of global *orthogonal* linear actions on a Euclidean space we obtain a concrete such equivariant tubular neighborhood by forming the union of Euclidean open balls of radius ε around each of the points, for any small enough positive real number ε. This kind of tubular neighborhood is indicated by the collection of dashed circles in *Figure A* and *Figure N*. Given this or any choice of equivariant tubular neighborhood, the extensions (39) in the proof of Theorem 3.10 apply to the vicinity of any one fixed points. This is a choice in $\{0,1\}$ for each element in $\mathrm{IsolSingPts}_G(X)$ (51), hence in total is the choice of an element in $\{0,1\}^{\mathrm{IsolSingPts}_G(X)}$, as it appears in (37). Since all these local extensions to the vicinity of any of the singularities "vanish at infinity" (11), i.e., at some distance $> \varepsilon$ from any and all of the well-isolated fixed points, they may jointly be further extended to a global cocycle $\mathbb{T}^V \xrightarrow{c} S^V$ by declaring that c sends every other point in $\mathbb{T}^V$ outside the given tubular neighborhood to $\infty \in S^V$ (shown in *Figure N*). From this induction onwards, the proof of [tD79, 8.4.1] applies verbatim and shows that on top of this initial Hopf degree choice in $\{0,1\}^{\mathrm{IsolSingPts}_G(X)}$ there may now be further $N_H \cdot |W_G(H)|$-worth of Hopf degree at the next higher Elmendorf stage H, and so on. □

Stable equivariant Hopf degree of representation tori. Note that the unstable equivariant Hopf degrees of representation spheres (Theorem 3.10) and of representation tori (Theorem 3.17) have the same form, (37) and (52), respectively, away from the unstable Hopf degrees in vanishing fixed space dimensions. It follows immediately that, up to equivariant homotopy, all brane charge may be thought of as concentrated in the vicinity of the "central" singularity (see *Figure O*):

Proposition 3.18 (Pushforward in unstable equivariant Cohomotopy). *Let* $\overset{\curvearrowright^{G}}{\mathbb{T}^V}$ *be a G-representation torus* (23) *with well-isolated singularities* (49)*, and write* $D_\varepsilon(\mathbb{R}^V) \overset{i}{\hookrightarrow} \mathbb{T}^V$, $0 \hookrightarrow x_0$, *for the inclusion of the G-equivariant tubular neighborhood around the fixed point* $x_0 \in \mathbb{T}^V$ *covered by* $0 \in \mathbb{R}^V$ *that is given by the open* ε*-ball around the point, for any small enough positive radius* ε*. Then pushforward along i from the unstable equivariant Cohomotopy of the vicinity of this fixed point (as in Theorem 3.10) to that of the full representation torus (as in Theorem 3.17)*

$$\begin{tikzcd}
\textbf{unstable equivariant Cohomotopy} \\ \textbf{of vicinity of } G\textbf{-singularity} & \textbf{identify with vicinity of } x_0 & \textbf{unstable equivariant Cohomotopy} \\ \textbf{of } G\textbf{-representation torus} \\
\pi^V_G\big((\mathbb{R}^V)^{\mathrm{cpt}}\big) \arrow[rr, hook, "i_*", "\simeq_{(d_{\mathrm{fix}}>0)}"'] & & \pi^V_G\big((\mathbb{T}^V)_+\big) \\
\left[\begin{array}{ccc} \mathbb{R}^V & \xrightarrow{\quad c \quad} & S^V \\ x & \longmapsto & \left\{\begin{array}{ll} c(x) & \text{if } d(x,0) < \varepsilon \\ \infty & \text{otherwise} \end{array}\right. \end{array}\right] & \longmapsto & \left[\begin{array}{ccc} \mathbb{T}^V & \xrightarrow{\quad i_*(c) \quad} & S^V \\ x & \longmapsto & \left\{\begin{array}{ll} c(x) & \text{if } d(x,x_0) < \varepsilon \\ \infty & \text{otherwise} \end{array}\right. \end{array}\right]
\end{tikzcd}$$

is an isomorphism on Hopf degrees at Elmendorf stages $H_{>0}$ *of non-vanishing fixed space dimension and an injection on the unstable Hopf degree set at Elemendorf stages* $H_{=0}$ *with vanishing fixed subspace dimension:*

$$i_* : \left\{ \begin{array}{lll} N_{H_{=0}}(c) & \hookrightarrow & N_{H_{=0}}(i_*(c)) \\ N_{H_{>0}}(c) & \mapsto & N_{H_{>0}}(i_*(c)) \,. \end{array} \right.$$

This is illustrated by *Figure O*:

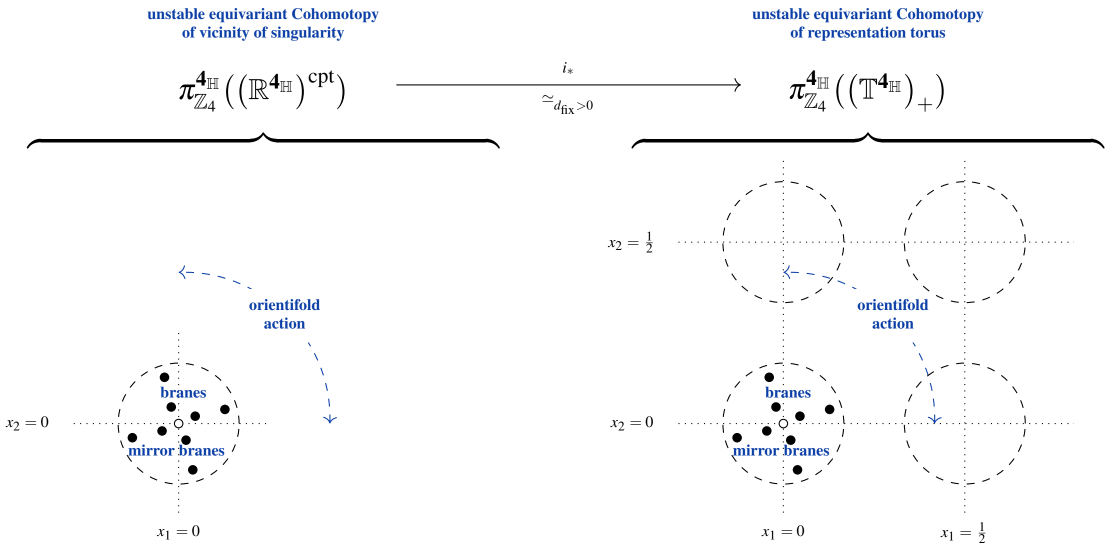


Figure O – Pushforward in equivariant Cohomotopy from the vicinity of a singularity to the full toroidal orientifold is an isomorphism on brane charges and an injection on O-plane charges, by Prop. 3.18. Shown is a case with $G = \mathbb{Z}_4$, as in *Figure M*. All integer number of branes (black dots) are in the image of the map, but only the O-plane at $(x_1, x_2) = (0,0)$ is in the image.

Local tadpole cancellation in toroidal ADE-orientifolds. Under the identification from Prop. 3.18, the stabilized equivariant Hopf degree theorem for representation spheres (Theorem 3.13) applies also to representation tori, and hence so does Prop. 3.14, showing now for the case of toroidal orbifolds with ADE-singularities that the brane charges classified by equivariant Cohomotopy are necessarily multiples of the regular representation. This result is visualized in *Figure P*:

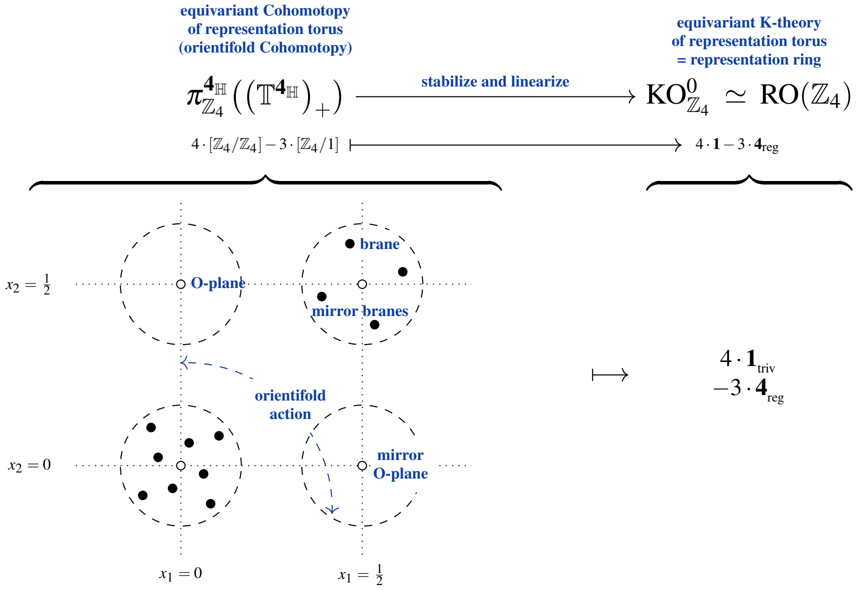


Figure P – Local/twisted tadpole cancellation in a toroidal ADE-orientifold is enforced by equivariant Cohomotopy according to Prop. 3.18, which reduces to the situation in the vicinity of a single singularity, as in §3.1. Shown is a case with $G = \mathbb{Z}_4$ as in *Figure O*.

