Equivariant Principal ∞-Bundles

Hisham Sati, Urs Schreiber

July 13, 2022

Hisham Sati, *Mathematics, Division of Science,*
and Center for Quantum and Topological Systems (CQTS), NYUAD Research Institute,
New York University Abu Dhabi, UAE,
hsati@nyu.edu

Urs Schreiber, *Mathematics, Division of Science,*
and Center for Quantum and Topological Systems (CQTS), NYUAD Research Institute,
New York University Abu Dhabi, UAE;
on leave from Czech Academy of Science, Prague.
us13@nyu.edu

What this book is about.

In this book we prove (Thm. 4.3.24) unified classification results for stable equivariant Γ-principal bundles when the underlying homotopy type $\int\Gamma$ of the topological structure group Γ is truncated, meaning that its homotopy groups vanish in and above some degree n. We discuss (Thm. 4.2.7.ii) how this coincides with the classification of equivariant higher non-abelian gerbes and generally of equivariant *principal ∞-bundles* [NSS12a] with structure n-group $\int\Gamma$; and we show how the equivariant homotopy groups of the respective classifying G-spaces are given by the non-abelian group cohomology of the equivariance group with coefficients in $\int\Gamma$ (Thm. 4.3.7).

The result is proven in a conceptually transparent manner as a consequence of a *smooth Oka principle* (Thm. 3.3.51, 4.1.55, based on [BBP19][Pv14]), which is available after faithfully embedding traditional equivariant topology into the singular-cohesive homotopy theory of globally equivariant higher stacks [SS20-Orb]. This works for discrete equivariance groups G acting properly on smooth manifolds ("proper equivariance" [DHLPS19]) with resolvable singularities (Ntn. 3.3.54), whence we are equivalently describing principal bundles on good orbifolds.

In setting up this proof, we re-develop the theory of equivariant principal bundles from scratch by systematic use of Grothendieck's *internalization* (Ntn. 1.0.23). In particular we prove (Thm. 4.2.7.i) that all the intricate equivariant local triviality conditions considered in the literature [tD69][Bi73][LM86] are automatically *implied* by regarding G-equivariant principal bundles as principal bundles internal to the BG-slice of the ambient cohesive ∞-topos. We also show that these conditions are all equivalent (Thm. 2.2.1). Generally we find (see §4.3) that the characteristic subtle phenomena of equivariant classifying theory all reflect basic *modal* properties of singular-cohesive homotopy theory (hence of cohesive "global" equivariant homotopy theory [Re14a], see e.g. Prop. 4.3.5).

A key example is the projective unitary structure group $\Gamma = \mathrm{PU}_\omega$, in which case (Ex. 4.4.2) we are classifying 3-twists of equivariant KU-theory (related to equivariant bundle gerbes, Ex. 4.1.26), recovering the results [TXLG04, Cor. 2.41][AS04, Thm. 6.3][BEJU14, Thm. 1.10][LU14, Thm. 15.17]. Our general theorem immediately enhances this to the conjugation-equivariant graded projective unitary structure group $\mathbb{Z}_{/2} \circlearrowright \mathrm{PU}^{\mathrm{gr}}_\omega$ with fixed locus $(\mathrm{PU}^{\mathrm{gr}}_\omega)^{\mathbb{Z}_{/2}} = \mathrm{PO}^{\mathrm{gr}}_\omega$ (Ex. 1.3.19), where we are classifying the twists of equivariant KR-theory in degrees 1 and 3 combined, restricting on "O-planes" to the twists of KO-theory in degrees 1 and 2 combined. This is the generality in which equivariant K-theory twists are conjectured to model quantum symmetries of topological phases of matter (Rem. 4.5.5) and the B-field in string theory on orbi-orientifolds (Rem. 4.5.6).

Our focus on the class of truncated structure groups is largely complementary to the classical literature on equivariant classifying spaces [tD69][Bi73][La82][LM86][May90], which is mostly concerned with the class of compact Lie structure groups Γ (see Tab. 1). The two classes intersect when $\Gamma = \mathbb{T}^d \rtimes K$ is the extension of a finite group K by a compact abelian Lie group, i.e., the case of equivariant sheeted torus bundles ("bundle 0-gerbes"). In this case our general classification recovers (Ex. 4.4.1) the results of [LMSe83, Lem. 1, Thm. 2][May90, Thm. 3, Thm. 10] and [Re18, Thm. 1.2] and generalizes them to the case of non-trivial G-action on Γ.

A classification result for general topological structure groups had previously been claimed in [MS95] but, as highlighted in [GMM17, §3.12], the proof had remained open. Besides characterizing its classification property for truncated structure groups, we show that the *Murayama-Shimakawa construction* of [MS95] generally produces the underlying equivariant homotopy type of the correct equivariant *moduli stacks* (Thm. 4.3.19).

While compact Lie structure groups Γ have received much attention due to the role of equivariant vector bundles as *cocycles* of equivariant K-theory, the complementary case of truncated structure groups Γ that we discuss will generally be relevant for equivariant bundles in their role as *twists* of equivariant generalized cohomology theories, since such twists will typically be considered in a finite number of degrees, with $\int\Gamma$ a truncation of the ∞-group of units of a ring spectrum. This combined *twisted & equivariant* enhancement of generalized cohomology theory has previously received little attention (except for a brief note in [Li14]) beyond the example of KU-theory. We mean this book to be laying previously missing foundations (§4.5).

In particular, via a twisted generalization of Elmendorf's Theorem (Thm. 4.5.3), the equivariant classifying spaces produced here provide the correct domains for twisted equivariant enhancements of Chern-Dold character maps (further generalizing [FSS20-TCD][SS20-ETw]) and hence allow the systematic definition and construction of twisted equivariant *differential* generalized cohomology theories in equivariant generalization of the construction in [FSS20-TCD, §8]. This will be discussed in [SS22-TEC][SS22-TED].

Contents

Part I

Introduction

We assume that the reader already appreciates principal bundles and their classifying theory (e.g., as in [Hu94][Coh17][RS17, §1.1][Tam21]) and is already motivated towards their equivariant generalization (due to [Stw61][tD69][Bi73][La82][LM86]), though we will effectively review and re-develop much of the classical theory.

The content of part II is a self-contained discussion of equivariant principal bundles entirely in point-set topology, which may be read independently of part III and may be of interest in itself, as a comprehensive, streamlined, and completed review.

The purpose of part III is to embed this classical topic into the relatively new context of higher geometry, where the nature of equivariant classifying spaces becomes more transparent and where *cohesive homotopy theory* ([SSS12, §3.1][Sc13][ScSh14][Sh15][SS20-Orb, §3]) provides new tools even for the classical theory, which we use to prove a new classification theorem. This higher geometric perspective on principal bundles (following [SSS12, §3][NSS12a][NSS12b]) and on equivariance (following [Re14a][SS20-Orb]) may conceptually be understood as follows.

0.1 Overview and Summary

Higher cohesive geometry as convenient topology. For the classical algebraic topologist, one way to appreciate the passage to higher geometric and to cohesive homotopy theory is as a completion of the pursuit of ever more *convenient categories* of topological spaces in the famous sense of [St67]. This starts out with compactly generated spaces (used in part II, see Prop. 1.0.16) but proceeds, along the general lines of [Vo71][Wy73][FR08], to the *really convenient category* [Sm][Dug03] of Delta-generated topological spaces (used in part III, see Ntn. 3.3.17).

At this point, we may observe [SS20-Orb, Ex. 3.18] with [SYH10, §3] (see Prop. 3.3.19 below) that D-topology faithfully embeds first into *diffeology* ([IZ13], see Ntn. 3.3.15 below) and from here further into the cohesive topos $\mathrm{SmthGrpd}_0$ over the site of smooth manifolds and then the cohesive ∞-topos $\mathrm{SmthGrpd}_\infty$ (Ntn. 3.3.26 below):

$$
\begin{tikzcd}[column sep=small]
\text{category of topological spaces} & & \text{cartesian closed category of compactly generated topological spaces} & & \text{cart. clsd. \& locl. pres. category of } \Delta\text{-generated topological spaces} & & \text{quasi-topos of diffeological spaces} & & \text{cohesive topos of smooth sets} & & \text{cohesive } \infty\text{-topos of smooth } \infty\text{-groupoids} \\
\mathrm{TopSpc} \arrow[rr, shift right, "k"'] \arrow[rr, phantom, "\perp"] \arrow[ddrrrrrrrrrr, bend right=10] & & \mathrm{kTopSpc} \arrow[ll, shift right] \arrow[rr, shift right, "\mathrm{Cdfflg}"'] \arrow[rr, phantom, "\perp"] \arrow[ddrrrrrrrr, bend right=10] & & \mathrm{DTopSpc} \arrow[ll, shift right] \arrow[rr, hook, shift right] \arrow[rr, phantom, "\perp"] \arrow[ddrrrrrr, bend right=10] & & \mathrm{DfflgSpc} \arrow[ll, shift right, "\mathrm{Dtplg}"'] \arrow[rr, hook, "i_{\sharp 1}"'] \arrow[ddrrrr, bend right=10, "\mathrm{Pth}"] & & \mathrm{SmthGrpd}_0 \arrow[rr, hook, "i_0"'] & & \mathrm{SmthGrpd}_\infty \arrow[dd, shift right, "\mathrm{Shp}"'] \\
& & & & & & & & \text{underlying homotopy types are preserved} & & \arrow[l, phantom, "\textstyle\int"] \\
& & & & & \text{form path } \infty\text{-groupoids (sing. simpl. compl.)} & & & & & \mathrm{Grpd}_\infty \arrow[uu, shift right, "\mathrm{Dsc}"'] \\
& & & & & & & & & & \text{base } \infty\text{-topos of bare } \infty\text{-groupoids}
\end{tikzcd}
$$

This diagram is homotopy-commutative, which says that the usual underlying homotopy type[1] $\int X$ of topological spaces X is preserved under all these embeddings (Props. 3.3.23, 3.3.48 below). Moreover, the underlying homotopy type of *mapping spaces* coincides with that of the corresponding *mapping stacks* (Prop. 3.3.24 below):

$$
X, Y \in \mathrm{kTopSpc} \quad \vdash \quad \underbrace{\int \mathrm{Map}(X,Y)}_{\text{shape of mapping space}} \;\simeq\; \underbrace{\int \mathrm{Map}\big(\mathrm{Cdfflg}(X), \mathrm{Cdfflg}(Y)\big)}_{\text{shape of topological mapping stack}}.
$$

[1]The "esh"-symbol "∫" (3.117) stands for *shape* [Sc13, 3.4.5][Sh15, 9.7], following [Bor75], which for the well-behaved topological spaces of algebraic topology is a convenient synonym for their *underlying weak homotopy type* [Lur09, 7.1.6][Wan17, 4.6]. It is in this sense that we use the term "shape" generalized to objects of cohesive ∞-toposes (Def. 3.3.1).

This is consistent (by Prop. 3.3.5 below) with the established generalization of shape theory to ∞-topos theory (Def. 3.2.19), in that the weak homotopy type of any topological space X is equivalently the shape of the slice of the ∞-topos $\mathrm{SmthGrpd}_\infty$ over its continuous-diffeological incarnation $\mathrm{Cdfflg}(X)$ (by Prop. 3.3.50 below).

The smooth Oka principle. After this embedding of differential topology into cohesive homotopy theory, a wonderful theorem becomes available, which we call the *smooth Oka principle*[2] (Thm. 3.3.51 below, based on [BBP19]), saying that, for smooth maps out of a smooth manifold into any smooth ∞-groupoid, the underlying homotopy type of the mapping stack is that of the mapping space of underlying homotopy types:

$$\left.\begin{array}{r}\mathrm{X}\text{ a smooth manifold}\\ \&\ A\text{ any smooth }\infty\text{-groupoid}\end{array}\right\} \quad\vdash\quad \overset{\text{shape of mapping stack}}{\int \mathrm{Map}(\mathrm{X}, A)} \underset{\substack{\text{smooth}\\ \text{Oka principle}}}{\simeq} \overset{\text{mapping space of shapes}}{\mathrm{Map}\big(\int \mathrm{X}, \int A\big)}\,.$$

When $A = \mathbf{B}\mathscr{G}$ is the moduli ∞-stack for $\mathscr{G}$-principal ∞-bundles (see Prop. 0.2.1, Thm. 3.2.97) of a smooth ∞-group $\mathscr{G}$, then the smooth Oka principle immediately implies that its shape

$$\overset{\substack{\text{moduli }\infty\text{-stack for}\\ \mathscr{G}\text{-principal }\infty\text{-bundles}}}{\mathbf{B}\mathscr{G}} \;:=\; *\!/\!\!/\mathscr{G} \xrightarrow[\substack{\text{send bundles to their}\\ \text{characteristic classes}}]{\eta^{\int}_{\mathbf{B}\mathscr{G}}} \overset{\substack{\text{classifying shape for concordance of}\\ \text{smooth }\mathscr{G}\text{-principal }\infty\text{-bundles}}}{B\mathscr{G} \;:=\; \int \mathbf{B}\mathscr{G} \;\simeq\; \mathbf{B}\!\int\!\mathscr{G}} \;\in\; \mathrm{Grpd}_\infty \overset{\mathrm{Dsc}}{\hookrightarrow} \mathrm{SmthGrpd}_\infty$$

is the homotopy type of a classifying space (the *classifying shape*) for concordance classes (Def. 4.1.5) of smooth $\mathscr{G}$-principal ∞-bundles (Thm. 4.1.12). When $\mathscr{G} = \Gamma$ is an ordinary Hausdorff-topological group, this $B\Gamma$ is equivalently its traditional Milgram classifying space (Prop. 4.1.11). But, since ordinary topological principal bundles are isomorphic if they are concordant (Thm. 2.2.8), this means that the smooth Oka principle specializes to an elegant cohesive re-proof of the classical classification theorem for principal bundles (Thm. 4.1.13).

An orbi-smooth Oka principle. Our strategy is to generalize this cohesive bundle theory to the equivariant context, using that G-equivariant principal ∞-bundles on $G \curvearrowright \mathrm{X}$ are equivalently principal ∞-bundles on the *cohesive orbispace* (e.g. *orbifold*) $\mathrm{X}/\!\!/G$ (§4.1.2, see [SS20-Orb] for background on orbifolds in cohesive homotopy theory). While the smooth Oka principle cannot hold for general domains X beyond smooth manifolds, it will hold with given codomain A for those ∞-stacks X that may be *approximated* by smooth manifolds to a degree of accuracy which the coefficients A do not resolve. Specifically, if Γ is a topological group whose shape $\int\Gamma$ is n-truncated, then $\mathbf{B}\Gamma$ does not distinguish (Lem. 4.1.31) the abstract G-orbifold singularity $*/\!\!/G$ from its "blowup" S^{n+2}_{sm}/G to a smooth spherical space form (Def. 3.3.52) of dimension above the truncation degree. Accordingly, an orbi-smooth Oka principle holds for truncated Γ-principal bundles on orbifolds with resolvable singularities which are "stable" (Ntn. 4.1.33) against the blowup (Thm. 4.1.55):

$$\left.\begin{array}{rl} & G\text{ discrete with resolvable singularities}\\ \& & \Gamma\text{ Hausdorff of truncated shape}\\ \& & G \curvearrowright \mathrm{X}\text{ a proper smooth }G\text{-manifold}\end{array}\right\} \;\vdash\; \overset{\text{shape of stable part of slice mapping stack}}{\int \mathrm{Map}\big(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G\big)^{\mathrm{stbl}}_{\mathbf{B}G}} \underset{\substack{\text{orbi-smooth}\\ \text{Oka principle}}}{\simeq} \overset{\text{slice mapping space of shapes}}{\mathrm{Map}\big(\int \mathrm{X}/\!\!/G, B\Gamma/\!\!/G\big)_{BG}}. \tag{1}$$

As before, this immediately specializes to the classification theorem for equivariant bundles up to concordance, and this – since also equivariant principal bundles are isomorphic if they are concordant (Lem. 4.2.6) – proves the full classification theorem (Thm. 4.2.7) under the given assumptions.

Notice that *every* ∞-group is the shape $\int\Gamma$ of some Hausdorff topological group Γ (Prop. 3.2.73), and that the assumption of discrete equivariance group G and truncated shape $\int\Gamma$ is generically verified in applications of equivariant principal bundles as twists for generalized cohomology of good orbifolds (Ex. 4.4.1, Ex. 4.4.2):

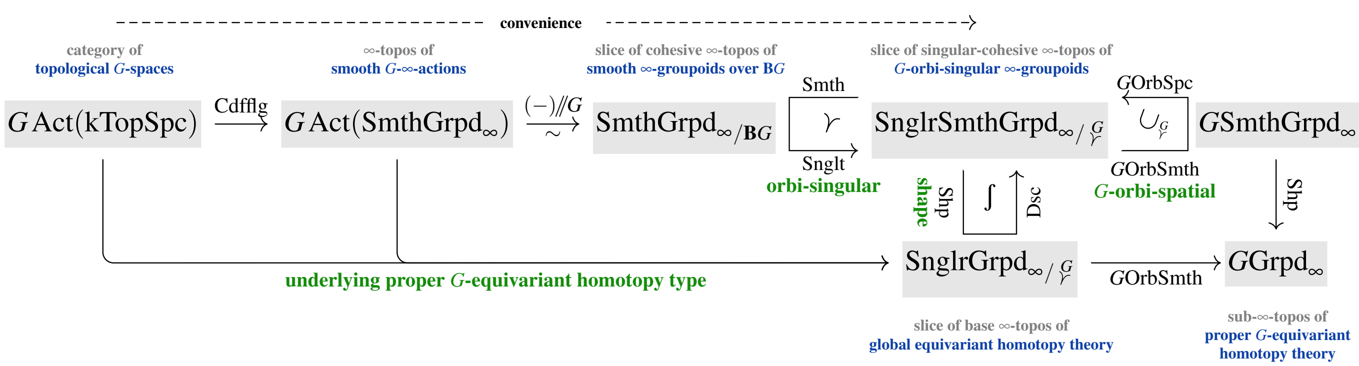
[2] Our terminology *smooth Oka principle* is meant to rhyme on the established *Oka-Grauert principle* in complex analysis, which says (reviewed in [FL11, Cor. 3.5, 3.2][FL13, (1.1)]) that over Stein manifolds: (a) the homotopy type of the space of holomorphic maps into an Oka manifold coincides with that of continuous maps between the underlying topological spaces; (b) the classification of holomorphic vector bundles coincides with that of topological vector bundles. In fact, also the ∞-topos over the site of complex-analytic manifolds is cohesive (this is implicit in [HQ15, §2.1]), suggesting that the classical Oka-Grauert principle and our smooth Oka principle are just two special cases of one unifying geometric homotopy principle in cohesive homotopy theory.

G-equivariant Γ-principal bundles: examples and applications		Structure group Γ: Compact Lie	Structure group Γ: Truncated
Equivariance group G	Discrete	– torus-principal bundles – finite covering spaces – K-theory cocycles – tangential twists	– torus-principal bundles – ordinary cohomology twists – K-theory twists – differential twists
		on good orbifolds	
	Compact Lie	as above, on bad effective orbifolds & in FHT theory	

Table 1.

Proper equivariant cohesive homotopy theory. A remarkable aspect of the classification result Thm. 4.2.7 is (see §4.3) that **(1)** the moduli ∞-stacks of equivariant principal bundles always exist in *Borel-equivariant* cohesive homotopy theory, and that **(2)** under the truncation condition on $\int \Gamma$ also the corresponding classifying spaces exist in Borel equivariant homotopy theory. While the first of these is a general fact, the second is a happy coincidence (often used without further amplification, e.g. in [FHT07]) conditioned on stability and the truncation of $\int \Gamma$.

General equivariant classifying spaces require more fine-grained proper-equivariant homotopy theory of spaces parameterized over the category of orbits (Ntn. 3.3.58) of the equivariance group (e.g. [tD79, §8][May96][Blu17]). Following [Re14a][SS20-Orb], we may understand this (see §3.3.2) through an embedding of the category of topological G-spaces (Ntn. 1.1.1) into, once again, a yet more convenient ambient category of G-actions, namely the slice of a *singular-cohesive ∞-topos* (Def. 3.3.65) over the generic G-orbi-singularity $\prec^{G}$ (Ntn. 3.3.57):



Elmendorf's theorem as a generalized equivariant Oka principle. On the right, $G\mathrm{Grpd}_\infty$ (Ex. 3.3.63) denotes the ∞-presheaves over the *category of G-orbits* (Def. 3.3.58). This is the modern context of G-equivariant homotopy theory, traditionally motivated by *Elmendorf's theorem* ([El83][DK84], recalled as Prop. 4.5.1 below). With the hindsight of cohesive homotopy theory, this classical theorem may conceptually be understood as *enforcing* an "equivariant Oka principle": For $G \circlearrowright \mathrm{X}$ a G-CW-complex (Ex. 1.3.6) and $G \circlearrowright Y$ any topological G-space, the simplicial enhancement of Elmendorf's theorem ([DK84, Thm. 3.1], see also [CP96][Blu17, Thm. 1.3.8]) says, after the above embedding into singular-cohesive homotopy theory (by Prop. 3.3.92 below), that one may take the shape operation inside the equivariant mapping stack *if* in the process one enhances G-orbifolds $\mathrm{X} /\!\!/ G$ to their *orbi-singularization* $\prec(\mathrm{X} /\!\!/ G)$ (Ex. 4.5.2 below):

$$\underbrace{\overbrace{\textstyle\int \mathrm{Map}\big(\mathrm{X}/\!\!/G,\, \mathrm{Y}/\!\!/G\big)_{\mathbf{B}G}}^{\text{shape of equivariant mapping stack}} \;\simeq\; \overbrace{\cup \mathrm{Map}\big(\textstyle\int \prec(\mathrm{X}/\!\!/G), \int \prec(\mathrm{Y}/\!\!/G)\big)_{\prec^{G}}}^{\text{mapping space of equivariant shapes}}}_{\text{Elmendorf-Dwyer-Kan theorem}} . \qquad (2)$$

Proper equivariant classifying spaces. Therefore, chasing a topological G-space through the above diagram produces its usual proper-equivariant homotopy type as a G-orbi-space (Prop. 3.3.92). But we may now also feed the above moduli stack $\mathbf{B}\Gamma /\!\!/ G$ of G-equivariant Γ-principal bundles through this machine, and we find (Thm. 4.3.19) that the resulting equivariant homotopy type is that of the Murayama-Shimakawa construction ([MS95][GMM17], §2.3):

$$\underbrace{B_G\Gamma}_{\substack{\text{proper equivariant classifying space}\\ \text{for equivariant principal bundles}}} := \cup_{\underset{\curlyvee}{G}} \int \curlyvee (\mathbf{B}\Gamma /\!\!/ G) \;:\; G/H \longmapsto \underbrace{\int \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G}}_{\substack{\text{concordances of equivariant}\\ \text{bundles over the point}}} \simeq \underbrace{\mathrm{Pth}\,|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)|^H}_{\text{Murayama-Shimakawa construction}}$$

In the special case when (G-singularities are resolvable and) $\int\Gamma$ is truncated, the above orbi-smooth Oka principle (1) applies to these values of the equivariant classifying space and gives (Prop. 4.3.9):

$$\left.\begin{array}{r} G \text{ discrete with resolvable singularities} \\ \& \; \Gamma \text{ topological of truncated shape} \\ \& \; G \circlearrowright \mathrm{X} \text{ a proper smooth } G\text{-manifold} \end{array}\right\} \vdash \quad B_G\Gamma \simeq B_G(\int\Gamma) \;:\; G/H \longmapsto \begin{array}{l} \mathrm{Map}(BH, B\Gamma /\!\!/ G)_{BG} \\ \quad \simeq \mathrm{Pth}\,\big|\mathrm{Map}(|\mathbf{E}G|, |\mathbf{B}\Gamma|)\big|^H. \end{array}$$

Classification statement in proper-equivariant cohomology. These proper-equivariant homotopy types are the coefficient systems that represent Borel-equivariant cohomology inside proper-equivariant cohomology (Prop. 4.3.21), so that our classification result may be re-stated in the following equivalent forms (Thm. 4.3.24):

$$\left.\begin{array}{r} G \text{ discrete with resolvable singularities} \\ \& \; \Gamma \text{ Hausdorff of truncated shape} \\ \& \; G \circlearrowright \mathrm{X} \text{ a proper smooth } G\text{-manifold} \end{array}\right\} \vdash \begin{cases} \big(G\mathrm{Equv}\,\Gamma\,\mathrm{PrnFibBdl}(\mathrm{DTopSpc})^{\mathrm{stbl}}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} & \text{isom. classes of equivariantly locally trivial stable equivariant principal topol. bundles} \\ \simeq H^1_G(X; \int\Gamma) = H^0_G(X; B\Gamma) & \text{Borel-equivariant cohomology with coefficients in classifying space} \\ \simeq H^0_{\underset{\curlyvee}{G}}(X; B_G(\int\Gamma)) \simeq H^0_{\underset{\curlyvee}{G}}\big(X; (B_G\Gamma)^{\mathrm{stbl}}\big) & \text{proper-equivariant cohomology with coefficients in equivariant classifying space.} \end{cases}$$

Specialized to trivial G action on Γ and the cases where Γ is either a 1-truncated compact Lie group or the projective unitary group PU_ω, this theorem reproduces (when $\mathrm{X} /\!\!/ G$ is a good orbifold with resolvable singularities) a series of statements found in the literature (Ex. 4.4.1, Ex. 4.4.2).

Outline. In order to make the presentation of these results reasonably self-contained also for the non-expert reader, we lay out a fair bit of the required background in chapter 1 (equivariant differential topology) and chapter 3 (equivariant cohesive homotopy theory). The new constructions and results are the content of chapter 2 (equivariant topological bundles) and, particularly, of chapter 4 (equivariant ∞-bundles).

Conclusion and outlook. In conclusion, we find that the unifying mechanism behind a large class of equivariant classification results and their generalization to higher truncated structure groups is a smooth Oka principle in cohesive homotopy theory, which seamlessly embeds the theory of equivariant bundles into a transparent modal homotopy theory of higher geometry, in particular into the context of higher principal bundles over orbifolds and more general cohesive orbispaces.

By way of outlook, notice that there is a large supply of equivariance groups with resolvable singularities (Prop. 3.3.53) including key examples of interest in applications (Ex. 3.3.55); and there is a large supply of truncated structure groups, as every ∞-group is the shape of some Hausdorff topological group (Prop. 3.2.73). In particular, for R a ring spectrum and $\mathrm{GL}(1,R)$ its ∞-group of units (see [FSS20-TCD, Ex. 2.37] for pointers), there is for every n a Hausdorff group Γ with shape its n-truncation $\int\Gamma \simeq \tau_n\mathrm{GL}(1,R)$. The corresponding G-equivariant Γ-principal bundles are candidate geometric twists for equivariant R-cohomology theory, generalizing the archetypical case of twisted complex K-theory (where $R = \mathrm{KU}$, $\Gamma = \mathrm{PU}^{\mathrm{gr}}_\omega$ with $\int\mathrm{PU}^{\mathrm{gr}}_\omega \simeq \tau_2\,\mathrm{GL}(1,\mathrm{KU})$).

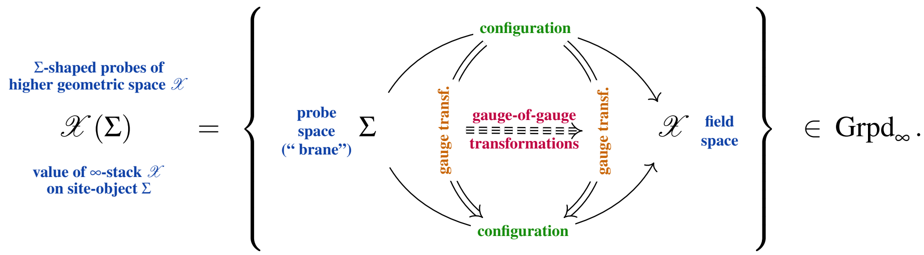
In all these cases, the classification theorem of §4.3 shows, in particular, that the equivariant classifying spaces of twists are, generically, equivariantly non-simply connected, which means that many traditional tools, notably of rational homotopy theory, do not apply without extra care. For example, in the case of twisted complex K-theory, this fact (see Ex. 4.1.57 and (4.144)), explains, we claim, the otherwise somewhat unexpected (cf. [FHT07, (3.22)]) appearance of local systems in the twisted equivariant Chern character ([TX06, Def. 3.10]). Generally, one may use the equivariant classifying theory developed here to give a general construction of twisted equivariant Chern-Dold character maps and hence of twisted equivariant differential cohomology theories, in equivariant generalization (along the lines of [SS20-ETw, §3]) of the construction in [FSS20-TCD, Def. 5.4]:

$$
\begin{tikzcd}
\underset{\substack{\text{universal local coefficient bundle} \\ \text{for } \Gamma\text{-twisted } G\text{-equivariant} \\ A\text{-cohomology theory}}}{} \int \curlyvee (A /\!\!/ (\Gamma /\!\!/ G)) \arrow[r] \arrow[d] & L_{\mathbb{Q}} \big(\int \curlyvee (A /\!\!/ (\Gamma /\!\!/ G)) \big) \arrow[d] \\
\underset{\substack{\text{equivariant classifying} \\ \text{space of twists } (\S 4.3)}}{B_G \Gamma} \arrow[r, "\substack{\text{proper equivariant sliced rationalization} \\ \text{representing the} \\ \text{twisted equivariant } A\text{-character map}}"] & L_{\mathbb{Q}} (B_G \Gamma)
\end{tikzcd}
$$

The homotopy pullback (formed in $\mathrm{SnglrSmthGrpd}_\infty$) of stacks of flat equivariant L_∞-algebra valued differential forms ([SS20-ETw, Def. 3.57]) along this rationalization operation (in direct equivariant generalization of [FSS20-TCD, Def. 4.38]) solves the open problem of providing a general construction of G-equivariant Γ-twisted *differential* A-cohomology. This application will be discussed in detail in [SS22-TEC][SS22-TED].

0.2 Tools and Techniques

Higher cohesive geometry as intrinsically equivariant geometry. The point of *higher homotopical geometry* is (see [SS20-Orb, p. 4-5]) that the notion and presence of *gauge transformations* (homotopies) and *higher gauge-of-gauge transformations* is natively built into the theory, so that absolutely every concept formulated in higher geometry is *intrinsically* equivariant with respect to all relevant symmetries. This makes higher geometry the natural context for laying foundations for equivariant algebraic topology.



(3)

The machinery of ∞-category theory. While higher stacks and their higher geometry are often perceived as an esoteric and convoluted mathematical subject, this is rather a property of their presentation by models in simplicial homotopy theory (reviewed in §3.1, [FSS20-TCD, §A]) and must be understood as a reflection of their precious richness, not as of their intractability. Indeed, with ∞*-category theory* [Jo08a][Jo08b][Lur09][Ci19][RV21] (see §3.1), and specifically with ∞*-topos theory* [Si00][Lur03][TV05][Jo08c][Lur09][Re10] (see §3.2), there is a high-level language, abstracting away from the zoo of models (e.g. [Be07b]), to admit efficient reasoning about higher stacks via elementary categorical logic. This point is made fully manifest by the existence of an elementary internal logic of ∞-toposes [Sh19], now known as Homotopy Type Theory [UFP13] (in our context see [ScSh14][SS20-Orb, p. 5]) which condenses all such reasoning to coding in a kind of programming language.

For example, once Cartesian (pullback) squares are understood as homotopy Cartesian squares, namely filled with a homotopy which exhibits the expected unique factorization property up to homotopy and in the sense of a contractible space of homotopy-factorizations:

$$
\underset{\substack{\text{homotopy Cartesian square /}\\ \text{homotopy pullback square}}}{\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\swarrow\text{(pb)}"] \\ X \arrow[r] & B \end{tikzcd}}
\quad \Rightarrow \quad
\left(
\begin{tikzcd} Q \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, phantom, "\swarrow"] & & \\ & & Y \arrow[d] \\ & X \arrow[r] & B \end{tikzcd}
\;\Leftrightarrow\;
\begin{tikzcd} Q \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed] & & \\ & X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ & X \arrow[r] & B \end{tikzcd}
\right),
\tag{4}
$$

then they follow patterns familiar from 1-category theory: For instance, the *pasting law* in 1-categories (recalled as Prop. 1.0.9 below) continues to hold verbatim in any ∞-category [Lur09, Lem. 4.4.2.1]:

$$
\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dl, phantom, "\swarrow\text{(pb)}"] & Z \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}
\quad \Rightarrow \quad
\left(
\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\swarrow\text{(pb)}"] \\ A \arrow[r] & C \end{tikzcd}
\;\Leftrightarrow\;
\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dl, phantom, "\swarrow\text{(pb)}"] & Z \arrow[d] \arrow[dl, phantom, "\swarrow\text{(pb)}"] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}
\right);
\tag{5}
$$

and it still makes sense, for instance, to say that a morphism f is a *monomorphism* if and only if its homotopy fiber product with itself is equivalently its domain ([Lur09, p. 575][Re19, p. 10], see also Ex. 3.1.16):

$$
X \underset{\text{monomorphism}}{\overset{f}{\hookrightarrow}} Y
\quad \Leftrightarrow \quad
\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\swarrow\text{(pb)}"] \\ X \arrow[r, "f"'] & Y . \end{tikzcd}
\tag{6}
$$

(In the following we will leave the homotopies filling these squares notationally implicit.)

As another example: For any ∞-category $\mathbf{C}$ and any pair X, A of its objects, we have their *hom(omorphism) ∞-groupoids* (see [DS11])

$$\underset{\substack{\text{hom-}\infty\text{-groupoid} \\ \text{from } X \text{ to } A \\ \text{in } \infty\text{-category } \mathbf{C}}}{\mathbf{C}(X,A)} \;=\; \left\{ \begin{tikzcd} \underset{\text{object}}{\text{domain}}\; X \arrow[rr, bend left=50, "\text{homomorphism}"{name=U}] \arrow[rr, bend right=50, "\text{homomorphism}"'{name=D}] & & A \;\underset{\text{object}}{\text{codomain}} \arrow[from=U, to=D, phantom, "\substack{\text{homotopy} \;\Rightarrow\; \text{higher homotopies} \;\Leftarrow\; \text{homotopy}}"] \end{tikzcd} \right\} \;\in\; \mathrm{Grpd}_\infty\,, \tag{7}$$

which are well-defined (model-independent) up to (weak homotopy-)equivalence, and homotopy functorial in the two arguments in the expected way.

For instance, an ∞-functor is *fully faithful* if it induces a natural equivalence on values of hom-∞-functors:

$$\mathbf{C} \xhookrightarrow[\text{fully faithful}]{F} \mathbf{D} \qquad \Leftrightarrow \qquad \mathbf{C}(-,-) \xrightarrow[\sim]{F_{(-,-)}} \mathbf{D}\big(F(-), F(-)\big)\,. \tag{8}$$

Moreover, the hom ∞-functor satisfies the expected category-theoretic properties in homotopy-theoretic form:
(i) hom-∞-groupoids respect homotopy (co)-limits via natural equivalences of the form

$$\mathbf{C}\Big(\underset{i\in\mathscr{I}}{\underset{\longrightarrow}{\lim}} X_i,\, \underset{j\in\mathscr{J}}{\underset{\longleftarrow}{\lim}} A_j\Big) \;\simeq\; \underset{i\in\mathscr{I}}{\underset{\longleftarrow}{\lim}}\, \underset{j\in\mathscr{J}}{\underset{\longleftarrow}{\lim}}\, \mathbf{C}\,(X_i, A_j)\,; \tag{9}$$

(ii) for a pair of *adjoint functors* between ∞-categories, there is a natural equivalence between these hom-∞-groupoids (7), of the usual form [Jo08b, p. 159][Lur09, Def. 5.2.2.7][RV21, Prop. F.5.6]:

$$\begin{tikzcd} \mathbf{D} \arrow[rr, bend right, "R"', "\text{right adjoint}"'{name=R}, leftarrow] \arrow[rr, phantom, "\perp"] & & \mathbf{C} \arrow[ll, bend right, "L"', "\text{left adjoint}"] \end{tikzcd} \qquad \Leftrightarrow \qquad \underbrace{\mathbf{C}\,(C, R(D)) \;\simeq\; \mathbf{D}\,(L(C), D)}_{\text{natural equivalence of hom } \infty\text{-groupoids}}\,; \tag{10}$$

(iii) in usual consequence, right (left) ∞-adjoint ∞-functors preserve ∞-limits (∞-colimits) via natural equivalences:

$$R\Big(\underset{i\in\mathscr{I}}{\underset{\longleftarrow}{\lim}} X_i\Big) \;\simeq\; \underset{i\in\mathscr{I}}{\underset{\longleftarrow}{\lim}}\, R(X_i)\,, \qquad\qquad L\Big(\underset{i\in\mathscr{I}}{\underset{\longrightarrow}{\lim}} X_i\Big) \;\simeq\; \underset{i\in\mathscr{I}}{\underset{\longrightarrow}{\lim}}\, L(X_i)\,. \tag{11}$$

Therefore, with a good supply of systems of adjoint ∞-functors (10) – which we gain by invoking *modal* and specifically *cohesive* homotopy theory ([SSS12, §3.1][Sc13][SS20-Orb], see §3.3 below) – many proofs in higher geometry, which may be formidable when done in simplicial components, are reduced to formal manipulations yielding strings of such natural equivalences. This is how we prove the main theorems in chapter 4.