This is the local/twisted tadpole cancellation in toroidal ADE-orentifolds according to *Table 1* and *Table 2*.

Global/untwisted tadpole cancellation from super-differential Cohomotopy. This concludes our discussion of local tadpole cancellation in global (i.e. toroidal) ADE-orientifolds implied by C-field charge quantization in equivariant Cohomotopy. Finally, we turn to discuss how the global/untwisted tadpole cancellation condition on toroidal orbifolds follows from charge quantization in super-differential equivariant Cohomotopy. We state the concrete condition below in (58), but first we explain how this condition arises from super-differential refinement:

Super-differential enhancement of unstable equivariant Cohomotopy theory. Given any generalized cohomology theory for charge quantization, it is its corresponding enhancement to a *differential cohomology theory* which classifies not just the topological soliton/instanton sectors, but the actual geometric higher gauge field content, hence including the flux densities. For stable/abelian cohomology theories this is discussed for instance in [Fre00][Bu12], while in the broader generality of unstable/non-abelian cohomology theories this is discussed in [FSS10][SSS09][FSS12][FSS15]. For example, ordinary degree-2 integral cohomology theory $BU(1) \simeq B^2\mathbb{Z}$ classifies magnetic charge sectors, but it is its differential cohomology enhancement $\mathbf{B}U(1)_{\mathrm{conn}}$ (Deligne cohomology) which is the universal moduli for actual electromagnetic field configurations. Similarly, plain (twisted) K-theory KU and KO classifies topological RR-charge sectors, but it is differential K-theory which classifies the actual RR-fields; see [GS17][GS19a][GS19b].

Hence with *Hypothesis H* we are ultimately to consider the refinement of ADE-equivariant Cohomotopy theory $\pi_G^{\mathbf{4}_{\mathbb{H}}}$, discussed so far, to some *differential* equivariant Cohomotopy theory, denoted $\big(\pi_G^{\mathbf{4}_{\mathbb{H}}}\big)_{\mathrm{conn}}$ and characterized as completing a homotopy pullback diagram of geometric unstable cohomology theories of the following form:

$$\begin{tikzcd}
\begin{array}{c}\text{super-differential}\\ \text{unstable equivariant Cohomotopy}\\ \text{[SS20] = [FSS15]} \wedge \text{[HSS18]}\end{array} & \big(\pi_G^{\bullet}\big)_{\mathrm{conn}} \arrow[r, "\begin{array}{c}\text{forget topological data,}\\ \text{retain only flux super-forms}\end{array}"] \arrow[d, "\begin{array}{c}\text{forget flux forms,}\\ \text{retain underlying cocycle in}\\ \text{plain equivariant Cohomotopy}\end{array}"'] \arrow[dr, phantom, "\begin{array}{c}\swarrow\!\!\!\swarrow\\ \text{universal homotopy}\end{array}"] & \Big\{ \big(\mu_{\mathrm{M2/M5}}\big)_G \Big\} \arrow[d, "\begin{array}{c}\text{inject this cocycle, thereby}\\ \text{enforce 11d SuGra torsion constraint}\end{array}"] & \begin{array}{c}G\text{-equivariant enhancement [HSS18, 5]}\\ \text{of M2/M5-brane super WZW-terms}\\ \text{jointly regarded as a cocycle in}\\ \text{super-rational 4-Cohomotopy}\\ \text{[FSS15, 3][FSS16a, 2.3][FSS19a, (57)]}\end{array} \\
\begin{array}{c}\text{unstable equivariant Cohomotopy}\\ \text{cohomology theory}\\ \text{(3)}\end{array} & \pi_G^{\bullet} \arrow[r, "\begin{array}{c}\text{rationalize, i.e.:}\\ \text{forget all torsion subgroups}\\ \text{in homotopy/cohomology groups}\end{array}"'] & \Omega_G\big(-, \mathfrak{l}S_G^4\big) & \begin{array}{c}\text{super-rational}\\ \text{unstable equivariant Cohomotopy theory}\\ \text{[HSS18, 3.2]}\end{array}
\end{tikzcd} \tag{55}$$

Discussing this construction $\big(\pi_G^{\mathbf{4}_{\mathbb{H}}}\big)_{\mathrm{conn}}$ in detail requires invoking concepts from ∞-stacks and L_∞-algebroids [FSS10][SSS09], as well as their application to super-geometric orbifolds [SS20], which is beyond the scope of this article. However, for the present purpose of seeing the global tadpole cancellation condition arise, all that matters are the following implications of super-differential refinement, which we make explicit by themselves:

Rational flux constraints from equivariant enhancement of M2/M5-cocycle. The homotopy pullback construction (55) amounts to equipping the rationalization of cocycles in plain unstable equivariant Cohomotopy (3) with equivalences (connection data) to prescribed flux super-forms in super-rational equivariant Cohomotopy theory [HSS18, 3.2]. The flux super-forms relevant for charge-quantization of the M-theory C-field according to *Hypothesis H* are G-equivariant enhancements of the joint M2/M5-brane cocycle [FSS15, 3][FSS16a, 2.3][HSS18, 3.42][FSS19a, (57)] with coefficients in the rationalized 4-sphere $\mathfrak{l}S^4$:

$$\underset{\substack{D = 11,\, \mathcal{N} = 1 \\ \text{super-Minkowski spacetime}}}{\mathbb{R}^{10,1|\mathbf{32}}} \xrightarrow[\substack{\text{M2/M5-brane super-cocycle} \\ \text{(joint M2/M5 WZW-term curvatures)}}]{\mu_{\mathrm{M2/M5}} := \left(\begin{array}{c} \frac{i}{2}\overline{\psi}\Gamma_{a_1 a_2}\psi \wedge e^{a_1} \wedge e^{a_2}\,, \\ \frac{1}{5}\overline{\psi}\Gamma_{a_1 \cdots a_5}\psi \wedge e^{a_1} \wedge \cdots \wedge e^{a_2} \end{array} \right)} \underset{\substack{\text{rationalized} \\ \text{4-sphere}}}{\mathfrak{l}S^4}\,. \tag{56}$$

Specifically, for G^{ADE}-equivariance (16) at ADE-singularities $\mathbf{4}_{\mathbb{H}}$ (17), a choice of equivariant extension of this cocycle is a choice of extension to an Elmendorf-stage diagram as in (36) – see [HSS18, 5]:[5]

$$\begin{tikzcd}
 & \left(\mathbb{R}^{10,1|\mathbf{32}}\right) \arrow[loop above, "G^{\mathrm{ADE}}"] \arrow[rr, "(\mu_{\mathrm{M2/M5}})_{G_{\mathrm{ADE}}}"] & & \left(\int S^{\mathbf{4}_{\mathbb{H}}}\right) \arrow[loop above, "G^{\mathrm{ADE}}"] & \text{G^{ADE}-equivariant enhancement of M2/M5-brane super-cocycle} & (57) \\
(-)^{H=1} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & & \int S^{\mathbf{4}_{\mathbb{H}}} & \text{M2/M5-brane super-cocycle (56)} & \\
\text{Elmendorf stages (32)} & & & & & \\
(-)^{H=G^{\mathrm{ADE}}} & \mathbb{R}^{6,1|\mathbf{16}} \arrow[u, hook] \arrow[rr, "\in\{0,1\}"] \arrow[d, phantom, "\text{MK6 super-embedding (see [HSS18, Thm. 4.3] and Rem. 4.7)}"] & & \int S^{0} \arrow[u, hook] & \text{charge at fixed planes} & \\
 & {} & & & &
\end{tikzcd}$$

This involves a binary choice at lowest (and hence any other, by Example 3.4) Elmendorf stage. The homotopy in the diagram (55) enforces this local choice of rationalized flux globally onto the rationalized fluxes of the equivariant Cohomotopy cocycles. This has two effects:

1. Super-differential enhancement at global Elmendorf stage implies vanishing total flux. Note the M2/M5-brane super-cocycle $\mu_{\mathrm{M2/M5}}$ (56) appearing at global Elmendorf stage in (57) has vanishing bosonic flux ($\mu_{\mathrm{M2/M5}}|_{\psi=0} = 0$ by (56)). Also, the infinitesimal fermionic component ψ does not contribute to the topology seen by plain equivariant Cohomotopy (see [SS20] for details). Hence the homotopy in (55) forces the underlying classes in plain equivariant Cohomotopy to be *pure torsion* at global Elmendorf stage. But, since in compatible RO-degree (as in Example 3.8) the Hopf degree theorem (13) implies non-torsion Cohomotopy groups at all positive Elmendorf stages (36), this means that super-differential refinement (55) of equivariant Cohomotopy in compatible RO-degree enforces *vanishing* Hopf degrees at global Elmendorf stage $H = 1$ (36).

Explicitly, this means that the super-differential enhancement (55) forces the underlying plain equivariant Cohomotopy cocycles of ADE-orientifolds in compatible RO-degree to be in the kernel of the forgetful map $(-)^1$ (36) from equivariant to ordinary Cohomotopy, which projects out the global Elmendorf stage at $H = 1$:

$$\begin{tikzcd}
\underset{\substack{\text{“super-differentiable”}\\ (55)}}{\overset{\substack{\text{unstable equivariant Cohomotopy}\\ \text{admitting super-differential refinement}}}{\pi^{\mathbf{4}_{\mathbb{H}}}_{G^{\mathrm{ADE}}}\left(\left(\mathbb{T}^{\mathbf{4}_{\mathbb{H}}}\right)_+\right)_{\mathrm{Sdiffble}}}} \arrow[rrr] \arrow[d, hook, "\text{kernel}"'] \arrow[drr, phantom, "\text{(pb)}"] & & & \{0\} \arrow[d, hook] & (58) \\
\underset{\substack{\text{equivariant Cohomotopy (3)}\\ \text{of toroidal orbifold (23)}\\ \text{with ADE-singularities (17)}}}{\pi^{\mathbf{4}_{\mathbb{H}}}_{G^{\mathrm{ADE}}}\left(\left(\mathbb{T}^{\mathbf{4}_{\mathbb{H}}}\right)_+\right)} \arrow[rr, "|Q_{\mathrm{tot}}|:=(-)^1", "\substack{\text{project out}\\ \text{global charge =}\\ \text{Hopf degree at global Elmendorf stage}\\ (36)}"'] & & \underset{\substack{\text{plain Cohomotopy}\\ \text{of plain 4-torus}\\ (5)}}{\pi^{4}\left(\left(\mathbb{T}^{4}\right)_+\right)} \arrow[r, phantom, "\simeq"] & \underset{\substack{\text{global Hopf degree}\\ \text{= net brane/O-plane charge}\\ (13)}}{\mathbb{Z}} &
\end{tikzcd}$$

It is now immediate, from Theorem 3.10 and Theorem 3.17, that this enforces the condition of vanishing net brane/O-plane charge, precisely in the form of the global/untwisted tadpole cancellation condition from *Table 1* and *Table 2* in the way illustrated in *Figure A*.

2. Super-differential enhancement at lower Elmendorf stage implies choice of O-plane charge. The globalization via (58) of the lower S^0-valued Elmendorf stage in the equivariantized M2/M5-brane cocycle (57) means to impose the chosen charge $\in \{0,1\}$ to all O-planes, via Prop. 3.14 as illustrated in *Figure H*. We will denote the ADE-equivariant Cohomotopy sets which admit super-differential refinement with the choice $-1 \in \{0,1\}$ in (57) by a subscript $(-)_{-}$:

---

[5] For more general actions this involves extension to a functor on the *orbit category*; see [HSS18, Lemma 5.4].

Example 3.19 (Super-differentiable equivariant Cohomotopy of ADE-orbifolds). Locally, the super-differentiable equivariant Cohomotopy of the vicinity of an ADE-singularity (*Table 5*) with respect to the choice $-1 \in \{-0, -1\}$ in the equivariant enhancement (57) of the super-flux form (56) is

$$\pi^{\mathbf{4}_{\mathbb{H}}}_{G^{\mathrm{ADE}}}\Big(\big(\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}\big)^{\mathrm{cpt}}\Big)_{-} \;=\; \Big\{ \underset{\substack{\textbf{local charge structure}\\ \textbf{(Prop. }(3.14))}}{1\cdot \mathbf{1}_{\mathrm{triv}} - N_{\mathrm{brane}}\cdot \mathbf{k}_{\mathrm{reg}}} \;\Big|\; N_{\mathrm{brane}} \in \mathbb{Z} \Big\}. \tag{59}$$

Globally, the super-differentiable equivariant Cohomotopy specifically of the Kummer surface ADE-orbifold $\mathbb{T}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}$ (Example 3.6) is

$$\underset{\substack{\textbf{ADE-equivariant Cohomotopy}\\ \textbf{admitting super-differential lift}\\ (55)}}{\pi^{\mathbf{4}_{\mathbb{H}}}_{\mathbb{Z}_2^{\mathrm{refl}}}\Big(\big(\mathbb{T}^{\mathbf{4}_{\mathbb{H}}}\big)_{+}\Big)_{\mathrm{Sdiffble}_-}} \;=\; \Big\{ \underset{\substack{\textbf{super-differentiability}\\ \textbf{at low Elmendorf stage}\\ (57)}}{16\cdot \mathbf{1}_{\mathrm{triv}}} \;-\; \underset{\substack{\textbf{local charge structure}\\ \textbf{(Prop. 3.14, Prop. 3.18)}}}{N_{\mathrm{brane}}\cdot \mathbf{2}_{\mathrm{reg}}} \;\Big|\; \underset{\substack{\textbf{super-differentiablity}\\ \textbf{at global Elmendorf stage}\\ (58)}}{2N_{\mathrm{brane}} - 16 = 0} \Big\}. \tag{60}$$

4 M5/MO5 anomaly cancellation

We now apply the general discussion of equivariant Cohomotopy in §3 to cohomotopical charge quantization of the M-theory C-field, according to *Hypothesis H*, for compactifications of heterotic M-theory on toroidal orbifolds with ADE-singularities. The resulting M5/MO5-anomaly cancellation is discussed in §4.2 below. In order to set the scene and to sort out some fine print, we first discuss in §4.1 relevant folklore regarding heterotic M-theory on ADE-orbifolds.

4.1 Heterotic M-theory on ADE-orbifolds

We now explain how the singularity structure (as in *Table 5*), which must really be meant when speaking of MO5-planes (61) coinciding with black M5-branes (62), is that of "$\frac{1}{2}$M5-branes" (65) [HSS18, 2.2.7][FSS19d, 4]; see *Figure S* below. This singularity structure goes back to [Sen97, 3] with further discussion and development in [FLO99][KSTY99][FLO00a][FLO00b][FLO00c][CHS19]; the type IIA perspective is considered in [GKST01] and also briefly in [KS02, p. 4]. We highlight the systematic picture behind the resulting *heterotic M-theory on ADE-orbifolds* and its string theory duals, further below in *Table 7*.

Critique of pure MO5-planes. We highlight the following:

- **(i)** Seminal literature on M-theoretic orientifolds speaks of M5-branes parallel and/or coincident to *MO5-singularities* [DM95][Wi95b, 3.3][Ho98, 2.1], namely to Euclidean $\mathbb{Z}_2$-orientifolds (19) of the form (see [HSS18, 2.2.2]):

$$\text{MO5} \qquad \mathbb{R}^{5,1} \hookrightarrow \mathbb{R}^{5,1} \times \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}}}\;, \tag{61}$$

where $\mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}}$ is the Euclidean singularity (19) of the 5-dimensional sign representation of the group $\mathbb{Z}_2$.

- **(ii)** But $1/2$BPS M5-brane solutions of $D = 11$ supergravity themselves have been classified [dMFO10, 8.3] and found to be given, in their singular far horizon limit [AFCS99, 3], by singularities for finite subgroups $G^{\mathrm{ADE}} \subset \mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ (16) of the type

$$\text{M5} \qquad \mathbb{R}^{5,1} \hookrightarrow \mathbb{R}^{5,1} \times \mathbb{R}^{1} \times \overset{G^{\mathrm{ADE}} \circlearrowright}{\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}}\;, \tag{62}$$

where the last factor is an ADE-singularity (17).

- **(iii)** As orbifold singularities, this coincides with the far horizon geometry of coincident KK-monopole solutions to 11d supergravity (e.g. [IMSY98, (47)][As00, (18)]; see [HSS18, 2.2.5])

$$\text{MK6} \qquad \mathbb{R}^{6,1} \hookrightarrow \mathbb{R}^{6,1} \times \overset{G^{\mathrm{ADE}} \circlearrowright}{\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}}\;, \tag{63}$$

which, from the perspective of type IIA theory, reflects the fact that NS5-branes are domain walls inside D6-branes (e.g. [EGKRS00, p. 5], see [Fa17, 3.3.1. 3.3.2]). This is illustrated by the central dot on the vertical axis in *Figure S*. Hence for the special case that $G^{\mathrm{ADE}} = \mathbb{Z}_2^{\mathrm{refl}}$ (18), this yields the product $\mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}_{\mathrm{sgn}}}$ of the 4-dimensional sign representation with the trivial 1-dimensional representation, instead of the 5-dimensional sign representation in (61).