Homotopy fibers and ∞-groups. While ∞-category theory thus parallels category theory in the abstract, a key difference is that fiber sequences in ∞-categories (whenever they exist) are, generically, *long*:

$$\cdots \xrightarrow{\mathrm{fib}^6(f)} \Omega^2 C \xrightarrow{\mathrm{fib}^5(f)} \Omega A \xrightarrow{\mathrm{fib}^4(f)} \Omega B \xrightarrow{\mathrm{fib}^3(f)} \Omega C \xrightarrow{\mathrm{fib}^2(f)} A \xrightarrow{\mathrm{fib}(f)} B \xrightarrow{f} C\,. \tag{12}$$

In particular, the (homotopy-)fiber of a point inclusion is not in general trivial (as it necessarily is in 1-category theory), but is the *looping*

$$
\begin{tikzcd}
\Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "x"] \\
* \arrow[r, "x"'] & X,
\end{tikzcd}
\tag{13}
$$

whence the long homotopy fiber sequences (12) follow by the pasting law (5):

$$
\begin{tikzcd}
\Omega^2 C \arrow[r] \arrow[d, "\mathrm{fib}^5(f)"] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] & & \\
\Omega A \arrow[r, "\mathrm{fib}^4(f)"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Omega B \arrow[r] \arrow[d, "\mathrm{fib}^3(f)"] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] & \\
* \arrow[r] & \Omega C \arrow[r, "\mathrm{fib}^2(f)"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[r] \arrow[d, "\mathrm{fib}(f)"] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] \\
& * \arrow[r] & B \arrow[r, "f"] & C.
\end{tikzcd}
$$

∞-Toposes. In ∞-categories $\mathbf{H}$ of higher stacks (3), namely in *∞-toposes* [Lur03][TV05][Lur09][Re10], every group object (∞-group) $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ arises, uniquely up to equivalence, as the looping (13) of its *delooping stack* $\mathbf{B}\mathscr{G} \in \mathbf{H}$ (recalled in Prop. 0.2.1):

$$
\mathscr{G} \;\simeq\; \Omega_* \mathbf{B}\mathscr{G}\,. \tag{14}
$$

Accordingly, the connected components $\tau_0(-)$ (see Prop. 3.2.53 below) of hom-∞-groupoids (7) in ∞-toposes $\mathbf{H}$ may be understood as (non-abelian, generalized) *cohomology theories* ([FSS20-TCD, §2][SS20-Orb, p. 6]):

$$
\underset{\substack{\text{domain}\\ \text{space}}}{X} \;,\; \underset{\substack{\text{coefficient stack/}\\ \text{classifying stack}}}{A} \;\in \mathbf{H} \qquad \vdash \qquad \underset{\substack{A\text{-cohomology of } X\\ \text{in degree } \bullet}}{H^\bullet(X,A)} \;:=\; \tau_0\, \mathbf{H}\big(X, \Omega^{-\bullet} A\big)\,.
$$

Specifically, for $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ we have *first non-abelian cohomology sets*:

$$
H^1(X;\mathscr{G}) \;=\; \tau_0 \mathbf{H}(X;\mathbf{B}\mathscr{G})\,.
$$

Moreover, the *fundamental theorem of ∞-topos theory* ([Lur09, Prop. 6.5.3.1], see around Prop. 3.2.48 below) says that for every object $B \in \mathbf{H}$ in an ∞-topos, the slice ∞-category $\mathbf{H}_{/B}$, with

$$
\underset{\substack{\text{slice hom } \infty\text{-groupoid}\\ \text{in H over } B}}{\mathbf{H}\big((X_1,p_1),(X_2,p_2)\big)_B} \;=\; \left\{
\begin{tikzcd}
X_1 \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \arrow[dl, phantom, shift right=4ex, "\Leftarrow"] \\
& B &
\end{tikzcd}
\right\} \;\in\; \mathrm{Grpd}_\infty\,, \tag{15}
$$

is itself an ∞-topos. When $B = \mathbf{B}\mathscr{G}$, the cohomology in the slice $\mathbf{H}_{/\mathbf{B}\mathscr{G}}$ is *Borel-$\mathscr{G}$-equivariant cohomology* of $\mathbf{H}$ (see Def. 4.3.20 below):

$$
\underset{\substack{\text{Borel-equivariant}\\ A\text{-cohomology of } X}}{H^\bullet_{\mathscr{G}}(X;A)} \;:=\; \tau_0\, \mathbf{H}\big(X /\!\!/ \mathscr{G}, \Omega^{-\bullet}_{\mathbf{B}\mathscr{G}}(A /\!\!/ \mathscr{G})\big)_{\mathbf{B}\mathscr{G}}\,.
$$

This is a consequence of the following fact:

Transformation groups in an ∞-topos. The key fact which propels our general theory in chapter 4 is (following [NSS12a] and [SS20-Orb, §2.2]): that the notions of *groups*, of *group actions*, of *principal bundles* and their *moduli stacks* and associated *fiber bundles* are all *native to ∞-topos theory*, in that these concepts and their pertinent properties are available internal to any ∞-topos without needing further axiomatization:

Proposition 0.2.1 (Groups, actions and principal bundles in any ∞-topos (Props. 3.2.70, 3.2.75, Thm. 3.2.97))**.** *Let* $\mathbf{H}$ *be an* ∞*-topos.*

(i) *The operation of forming loop space objects* (13) *constitutes an equivalence*[3] *of group objects with pointed connected objects in* $\mathbf{H}$ *(Ntn. 3.2.54):*

$$
\begin{tikzcd}
\mathrm{Grp}(\mathbf{H}) \arrow[r, shift right, "\mathbf{B}"'] \arrow[r, phantom, "\simeq"] & \mathbf{H}^{*/}_{\geq 1} \arrow[l, shift right, "\Omega"'] \\[-15pt]
\mathscr{G} \arrow[r, mapsto] & * /\!\!/ \mathscr{G}
\end{tikzcd}
\qquad \text{i.e.} \qquad
\begin{tikzcd}
\overset{\text{group stack}}{\mathscr{G}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{base}\\ \text{point}}}{*} \arrow[d, "\mathrm{pt}_{\mathbf{B}G}"] \\
* \arrow[r, "\mathrm{pt}_{\mathbf{B}G}"] & \mathbf{B}\mathscr{G}
\end{tikzcd}
\tag{16}
$$

hence

$$
X \in \mathbf{H}^{*/} \text{ is connected} \qquad \Leftrightarrow \qquad X \simeq \mathbf{B}\Omega X\,. \tag{17}
$$

(ii) *For* $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$*, the* $\mathscr{G}$*-actions (Def. 3.2.74) and* $\mathscr{G}$*-principal bundles (Def. 3.2.94) are both identified with the slice objects* (15) *over the delooping* $\mathbf{B}\mathscr{G}$ (0.2.1)*, as follows:*

$$
\begin{tikzcd}
\mathscr{G}\,\mathrm{Act}(\mathbf{H}) \arrow[r, "\sim", "(-)/\!\!/G"'] & \mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H}) & \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[l, "\sim"', "\mathrm{fib}"] \\
G \circlearrowright P \arrow[r, mapsto] & \left(\begin{smallmatrix} P \\ \downarrow \\ P/\!\!/\mathscr{G} \end{smallmatrix}\right) & \left(\begin{smallmatrix} P/\!\!/\mathscr{G} \\ \downarrow \\ \mathbf{B}\mathscr{G} \end{smallmatrix}\right) \arrow[l, mapsto]
\end{tikzcd}
\tag{18}
$$

i.e.,

$$
\begin{tikzcd}
\text{action } \mathscr{G} \circlearrowright \overset{\substack{\mathscr{G}\text{-principal}\\ \text{bundle}}}{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{universal}\\ \mathscr{G}\text{-principal bundle}}}{* \simeq \mathscr{G}/\!\!/\mathscr{G}} \arrow[d, "\mathrm{pt}_{\mathbf{B}\mathscr{G}}"] \\
\substack{\text{action quotient-}\\ \text{/ base-stack}} \; P/\!\!/\mathscr{G} \arrow[r, "\vdash P", "\text{cocycle}"'] & \mathbf{B}\mathscr{G} \; \substack{\text{universal}\\ \text{moduli stack}}
\end{tikzcd}
\tag{19}
$$

Equivariant principal ∞-bundles. With the conceptualization of Prop. 0.2.1 in hand, there is an evident general-abstract definition of G-equivariant principal ∞-bundles (Def. 4.1.22): These must simply be the principal ∞-bundles internal to a slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$ over the delooping $\mathbf{B}G$:

$$
\underset{}{\overset{\substack{G\text{-equivariant } \Gamma\text{-principal } \infty\text{-bundles in } \mathbf{H}}}{G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnBdl}(\mathbf{H})_X}} \quad := \quad \overset{\substack{\Gamma/\!\!/G\text{-principal } \infty\text{-bundles in } \mathbf{H}_{/\mathbf{B}G}}}{(\Gamma/\!\!/G)\,\mathrm{PrnBdl}(\mathbf{H}_{/\mathbf{B}G})_{X/\!\!/G}}\,.
$$

We prove (in (4.110) of Thm. 4.2.7) that, for the case $\mathbf{H} := \mathrm{SmthGrpd}_\infty$ and restricted to topological G-spaces X and topological structure groups Γ, this canonical ∞-topos-theoretic definition recovers the traditional definition of topological equivariant principal bundles, including their equivariant local triviality property (which are all reviewed and developed in Chapter 2).

By extension, this means that for more general $\mathbf{H}$ and/or more general $X, \Gamma \in \mathbf{H}_{/\mathbf{B}G}$, we obtain sensible generalizations of these classical definitions. For example, by taking $\mathbf{H}$ to be the cohesive ∞-topos of super-geometric ∞-groupoids ([SS20-Orb, §3.1.3]) the general theory developed here immediately produces a good theory of equivariant higher super-gerbes (as needed, e.g., in super-string theory on super-orbifolds [FSS13][HSS18]).

Finally, this abstract definition in combination with the orbi-smooth Oka principle (1) implies the classification of stable G-equivariant Γ-principal bundles, at least in the case that G-singularities are cover-resolvable (Ntn. 3.3.54) and Γ is a truncated Hausdorff group. This is the content of our main Theorems 4.1.13 ands 4.3.24 below.

Even though this classification result concerns only equivariant "1-bundles" instead of more general equivariant ∞-bundles, the cohesive ∞-topos theory drives the proof: For example, our generalization of the existing classification results to non-trivial discrete G-action on the structure group Γ is a direct consequence (see the end of the proof of Thm. 3.3.90) of the general fact (Prop. 3.3.8) that the shape operation in cohesive ∞-toposes preserves homotopy fiber products over geometrically discrete groupoids (such as $\mathbf{B}G$ for discrete G).

[3] This is the *May recognition theorem* [May72] generalized from Grpd_∞ to ∞-toposes by [Lur09, 7.2.2.11][Lur17, 6.2.6.15].

0.3 Notation and Terminology

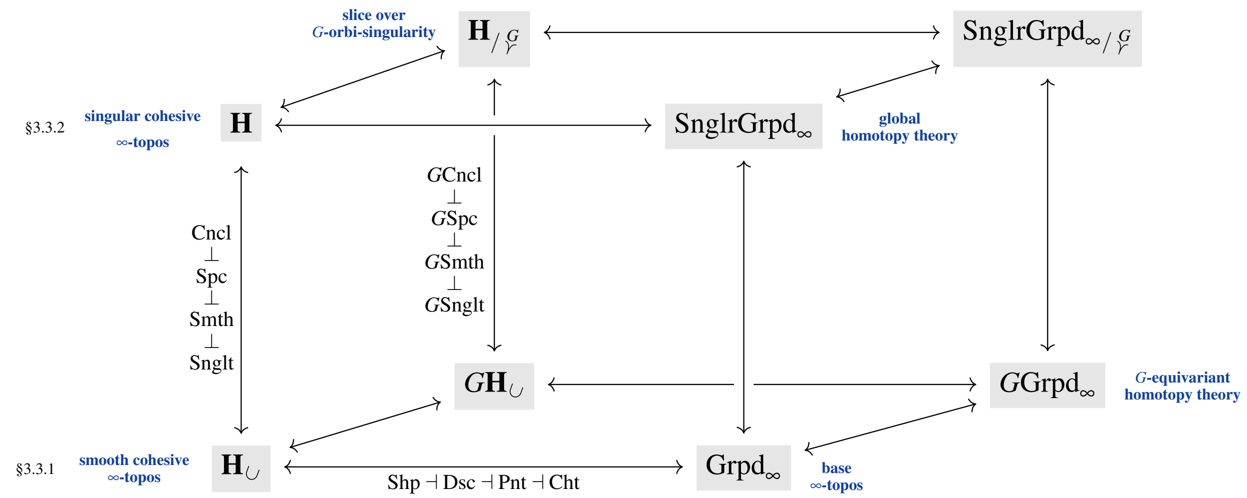
Categories and functors.

Category	**of**	
kTopSpc	cg topological spaces	Ntn. 1.0.16
kHausSpc	cg Hausdorff spaces	Ntn. 1.0.16
$G\,\mathrm{Act}(\mathrm{kTopSpc})$	topological G-spaces	Ntn. 1.1.1
$\mathrm{Grp}(\mathrm{Set})$	discrete groups	
$\mathrm{Grp}(\mathrm{kHausSpc})$	Hausdorff groups	
$\mathrm{Frm}\Gamma\mathrm{PrnBdl}(\mathscr{C})$	formally principal internal bundles	Ntn. 1.0.25
$G\,\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}$	equivariant principal bundles	Def. 2.1.3
$G\,\mathrm{Equv}\,\Gamma\mathrm{PrnFibBdl}$	... equivariantly locally trivial	Def. 2.2.2
CartSpc	Cartesian spaces	
DfflgSpc	diffeological spaces	Ntn. 3.3.15
DTopSpc	D-topological spaces	Ntn. 3.3.18
ΔSet	simplicial sets	Ntn. 3.1.1
ΔCat	simplicial categories	Ntn. 3.1.3
ΔPSh	simplicial presheaves	Ntn. 3.2.26
2-category	**of**	
Grpd	groupoids	
$\mathrm{Grpd}(\mathrm{kTopSpc})$	topological groupoids	Ntn. 1.2.1
$\mathrm{Ho}_2(\mathrm{PresCat}_\infty)$	presentable ∞-categories	Prop. 3.1.6
∞-category	**of**	
Grpd_∞	∞-groupoids	Ntn. 3.1.10
$\mathrm{SmthGrpd}_\infty$	smooth ∞-groupoids	Ntn. 3.3.26
$\mathrm{SnglrSmthGrpd}_\infty$	singular smooth ∞-groupoids	Ntn. 3.3.66
Functor	**producing**	
N	simplicial nerve	Ntn. 1.2.24
$\lvert - \rvert$	topological realization	Ntn. 1.2.28
Cdfflg	continuous diffeology	Ex. 3.3.17
Dtplg	D-topology	Ex. 3.3.17

Types of groups.

G	$\in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathbf{H})$	equivariance group	Ntn. 1.1.1
$G \circlearrowright \Gamma$	$\in \mathrm{Grp}(\mathrm{kTopSpc})$	equivariant structure group	Def. 2.1.2, Lem. 2.1.4
$\mathscr{G}$	$\in \mathrm{Grp}(\mathbf{H})$	structure ∞-group	Prop. 0.2.1
$G \circlearrowright \Gamma$	$\in \mathrm{Grp}\big(G\,\mathrm{Act}(\mathbf{H})\big)$	equivariant structure ∞-group	Def. 4.1.16
$\Gamma /\!\!/ G$	$\in \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}G})$	equivariant structure ∞-group	Def. 4.1.16

Ambient ∞-toposes.



Here:

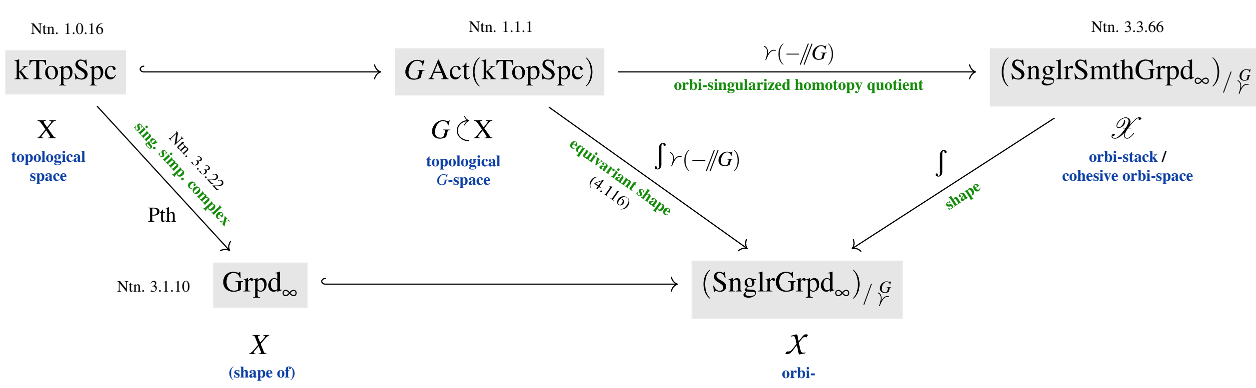
$$\mathbf{H}_\cup := \mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CartSpc}) \quad \text{(Ntn. 3.3.26, [SS20-Orb, Ex. 3.18]),}$$

$$\mathbf{H} := \mathrm{SnglrSmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CartSpc} \times \mathrm{Snglrt}) \quad \text{(Ntn. 3.3.66, [SS20-Orb, Ex. 3.56]).}$$

Modalities.

Cohesion Def. 3.3.1	shape $\int := \mathrm{Dsc} \circ \mathrm{Shp}$	discrete $\flat := \mathrm{Dsc} \circ \mathrm{Pnt}$	sharp $\mathrm{chaotic} := \mathrm{Cht} \circ \mathrm{Pnt}$
Singularities Def. 3.3.62	conical $\vee := \mathrm{Spc} \circ \mathrm{Cncl}$	smooth $\cup := \mathrm{Spc} \circ \mathrm{Smth}$	orbisingular $\prec := \mathrm{Snglt} \circ \mathrm{Smth}$
***G*-singularities** Def. 3.3.78	*G*-conical $\vee_{\prec^G} := G\mathrm{Spc} \circ G\mathrm{Cncl}$	*G*-smooth $\cup_{\prec^G} := G\mathrm{Spc} \circ G\mathrm{Smth}$	*G*-orbisingular $\prec_{\prec^G} := G\mathrm{Snglt} \circ G\mathrm{Smth}$

Notions of space.



Part II

In topological spaces

Chapter 1

Equivariant topology

We recall and develop basics of equivariant algebraic topology (see [Il72][Bre72][tD87][May96][Blu17]) that we invoke below in chapter 2. The expert reader may want to skip this chapter and refer back to it just as need be.

§1.1 recalls basics of equivariant point-set topology, highlighting how the change-of-group adjoint triple governs the theory, and establishing lemmas needed in chapter 2.

§1.2 recalls basics of topological groupoids, generalizing to equivariant topological groupoids, and establishing lemmas needed in §2.3.

§1.3 recalls basics of proper equivariant homotopy theory, recording lemmas needed in §2.3 and §4.2.

Throughout, we make extensive use of a hierarchy of *internalizations* of mathematical structures (Ntn. 1.0.23) into *categories with pullbacks* (Ntn. 1.0.5), starting in a *convenient category of topological spaces* (Ntn. 1.0.16). Here is the basic terminology and notation that we are using:

Categories. In §1.1 and chapter 2 we need just basic notions of (co)limits and adjoint functors in plain category theory (e.g., [AHS90][Bo94I][Bo94II]), while in §1.2 and §4.2 we need these notions in their Grpd-enriched enhancement (e.g., [Bo94II, §6][Rie14, §3][JY21, §1.3]) – but we only need the most basic concepts: enriched adjunctions and conical (i.e., non-weighted) enriched (co-)limits, and here mostly just finite ones.

Notation 1.0.1 (Basic categories)**.** We write
(i) Set for the category of sets with functions between them;
(ii) Grpd $:=$ Grpd(Set) for the category of small groupoids with functors between them.

Notation 1.0.2 (Morphisms)**.** Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$ a pair of its objects. We write
(i) $\mathscr{C}(X,Y) \in$ Set for the set of morphisms $X \to Y$ in $\mathscr{C}$ (the *hom-set*);
(ii) $X \xrightarrow{\sim} Y$ to indicate *isomorphisms*.

Notation 1.0.3 (Grpd-enriched categories)**.** With Grpd (Ntn. 1.0.1) regarded as a cartesian monoidal category, a *strict (2,1)-category*, namely a Grpd-*enriched category* [FM83], has for each pair of objects $X, Y \in \mathscr{C}$ a *hom-groupoid* $\mathscr{C}(X,Y) \in$ Grpd. Equivalently, this is a *strict 2-category* or Cat-*enriched category* (e.g. [JY21, §2.3][Ric20, §9.5]) whose hom-categories happen to be groupoids.

Notation 1.0.4 (Adjoint functors)**.** We denote pairs of adjoint functors as shown on the left here:

$$\mathscr{D} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathscr{C} \qquad \Leftrightarrow \qquad \begin{array}{ccc} \mathscr{C}(c, R(d)) & \simeq & \mathscr{D}(L(c), d) \\ c \xrightarrow{f} R(d) & \leftrightarrow & L(c) \xrightarrow{\tilde{f}} d \end{array} \tag{1.1}$$

meaning that for all objects $c \in \mathscr{C}$ and $d \in \mathscr{C}$ there is a natural isomorphism ("forming adjuncts") between the hom-objects (Ntn. 1.0.2, 1.0.3) out of the image of left adjoint functor L and into the image of the right adjoint functor E, as shown on the right.

Notation 1.0.5 (Cartesian/pullback squares)**.** We indicate that a commuting square of morpisms in some category $\mathscr{C}$ (Ntn. 1.0.2) – here typically in the category of $G\mathrm{Act}(\mathrm{kTopSpc})$ (1.8) – is a *pullback square* (also: *Cartesian square* or *fiber product*) by putting the symbols "(pb)" at its center. This means that each pair of morphisms forming another commuting square with its right and bottom morphism (a "cone") factors uniquely through the its top left object such that the resulting triangles commute:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\forall"] \arrow[ddr, bend right, "\forall"'] & & \\
& \mathrm{X} \times_{\mathrm{B}} \mathrm{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P} \arrow[d] \\
& \mathrm{X} \arrow[r] & \mathrm{B}
\end{tikzcd}
\qquad \text{e.g.} \qquad
\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\forall"] \arrow[ddr, bend right, "\forall"'] & & \\
& \mathrm{X} \times \mathrm{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P} \arrow[d] \\
& \mathrm{X} \arrow[r] & *
\end{tikzcd}
$$

fiber product / pullback **product**

Over the *terminal object*, denoted by a point: $Q \overset{\exists!}{\dashrightarrow} *$, a fiber product is a plain *product*, as shown on the right.

Proposition 1.0.6 (Finite limits, e.g. [Bo94I, Prop. 2.8.2])**.** *In the presence of a terminal object, every finite limit is an iteration of pullbacks (Ntn. 1.0.5).*

Example 1.0.7 (Pullback preserves isomorphisms)**.** A commuting square with a bottom isomorphism (Ntn. 1.0.2) is a pullback square (Ntn. 1.0.5) if and only if also the top morphism is an isomorphism:

$$
\begin{tikzcd}
\mathrm{A} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P} \arrow[d] \\
\mathrm{X} \arrow[r, "\sim"'] & \mathrm{B}
\end{tikzcd}
\quad \Leftrightarrow \quad
\mathrm{A} \xrightarrow[\sim]{f} \mathrm{P}\,.
$$

Proposition 1.0.8 (Right adjoint functors preserve limits)**.** *A right adjoint functor R (Ntn. 1.0.4) preserves all limits, in particular it preserves all finite limits (Prop. 1.0.6) and hence terminal objects and pullbacks (Ntn. 1.0.5):*

$$R(\mathrm{X} \times_{\mathrm{B}} \mathrm{P}) \simeq R(\mathrm{X}) \times_{R(\mathrm{B})} R(\mathrm{P})\,.$$

Proposition 1.0.9 (Pasting law (e.g. [AHS90, Prop. 11.10]))**.** *Given two adjacent commuting squares where the right one is a pullback (Ntn. 1.0.5) then the left square is a pullback if and only if the total rectangle is:*

$$
\begin{tikzcd}
\mathrm{P}_1 \arrow[r] \arrow[d] & \mathrm{P}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_3 \arrow[d] \\
\mathrm{X}_1 \arrow[r] & \mathrm{X}_2 \arrow[r] & \mathrm{X}_3
\end{tikzcd}
$$

Notation 1.0.10 (Effective epimorphism [Gr61III, p. 101][Bo94II, Def. 2.5.3])**.** A morphism p is called a *regular epimorphism* if it is the coequalizer of *some* parallel pair of morphisms, and an *effective epimorphism*, to be denoted by double-headed arrows, if it is the coequalizer specifically of the two projections out of its pullback (Ntn. 1.0.5) along itself:

$$
\begin{tikzcd}
\mathrm{P} \times_{\mathrm{X}} \mathrm{P} \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & \mathrm{P} \arrow[r, two heads, "p", "\mathrm{coeq}"'] & \mathrm{X}\,.
\end{tikzcd}
$$

Definition 1.0.11 (Regular categories ([BGvO71][Bo94II, §2][Gra21]))**.** A category is called *regular* if

(i) for every morphism $X \to Y$,

- **(a)** the fiber product $X \times_Y X$ exists (the "kernel pair");
- **(b)** the coequalizer $X \times_Y X \rightrightarrows X \xrightarrow{\mathrm{coeq}} X/(X \times_Y X)$ exists (the *image*);

(ii) pullbacks of regular epimorphisms (Ntn. 1.0.10) exist and are again regular epimorphisms.

In this Part I we are interested in the exceptional example of the category of compactly-generated topological spaces (Prop. 1.0.20 below). A generic class of examples of regular categories are toposes (such as the category of simplicial sets, which is of interest in Part II, see Ntn. 3.1.1 below):

Example 1.0.12 (Toposes are regular [BGvO71, p. 17][Bo94III, Prop. 3.4.14][Jo02a, p. 92])**.** Every topos (hence in particular every category of presheaves) is a regular category (Def. 1.0.11). Moreover, in toposes the classes of (i) epimorphisms, (ii) regular epimorphisms and (iii) effective epimorphisms (Ntn. 1.0.10) all coincide (e.g. [MLM92, §IV.7, Thm. 8 (p. 197)][Bo94III, Prop. 3.4.13, 3.4.15]).

Lemma 1.0.13 (Effective epimorphisms in regular categories (e.g. [Bo94I, Prop. 2.5.7][Bo94I, Prop. 2.3.3]))**.** *In a regular category (Def. 1.0.11), the notions of regular and of effective epimorphisms (Ntn. 1.0.10) coincide, and pullback along any morphism f preserves effective epimorphisms p together with their coequalizer diagrams:*

$$
\begin{tikzcd}
(f^*\mathrm{P}) \times_{\mathrm{X}'} (f^*\mathrm{P}) \arrow[r] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P} \times_{\mathrm{X}} \mathrm{P} \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] \\
f^*\mathrm{P} \arrow[r] \arrow[d, two heads, "\mathrm{coeq}"', "f^*p"] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P} \arrow[d, two heads, "\mathrm{coeq}"', "p"] \\
\mathrm{X}' \arrow[r, "f"'] & \mathrm{X}
\end{tikzcd}
$$

In regular categories, there are partial reverses to the implications of Ex. 1.0.7 and Prop. 1.0.9 (see also the ∞-category theoretic version in Lem. 3.3.9 below):

Lemma 1.0.14 (Reverse pasting law in regular categories (e.g. [Gra21, Lem. 1.15]))**.** *Given a commuting diagram in a regular category (Def. 1.0.11) of the form*

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, two heads] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

where the left square is Cartesian (Ntn. 1.0.5) and the bottom left morphism is an effective epimorphism (Ntn. 1.0.10), then the right square is Cartesian if and only if the total rectangle is Cartesian.

Lemma 1.0.15 (Local recognition of isomorphisms in regular categories)**.** *In a regular category (Def. 1.0.11), if the pullback p^*f of a morphism f along an effective epimorphism p (Ntn. 1.0.10) is an isomorphism, then f was already an isomoprhism itself.*

Proof. By assumption, we have a pullback square as on the bottom of the following diagram:

$$
\begin{tikzcd}
\widehat{X} \times_X \widehat{X} \arrow[r, "\sim"] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{Y} \times_Y \widehat{Y} \arrow[d, shift right] \arrow[d, shift left] \\
\widehat{X} \arrow[r, "p^*f", "\sim"'] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{Y} \arrow[d, two heads, "p"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Here the bottom left morphism is an effective epimorphism by Lem. 1.0.13, since p is so by assumption. Since limits commute with each other, we get the top pullback squares, where the topmost morphism is an isomorphism as the pullback of the isomorphism p^*f (by Ex. 1.0.7). By the nature of effective epimorphisms (Ntn. 1.0.10), this now exhibits f as the image under passage to coequalizers of an isomorphism of coequalizer diagrams, hence as an isomorphism. □

Topological spaces. We use the following *convenient category of topological spaces* [St67] which has become the standard foundation for algebraic topology:

Notation 1.0.16 (Category of compactly-generated topological spaces)**.** We write

$$\begin{tikzcd} \underset{\text{topological spaces}}{\mathrm{TopSpc}} \arrow[r, shift right, "k"'] \arrow[r, phantom, "\perp"] & \underset{\text{topological k-spaces}}{\mathrm{kTopSpc}} \arrow[l, hook', shift right] & \underset{\text{Hausdorff k-spaces}}{\mathrm{kHausSpc}} \arrow[l, hook'] & \underset{\substack{\text{locally compact}\\ \text{Hausdorff spaces}}}{\mathrm{LCHausSpc}} \arrow[l, hook'] \end{tikzcd} \tag{1.2}$$

for the coreflective subcategory of those topological spaces which are the colimits of all images of compact spaces inside them (*k-spaces* [Ga50, §1]), and its full subcategory of Hausdorff spaces among these (subsuming all locally compact Hausdorff spaces), hence of *compactly generated* topological spaces (e.g. [Du66, §XI.9][St67][Le78] [HSt97, §3.4], concise practical review is in [FHTh00, §0], and specifically for the equivariant context in [LU14, §16]).

Remark 1.0.17 (Mapping spaces)**.** In particular, the category of k-spaces is Cartesian closed, which means that, for $\mathrm{X}, \mathrm{Y} \in \mathrm{kTopSpc}$ (1.2), the *mapping space*

$$\mathrm{Maps}(\mathrm{X}, \mathrm{Y}) \quad \in \ \mathrm{kTopSpc} \tag{1.3}$$

(namely the set of continuous functions $\mathrm{X} \to \mathrm{Y}$ equipped with the k-ified compact-open topology) serves as an *exponential object* or *cartesian internal hom*, (e.g., [Nie78, §9][Pic92, Thm. B.10][Bo94II, §7.1-7.2]). That is, $\mathrm{Maps}(\mathrm{X}, -)$ is right adjoint (Ntn. 1.0.4) to forming the categorical product (the k-ified product topological space) with X:

$$\begin{tikzcd} \mathrm{kTopSpc} \arrow[r, shift right, "\mathrm{Maps}(\mathrm{X},-)"'] \arrow[r, phantom, "\perp"] & \mathrm{kTopSpc} \arrow[l, shift right, "\mathrm{X}\times(-)"'] \end{tikzcd}. \tag{1.4}$$

Besides making the adjunction (1.4) work, compactly generated topological spaces behave essentially like plain topological spaces:

Remark 1.0.18 (Colimits of compactly generated topological spaces)**.** Colimits of k-spaces (Ntn. 1.0.16) are computed as usual colimits of topological spaces. For instance:
(i) Orbit spaces – i.e., the usual quotient topological spaces of continuous group actions (see Lem. 1.1.16 and Cor. 1.1.23 below) – are (split) coequalizers in kTopSpc (1.2).
(ii) It is still true that the functor $\pi_0 : \mathrm{kTopSpc} \to \mathrm{Set}$ assigning sets of path-connected components preserves coequalizers (quotients by relations) and it preserves finite products:

$$\mathrm{X}, \mathrm{Y} \ \in \ \mathrm{kTopSpc} \quad \Rightarrow \quad \begin{cases} \pi_0\big(\mathrm{coeq}(\mathrm{R} \rightrightarrows \mathrm{X})\big) & \simeq \ \mathrm{coeq}\big(\pi_0(\mathrm{R}) \rightrightarrows \pi_0(\mathrm{X})\big) \\ \pi_0(\mathrm{X}\times\mathrm{Y}) & \simeq \ \pi_0(\mathrm{X})\times\pi_0(\mathrm{Y}) \end{cases} \quad \in \ \mathrm{Set}. \tag{1.5}$$

Remark 1.0.19 (Relation to locally compact Hausdorff spaces)**.**
(i) If $\mathrm{X} \in \mathrm{LCHausSpc} \hookrightarrow \mathrm{kTopSpc}$ (1.2) then, for any $\mathrm{Y} \in \mathrm{kTopSpc}$, the usual product topological space is the category-theoretic product $\mathrm{X}\times\mathrm{Y} \in \mathrm{kTopSpc}$ in k-spaces ([Le78, Lem. 2.4][Pic92, Thm. B.6]).
(ii) Any topological space is a k-space if and only if it is a quotient topological space of a locally compact Hausdorff space. ([ELS04, Lem. 3.2 (v)], strengthening [Du66, §XI, Thm. 9.4][Pic92, Thm. B.4]).

Moreover:

Proposition 1.0.20 (Compactly-generated topological spaces form a regular category [CMV95, p. 3])**.**
The categories kTopSpc *and* kHausSpc *(Ntn. 1.0.16) are regular (Def. 1.0.11).*

Example 1.0.21 (Open covers are effective epimorphisms)**.** For $\mathrm{X} \in \mathrm{kTopSpc}$ and $\{\mathrm{U}_i \hookrightarrow \mathrm{X}\}_{i\in I}$ an open cover, then the canonical map $\underset{i\in I}{\sqcup}\mathrm{U} \twoheadrightarrow \mathrm{X}$ is an effective epimorphism (Def. 3.2.11), in that the canonical maps

$$\underset{j_1,j_2\in I}{\sqcup}\mathrm{U}_{j_1}\cap\mathrm{U}_{j_2} \rightrightarrows \coprod_{i\in I}\mathrm{U}_i \longrightarrow \mathrm{X}$$

make a coequalizer diagram. (Namely, this is the case on underlying sets, by the fact that the covering is surjective; hence the remaining condition is that a subset of X is open precisely if its intersection with each U_i is open, which is the case by the fact that the covering is by open subsets.)

Example 1.0.22 (Isomorphism of bundles is detected on covers)**.** Given a commuting diagram in kTopSpc (Ntn. 1.0.16) of the form

$$\begin{tikzcd} \mathrm{P} \arrow[rr, "f"] \arrow[dr] & & \mathrm{P}' \arrow[dl] \\ & \mathrm{X} & \end{tikzcd}$$

then f is an isomorphism as soon as it is locally so, hence if its pullback to any open cover $\widehat{X} := \coprod_{i \in I} \mathrm{U}_i$ is an isomorphism:

$$\begin{tikzcd} \mathrm{P}|_{\widehat{X}} \arrow[rr] \arrow[dr, "\sim"] \arrow[ddr, bend right] \arrow[drr, phantom, "\scriptstyle(\mathrm{pb})"] & & \mathrm{P} \arrow[dr, "f"] \arrow[ddr, bend left] & \\ & \mathrm{P}'|_{\widehat{\mathrm{X}}} \arrow[rr, two heads] \arrow[d] \arrow[drr, phantom, "\scriptstyle(\mathrm{pb})"] & & \mathrm{P}' \arrow[d] \\ & \widehat{X} \arrow[rr, two heads] & & \mathrm{X}\,. \end{tikzcd}$$

Namely, here the bottom morphism is an effective epimorphism by Ex. 1.0.21, hence the middle morphism is an effective epimorphism by regularity of kTopSpc (Prop. 1.0.20). Now with the rear square also the top square is Cartesian, by the pasting law (Prop. 1.0.9), whence the top square exhibits the pullback of f along an effective epimorphism as an isomorphism, so that Lem. 1.0.15, implies that it is itself an isomorphism.

Internal mathematical structures. We find below that equivariant topology is both a beautiful example of and is itself beautified by a systematic use of *internalization* of mathematical structures into ambient categories other than Set; a basic point that seems not to have received due attention.

Notation 1.0.23 (Internal mathematical structures)**.** For S a mathematical structure expressible in terms of finite limits (a "finite limit sketch" [BE72][BW83, §4][AR94, §1.49]), hence by operations on fiber products (Prop. 1.0.6), and for $\mathscr{C}$ any category,
(i) we write $S(\mathscr{C})$ for the category of S-models in $\mathscr{C}$, hence the category of S-structures *internal* to $\mathscr{C}$ in the original sense of [Gr60II, p. 370].
(ii) For F a functor that preserves finite limits (denoted *lex*), there is the evident induced functor on S-structures, which we denote as follows:

$$F : \mathscr{C} \xrightarrow{\text{lex}} \mathscr{D} \qquad \vdash \qquad S(F) \,:\, S(\mathscr{C}) \longrightarrow S(\mathscr{D})\,. \tag{1.6}$$

Example 1.0.24 (Archetypical examples of internal structures)**.** We have the categories:

- $\mathrm{Grp}(\mathscr{C})$ of internal groups (e.g., [EH61][EH62][EH63][BW83, §4.1], see Def. 2.1.2 below);
- $G\mathrm{Act}(\mathscr{C})$ of internal group actions, (e.g., [Boa95, §7][BJK05, p. 8], see Def. 2.1.3 below);
- $\mathrm{Grpd}(\mathscr{C})$ of internal groupoids (e.g., [Bo94I, §8][NP19, §1], see Ntn. 1.2.1 below);

all originally due to [Gr61III, §4]. We consider these notions mainly internal to the category $\mathscr{C} = G\mathrm{Act}(\mathrm{kTopSpc})$ (1.8) of, in turn, group actions internal to topological spaces (1.2).

Notation 1.0.25 (Internalization of principal bundle theory)**.** The key example of internal structures for our purposes is the category

- $\mathrm{Frm}\Gamma\mathrm{PrnBdl}(\mathscr{C})$ of *formally principal bundles* ([Gr60I, p. 312 (15 of 30)][Gr71, p. 9 (293)],
also: *pseudo-torsors* [Gr67, §16.5.15]),

which is the subcategory of $\Gamma\mathrm{Act}(\mathscr{C})$ (Ntn. 1.0.23) on those actions that are either fiberwise principal or empty (2.4), see Rem. 2.1.8 below.

We observe in Cor. 2.1.6 that these (formally) principal bundles, when internalized in $\mathscr{C} = G\mathrm{Act}(\mathrm{kTopSpc})$, are equivalently equivariant principal bundles in the original and general sense of [tD69], see Rem. 2.1.7 below. While this might not be a surprising observation for experts with the relevant background, it is, we find, absolutely foundational to the subject of equivariant bundle theory, and seems not to have been made before in existing literature.

1.1 G-Actions on topological spaces

Notation 1.1.1 (Equivariant topology ("transformation groups", "G-spaces", e.g.[Bre72][tD79][tD87])**.** Throughout:

- $G \in \mathrm{Grps}(\mathrm{kHausSpc})$ denotes a Hausdorff topological group, with group operation denoted $(-)\cdot(-)$;
- $H \subset G$ denotes a topological subgroup (necessarily Hausdorff, since G is);
- $N(H) \subset G$ denotes its *normalizer subgroup*, and
- $W(H) := N(H)/H$ its *Weyl group* (e.g. [May96, p. 13]):

$$\underset{\text{subgroup}}{\text{normalizer}} \;\; N(H) \longrightarrow\!\!\!\!\!\!\twoheadrightarrow N(H)/H =: W(H) \;\; \text{Weyl group} \tag{1.7}$$

- We write

$$G\mathrm{Act}(\mathrm{kTopSpc}) \;:=\; \left\{ \left(\begin{array}{l} \mathrm{X} \in \mathrm{kTopSpc}\,, \\ \rho \;:\; G \xrightarrow{\rho} \mathrm{Aut}(\mathrm{X}) \end{array} \right) \right\} \;=\; \left\{ G \circlearrowright \mathrm{X} \;:=\; \left(G \times \mathrm{X} \xrightarrow[\substack{\text{continuous}\\ \text{actions}}]{(-)\cdot(-)} \mathrm{X} \right) \right\} \tag{1.8}$$

  for the category whose objects $G \circlearrowright \mathrm{X}$ are topological spaces X equipped with continuous left G-actions and whose morphisms are G-equivariant contionuous functions between these (often: "G-spaces", for short).
- For $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$, we write
  - $\mathrm{X}^G := \left\{ x \in \mathrm{X} \,\middle|\, \underset{g \in G}{\forall}\; g \cdot x = x \right\} \hookrightarrow \mathrm{X} \quad \in \mathrm{kTopSpc}$ for the *G-fixed subspace*;
  - $\mathrm{X} \longrightarrow\!\!\!\!\twoheadrightarrow X_G := X/G := \{[x] := G \cdot x \mid \mathrm{X} \in \mathrm{X}\} \;\in \mathrm{kTopSpc}$ for the *G-quotient space* (*G-orbit space*).
- For $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ and $x \in \mathrm{X}$, its *isotropy subgroup* is denoted

$$G_x \;:=\; \mathrm{Stab}_G(x) \;:=\; \{g \in G \,|\, g \cdot x = x\} \;\subset\; G\,. \tag{1.9}$$

From §2.2 on, we make the following further assumptions on the equivariance group:

Assumption 1.1.2 (Proper equivariant topology (following [DHLPS19][SS20-Orb])**.** We speak of *proper equivariant topology* if:

- equivariance groups G are Lie groups with compact connected components;
- subgroups $H \subset G$ are compact;
- domain spaces X are locally compact and Hausdorff;
- equivariance actions $G \circlearrowright \mathrm{X}$ are proper. [1]

Lemma 1.1.3 (Equivariance subgroups in proper equivariant topology)**.** *Under Assumption 1.1.2,*

(i) *every $H \subset G$ (namely every compact subgroup of a Lie group with compact connected components) is:*
  - **(a)** *a closed subgroup;*
  - **(b)** *a compact Lie group;*

(ii) *every $G_x \subset G$ (namely the isotropy subgroup* (1.9) *of a proper action at any point x) is of this form.*

Proof. For the first statement, it is sufficient to consider the connected components of the neutral element. Here statement (a) follows since Lie groups are Hausdorff spaces and compact subspaces of compact Hausdorff spaces are equivalently closed subspaces. With this, statement (b) follows from Cartan's closed-subgroup theorem (e.g. [Le12, Thm. 10.12]). The assumption that X is locally compact and Hausdorff ensures that all notions of proper action agree, and it follows that all stabilizer subgroups of points are compact. With this, the second statement follows from the first. □

Basic examples of G-actions. To fix notation and conventions, we make explicit the following basic G-actions.

[1] Under the previous assumption that domain spaces are locally compact and Hausdorff, all notions of proper actions agree [Pa61, Thm. 1.2.9]; see also [Ka16, Rem. 5.2.4].

Example 1.1.4 (Left and right-inverse multiplication action)**.** Each $G \in \mathrm{Grp}(\mathrm{kTopSpc})$ carries canonical left G-actions (1.8), by left multiplication and by inverse right multiplication, respectively,

$$G \circlearrowright G^L \in G\mathrm{Act}(\mathrm{kTopSpc}) \qquad\qquad G \circlearrowright G^R \in G\mathrm{Act}(\mathrm{kTopSpc}) \tag{1.10}$$

$$\begin{array}{ccc} G \times G^L & \longrightarrow & G^L \\ (g,h) & \longmapsto & g \cdot h \end{array} \qquad\qquad \begin{array}{ccc} G \times G^R & \longrightarrow & G^R \\ (g,h) & \longmapsto & h \cdot g^{-1}\,. \end{array}$$

Under inversion, these two actions are isomorphic:

$$G^L \xrightarrow{(-)^{-1}} G^R \quad \in G\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{1.11}$$

Example 1.1.5 (Diagonal action)**.** For $G \circlearrowright \mathrm{X}_1, G \circlearrowright \mathrm{X}_2 \in G\mathrm{Act}(\mathrm{kTopSpc})$, one can consider the *diagonal action* $G \circlearrowright (\mathrm{X}_1 \times \mathrm{X}_2)$ on the product space of the underlying spaces:

$$\begin{array}{ccc} G \times (\mathrm{X}_1 \times \mathrm{X}_2) & \longrightarrow & \mathrm{X}_1 \times \mathrm{X}_2 \\ (g,(x_1,x_2)) & \longmapsto & (g \cdot x_1, g \cdot x_2)\,. \end{array}$$

Example 1.1.6 (Conjugation action on mapping space (e.g. [GMR19, p. 5]))**.** Let $G \circlearrowright \mathrm{X}_1, G \circlearrowright \mathrm{X}_2 \in G\mathrm{Act}(\mathrm{kTopSpc})$.
(i) The mapping space (1.3) of the underlying topological spaces carries a G-action given by *conjugation*:

$$G \circlearrowright \mathrm{Maps}(\mathrm{X}_1,\mathrm{X}_2) \quad \in G\mathrm{Act}(\mathrm{kTopSpc}) \tag{1.12}$$

$$f \in \mathrm{Maps}(\mathrm{X}_1,\mathrm{X}_2) \quad \vdash \quad \underset{g \in G}{\forall} \begin{tikzcd} \mathrm{X}_1 \arrow[r, "g\cdot f"] \arrow[d, "g^{-1}\cdot(-)"'] & \mathrm{X}_2 \\ \mathrm{X}_1 \arrow[r, "f"'] & \mathrm{X}_2 \arrow[u, "g\cdot(-)"'] \end{tikzcd} \quad \text{i.e.,} \quad \underset{x \in \mathrm{X}_1}{\forall} (g \cdot f)(x) = g \cdot \big(f(g^{-1} \cdot x)\big)\,. \tag{1.13}$$

(ii) The fixed locus of the conjugation action (1.13) is the subspace of *G-equivariant functions*

$$\underbrace{\big\{ f \in \mathrm{Maps}(\mathrm{X}_1,\mathrm{X}_2) \,\big|\, f(-) = g^{-1} \cdot f(g \cdot -) \big\}}^{\text{subspace of } G\text{-equivariant maps}} = \overbrace{\mathrm{Maps}(\mathrm{X}_1,\mathrm{X}_2)^G}^{\substack{G\text{-fixed subspace} \\ \text{of conjugation action}}} \hookrightarrow \overbrace{\mathrm{Maps}(\mathrm{X}_1,\mathrm{X}_2)}^{\substack{\text{space of all} \\ \text{continuous maps}}}\,. \tag{1.14}$$

(iii) This construction (1.12) is functorial in both arguments, contravariantly in the first. With (1.11) with means, in particular, for $G \circlearrowright X \in G\mathrm{Act}(\mathrm{kTopSpc})$ that

$$\mathrm{Maps}(G^L,\mathrm{X}) \simeq \mathrm{Maps}(G^R,\mathrm{X}) \quad \in G\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{1.15}$$

(iv) With the first argument fixed, this construction (1.12) is a right adjoint to the product operation from Ex. 1.1.5:

$$\begin{tikzcd} G\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, leftarrow, "G \circlearrowright \mathrm{X} \times (-)"'] & G\mathrm{Act}(\mathrm{kTopSpc}) \arrow[l, shift left=1.5ex, leftarrow, "G \circlearrowright \mathrm{Maps}(\mathrm{X},-)"] \end{tikzcd}\,. \tag{1.16}$$

Change of equivariance group. Much of our formulation of equivariant topology proceeds by applying the *change of equivariance group* adjoint triple from the following Lem. 1.1.7, in numerous ways.

Lemma 1.1.7 (Change of equivariance group (e.g. [May96, §I.1][DHLPS19, p. 9]))**.** *Given a continuous homomorphism of topological groups*

$$G_1 \xrightarrow{\phi} G_2\,,$$

we have a triple of adjoint functors (Ntn. 1.0.4) between their categories of continuous actions (Ntn. 1.1.1):

$$\begin{tikzcd}[column sep=huge] G_1\mathrm{Act}(\mathrm{TopSp}) \arrow[r, shift left=3ex, "G_2 \times_{G_1} (-) := \left(\phi^*(G_2^R) \times (-)\right)_{G_1}"] \arrow[r, shift left=1.5ex, phantom, "\perp"] \arrow[r, shift right=1.5ex, phantom, "\perp"] \arrow[r, shift right=3ex, "\mathrm{Maps}(G_2,-)^{G_1} := \mathrm{Maps}\left(\phi^*(G_2^L),-\right)^{G_1}"'] & G_2\mathrm{Act}(\mathrm{TopSp}) \arrow[l, "\phi^*" description] \end{tikzcd}\,, \tag{1.17}$$

where:

- *the* G_1-pullback action *on $\phi^* Y$ is through ϕ, on the same underlying topological space;*
- *the* induced G_2-action *on $G_2 \times_{G_1} X$ is that given by left multiplication of G_2 on the G_2-factor;*
- *the* co-induced G_2-action *on* $\mathrm{Maps}(G_2,X)^{G_1}$ *is given by right multiplication on the G_2-argument.*

Example 1.1.8 (Quotient spaces, fixed loci and trivial action)**.** For $G \to 1$ the unique group homomorphism to the trivial group, the corresponding pullback action (Lemma 1.1.7) is the trivial G-action, whose adjoints (1.17) form the quotient space $(-)_G$ and the G-fixed space $(-)^G$ (Ntn. 1.1.1), respectively:

$$
\begin{tikzcd}
G\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, shift left=6, "\underset{\text{quotient space}}{(-)/G}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\underset{(-)^G}{\text{fixed locus}}"'] & \mathrm{kTopSpc} \arrow[l, "\text{trivial } G\text{-action}"']
\end{tikzcd} . \tag{1.18}
$$

Example 1.1.9 (Underlying topological spaces and (co-)free actions)**.** For $1 \hookrightarrow G$ the unique inclusion of the trivial group, the corresponding pullback action (Lemma 1.1.7) is the forgetful functor from continuous G-actions to their underlying topological spaces, whose adjoints (1.17) form the free action and cofree action, respectively:

$$
\begin{tikzcd}
\mathrm{kTopSpc} \arrow[r, shift left=6, "\underset{\text{free action}}{G\times(-)}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\underset{\mathrm{Maps}(G,-)}{\text{co-free action}}"'] & G\mathrm{Act}(\mathrm{kTopSpc}) \arrow[l, "\text{forget } G\text{-action}"']
\end{tikzcd} . \tag{1.19}
$$

Lemma 1.1.10 (Forgetting G-action creates limits and colimits)**.** *The forgetful functor from topological G-actions to underlying topological spaces (Example 1.1.9) creates limits and colimits, in that a diagram of topological G-actions is a (co)limiting (co)cone diagram precisely if its underlying diagram of topological spaces is:*

$$
\begin{array}{ccc}
G\mathrm{Act}(\mathrm{kTopSpc}) & \xrightarrow{\text{forget } G\text{-action}} & \mathrm{kTopSpc} \\
G \circlearrowright \mathrm{X} \simeq \underset{\underset{i}{\longleftarrow}}{\lim}\,(G \circlearrowright \mathrm{X}_i) & \Leftrightarrow & \mathrm{X} \simeq \underset{\underset{i}{\longleftarrow}}{\lim}\,(\mathrm{X}_i) \\
G \circlearrowright \mathrm{X} \simeq \underset{\underset{i}{\longrightarrow}}{\lim}\,(G \circlearrowright \mathrm{X}_i) & \Leftrightarrow & \mathrm{X} \simeq \underset{\underset{i}{\longrightarrow}}{\lim}\,(\mathrm{X}_i)\,.
\end{array}
$$

Proof. That the forgetful functor *preserves* all limits and colimits follows (Prop. 1.0.8) from it being a right and a left adjoint (1.19). A general abstract way to see that it also reflects and hence creates all limits and colimits is to notice that G-actions are algebras for the monad $G \times (-)$, and that monadic functors create all limits which exist in their codomain, and create all colimits which exist and are preserved by the monad (e.g., [ML70, pp. 137-138]). But the monad here is the composite of the two left adjoints in the change of group adjoint triple (1.17) along $1 \to G$ and hence preserves all colimits. The resulting claim also appears in [Schw18, §B]. For the record, we spell out the reflection of pullbacks/fiber products (Ntn. 1.0.5, the general proof is directly analogous), which is the main case of interest in chapter 2:

Consider a commuting square of topological G-actions whose underlying square of topological spaces is a pullback, and consider a cone with tip $G \circlearrowright Q$ over this square, as shown on the left here:

$$
\begin{tikzcd}
\overset{G}{\circlearrowright} Q \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \overset{G}{\circlearrowright}\mathrm{X} \times_{\mathrm{B}} \mathrm{P} \arrow[r] \arrow[d] & \overset{G}{\circlearrowright}\mathrm{P} \arrow[d] \\
& \underset{G}{\circlearrowleft}\mathrm{X} \arrow[r] & \underset{G}{\circlearrowleft}\mathrm{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times \mathrm{Q} \arrow[r, dashed, "\mathrm{id}\times f"] \arrow[dd] & G \times (\mathrm{X} \times_{\mathrm{B}} \mathrm{P}) \arrow[rr] \arrow[dr] \arrow[dd] & & G \times \mathrm{P} \arrow[dr] \arrow[dd] & \\
& & G \times \mathrm{X} \arrow[rr] \arrow[dd] & & G \times \mathrm{B} \arrow[dd, "\text{group action}"] \\
\mathrm{Q} \arrow[r, dashed, "f"] & \mathrm{X} \times_{\mathrm{B}} \mathrm{P} \arrow[rr] \arrow[dr] & & \mathrm{P} \arrow[dr] & \\
& & \mathrm{X} \arrow[rr] & & \mathrm{B}
\end{tikzcd}
\tag{1.20}
$$

We need to show that there exists a unique dashed morphism making the full diagram on the left commute. But since the underlying square of topological spaces is a pullback, there exists a unique such continuous function, in the bottom part of the diagram on the right of (1.20). Hence it remains to show that this unique function is necessarily G-equivariant. But the functor $G \times (-)$, preserves limits, being itself a limit, so that also the top square on the right is a pullback. Therefore also the top dashed morphism exists uniquely, and makes the square commute as shown, by functoriality of limits.

The argument for colimits is analogous, now using that $G \times (-)$ also preserves colimit diagrams, by (1.4). $\square$

Proposition 1.1.11 (Compactly generated topological G-actions form a regular category)**.** *For $G \in \mathrm{Grp}(\mathrm{kTopSpc})$, the category $G\,\mathrm{Act}(\mathrm{kTopSpc})$ (1.1.1) is regular (Def. 1.0.11).*

Proof. Since regularity is entirely a condition on limits and colimits of a category, it transfers through any forgetful functor which creates all limits and colimits. Therefore the statement follows by the combination of Lem. 1.1.10 with Prop. 1.0.20. ☐

In generalization of Example 1.1.9, we have:

Example 1.1.12 (Restricted actions)**.** Let $H \hookrightarrow G$ any subgroup inclusion. Then the corresponding pullback H-action (Lemma 1.1.7) of a $G \circlearrowright \mathrm{Y} \in G\,\mathrm{Act}(\mathrm{kTopSpc})$ is just its restriction to the action of $H \subset G$

$$
\begin{tikzcd}
H\mathrm{Act}(\mathrm{TopSp}) \arrow[r, shift left=1.1ex, "G\times_H(-)"] \arrow[r, phantom, "\perp"] & G\mathrm{Act}(\mathrm{TopSp}) \arrow[l, shift left=1.1ex, "\text{restricted action}"] \\
H \circlearrowright \mathrm{Y} & G \circlearrowright \mathrm{Y}\,, \arrow[l, mapsto]
\end{tikzcd}
\tag{1.21}
$$

and the adjunction (1.17) corresponds to a natural bijection (1.1) of hom-sets of the following form:

$$
\left\{
\begin{array}{ccc}
\overset{H}{\circlearrowright}\mathrm{X} & \longrightarrow & \overset{H}{\circlearrowright}\mathrm{Y} \\
x & \longmapsto & f(x)
\end{array}
\right\}
\quad
\xleftrightarrow[\text{induction/restriction}]{\widetilde{(-)}}
\quad
\left\{
\begin{array}{ccc}
\overset{G}{\circlearrowright}G \times_H \mathrm{X} & \longrightarrow & \overset{G}{\circlearrowright}\mathrm{Y} \\
[g,x] & \longmapsto & g \cdot f(x)
\end{array}
\right\}.
\tag{1.22}
$$

Example 1.1.13 (Fixed loci with residual Weyl-group action)**.** Let $H \subset G$ be a subgroup inclusion. Consider the functor which sends a G-action to its H-fixed subspace equipped with its residual Weyl group action (Ntn. 1.1.1)

$$
\begin{array}{ccc}
N(H)\mathrm{Act}(\mathrm{TopSp}) & \xrightarrow{\;(-)^H := \mathrm{Maps}(N(H)/H,X)^{N(H)}\;} & W(H)\mathrm{Act}(\mathrm{TopSp}) \\
N(H) \circlearrowright \mathrm{X} & \longmapsto & W(H) \circlearrowright \mathrm{X}^H =: \big\{ x \in \mathrm{X} \,\big|\, \underset{h \in H \subset G}{\forall} h \cdot x = x \big\}.
\end{array}
\tag{1.23}
$$

This fixed locus functor $(-)^H$ is, equivalently, the pull-push of change-of-equivariance-groups (Lemma 1.1.7) through the normalizer correspondence

$$
\begin{tikzcd}
 & N(H) \arrow[dl, hook'] \arrow[dr, two heads] & \\
G & & W(H)\,,
\end{tikzcd}
$$

in that it is the composite right adjoint in the following composite of change-of-equivariance group adjunction (1.17):

$$
\begin{tikzcd}[column sep=huge]
G\mathrm{Act} \arrow[r, shift right=1.1ex, "(N(H)\hookrightarrow G)^*"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right=30, "(-)^H"'] & N(H)\mathrm{Act} \arrow[l, shift right=1.1ex, "G\times_{N(H)}(-)"'] \arrow[r, shift right=1.1ex, "\mathrm{Maps}(N(H)/H,-)^{N(H)}"'] \arrow[r, phantom, "\perp"] & \big(N(H)/H\big)\mathrm{Act}\,. \arrow[l, shift right=1.1ex, "(N(H)\twoheadrightarrow N(H)/H)^*"'] \arrow[ll, bend right=30, "G/H\times_{N(H)/H}(-)"']
\end{tikzcd}
\tag{1.24}
$$

Example 1.1.14 (Coset spaces (e.g. [Bre72, p. 34]))**.** For $H \hookrightarrow G$ a subgroup inclusion, the induced G-action (1.21) of the unique trivial H-action on the point is the coset space

$$
G \times_H * \;=\; G/H \;:=\; \{gH \subset G \,|\, g \in G\} \;\in\; G\,\mathrm{Act}(\mathrm{kTopSpc})
\tag{1.25}
$$

equipped with its G-action by left multiplication of representatives in G.

Quotient spaces. In generalization of Example 1.1.8 we have:

Example 1.1.15 (Partial quotient spaces)**.** For $G, G' \in \mathrm{Grps}(\mathrm{HausSp})$ and $G \times G' \xrightarrow{\mathrm{pr}_2} G'$ the projection homomorphism out of their direct product, the corresponding pullback action in Lemma 1.1.7 assigns trivial G-actions

$$
\begin{tikzcd}
(G \times G')\mathrm{Act}(\mathrm{TopSp}) \arrow[r, shift left=1.1ex, "(-)/G"] \arrow[r, phantom, "\perp"] & G'\mathrm{Act}(\mathrm{TopSp}) \arrow[l, shift left=1.1ex, "\text{trivial } G\text{-action}"]
\end{tikzcd}
$$

and its left adjoint (1.17) forms G-quotients $G' \times_{G\times G'} (-) = (-)/G$. The unit of this adjunction is the natural transformation which sends any $G \times G'$-action to the coprojection $q_{\mathrm{X}} : \mathrm{X} \to \mathrm{X}/G$ onto its G-quotient space, and any G-equivariant continuous function $f : \mathrm{X} \to \mathrm{Y}$ to a commuting square of G'-actions:

$$
\begin{tikzcd}
\overset{G' \circlearrowright}{\mathrm{X}} \arrow[r, "f"] \arrow[d, "q_{\mathrm{X}}"'] & \overset{G' \circlearrowright}{\mathrm{Y}} \arrow[d, "q_{\mathrm{Y}}"] \\
\underset{G' \circlearrowleft}{\mathrm{X}/G} \arrow[r, "f/G"'] & \underset{G' \circlearrowleft}{\mathrm{Y}/G}
\end{tikzcd}
\tag{1.26}
$$

Lemma 1.1.16 (Hausdorff quotient spaces (e.g. [Bre72, Thm. 3.1]))**.** *If the equivariance group G is compact and the topological space underlying* $\mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ *(Ntn. 1.1.1) is Hausdorff, then the quotient space X/G is Hausdorff.*

Remark 1.1.17 (Recognition of pullbacks of quotient coprojections)**. (i)** The quotient coprojection squares (1.26) are not in general pullbacks (Ntn. 1.0.5); and it is important to recognize those situations in which they are.

(ii) A general recognition principle applies to *compact* quotient groups (Lemma 1.1.18 below) which is however of little value in the applications to twisted cohomology theory, where the quotient groups generically are topological representatives of general ∞-groups (topological realizations of general simplicial groups) and thus rarely compact.

(iii) Without assuming compactness of the quotient group we may still recognize pullbacks of *free* (principal) quotients (Lemma 1.1.19) from just the fact that the left one is locally trivial below (compare Ntn. 2.2.2). This is a basic fact of principal bundle theory, but rarely, if ever, stated in the general form of Lemma 1.1.19 in which it drives much of the proofs in §2.2.

Lemma 1.1.18 (Recognition of pullbacks of compact group quotients ([BF15, Prop. 4.1]))**.** *If G is compact and the underlying topological spaces of* $\mathrm{X}, \mathrm{Y} \in G\mathrm{Act}(\mathrm{kTopSpc})$ *are Hausdorff then, for any morphism* $f : \mathrm{X} \to \mathrm{Y}$, *its quotient naturality square* (1.26) *is a pullback square (Ntn. 1.0.5) if and only if f preserves isotropy groups (as subgroups of G):*

$$
\begin{tikzcd}
\mathrm{X} \arrow[r, "f"] \arrow[d, "q_{\mathrm{X}}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Y} \arrow[d, "q_{\mathrm{Y}}"] \\
\mathrm{X}/G \arrow[r, "f/G"'] & \mathrm{Y}/G
\end{tikzcd}
\quad \Leftrightarrow \quad
\underset{x \in \mathrm{X}}{\forall} \big(G_x \simeq G_{f(x)} \big) \,.
$$

Lemma 1.1.19 (Recognition of pullbacks of principal quotients)**.** *A homomorphism of G-principal fibrations (covering any morphism of base spaces) is a pullback square (Ntn. 1.0.5) as soon as the domain is a locally trivializable fiber bundle:*

$$
\begin{tikzcd}
\overset{G \circlearrowright}{\mathrm{P}_1} \arrow[r, "f"] \arrow[d, "p_1"'] & \overset{G \circlearrowright}{\mathrm{P}_2} \arrow[d, "p_2"] \\
\underset{\substack{G\text{-principal \&} \\ \text{locally trivial}}}{\mathrm{X}_1} \arrow[r, "f/G"] & \underset{G\text{-principal}}{\mathrm{X}_2}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathrm{P}_1 \arrow[r] \arrow[d, "p_1"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_2 \arrow[d, "p_2"] \\
\mathrm{X}_1 \arrow[r] & \mathrm{X}_2
\end{tikzcd}
\tag{1.27}
$$

Proof. First, consider the special case when the domain bundle is actually trivial and that the base morphism is the identity

$$
\begin{tikzcd}
G \times \mathrm{X} \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"'] & \mathrm{P} \arrow[d, "p"] \\
\mathrm{X} \arrow[r, equal] & \mathrm{X}
\end{tikzcd}
$$

(equivalently a global section $\sigma(\mathrm{e})(-)$). In that case, a continuous inverse of σ is given by the composite

$$
G \times \mathrm{X} \xleftarrow{\mathrm{id} \times p} G \times \mathrm{P} \xleftarrow{\sim} \mathrm{P} \times_{\mathrm{X}} \mathrm{P} \xleftarrow{(\sigma(\mathrm{e}, p(-)), \mathrm{id})} \mathrm{P}\,,
$$

where the second map is the inverse of the shear map (see (2.4)) of the codomain bundle, and where $\mathrm{e} \in G$ denotes the neutral element. From this it follows that morphisms out of any locally trivial principal bundle over the identity base morphism are isomorphisms, by local recognition of homeomorphisms (Ex. 1.0.22). Finally, in the general case the universal comparison morphism

$$\begin{tikzcd} \mathrm{P}_1 \arrow[r, dashed, "\sim"] \arrow[dr] & \mathrm{P}_2 \times_{\mathrm{X}_1} \mathrm{X}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_2 \arrow[d] \\ & \mathrm{X}_1 \arrow[r] & \mathrm{X}_2 \end{tikzcd}$$

from P_1 to the pullback of P_2 in (1.27) is such a homomorphism over a common base space X_1, hence is an isomorphism, thus exhibiting P_1 as a pullback. □

Slices of G-orbits. The existence of local *slices through families of orbits* of group actions is guaranteed by the Slice Theorem (Prop. 1.1.21) below and serves to ensure or detect local triviality of plain principal bundles (Cor. 1.1.23) below and of equivariant principal bundles (around Ntn. 2.2.32 below).

Definition 1.1.20 (Slices of G-orbits)**.** For $G \curvearrowright \mathrm{U} \in G\mathrm{Act}(\mathrm{kTopSpc})$ and $H \subset G$ a subgroup, an H-subspace

$$\overset{H \circlearrowright}{\mathrm{S}} \overset{\iota}{\hookrightarrow} \overset{H \circlearrowright}{\mathrm{U}} \tag{1.28}$$

is called a *slice* through its G-orbit modulo H if its induction/restriction-adjunct (1.22) is an isomorphism

$$\begin{array}{ccc} \overset{G \circlearrowright}{G \times_H \mathrm{S}} & \xrightarrow[\sim]{\tilde{\iota}} & \overset{G \circlearrowright}{\mathrm{U}} \\ {[g,s]} & \longmapsto & g \cdot s\,. \end{array} \tag{1.29}$$

Specifically, for $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ and $x \in \mathrm{X}$ a point, by a *slice through* x one means (e.g. [Bre72, §II, Def. 4.1]) a slice (1.28) relative to the isotropy group G_x (1.9) through x of an open G-neighborhood $G \curvearrowright \mathrm{U}_x$ of x

$$x \;\in\; \overset{G_x \circlearrowright}{\mathrm{S}_x} \hookrightarrow \overset{G_x \circlearrowright}{\mathrm{U}_x}\,. \tag{1.30}$$

Proposition 1.1.21 (Slice Theorem ([Pa61, Prop. 2.3.1][Ka16, Thm. 6.2.7]))**.** *Under the assumption 1.1.2 of proper equivariance, given $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ then for every $x \in \mathrm{X}$ there exists a slice through x (Def. 1.1.20).*

Remark 1.1.22 (Technical conditions in the slice theorem)**.** The slice theorem for *compact* Lie group actions is due to [Mo57, Thm. 2.1][Pa60, Cor. 1.7.19], and for proper actions of general Lie groups it is due to [Pa61], reviewed in [Ka16]. Beware that [Pa61] goes to some length to further generalize beyond proper actions, which leads to a wealth of technical conditions that, it seems, have been of rare use in practice. But, under the assumption 1.1.2 that all G-spaces are locally compact, all these conditions reduce to properness [Pa61, Thm 1.2.9][Ka16, Rem. 5.2.4], and thus the theorem reduces to the statement in Prop. 1.1.21.

Corollary 1.1.23 (Quotient coprojection of free proper action is locally trivial [Pa61, §4.1])**.** *Under the assumption 1.1.2 of proper equivariance, the quotient space coprojection $P \xrightarrow{q} P/G$ of a free action $G \curvearrowright P$ admits local sections.*

Equivariant open covers.

Definition 1.1.24 (Properly equivariant open cover)**.** Given $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ (1.1.1), we say that an open cover (Ex. 1.0.21) of the underlying space

$$\widehat{\mathrm{X}} = \coprod_{i \in I} \mathrm{U}_i \twoheadrightarrow \mathrm{X} \tag{1.31}$$

is

(i) *equivariant* if the G-action on X pulls back to $\widehat{\mathrm{X}}$

$$G \curvearrowright \widehat{\mathrm{X}} \overset{p}{\twoheadrightarrow} G \curvearrowright \mathrm{X} \quad \in G\mathrm{Act}(\mathrm{kTopSpc})\,; \tag{1.32}$$

(ii) *regular* if there is a G-action on the index set such that

$$
\begin{aligned}
&(\mathbf{a}) \quad \underset{i,j\in I}{\forall} \left(\begin{array}{ccc} \mathrm{U}_i & \xrightarrow[\sim]{} & \mathrm{U}_{g\cdot j} \\ \downarrow & & \downarrow \\ \mathrm{X} & = & \mathrm{X} \end{array} \quad \Rightarrow \quad i = j \right); \\
&(\mathbf{b}) \quad \underset{i\in I}{\forall}\ \underset{g\in G}{\forall} \left(U_i \cap g\cdot U_j \neq \varnothing \ \Rightarrow \ \begin{array}{ccc} \mathrm{U}_i & \xrightarrow[\sim]{} & \mathrm{U}_i \\ \downarrow & & \downarrow \\ \mathrm{X} & \underset{\sim}{\xrightarrow{g}} & \mathrm{X} \end{array} \right); \qquad (1.33) \\
&(\mathbf{c}) \quad \underset{n\in\mathbb{N}}{\forall}\ \underset{\substack{i_0,\cdots,i_n\in I \\ g_0,\cdots,g_n\in G}}{\forall} \left(\begin{array}{rcl} U_{i_0}\cap\cdots\cap U_{i_n} & \neq & \varnothing, \\ g_0\cdot U_{i_0}\cap\cdots g_n\cdot U_{i_n} & \neq & \varnothing \end{array} \ \Rightarrow \ \underset{g\in G}{\exists}\ \underset{0\leq k\leq n}{\forall} g\cdot U_{i_k} = g_k\cdot U_{i_k} \right);
\end{aligned}
$$

(iii) *properly equivariant* if, in addition, each H-fixed locus of $\widehat{\mathrm{X}}$ is an open cover of that of X:

$$
\underset{\substack{H\subset G \\ \text{clsd}}}{\forall} \quad \widehat{\mathrm{X}}^H \xrightarrow[\text{open cover}]{p^H} \mathrm{X}^H; \qquad (1.34)
$$

(iv) properly equivariantly *good* if all the restrictions (1.34) are good open covers (Def. 3.3.14).

Proposition 1.1.25 (Smooth G-manifolds admit properly equivariant regular good open covers)**.** *At least for* $G \in \mathrm{Grp}(\mathrm{FinSet}) \xhookrightarrow{\mathrm{Grp(Dsc)}} \mathrm{Grp}(\mathrm{kTopSpc})$, *every smooth G-manifold* $G \circlearrowright \mathrm{X} \in \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ *admits a regular properly equivariant and good open cover (Def. 1.1.24).*

Proof. This follows with the equivariant triangulation theorem [Il72, Thm. 3.1][Il83]; see [Yan14, Thm. 2.11]. □

Remark 1.1.26. If an equivariant open cover (2.2.12) is regular (1.33) then for $x \in \mathrm{U}_i \hookrightarrow \mathrm{X}$ the stabilizer group $G_x := \mathrm{Stab}_G(x)$ of $x \in X$ also fixes the index i:

$$
\begin{array}{ccc} x \in \mathrm{U}_i & \xrightarrow[\sim]{} & \mathrm{U}_i \\ \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{g\in G_x} & \mathrm{X}. \end{array}
$$

1.2 G-Actions on topological groupoids

The theory of *universal* equivariant bundles turns out to be most naturally formulated in the language not just of topological spaces equipped with G-actions, but of topological *groupoids* equipped with G-actions. This observation, which is due to [MS95] and was only more recently amplified again in [GMM17], we turn in §2.3 and then especially in part III below. Here we recall and develop some basics of equivariant topological groupoids that will make the theory of universal equivariant bundles in §2.3 be transparent and run smoothly. (The material here is not needed for chapter 2.)

Topological groupoids.

Notation 1.2.1 (Topological groupoids)**.** We write Grpd(kTopSpc) for the strict (2,1)-category (Ntn. 1.0.3) of groupoid objects internal (Ntn. 1.0.23) to kTopSpc (1.2), hence of *topological groupoids* ([Eh59], survey in [Mk87, §II], exposition in [We96, p. 6]):
(i) Its objects are diagrams of topological spaces

$$
\begin{array}{ccccc}
\substack{\textbf{space of composable} \\ \textbf{pairs of morphisms}} & \substack{\textbf{composition} \\ \textbf{map}} & \substack{\textbf{space of} \\ \textbf{morphisms}} & \substack{\textbf{source, target \& unit} \\ \textbf{maps}} & \substack{\textbf{space of} \\ \textbf{objects}} \\
(\mathrm{X}_1)\underset{\mathrm{X}_0}{{}_t\times_s}(\mathrm{X}_1) & \xrightarrow{\ \circ\ } & \mathrm{X}_1 \circlearrowleft (-)^{-1} & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\mathrm{e}} \\ \xrightarrow[t]{} \end{array} & \mathrm{X}_0
\end{array}
\qquad (1.35)
$$

$\substack{\textbf{inversion} \\ \textbf{map}}$

such that the composition operation $\circ$ is associative, unital with respect to e and with inverses given by $(-)^{-1}$. We will mostly denote such an object by

$$\mathrm{X}_1 \rightrightarrows \mathrm{X}_0 \quad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})\,,$$

with the rest of the structure understood from the given context.
(ii) Its morphisms are *continuous functors* hence continuous functions F_0, F_1 compatible with all this structure:

$$\begin{tikzcd}
(\mathrm{X}_1)\,{}_t\!\underset{\mathrm{X}_0}{\times}\!{}_s\,(\mathrm{X}_1) \arrow[r, "\circ"] \arrow[d, "F_1\,{}_t\!\underset{F_0}{\times}\!{}_s\,F_1"'] & \mathrm{X}_1 \arrow[loop above, "(-)^{-1}"] \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] \arrow[d, "F_1"] & \mathrm{X}_0 \arrow[l, "\mathrm{e}" description] \arrow[d, "F_0"] \\
(\mathrm{Y}_1)\,{}_t\!\underset{\mathrm{X}_0}{\times}\!{}_s\,(\mathrm{Y}_1) \arrow[r, "\circ"] & \mathrm{Y}_1 \arrow[loop below, "(-)^{-1}"] \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & \mathrm{Y}_0 \arrow[l, "\mathrm{e}" description]
\end{tikzcd} \tag{1.36}$$

(iii) Its *2-morphism* $\eta : F \Rightarrow F'$ are *continuous natural transformations*, hence continuous functions $\eta(-) : \mathrm{X}_0 \to \mathrm{Y}_1$ making all the naturality squares commute:

$$\begin{tikzcd}
(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\Downarrow\eta"] & & (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)
\end{tikzcd}
\quad : \quad
\begin{tikzcd}
x \arrow[d, "\gamma"] \\ x'
\end{tikzcd}
\overset{\text{cts.}}{\longmapsto}
\begin{tikzcd}
F(x) \arrow[r, "\eta(x)"] \arrow[d, "F(\gamma)"] & F'(x) \arrow[d, "F'(\gamma)"] \\
F(x') \arrow[r, "\eta(x')"] & F'(x')\,.
\end{tikzcd} \tag{1.37}$$

(iv) An *isomophism of topological groupoids* $(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \simeq (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)$ is an isomorphsim in the underlying 1-category (ignoring the 2-morphisms).
(v) An *equivalence of topological groupoids* is a pair of morphisms going back and forth between them, together with 2-morphisms (1.37) relating their composites to the identity morphism:

$$(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \underset{\mathrm{hmtpy}}{\simeq} (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0) \qquad \Leftrightarrow \qquad (\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \underset{R}{\overset{L}{\rightleftarrows}} (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)\,, \quad L \circ R \Rightarrow \mathrm{id}\,, \quad \mathrm{id} \Rightarrow R \circ L\,. \tag{1.38}$$

Example 1.2.2 (Topological spaces as constant topological groupoids)**.** Each $\mathrm{X} \in \mathrm{kTopSpc}$ (1.2) becomes a topological groupoid (Ntn. 1.2.1)

$$\mathrm{Cnst}(\mathrm{C}) \;:=\; \big(\mathrm{X} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{X}\big) \quad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})$$

by taking all structure maps (1.35) to be the identity on X. This construction constitutes a full subcategory inclusion

$$\mathrm{kTopSpc} \xhookrightarrow{\;\mathrm{Cnst}\;} \mathrm{Grpd}(\mathrm{kTopSpc})\,. \tag{1.39}$$

Example 1.2.3 (Topological pair groupoid)**.** For $\mathrm{X} \in \mathrm{kTopSpc}$, its *chaotic groupoid* or *pair groupoid* is the topological groupoid (Ntn. 1.2.1) whose space of morphisms is the product of X with itself (the space of pairs of elements of X), with source and target given by the two canonical projection maps

$$\mathrm{Cht}(\mathrm{X}) \;:=\; \big(\mathrm{X} \times \mathrm{X} \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathrm{X}\big) \quad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})$$

and equipped with the unique admissible composition operation:

$$(\mathrm{X} \times \mathrm{X})\,{}_t\!\underset{\mathrm{X}}{\times}\!{}_s\,(\mathrm{X} \times \mathrm{X}) \;=\; \mathrm{X} \times \mathrm{X} \times \mathrm{X} \xrightarrow{(\mathrm{pr}_1, \mathrm{pr}_3)} \mathrm{X} \times \mathrm{X}\,.$$

This construction constitutes *another* full subcategory inclusion

$$\mathrm{kTopSpc} \xhookrightarrow{\;\mathrm{Cht}\;} \mathrm{Grpd}(\mathrm{kTopSpc})\,.$$

Definition 1.2.4 (Space of components of a topological groupoid)**.** For $(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)$ a topological groupoid (Ntn. 1.2.1), its *space of connected components* (or: *0-truncation*)

$$\tau_0(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \quad \in \mathrm{kTopSpc}$$

is the quotient space by the source/target relation, hence the coequalizer of its source and target maps:

$$\mathrm{X}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{X}_0 \xrightarrow{\;\mathrm{coeq}(s,t)\;} \tau_0(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)\,. \tag{1.40}$$

All these basic notions are unified as follows:

Proposition 1.2.5 (Adjunctions between topological groupoids and topological spaces)**.** *The 1-category of topological groupoids (Ntn. 1.2.1) is related to that of topological spaces (Ntn. 1.0.16) by a quadruple of adjoint functors (Ntn. 1.0.4)*

$$\mathrm{Grpd}(\mathrm{kTopSpc}) \;\underset{\underset{\longleftarrow\,\mathrm{Cht}\,\hookleftarrow}{\perp}}{\overset{\overset{\longrightarrow\,\tau_0\,\longrightarrow}{\perp}}{\underset{\longrightarrow\,(-)_0\,\longrightarrow}{\overset{\longleftarrow\,\mathrm{Cnst}\,\hookleftarrow}{\perp}}}}\; \mathrm{kTopSpc}\,,$$

where

- Cnst *assigns constant groupoids in the sense of Ex. 1.2.2;*
- Cht *assigns pair groupoids in the sense of Ex. 1.2.3;*
- $(-)_0$ *assigns spaces of objects* (1.35)*;*
- τ_0 *assigns spaces of connected components (Def. 1.2.4).*

Proof. The hom-isomorphisms (1.1) are readily seen by unwinding the definitions:
(1) For $\tau_0 \dashv \mathrm{Cnst}$, the natural bijection

$$\mathrm{kTopSpc}\big(\tau_0(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathrm{Y}\big) \;\simeq\; \mathrm{Grpd}(\mathrm{kTopSpc})\big((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathrm{Cnst}(\mathrm{Y})\big)$$

exhibits the universal property of the coequalizer:

$$\begin{tikzcd}
\mathrm{X}_1 \arrow[rr] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & & \mathrm{Y} \arrow[d, equal] \\
\mathrm{X}_0 \arrow[rr] \arrow[d, "\mathrm{coequ}(s,t)"'] & & \mathrm{Y} \\
\tau_0(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \arrow[urr, dashed, "\exists !"'] & &
\end{tikzcd}$$

(2) For $\mathrm{Cnst} \dashv (-)_0$, the natural bijection

$$\mathrm{Grpd}(\mathrm{kTopSpc})\big(\mathrm{Cnst}(\mathrm{X}), (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)\big) \;\simeq\; \mathrm{kTopSpc}(\mathrm{X}, \mathrm{Y}_0)$$

reflects the unitality of the groupoid composition:

$$\begin{tikzcd}
\mathrm{X} \arrow[r, dashed, "\exists !"] & \mathrm{Y}_1 \\
\mathrm{X} \arrow[r] \arrow[u, equal, "\mathrm{e}"] & \mathrm{Y}_0 \arrow[u, "e"']
\end{tikzcd}$$

(3) For $(-)_0 \dashv \mathrm{Cht}$, the natural bijection

$$\mathrm{kTopSpc}(\mathrm{X}_0, \mathrm{Y}) \;\simeq\; \mathrm{Grpd}(\mathrm{kTopSpc})\big((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathrm{Cht}(\mathrm{Y})\big)$$

reflects the universal property of the Cartesian product:

$$\begin{tikzcd}
\mathrm{X}_1 \arrow[r, dashed, "\exists !"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathrm{Y} \times \mathrm{Y} \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] \\
\mathrm{X}_0 \arrow[r] & \mathrm{Y}\,.
\end{tikzcd}$$

□

We continue to list some classes of examples of topological groupoids that we need later on.