(iv) In order to allow M5-singularities (62) to coincide with MO5-singularities (61) we have to consider intersecting a 1/2BPS 5-brane solution with an MO9 locus fixed by a Hořava-Witten involution $\mathbb{Z}_2^{\mathrm{HW}}$ ([HW95], see [HSS18, 2.2.1]):

$$\begin{tikzcd} \mathrm{MO9} & \mathbb{R}^{9,1} \arrow[r, hook] & \mathbb{R}^{9,1} \times \mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}} \arrow[loop above, "\mathbb{Z}_2^{\mathrm{HW}}"] . \end{tikzcd} \tag{64}$$

(v) This intersection is called the $\frac{1}{2}$M5 in [HSS18, 2.2.7][FSS19d, 4]

$$\begin{tikzcd} \tfrac{1}{2}\mathrm{M5} = \mathrm{MK6} \cap \mathrm{MO9} & \mathbb{R}^{5,1} \arrow[r, hook] & \mathbb{R}^{5,1} \times \mathbb{T}^{\mathbf{1}_{\mathrm{sgn}}} \arrow[loop above, "\mathbb{Z}_2^{\mathrm{HW}}"] \times \mathbb{T}^{\mathbf{4}_{\mathbb{H}}} \arrow[loop above, "\times\; G^{\mathrm{ADE}}"] \end{tikzcd} \tag{65}$$

since its type IIA incarnation is known as the $\frac{1}{2}$NS5 [GKST01, 6][AF17, p. 18]. This is the brane configuration thought to geometrically engineer $D = 6$, $\mathcal{N} = (1,0)$ field theories [HZ97][HKLY15][DHTV14, 6].

Since the fixed point set of the toroidal orbifolds (23) for both the $\frac{1}{2}$M5 (65) and the MO5 (61) is the same set (27) of 32 points, all arguments about MO5 (61) which depend only on the set of isolated orientifold fixed points, such as in [DM95][Wi95b, 3.3][Ho98, 2.1], apply to $\frac{1}{2}$M5 (65) as well. But the $\frac{1}{2}$M5 orientifold has in addition fixed lines, namely the MK6 loci, and fixed 4-planes, namely the MO9, as shown on the right of Figure S. This reflects the fact that, by the classification of [dMFO10, 8.3], the black M5 not only may, but must appear as a domain wall inside an MK6 singular locus.

We **conclude** from this that: *The* $\frac{1}{2}$M5 (65) *orientifold is the correct model of orientifolded M5/MO5 geometry, while the pure* MO5 (61) *is just its restriction along the diagonal subgroup inclusion* (66)*, as shown in Figure R*

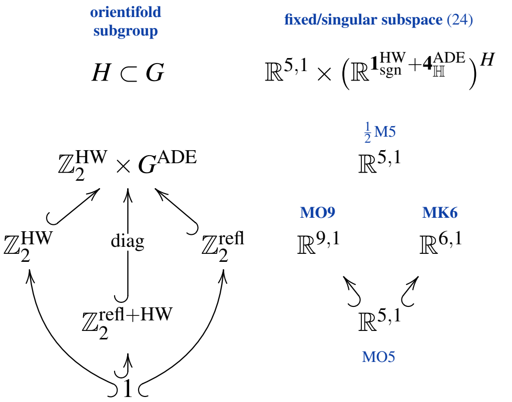


Figure R – Fixed subspaces in the $\frac{1}{2}$M5-singularity (61) with MO5 (61) in the intersection of MK6 (63) with MO9 (64), illustrated in *Figure S*.

$$\begin{tikzcd} \underset{(61)}{\mathbb{Z}_2^{\mathrm{refl+HW}}} \arrow[r, hook, "\mathrm{diag}"] & \underset{(64)}{\mathbb{Z}_2^{\mathrm{HW}}} \times \underset{(18)}{\mathbb{Z}_2^{\mathrm{refl}}} \arrow[r, hook] & \underset{(64)}{\mathbb{Z}_2^{\mathrm{HW}}} \times \underset{(16)}{G^{\mathrm{ADE}}} . \end{tikzcd} \tag{66}$$

In summary, this data arranges into a short exact sequence of orbi-/orienti-fold group actions (as in [DFM11, p. 4])

$$\begin{tikzcd}[column sep=small] 1 \arrow[r] & G^{\mathrm{ADE}} \arrow[r, hook, "\text{index-2 subgroup}"] & \overbrace{\mathbb{Z}_2^{\mathrm{HW}}}^{\{\mathrm{e},\sigma\}} \times G^{\mathrm{ADE}} \arrow[r, "{\substack{(\mathrm{e},q) \mapsto (\mathrm{e},+q) \\ (\sigma,q) \mapsto (R,-q)}}", "\simeq"'] & \overbrace{\mathbb{Z}_2^{\mathrm{HW+refl}}}^{\{\mathrm{e},R\}} \times G^{\mathrm{ADE}} \arrow[r, two heads] & \mathbb{Z}_2^{\mathrm{HW+refl}} \arrow[r] & 1 \\ & \textbf{orbifold} & \multicolumn{2}{c}{\text{——— \textbf{orbi-orientifold} ———}} & \textbf{orientifold} & \\ & \mathbb{R}^{\mathbf{1}_{\mathrm{triv}} + \mathbf{4}_{\mathbb{H}}} & \mathbb{R}^{\mathbf{1}_{\mathrm{sgn}} + \mathbf{4}_{\mathbb{H}}} & \mathbb{R}^{\mathbf{5}} & \mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}} & \end{tikzcd} \tag{67}$$

This situation is illustrated by the following figure:

Orientifold	**MO5**	$\frac{1}{2}$**M5**
Global quotient group $G =$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\mathrm{HW}} \times G^{\mathrm{ADE}}$
Global quotient group action $\overset{G}{\overset{\curvearrowright}{\mathbb{T}^V}} =$	$\overset{\mathbb{Z}_2}{\overset{\curvearrowright}{\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}}}}$	$\overset{\mathbb{Z}_2^{\mathrm{HW}}}{\overset{\curvearrowright}{\mathbb{T}^{\mathbf{1}_{\mathrm{sgn}}}}} \times \overset{G^{\mathrm{ADE}}}{\overset{\curvearrowright}{\mathbb{T}^{\mathbf{4}_{\mathrm{sgn}}}}}$
Fixed/singular points $\left(T^V\right)^G =$	$\{0, \frac{1}{2}\}^5 = \overline{32}$	
Far horizon-limit of M5 SuGra solution?	no	yes

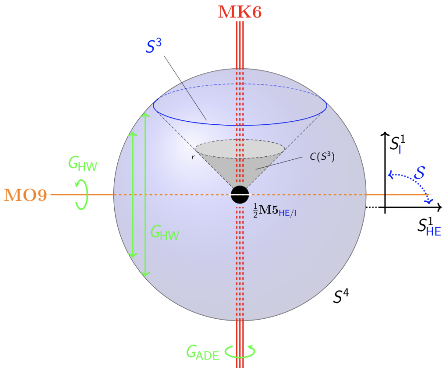


Figure S – Singularity structure of heterotic M-theory on ADE-singularities, as in Figure R, [HSS18, 2.2.2, 2.2.7]. The corresponding toroidal orbifolds (as per *Table 5*) are illustrated in *Figure V* and *Table 8*.

O^0-planes and M2-brane CS level. There is one more ingredient to the G-space structure of heterotic M-theory on ADE-orbifolds (see *Table 7* below for the full picture): While the MO5-planes (61) are supposed to be the M-theory lifts of the charged $\mathrm{O4}^{\pm}$-planes [Ho98, 3][Gi98, III.A][HK00, 3.1.1], the M-theory lift of the un-charged $\mathrm{O4}^0$-planes (see *Figure OP*) involves one more group action on spacetime [Gi98, III.B], being rotation of the circle fiber in M/IIA-duality, which we hence indicate as follows:

$$\mathrm{IIA}^{\mathbf{0}} \qquad \mathbb{R}^{9,1} \times \varnothing \hookrightarrow \mathbb{R}^{9,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{S^1}} . \tag{68}$$

Here on the right we have the circle regarded as a $\mathbb{Z}_k$-space (§3) via rigid rotation by multiples of $2\pi/k$, for any $k \in \mathbb{Z} \setminus \{0\}$. This is of course a free action (in particular, not a representation sphere (21)) hence with empty fixed subspace (24), whence the superscript $(-)^0$ and the empty set $\varnothing$ of fixed points in (68). But passing along the unique $\mathbb{Z}_k^{\mathrm{rot}}$-equivariant function (31)

$$\begin{tikzcd} \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{S^1}} \arrow[r, "\mathrm{KK}_{S^1_{\mathrm{rot}}}", "\text{KK-reduction on } S^1_{\mathrm{rot}}"'] & \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{*}} \end{tikzcd} \tag{69}$$

from the circle to the point $*$ with its necessarily trivial $\mathbb{Z}_k^{\mathrm{rot}}$-action, as befits KK-reduction from M-theory to type IIA string theory (see [BMSS18] for discussion in the context of *Hypothesis H*), we obtain a non-empty fixed subspace:

$$\mathrm{IIA} \qquad \mathbb{R}^{9,1} \hookrightarrow \mathbb{R}^{9,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{*}} . \tag{70}$$

In these terms, we may phrase the core of M/IIA duality as saying that

The lift of IIA (70) *through* $\mathrm{KK}_{S^1_{\mathrm{rot}}}$ (69) *is* IIA^0 (68).

Notice in the case that the global 11d-spacetime is AdS_3 times S^7 regarded as an S^1_{rot}-fibration

$$\begin{tikzcd} \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{S^1}} \arrow[r] & \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{S^7}} \arrow[r] & \mathbb{C}P^3 \end{tikzcd}$$

the order k of $\mathbb{Z}_k^{\mathrm{rot}}$ in (68) is the level of the dual 3d Chern-Simons-matter theory [ABJM08].

The argument in [Gi98, III.B], together with our discussion above, suggests that the analogous statement for $\mathrm{O4}^0$-planes is this:

The lift of $\mathrm{O4}^0$ *through* $\mathrm{KK}_{S^1_{\mathrm{rot}}}$ (69) *is* $\mathrm{MO5}^0$ (71) .

Hence we take $\mathrm{MO5}^0$ to be the following G-space/orbifold, combining MO5 (61) with IIA^0 (68):

$$\mathrm{MO5}^{\mathbf{0}} \qquad \mathbb{R}^{4,1} \times \varnothing \hookrightarrow \mathbb{R}^{4,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}}}{\overset{\curvearrowright}{S^1}} \times \overset{\mathbb{Z}_2^{\mathrm{refl+HW}}}{\overset{\curvearrowright}{\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}}}} . \tag{71}$$

As before in (68), the fixed subspace of the diagonal group action (now for $k = 2$, as in [Gi98, (3.2)])

$$\mathbb{Z}_2^{\mathrm{refl+rot+HW}} \xhookrightarrow{\;\mathrm{diag}\;} \mathbb{Z}_2^{\mathrm{refl+HW}} \times \mathbb{Z}_2^{\mathrm{rot}}$$

in (71) is actually empty, since the action of $\mathbb{Z}_2^{\mathrm{rot}}$ and hence that of $\mathbb{Z}_2^{\mathrm{refl+HW+rot}}$ is free, whence the superscript $(-)^0$. But, as before in (70), under M/IIA KK-reduction (69) we have an equivariant projection map to the orbifold

$$\mathrm{O4}^0 \qquad \mathbb{R}^{4,1} \hookrightarrow \mathbb{R}^{4,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}} \circlearrowright}{*} \times \overset{\mathbb{Z}_2^{\mathrm{refl+HW}} \circlearrowright}{\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}}} \;, \tag{72}$$

with non-empty fixed/singular subspace being the O4-worldvolume – which is thereby exhibited as being uncharged, as its lift to M-theory in in fact non-singular.

In the same manner, there is the analogous $\mathbb{Z}_k^{\mathrm{rot}}$-resolution of the MK6-singularity (63)

$$\mathbf{MK6^0} \qquad \mathbb{R}^{6,1} \times \varnothing \hookrightarrow \mathbb{R}^{5,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}} \circlearrowright}{S^1} \times \overset{G^{\mathrm{ADE}} \circlearrowright}{\mathbb{T}^{\mathbf{4}_{\mathbb{H}}}} \;, \tag{73}$$

as well as of the MO9-singularity (64):

$$\mathbf{MO9^0} \qquad \mathbb{R}^{9,1} \times \varnothing \hookrightarrow \mathbb{R}^{8,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}} \circlearrowright}{S^1} \times \overset{\mathbb{Z}_2^{\mathrm{HW}} \circlearrowright}{\mathbb{T}^{\mathbf{1}_{\mathrm{sgn}}}} \;. \tag{74}$$

The reduction of the latter along $\mathrm{KK}_{S^1_{\mathrm{rot}}}$ (69) is

$$\mathbf{O8^0} \qquad \mathbb{R}^{8,1} \hookrightarrow \mathbb{R}^{8,1} \times \overset{\mathbb{Z}_k^{\mathrm{rot}} \circlearrowright}{*} \times \overset{\mathbb{Z}_2^{\mathrm{HW}} \circlearrowright}{\mathbb{T}^{\mathbf{1}_{\mathrm{sgn}}}} \;. \tag{75}$$

In summary, *the full singularity structure of heterotic M-theory on ADE-orbifolds*, such as to admit

(i) black M5-branes coinciding with MO5-planes and

(ii) the MO5^0-lift of O4^0-planes

is as shown in *Table 7*.

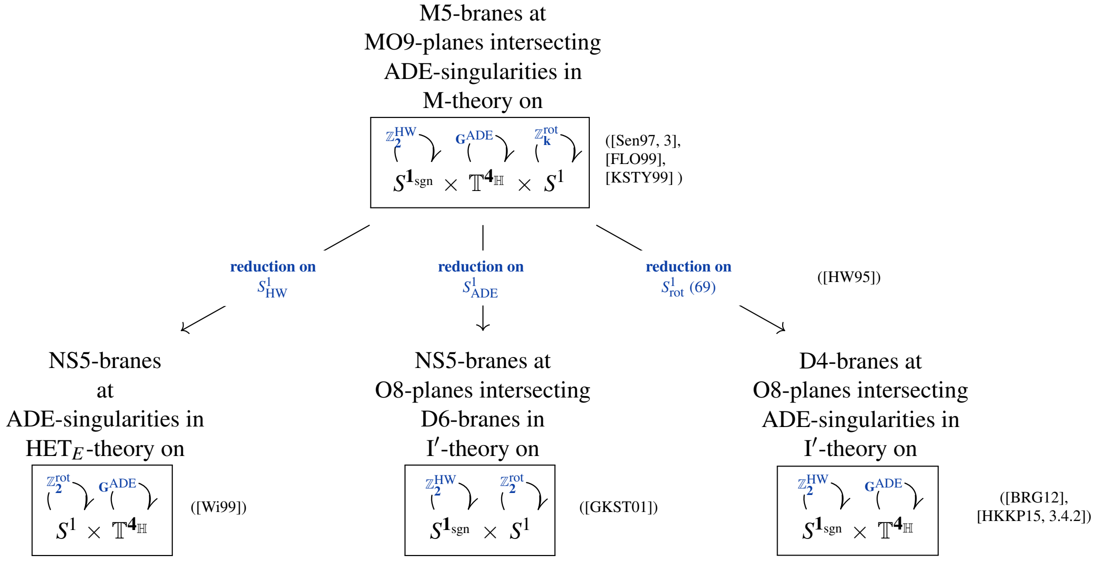


Table 7 – Singularity structure of heterotic M-theory on ADE-orbifolds and its string theory duals given by combining the $\frac{1}{2}$M5-structure of *Figure S* with IIA^0-structure (68), hence admitting also MO5^0-structure (71).

The following *Figure T* shows the corresponding subgroup lattice with its associated fixed/singular spaces:

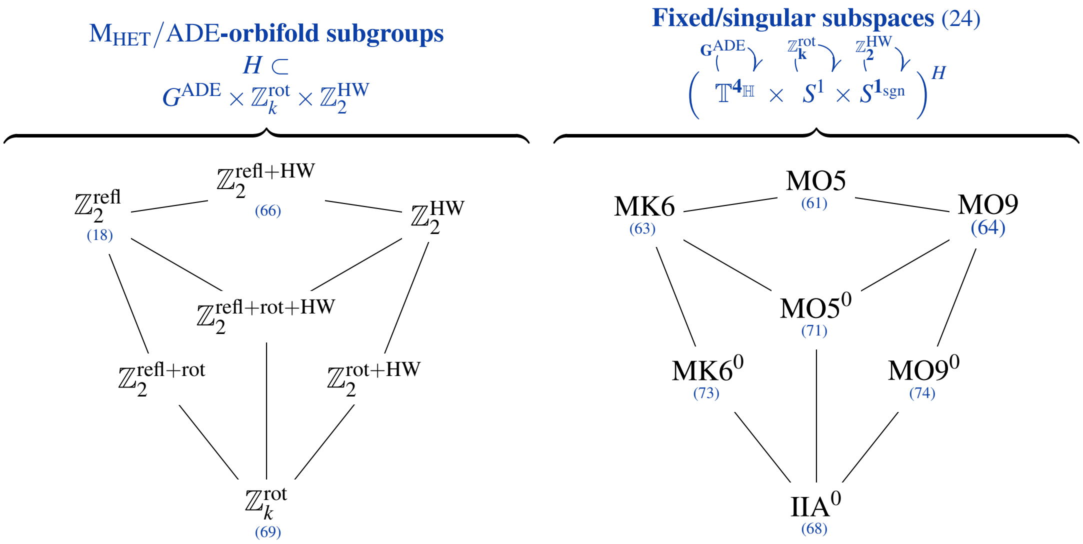


Figure T – Subgroup lattice and fixed/singular subspaces in the singularity structure for heterotic M-theory on ADE-orbifolds from *Table 7*. On the left, groups associated to the middle of a sub-simplex are diagonal subgroups inside the direct product of subgroups associated to the vertices, as indicated by the superscripts. On the right, all fixed loci with superscript $(-)^0$ are actually empty, but appear as superficially non-empty (un-charged) singularities after M/IIA KK-reduction (69), e.g. $\mathrm{O4}^0$ (72), $\mathrm{O8}^0$ (75), as on the right of *Figure OP*. The numbered subscripts (xx) indicate the corresponding expression in the text.