Example 1.2.6 (Topological action groupoid)**.** For $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$ (1.8), the corresponding *action groupoid* is the topological groupoid (Ntn. 1.2.1) given by

$$\begin{array}{ccccccc}
\mathrm{X} \times G^{\mathrm{op}} \times G^{\mathrm{op}} & \xrightarrow{\mathrm{id}\times((-)\cdot(-))} & \mathrm{X} \times G^{\mathrm{op}} & \underset{\xrightarrow[(-)\cdot(-)]{}}{\overset{\xrightarrow{\mathrm{pr}_1}}{\xleftarrow{\mathrm{id}\times e}}} & \mathrm{X} & \in\; \mathrm{Grpd}(\mathrm{kTopSpc}) \\
(x, g_1, g_2) & \longmapsto & (x, g_2 \cdot g_1) & \longmapsto & (g_2 \cdot g_1 \cdot x) &
\end{array} \tag{1.41}$$

with composition given by the *reverse* of the group operation in G.

Example 1.2.7 (Topological delooping groupoid)**.** For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$, its *delooping groupoid* is the topological left action groupoid (Ex. 1.2.6) of the unique Γ^{op}-action $\Gamma^{\mathrm{op}} \circlearrowright * \in \Gamma\mathrm{Act}(\mathrm{kTopSpc})$ on the point space:

$$\mathbf{B}\Gamma \;:=\; (* \times \Gamma \rightrightarrows *) \;=\; (\Gamma \rightrightarrows *) \quad \in\; \mathrm{Grpd}(\mathrm{kTopSpc})\,,$$

with composition given by the group operation in Γ:

$$\begin{tikzcd} \bullet \arrow[r, "g_1"] \arrow[rr, bend left, "g_1 \cdot g_2"] & \bullet \arrow[r, "g_2"] & \bullet \end{tikzcd} \quad \in\; \mathbf{B}\Gamma\,.$$

Example 1.2.8 (Action groupoid of group multiplication is pair groupoid)**.** For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$, the topological pair groupoid (Ex. 1.2.3) on its underlying topological space is isomorphic to the action groupoid (Ex. 1.2.6) of both the left and inverse-right action of G^{op} on itself (1.10)

$$\begin{tikzcd}[row sep=small] \substack{\text{left multiplication}\\ \text{action groupoid}} & & \text{pair groupoid} & & \substack{\text{right multiplication}\\ \text{action groupoid}} \\ (G \times G \rightrightarrows G) & & (G \times G \rightrightarrows G) \arrow[ll, "\sim"'] \arrow[rr, "\sim"] & & (G \times G \rightrightarrows G) \\ \big(g_1 \xrightarrow{(g_1,\, g_1 \cdot g_2^{-1})} g_2\big) & & \big(g_1 \xrightarrow{(g_1,\, g_2)} g_2\big) \arrow[ll, mapsto] \arrow[rr, mapsto] & & \big(g_1 \xrightarrow{(g_1,\, g_1^{-1} \cdot g_2)} g_2\big) \end{tikzcd}$$

Example 1.2.9 (Topological mapping groupoid)**.** Given a pair of topological groupoids (Ntn. 1.2.1), their *mapping groupoid* or *functor groupoid* (e.g. [NP19, §2])

$$\mathrm{Maps}(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0,\, \mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0) \quad \in\; \mathrm{Grpd}(\mathrm{kTopSpc}) \tag{1.42}$$

has as object space the subspace of the product of mapping spaces $\mathrm{Maps}(\mathrm{X}_0, \mathrm{Y}_0) \times \mathrm{Maps}(\mathrm{X}_1, \mathrm{Y}_1)$ (1.3) on the elements that satisfy the functoriality condition (1.36), and as morphism space the subspace on the product of that space with the mapping space $\mathrm{Maps}(\mathrm{X}_0, \mathrm{X}_1)$ on those elements which satisfy the naturality condition (1.37).

This construction is a Grpd-enriched functor in both arguments contravariantly so in the first:

$$\mathrm{Maps}(-,-) \;:\; \mathrm{Grpd}(\mathrm{kTopSpc})^{\mathrm{op}} \times \mathrm{Grpd}(\mathrm{kTopSpc}) \longrightarrow \mathrm{Grpd}(\mathrm{kTopSpc})\,.$$

With the first argument fixed, it constitutes a Grpd-enriched right adjoint (Ntn. 1.0.4) to the product functor (e.g. [NP19, Prop. 3.1]):

$$\begin{tikzcd} \mathrm{Grpd}(\mathrm{kTopSpc}) \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathrm{Maps}((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0),\, -)"'] & \mathrm{Grpd}(\mathrm{kTopSpc}) \arrow[l, shift right=1.5ex, "(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \times (-)"'] \end{tikzcd}. \tag{1.43}$$

Example 1.2.10 (Mapping groupoid between delooping groupoids)**.** For $G, \Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$, the mapping groupoid (Ex. 1.2.9) between their topological delooping groupoids (Ex. 1.2.7) is isomorphic to the topological action groupoid (Ex. 1.2.6) of the adjoint action of Γ on the hom-set of group homomorphisms $G \to \Gamma$ (topologized as a subspace of $\mathrm{Maps}(G, \Gamma)$):

$$\mathrm{Maps}(G \rightrightarrows *,\, \Gamma \rightrightarrows *) \;\simeq\; \big(\mathrm{Grp}(G, \Gamma) \times \Gamma^{\mathrm{op}} \;\rightrightarrows\; \mathrm{Grp}(G, \Gamma)\big)\,. \tag{1.44}$$

$$\begin{tikzcd} \bullet \arrow[d, "g_1"] \arrow[dd, bend right=60, "g_1 \cdot g_2"'] & & \bullet \arrow[r, "\gamma"] \arrow[d, "\phi(g_1)"] \arrow[dd, bend right=60, "\phi(g_1 \cdot g_2)"'] & \bullet \arrow[d, "\phi'(g_1)"'] \\ \bullet \arrow[d, "g_2"] & & \bullet \arrow[r, "\gamma"] \arrow[d, "\phi(g_2)"] & \bullet \arrow[d, "\phi'(g_2)"'] & \longmapsto & \big(\phi \xrightarrow{(\gamma, \phi)} \phi' = \mathrm{ad}_\gamma \circ \phi\big) \\ \bullet & & \bullet \arrow[r, "\gamma"] & \bullet \end{tikzcd}$$

Crossed homomorphisms and first non-abelian group cohomology. The classical notion of *crossed homomorphisms*, recalled as Def. 1.2.11 below, turns out to play a pivotal role in equivariant bundle theory (e.g., Lem. 2.2.11 and Prop. 2.3.17), often secretly so (Rem. 1.2.15 below). Here we highlight a transparent groupoidal understanding of crossed homomorphisms with crossed conjugations between them (Prop. 1.2.18 below).

Definition 1.2.11 (Crossed homomorphisms and first non-abelian group cohomology)**.** Let $G \in \mathrm{Grp}(\mathrm{kTopSpc})$ and $G \circlearrowright \Gamma \in G\mathrm{Act}(\mathrm{kTopSpc})$, with $\alpha : G \to \mathrm{Aut}_{\mathrm{Grp}}(\Gamma)$ the underlying automorphism action (Lem. 2.1.4).
(i) A continuous *crossed homomorphism* from G to Γ is a continuous map $\phi : G \to \Gamma$ which satisfies the following *G-crossed* homomorphism property:

$$\underset{g_1, g_2 \in G}{\forall} \; \phi(g_1 \cdot g_2) \;=\; \phi(g_1) \cdot \alpha(g_1)\big(\phi(g_2)\big)\,. \tag{1.45}$$

We write

$$\mathrm{CrsHom}(G, G \circlearrowright \Gamma) \;\subset\; \mathrm{Maps}(G, \Gamma) \quad \in \; \mathrm{kTopSpc} \tag{1.46}$$

for the subspace of the mapping space (1.3) on the crossed homomorphisms.
(ii) A *crossed conjugation* between two crossed homomorphisms $\phi \to \phi'$ is an element $\gamma \in \Gamma$ such that

$$\underset{g \in G}{\forall} \; \phi'(g) \;=\; \gamma^{-1} \cdot \phi(g) \cdot \alpha(g)(\gamma)\,. \tag{1.47}$$

We denote the continuous Γ-action by crossed conjugation by

$$\Gamma \circlearrowright \mathrm{CrsHom}(G, G \circlearrowright \Gamma) \quad \in \; \Gamma\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{1.48}$$

We write $\phi \sim_{\mathrm{ad}} \phi'$ for the corresponding equivalence relation.
(iii) The *non-abelian group cohomology* of G in degree 1 with coefficients in $G \circlearrowright \Gamma$ is – at least when G is discrete[2] – the set of connected components[3] of the quotient space by crossed conjugation classes (1.47) of the space (1.46) of crossed homomorphisms (1.45):

$$H^1_{\mathrm{Grp}}(G, G \circlearrowright \Gamma) \;:=\; \pi_0\big(\mathrm{CrsHom}(G, G \circlearrowright \Gamma)/\sim_{\mathrm{ad}}\big) \quad \in \; \mathrm{Set}^{*/}\,. \tag{1.49}$$

Remark 1.2.12 (Crossed homomorphisms in the literature)**.** Since the notion of crossed homomorphisms, in the generality that we need them here (Def. 1.2.11), tends to be neglected in the literature, we record some pointers: Crossed homomorphisms (1.45) appear first, already in full non-abelian generality, in [Wh49, (3.1)]. Much later, following [ML75, §IV.2], they became widely appreciated only in the special case when Γ is an abelian group, as a tool in ordinary group cohomology (e.g. [Br82, p. 45]). Crossed homomorphisms in their non-abelian generality appear again in [tD69, §2.1] (not using the "crossed" terminology, though) and in [MS95, p. 2][GMM17, Def. 4.1], all in the context of equivariant bundle theory (in which we consider them in §2.3). Textbook accounts in this generality are in [NSW08, p. 16][Mi17, §15.a-b]. The corresponding definition (1.49) of non-abelian group 1-cohomology is rarely made explicit; exceptions are [GiS06, Def. 2.3.2][GMM17, Def. 4.17][Mi17, §3.k][4].

We also need to recall the following standard fact (e.g. [Mi17, Ex. 15.1]):

Lemma 1.2.13 (Crossed homomorphisms are sections of the semidirect product projection)**.** **(i)** *Crossed homomorphisms $\phi : G \to \Gamma$ (Def. 1.2.10) are in bijective correspondence to homomorphic sections of the semidirect group projection* (2.7)*:*

$$\begin{tikzcd} & \Gamma \rtimes G \arrow[d, "\mathrm{pr}_2"] \\ G \arrow[ur, dashed, "g \mapsto (\phi(g), g)"] \arrow[r, equal] & G \end{tikzcd}$$

(ii) *Under this identification, crossed conjugations* (1.47) *are equivalently plain conjugations with elements in* $\Gamma \xrightarrow{i} \Gamma \rtimes G$.

[2] For non-discrete domain groups the notion of crossed homomorphisms need no longer capture all 1-cocycles in non-abelian group cohomology, when the latter is formulated in proper stacky generality; see [WW15] for pointers.

[3] The passage to connected components in (1.49) seems not to be considered in existing literature. It makes no difference when the coefficient group is discrete, (which tends to be tacitly understood in this context, but is not the most general case of interest), as well as under other sufficient conditions discussed in Prop. 1.2.20 below. But in general the correct homotopy-meaningful definition of non-abelian group 1-cohomology (see Rem. 1.2.19 below) is only obtained with passage to connected components included. (The statement in [GMM17, §4.3], that any groupoid is equivalent to the coproduct of its automorphism sub-groupoids, is patently false for topological groupoids, in general. It does hold under suitable extra conditions, such as in Prop. 1.2.20 below). For further discussion of this point see the companion article (cite).

[4] These are sections 16.a-b & 27.a in the expanded version of Milne's book at `www.jmilne.org/math/CourseNotes/iAG200.pdf`

Proof. Having a section means that $g \mapsto (\phi(g), g) \in \Gamma \rtimes G$, and this being a homomorphism means that

$$(\phi(g_1 \cdot g_2), g_1 \cdot g_2) \;=\; (\phi(g_1), g_1) \cdot (\phi(g_2), g_2) \;=\; \big(\phi(g_1) \cdot \alpha(g_2)\big(\phi(g_2)\big), g_1 \cdot g_2\big), \tag{1.50}$$

where the second equality on the right is the definition of the semidirect product group operation, evidently reproducing the defining condition (1.45) in the first argument.

Analogously, plain conjugation in the semidirect product group with elements of the form $(\gamma, \mathrm{e}) \in \Gamma \rtimes G$ gives

$$(\gamma, \mathrm{e})^{-1} \cdot (\phi(g), g) \cdot (\gamma, \mathrm{e}) \;=\; \big(\gamma^{-1} \cdot \phi(g) \cdot \alpha(g)(\gamma), g\big), \tag{1.51}$$

reproducing the formula (1.47) in the first argument. □

Similarly elementary, but maybe less widely appreciated (see Rem. 1.2.15), is the following:

Lemma 1.2.14 (Graphs of crossed homomorphisms [tD69, §2.1][GMM17, Lem. 4.5])**.**
(i) *The graph of a crossed homomorphism $\phi : G \to \Gamma$ (Def. 1.2.11) is a subgroup*

$$\widehat{G} \in \Gamma \rtimes G, \qquad \text{such that} \quad \mathrm{pr}_2(\widehat{G}) \simeq G \quad \text{and} \quad \widehat{G} \cap i(\Gamma) \simeq \{(\mathrm{e}, \mathrm{e})\}, \tag{1.52}$$

where $i : \Gamma \hookrightarrow \Gamma \rtimes G$ is the canonical subgroup inclusion (2.7)*.*
(ii) *Every such subgroup* (1.52) *is the graph of a unique crossed homomorphism.*

Proof. The first statement is immediate from the definitions. For the converse statement (ii), consider a subgroup $\widehat{G}$ as in (1.52). Then the subgroup property implies that

$$(\gamma, g), (\gamma', g) \in \widehat{G} \qquad \Rightarrow \qquad (\gamma', g) \cdot (\gamma, g)^{-1} \;=\; (\gamma', g) \cdot \big(\alpha(g^{-1})(\gamma^{-1}), g^{-1}\big) \;=\; \big(\gamma' \cdot \gamma^{-1}, \mathrm{e}\big) \;\; \in \widehat{G}.$$

From this, the second condition in (1.52) implies that

$$(\gamma, g), (\gamma', g) \in \widehat{G} \qquad \Rightarrow \qquad \gamma = \gamma'.$$

Together with the first condition in (1.52), this implies that $\widehat{G}$ is the graph of a function $\phi : G \times \Gamma$. From this the claim follows by Lem. 1.2.13. □

Remark 1.2.15 (Graphs of crossed homomorphisms in the literature)**.** Subgroups of the form (1.52) were used in early articles on equivariant bundle theory (e.g. [LM86, Thm. 10][May90, Thm. 7]). That these are equivalently (graphs of) crossed homomorphisms (Lem. 1.2.14) and hence homomorphic sections of the semidirect product group (Lem. 1.2.13) may have been (in view of Prop. 1.2.18 below) one of the key observations that led to the construction in [MS95] (discussed in §2.3 below); however, Lem. 1.2.14 is still not made explicit there.

Notation 1.2.16 (Conjugation groupoid of crossed homomorphisms)**.** We write

$$\mathrm{CrsHom}(G, G \circlearrowright \Gamma) /\!\!/_{\mathrm{ad}} \Gamma \;:=\; \big(\mathrm{CrsHom}(G, G \circlearrowright \Gamma) \times \Gamma \;\rightrightarrows\; \mathrm{CrsHom}(G, G \circlearrowright \Gamma)\big)$$

for the topological action groupoid (Ex. 1.2.6) of crossed conjugations (1.47) acting on the space (1.46) of crossed homomorphisms (1.45).

Definition 1.2.17 (Topological groupoid of sections of delooped semidirect product projection)**.** For $G \in \mathrm{Grp}(\mathrm{kTopSpc})$ and $G \circlearrowright \Gamma \in G\mathrm{Act}(\mathrm{kTopSpc})$, consider the topological mapping groupoid (Ex. 1.2.9) from the delooping groupoid (Ex. 1.2.7) of G into that of the semidirect product group $\Gamma \rtimes G$ (Lem. 2.1.4) restricted to those functors and natural transformations which are sections, in that their composition with the projection (2.7) back to $\mathbf{B}G$ is the identity:

$$\mathrm{Map}\big(\mathbf{B}G, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} \;:=\; \mathrm{Maps}\left(\mathbf{B}G, \mathbf{B}(\Gamma \rtimes G)\right) \underset{\mathrm{Maps}(\mathbf{B}G,\, \mathbf{B}G)}{\times} \{\mathrm{id}\} \;=\; \left\{ \begin{tikzcd} & \mathbf{B}(\Gamma \rtimes G) \arrow[d, "\mathbf{B}\mathrm{pr}_2"] \\ \mathbf{B}G \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] \arrow[ur, phantom, "\Downarrow"] \arrow[r, equal] & \mathbf{B}G \end{tikzcd} \right\}. \tag{1.53}$$

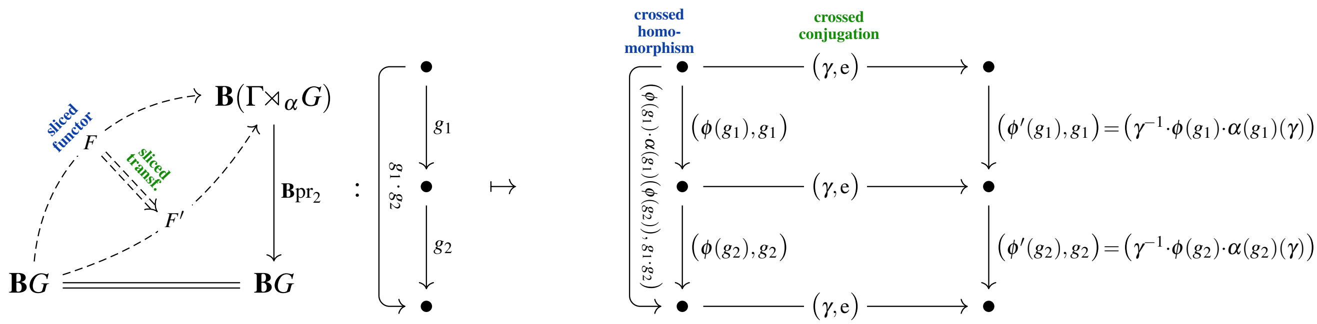
Proposition 1.2.18 (Conjugation groupoid of crossed homomorphisms is sections of delooped semidirect product)**.** *The groupoid of sections of the delooped semidirect product projection (Def. 1.2.17) is isomorphic to the conjugation groupoid of crossed homomorphisms (Ntn. 1.2.16):*

$$\mathrm{Map}(\mathbf{B}G, \mathbf{B}(\Gamma \rtimes G))_{\mathbf{B}G} \;\simeq\; \mathrm{CrsHom}(G, G \circlearrowright \Gamma) \,/\!\!/_{\mathrm{ad}}\, \Gamma\,. \tag{1.54}$$

Hence its connected components (Def. 1.2.4) are bijective to the non-abelian group 1-cohomology (1.49)*:*

$$\tau_0 \mathrm{Map}\big(\mathbf{B}G, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} \;\simeq\; H^1_{\mathrm{Grp}}(G, G \circlearrowright \Gamma)\,.$$

Proof. By definition, a morphism in the groupoid (1.53) is a commuting diagram of functors and natural transformations as shown on the left of the following:



By unwinding the definition and using Lem. 1.2.13 the claim follows, as indicated on the right. □

Remark 1.2.19 (General abstract perspective on non-abelian group 1-cohomology)**.** Prop. 1.2.18 says that non-abelian group cohomology (here in degree 1) is an example of the general notion of twisted non-abelian cohomology in [SS20-Orb, §2.2], for local coefficient bundle being the fibration $B(\Gamma \rtimes G) \to BG$.

Proposition 1.2.20 (Discrete spaces of crossed-conjugacy classes of crossed homomorphisms)**.** *let*
(a) G be a compact Lie group (e.g. a finite group),
(b) Γ be any Lie group, and
(c) $\alpha : G \to \mathrm{Aut}_{\mathrm{Grp}}(\Gamma)$ be trivial on the center of G.
Then:
(i) *the crossed-conjugation quotient space in* (1.49) *is discrete, hence already is the group 1-cohomology as a set:*

$$H^1_{\mathrm{Grp}}(G, G \circlearrowright \Gamma) \;\simeq\; \mathrm{CrsHom}(G, G \circlearrowright \Gamma)/\sim_{\mathrm{ad}} \quad \in \mathrm{Set} \xhookrightarrow{\mathrm{Disc}} \mathrm{kTopSpc}\,. \tag{1.55}$$

(ii) *The crossed homomorphism space equipped with its crossed conjugation action* (1.48)*, decomposes as a disjoint union, indexed by this group 1-cohomology set* (1.55)*, of the coset spaces (Ex. 1.1.14) of Γ by the crossed conjugation stabilizer subgroups $C_\Gamma(\phi) \subset \Gamma$:*

$$\mathrm{CrsHom}(G, G \circlearrowright \Gamma) \;\simeq\; \coprod_{\substack{[\phi] \in \\ H^1_{\mathrm{Grp}}(G, G \circlearrowright \Gamma)}} \Gamma / C_\Gamma(\phi) \quad \in \Gamma\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{1.56}$$

Proof. For the special case when the action α is entirely trivial, the statement is observed in [Re14a, Rem. 2.2.1]. To deduce from this the more general statement, identify crossed homomorphisms $\phi : G \to \Gamma$ with constrained plain homomorphisms $G \to \Gamma \rtimes G$ according to Lem. 1.2.13. Applied to these, the previous version of the statement guarantees for nearby ϕ, ϕ' an element $(\gamma, h) \in \Gamma \rtimes G$, such that:

$$\underset{g \in G}{\forall} \quad \big(\phi'(g), (g)\big) \;=\; (\gamma, h)^{-1} \cdot (\phi(g), g) \cdot (\gamma, h)\,.$$

In the second component this implies that $h \in C(G)$ is in the center of G. Therefore, further conjugation of this equation with $(\mathrm{e}, h) \in \Gamma \rtimes G$, and using assumption (iii), yields

$$\underset{g \in G}{\forall} \quad \big(\phi'(g), (g)\big) \;=\; (\gamma, \mathrm{e})^{-1} \cdot (\phi(g), g) \cdot (\gamma, \mathrm{e})\,,$$

which implies the claim as in (1.51). □

Proposition 1.2.21 (Weyl group action on group 1-cohomology of subgroup)**.** *For* $G \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $G \curvearrowright \Gamma \in \mathrm{Grp}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$*, let* $H \subset G$ *be a subgroup, with* $H \curvearrowright \Gamma \in \mathrm{Grp}(H\,\mathrm{Act}(\mathrm{kTopSpc}))$ *denoting the restricted action (Ex. 1.1.12). Then:*
(i) *The assignment*

$$\begin{array}{ccc} N(H) \times \mathrm{CrsHom}(H, H \curvearrowright \Gamma) & \longrightarrow & \mathrm{CrsHom}(H, H \curvearrowright \Gamma) \\ (n,\phi) & \longmapsto & \phi_n \,:\, h \mapsto \alpha(n)(n^{-1}\cdot h \cdot n) \end{array} \tag{1.57}$$

is a continuous action of the normalizer subgroup $N(H)$ *(Ntn. 1.1.1) on the space* (1.46) *of crossed homomorphisms out of* H.
(ii) *This descends to the quotient by crossed conjugations, hence to the non-abelian group 1-cohomology set* (1.49)*,*
(iii) *where it passes through an action of the Weyl group* $W(H) := N(H)/H$ *(Ntn. 1.1.1):*

$$G \curvearrowright \Gamma \in \mathrm{Grp}\big(H\,\mathrm{Act}(\mathrm{kTopSpc})\big) \qquad \vdash \qquad \underset{H \subset G}{\forall} \;\; W(H) \curvearrowright H^1_{\mathrm{Grp}}(H, H \curvearrowright \Gamma) \;\; \in W(H)\,\mathrm{Act}(\mathrm{Set})\,. \tag{1.58}$$

Proof. The first two statements are immediate in the equivalent incarnation of crossed homomorphisms $\phi : H \to \Gamma$ as plain homomorphisms of the form $(\phi(-),(-)) : G \to \Gamma \rtimes G$ (Lem. 1.2.13), observing that, under this identification, the assignment (1.57) is just the "conjugation action by the adjoint action":

$$\big(\phi_n(-),(-)\big) \;=\; (n,\mathrm{e})\cdot\Big(\phi_n\big(n^{-1}\cdot(-)\cdot n\big),\,\big(n^{-1}\cdot(-)\cdot n\big)\Big)\cdot(n,\mathrm{e})\,.$$

For the last statement we need to exhibit a crossed conjugation between ϕ_n and ϕ whenever $n \in H \subset N(H)$. The following direct computation shows that this is given by $\phi(n)$ (which is defined for such n):

$$\begin{aligned} \phi_n(h) \;&=\; \alpha(n)\big(\phi(n^{-1}\cdot h\cdot n)\big) && \text{by definition (1.57)} \\ &=\; \alpha(n)\Big(\phi(n^{-1})\cdot\alpha(n^{-1})\big(\phi(h)\cdot\alpha(h)(\phi(n))\big)\Big) && \text{by crossed hom. property (1.45)} \\ &=\; \phi(n^{-1})\cdot\phi(h)\cdot\alpha(h)\,(\phi(n)) && \text{by action property of } \alpha\,. \end{aligned}$$

□

Simplicial topological spaces.

Notation 1.2.22 (Simplicial topological spaces)**.** We denote by

$$\Delta\mathrm{kTopSpc} \;:=\; \mathrm{Fnctr}(\Delta^{\mathrm{op}}, \mathrm{kTopSpc}) \tag{1.59}$$

the category of simplicial topological spaces, hence of simplicial objects internal to (Ntn. 1.0.23) the category of compactly generated weak Hausdorff spaces (1.2).

Definition 1.2.23 (Homotopy Kan fibrations [Lur11, Def. 3][MG14, p. 2])**.** We say that a morphism of simplicial topological spaces (1.59) $f_\bullet : \mathrm{X}_\bullet \to \mathrm{Y}_\bullet$ is a *homotopy Kan fibrations*, to be denoted

$$f_\bullet \;\in\; \mathrm{hKanFib} \;\subset\; \mathrm{Mor}(\Delta\mathrm{kTopSpc})\,,$$

if for all positive $n \in \mathbb{N}_+$ every solid homotopy-commutative diagram as follows has a dashed lift up to homotopy, as shown:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, hook] \arrow[d] & \mathrm{X}_\bullet \arrow[d, "f_\bullet"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Y}_\bullet \end{tikzcd} \qquad \Leftrightarrow \qquad \pi_0\Big(X(\Delta^n) \to X(\Lambda^n_k) \underset{Y(\Lambda^n_k)}{\times^h} Y(\Delta^n)\Big) \text{ is surjective,} \tag{1.60}$$

where the morphism on the far left is the (n,k)-horn inclusion of simplicial sets regarded as degreewise discrete simplicial topological spaces. The equivalent condition on the right of (1.60) simplifies when horn filling in $\mathrm{Y}_\bullet$ is a Serre fibration, in which case the homotopy fiber product in (1.60) is given by the plain fiber product, so that:

$$\left(\underset{\substack{n\in\mathbb{N}_+ \\ 0\leq k\leq n}}{\forall} \mathrm{X}(\Delta^n) \xrightarrow{\in \mathrm{SerFib}} \mathrm{Y}(\Lambda^n_k)\right) \;\Rightarrow\; \left(\big(f_\bullet \,\in\, \mathrm{hKanFib}\big) \Leftrightarrow \left(\underset{\substack{n\in\mathbb{N}_+ \\ 0\leq k\leq n}}{\forall} \pi_0\big(\mathrm{X}(\Delta^n) \to \mathrm{X}(\Lambda^n_k) \underset{\mathrm{Y}(\Lambda^n_k)}{\times} \mathrm{Y}(\Delta^n)\big)\,\text{surj.}\right)\right). \tag{1.61}$$

Notation 1.2.24 (Nerve of topological groupoids)**.** We denote the *nerve* functor from topological groupoids (Ntn. 1.2.1) to simplicial topological spaces (1.2.22) by

$$
\begin{array}{rcl}
N \;:\; \mathrm{Grpd}(\mathrm{kTopSpc}) & \longrightarrow & \Delta\mathrm{kTopSpc} \\
(\mathrm{X}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{X}_1) & \longmapsto & \Big([n] \mapsto \underbrace{(\mathrm{X}_1)\, {}_t{\times_s}_{\mathrm{X}_0} \cdots {}_t{\times_s}_{\mathrm{X}_0} \mathrm{X}_1}_{n \text{ factors}}\Big)
\end{array}
\tag{1.62}
$$

Example 1.2.25 (Nerves of action groupoids have horn filling by Serre fibrations)**.** For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ and $\mathrm{X} \circlearrowright \Gamma \in \Gamma^{\mathrm{op}}\mathrm{Act}(\mathrm{kTopSpc})$, the nerve (1.62) of the corresponding action groupoid (Ex. 1.2.6) is of the form

$$
N(\mathrm{X} \times \Gamma \rightrightarrows \Gamma)_n \;\;\simeq\;\; \mathrm{X} \times \Gamma^{\times n} \;\;\in\; \mathrm{kTopSpc}\,.
$$

from which it is readily seen that the horn filler maps are all Serre fibrations (for $n = 1$ by Lem. 1.3.2):

$$
\begin{array}{l}
N(\mathrm{X} \times \Gamma \rightrightarrows \Gamma)(\Delta^1) \;\simeq\; \mathrm{X} \times \Gamma \;\xrightarrow[\in\mathrm{SerFib}]{\mathrm{pr}_1}\; \mathrm{X} \quad\;\; \simeq N(\mathrm{X} \times \Gamma \rightrightarrows \Gamma)(\Lambda^1_1) \\
\underset{\substack{n \geq 2 \\ 0 \leq k \leq n}}{\forall} \quad N(\mathrm{X} \times \Gamma \rightrightarrows \Gamma)(\Delta^n) \;\simeq\; \mathrm{X} \times \Gamma^{\times n} \;\xrightarrow[\in\mathrm{SerFib}]{\sim}\; \mathrm{X} \times \Gamma^{\times n} \;\simeq\; N(\mathrm{X} \times \Gamma \rightrightarrows \Gamma)(\Lambda^n_k)\,.
\end{array}
\tag{1.63}
$$

Example 1.2.26 (Nerves of morphisms of delooping groupoids are homomotopy Kan fibrations)**.** Let $\Gamma_1 \xrightarrow{\phi} \Gamma_2 \;\in$ $\mathrm{Grp}(\mathrm{kTopSpc})$ be a homomorphism of topological groups which is surjective on connected components of underlying topological spaces:

$$
\pi_0(\Gamma_1) \xrightarrow{\;\pi_0(\phi)\;}\!\!\!\!\rightarrow \pi_0(\Gamma_2)\,.
$$

Then its image under taking nerves (1.62) of delooping groupoids (Ex. 1.2.7) is a homotopy Kan fibration (Def. 1.2.23):

$$
N(\Gamma_1 \rightrightarrows *)_\bullet \xrightarrow[\in\,\mathrm{hKanFib}]{N(\phi \rightrightarrows *)_\bullet} N(\Gamma_2 \rightrightarrows *)_\bullet\,.
\tag{1.64}
$$

Proof. By (1.63) in Ex. 1.2.25 the condition to be checked is that on the right of (1.61), which in degree 1 is satisfied by assumption, since here the comparison map is ϕ itself:

$$
N(\Gamma_1 \rightrightarrows *)(\Delta^1) \;\simeq\; \Gamma_1 \;\;\xrightarrow{\;\;\phi\;\;}\;\; \Gamma_2 \;\simeq\; N(\Gamma_1 \rightrightarrows *)(\Delta^0) \underset{N(\Gamma_2 \rightrightarrows *)(\Delta^0)}{\times} (\Gamma_2 \rightrightarrows *)(\Delta^1)\,,
$$

while in higher degrees the condition is trivial, as here the comparison map is an isomorphism (since the pullbacks of the isomorphisms in (1.63) are isomorphisms):

$$
\underset{\substack{n \geq 2 \\ 0 \leq k \leq n}}{\forall} \; N(\Gamma_1 \rightrightarrows *)(\Delta^n) \;\simeq\; \Gamma_1^{\times n} \;\;\xrightarrow{\;\;\sim\;\;}\;\; \Gamma_1^{\times n} \;\simeq\; N(\Gamma_1 \rightrightarrows *)(\Lambda^n_k) \underset{N(\Gamma_2 \rightrightarrows *)(\Lambda^n_k)}{\times} (\Gamma_2 \rightrightarrows *)(\Delta^n)\,.
$$

□

Example 1.2.27 (Base morphisms out of nerves of action groupoids are homotopy Kan fibrations)**.**
For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ and $\mathrm{X} \circlearrowright \Gamma \in \Gamma^{\mathrm{op}}\mathrm{Act}(\mathrm{kTopSpc})$, the image under taking nerves (1.62) of the canonical morphism from the action groupoid (Ex. 1.2.6) to the delooping groupoid (Ex. 1.2.7) is a homotopy Kan fibration (Def. 1.2.23):

$$
N(\mathrm{X} \times \Gamma \rightrightarrows \mathrm{X}) \xrightarrow[\in\mathrm{hKanFib}]{} N(\Gamma \rightrightarrows *)\,.
$$

Proof. Again, by (1.63) in Ex. 1.2.25 the condition to be checked is that on the right of (1.61). One readily sees that the comparison maps are in fact isomorphisms, hence in particular surjective on connected components. □

Topological realization of topological groupoids.