4.2 Equivariant Cohomotopy charge of M5 at $\mathrm{MO5_{ADE}}$

Applying the general mathematical results of §3 to the $\mathrm{M_{HET}}/\mathrm{ADE}$-singularities from §4.1, we finally show here (see *Figure V*) that *Hypothesis H* formalizes and validates the following widely accepted but informal Folklore 4.1, concerning the nature of M-theory:

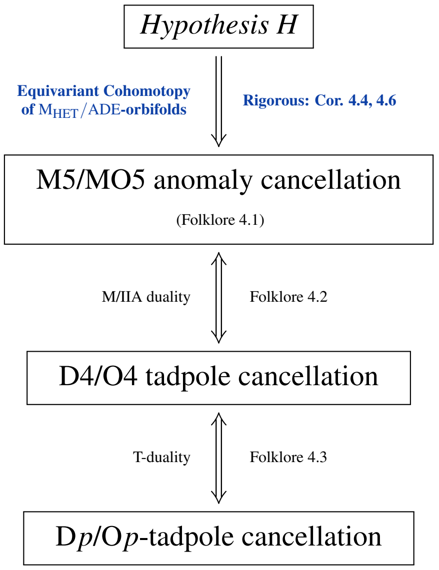


Figure U – Structure of the argument. We demonstrate that *Hypothesis H* on C-field charge quantization in Cohomotopy, applied to heterotic M-theory on toroidal ADE-oribolds, implies M5/MO5-anomaly cancellation in M-theory. This directly subsumes and implies the statement of tadpole cancellation for D4/O4 branes in string theory.

Folklore 4.1 (M5/MO5 anomaly cancellation [DM95][Wi95b, 3.3][Ho98, 2.1])**.** *For M-theory on the toroidal orientifold* $\mathbb{R}^{5,1} \times \mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}} /\!\!/ \mathbb{Z}_2$ *(Table 5) with MO5-singularities* (61)*, consistency requires the situation shown in Table 2:*

- **(i)** *a charge of* $q_{\mathrm{MO5}}/q_{\mathrm{M5}} = -1/2$ *is carried by each of the fixed/singular MO5-planes* (61)*;*
- **(ii)** *the M5-brane charge is integral in natural units, hence on the covering* $\mathbb{Z}_2$*-space* $\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}}$ *the M5-branes appear in* $\mathbb{Z}_2$*-mirror pairs around the MO5-planes, as in Figure L and Figure N;*
- **(iii)** *the total charge of the* N_{M5} *M5-branes has to cancel that of the 32 O-planes* (27)*,* $N_{\mathrm{M5}} q_{\mathrm{M5}} + 32 q_{\mathrm{MO5}} = 0$*, as indicated in Figure A.*

Via the similarly widely accepted Folklore 4.2, the statement of Folklore 4.1 implies tadpole anomaly cancellation in string theory. Notice that this is not so much a claim than part of the defining criterion for M-theory:

Folklore 4.2 (Double dimensional reduction of M5/MO5 to D4/O4 [Ho98, 3][Gi98, III.A][HK00, 3.1.1])**.** *Under M/IIA duality, the situation of Folklore 4.1 becomes the string-theoretic tadpole cancellation condition from Table 1 for D4-branes and* $\mathrm{O4}^-$*-planes.*

Folklore 4.3 (T-duality relating O-planes, e.g. [BLT13, p.317-318])**.** *By iterative T-duality, the situation of Folklore 4.2 implies general tadpole cancellation for* Dp*-branes and* Op^-*-planes (Table 3).*

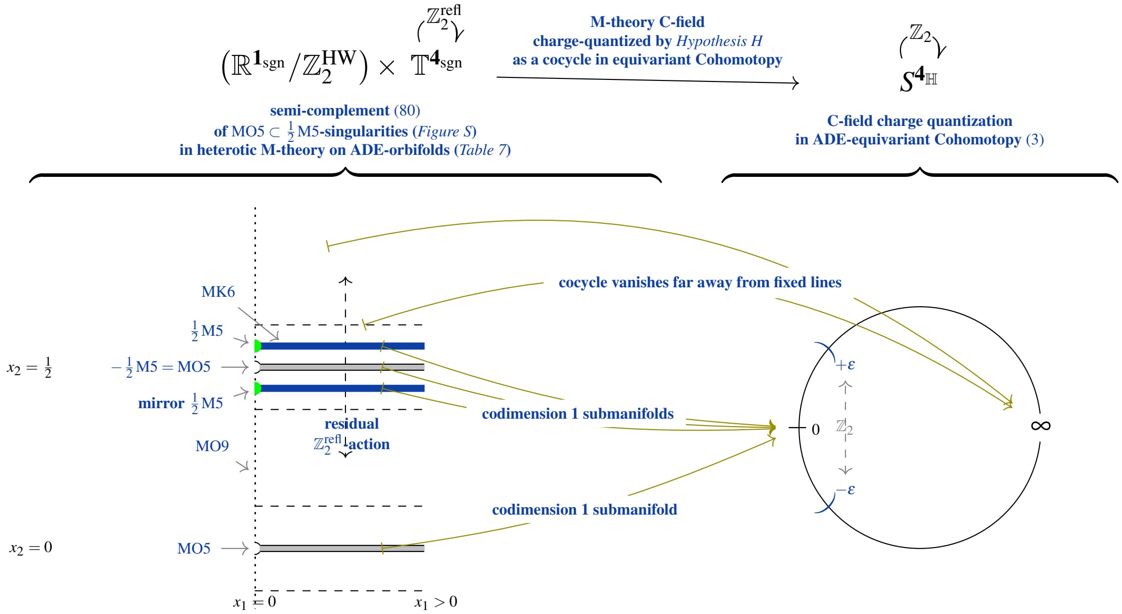


Figure V – Equivariant Cohomotopy of ADE-orbifolds in heterotic M-theory with singularity structure as in *Figure S*. The resulting charge classification (Cor. 4.4) implies, via the unstable PT isomorphism (§2.1), the $\frac{1}{2}$M5 = MO9 ∩ MK6-brane configurations (65) similarly shown in [FLO99, Fig. 1][KSTY99, p. 7][FLO00a, Fig. 1][FLO00b, Fig. 2][FLO00c, Fig. 1][GKST01, p. 4, 68, 71]. This is as in *Figure L* but with points (M5s) extended to half-line (MK6s), see Remark 4.7 and *Table 8*.

C-Field flux quantization at pure MO5-Singularities. To put the discussion below in perspective, it is instructive to first recall the success and the shortcoming of the existing argument [Ho98, 2] for M5/MO5-brane charge quantization around a *pure* MO5-singularity (61) (see the left column of*Table 8*): Following the classical argument of [Dir31], we consider removing the locus of the would-be M5-brane from spacetime and then computing the appropriate cohomology of the remaining complement. For the *pure* MO5-singularity (61) the complement spacetime is, up to homotopy equivalence, the 4-dimensional real projective space:

$$\underset{\substack{\text{complement spacetime}\\ \text{around pure MO5-singularity}}}{X^{11}_{\mathrm{MO5}}} \;=\; \underset{\substack{\text{full Euclidean orientifold (19)}\\ \text{with pure MO5-singularity (61)}}}{\overset{\substack{\text{pure (66)}\\ \text{orbifold quotient}}}{\Big(\mathbb{R}^{5,1}\times\mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}}/\!\!/\mathbb{Z}_2^{\mathrm{refl+HW}}\Big)}} \setminus \underset{\substack{\mathbb{Z}_2^{\mathrm{refl+HM}}\text{-fixed}\\ \text{subspace (24)}}}{\big\{\mathbb{R}^{5,1}\times\{0\}\big\}} \;\simeq_{\mathrm{homotopy}}\; \underset{\substack{\text{unit 4-sphere}\\ \text{around MO5}}}{S\big(\mathbb{R}^{\mathbf{5}_{\mathrm{sgn}}}\big)}\underset{\substack{\text{pure (66)}\\ \text{MO5-quotient}}}{/\mathbb{Z}_2^{\mathrm{refl+HW}}} \;\simeq\; \underset{\substack{\text{real projective}\\ \text{4-space}}}{\mathbb{R}P^4}. \tag{76}$$