Notation 1.2.28 (Topological realization functors)**.** We denote
(i) the cosimplicial topological space of *standard topological simplices* by

$$
\begin{array}{rcl}
\Delta^\bullet_{\mathrm{top}} \;:\; \Delta & \longrightarrow & \mathrm{kTopSpc} \\
{[n]} & \longmapsto & \big\{\vec{x} \in \mathbb{R}^n \,\big|\, \sum_i x_i = 1 \text{ and } \forall_i (0 \leq x_i \leq 1)\big\}
\end{array}
$$

(ii) the *topological realization of simplicial topological spaces* ([ML70, §6][May72, §11]) by

$$\begin{array}{rcl} |-| \,:\, \Delta\mathrm{kTopSpc} & \longrightarrow & \mathrm{kTopSpc} \\ \mathrm{X}_\bullet & \longmapsto & \int^{[n]\in\Delta} \mathrm{X}_n \times \Delta^n_{\mathrm{top}} \end{array} \tag{1.65}$$

(iii) the *topological realization of topological groupoids* by the same symbol, leaving the nerve (1.62) notationally implicit:

$$|-| \;:\; \mathrm{Grpd}(\mathrm{kTopSpc})\,. \xrightarrow{N} \Delta\mathrm{kTopSpc} \xrightarrow{|-|} \mathrm{kTopSpc}\,. \tag{1.66}$$

Remark 1.2.29 (Classifying spaces and realization)**.** The realization (1.66) of topological groupoids has traditionally been denoted $B(-)$. This may seem like a good idea if one already notationally conflates topological groups Γ with their delooping groupoids $\Gamma \rightrightarrows *$ (Ex. 1.2.7), because then $B\Gamma$ denotes the intended classifying space. But, since a group is crucially not the same as its delooping groupoid, we give the latter its own symbol (following [NSS12a][SS20-Orb]): $\mathbf{B}\Gamma := (\Gamma \rightrightarrows *)$, and denote topological realization of topological groupoids by the same symbol $|-|$ as that of topological spaces (see corresponding discussion in [GMM17, p. 4]). Therefore, in our notation, the classifying space (2.65) of a topological group Γ (rather: of Γ-principal bundles, when Γ is well-behaved) arises as

$$B\Gamma \;=\; |\mathbf{B}\Gamma|\,.$$

This typesetting serves to express that the classifying space BG is but a shadow (namely: the *shape*) of the richer *moduli stack* $\mathbf{B}G$ (see [SS20-Orb]).

Example 1.2.30 (Topological realization of constant groupoids)**.** The topological realization (1.66) of constant topological groupoids (Ex. 1.2.2) is the topological realization (1.65) of the corresponding constant simplicial space, and hence is homeomorphic to the underlying topological space:

$$|\mathrm{Cnst}(\mathrm{X})| \;\simeq\; \mathrm{X} \quad \in \mathrm{kTopSpc}\,. \tag{1.67}$$

Lemma 1.2.31 (Topological realization preserves finite limits)**.** *The topological realization functors (Ntn. 1.2.28), both (i) of simplicial topological spaces* (1.65) *and (ii) of topological groupoids* (1.66)*, preserve finite limits:*

$$|-| \;:\; \mathrm{Grpd}(\mathrm{kTopSpc}) \xrightarrow{\mathrm{lex}} \mathrm{kTopSpc}\,.$$

Proof. The second statement follows from the first by the fact that the nerve (1.62) is a right adjoint ([Kan58, §3]) and hence preserves all limits.

For the first statement it is sufficient (e.g. [Bo94I, Prop. 2.8.2]) to see that topological realization (a) preserves the terminal object (here: the point) and (b) preserves all pullbacks. Here (a) follows by Ex. 1.2.30 and (b) is the statement of [May72, Cor. 11.6] (making crucial use of compact generation (1.2), see [May72, p. 1]). □

Lemma 1.2.32 (Topological realization of equivalence is homotopy equivalence)**.** *An equivalence* (1.38) *of topological groipoids induces a homotopy equivalence between their topological realizations* (1.66)*:*

$$(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \underset{\mathrm{htpy}}{\simeq} (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0) \qquad \Rightarrow \qquad |(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)| \underset{\mathrm{htpy}}{\simeq} |(\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)|\,.$$

Proof. Observe that a 2-morphism $F \overset{\eta}{\Rightarrow} F'$ (1.37) of topological groipoids is equivalently a morphism $\widetilde{\eta}$ out of the product of the domain groupoid with the codiscrete groupoid (Ex. 1.2.3) on the 2-element set:

$$\begin{array}{rcl} \widetilde{\eta} \,:\, (\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \times \mathrm{Cht}(\{*,*'\}) & \longrightarrow & (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0) \\ (x \xrightarrow{\gamma} x') \times * & \longmapsto & F(x) \xrightarrow{F(\gamma)} F(x') \\ (x \xrightarrow{\gamma} x') \times *' & \longmapsto & F'(x) \xrightarrow{F'(\gamma)} F'(x') \\ x \times (* \to *') & \longmapsto & F(x) \xrightarrow{\eta(x)} F'(x)\,. \end{array}$$

The topological realization of this morphism is, by Lem. 1.2.31, of the following form

$$
\begin{tikzcd}
\left|(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)\right| \arrow[d, "\mathrm{id}(-)\times *"'] \arrow[drr, "|F|"] & & \\
\left|(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)\right| \times \left|\mathrm{Cht}(\{*,*'\})\right| \arrow[rr, "|\widetilde{\eta}|"] & & \left|(\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)\right| . \\
\left|(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)\right| \arrow[u, "\mathrm{id}(-)\times *'"] \arrow[urr, "|F'|"'] & &
\end{tikzcd}
$$

Observing that

$$
\begin{tikzcd}
\{*,*'\} \arrow[r, hook] \arrow[rr, bend left, "\nabla_*"] & \left|\mathrm{Cht}(\{*,*'\})\right| \arrow[r] & *
\end{tikzcd}
$$

is a cylinder object for the point in the classical model structure on topological spaces, this exhibits a homotopy of continuous functions

$$|F| \Rightarrow |F'| .$$

Applying this to the two homotopies involved in the data (1.38) constituting an equivalence of topological groupoids, this yields a homotopy equivalence of topological spaces as claimed. ☐

Topological 2-groups.

Notation 1.2.33 (Topological strict 2-groups)**.** An internal group (Ntn. 1.0.23) in topological groupoids (Ntn. 1.2.1) is also called a topological *strict 2-group*:

$$2\mathrm{Grp}(\mathrm{kTopSpc}) \;:=\; \mathrm{Grp}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right) . \tag{1.68}$$

By Lem. 1.2.31, topological realization (1.66) induces (1.6) a functor from topological 2-groups to topological groups

$$2\mathrm{Grp}(\mathrm{kTopSpc}) \;=\; \mathrm{Grp}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right) \xrightarrow{\;|-|\;} \mathrm{Grp}(\mathrm{kTopSpc}) . \tag{1.69}$$

Example 1.2.34 (Topological groups as topological 2-groups)**.** For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$, its constant groupoid (Ex. 1.2.2) carries the structure of a topological strict 2-group (Ntn. 1.2.33)

$$(\Gamma \rightrightarrows \Gamma) \quad \in \; 2\mathrm{Grp}(\mathrm{kTopSpc})$$

with group structure given degreewise by that of $\widehat{\Gamma}$.

Example 1.2.35 (Strict 2-groups delooping abelian groups)**.** For $A \in \mathrm{AbGrp}(\mathrm{kTopSpc})$ an abelian topological group, its delooping groupoid (Ex. 1.2.7) carries the structure of a topological strict 2-group (Ntn. 1.2.33)

$$(A \rightrightarrows 1) \quad \in \; 2\mathrm{Grp}(\mathrm{kTopSpc})$$

with group structure given degreewise by that of A. The topological group which is the topological realization (1.69) of this 2-group

$$BA \;:=\; |A \rightrightarrows *| \quad \in \; \mathrm{Grp}(\mathrm{kTopSpc})$$

has as underlying space the Milgram classifying space (2.65) of A.

Example 1.2.36 (Long exact sequence of 2-groups from central extension of groups)**.** Given a central extension of topological groups

$$
\begin{tikzcd}
1 \arrow[r] & A \arrow[rr, hook, "i"] \arrow[dr, hook] & & \widehat{\Gamma} \arrow[r, two heads, "p"] & \Gamma \arrow[r] & 1 \\
& & Z(\widehat{\Gamma}) \arrow[ur, hook] & & &
\end{tikzcd}
\tag{1.70}
$$

the action groupoid (Ex. 1.2.6) of the canonical A action on $\widehat{\Gamma}$ on becomes a strict 2-group (Ntn. 1.2.33) under the direct product group structure:

$$(\widehat{\Gamma} \times A \rightrightarrows \widehat{\Gamma}) \quad \in \; 2\mathrm{Grp}(\mathrm{kTopSpc}) .$$

This sits in the following diagram of 2-groups

$$
\begin{tikzcd}
 & & (\widehat{\Gamma} \times A \rightrightarrows \widehat{\Gamma}) \arrow[r, "(\mathrm{pr}_2 \rightrightarrows 1)"] \arrow[d, "((p \circ \mathrm{pr}_1) \rightrightarrows p)"] & (A \rightrightarrows 1) \\
(A \rightrightarrows A) \arrow[r, "(i \rightrightarrows i)"] & (\widehat{\Gamma} \rightrightarrows \widehat{\Gamma}) \arrow[r, "(p \rightrightarrows p)"] \arrow[ur, "((\mathrm{id},\mathrm{e}) \rightrightarrows \mathrm{id})"] & (\Gamma \rightrightarrows \Gamma)
\end{tikzcd}
\tag{1.71}
$$

with the constant 2-groups from Ex. 1.2.34 on the bottom, and the delooping 2-group from Ex. 1.2.35 on the top right.

Equivariant topological groupoids.

Definition 1.2.37 (Equivariant topological groupoids)**.** We write $\mathrm{Grpd}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$ for the category of internal groupoids (Ntn. 1.0.23) internal to equivariant topological spaces (1.8), hence of diagrams of the form (1.35) but where all spaces involved are equipped with continuous G-actions and all maps involved are G-equivariant. With the equivariance group regarded as a group object in topological groupoids via the inclusion (1.39) of Ex. 1.2.2

$$G \in \mathrm{Grp}(\mathrm{kTopSpc}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Cnst})} \mathrm{Grp}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right),$$

this is equivalent to the category of internal G-actions (Ntn. 1.0.23) in $\mathrm{Grpd}(\mathrm{kTopSpc})$ (Ntn. 1.2.1):

$$\mathrm{Grpd}\left(G\,\mathrm{Act}(\mathrm{kTopSpc})\right) \;\simeq\; G\,\mathrm{Act}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right). \tag{1.72}$$

Example 1.2.38 (Right action groupoid inherits left group actions)**.** Let $G_L, G_R \in \mathrm{Grp}(\mathrm{kTopSpc})$ and

$$G_L \circlearrowright \mathrm{X} \circlearrowleft G_R \quad \in \; \left(G_L \times G_R^{\mathrm{op}}\right)\mathrm{Act}(\mathrm{kTopSpc}).$$

Then the G_R^{op}-action groupoid of X (Ex. 1.2.6) inherits the left G_L-action to become a G_L-eqivariant topological groupoid (Def. 1.2.37):

$$G_L \circlearrowright (\mathrm{X} \times G_R \rightrightarrows \mathrm{X}) \quad \in \; G_L\,\mathrm{Act}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right).$$

In generalization of Ex. 1.1.6, we have:

Example 1.2.39 (Conjugation action on mapping groupoid)**.** Given a pair of G-equivariant topological groupoids (Def. 1.2.37)

$$G \circlearrowright (\mathrm{X}_1 \rightrightarrows \mathrm{X}_0),\; G \circlearrowright (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0) \quad \in \; G\,\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))$$

the mapping groupoid (1.42) of their underlying topological groupoids (Ntn. 1.2.1) inherits the conjugation action

$$G \circlearrowright \mathrm{Maps}\big((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), (\mathrm{Y}_1 \rightrightarrows \mathrm{Y}_0)\big) \quad \in G\,\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))$$

given on the spaces of morphisms and of objects by the restriction of the ordinary conjugation action (1.13). With the first argument fixed, this construction constitutes a right adjoint to the product operation, in joint generalization of (1.16) and (1.43):

$$
\begin{tikzcd}
G\,\mathrm{Act}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right) \arrow[r, shift right, "G \circlearrowright \mathrm{Maps}((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), -)"'] \arrow[r, phantom, "\perp"] & G\,\mathrm{Act}\left(\mathrm{Grpd}(\mathrm{kTopSpc})\right). \arrow[l, shift right, "G \circlearrowright (\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \times (-)"']
\end{tikzcd}
\tag{1.73}
$$

Equivariant topological realization of equivariant topological groupoids. The simplicial nerve functor (1.62) is a right adjoint and as such preserves all limits; but it preserves only a small class of colimits, among them the class of the following Lem. 1.2.40, where this preservation secretly underlies classical constructions (Prop. 2.3.4 below) of universal principal bundles via realization of topological groupoids (as amplified in [GMM17, p. 11]):

Lemma 1.2.40 (Nerve preserves left quotients of right action groupoids)**.** *Let* $\Gamma_L, \Gamma_R \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and*

$$\Gamma_L \circlearrowright \mathrm{X} \circlearrowleft \Gamma_R \quad \in \; \left(\Gamma_L \times \Gamma_R^{\mathrm{op}}\right)\mathrm{Act}(\mathrm{kTopSpc}).$$

Then the canonical comparison morphism from the nerve (1.62) *of the left* Γ_L*-quotient of the equivariant right action groupoid (Ex. 1.2.38) to the* Γ_L*-quotient of the nerve is an isomorphism:*

$$N\left(\Gamma_L \backslash (\mathrm{X} \times \Gamma_R \rightrightarrows \mathrm{X})\right) \;\simeq\; \Gamma_L \backslash \left(N(\mathrm{X} \times \Gamma_R \rightrightarrows \mathrm{X})\right).$$

Proof. The point is that in each degree $n \in \mathbb{N}$, the Γ_L-action is non-trvial only on the leftmost factor of the n-th component space of the nerve:

$$
\begin{array}{ccccc}
N(\mathrm{X} \times \Gamma \rightrightarrows \mathrm{X})_n & \simeq & \mathrm{X} \times \overbrace{\Gamma \times \cdots \times \Gamma}^{n \text{ factors}} & & \\
{\scriptstyle \text{quotient coprojection}} \big\downarrow & & \big\downarrow & & \\
\Gamma \backslash \big(N(\mathrm{X} \times \Gamma \rightrightarrows \mathrm{X})_n\big) & \simeq & (\Gamma \backslash \mathrm{X}) \times \underbrace{\Gamma \times \cdots \times \Gamma}_{n \text{ factors}} & \simeq & N\big(\Gamma \backslash (\mathrm{X} \times \Gamma \rightrightarrows \mathrm{X})\big)_n \,.
\end{array}
$$

□

Proposition 1.2.41 (Topological realization of topological groupoids respects equivariance)**.** *For any $G \in$ Grp(kTopSpc), topological realization* (1.66) *of topological groupoids extends to a functor from G-actions on topological groupoids (Def. 2.1.2) to G-action on topological spaces (Ntn. 1.1.1):*

$$|-| \,:\, G\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big) \longrightarrow G\,\mathrm{Act}(\mathrm{kTopSpc})\,.$$

Proof. Under the equivalence (1.72) the functor in question is of the form

$$G\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big) \xrightarrow{(|-|)\,\mathrm{Act}(|-|)} G\,\mathrm{Act}\,(\mathrm{kTopSpc})\,,$$

in the notation (1.6), and hence exists since $|-|$ preserves finite limits (here in particular: finite products), by Lemma 1.2.28. (Here we are using that form of the group object is indeed preserved, by Ex. 1.2.30). □

Equivariantly equivariant topological groupoids. In equivariant generalization of Def. 2.1.2, we have:

Definition 1.2.42 (Equivariant topological groupoid)**.** Let $G \in$ Grp(kTopSpc) and $G \circlearrowright \Gamma_L \in \mathrm{Grp}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))$ be a G-equivariant group (Def. 2.1.2). With G-equivariant topological groups regarded as constant G-equivariant topological groupoids

$$(G \circlearrowright \Gamma) \;\in\; \mathrm{Grp}\,\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big) \xhookrightarrow{\mathrm{Grp}((\mathrm{Cnst}(G))\,\mathrm{Act}(\mathrm{Cnst}(-)))} \mathrm{Grp}\,\big(G\,\mathrm{Act}\,(\mathrm{Grpd}(\mathrm{kTopSpc}))\big)\,,$$

we obtain the category of $(G \circlearrowright \Gamma)$-equivariant G-equivariant topological groupoids:

$$(G \circlearrowright \Gamma)\,\mathrm{Act}\big(\mathrm{Grpd}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))\big) \;\simeq\; (G \circlearrowright \Gamma)\,\mathrm{Act}\big(G\,\mathrm{Act}\,(\mathrm{Grpd}(\mathrm{kTopSpc}))\big)\,.$$

In equivariant generalization of Ex. 1.2.38, we get:

Example 1.2.43 (Right equivariant action groupoid inherits left equivariant group action)**.** For $G \in$ Grp(kTopSpc), let

$$G \circlearrowright \Gamma_L,\, G \circlearrowright \Gamma_R \quad \in \mathrm{Grp}\,\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)$$

be two G-equivariant groups (Def. 2.1.2) and consider a topological G-space X (1.8) equipped with commuting equivariant left and right actions by these, respectively:

$$(G \circlearrowright \Gamma_L) \circlearrowright (G \circlearrowright \mathrm{X}) \circlearrowleft (G \circlearrowright \Gamma_R) \quad \in \; \big((G \circlearrowright \Gamma_1) \times (G \circlearrowright \Gamma_2)^{\mathrm{op}}\big)\,\mathrm{Act}\,\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\,.$$

Then the evident $(G \circlearrowright \Gamma_R)$-action G-groupoid (formed just as in Ex. 1.2.6 with Ex. 1.2.38, only that now all component spaces carry G-action and all structure maps are G-equivariant) inherits a (further, G-equivariant) $(G \circlearrowright \Gamma_L)$-action:

$$\underbrace{(G \circlearrowright \Gamma_L) \circlearrowright}_{\substack{G\text{-equivariant} \\ \text{left } \Gamma\text{-action}}} \underbrace{\big((G \circlearrowright \mathrm{X}) \times (G \circlearrowright \Gamma_R) \rightrightarrows (G \circlearrowright \mathrm{X})\big)}_{\substack{G\text{-equivariant} \\ \text{right action groupoid}}} \;\in\; (G \circlearrowright \Gamma_L)\,\mathrm{Act}\big(\mathrm{Grpd}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))\big)\,.$$

In equivariant generalization of Ex. 1.2.7, we get:

Example 1.2.44 (Equivariant topological delooping groupoid)**.** For $(G \circlearrowright \Gamma) \in \mathrm{Grp}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$, its *equivariant delooping groupoid* is the G-equivariant topological left action groupoid (Ex. 1.2.43) of the unique $(G \circlearrowright \Gamma)$-action on $(G \circlearrowright *)$:

$$G \circlearrowright \mathbf{B}\Gamma \;:=\; G \circlearrowright (* \times \Gamma^{\mathrm{op}} \rightrightarrows *) \;=\; G \circlearrowright (\Gamma^{\mathrm{op}} \rightrightarrows *) \quad \in \mathrm{Grpd}\,\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\,.$$

1.3 G-Equivariant homotopy types

We recall and develop some of equivariant homotopy theory needed in §2.3. for identifying equivariant homotopy groups of equivariant classifying spaces.

Notation 1.3.1 (Classical model structure on topological spaces)**. (i)** We write

$$\mathrm{kTopSpc}_{\mathrm{Qu}} \quad \in \ \mathrm{MdlCat} \tag{1.74}$$

for the classical model category on (compactly generated, Ntn. 1.0.16) topological spaces, whose weak equivalences are the weak homotopy equivalences WHmtpEq and whose fibrations are the Serre fibrations SerFib (e.g., [Ho99, Thm. 2.4.23]).
(ii) We write

$$\mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}}) \quad \in \ \mathrm{Cat} \tag{1.75}$$

for the classical homotopy category; the localization of (1.74) at the weak equivalences.

For example:

Lemma 1.3.2 (Locally trivial bundles are Serre fibrations [tD08, p. 130, Thm. 6.3.3])**.** *A map* $\mathrm{E} \xrightarrow{p} \mathrm{X} \ \in \mathrm{kTopSpc}$*, which over some open cover* $\widehat{\mathrm{X}} = \underset{i \in I}{\sqcup} \mathrm{U}_i \twoheadrightarrow \mathrm{X}$ *(Ex. 1.0.21) restricts to a Cartesian product, is a Serre fibration:*

$$\begin{tikzcd} \mathrm{U} \times \mathrm{F} \arrow[r] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\scriptstyle (\mathrm{pb})"] & \mathrm{E} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, two heads, "\mathrm{open}"'] & \mathrm{X} \end{tikzcd} \qquad \Rightarrow \qquad p \in \mathrm{SerFib}\,.$$

Notation 1.3.3 (Homotopy fiber sequence)**.** A sequence of morphisms

$$F \xrightarrow{i} \mathrm{E} \xrightarrow{p} \mathrm{X} \quad \in \mathscr{C}$$

in a model category $\mathscr{C}$ is a *homotopy fiber sequence* for a given point $* \xrightarrow{x} \mathrm{X}$, to be denoted

$$i \simeq \mathrm{hofib}x(p)\,,$$

if, for any choice of fibration replacement $\hat{p} : \widehat{\mathrm{X}} \to \mathrm{X}$ of p, the sequence is isomorphic, in the homotopy category, to the ordinary x-fiber mapping into the fibration:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, dashed, "\exists"', "\in \mathrm{W}"] & \mathrm{E} \arrow[r, "p"] \arrow[d, "\in \mathrm{W}"] & \mathrm{Y} \arrow[d, equal] \\ \widehat{E}_{x_0} \arrow[r, "\mathrm{fib}_x(\hat{p})"] & \widehat{\mathrm{E}} \arrow[r, "\hat{p}", "\in \mathrm{Fib}"'] & \mathrm{X} \end{tikzcd} \quad \in \ \mathrm{Ho}(\mathscr{C})\,. \tag{1.76}$$

Proper equivariant homotopy theory of G-spaces.

Proposition 1.3.4 (Proper equivariant model category of G-spaces [DK84, §1.2])**.** *For* $G \in \mathrm{Grp}(\mathrm{CptSmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ *the topological group underlying a compact Lie group, there exists a model category structure on the category of topological G-spaces (Ntn. 1.1.1), to be denoted*

$$G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}} \quad \in \ \mathrm{MdlCat} \tag{1.77}$$

whose weak equivalences $G\mathrm{WHmtpEq}$ *and fibrations* $G\mathrm{SerFib}$ *are those morphisms whose underlying continuous maps between H-fixed loci, for all closed* [5] *subgroups* $H \underset{\mathrm{clsd}}{\subset} G$*, are weak equivalences or fibrations in the Serre-Quillen model structure on topological spaces, (Ntn. 1.3.1) hence are weak homoropy equivalences or Serre fibrations, respectively.*

[5] This model category structure itself exists for any topological group and any non-empty subset of subgroups, but for it to be equivalent to G-CW-complexes localized at the G-equivariant homotopy equivalences one needs to make sufficient restrictions, such as to compact Lie groups and their closed subgroups.

Proposition 1.3.5 (Cofibrant generation of the proper equivariant model structure)**.**
The model category $G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}}$ *from Prop. 1.3.4 is*

(i) *a proper model category;*

(ii) *a cofibrantly generated model category (see e.g. [Hi02, §11]) whose class of generating (acyclic) cofibrations is the product of coset spaces (Ex. 1.1.14) with the generating (acyclic) cofibrations of the Serre-Quillen nodel strutucre*

$$I_{\mathrm{prop}} \;:=\; \big\{ G/H \times S^{n-1} \xhookrightarrow{\mathrm{id}_{G/H}\times i_n} G/H \times D^n \big\}_{n\in\mathbb{N},\, H \underset{\mathrm{clsd}}{\subset} G} \tag{1.78}$$

$$I_{\mathrm{prop}} \;:=\; \big\{ G/H \times D^n \times \{0\} \xhookrightarrow{\mathrm{id}_{G/H}\times j_n} G/H \times D^n \times [0,1] \big\}_{n\in\mathbb{N},\, H \underset{\mathrm{clsd}}{\subset} G;} \tag{1.79}$$

(iii) *an enriched model category (see e.g. [GM11, §4.3]) over* $\mathrm{kTopSpc}_{\mathrm{Qu}}$ *(Ntn. 1.3.1), in that*

(a) *the functor assigning spaces* (1.14) *of equivariant maps*

$$\mathrm{Maps}(-,-)^G \;:\; G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})^{\mathrm{op}} \times G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}}) \longrightarrow \mathrm{kTopSpc}_{\mathrm{Qu}}$$

is a right Quillen bifunctor;

(b) *for* $G \circlearrowright \mathrm{X} \xrightarrow{c\in\mathrm{Cof}} G \circlearrowright \mathrm{Y}$ *a cofibration, and* $G \circlearrowright \mathrm{A} \xrightarrow{f\in\mathrm{Fib}} G \circlearrowright \mathrm{B}$ *a fibration, the morphism*

$$\mathrm{Maps}(\mathrm{Y},\mathrm{A}) \xrightarrow{(c^*,f_*)} \mathrm{Maps}(\mathrm{X},\mathrm{A}) \underset{\mathrm{Maps}(\mathrm{X},\mathrm{B})}{\times} \mathrm{Maps}(\mathrm{Y},\mathrm{B}) \tag{1.80}$$

is a fibration,

(c) *and is in addition a weak equivalence (in* $\mathrm{kTopSpc}_{\mathrm{Qu}}$*) if* c *or* f *is a weak equivalence.*

Proof. The first two statement may be found as [Fa08, Prop. 2.11][Ste16, Prop. 2.6]. With the third statement included this appears in [GMR19, Thm. 3.7][Schw18, Prop. B.7]. ☐

Example 1.3.6 (G-CW complexes are cofibrant objects in the proper equivariant model category)**.**
(i) A *G-CW complex* [Mat71a] is, by definition, a G-space obtained by a sequence, monotone in the dimension n, of cell attachments with the generating cofibrations (1.83). Hence Prop. 1.3.5 implies that G-CW complexes are cofibrant objects in the proper equivariant model structure (Prop. 1.3.4).
(ii) By the equivariant triangulation theorem [Il72, Thm. 3.1][Il83], every smooth manifold with a smooth action by a compact Lie group G admits an equivariant triangulation, hence is a G-CW complex and hence a cofibrant object in the proper equivariant model category:

$$G\,\mathrm{Act}(\mathrm{SmthMfd}) \hookrightarrow G\mathrm{CWCplx} \;\hookrightarrow\; \big(G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}}\big)^{\mathrm{cof}}.$$

Proposition 1.3.7 (Monoidalness of the proper equivariant model structure)**.** *The model category* $G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}}$ *from Prop. 1.3.4 is a monoidal model category, in that its Cartesian product is a left Quillen bifunctor, meaning that for for any pair of cofibrations* $G \circlearrowright \mathrm{X}_1 \xrightarrow{f_1\in\mathrm{Cof}} \mathrm{Y}_1$, $G \circlearrowright \mathrm{X}_2 \xrightarrow{f_1\in\mathrm{Cof}} G \circlearrowright \mathrm{Y}_2$ *the pushout-product morphism*

$$\mathrm{X}_1 \times \mathrm{Y}_2 \coprod_{\mathrm{Y}_1\times\mathrm{Y}_2} \mathrm{Y}_1 \times \mathrm{X}_2 \xrightarrow[\in\mathrm{Cof}]{f_1 \square f_2} \mathrm{X}_1 \times \mathrm{X}_2 \tag{1.81}$$

is a cofibration, and is in addition a weak equivalence if f_1 *or* f_2 *is such.*

Proof. It is folklore that this follows from the fact that products of coset spaces $G/H_1 \times G/H_2$ admit a G-CW complex structure, by Ex. 1.3.6 (making crucial use of the assumption that G is a Lie group, so that its coset spaces are smooth manifolds with smooth G-action). The conclusion has more recently been made explicit in [DHLPS19, Prop. 1.1.3 (iii)]. ☐

Proposition 1.3.8 (Equivariant mapping space Quillen adjunction)**.** *For* $\mathrm{X} \in \big(G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}}\big)^{\mathrm{cof}}$ *a cofibrant G-space, the equivariant mapping space adjunction* (1.16) *for* X *is a Quillen adjunction from the proper equivariant model structure (Prop. 1.3.4) to itself:*

$$G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}} \underset{G \circlearrowright \mathrm{Maps}(\mathrm{X},-)}{\overset{G \circlearrowright \mathrm{X}\times(-)}{\underset{\perp_{\mathrm{Qu}}}{\rightleftarrows}}} G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{prop}}\,.$$

Proof. This is a standard consequence of the monoidal model category structure due to Prop. 1.3.7, which immediately implies that the operation of forming the cartesian products with a cofibrant object is a left Quillen functor. □

Example 1.3.9 (Equivariant classifying shapes for pure shape structure)**.**
For $G \in \mathrm{Grp}(\mathrm{FinSet})$ and $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{kTopSpc})_{\mathrm{wellpt}}$ (Ntn. 1.3.17), the topological G-space

$$\mathrm{Map}(|\mathbf{E}G|, |\mathbf{B}\mathscr{G}|) \;\in\; G\,\mathrm{Act}(\mathrm{kTopSpc})$$

has fixed loci at $H \subset G$ naturally weakly homotopy equivalent to

$$\begin{aligned}\mathrm{Pth}\big(\mathrm{Map}(|\mathbf{E}G|, |\mathbf{B}\mathscr{G}|)^{H}\big) &\simeq \mathrm{Pth}\big(\mathrm{Map}(|\mathbf{E}G|/H, |\mathbf{B}\mathscr{G}|)\big)\\ &\simeq \mathrm{Pth}\big(\mathrm{Map}(|\mathbf{B}H|, |\mathbf{B}\mathscr{G}|)\big)\\ &\simeq \mathrm{Shp}\,\mathrm{Map}(BH, B\mathscr{G})\,.\end{aligned}$$

Hence

$$\begin{array}{ccccc} G\,\mathrm{Act}(\mathrm{DTopSpc}) & \xrightarrow{\mathrm{FxdLoc}} & G\mathrm{SmthGrpd}_\infty & \xrightarrow{\mathrm{Shp}} & G\mathrm{Grpd}_\infty\,. \\ G \circlearrowright \mathrm{Map}(|\mathbf{E}G|, |\mathbf{B}\mathscr{G}|) & \longmapsto & \mathrm{Map}(|\mathbf{B}(-)|, |\mathbf{B}\mathscr{G}|) & \longmapsto & \mathrm{Map}(B(-), B\mathscr{G}) \end{array}$$

Borel equivariant homotopy theory of G-spaces. In fact, the above statements and proofs about the proper equivariant model structure apply more generally to any choice of subset of closed subgroups of G containing the trivial subgroup, the above being the case of the maximal such subset. For the mininal subset containing only the trivial group one obtains the *coarse* or *Borel* equivariant model structure:

Proposition 1.3.10 (Borel model structure on G-spaces)**.** *For $G \in \mathrm{Grp}(\mathrm{kTopSpc})$, there exists a model category structure on the category of topological G-spaces (Ntn. 1.1.1), to be denoted*

$$G\,\mathrm{Act}(\mathrm{kTopSpc}_{\mathrm{Qu}})_{\mathrm{coarse}} \;\;\in\; \mathrm{MdlCat}\,, \tag{1.82}$$

whose weak equivalences and fibrations are those G-equivariant functions whose underlying maps are weak equivalences or fibrations, respectively, in the classical model structure on topological spaces (Ntn. 1.3.1).

Proof. This is also the special case of the general projective model structure on $\mathrm{kTopSpc}_{\mathrm{Qu}}$-enriched presheaf categories ([Pi91, Thm, 5.4]) for the site being the topological delooping groupoid ($G \rightrightarrows *$); see Ex. 1.2.7. □

Proposition 1.3.11 (Cofibrant generation of the Borel equivariant model structure)**.**
The Borel model structure $G\,\mathrm{Act}(\mathrm{kTopSpc})_{\mathrm{coarse}}$ from Prop. 1.3.10 is
(i) *Cofibrantly generated with generating cofibrations*

$$I_{\mathrm{coarse}} \;:=\; \big\{ G \times S^{n-1} \xhookrightarrow{\mathrm{id}_G \times i_n} G \times D^n \big\}_{n \in \mathbb{N}}. \tag{1.83}$$

(ii) *Cartesian monoidal, in that the pushout-product $f_1 \,\square\, f_2$* (1.81) *of a pair of cofibrations is itself a cofibration, and in addition is a weak if f_1 or f_2 is so.*

Good simplicial topological spaces.

Notation 1.3.12 (H-cofibrations)**.** We write

$$\mathrm{hCof} \;\subset\; \mathrm{Mor}(\mathrm{kTopSpc})$$

for the class of *Hurewicz cofibrations* – to be called *h-cofibrations*, following [MMSS01, p. 16] – in the category (1.2) of compactly generated weak Hausdorff spaces, hence for those maps $i\colon\, \mathrm{A} \to \mathrm{X}$, such that every solid commuting diagram as follows admits a dashed lift, as shown (e.g. [May99, p. 43]):

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d, "i"'] & \mathrm{Maps}([0,1], \mathrm{Y}) \arrow[d, "\mathrm{Maps}(i_0, \mathrm{Y})"] \\ \mathrm{X} \arrow[r] \arrow[ur, dashed, "\exists"] & \mathrm{Y}\,. \end{tikzcd}$$

Notice that, due to the weak Hausdorff property assumed in (1.2), such maps are necessarily injections with closed images, hence these are equivalently *closed cofibrations* (e.g. [May99, p. 44]).

Proposition 1.3.13 (Equivalent characterizations of h-cofibrations)**.**
A closed subspace inclusion $i : \mathrm{A} \hookrightarrow \mathrm{X}$ *is a Hurewicz cofibration,* $i \in \mathrm{hCof}$ *(Ntn. 1.3.12) if and only if the following equivalent conditions hold:*

(i) *([Str68, Thm. 2], review in [Bre93, §VII, Thm. 1.3][May99, p. 45]): The pushout-product* $i\Box 0$ *of* i *with the endpoint inclusion* $0 : \ast \to [0,1]$ *admits a retraction map r:*

$$\begin{tikzcd}
\mathrm{X}\times\{0\} \cup \mathrm{A}\times[0,1] \arrow[r, "i\Box 0"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{X}\times[0,1] \,. \arrow[r, dashed, "r"] & \mathrm{X}\times\{0\} \cup \mathrm{A}\times[0,1] \,.
\end{tikzcd}$$

(ii) *([Str66, Thm. 2], review in [Bre93, §VII, Thm. 1.5]): There exists*

- **(a)** *a neighborhood* $\mathrm{A} \hookrightarrow \mathrm{U} \hookrightarrow \mathrm{X}$ *of* A *in* X*;*
- **(b)** *a homotopy* η *deforming this neighborhood into A relative to A, in that its makes the following diagram commute:*

$$\begin{tikzcd}
& A\times[0,1] \arrow[ddl, hook] \arrow[dr, ->>] & \\
U \arrow[d, hook, "{(\mathrm{id},0)}"'] \arrow[drr, hook] & & A \arrow[d, hook] \\
U\times[0,1] \arrow[rr, dashed, "\eta"] & & X \\
U \arrow[u, hook, "{(\mathrm{id},1)}"] \arrow[rr, dashed] & & A \arrow[u, hook]
\end{tikzcd}$$

- **(c)** *a continuous function* $\phi : \mathrm{X} \to [0,1]$ *making the following diagram commute:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & \{0\} \arrow[d, hook] \\
X \arrow[r, dashed, "\phi"] & [0,1] \\
X\setminus U \arrow[r] \arrow[u, hook] & \{1\}\,. \arrow[u, hook]
\end{tikzcd}$$

Lemma 1.3.14 (Composition preserves h-cofibrations (e.g. [AGP02, Ex. 4.2.17]))**.** *If* $\mathrm{A} \xrightarrow{i_1} \mathrm{B}$ *and* $\mathrm{B} \xrightarrow{i_2} \mathrm{X}$ *are h-cofibrations (Ntn. 1.3.12) then so is their composite* $i_2 \circ i_1$*.*

Lemma 1.3.15 (Products preserve h-cofibrations)**.** *For* $\mathrm{A} \xrightarrow{i\in\mathrm{hCof}} \mathrm{X}$ *an h-cofibration (Ntn. 1.3.12) and any* $\mathrm{Y} \in \mathrm{kTopSpc}$*, also their product is a closed cofibration:*

$$\mathrm{Y}\times\mathrm{A} \xhookrightarrow{\mathrm{id}\times i\in\mathrm{hCof}} \mathrm{Y}\times\mathrm{X}\,.$$

Proof. By Prop. 1.3.13 it is sufficient to show that from a retraction r of $i\Box 0$ we obtain a retraction of $(\mathrm{id}_{\mathrm{Y}}\times i)\Box 0$. But since the product operation $\mathrm{Y}\times(-)$ is a left adjoint (1.4), it preserves the pushout-product:

$$(\mathrm{id}_{\mathrm{Y}}\times i)\Box 0 \;=\; \mathrm{id}_{\mathrm{Y}}\times(i\Box 0)\,.$$

Therefore, the required retraction is given by $\mathrm{id}_{\mathrm{Y}}\times r$. □

Definition 1.3.16 (Good simplicial topological space [Se74, Def. B.4])**. (i)** A simplicial topological space $\mathrm{X}_\bullet \in \Delta\mathrm{kTopSpc}$ (1.59) is called *good* if all its degeneracy maps are h-cofibrations (Ntn. 1.3.12, Prop. 1.3.13).

$$\underset{\substack{n\in\mathbb{N}\\ 0\leq i\leq n}}{\forall}\ \mathrm{X}_n \xrightarrow{\sigma_i\in\mathrm{hCof}} \mathrm{X}_{n+1}\,.$$

(ii) We denote the full subcategory of good simplicial topological spaces by:

$$\Delta\mathrm{kTopSpc}_{\mathrm{good}} \hookrightarrow \Delta\mathrm{kTopSpc}\,.$$

Notation 1.3.17 (Well-pointed simplicial topological group)**. (i)** A topological group $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ is called *well-pointed* (e.g., [Bre93, §VII, Def. 1.8]) if the inclusion of its neutral element is an h-cofibration (Ntn. 1.3.12):

$$\{\mathrm{e}\} \xhookrightarrow{\in \mathrm{hCof}} \Gamma. \tag{1.84}$$

(ii) We denote the full subcategory of well-pointed topological groups by:

$$\mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}} \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc}).$$

(iii) More generally (see Rem. 1.3.22), a simplicial topological group $\mathscr{G}_\bullet \in \mathrm{Grp}(\Delta\mathrm{kTopSpc})$ is called *well-pointed* if all component maps of the the unique homomorphism from $1 := \mathrm{const}(\{\mathrm{e}\})_\bullet \in \mathrm{Grp}(\Delta\mathrm{kTopSpc})$ are h-cofibrations:

$$\underset{n\in\mathbb{N}}{\forall} \;\; \{e\} \xhookrightarrow{\in \mathrm{hCof}} \mathscr{G}_n . \tag{1.85}$$

(iv) We denote the full subcategory of well-pointed simplicial topological groups by

$$\mathrm{Grp}(\Delta\mathrm{kTopSpc})_{\mathrm{wellpt}} \hookrightarrow \mathrm{Grp}(\Delta\mathrm{kTopSpc}).$$

Proposition 1.3.18 (Banach Lie groups are well-pointed)**.** *Every (paracompact Hausdorff) Banach Lie group, hence in particular every ordinary Lie group, is well-pointed (Ntn. 1.3.17).*

Proof. Paracompact Banach manifolds are "absolute neighborhood retracts" (ANRs), by [Pa66, Cor. to Thm. 5, p. 3]; and closed inclusions of ANRs are h-cofibrations, by [AGP02, Thm. 4.2.15]. The claim follows since points are ANRs, trivially, and are closed by assumption of Hausdorffness. □

Infinite projective groups The following infinite-projective groups are examples of a well-pointed groups which are *not* special cases of the class in Prop. 1.3.18:

Example 1.3.19 (Projective groups on countably-dimensional Hilbert spaces)**.** We write

$$\mathrm{U}_\omega \;:=\; \mathrm{U}(\mathscr{H}) \;\; \in \mathrm{Grp}(\mathrm{kTopSpc}) \tag{1.86}$$

for the group of unitary operators on any countably infinite-dimensional complex Hilbert space equipped with its strong operator topology (which is equal to the weak operator topology [HN93, Cor. 9.4] as well as to the compact-open topology [EU14][Scho18], but strictly coarser than the norm topology that would make it a Banach Lie group). Notice that this group (which is famously contractible in the norm topology by Kuiper's theorem) is still contractible in the strong operator topology ([DD63, Lem. 3 on p. 251][Scho18, p. 4]):

$$\mathrm{U}_\omega \underset{\mathrm{htpy}}{\simeq} \ast . \tag{1.87}$$

(i) We write

$$\mathrm{PU}_\omega \;:=\; \mathrm{U}_\omega/\mathrm{U}_1 \;\;\in\; \mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}} \tag{1.88}$$

for the topological quotient group of (1.86) by its subgroup of operators acting by multiplication with a complex number. This *projective unitary group* (1.88) is well-pointed (Ntn. 1.3.17), see [HSa20, p. 23]. The quotient coprojection is a locally trivial $\mathrm{U}(1)$-principal bundle [Si70, Thm. 1]:

$$\begin{tikzcd} \mathrm{U}_1 \arrow[r, hook] & \mathrm{U}_\omega \arrow[d] \\ & \mathrm{PU}_\omega . \end{tikzcd} \tag{1.89}$$

(ii) Under the canonical $\mathbb{Z}_{/2}$-action by complex conjugation, these are compatibly $\mathbb{Z}_{/2}$-equivariant topological groups (Def. 2.1.2)

$$\overset{\mathbb{Z}_{/2}}{\circlearrowright}\!\!\mathrm{U}_1 \hookrightarrow \overset{\mathbb{Z}_{/2}}{\circlearrowright}\!\!\mathrm{U}_\omega \twoheadrightarrow \overset{\mathbb{Z}_{/2}}{\circlearrowright}\!\!\mathrm{PU}_\omega \;\;\in\; \mathbb{Z}_{/2}\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big). \tag{1.90}$$

Their fixed locus is the group of orthogonal operators on any countably infinity-dimensional *real* Hilbert space equipped with its operator topology,

$$\mathrm{O}_\omega \;\simeq\; (\mathrm{U}_\omega)^{\mathbb{Z}_{/2}} \;\in \mathrm{Grp}(\mathrm{kTopSpc})\,. \tag{1.91}$$

In turn, the quotient of the latter by the subgroup of operators acting by multiplication with real units, is the infinite *projective orthogonal group* [Ro89, §3][MMS03]:

$$\mathrm{PO}_\omega \;=\; (\mathrm{PU}_\omega)^{\mathbb{Z}_{/2}} \;=\; \mathrm{O}_\omega/\mathbb{Z}_{/2} \;\;\in\; \mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}}\,. \tag{1.92}$$

This is again well-pointed, see [HSa20, p. 23].

(iii) These statements generalize to the $\mathbb{Z}_{/2}$-graded projective group $\mathrm{PU}^{\mathrm{gr}}_\omega$ [MP88, Prop. 2.2] (see also [CW08, p. 5]), which is obtained from the graded unitary group

$$\mathrm{U}^{\mathrm{gr}}_\omega \;:=\; \left\{ \begin{pmatrix} U_{++} & 0 \\ 0 & U_{--} \end{pmatrix}, \begin{pmatrix} 0 & U_{+-} \\ U_{-+} & 0 \end{pmatrix} \,\middle|\, U_{\bullet\bullet} \in \mathrm{U}(\mathscr{H}) \right\} \;\subset\; \mathrm{U}\big(\mathscr{H} \otimes \mathbb{C}^2\big)\,,$$

that itself is a $\mathrm{U}_\omega \times \mathrm{U}_\omega$-extension of $\mathbb{Z}_{/2}$

$$\begin{array}{ccccc}
\mathrm{U}^2_\omega & \hookrightarrow & \mathrm{U}^{\mathrm{gr}}_\omega & \xrightarrow{\;\;c\;\;} & \mathbb{Z}_{/2} \\
(U_{++}, U_{--}) & \longmapsto & \begin{pmatrix} U_{++} & 0 \\ 0 & U_{--} \end{pmatrix} & \longmapsto & \mathrm{e} \\
 & & \begin{pmatrix} 0 & U_{-+} \\ U_{+-} & 0 \end{pmatrix} & \longmapsto & \mathrm{odd}
\end{array} \tag{1.93}$$

by quotienting out the diagonal subgroup $\mathrm{U}_1 \hookrightarrow \mathrm{U}_\omega \hookrightarrow \mathrm{U}^{\mathrm{gr}}_\omega$:

$$\mathrm{U}_1 \hookrightarrow \mathrm{U}^{\mathrm{gr}}_\omega \twoheadrightarrow \mathrm{PU}^{\mathrm{gr}}_\omega\,. \tag{1.94}$$

(iv) The $\mathbb{Z}_{/2}$-action (1.90) evidently extends to these graded groups (1.94) (acting trivially on the $\{\pm 1\}$-grading)

$$\mathbb{Z}_{/2} \circlearrowright \mathrm{PU}^{\mathrm{gr}}_\omega \;\;\in\; \mathbb{Z}_{/2}\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}}\big)$$

with fixed locus being the analogous graded projective orthogonal group:

$$\mathrm{PO}^{\mathrm{gr}}_\omega \;=\; (\mathrm{PU}^{\mathrm{gr}}_\omega)^{\mathbb{Z}_{/2}} \;\simeq\; \mathrm{O}^{\mathrm{gr}}_\omega/\mathbb{Z}_{/2} \;\;\in\; \mathrm{Grp}(\mathrm{kTopSpc})\,.$$

(v) This allows to form semidirect products of all these (projective, graded) unitary groups with their complex conjugation action, yielding the following system of short exact sequences of well-pointed topological groups:

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[r, equal] \arrow[d, hook] & \mathrm{U}_1 \arrow[r, two heads] \arrow[d, hook] & 1 \arrow[d] \\
\mathrm{U}^2_\omega \arrow[r, hook] \arrow[d, two heads] & \mathrm{U}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2} \arrow[r, two heads, "{(c,\mathrm{pr}_2)}"] \arrow[d, two heads] & \mathbb{Z}_{/2} \times \mathbb{Z}_{/2} \arrow[d, equal] \\
\mathrm{U}^2_\omega/\mathrm{U}_1 \arrow[r, hook] & \mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2} \arrow[r, two heads, "{(c,\mathrm{id})}"] & \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}\,.
\end{tikzcd} \tag{1.95}$$

(vi) The projective graded unitary group (1.94) (and thus its semidirect product (1.95)) has a canonical continuous action on the following space of Fredholm operators[6] ([AS04, Def. 3.2], see also [FHT11, Def. A.39]):

$$\mathrm{Fred}^{(0)} := \left\{ F \in \mathscr{B}\big(\mathscr{H}_+ \oplus \mathscr{H}_-\big) \,\middle|\, \begin{array}{l} F^\dagger = F, \\ \deg(F) = \mathrm{odd} \in \mathbb{Z}_{/2}, \\ F^2 - 1 \in \mathscr{K}\big(\mathscr{H}_+ \oplus \mathscr{H}_-\big) \end{array} \right\} \xhookrightarrow{F \mapsto (F, F^2 - 1)} \mathscr{B}_{\mathrm{co}} \times \mathscr{K}_{\mathrm{nrm}}\big(\mathscr{H}_+ \oplus \mathscr{H}_-\big) \tag{1.96}$$

[6] The invertibility up to compact operators, in (1.96), is equivalent to the more traditional definition of Fredholm operators, by Atkinson's theorem, e.g., [Mu94, Thm. 2.1].

(where "$\mathscr{B}_{\mathrm{co}}$" denotes the space of bounded operators with the compact-open topology and "$\mathscr{K}_{\mathrm{nrm}}$" denotes space of compact operators with the normal topology), namely by conjugation:

$$\begin{array}{ccc} \big(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_{/2}\big) \times \mathrm{Fred}^{(0)} & \longrightarrow & \mathrm{Fred}^{(0)} \\ \left(\left(\begin{bmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--}\end{bmatrix}, \sigma\right), \begin{pmatrix} 0 & F_{+-} \\ F^{\dagger}_{+-} & 0 \end{pmatrix}\right) & \longmapsto & \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--}\end{pmatrix} \circ \begin{pmatrix} 0 & F^{\sigma}_{+-} \\ F^{\dagger\sigma}_{+-} & 0 \end{pmatrix} \circ \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--}\end{pmatrix}^{-1}. \end{array} \tag{1.97}$$

(Here either $U_{+-}, U_{-+} = 0$ or $U_{++}, U_{--} = 0$; the square bracket denotes the $\mathrm{diag}(\mathrm{U}_1)$-equivalence class of the matrix; and F^{σ} equals F when $\sigma = \mathrm{e}$ and equals its complex conjugate operator otherwise, see also [Mat71b, §5.B].)

Lemma 1.3.20 (Good connected simplicial groups are well-pointed)**.** *If $\mathscr{G}_\bullet \in \mathrm{Grp}(\Delta\mathrm{kTopSpc})$ is trivial in degree 0, $\mathscr{G}_0 = 1$, and such that its underlying simplicial space is good (Def. 1.3.16), then it is well-pointed (Ntn. 1.3.17).*

Proof. For all $n \in \mathbb{N}$, the inclusion of the neutral element in that degree is the composition of a sequence of degeneracy maps with the inclusion of the neutral element in degree 0:

$$\{\mathrm{e}\} = \mathscr{G}_0 \xrightarrow[\in \mathrm{hCof}]{\sigma} \mathscr{G}_1 \xrightarrow[\in \mathrm{hCof}]{\sigma} \cdots \xrightarrow[\in \mathrm{hCof}]{\sigma} \mathscr{G}_n\,.$$

By assumption, all these morphisms are h-cofibrations, and hence so is their composite, by Lem. 1.3.14. □

Proposition 1.3.21 (Nerves of action groupoids of well-pointed topological group actions are good)**.**
Let $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ be well-pointed (Ntn. 1.3.17). Then for every $\mathrm{X} \in \Gamma\mathrm{Act}(\mathrm{kTopSpc})$ the nerve (1.62) *of the topological action groupoid (Ex. 1.2.6) is a good simplicial space (Def. 1.3.16).*

Proof. This follows by Lemma 1.3.15 with the observation that all degeneracy maps in the action groupoid are of the form

$$\mathrm{id}_{\mathrm{X}\times\Gamma^{\times n}} \times \big(\{\mathrm{e}\} \xhookrightarrow{\in \mathrm{hCof}} \Gamma\big) \quad \in \in \mathrm{hCof}\,.$$ □

Remark 1.3.22 (Well-pointed groups are those with good delooping)**.** Specialized to the trivial action on the point, Prop. 1.3.21 says that a topological group is well-pointed (Ntn. 1.3.17) if and only if the nerve of its delooping groupoid (Ex. 1.2.7) is good (Def. 1.3.16), and if and only if it is well-pointed in the sense (1.85) of a (constant) simplicial topological group.

Moreover:

Lemma 1.3.23 (Well-pointed simplicial groups are good simplicial spaces [RS12, Prop. 3 (1)])**.** *Underlying any well-pointed simplicial topological group (Ntn. 1.3.17) is a good simplicial topological space (Ntn. 1.3.16).*

$$\begin{tikzcd} \mathrm{Grp}(\Delta\mathrm{kTopSpc})_{\mathrm{wellpt}} \arrow[r, dashed] \arrow[d, hook] & \Delta\mathrm{kTopSpc}_{\mathrm{good}} \arrow[d, hook] \\ \mathrm{Grp}(\Delta\mathrm{kTopSpc}) \arrow[r, "\text{underlying}"] & \Delta\mathrm{kTopSpc} \end{tikzcd}$$

Proposition 1.3.24 (Good resolutions of simplicial topological spaces [Se74, p. 308-309])**.** *For every $\mathrm{X}_\bullet \in \Delta\mathrm{kTopSpc}$ there exists a good simplicial space $\mathrm{X}^{\mathrm{good}}_\bullet \in \Delta\mathrm{kTopSpc}_{\mathrm{good}}$ (Def. 1.3.16) and a morphism*

$$\mathrm{X}^{\mathrm{good}}_\bullet \xrightarrow{\in \mathrm{WEqs}} \mathrm{X}_\bullet$$

which is degreewise a weak homotopy equivalence.

Topological realization of good simplicial spaces.

Proposition 1.3.25 (Realization of well-pointed simplicial group is well-pointed [RS12, Prop. 3][BS09, Lem. 1]). *If a simplicial topological group is well-pointed (Ntn. 1.3.17), then its topological realization* (1.65) *– with its induced structure* (1.6) *of a topological group, by Lem. 1.2.31 – is again well-pointed* (1.84)*:*

$$\begin{tikzcd} \mathrm{Grp}(\Delta\mathrm{kTopSpc})_{\mathrm{wellpt}} \arrow[r, dashed] \arrow[d, hook] & \mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}} \arrow[d, hook] \\ \mathrm{Grp}(\Delta\mathrm{kTopSpc}) \arrow[r, "\mathrm{Grp}(|-|)"] & \mathrm{Grp}(\mathrm{kTopSpc})\,. \end{tikzcd}$$

Remark 1.3.26 (Topological realization of good simplicial spaces models their homotopy colimit). The point of good simplicial spaces (Def. 1.3.16) is that their topological realization (Ntn. 1.2.28) is weakly homotopy equivalent to the "fat" realization [Se74, Prop. A.1 (iv)] (also [tD74, Prop. 1] with [RS12, §A]) which, in turn, is a standard model for their homotopy colimit (e.g., [AØ18, Ex. 6.4]):

$$\mathrm{X}_\bullet \;\in\; \Delta\mathrm{kTopSpc}_{\mathrm{good}} \qquad \Rightarrow \qquad \mathrm{Pth}\,|\mathrm{X}_\bullet| \;\simeq\; \underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\big(\mathrm{Pth}(\mathrm{X}_n)\big) \;\;\in\; \mathrm{Ho}(\Delta\mathrm{Set}_{\mathrm{Qu}})\,. \tag{1.98}$$

In view of this fact (1.98), we quote the following statements about homotopy colimits of simplicial spaces (Prop. 1.3.27) in terms of topological realizations of good simplicial spaces:

Proposition 1.3.27 (Sufficient conditions for topological realization to preserve homotopy fibers). *Sufficient conditions for a homotopy fiber sequence (Ntn. 1.3.3) of good simplicial topological spaces (Def. 1.3.16), hence a degreewise homotopy fiber product of topological spaces, to remain a homotopy fiber sequence under topological realization (Ntn. 1.2.28)*

$$\Big(\mathrm{F}_\bullet \xrightarrow[\simeq\mathrm{hofib}(p_\bullet)]{i_\bullet} \mathrm{E}_\bullet \xrightarrow{p_\bullet} \mathrm{X}_\bullet \quad \in\; \Delta\mathrm{kTopSpc}_{\mathrm{good}}\Big) \;\Rightarrow\; \Big(|\mathrm{F}_\bullet| \xrightarrow[\simeq\mathrm{hofib}(|p_\bullet|)]{|i_\bullet|} |\mathrm{E}_\bullet| \xrightarrow{|p_\bullet|} |\mathrm{X}_\bullet| \quad \in\; \mathrm{kTopSpc}\Big)$$

include any of the following conditions on $p_\bullet$:

(i) *[An78, p. 2]:*
- *on simplicial sets of connected components, $\pi_0(p)_\bullet$ is a Kan fibration,*
- *the component spaces E_n, Y_n ($n \in \mathbb{N}$) are connected or discrete.*

(ii) *[Re14b, Prop. 5.4][Lur17, Lem. 5.5.6.17]:*
- *the simplicial set of connected components of the base is constant: $\pi_0(Y)_\bullet \simeq \mathrm{const}\big(\pi_0(Y_0)\big)_\bullet$.*

(iii) *[MG14, Cor. 6.7][Lur11, Prop. 10]:*
- *$p_\bullet$ is a homotopy Kan fibration: $p_\bullet \in \mathrm{hKanFib}$ (Def. 1.2.23).*

Chapter 2

Equivariant principal bundles

We give a streamlined account of basic notions of topological equivariant principal bundles:
- §2.1 discusses the basic definitions.
- §2.2 discusses equivariant local triviality.
- §2.3 constructs universal equivariant principal bundles.

2.1 As bundles internal to G-actions

Definition of equivariant principal topological bundles. We consider the definition of equivariant principal topological bundles as principal bundles *internal* (Ntn. 1.0.23) to topological G-actions (Def. 2.1.3 below), extract what this means externally (Cor. 2.1.6 below), and explain how this compares to previous definitions found in the literature (Remark 2.1.7 below).

Remark 2.1.1 (Assumption of local trivalizability)**.** Here we consider (formally) principal bundles which are not required to be locally trivial, and we shall say "principal *fiber* bundles" for those that are, discussed further below in §2.2, see Def. 2.2.2. Notice that this terminology in line with:
(i) Cartan's original definition of "principal bundle", which did *not* include the local triviality clause (whence [Pa61, Def. 1.1.2] speaks of "Cartan principal bundles" in this more general case),
(ii) tradition in equivariant topology, where the equivariant local triviality condition is more subtle (see §2.2) and not included (see [tD69][Bi73][La82]) in the bare definition of equivariant principal bundles.

In part III we show that this issue is an artefact of internalization into a 1-category of topological spaces, instead of into an ∞-topos of higher geometric spaces: When (G-equivariant) ordinary principal bundles are regarded internal to (the $\mathbf{B}G$-slice of) the ∞-topos $\mathrm{SmthGrpd}_\infty$, then their (equivariant) local triviality is automatically implied (Thm. 4.1.2, Thm. 4.2.7 below).

Definition 2.1.2 (Equivariant topological group)**.** We say that an *equivariant topological group* is a group object internal (Ntn. 1.0.23) to $G\mathrm{Act}(\mathrm{kTopSpc})$ (Ntn. 1.1.1), hence:

(i) an object $\overset{\curvearrowright^{G}}{\Gamma} \in G\mathrm{Act}(\mathrm{kTopSpc})$;

(ii) morphisms $\ast \xrightarrow[\text{neutral element}]{\mathrm{e}} \overset{\curvearrowright^{G}}{\Gamma}$, $\qquad \overset{\curvearrowright^{G}}{\Gamma} \times \overset{\curvearrowright^{G}}{\Gamma} \xrightarrow[\text{multiplication}]{m} \overset{\curvearrowright^{G}}{\Gamma}$, $\qquad \overset{\curvearrowright^{G}}{\Gamma} \xrightarrow[\text{inverses}]{(-)^{-1}} \overset{\curvearrowright^{G}}{\Gamma}$,

such that the following diagrams commute:

(a) (Unitality)

$$
\begin{tikzcd}
\overset{\curvearrowright^{G}}{\Gamma} \times \ast \arrow[r, "\mathrm{id}\times\mathrm{e}"] \arrow[d, "\simeq"] & \overset{\curvearrowright^{G}}{\Gamma} \times \overset{\curvearrowright^{G}}{\Gamma} \arrow[d, "m"] \\
\underset{\curvearrowright_{G}}{\Gamma} \arrow[r, equal] & \underset{\curvearrowright_{G}}{\Gamma}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\ast \times \overset{\curvearrowright^{G}}{\Gamma} \arrow[r, "\mathrm{id}\times\mathrm{e}"] \arrow[d, "\simeq"] & \overset{\curvearrowright^{G}}{\Gamma} \times \overset{\curvearrowright^{G}}{\Gamma} \arrow[d, "m"] \\
\underset{\curvearrowright_{G}}{\Gamma} \arrow[r, equal] & \underset{\curvearrowright_{G}}{\Gamma}
\end{tikzcd}
$$

(b) (Associativity)

$$
\begin{tikzcd}
\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"] & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[d, "m"] \\
\underset{G\circlearrowleft}{\Gamma} \times \underset{G\circlearrowleft}{\Gamma} \arrow[r, "m"] & \underset{G\circlearrowleft}{\Gamma}
\end{tikzcd}
$$

(c) (Invertibility)

$$
\begin{tikzcd}
 & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[r, "(-)^{-1}\times\mathrm{id}"] & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[d, "m"] \\
\overset{G\circlearrowright}{\Gamma} \arrow[ur, "\mathrm{diag}"] \arrow[r] & * \arrow[r, "\mathrm{e}"] & \underset{G\circlearrowleft}{\Gamma}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[r, "\mathrm{id}\times(-)^{-1}"] & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \arrow[d, "m"] \\
\overset{G\circlearrowright}{\Gamma} \arrow[ur, "\mathrm{diag}"] \arrow[r] & * \arrow[r, "\mathrm{e}"] & \underset{G\circlearrowleft}{\Gamma}
\end{tikzcd}
$$

A homomorphism of equivariant groups is a morphism $\overset{G\circlearrowright}{\Gamma_1} \xrightarrow{\;f\;} \overset{G\circlearrowright}{\Gamma_1} \in G\mathrm{Act}(\mathrm{kTopSpc})$ which makes commuting diagrams with the structure morphisms, in the evident way. This yields the category

$$G\mathrm{EquivTopGrp} := \mathrm{Grps}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\,. \tag{2.1}$$

Definition 2.1.3 (Equivariant principal bundle)**.** We say that a (topological) *G-equivariant princial bundle* with *equivariant structure group* $G \circlearrowright \Gamma \in \mathrm{Grp}\big(G\mathrm{Actions}(\mathrm{kTopSpc})\big)$ (Def. 2.1.2) is a *formally principal* bundle ([Gr60I, p. 312 (15 of 30)][Gr71, p. 9 (293)], also: *pseudo-torsor* [Gr67, §16.5.15], see Rem. 2.1.8) internal (Ntn. 1.0.23) to $G\,\mathrm{Act}(\mathrm{kTopSpc})$ (Ntn. 1.1.1), hence:

(i) (Bundle) a morphism

$$\overset{G\circlearrowright}{\mathrm{P}} \xrightarrow{\quad p \quad} \overset{G\circlearrowright}{\mathrm{X}} \qquad \in G\,\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{2.2}$$

(ii) (Action) an internal left $(G \circlearrowright \Gamma)$-action (e.g. [MLM92, §V.6][BJK05, p. 8]), on its total space, namely a morphism

$$\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\mathrm{P}} \xrightarrow{\;\rho\;} \overset{G\circlearrowright}{\mathrm{P}} \qquad \in G\,\mathrm{Act}(\mathrm{kTopSpc}) \tag{2.3}$$

making the following diagrams commute:

$$
\textbf{(unitality)}\quad
\begin{tikzcd}
* \times \overset{G\circlearrowright}{P} \arrow[r, "\mathrm{e}\times\mathrm{id}"] \arrow[d, "\simeq"] & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{P} \arrow[d, "\rho"] \\
\underset{G\circlearrowleft}{P} \arrow[r, equal] & \underset{G\circlearrowleft}{P}
\end{tikzcd}
\qquad
\textbf{(action property)}\quad
\begin{tikzcd}
\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{P} \arrow[r, "\mathrm{id}\times\rho"] \arrow[d, "m\times\mathrm{id}"] & \overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{P} \arrow[d, "\rho"] \\
\underset{G\circlearrowleft}{\Gamma} \times \underset{G\circlearrowleft}{P} \arrow[r, "\rho"] & \underset{G\circlearrowleft}{P}
\end{tikzcd}
$$

such that, moreover:

(a) (Fiberwise action) the action ρ (2.3) is fiberwise relative to p (2.2), in that the following diagram commutes:

$$
\begin{tikzcd}
\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\mathrm{P}} \arrow[rr, "\rho"] \arrow[dr, "p\circ\mathrm{pr}_2"'] & & \overset{G\circlearrowright}{P} \arrow[dl, "p"] \\
 & \underset{G\circlearrowleft}{\mathrm{X}} &
\end{tikzcd}
$$

(b) (Principality) The resulting commuting square is a pullback square (shown on the left), meaning equivalently that the fiberwise *shear map* (the universal morphism factoring through the fiber product, shown on the right) is an isomorphism:

$$
\begin{tikzcd}
\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\mathrm{P}} \arrow[r, "\rho"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G\circlearrowright}{\mathrm{P}} \arrow[d, "p"] \\
\underset{G\circlearrowleft}{\mathrm{P}} \arrow[r, "p"'] & \underset{G\circlearrowleft}{\mathrm{X}}
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\overset{G\circlearrowright}{\Gamma} \times \overset{G\circlearrowright}{\mathrm{P}} \arrow[dr, dashed, "\sim", "\text{shear map}"'] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\mathrm{pr}_2"'] & & \\
 & \overset{G\circlearrowright}{P} \times_{\mathrm{X}} \overset{G\circlearrowright}{\mathrm{P}} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G\circlearrowright}{\mathrm{P}} \arrow[d, "p"] \\
 & \underset{G\circlearrowleft}{\mathrm{P}} \arrow[r, "p"'] & \underset{G\circlearrowleft}{\mathrm{X}}
\end{tikzcd}
\tag{2.4}
$$

We denote the category of these objects by:

$$G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathrm{kTopSpc}) \;:=\; \mathrm{Frm}(G \circlearrowright \Gamma)\,\mathrm{PrnBdl}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\,. \tag{2.5}$$

We extract what the internal definition (Def. 2.1.3) of equivariant principal bundles means externally:

Lemma 2.1.4 (G-Equivariant groups are semidirect products with G)**.** *There is a fully faithful functor* (8)

$$\begin{array}{ccc} G\mathrm{EquivTopGrp} & \hookrightarrow & \mathrm{Grp}(\mathrm{kTopSpc})^{G/}_{/G} \\ {}_{\alpha}\!\circlearrowright_{G}\Gamma & \longmapsto & \Gamma \rtimes_\alpha G \end{array} \tag{2.6}$$

which identifies G-equivariant topological groups (Def. 2.1.2) with the topological semidirect product groups with G, regarded as pointed objects in the slice over G, via the canonical group homomorphisms

$$\begin{array}{ccccc} g & \longmapsto & (\mathrm{e}, g) & \longleftarrow & g \\ G & \xhookrightarrow{\;i\;} & \Gamma \rtimes_\alpha G & \underset{\mathrm{pr}_2}{\overset{s}{\rightleftarrows}} & G \\ & & (\gamma, g) & \longmapsto & g \end{array} \tag{2.7}$$

that jointly exhibit the semidirect product as a split group extension of G by Γ.

Proof. This is a matter of straightforward unwinding of the definitions:

First, observe that an equivariant topological group in the internal sense of Def. 2.1.2 is equivalently a topological group Γ equipped with a continuous left G-action by group automorphisms $\alpha : G \to \mathrm{Aut}_{\mathrm{Grp}}(\Gamma)$.

In terms of this identification, the functor is defined on objects by sending the pair (Γ, α) to the semidirect product $\Gamma \rtimes_\alpha G$. The functor sends a morphism $f : (\Gamma, \alpha) \to (\Gamma', \alpha')$, determined by an equivariant continuous homomorphism $f : \Gamma \to \Gamma'$, to the map $f_* : \Gamma \rtimes_\alpha G \to \Gamma' \rtimes_{\alpha'} G$, defined by $f_*(\gamma, g) = (f(\gamma), g)$, which is a morphism in the pointed slice over G. This association is clearly faithful. Any morphism $h : \Gamma \rtimes_\alpha G \to \Gamma' \rtimes_{\alpha'} G$ in the slice over G determines a map $h' : \Gamma \to \Gamma'$. The group structure on the semidirect product forces h' to be an equivariant homomorphism. Hence the functor is also full. □

Lemma 2.1.5 (Equivariant group actions seen externally)**.** *Under the identification from Lemma 2.1.4, actions of G-equivariant topological groups* (Γ, α) *(Def. 2.1.2) on G-equivariant topological spaces* (X, ρ) *(Ntn. 1.1.1) are equivalently actions of the corresponding semidirect product groups* (2.6) *on the underlying plain topological space:*

$$\begin{array}{ccc} (G \circlearrowright \Gamma)\mathrm{Act}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big) & \xrightarrow{\;\sim\;} & (\Gamma \rtimes_\alpha G)\mathrm{Act}(\mathrm{kTopSpc}) \\ (\Gamma, \alpha) \xrightarrow{R} \mathrm{Aut}(\mathrm{X}, \rho) & \longmapsto & \Gamma \rtimes_\alpha G \xrightarrow{(R,\rho)} \mathrm{Aut}(X)\,, \end{array} \tag{2.8}$$

where the semidirect product group action on the right is $(R, \rho)(\gamma, g)(x) \;:=\; R(\gamma)\big(\rho(g)(x)\big)\,.$

Proof. This is again a matter of straightforward unwinding of the definitions. The morphism R can be identified with a continuous homomorphism $R : \Gamma \to \mathrm{Aut}(X)$ that is equivariant with respect to α and the conjugacy action on automorphisms. Given an action by the semidirect product $\Gamma \rtimes_\alpha G$ on X, we can recover R by restricting the action to elements $(\gamma, 1) \in \Gamma \rtimes_\alpha G$. This map is indeed equivariant since for all $x \in X$,

$$(\alpha(g)\gamma)x = (\alpha(g)\gamma, 1)x = (1, g)\left(\gamma, g^{-1}\right)x = g\gamma\left(g^{-1}x\right)\,.$$

With these identifications, it is trivial to verify that the functor (2.8) is fully faithful. □

Corollary 2.1.6 (External description of equivariant principal bundles)**.** *For* $(G \circlearrowright \Gamma)$ *a G-equivariant group, hence a topological group* Γ *equipped with an action* $\alpha : G \to \mathrm{Aut}(\Gamma)$*, the category of equivariant* $(G \circlearrowright \Gamma)$*-principal bundles in the internal sense of Def. 2.1.3 is equivalently described as follows:*

(i) (Bundle) *Objects are topological bundles, i.e., continuous functions* $\mathrm{P} \xrightarrow{p} \mathrm{X}$.

(ii) (Action) *equipped with continuous actions*

$$\begin{aligned} (\Gamma \rtimes_\alpha G) \times \mathrm{P} &\xrightarrow{(-)\cdot(-)} \mathrm{P}\,, \\ G \times \mathrm{X} &\xrightarrow{(-)\cdot(-)} \mathrm{X}\,, \end{aligned}$$

such that

(a) (Fiberwise action) *the bundle projection is G-equivariant and the* Γ*-action is G-equivariant and fiberwise, in that this square commutes:*

$$
\begin{tikzcd}
(\Gamma \rtimes_\alpha G) \times \mathrm{P} \arrow[r, "(-)\cdot(-)"] \arrow[d, "\mathrm{pr}_2 \times p"'] & \mathrm{P} \arrow[d, "p"] \\
G \times \mathrm{X} \arrow[r, "(-)\cdot(-)"] & \mathrm{X}
\end{tikzcd}
\tag{2.9}
$$

(b) (Principality) *the fiberwise shear map is an isomorphism (i.e., a homeomorphism)*

$$
\begin{array}{ccc}
\Gamma \times \mathrm{P} & \xrightarrow{\sim} & \mathrm{P} \times_{\mathrm{X}} \mathrm{P} \\
(\gamma, p) & \longmapsto & (\gamma \cdot p, p)\,.
\end{array}
\tag{2.10}
$$

Morphisms between these objects are $\Gamma \rtimes_\alpha G$*-equivariant continuous functions:*

$$
\begin{tikzcd}
\mathrm{P}_1 \arrow[r, "f_{\mathrm{P}}"] \arrow[d] \arrow[loop above, "\Gamma \rtimes_\alpha G"] & \mathrm{P}_2 \arrow[d] \arrow[loop above, "\Gamma \rtimes_\alpha G"] \\
\mathrm{X}_1 \arrow[r, "f_{\mathrm{X}}"'] \arrow[loop below, "G"] & \mathrm{X}_2 \arrow[loop below, "G"]
\end{tikzcd}
\tag{2.11}
$$

For archetypical classes of examples of equivariant principal bundles, see Lemma 2.2.11 and Prop. 2.3.16 below.

Remark 2.1.7 (Comparison to the literature on equivariant principal bundles)**.** The external description of equivariant principal bundles obtained in Cor. 2.1.6 from the canonical internal definition in Def. 2.1.3:

(i) recovers the definition of equivariant bundles due to [tD69, §1.1][tD87, §I.8], followed in e.g. [Ni78][MS95];

(ii) while much of the literature (e.g. [Bi73][La82][LU14]) considers the definition recovered in Cor. 2.1.6 only in the special case when α (2.6) is trivial, hence when the semidirect product group $\Gamma \rtimes_\alpha G$ reduces to the direct product group $\Gamma \times G$, hence when the action of the structure group Γ and the equivariance group G commute with each other.

(iii) In the other direction, [LM86][LMSt86, IV.1] proposed to consider group extensions of G by Γ more general than semidirect products (which are the split extensions); but followups [May90][GMM17] fall back to the semidirect product group actions originally considered in [tD69] and recovered in Cor. 2.1.6 from our Def. 2.1.3.

In fact, the internal definition subsumes one degenerate boundary case which is traditionally disregarded, but which is important to include for the equivariant theory to work well:

Remark 2.1.8 (The role of formal principality)**.** Beware that the formal principality condition [Gr60I, p. 312 (15 of 30)][Gr71, p. 9 (293)] in Def. 2.1.3 does not explicitly demand that the base space X be the quotient P/Γ of the total space by the structure group action, as would be demanded for actual principal bundles/torsors. We claim that this is the right definition for equivariant principal bundles, for two reasons:

(i) The internal Def. 2.1.3 involves only finite limits (products and fiber product, but no colimits such as quotients), which guarantees (Prop. 1.0.8) that every right adjoint functor (Ntn. 1.0.4) on the ambient category of $G\mathrm{Act}(\mathrm{kTopSpc})$, hence notably the fixed locus functor (Ex. 1.1.13), preserves these principal bundle objects (crucial in the form of Cor. 2.1.9 below).

(ii) Once equivariant local triviality is imposed (in §2.2 below) the quotient condition is both implied as well as circumvented, as need be. Namely, in that case the underlying bundle $\mathrm{P} \to \mathrm{X}$ (Cor. 2.1.10) is a locally trivial fiber bundle (Prop. 2.2.19), whence there are two cases:

(a) If fibers are inhabited (i.e., not empty), it follows that $\mathrm{P} \to \mathrm{X}$ is an effective epimorphism (Ntn. 1.0.10), in which case the shear map isomorphism (2.10) *implies* the quotient condition: $\mathrm{X} \simeq \mathrm{P}/\mathrm{X}$ (Lem. 2.1.11).

(b) If the fibers are empty then, while the quotient condition is of course not met (unless X itself is empty), the bundle is still formally principal in the sense of Def. 2.1.3, Cor. 2.1.6, as the shear map is indeed an isomorphism

$$\Gamma \times \varnothing \xrightarrow[\sim]{\exists !} \varnothing \times_{\mathrm{X}} \varnothing$$

(since both its domain and codomain are the empty space). Hence: *Empty bundles are formally principal.* While this degenerate case is irrelevant in ordinary topological bundle theory and hence traditionally ignored (not so in algebraic geometry [Gr67, §16.5.15]), its inclusion is crucial for equivariant principal bundle theory to work well, since the fixed loci (Cor. 2.1.9) of equivariant principal bundles are frequently empty (see Example 2.2.29).

We see this effect clearly brought out in the universal equivariant principal bundles, Rem. 2.3.21 below.