Since this ambient spacetime (76) is a smooth but curved (i.e. non-parallelizable) manifold, the flavor of Cohomotopy theory that measures its M-brane charge, according to *Hypothesis H*, is, according to *Table 4*, the *J*-twisted Cohomotopy theory of [FSS19b, 3]. This implies, by [FSS19b, Prop. 4.12], that rationalized brane charge (bottom of (55)) is measured by the integral of a differential 4-form $G_4 \in \Omega^4\big(X^{11}\big)$ (the C-field 4-flux density) which satisfies the half-integral shifted flux quantization condition

$$[G_4] + \big[\tfrac{1}{4}p_1\big] \;\in\; H^4\big(X^{11},\mathbb{Z}\big) \to H^4\big(X^{11},\mathbb{R}\big) \tag{77}$$

as is expected from the M-theory folklore (recalled in [FSS19b, 2.2]). Applying this to the complement X^{11}_{MO5} (76) around a pure MO5-plane implies, as pointed out in [Ho98, 2.1], that there must be an *odd integer* of brane charge in the pure MO5-spacetime

$$\underset{\substack{J\text{-twisted Cohomotopy ([FSS19b, 3.1])}\\ \text{of pure MO5-complement (76)}}}{\pi^{\overset{\text{twist}}{T\mathbb{R}P^4}}\big(X^{11}_{\mathrm{MO5}}\big)_{\mathbb{R}}} \;=\; \underset{\substack{\text{due to half-integral } G_4\text{-flux quantization (77)}\\ \text{implied by twisted Cohomotopy [FSS19b, Prop. 4.12]}}}{\left\{ 2\int_{\mathbb{R}P^4} G_4 \;=\; \overset{\substack{\text{odd integer}\\ \text{net charge}}}{1-2N} \;\middle|\; N\in\mathbb{N}\right\}} \;. \tag{78}$$

The need to resolve further microscopic details. If one could identify in (78) the offset of 1 mod 2 in (78) with the charge carried by the pure MO5-plane (61), and the remaining even charge $2N$ with that of N M5-branes in its vicinity

$$1 - N \cdot 2 \overset{?}{=} Q_{\mathrm{MO5}} - N_{\mathrm{brane}} \cdot Q_{\mathrm{M5}} \tag{79}$$

this would be the local/twisted M5/MO5-anomaly cancellation condition of *Table 2*. Without such further information, the charge quantization (78) around *pure* MO5-planes (61) is only *consistent with* the local/twisted M5/MO5-anomaly cancellation from *Table 2*, as noticed in [Ho98, bottom of p. 5].

But with the results of §3 and in view of §4.1, we may now complete this old argument (see the right column of *Table 8*):

Equivariant Cohomotopy implies local/twisted M5/MO5-anomaly cancellation at $\frac{1}{2}$M5-singularities. We know from §3.1 that the identification (79) missing from the result (78) for twisted Cohomotopy on smooth but curved spacestimes *is* implied by the result of Prop. 3.14 for equivariant Cohomotopy of singular but flat spacetimes. Moreover, we have argued in §4.1 that having black M5-branes actually coinciding with MO5-planes requires/implies that the pure MO5-planes are but the diagonally fixed sub-loci (shown in *Table 6*) inside the richer $\frac{1}{2}\mathrm{M5} = \mathrm{MK6} \cap \mathrm{MO9}$-singularities (65) of heterotic M-theory on ADE-orbifolds (*Figure S*). Hence for a rigorous M5/MO5-anomaly cancellation result not just consistent with (as in (78)), but actually *implying* Folkore 4.1, we need to compute the M-brane charge at MO5-singularities inside $\frac{1}{2}$M5-singularities (65). Concretely, this means with *Hypothesis H* that M5/MO5-charge at a single MO5-singularity is measured by the equivariant Cohomotopy of the following $\frac{1}{2}$M5-refinement of the naive MO5-complement spacetime (76):

$$\begin{aligned}
\underset{\substack{\textbf{semi-complement spacetime}\\ \textbf{around MO5 in } \frac{1}{2}\textbf{M5-singularity}}}{X^{11}_{\frac{1}{2}\mathrm{M5}}} \quad &:= \quad \underset{\substack{\textbf{full Euclidean orientifold (19)}\\ \textbf{with } \frac{1}{2}\textbf{M5-singularity (65)}}}{\Big(\mathbb{R}^{5,1} \times \mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}} \times \mathbb{R}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{HW}} \times \mathbb{Z}_2^{\mathrm{refl}}\Big)} \setminus \underset{\substack{\mathbb{Z}_2^{\mathrm{HW}}\textbf{-fixed subspace (24)}\\ \textbf{with residual } \mathbb{Z}_2^{\mathrm{het}}\textbf{-action (28)}}}{\Big(\mathbb{R}^{5,1} \times \{0\} \times \mathbb{R}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}\Big)} \\
\underset{\text{homotopy}}{\simeq} \quad & \underbrace{\underset{\substack{\textbf{unit 0-sphere}\\ \textbf{around MO9}}}{S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}})} / \underset{\substack{\textbf{HW-quotient}\\ \text{(64)}}}{\mathbb{Z}_2^{\mathrm{HW}}}}_{\simeq *} \times \underset{\substack{\textbf{ADE-singularity}\\ (\textit{Table 5})}}{\mathbb{R}^{\mathbf{4}_{\mathbf{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}} \, .
\end{aligned} \tag{80}$$

As shown in the second line, this is homotopy-equivalent to a residual ADE-singularity (*Table 5*). Therefore, the discussion from §3.1 applies:

Corollary 4.4 (**Equivariant Cohomotopy implies local/twisted M5/MO5-anomaly cancellation).** *The super-differentiable* (55) *equivariant Cohomotopy charge of the vicinity (Def. 3.9) of the semi-complement spacetime of a single charged* MO5-*singularity* (80)

$$\pi^{\mathbf{4}_{\mathbb{H}}}\Big(\Big(X^{11}_{\frac{1}{2}\mathrm{MO5}}\Big)^{\mathrm{cpt}}\Big)_{-} = \Big\{ \underset{\substack{\textbf{MO5-plane}\\ \textbf{charge}}}{1 \cdot \mathbf{1}_{\mathrm{triv}}} - \underset{\substack{\textbf{M5-brane}\\ \textbf{charge}}}{N_{\mathrm{M5}} \cdot \mathbf{2}_{\mathrm{reg}}} \;\Big|\; N_{\mathrm{M5}} \in \mathbb{Z} \Big\}$$

as in Folklore 4.1, Table 2, regarding the local/twisted form of M5/MO5-anomaly cancellation.

Proof. By G-homotopy invariance of G-equivariant homotopy theory, this follows as the special case of Prop. 3.14 with (59) in Example 3.19, for $G = \mathbb{Z}_2$, hence with $k = |W_G(1)| = 2$. ☐

Remark 4.5 (Super-exceptional geometry of MO5 semi-complement)**.** While here we consider only topological orientifold structure, the full super-exceptional geometry corresponding to (80) is introduced in [FSS19d, 4]; shown there to induce the M5-brane Lagrangian on any super-exceptional embedding of the $\frac{1}{2}$M5-locus.

Equivariant Cohomotopy implies local/untwisted M5/MO5-anomaly cancellation at $\frac{1}{2}$M5-singularities. It is immediate to consider the globalization of this situation to the semi-complement around one MO9 in heterotic M-theory compactified on the toroidal $\mathbb{Z}_2^{\mathrm{refl}}$-orbifold $\mathbb{T}^{\mathbf{5}_{\mathrm{sgn}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl+HW}}$ with MO5-singularities:

$$\underset{\substack{\textbf{semi-complement spacetime}\\ \textbf{around MO5}s\textbf{ in } \mathrm{M}_{\mathrm{HET}}/\mathbb{Z}_2^{\mathrm{refl}}}}{X^{11}_{\mathrm{M}_{\mathrm{HET}}/\mathbb{Z}_2^{\mathrm{refl}}}} := \mathbb{R}^{5,1} \times \overbrace{\underset{\substack{\textbf{unit 0-sphere}\\ \textbf{around MO9}}}{S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}})} / \underset{\substack{\textbf{HW-quotient}\\ \text{(64)}}}{\mathbb{Z}_2^{\mathrm{HW}}}}^{\simeq *} \times \underset{\substack{\textbf{toroidal reflection-orbifold (18)}\\ (\textit{Table 5})}}{\mathbb{T}^{\mathbf{4}_{\mathbf{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}} \, . \tag{81}$$

To this toroidal ADE-orbifold the discussion in §3.2 applies as follows.

Corollary 4.6 (Equivariant Cohomotopy implies global/untwisted M5/MO5-anomaly cancellation). *The super-differentiable* (55) *equivariant Cohomotopy charge* (3) *of the semi-complement spacetime* (81) *of heterotic M-theory on a toroidal MO5-orientifold (§4.1) with charged MO5-planes in compatible RO-degree (Example 3.8) and admitting equivariant super-differential refinement* (58) *is*

$$\pi^{\mathbf{4}_{\mathbb{H}}}\Big(\big(X^{11}_{\mathrm{M_{HET}}/\mathbb{Z}_2^{\mathrm{refl}}}\big)_+\Big)_{\mathrm{Sdiffble}\_} \;=\; \Big\{\underset{\textbf{MO5-plane charge}}{16\cdot\mathbf{1}_{\mathrm{triv}}} - \underset{\textbf{M5-brane charge}}{8\cdot\mathbf{2}_{\mathrm{reg}}}\Big\}$$

as expected from Folklore 4.1, Table 2, regarding the global/untwisted form of M5/MO5-anomaly cancellation (recalling that the semi-complement (81) *is that around* one *of the two MO9-planes).*

Proof. By G-homotopy invariance of equivariant Cohomotopy, this follows from the statement (60) in Example 3.19. ☐

More generally we have the following:

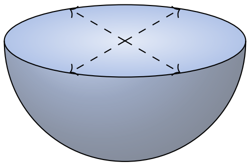
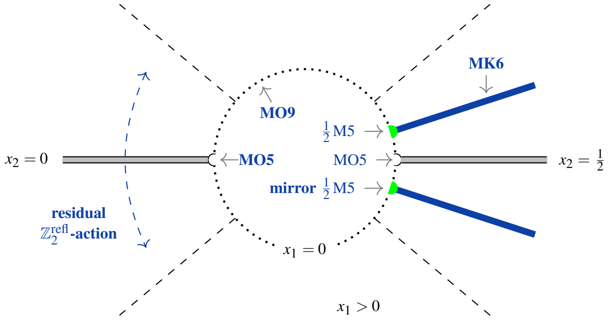
M5/MO5-anomaly cancellation in heterotic M-theory on general ADE-orbifolds. The statements and proofs of Corollary 4.4 and Cor. 4.6 directly generalize to heterotic M-theory on general G^{ADE}-singularities $\mathbb{R}^{\mathbf{4}_{\mathbb{H}}}$ §4.1, because the underlying results in §3 apply in this generality. Hence *Hypothesis H* implies that on the semi-complement spacetime of an MO9 intersecting a toroidal ADE-orbifold

$$\underset{\substack{\textbf{semi-complement spacetime}\\ \textbf{around } \frac{1}{2}\mathrm{M5}_{\mathrm{ADE}} \textbf{ in } \mathrm{M_{HET}}/\mathrm{ADE}\\ (\textit{Figure S})}}{X^{11}_{\mathrm{M_{HET}}/G^{\mathrm{ADE}}}} \;:=\; \mathbb{R}^{5,1} \times \overbrace{\underset{\substack{\textbf{unit 0-sphere}\\ \textbf{around MO9}}}{S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}})}/\underset{\substack{\textbf{HW-quotient}\\ (64)}}{\mathbb{Z}_2^{\mathrm{HW}}}}^{\simeq *} \times \underset{\substack{\textbf{toroidal ADE-orbifold}\\ (\textit{Table 5})}}{\mathbb{T}^{\mathbf{4_H}} /\!\!/ G^{\mathrm{ADE}}} \tag{82}$$

the M5/MO5 charge, measured in equivariant Cohomotopy, is

$$Q_{\mathrm{tot}} \;=\; 16\cdot\mathbf{1}_{\mathrm{triv}} - N_{\mathrm{M5}}\cdot\mathbf{k}_{\mathrm{reg}} \qquad |Q_{\mathrm{tot}}| = 0\,,$$

for $k = \left|G^{\mathrm{ADE}}\right|$ the order of the global quotient group. Under double dimensional reduction to type IIA string theory according to *Table 7*, this implies the tadpole cancellation conditions for D4-branes in ADE-orientifolds, from *Table 1*.

<table>
<tr><th colspan="3">Spacetimes on which to measure flux sourced by M5/MO5-charge</th></tr>
<tr><td>Definition</td><td>$X_{\mathrm{MO5}} \simeq_{\mathrm{htpy}} S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}+\mathbf{4}_{\mathrm{sgn}}})/\mathbb{Z}_2^{\mathrm{het+HW}}$
(76)</td><td>$X_{1/2\mathrm{M5}} \simeq_{\mathrm{htpy}} S(\mathbb{R}^{\mathbf{1}_{\mathrm{sgn}}})/\mathbb{Z}_2^{\mathrm{HW}} \times \mathbb{T}^{\mathbf{4}_{\mathbb{H}}} /\!\!/ \mathbb{Z}_2^{\mathrm{refl}}$
(80)</td></tr>
<tr><td>Illustration</td><td></td><td> </td></tr>
<tr><td>Geometry</td><td>smooth but curved</td><td>singular but flat</td></tr>
</table>

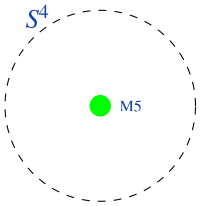
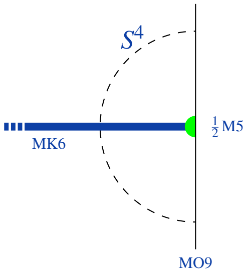
<table>
<tr><th colspan="4">Cohomological charge quantization
by <i>Hypothesis H</i></th></tr>
<tr><td>Cohomology theory
(by <i>Table 4</i>)</td><td>J-twisted Cohomotopy $\pi^{TX}(X)$
[FSS19b][FSS19c]</td><td colspan="2">equivariant Cohomotopy $\pi^{V}_{\mathbb{Z}_2}(\mathbb{T}^V)$
§3</td></tr>
<tr><td>Illustration
(Remark 4.7)</td><td> </td><td colspan="2"> </td></tr>
<tr><td>Charge classification</td><td>$c_{\mathrm{tot}} = 1 - N \cdot 2$
(78)</td><td>$c_{\mathrm{tot}} = N_{\mathrm{MO5}} \cdot \mathbf{1}_{\mathrm{triv}} - N_{\mathrm{M5}} \cdot \mathbf{2}_{\mathrm{reg}}$
(Cor. 4.4)</td><td>$\Leftrightarrow \begin{array}{ll} \lvert Q_{\mathrm{tot}} \rvert & = 0 \\ N_{\mathrm{M5}} & = 8 \end{array}$
(Cor. 4.6)</td></tr>
</table>

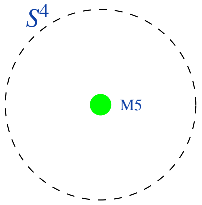
Table 8 – Two ways of measuring M5/MO5-charge. On the left is the traditional approach not resolving the singularities. On the right (which shows the same situation as in *Figure V* but with the periodic identification indicated more explicitly) the fine-grained microscopic picture seen by C-field charge quantization in equivariant Cohomotopy.

With these result in hand, we highlight that not only did equivariant Cohomotopy inform us about M-theory, but M-theory also shed light on a subtle point regarding the interpretation of equivariant Cohomotopy:

Remark 4.7 (Equivariant Cohomotopy and MK6 ending on M5). (i) The heuristic way to see that ordinary Cohomotopy π^4 from (5) canonically measures charges of 5-branes inside 11-dimensional spacetime is that the '*classifying space*' S^4 of π^4 gets essentially identified with the (any) *spacetime* 4-sphere *around* a 5-brane in an 11-dimensional ambient space (see [HSS18, (6)] for the heuristic picture, and [FSS19b, 4.5] for the full mathematical detail).

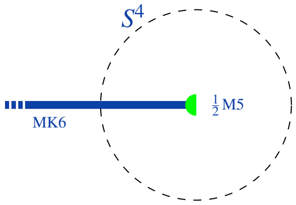
(ii) But as we pass from plain to equivariant Cohomotopy, this picture

Brane charge sourced in the center of S^4



may superficially appear to be in tension with the picture provided by the Pontrjagin-Thom theorem as in *Figure D* and *Figure L*, where instead

Brane charge is sourced at the 0-pole of S^4



However, in the orbi-geometry of heterotic M-theory on ADE-singularities §4.1 indeed *both pictures apply simultaneously*, witnessing different but closely related brane species (see *Table 8*):

(iii) The black $\frac{1}{2}$M5-brane locus (*Figure S*) is the terminal point of an MK6-singularity which extends radially away from the M5. Hence, given any radial 4-sphere with the $\frac{1}{2}$M5 at its center, the MK6 will pierce this 4-sphere at one point. Since the $\frac{1}{2}$M5 and the MK6 are *necessarily* related this way, the 5-brane charge inside the S^4 may equivalently be measured by 6-brane charge piercing through S^4. This is exactly what the Pontrjagin-Thom theorem says happens in Corollary 4.4, as shown in *Figure V* and on the right of *Table 8*.

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