Corollary 2.1.9 (Fixed loci bundles of equivariant principal bundles)**.** *For $H \subset G$ a subgroup inclusion, passage to H-fixed loci constitutes a functor from G-equivariant Γ-principal bundles to $W(H)$-equivariant (Ntn. 1.1.1) Γ^H-principal bundles:*

$$\mathrm{Frm}(G \circlearrowright \Gamma)\,\mathrm{PrnBdl}(G\mathrm{Actions}) \xrightarrow{(-)^H} \mathrm{Frm}\big(W(H) \circlearrowright \Gamma^H\big)\,\mathrm{PrnBdl}\,(W(H)\mathrm{Act})\ .$$

Proof. By Example 1.1.13, the functor $(-)^H : G\mathrm{Actions} \to W(H)\mathrm{Actions}$ is a right adjoint, hence preserves finite limits (Prop. 1.0.8), hence induces (1.6) a functor of internal groups (Def. 2.1.2), and internal formally principal bundles (Def. 2.1.3). □

For completeness, we also record:

Corollary 2.1.10 (Underlying principal bundles of equivariant principal bundles)**.** *Passage to underlying topological spaces (forgetting the equivariance group action) (Example 1.1.9) constitutes a functor from equivariant to ordinary principal bundles:*

$$\mathrm{Frm}(G \circlearrowright \Gamma)\,\mathrm{PrnBdl}\,(G\,\mathrm{Act}(\mathrm{kTopSpc})) \xrightarrow[\text{forget } G\text{-action}]{} \mathrm{Frm}\Gamma\,\mathrm{PrnBdl}(\mathrm{kTopSpc})\,. \tag{2.12}$$

Proof. This is immediate from inspection of the definitions, but it also follows abstractly, as in the proof of Cor. 2.1.9, from the fact that the forgetful functor (1.19) is a right adjoint, hence preserves limits (Prop. 1.0.8), and therefore (1.6) preserves internal groups and internal principal bundles. □

Basic properties of equivariant principal topological bundles. We discuss some basic aspects of equivariant bundles related to effective epimorphy.

The following Lem. 2.1.11 is a slight strengthening of the fact that effective epimorphisms in topological spaces are equivalently quotient maps, which is worth recording (and holds more generally for formally principal bundles internal to any ambient category):

Lemma 2.1.11 (Effective equivariant principal bundles)**.** *Given an equivariant principal bundle $G \circlearrowright \mathrm{P} \xrightarrow{p} G \circlearrowright \mathrm{X} \in \mathrm{Frm}\Gamma\,\mathrm{PrnBdl}(\mathrm{kTopSpc})$ in the sense of Def. 2.1.3 (i.e. not requiring any local trivializability) the following two conditions are equivalent:*

(i) $\mathrm{P} \xrightarrow{p} \mathrm{X}$ *is the Γ-quotient coprojection.*

(ii) $\mathrm{P} \xrightarrow{p} \mathrm{X}$ *is an effective epimorphism (Ntn. 1.0.10).*

Proof. By definition, the two conditions mean equivalently that the following vertical diagrams on the left or right, respectively, are coequalizer diagrams

$$\begin{array}{ccc}
\Gamma \times \mathrm{P} & \xrightarrow[\text{shear map}]{\sim} & \mathrm{P} \times_{\mathrm{X}} \mathrm{P} \\
{\scriptstyle \rho}\downarrow\ \downarrow{\scriptstyle \mathrm{pr}_2} & & {\scriptstyle \mathrm{pr}_1}\downarrow\ \downarrow{\scriptstyle \mathrm{pr}_2} \\
\mathrm{P} & = & \mathrm{P} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathrm{X} & = & \mathrm{X}
\end{array}$$

But the shear map (2.4) is readily seen to provide a morphism between the top diagrams (makes the two parallel squares commute) and is an isomorphism by principality. Therefore the two vertical diagrams are equivalent, and one is a coequalizer diagram if and only if the other is. □

The following may seem obvious, but does need an argument given that we define principality via shear isomorphisms:

Lemma 2.1.12 (Locally trivial Γ-actions are principal bundles)**.** *Let* $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $\Gamma \circlearrowright \mathrm{P} \in \Gamma\,\mathrm{Act}(\mathrm{kTopSpc})$ *such that* $\mathrm{P} \xrightarrow{q} \mathrm{P}/\Gamma =: \mathrm{X}$ *is locally, over an open cover,* Γ*-equivariantly isomorphic to a trivial* Γ*-principal bundle. Then* P *is itself a (locally trivial)* Γ*-principal bundle, in that its shear map is an isomorphism.*

Proof. Let $\{\mathrm{U}_i \hookrightarrow \mathrm{X}\}_{i\in I}$ be a trivializing open cover and abbreviate $\widehat{\mathrm{X}} := \underset{i\in I}{\sqcup}\mathrm{U}_i$. Consider the following pasting diagram of pullbacks, whose left side follows by the assumption of local triviality:

$$
\begin{tikzcd}
\Gamma\times(\widehat{\mathrm{X}}\times\Gamma) \arrow[rr] \arrow[dr, "\sim"] \arrow[ddr, bend right] & & \Gamma\times\mathrm{P} \arrow[dr, "shear"] \arrow[ddr, bend left] \arrow[d, phantom, "\text{(pb)}"] & \\
& (\widehat{\mathrm{X}}\times\Gamma)\underset{\widehat{\mathrm{X}}}{\times}(\widehat{\mathrm{X}}\times\Gamma) \arrow[rr, two heads] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathrm{P}\times_{\mathrm{X}}\mathrm{P} \arrow[d] \\
& \widehat{X} \arrow[rr, two heads] & & \mathrm{X}\,.
\end{tikzcd}
$$

By local recognition of homeomorphisms (Ex. 1.0.22) it follows that the shear map itself is an isomorphism. □

2.2 Equivariant local triviality

We consider equivariant principal bundles which are equivariantly locally trivial (in Thm. 2.2.1, Def. 2.2.2 below, recall Rem. 2.1.1) and prove that isomorphism classes these equivariant principal *fiber bundles* coincide with concordance classes (Thm. 2.2.8 below). This is a key ingredient in the proof of the classification theorem for equivariant principal bundles fiber bundles in Thm. 4.2.7 below.

Notions of equivariant local triviality. There is a curious subtlety in the notion of equivariant local trivializability (whose resolution is provided by the perspective of higher geometry in §4.1 below): While internalization (Ntn. 1.0.23) of the notion of principal bundles into equivariant topology is the method of choice for understanding the plain definition of equivariant principal bundles (as discussed in §2.1), it *fails* when it comes to understanding their local triviality.

That is, using that $G\,\mathrm{Act}(\mathrm{kTopSpc})$ is a regular category (Prop. 1.1.11), internal local triviality of an equivariant principal bundle $G \circlearrowright \mathrm{P}$ would be the existence of an effective epimorphism $G \circlearrowright \widehat{\mathrm{X}} \twoheadrightarrow G \circlearrowright \mathrm{X}$ and of a pullback diagram of the form

$$
\begin{tikzcd}
\overset{{}^{\alpha}\!\curvearrowright^{G}\;\curvearrowright^{G}}{\Gamma\times\widehat{\mathrm{X}}} \arrow[r] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\curvearrowright^{G}}{\mathrm{P}} \arrow[d] \\
\underset{\curvearrowleft_{G}}{\widehat{\mathrm{X}}} \arrow[r, two heads] & \underset{\curvearrowleft_{G}}{\mathrm{X}}
\end{tikzcd}
\;\;\in\; G\,\mathrm{Act}(\mathrm{kTopSpc})\,, \tag{2.13}
$$

(such that the top morphism is also Γ-equivariant). That this notion of local triviality would be too restrictive in practice is readily seen in the special case when the action α of G on Γ is trivial: In this case (2.13) essentially says that the G-action is trivial on total spaces of equivariant principal bundles, relative to the action on the base.

Putting the systematics of internalization aside for the moment, one may guess what the "correct" equivariant local trivialization condition should be, and several authors have done so. Below we review the existing definitions in streamlined form, generalize them where necessary to the case when the structure group carries a non-trivial G-action, and then show that the resulting definitions are all equivalent to each other (Thm. 2.2.1). Further below in §4.1 we show that these equivariant local triviality conditions do follow from systematic internalization after all: though not in the 1-category of topological spaces but in a higher category of, in particular, D-topological stacks.

One way to motivate the correct definition of equivariant local trivialization (without yet passing to ∞-topos theory) is to observe that equivariantly there is no longer a single local model for principal bundles: A trivial underlying Γ-principal bundle $\Gamma\times\mathrm{U}\to\mathrm{U}$ in general carries several inequivalent G-actions that make it an equivariant

$(G \circlearrowleft \Gamma)$-principal bundle. We consider three different ways of parametrizing the possible equivariant local model bundles, which we name after the original authors who put these models into focus:

Local model equivariant principal bundles		
tom Dieck §2.2.1	Lashof-May §2.2.2	Bierstone §2.2.3
$\widehat{G}/\widehat{H}$ $\downarrow$ G/H	$\widehat{G} \times_{\widehat{H}} \mathrm{S}$ $\downarrow$ $G \times_H \mathrm{S}$	$\widehat{G}_x \times_{\widehat{G_x}} \mathrm{U}_x$ $\downarrow$ $G_x \times_{G_x} \mathrm{U}_x$

Arguably, tom Dieck's definition is the most conceptual (here one derives the general form of the local model bundles, instead of prescribing them, see Prop. 2.2.17 below) and also the most general (its underlying local triviality follows for G any topological group, by using the first item of Lemma 2.2.14 in Lemma 2.2.15), while Bierstone's definition makes manifest (Lem. 2.2.27 below) that the underlying bundles are indeed locally trivial, only that the G-action on the local models is not, in general. In particular, this makes manifest that equivariant local triviality is preserved by passage to fixed loci (Prop. 2.2.30 below). Finally, Bierstone's definition (in the special case when α is trivial) is the one traditionally used (albeit without attribution to Bierstone) in the seminal example of the equivariant degree-3 twisting bundle of complex K-theory, following [AS04, p. 28].

Notice that [Bi73] and [La82] considered only the special case when α in (2.6) is trivial; we generalize their definitions and proofs; see Remark 2.2.28 and Remark 2.2.34 below. Moreover, the relation to tom Dieck's original notion of equivariant local trivialization seems to not have been discussed before.

Theorem 2.2.1 (Equivalent notions of equivariant local triviality)**.** *Under the assumption of proper equivariance (Assump. 1.1.2), the three notions of local trivializability of equivariant principal bundles (Def. 2.1.3) are all equivalent:*

$$
\begin{tikzcd}
\boxed{\text{tom Dieck}} \arrow[rr, Rightarrow, "\text{Prop. 2.2.36}"] & & \boxed{\text{Bierstone}} \arrow[dl, Rightarrow, "\text{Prop. 2.2.35}"] \\
\text{(Def. 2.2.13)} & \boxed{\text{Lashof-May}} \arrow[ul, Rightarrow, "\text{Prop. 2.2.23}"] & \text{(Def. 2.2.26)} \\
& \text{(Def. 2.2.21)} &
\end{tikzcd}
$$

Definition 2.2.2 (Equivariant principal fiber bundles)**.** We write

$$
\underbrace{G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnFibBdl}(\mathrm{kTopSpc})}_{\text{equivariantly locally trivial equivariant principal bundles}} \hookrightarrow \underbrace{G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnBdl}(\mathrm{kTopSpc})}_{\text{equivariant principal bundles}} := \underbrace{\mathrm{Frm}(G \circlearrowleft \Gamma)\,\mathrm{PrnBdl}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)}_{\text{formally principal internal actions}}
$$

for the full subcategory of (2.5) on those equivariant principal bundles (Def. 2.1.3, see Rem. 2.1.1) which satisfy the equivalent equivariant local triviality conditions of Thm. 2.2.1.

Immediate consequences of equivariant local trivialization.

Proposition 2.2.3 (Passage to fixed loci preserves equivariant local triviality)**.** *For $H \subset G$ any closed subgroup, the functor (Cor. 2.1.9) sending G-equivariant principal bundles to their $\mathrm{W}_G H$-equivariant H-fixed loci bundles respects the subcategories in Def. 2.2.2 of equivariant principal fiber bundles (Def. 2.2.2):*

$$
\begin{tikzcd}
\overset{\text{equivariantly locally trivial}}{G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnFibBdl}(\mathrm{kTopSpc})} \arrow[r, hook] \arrow[d, "(-)^H"] & \overset{\text{equivariant principal bundles}}{G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnBdl}(\mathrm{kTopSpc})} \arrow[d, "(-)^H", "\text{passage to } H\text{-fixed loci}"'] \\
\mathrm{W}_G(H)\,\mathrm{Equv}\,\Gamma^H\,\mathrm{PrnFibBdl}(\mathrm{kTopSpc}) \arrow[r, hook] & \mathrm{W}_G(H)\,\mathrm{Equv}\,\Gamma^H\,\mathrm{PrnBdl}(\mathrm{kTopSpc})
\end{tikzcd}
\tag{2.14}
$$

Proof. This is manifest in Bierstone's formulation (Prop. 2.2.30) and hence holds generally, by Theorem 2.2.1. $\square$

Lemma 2.2.4 (Equivariant principal fiber bundles are proper equivariant Serre fibrations)**.** *An equivariant principal bundle $G \curvearrowright \mathrm{P} \xrightarrow{p} G \curvearrowright \mathrm{X}$ which is equivariantly locally trivial (Def. 2.2.2) is a fibration in the proper equivariant model structure (Prop. 1.3.4):*

$$G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}(\mathrm{kTopSpc}) \subset G\mathrm{SerFib}\,.$$

Proof. By Thm. 2.2.1 and Prop. 2.2.3, we have for all $H \subset G$ that $\mathrm{W}_G H \curvearrowright \mathrm{P}^H \to \mathrm{W}_G H \curvearrowright \mathrm{X}$ satisfies Bierstone's condition (Def. 2.2.26). By Lem. 2.2.27, this implies that the underlying principal bundle $\mathrm{P}^H \to \mathrm{X}^H$ (2.12) is locally trivial, hence a Serre fibration by Lem. 1.3.2. $\square$

Lemma 2.2.5 (Quotiented fiber product of equivariant principal fiber bundles is proper equivariant fibration)**.** *Given a pair of equivariant principal fiber bundles $G \curvearrowright \mathrm{P}_1, G \curvearrowright \mathrm{P}_2 \in G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}(\mathrm{kTopSpc})$ (Def. 2.2.2), the $G \curvearrowright \Gamma$-quotient of their fiber product is a proper equivariant fibration (according to Prop. 1.3.4):*

$$\begin{array}{c} \overset{\curvearrowright G}{(\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2)/\Gamma} \\ \Big\downarrow \\ \underset{\curvearrowright G}{\mathrm{X}} \end{array} \quad \in G\mathrm{SerFib}\,. \tag{2.15}$$

Proof. By Prop. 2.2.3 and Lem. 2.2.27, we have for all $H \underset{\mathrm{cpt}}{\subset} G$ that the underlying Γ^H-principal bundle of the fixed locus bundle P_i^H is locally trivial, for $i \in \{0,1\}$. Let $\widehat{\mathrm{X}^H} \twoheadrightarrow \mathrm{X}^H$ be an open cover over which both trivialize and consider the following commuting diagram:

$$\begin{tikzcd}
(\widehat{\mathrm{X}^H} \times \Gamma^H) \times_{\mathrm{X}^H} (\widehat{\mathrm{X}^H} \times \Gamma^H) \times \Gamma^H \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "\text{(pb)}"] & (\mathrm{P}_1^H \times_{\mathrm{X}^H} \mathrm{P}_2^H) \times \Gamma \arrow[d, shift left] \arrow[d, shift right] \\
(\widehat{\mathrm{X}^H} \times \Gamma^H) \times_{\mathrm{X}^H} (\widehat{\mathrm{X}^H} \times \Gamma^H) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_1^H \times_{\mathrm{X}^H} \mathrm{P}_2^H \arrow[d, two heads] \\
\widehat{\mathrm{X}^H} \times \Gamma^H \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\mathrm{P}_1^H \times_{\mathrm{X}^H} \mathrm{P}_2^H)/\Gamma^H \arrow[d] \\
\widehat{\mathrm{X}^H} \arrow[r, two heads] & \mathrm{X}^H
\end{tikzcd}$$

Here, the right vertical column is given and we are pulling back to the cover. The form of the top square follows by the pasting law (Prop. 1.0.9) and the left middle morphism is an effective epimorphism by regularity of $G\mathrm{Act}(\mathrm{kTopSpc})$ (Prop. 1.1.11). This identifies is codomain with the quotient $\widehat{\mathrm{X}^H} \times \Gamma^H$, as shown. In conclusion, the bottom square shows that each H-fixed locus of (2.15) is a locally trivial fiber bundle, hence a Serre fibration by Lem. 1.3.2. $\square$

Concordance of equivariant principal bundles. Below, in chapter 4, we find that the classification of equivariant principal bundles *up to concordance* follows on general abstract grounds, after embedding them into cohesive ∞-topos theory, from the *orbi-smooth Oka principle* (Thm. 3.3.51, Thm. 4.1.55). Here we show that for topological equivariant principal fiber bundles, their concordance classes actually coincide with their isomorphism classes (Thm. 2.2.8 below). Together with the orbi-smooth Oka principle, this hence implies the full classification theorem (Thm. 4.2.7 below).

Definition 2.2.6 (Concordance of equivariant principal bundles)**.** For $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$,
(i) we say that a pair of equivariant principal fiber bundles $G \curvearrowright \mathrm{P}_0, G \curvearrowright \mathrm{P}_1 \in G\mathrm{Equiv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}}$ (Def. 2.2.2) are *concordant* if there exists an equivariant principal fiber bundle

$$G \curvearrowright \widehat{\mathrm{P}} \in G\mathrm{Equiv}\Gamma\mathrm{PrnBdl}_{\mathrm{X}\times[0,1]}$$

on the cylinder $G \curvearrowright \mathrm{X} \times [0,1]$ with trivial G-action on the topological interval, whose restriction to the endpoints is isomorphic to these bundles:

$$G \curvearrowright \mathrm{P}_0 \simeq G \curvearrowright \widehat{\mathrm{P}}|_{\mathrm{X}\times\{0\}} \quad \text{and} \quad G \curvearrowright \mathrm{P}_1 \simeq G \curvearrowright \widehat{\mathrm{P}}|_{\mathrm{X}\times\{1\}}\,.$$

(ii) We denote the sets of equivalences classes of equivariant principal bundles under isomorphisms and under concordance, respectively, by:

$$\underbrace{\big(G\mathrm{Equiv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}}}_{\text{isomorphism classes}} \longrightarrow\!\!\!\!\to \underbrace{\big(G\mathrm{Equiv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}}}_{\text{concordance classes}}\,. \tag{2.16}$$

We are going to show (Thm. 2.2.8) that (2.20) is in fact a bijection. For this purpose, we need the following Lemma 2.2.7, whose statement is not surprising, but whose proof requires some care:

Lemma 2.2.7 (Isomorphisms of equivariant principal bundles as sections of their quotiented fiber product)**.** *Given* $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$*, and* $G \curvearrowright \mathrm{P}_1, G \curvearrowright \mathrm{P}_2 \in G\mathrm{Equiv}\Gamma\mathrm{PrnBdl}(\mathrm{kTopSpc})_{\mathrm{X}}$ *a pair of* G*-equivariant* Γ*-principal fiber bundles* [1] *(Def. 2.2.2), then there is a natural bijection*

$$\left\{ \begin{tikzcd} \underset{}{\overset{\Gamma\times G \curvearrowright}{\mathrm{P}_1}} \arrow[rr, dashed, "f", "\sim"'] \arrow[dr] & & \overset{\Gamma\times G \curvearrowright}{\mathrm{P}_2} \arrow[dl] \\ & \underset{G \curvearrowleft}{\mathrm{X}} & \end{tikzcd} \right\} \leftrightarrow \left\{ \begin{tikzcd} & \overset{G \curvearrowright}{(\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2)/\Gamma} \arrow[d] \\ \underset{G \curvearrowleft}{\mathrm{X}} \arrow[ur, dashed, "\sigma"] \arrow[r, equals] & \underset{G \curvearrowleft}{\mathrm{X}} \end{tikzcd} \right\} \tag{2.17}$$

between their homomorphisms (necessarily isomorphisms, by Lem. 1.1.19) and the continuous sections of the diagonal Γ*-quotient of their fiber product.*

Proof. We spell out a proof in the non-equivariant case, i.e. for the case $G = 1$, in a way that uses nothing but the fact (Prop. 1.0.20) that kTopSpc is a regular category. The general case then follows by observing that these arguments lift through the forgetful functor (since this creates limits and colimits, Lem. 1.1.10) to the regular category $G\mathrm{Act}(\mathrm{kTopSpc})$ (Prop. 1.1.11). In other words, the general proof is verbatim the following proof with $G \curvearrowright (-)$ adjoined to all objects appearing.

Noticing that the underlying bundles of $\mathrm{P}_1, \mathrm{P}_2$ are locally trivial (Lem. 2.2.27), first observe that the quotient coprojection

$$\begin{tikzcd} \mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2 \arrow[d, two heads] \\ (\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2)/\Gamma \end{tikzcd} \qquad \text{is locally trivial and an effective epimorphism.} \tag{2.18}$$

Namely, restriction of the fiber product bundle to any patch $\mathrm{U} \hookrightarrow \mathrm{X}$ over which both factor bundles P_1 and P_2 trivialize gives the pullback diagram shown on the left here:

$$\begin{tikzcd} \mathrm{U} \times \Gamma^{\times^3} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "\text{(pb)}"] & (\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2) \times \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ \mathrm{U} \times \Gamma^{\times^2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2 \arrow[d] \\ \mathrm{U} \arrow[r, hook] & \mathrm{X} \end{tikzcd} \simeq \begin{tikzcd} \mathrm{U} \times \Gamma^{\times^3} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "\text{(pb)}"] & (\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2) \times \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ \mathrm{U} \times \Gamma^{\times^2} \arrow[r] \arrow[d, two heads, "\mathrm{coeq}"] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2 \arrow[d, two heads, "\mathrm{coeq}"] \\ \mathrm{U} \times \Gamma \arrow[r, dashed] \arrow[d] & (\mathrm{P}_1 \times_{\mathrm{X}} \mathrm{P}_2)/\Gamma \arrow[d] \\ \mathrm{U} \arrow[r, hook] & \mathrm{X}\,. \end{tikzcd}$$

[1] In fact, the proof of Lem. 2.2.7 only uses that the underlying principal bundles are locally trivial, not that this also the case for all H-fixed loci.

Passage to coequalizers of the top morphism pairs factors the left diagram as shown on the right, which exhibits the resulting middle square as a homomorphism of Γ-torsors over X, whose domain is a trivial principal bundle. This implies that the middle square on the right is a pullback, by Lem. 1.1.19. Therefore regularity of kTopSpc (Prop. 1.0.20) implies with Lem. 1.0.14 that also the bottom square on the right is a pullback. Since this holds for all patches U in any joint trivializing cover of the two bundles, this proves the local triviality in (2.18). From this, the effective epimorphy follows with Lem. 2.1.12 and Lem. 2.1.11.

Now, given a section σ as on the right of (2.17), complete it to the following commuting diagram, for $i \in \{1,2\}$:

$$
\begin{tikzcd}
\mathrm{P}\times\Gamma \arrow[r] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\rho"] \arrow[dr, phantom, "\text{(pb)}"] & (\mathrm{P}_1\times_{\mathrm{X}}\mathrm{P}_2)\times\Gamma \arrow[r, "\mathrm{pr}_i\times\mathrm{id}"] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\rho"] & \mathrm{P}_i\times\Gamma \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\rho"] \\
\mathrm{P} \arrow[r, "(\phi, f\circ\phi^{-1})"] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_1\times_{\mathrm{X}}\mathrm{P}_2 \arrow[r, "\mathrm{pr}_i"] \arrow[d, two heads] & \mathrm{P}_i \arrow[d, two heads] \\
\mathrm{X} \arrow[r, dashed, "\sigma"] \arrow[rr, bend right, "\mathrm{id}"'] & (\mathrm{P}_1\times_{\mathrm{X}}\mathrm{P}_2)/\Gamma \arrow[r, "\mathrm{pr}_i/\Gamma"] & \mathrm{X}\,.
\end{tikzcd}
\tag{2.19}
$$

Here the squares on the left are pullbacks by definition, the top right square commutes by nature of the diagonal action, and the bottom left square is induced by passage to quotients.

But the right part of this diagram exhibits the projection pr_i as a homomorphism of Γ-torsors over X out of a locally trivial principal bundle (2.18), whereby Lem. 1.1.19 implies that also the bottom right square is a pullback. From this the pasting law (Prop. 1.0.9) implies that the total bottom rectangle is a pullback, hence (by Ex. 1.0.7), that the total horizontal middle morphisms are isomorphism of Γ-torsors over X, for $i \in \{1,2\}$, which we may suggestively denote by:

$$
\mathrm{P} \xrightarrow[\sim]{\phi} \mathrm{P}_1 \;,\quad \mathrm{P} \xrightarrow[\sim]{f\circ\phi^{-1}} \mathrm{P}_2 \;,\quad \text{hence:}\quad \mathrm{P}_1 \xrightarrow[\sim]{f} \mathrm{P}_2\,.
$$

Conversely, given such f, form the top part of the diagram (2.19) with $\mathrm{P} := \mathrm{P}_1$ and $\phi := \mathrm{id}$. Then passage to coequalizers as in the bottom part of (2.19) yields a section σ. By effective epimorphy of the bottom vertical morphisms, these two constructions are inverse to each other, as claimed. □

Theorem 2.2.8 (Concordant equivariant principal fiber bundles bundles are isomorphic)**.** *For*
- $G \circlearrowright \Gamma \in \mathrm{Grp}(G\mathrm{Act}(\mathrm{kTopSpc}))$,
- $G \circlearrowright \mathrm{X} \in G\mathrm{CWCplx} \hookrightarrow G\mathrm{Act}(\mathrm{kTopSpc})$ *(e.g., a smooth G-manifold),*

concordance classes (Def. 2.2.6) of topological G-equivariant Γ-principal fiber bundles over X *(Def. 2.2.2) coincide with their isomorphism classes in that the quotient projection* (2.20) *is a bijection:*

$$
\big(G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}(\mathrm{kTopSpc})_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;\sim\;} \big(G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}(\mathrm{kTopSpc})_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}}\,.
\tag{2.20}
$$

The following proof adapts the idea of the proof of [RS12, Cor. 15] to equivariant bundles.

Proof. Given a concordance, as shown on the left, we need to produce an isomorphism, as shown on the right here:

$$
\begin{tikzcd}
\overset{G}{\circlearrowright}\mathrm{P}_0 \arrow[r] \arrow[d, "p_0"] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G}{\circlearrowright}\mathrm{P} \arrow[d, "p"] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G}{\circlearrowright}\mathrm{P}_1 \arrow[l] \arrow[d, "p_1"] \\
\mathrm{X}\times\{0\} \arrow[r, hook] & \mathrm{X}\times[0,1] & \mathrm{X}\times\{1\} \arrow[l, hook']
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
\overset{\Gamma\times G}{\circlearrowright}\mathrm{P}_0 \arrow[rr, "\sim"] \arrow[dr, "p_0"'] & & \overset{\Gamma\times G}{\circlearrowright}\mathrm{P}_1 \arrow[dl, "p_1"] \\
& \underset{G}{\circlearrowright}\mathrm{X} &
\end{tikzcd}
\tag{2.21}
$$

Consider the following solid commuting square in $G\mathrm{Act}(\mathrm{kTopSpc})$, showing a local section σ_0 which exhibits, under Lem. 2.2.7, that the restriction of $P_0 \times [0,1]$ to $\{0\} \subset [0,1]$ is isomorphic to P_0, by construction:

$$
\begin{tikzcd}
\mathrm{X} \times \{0\} \arrow[rr, "\sigma_0"] \arrow[d, "\begin{smallmatrix} G\mathrm{SerCof} \\ \cap G\mathrm{WHmtpEq} \end{smallmatrix} \ni"'] && \big(\mathrm{P} \underset{\mathrm{X}\times[0,1]}{\times} (\mathrm{P}_0 \times [0,1])\big)/\Gamma \arrow[d, "\in G\mathrm{SerFib}"] \\
\mathrm{X} \times [0,1] \arrow[rr, equal] \arrow[urr, dashed, "\exists"] && \mathrm{X} \times [0,1]
\end{tikzcd}
\tag{2.22}
$$

Now observe that, with respect to the proper equivariant model structure (from Prop. 1.3.4):

(i) the left morphism is an acyclic cofibration, since it is the image of the generating acyclic cofibration $G/G \times (D^0 \hookrightarrow D^0 \times [0,1])$ (1.79) under the the product functor $\mathrm{X} \times (-)$, which is left Quillen (by Prop. 1.3.8) since G-CW-complexes are cofibrant (Ex. 1.3.6);

(ii) the right vertical morphism is a fibration, by Lem. 2.15.

Therefore, the proper equivariant model structure implies that a dashed lift in (2.22) exists. The resulting commutativity of the bottom right triangle in (2.22) means that this lift is a section which, under Lem. 2.2.7, exhibits an isomorphism $G \circlearrowright \mathrm{P} \simeq G \circlearrowright \mathrm{P}_0 \times [0,1]$ of equivariant principal bundles over $G \circlearrowright \mathrm{X} \times [0,1]$. The restriction of this isomorphism to $\{1\} \subset [0,1]$ is of the required form (2.21). □

Remark 2.2.9 (Generalized conditions such that concordance classes coincide with isomorphism classes)**.** The proof of Thm. 2.2.8 clearly works internal to any (regular) model category in which fiber bundles are fibrations and $X \times \{0\} \to X \times [0,1]$ exists and is an acyclic cofibration. In particular, there ought to be a suitable such model structure on diffeological spaces[2] which would imply that concordance classes coincide with isomorphism classes also for the case of diffeological principal bundles – in particular of smooth principal bundles with structure Lie groups. Moreover, if a proper equivariant version of such a model structure on diffeological spaces existed, it would imply the identification of concordance classes with isomorphism classes also for smooth equivariant principal bundles. When used further below in the proof of Thm. 4.2.7, this would imply, under the assumptions used there, the classification if smooth equivariant principal bundles not just for D-topological but for more general diffeological structure.

Proof of equivalence of notions of equivariant local triviality. We now turn to the proof of equivalence of the three notions of equivariant local triviality (Thm. 2.2.1). The following notion plays a pivotal role in all three perspectives:

Notation 2.2.10 (Lifts of equivariance subgroups to semidirect product with structure group)**.** Given a G-equivariant topological group (Γ, α) (Def. 2.1.2), we will now abbreviate the semidirect product group (2.6) as $\widehat{G}$. Moreover, given a closed subgroup $H \subset G$, we will write $\widehat{H} \subset \widehat{G}$ for any *choice* of lift of H to a subgroup of $\widehat{G}$:

$$
\begin{tikzcd}
\widehat{H} \arrow[d, shift left, "\simeq"] & \subset & \widehat{G} := \Gamma \rtimes_\alpha G \arrow[d, "\mathrm{pr}_2"] & \widehat{G}/\widehat{H} \arrow[d, "\mathrm{pr}_2/\widehat{H}"] \\
H \arrow[u, shift left, "\widehat{(-)}"] & \subset & G & G/H
\end{tikzcd}
\tag{2.23}
$$

Equivalently, such $\widehat{H}$ is (by Lem. 1.2.14) the graph of a crossed homomorphism $H \to \Gamma$ (Def. 1.2.11).

These choices of lifts $\widehat{H}$ are at the heart of the local characterization of equivariant principal bundles:

Lemma 2.2.11 (Equivariant principal $\widehat{H}$-twisted product bundles)**.** *Consider $H \subset G$ a subgroup, with a lift $\widehat{H} \subset \widehat{G} := \Gamma \rtimes_\alpha G$ (Ntn. 2.2.10) and* $\mathrm{S} \in H\mathrm{Act}(\mathrm{TopSp})$. *Then sufficient conditions for the canonical projection of twisted product spaces*

[2] This is claimed to be the case by Dmitri Pavlov, in private conversation.

$$
\begin{tikzcd}
\underset{}{\overset{\curvearrowright^{\widehat{G}}}{\widehat{G}}} \times_{\widehat{H}} \mathrm{S} \arrow[d, "\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_{\mathrm{S}}"'] & := & \big\{(\gamma, g), s\big\} \big/ \Big(\big((\gamma,g),s\big) \sim \big((\gamma,g)\cdot\hat{h}, h^{-1}\cdot s\big) \Big) \arrow[d, mapsto, "{[(\gamma,g),s] \mapsto [g,s]}"'] \\
\underset{\curvearrowleft_G}{G} \times_H \mathrm{S} & := & \big\{(g,s)\big\} \big/ \big((g,s) \sim (g\cdot h, h^{-1}\cdot s) \big)
\end{tikzcd}
\tag{2.24}
$$

(equipped with their canonical left multiplication actions) to be a G-equivariant (Γ, α)-principal bundle, according to Def. 2.1.3 and via Cor. 2.1.6, are:

- *the underlying Γ-principal bundle is locally trivial;*
- *or Γ is compact.*

Proof. The equivariance conditions (2.9) are immediate by construction. Principality (2.10), follows from consideration of the following diagram:

$$
\begin{tikzcd}
\Gamma \times \big((\Gamma \rtimes_\alpha G) \times_{\widehat{H}} \mathrm{S}\big) \arrow[r, shift left, "{(\gamma',[(\gamma,g),s]) \mapsto [(\gamma'\cdot\gamma,\gamma,g),s]}"] \arrow[rrd, bend right, "\mathrm{pr}_2"'] & ((\Gamma \times \Gamma) \rtimes_\alpha G) \times_{\widehat{H}} \mathrm{S} \arrow[l, shift left, "{\big(\gamma_1\cdot\gamma_2^{-1},[(\gamma_2,g),s]\big) \mapsfrom [(\gamma_1,\gamma_2,g),s]}", "\sim"'] \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\text{(pb)}"] & (\Gamma \rtimes_\alpha G) \times_{\widehat{H}} \mathrm{S} \arrow[d, "\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_{\mathrm{S}}"] \\
& (\Gamma \rtimes_\alpha G) \times_{\widehat{H}} \mathrm{S} \arrow[r, "\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_{\mathrm{S}}"'] & G \times_H \mathrm{S}\,.
\end{tikzcd}
\tag{2.25}
$$

Here the square on the right, regarded in $\Gamma\mathrm{Act}(\mathrm{TopSp})$ with respect to the left multplication action on the left of the two Γ-factors, is recognized as a pullback:

- if the underlying Γ-principal bundle is locally trivial: by Lemma 1.1.19, observing that any local trivialization of $\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_{\mathrm{S}}$ induces one of p
- if Γ is compact: by Lemma 1.1.18, observing that all spaces are Hausdorff by assumption on G, H, and Γ and by Lemma 1.1.16.

With this, the shear map (2.10) has an evident continuous inverse, as shown on the top left of (2.25); where the point is to observe that these formulas are indeed compatible with the given quotient space structures. □

Accordingly, in order to apply Lemma 2.2.11 we consider now classes of choices of H and S such that the twisted product projections (2.25) are locally trivial as Γ-principal bundles.

2.2.1 tom Dieck local trivialization

Definition 2.2.12 (Equivariant open cover)**.** For $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$, a *$G$-equivariant open cover* is an index set $I \in \mathrm{Set}$, and an I-indexed set of open G-subspaces

$$
\Big\{ \overset{\curvearrowright^G}{\mathrm{U}_i} \xhookrightarrow[\text{open}]{\iota_i} \overset{\curvearrowright^G}{\mathrm{X}} \Big\}_{i \in I}\,, \qquad \underset{i\in I}{\bigsqcup} \overset{\curvearrowright^G}{\mathrm{U}_i} \xrightarrow[\text{open}]{} \!\!\!\!\rightarrow \overset{\curvearrowright^G}{\mathrm{X}}
$$

such that, forgetting the G-action, the underlying open subsets cover X.

Definition 2.2.13 (tom Dieck's equivariant local trivializability)**.** **(i)** A G-equivariant principal bundle $\mathrm{P} \xrightarrow{p} \mathrm{X}$ (Def. 2.1.3, Cor. 2.1.6) is *locally trivial* in the sense of [tD69, §2.1][3] if there exists a G-equivariant open cover (Def. 2.2.12)

$$
\underset{i\in I}{\bigsqcup} \overset{\curvearrowright^G}{\mathrm{U}_i} \xrightarrow[\text{open}]{} \!\!\!\!\rightarrow \overset{\curvearrowright^G}{\mathrm{X}}
$$

[3] In fact, [tD69] does not require the square in (2.26) to be a pullback, but instead adds to the definition of equivariant bundles in the sense of Cor. 2.1.6 the condition that the underlying ordinary principal bundles are locally trivial. These conditions are immediately equivalent to the ones we use, as shown by Prop. 2.2.19 and Prop. 2.2.20. We find the order of the conditions as used here a bit more systematic.

of its base space such that for each $i \in I$ the restriction of P to U_i is the pullback of an equivariant principal bundle P_i over a coset space G/H_i for some closed subgroup $H_i \subset G$:

$$\begin{tikzcd} \mathrm{P}_{|\mathrm{U}_i} \arrow[loop above, "\Gamma\rtimes_\alpha G"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{P}_i \arrow[loop above, "\Gamma\rtimes_\alpha G"] \arrow[d, "\exists" left, "\text{principal}"] \\ \mathrm{U}_i \arrow[loop below, "G"] \arrow[r, "\exists"'] & G/H_i \,. \arrow[loop below, "G"] \end{tikzcd} \tag{2.26}$$

Lemma 2.2.14 (Coset space coprojections admitting local sections)**.** *Let G be a topological group and H a topological subgroup. Then the following are sufficient conditions*[4] *for the coset space coprojection $G \xrightarrow{q} G/H$ to admit local sections:*

- **(i)** *G is arbitrary, and*
  H is a compact Lie group;
- **(ii)** *G is a locally compact separable metric space of finite dimension, and*
  H is a closed subgroup;
- **(iii)** *G is a Lie group;*
  H is a closed subgroup.

Proof. The first statement is [Gl50, Thm. 4.1], the second is [Mos53, Thm. 3], see also [Ka58, Thm. 2]. The third statement may be found in [tDB85, Thm. 4.3]. □

Lemma 2.2.15 (Semidirect product coset bundles [tD69, Lem. 2.1] are locally trivial)**.** *Let $H \subset G$ a compact Lie group and $\widehat{H}$ any lift of H to $\widehat{G} := \Gamma \rtimes_\alpha G$ (Ntn. 2.2.10). Then the ordinary Γ-principal bundle, underlying (by Cor. 2.1.10) the equivariant principal $\widehat{H}$-twisted coset space bundle from Lemma 2.2.11 for $\mathrm{S} = *$ (Ex. 1.1.15):*

$$\begin{tikzcd} \widehat{G} \times_{\widehat{H}} * \arrow[r, equal] \arrow[d, "\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_*"'] & \widehat{G}/\widehat{H} \arrow[loop above, "\Gamma\rtimes_\alpha G"] \arrow[d, "\mathrm{pr}_2/\widehat{H}"] & \text{local model } G\text{-equivariant } \Gamma\text{-principal bundle} \\ G \times_H * \arrow[r, equal] & G/H \arrow[loop below, "G"] \end{tikzcd}$$

is locally trivial.

(Compare the following proof with its re-derivation in geometric homotopy theory, below in Ex. 4.2.8.)

Proof. Under the given assumption on $H \subset G$, Lemma 2.2.14 says that there exists an open cover of G/H such that over each of its charts $U \subset G/H$ we have a continuous section σ

$$\begin{tikzcd} & G_{|U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & G \arrow[d] \\ U \arrow[ur, dashed, "\sigma"] \arrow[r, equal] & U \arrow[r, hook] & G/H \end{tikzcd}$$

With this, observe that

$$\begin{tikzcd}[column sep=huge] \Gamma \times U \arrow[r, shift left, "(\gamma,u) \mapsto [\gamma,\sigma(u)]"] \arrow[d, "\mathrm{pr}_1"'] & \Big((\Gamma \rtimes_\alpha G)/\widehat{H}\Big)\Big|_U \arrow[l, shift left, "\left(\mathrm{pr}_1\left((\gamma,g)\cdot \widehat{g^{-1}\sigma([g])}\right),[g]\right) \mapsfrom [\gamma,g]"] \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd} \tag{2.27}$$

is a Γ-equivariant homeomorphism over U: The top map is clearly Γ-equivariant, the bottom map is clearly its inverse and both are continuous, by the continuity of all the maps from which they are composed, as shown. □

[4] Regarding necessity of these conditions, see counter-examples given in [Ka58, §3].

Lemma 2.2.16 (Isomorphisms of local model equivariant principal bundles [tD69, §2.2])**.** *Let $H \subset G$ be any subgroup and $\widehat{H}_1, \widehat{H}_2 \subset \widehat{G} := \Gamma \rtimes_\alpha G$ be two lifts of H as in* (2.23)*. There exists an isomorphism of equivariant principal bundles between the corresponding semidirect product coset space bundles from Lemma 2.2.15, covering an isomorphism of their base spaces, precisely if $\widehat{H}_1$ and $\widehat{H}_2$ are conjugate in $\widehat{G}$ by an element $\hat{g} \in \widehat{G}$:*

$$
\begin{tikzcd}
\widehat{G}/\widehat{H}_1 \arrow[loop above, "\widehat{G}"] \arrow[r, "\sim", "{[\hat{g}] \mapsto [\hat{g}\hat{g}_0]}"'] \arrow[d] & \widehat{G}/\widehat{H}_2 \arrow[loop above, "\widehat{G}"] \arrow[d] \\
G/H_1 \arrow[loop below, "G"] \arrow[r, "\sim", "{[g] \mapsto [gg_0]}"'] & G/H_2 \,. \arrow[loop below, "G"]
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\widehat{H}_1 \arrow[d, "\mathrm{pr}_2"] & \subset & \hat{g}_0 \widehat{H}_2 \hat{g}_0^{-1} & \subset & \widehat{G} \\
H_1 & = & g_0 H_2 g_0^{-1} & \subset & G
\end{tikzcd}
$$

Proof. Any $\widehat{G}$-equivariant map of $\widehat{G}$-coset spaces is determined, via equivariance, by its image $[\hat{g}_0]$ of the element $[e_{\widehat{G}}]$; and the condition that this assignment is consistent, in that it descends to the quotient space

$$
\begin{tikzcd}
\widehat{G} \arrow[loop above, "\widehat{G}"] \arrow[r, "{\hat{g} \mapsto [\hat{g}\hat{g}_0]}"] \arrow[d] & \widehat{G}/\widehat{H}_2 \,, \arrow[loop above, "\widehat{G}"] \\
\widehat{G}/\widehat{H}_1 \arrow[ur, dashed]
\end{tikzcd}
$$

is equivalent to the condition $\widehat{H}_1 \hat{g}_0 \subset \hat{g}_0 \widehat{H}_2$. If this inclusion is an isomorphism of subgroups, as assumed and as shown on the left here:

$$
\widehat{H}_2 \;=\; \hat{g}_0^{-1} \cdot \widehat{H}_1 \cdot \hat{g}_0 \quad \overset{\mathrm{pr}_2}{\Longrightarrow} \quad H_2 \;=\; g_0^{-1} \cdot H_1 \cdot g_0 \,,
$$

then the analogous argument (with $\widehat{g}$ replaced by $\widehat{g}^{-1}$) shows that we have an inverse map. Under the equivariant projection map, pr_2, the analogous statement holds for the image subgroups in G. □

Proposition 2.2.17 (Characterization of equivariant principal bundles over orbits [tD69, §2.1])**.** *Given a subgroup $H \subset G$, every G- equivariant Γ-principal bundle $\mathrm{P} \to G/H$ (Def. 2.1.3, Cor. 2.1.6) whose base space is the coset space G/H is either empty (Rem. 2.1.8) or isomorphic to a semidirect product coset bundle from Lemma 2.2.15, for some lift $\hat{H}$* (2.23)*:*

$$
\exists \quad
\begin{tikzcd}
(\Gamma \rtimes_\alpha G)/\widehat{H} \arrow[loop above, "\Gamma \rtimes_\alpha G"] \arrow[r, "\sim"] \arrow[d] & P \arrow[loop above, "\Gamma \rtimes_\alpha G"] \arrow[d, "p"] \\
G/H \arrow[r, equal] & G/H
\end{tikzcd}
\tag{2.28}
$$

Proof. In the case when the bundle is not empty, there exists an element in some fiber; and hence, by G-equivariance and transitivity of the G-action on G/H, we may find a point $\widehat{[e_G]} \in P_{[H]}$ in the fiber over $[e_G] \in G/H$. Let then

$$
\widehat{H} \;:=\; \mathrm{Stab}_{\Gamma \rtimes_\alpha G}\Big(\widehat{[e_G]}\Big) \;\subset\; \widehat{G} \;:=\; \Gamma \rtimes_\alpha G \tag{2.29}
$$

be its isotropy group under the action of the full semidirect product group. We observe that this $\widehat{H}$ is a lift of H as required in (2.23):

(i) $\widehat{H} \subset \widehat{G}$ is a closed subgroup, since it is the preimage of the closed (by Hausdorffness, Assump. 1.1.2) singleton subset $\{\widehat{[e_G]}\} \subset \mathrm{P}$ under the continuous function $\widehat{G} \xrightarrow{\hat{g} \mapsto \hat{g} \cdot \widehat{[e_G]}} \mathrm{P}$.

(ii) The restriction of $\widehat{G} \xrightarrow{\mathrm{pr}_2} G$ to $\widehat{H}$ factors through $H \subset G$, since for all $\hat{h} \in H$ we have

$$
\begin{aligned}
[e_G] &= p\big(\hat{h} \cdot \widehat{[e_G]}\big) \\
&= \mathrm{pr}_2(\hat{h}) \cdot p\big(\widehat{[e_G]}\big) \\
&= \mathrm{pr}_2(\hat{h}) \cdot [e_G] \,.
\end{aligned}
$$

(iii) This map $\mathrm{pr}_2 : \widehat{H} \to H$ is an isomorphism since (Example 1.0.7) it is a pullback of (a restriction of) the Γ-principality isomorphism (2.10):

$$
\begin{tikzcd}
\widehat{H} \arrow[rr, "(\gamma{,}g)\mapsto g"] \arrow[d, "{(\gamma,g) \atop \mapsto \atop \gamma^{-1}}"'] \arrow[drr, phantom, "\text{(pb)}"] && \{\widehat{[e_G]}\} \times H \arrow[d, "{g \atop \mapsto \atop (e_\Gamma,g)\cdot\widehat{[e]}}"] \\
\{\widehat{[e_G]}\} \times \Gamma \arrow[rr, "\gamma\mapsto(\gamma{,}e_G)\cdot\widehat{[e_G]}", "\simeq"'] && P_{[e]} \,.
\end{tikzcd}
$$

□

Example 2.2.18 (Adjusting the classifying maps in a tom Dieck local trivialization)**.** Given a tom Dieck local trivialization (Def. 2.2.13) over a G-equivariant patch U_i, consider any point $x \in \mathrm{U}_i$ and write $[g_0] \in G/H_i$ for its image under the given classifying map (2.26). Now choosing any identification of the local model bundle with a semidirect product coset projection $\widehat{G}/\widehat{H}_i$ (Lemma 2.2.17) and choosing any lift $[\hat{g}_0] \in \widehat{G}/\widehat{H}_i$ (Ntn. 2.2.24), we may then form the pasting composite of this pullback square with the corresponding isomorphism of local model bundles from Lemma 2.2.16:

$$
\begin{tikzcd}
\mathrm{P}_{|\mathrm{U}_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{G}/\widehat{H}_i \arrow[r, "{[\hat g]\mapsto[\hat g\hat g_0^{-1}]}", "\sim"'] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{G}/\widehat{H'_i} \arrow[d] \\
\mathrm{U}_i \arrow[r, "x\mapsto[g_0]"'] & G/H_i \arrow[r, "\sim", "{[g]\mapsto[gg_0^{-1}]}"'] & G/H'_i
\end{tikzcd}
\quad \text{with} \quad
\begin{array}{ccccc}
\widehat{H'_i} & := & \hat{g}_0\widehat{H}_i\hat{g}_0^{-1} & \subset & \widehat{G} \\
 & & \downarrow & & \\
H'_i & := & g_0 H_i g_0^{-1} & \subset & G
\end{array}
\tag{2.30}
$$

Since the square on the right is a pullback (by Example 1.0.7), the total rectangle is a pullback (by the pasting law, Prop. 1.0.9) and hence exhibits another, equivalent, tom Dieck local trivialization, whose classifying map takes x to $[e_G]$.

Proposition 2.2.19 (tom Dieck's local triviality implies underlying ordinary local triviality)**.** *An equivariant principal bundle (Def. 2.1.3, Cor. 2.1.6), which is locally trivial in the sense of tom Dieck (Def. 2.2.13), has an underlying principal bundle (Cor. 2.1.10) which is locally trivial in the ordinary sense:*

$$
\mathrm{Frm}(G \circlearrowright \Gamma)\,\mathrm{PrnFibBdl}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big) \xrightarrow{\text{forget } G\text{-action}} \mathrm{Frm}\Gamma\mathrm{PrnFibBdl}(\mathrm{kTopSpc})\,.
$$

Proof. By Prop. 2.2.17 and Lemma 2.2.15, the statement is true for the local model bundles over coset spaces. By Lemma 1.1.10 the pullback of this local model bundle is given by the pullback of the underlying topological spaces. With this, the condition (2.26) implies the claim, since any pullback bundle of an ordinary locally trivial bundle is again locally trivial. □

Proposition 2.2.20 (Homomorphisms of tom Dieck-locally trivial equivariant bundles are pullbacks)**.** *Let G be a Lie group. Then every homomorphism* f (2.11) *between tom Dieck-locally trivial G-equivariant* Γ*-principal bundles (Def. 2.2.13) is a pullback square:*

$$
\begin{tikzcd}
\overset{\Gamma\rtimes_\alpha G}{\circlearrowright}\mathrm{P}_1 \arrow[r, "f"] \arrow[d] & \overset{\Gamma\rtimes_\alpha G}{\circlearrowright}\mathrm{P}_2 \arrow[d] \\
\underset{G}{\circlearrowright}\mathrm{X}_1 \arrow[r, "f/\Gamma"'] & \underset{G}{\circlearrowright}\mathrm{X}_2
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overset{G}{\circlearrowright}\mathrm{P}_1 \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G}{\circlearrowright}\mathrm{P}_2 \arrow[d] \\
\underset{G}{\circlearrowright}\mathrm{X}_1 \arrow[r, "f/\Gamma"'] & \underset{G}{\circlearrowright}\mathrm{X}_2
\end{tikzcd}
$$

Proof. Prop. 2.2.19 says that, under the given assumptions, the ordinary principal bundles underlying the given equivariant principal bundles are locally trivial. With this, the claim follows from the fact that morphisms of ordinary locally trivial bundles are pullbacks (Lemma 1.1.19) and that underlying pullbacks reflect equivariant pullbacks (Lemma 1.1.10). □

2.2.2 Lashof-May local trivialization

Definition 2.2.21 (Lashof-May's equivariant local trivialization)**.** An equivariant principal bundle $\mathrm{P} \xrightarrow{p} \mathrm{X}$ (Def. 2.1.3, Cor. 2.1.6) is *locally trivial* in the sense of [La82, p. 258][LM86, p. 267] if there exists a G-equivariant open cover (Def. 2.2.12) $\underset{i \in I}{\sqcup} \overset{\circlearrowright G}{\mathrm{U}_i} \xrightarrow[\text{open}]{} \overset{\circlearrowright G}{\mathrm{X}}$ by orbits of H_i-slices (Def. 1.1.20)

$$\mathrm{U}_i \;=\; G \cdot \mathrm{S}_i \xleftarrow[(-)\cdot(-)]{\sim} G \times_{H_i} \mathrm{S}_i\,, \qquad \overset{\circlearrowright H_i}{\mathrm{S}_i} \hookrightarrow \overset{\circlearrowright H}{\mathrm{X}} \tag{2.31}$$

for closed subgroups

$$H_i \xhookrightarrow{\text{closed}} G\;, \tag{2.32}$$

such that, for each $i \in I$,

- either the restriction $\mathrm{P}_{|\mathrm{U}_i}$ is empty (Rem. 2.1.8),
- or there is a lift $\widehat{H}_i$ (2.23) of H_i (2.32) to $\widehat{G} := \Gamma \rtimes_\alpha G$ (Ntn. 2.2.10), such that the the restriction of P to U_i is isomorphic, as a G-equivariant Γ-principal bundle, to the equivariant principal $\widehat{H}_i$-twisted product bundle of the slice, from Lemma 2.2.11:

$$\begin{tikzcd}
\overset{\circlearrowright \widehat{G}}{\mathrm{P}_{|\mathrm{U}_i}} \arrow[r, "\sim"] \arrow[d, "p_{|\mathrm{U}_i}"] & \overset{\circlearrowright \widehat{G}}{\widehat{G} \times_{\widehat{H}_i} \mathrm{S}_i} \arrow[d, "\mathrm{pr}_2 \times_{\widehat{H}} \mathrm{id}_{\mathrm{S}_i}"] \\
\underset{\circlearrowleft G}{\mathrm{U}_i} \arrow[r, "\sim"] & \underset{\circlearrowleft G}{G \times_H \mathrm{S}_i}
\end{tikzcd} \tag{2.33}$$

Lemma 2.2.22 (Local triviality of $\widehat{H}$-twisted product bundles)**.** *Let $H \subset G$ be a compact subgroup (Assump. 1.1.2) and $H \circlearrowright S \in H\mathrm{Act}(\mathrm{TopSp})$.*

(i) *There exists a tubular neighborhood $D \times \mathrm{S}$ of S in $G \times_H S$ (D an open ball), and hence an open cover of $G \times_H \mathrm{S}$ by its G-translates.*

(ii) *For every $\widehat{H} \subset \widehat{G} := \Gamma \times_\alpha G$ (Ntn. 2.2.10), the induced $\widehat{H}$-twisted product bundle (Lemma 2.2.11) has underlying Γ-principal bundle (Cor. 2.1.10) which is locally trivial over this open cover:*

$$\begin{tikzcd}
\Gamma \times \overset{\circlearrowright \Gamma}{D} \times \mathrm{S} \arrow[r] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\circlearrowright \Gamma}{\widehat{G} \times_{\widehat{H}} \mathrm{S}} \arrow[d] \\
D \times \mathrm{S} \arrow[r, hook, "\text{open}"'] & G \times_H \mathrm{S}
\end{tikzcd}$$

Proof. The core of this argument is indicated inside the proof of [La82, Lem. 1.1] for the special case when α (2.6) is trivial. We spell it out and generalize it.

Since the connected component of G is a compact Lie group (Assump. 1.1.2), we may find a bi-invariant Riemannian metric on G (by [Mi76, Cor. 1.4]). With respect to this metric, let

$$D_e^\varepsilon \;:=\; D^\varepsilon N_e H \tag{2.34}$$

be an open ε-ball for any $\varepsilon \in \mathbb{R}_+$ in the normal bundle at the neutral element, and hence in a tubular neighborhood of H. By right invariance of the metric, the multiplication action yields a homomorphism

$$\begin{array}{ccc}
D \times H & \xrightarrow[\sim]{\kappa} & D \cdot H \\
(d,h) & \longmapsto & d \cdot h
\end{array} \tag{2.35}$$

which is surjective (since for each h it restricts to an isometry $D \simeq D_h$) and continuously invertible (with inverse given by right multiplication with the inverse of the normal projection into H) so that by dimensional reasons we have that

$$D \times \mathrm{S} \;\simeq\; (D \cdot H) \times_H S \;\subset\; G \times_H S$$

is an open neighborhood of $S \subset G \times_H S$. This proves the first statement.

For the second statement, consider the following diagram, whose top morphisms are Γ-equivariant:

$$
\begin{tikzcd}
\Gamma \times D \times \mathrm{S} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\Gamma \rtimes_\alpha (D \times H)) \times_{\widehat{H}} \mathrm{S} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\Gamma \rtimes_\alpha D \cdot H) \times_{\widehat{H}} \mathrm{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{G} \times_{\widehat{H}} \mathrm{S} \arrow[d] \\
D \times \mathrm{S} \arrow[r, "\sim"] & D \times H \times_H \mathrm{S} \arrow[r, "\sim", "\kappa \times_H \mathrm{id}_{\mathrm{S}}"'] & (D \cdot H) \times_H \mathrm{S} \arrow[r, hook, "\text{open}"'] & G \times_H \mathrm{S} .
\end{tikzcd}
\tag{2.36}
$$

Here the two squares on the right are pullback squares since they have parallel isomorphisms (by Example 1.0.7), induced from (2.35). This witnesses the second vertical morphism from the right as a trivialized Γ-principal bundle. With this, the right square is a pullback by Lemma 1.1.19. In conclusion, the total rectangle in (2.36) is a pullback by the pasting law (Prop. 1.0.9), and hence exhibits the restriction of the twisted product bundle to the open subset $D \times \mathrm{S}$ of its base space as a Cartesian product projection. $\square$

Proposition 2.2.23 (Lashof-May's local trivializability implies tom Dieck's)**.** *If a G-equivariant $(G \circlearrowright \Gamma)$-principal bundle (Def. 2.1.3, Cor. 2.1.6, Assump. 1.1.2) is locally trivial in the sense of Lashof-May (Def. 2.2.21) then it is also locally trivial in the sense of tom Dieck (Def. 2.2.13).*

Proof. Observe the evident commuting square of equivariant maps shown on the left in the following (2.37), where $\widehat{G}$ acts by left multiplication on itself (and on G, through pr_2), while $\widehat{H}_i$ acts by right inverse multiplication on $\widehat{G}$ (and on G through pr_2), by its given action on S_i and diagonally on the Cartesian products:

$$
\begin{tikzcd}
\overset{\widehat{G}\times\widehat{H}_i}{\circlearrowright}\; \widehat{G} \times \mathrm{S}_i \arrow[r, "\mathrm{id}_{\widehat{G}} \times \mathrm{pr}"] \arrow[d, "\mathrm{pr}_2 \times \mathrm{id}"] & \overset{\widehat{G}\times\widehat{H}_i}{\circlearrowright}\; \widehat{G} \times * \arrow[d, "\mathrm{pr}_2"] & & \overset{\widehat{G}}{\circlearrowright}\; \widehat{G} \times_{\widehat{H}_i} \mathrm{S}_i \arrow[r] \arrow[d] & \overset{\widehat{G}}{\circlearrowright}\; \widehat{G}/\widehat{H}_i \arrow[d] \\
\underset{G\times\widehat{H}_i}{\circlearrowleft}\; G \times \mathrm{S}_i \arrow[r, "\mathrm{id}_G \times \mathrm{pr}"] & \underset{G\times\widehat{H}_i}{\circlearrowleft}\; G \arrow[r, phantom, "\xmapsto{(-)/\widehat{H}_i}"] & & \underset{G}{\circlearrowleft}\; G \times_{H_i} \mathrm{S}_i \arrow[r] & \underset{G}{\circlearrowleft}\; G/H_i
\end{tikzcd}
\tag{2.37}
$$

Passage to $\widehat{H}_i$-quotients (Example 1.1.15) yields the commuting square shown on the right, where the vertical map on the left is the Lashof-May local model bundle (2.33), while the vertical map on the right is a tom Dieck local model bundle (2.26), by Lemma 2.2.15.

Hence it is now sufficient to see that the square on the right of (2.37) is in fact a pullback square, as required in tom Dieck's local trivialization condition (2.26). By Lemma 1.1.10, this is equivalent to showing that the underlying square of topological spaces is a pullback. But the square is a homomorphism of ordinary Γ-principal bundles (since $\Gamma \subset \widehat{G}$) which are both locally trivial, by Prop. 2.2.19 and by Lemma 2.2.22. Therefore its pullback property follows by Lemma 1.1.19. $\square$

2.2.3 Bierstone local trivialization

Notation 2.2.24 (Lift of isotropy groups to semidirect product with structure group)**.** Given an action $G \circlearrowright \mathrm{X}$ and a point $x \in \mathrm{X}$, recall from (1.9) the isotropy group $G_x \subset X$. Thinking of the semidirect product group $\widehat{G} := \Gamma \rtimes_\alpha G$ (Ntn. 2.2.10) as acting on X through the projection pr_2, we write

$$
\widehat{G}_x \;:=\; \mathrm{Stab}_{\widehat{G}}(x) \;:=\; \Gamma \rtimes_\alpha G_x \,.
\tag{2.38}
$$

Hence in further following Ntn. 2.2.10, a choice of lift (2.23) of $G_x \subset G$ to $\widehat{G}_x$ is to be denoted $\widehat{G_x}$ (note the scope of the hat):

$$
\begin{tikzcd}
\widehat{G_x} \arrow[d, shift left, "\simeq"] & \subset & \widehat{G}_x := \Gamma \rtimes_\alpha G_x \arrow[d, "\mathrm{pr}_2"] \\
G_x \arrow[u, shift left, "\widehat{(-)}"] & = & G_x
\end{tikzcd}
\tag{2.39}
$$

Lemma 2.2.25 (Beck-Chevalley for lift of isotropy groups)**.** *Given an isotropy subgroup $G_x \hookrightarrow G$, its two inclusions into $\widehat{G}$ (via Ntn. 2.2.24) form a commuting square as shown on the left here:*

$$
\begin{tikzcd}[column sep=small]
& G_x \arrow[dl, hook', "s"'] \arrow[dr, hook, "i_x"] \arrow[dd, phantom, "\text{(pb)}"] & \\
\widehat{G}_x \arrow[dr, hook, "\hat{i}_x"'] & & G \arrow[dl, hook', "s"] \\
& \widehat{G} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& G_x\mathrm{Actions} \arrow[dr, "G\times_{G_x}(-)"] & \\
\widehat{G}_x\mathrm{Actions} \arrow[ur, "s^*"] \arrow[dr, "\widehat{G}\times_{\widehat{G}_x}(-)"'] & \Leftarrow\!\!\!\sim & G\mathrm{Actions} \\
& \widehat{G}\mathrm{Actions} \arrow[ur, "s^*"'] &
\end{tikzcd}
\tag{2.40}
$$

and the pull-push change-of-group operations (Lemma 1.1.7) via inductions/restriction (Example 1.1.12) from left to right through this square coincide to yield the commuting diagram of functors shown on the right of (2.40).

Proof. This is a direct consequence of the basic fact that $\widehat{G}_x/\Gamma = (\Gamma \rtimes_\alpha G_x)/\Gamma \simeq G_x$. Explicitly, we claim that the natural transformation filling the square on the right of (2.40) is

$$
\overset{\widehat{G}_x \circlearrowright}{\mathrm{P}} \longrightarrow \overset{G \circlearrowright}{G \times_{G_x} \mathrm{P}} \xrightarrow[\sim]{\quad} \overset{G \circlearrowright}{\widehat{G} \times_{\widehat{G}_x} \mathrm{P}}
\qquad
[g,p] \longmapsto [(\mathrm{e}_\Gamma, g), p]\,.
\tag{2.41}
$$

This is manifestly equivariant under the left multiplication action by $G \overset{s}{\hookrightarrow} \Gamma \rtimes_\alpha G$ and it is manifestly natural under $\widehat{G}_x$-equivariant maps $\mathrm{P} \to \mathrm{P}'$ on the right. Moreover, (2.41) is surjective because the general element $[(\gamma, g), p]$ on the right is equal to

$$
[(\gamma,g),\, p] = \big[(\mathrm{e}_\Gamma,\, g)\cdot(\alpha(g^{-1})(\gamma),\, \mathrm{e}_G),\, p\big] = \big[(\mathrm{e}_\Gamma,\, g),\, (\alpha(g^{-1})(\gamma),\, \mathrm{e}_G)\cdot p'\big]\,,
$$

and it is injective because

$$
\begin{aligned}
[(\mathrm{e}_\Gamma, g_1), p_1] = [(\mathrm{e}_\Gamma, g_2), p_2] \;\;\Leftrightarrow\;\; \underset{\substack{g' \in G_x \\ \gamma \in \Gamma}}{\exists} \left\{ \begin{array}{l} (\mathrm{e}_\Gamma, g_1)\cdot(\gamma, g') = (\mathrm{e}_\Gamma, g_1)\,, \\ (\gamma, g')^{-1}\cdot p_1 = p_2 \end{array} \right. \;\;&\Rightarrow\;\; \underset{g' \in G_x}{\exists} \left\{ \begin{array}{l} (\mathrm{e}_\Gamma, g_1)\cdot(\mathrm{e}_\Gamma, g') = (\mathrm{e}_\Gamma, g_1)\,. \\ (\mathrm{e}_\Gamma, g')^{-1}\cdot p_1 = p_2 \end{array} \right. \\
&\Leftrightarrow\;\; [g_1,\, g_1] = [g_2,\, p_2]
\end{aligned}
$$

In conclusion, we have a G-equivariant natural bijection, as claimed. ☐

Definition 2.2.26 (Bierstone's equivariant local trivializability)**.** A G-equivariant $(G \circlearrowright \Gamma)$-principal bundle $\mathrm{P} \xrightarrow{p} \mathrm{X}$ (Def. 2.1.3, Cor. 2.1.6) is *locally trivial* in the sense of [Bi73, §4] generalized to non-trivial α (2.6), if for every point $x \in \mathrm{X}$ there exists a G_x-equivariant (1.9) open neighborhood $x \in \overset{G_x \circlearrowright}{\mathrm{U}_x} \xhookrightarrow[\text{open}]{} \overset{G_x \circlearrowright}{\mathrm{X}}$, and a lift $\widehat{G_x} \subset \widehat{G}_x := \Gamma \rtimes_\alpha G_x$ (2.39) such that the restriction of P to U_x is

- either empty (Rem. 2.1.8),
- or isomorphic, as a G_x-equivariant (Γ, α)-principal bundle under restriction along $G_x \hookrightarrow G$ (Ex. 1.1.12), to the equivariant direct product bundle from Lemma 2.2.27:

$$
\begin{tikzcd}
\overset{\widehat{G_x} \circlearrowright}{\mathrm{P}_{|\mathrm{U}_x}} \arrow[r, "\exists", "\sim"'] \arrow[d] & \overset{\widehat{G_x} \circlearrowright}{\widehat{G}_x \times_{\widehat{G_x}} \mathrm{U}_x} \arrow[d] \\
\underset{G_x \circlearrowleft}{\mathrm{U}_x} \arrow[r, equal] & \underset{G_x \circlearrowleft}{\mathrm{U}_x}\,.
\end{tikzcd}
\tag{2.42}
$$

Lemma 2.2.27 (Underlying principal bundles of Bierstone local model bundles are trivial)**.** *Any Bierstone local model bundle* (2.42) *is isomorphic, as a bundle of topological G_x-actions, to the projection out of the Cartesian product of $G_x \circlearrowright \mathrm{U}_x$ with a G_x-action ρ_x on Γ:*

$$
\begin{tikzcd}
\overset{G_x \circlearrowright}{\widehat{G}_x \times_{\widehat{G_x}} \mathrm{U}_x} \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_2 \times_{\widehat{G_x}} \mathrm{id}_{\mathrm{U}_x}"'] & \overset{G_x \circlearrowright_{\rho_x}}{\Gamma} \times \overset{G_x \circlearrowright}{\mathrm{U}_x} \arrow[d, "\mathrm{pr}_2"] \\
\underset{G_x \circlearrowleft}{G_x \times_{G_x} \mathrm{U}_x} \arrow[r, "\sim"] & \underset{G_x \circlearrowleft}{\mathrm{U}_x}
\end{tikzcd}
\tag{2.43}
$$

Proof. The point is that the $\widehat{G_x}$-action is transitive on the G_x-factor in $\widehat{G}_x$, which implies that there are canonical representatives of the elements in the twisted product quotient (namely those with the neutral element in the G_x-factor) passing to which yields, first of all, an isomorphism of underlying topological spaces:

$$
\begin{array}{ccc}
\widehat{G}_x \times_{\widehat{G_x}} \mathrm{U}_x & \xrightarrow{\quad\sim\quad} & \Gamma \times S \\
{[(\gamma, \mathrm{e}_G), s]} & \longmapsto & (\gamma, s)\,.
\end{array}
\tag{2.44}
$$

One readily checks that, under this identification, the G_x-action on the left turns into the claimed Cartesian product action:

$$
g \cdot [(\gamma, \mathrm{e}_G), u] \;=\; [(\alpha(g)(\gamma), g), u] \;=\; \big[(\alpha(g)(\gamma), g) \cdot \hat{g}^{-1}, g \cdot u\big] \;=\; [(\rho_x(g)(\gamma), \mathrm{e}_G), g \cdot u]\,,
\tag{2.45}
$$

with $(\gamma, g) \mapsto \rho_x(g)(\gamma) \in \Gamma$ defined by the rightmost equation. □

Remark 2.2.28 (Bierstone's local model bundles and generalization)**.** In the special case when α (2.6) is trivial, the equivariant bundles appearing on the right of (2.43) are the local model principal bundles considered in [Bi73]. We may regard Lemma 2.2.27 as providing the consistent generalization of Bierstone's notion to equivariant principal bundles with nontrivial action α of the equivariance group on the structure group.

Example 2.2.29 (Fixed loci in Bierstone local model bundle for trivial α)**.** Consider the special case when α (2.6) is trivial. Then any lift $\widehat{G_x}$ in (2.38) is equivalently the graph of an injective group homomorphism ϕ

$$
\widehat{G_x} \;=\; \{(\phi(g), g) \,|\, g \in G_x\} \;\subset\; \Gamma \times G_x\,, \qquad G_x \overset{\phi}{\hookrightarrow} \Gamma
$$

and the action ρ_x in (2.45) is just the right inverse multiplication action through this homomorphism

$$
\begin{array}{ccc}
G \times (\Gamma \times \mathrm{U}_x) & \xrightarrow{\quad\rho_x\quad} & \Gamma \times \mathrm{U}_x \\
(g, (\gamma, s)) & \longmapsto & (\gamma \cdot \phi(g)^{-1}, g \cdot s)\,.
\end{array}
$$

Since this action is free, it has no H-fixed points as soon as $H \subset G_x$ is nontrivial. Therefore the H-fixed locus bundle (Cor. 2.1.9) of a Bierstone local model bundle for trivial α is necessarily of the following form:

$$
\big(\widehat{G}_x \times_{\widehat{G_x}} \mathrm{S}\big)^H \;\simeq\; \left\{
\begin{array}{lcl}
\overset{G_x \circlearrowright_{\rho_x}}{\Gamma} \times \overset{G_x \circlearrowright}{\mathrm{U}_x} & | & H = 1 \\
\varnothing \times \overset{W(H) \circlearrowright}{\mathrm{U}_x^H} & | & \text{otherwise}
\end{array}
\right.
$$

Proposition 2.2.30 (Bierstone local triviality is preserved by passage to H-fixed loci)**.** *If a G-equivariant principal bundle is locally trivial in the sense of Bierstone (Def. 2.2.26), then for all $H \subset G$ so is its $W(H)$-equivariant H-fixed locus bundle (Cor. 2.1.9).*

Proof. Let $x \in \mathrm{X}^H \subset \mathrm{X}$ be an H-fixed point and consider a Bierstone local trivialization (2.42) over an open neighborhood $\mathrm{U}_x \subset \mathrm{X}$ with given lift $\widehat{G_x}$. It is sufficient to show that the H-fixed locus of this G-equivariant Bierstone local model bundle is $W(H)$-equivariantly isomorphic to a $W(H)$-equivariant Bierstone local model bundle:

Since the fixed locus functor $(-)^H$ is a right adjoint (1.24), it preserves (Prop. 1.0.8) the Cartesian product on the right hand side of the identification (2.43) of the Bierstone local model model with a Cartesian product (Lemma 2.2.27). Hence we just have to observe that this isomorphism (2.44) passes to the fixed locus:

$$
\begin{tikzcd}
\widehat{G}_x \times_{\widehat{G_x}} \mathrm{U}_x \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_2 \times_{\widehat{G_x}} \mathrm{id}_{\mathrm{U}_x}"'] & \Gamma \times \mathrm{U}_x \arrow[d, "\mathrm{pr}_2"] \\
G_x \times_{G_x} \mathrm{S} \arrow[r, "\sim"] & \mathrm{U}_x
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
(\widehat{W(H)})_x \times_{\widehat{W(H)_x}} \mathrm{U}_x^H \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_2 \times_{\widehat{W(H)_x}} \mathrm{id}_{\mathrm{U}_x^H}"'] & \Gamma^H \times \mathrm{U}_x^H \arrow[d, "\mathrm{pr}_2^H"] \\
W(H) \times_{W(H)_x} \mathrm{U}_x^H \arrow[r, "\sim"] & \mathrm{U}_x^H
\end{tikzcd}
\tag{2.46}
$$

$$
[(\gamma, e_{G_x}), u] \longmapsto (\gamma, u) \qquad\qquad \left[(\gamma, e_{W(H)_x}), u\right] \longmapsto (\gamma, u)
$$

Here in the left column of the right square we systematically follow our notation conventions (Ntn. 1.1.1, 2.2.10, 2.2.24) to the new equivariance group $W(H)$ at the fixed locus

$$
W(H)_x := \mathrm{Stab}_{W(H)}(x) = (G_x \cap N(H))/(G_x \cap H), \qquad (\widehat{W(H)})_x := \Gamma^H \rtimes_\alpha W(H)_x
$$

and take the lift to be the subquotient of the given lift

$$
\widehat{W(H)_x} := \widehat{G_x \cap N(H)}/H .
$$

The computation (2.45) now applies verbatim to check that the homeomorphism (2.46) is indeed $W(H)_x$-equivariant

$$
w \cdot \left[(\gamma, e_{W(H)}), u\right] = \left[(\alpha(w)(\gamma), w), u\right] = \left[(\alpha(w)(\gamma), w) \cdot \hat{w}^{-1}, w \cdot u\right] = \left[\left(\rho_x(w)(\gamma), e_{W(H)}\right), w \cdot u\right] . \tag{2.47}
$$

In conclusion, this shows that if a G-equivariant principal bundle is isomorphic in a neighborhood U_x of an H-fixed point x to a Bierstone local model bundle of the form (2.43), then its H-fixed locus is $W(H)$-equivariantly isomorphic over U_x^H to the Bierstone local model (2.46). □

We proceed to discuss the relation of Bierstone's local trivalizability condition to that of tom Dieck and of Lashof-May.

Lemma 2.2.31 (Intersection with open neighborhoods preserves slices through points)**.** *Let* $x \in \mathrm{U}_x \xhookrightarrow{\text{open}} \mathrm{X}$ *be a G_x-equivariant open neighborhood in a given* $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$*, of some* $x \in \mathrm{X}$*. Then, with every slice* $x \in \mathrm{S}'_x \xhookrightarrow{\iota} G \cdot \mathrm{S}'_x$ *(1.30) through x, the intersection* $\mathrm{S}_x := \mathrm{S}'_x \cap \mathrm{U}_x$ *is also a slice through x (Def. 1.1.20).*

Proof. We need to see that the image

$$
G \times_{G_x} (\mathrm{S}'_x \cap \mathrm{U}_x) \xrightarrow[\sim]{\widetilde{\iota_{|\mathrm{U}_x}}} G \cdot (\mathrm{S}'_x \cap \mathrm{U}_x) \hookrightarrow G \cdot \mathrm{S}'_x \xhookrightarrow{\text{open}} \mathrm{X}
$$

is still an open subset of X, hence an open neighborhood of x. Observe that for any slice S, under Assump. 1.1.2, the map $G \times \mathrm{S} \xrightarrow{(g,s) \mapsto g \cdot s} G \cdot \mathrm{S}$ is an open map ([An17, Thm. 2.1 (1)]), hence sends open subsets to open subsets. With this, and since $G \times (\mathrm{S}'_x \cap \mathrm{U}_x) \xhookrightarrow{\text{open}} G \times \mathrm{S}'_x$ is open by assumption on U_x, it follows that its image is open in $G \cdot \mathrm{S}'_x$, which in turn is open in X by assumption on S'_x. □

Notation 2.2.32 (Slices inside Bierstone patches)**.** Given an equivariant local trivialization (2.42) in the sense of Bierstone (Def. 2.2.26), we may choose (by the slice theorem, Prop. 1.1.21, using Assump. 1.1.2) for each point $x \in \mathrm{X}$ a G_x-slice S'_x through that point. Furthermore, by Lemma 2.2.31, we obtain from this a slice $\mathrm{S}_x := \mathrm{S}'_x \cap \mathrm{U}_x$ which is still through x but also contained in U_x:

$$
\begin{tikzcd}
 & \mathrm{U}_x \arrow[dr, hook] & \\
x \in \mathrm{S}_x \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{X} \\
 & G \cdot \mathrm{S}'_x \arrow[ur, hook] &
\end{tikzcd}
\; . \tag{2.48}
$$

Lemma 2.2.33 (Bierstone local trivializations restricted to slices of the base are slices in the total space)**.** *Given an equivariant local trivialization* (2.42) *in the sense of Bierstone (Def. 2.2.26), then the further restrictions of the equivariant bundle to slices* $\mathrm{S}_x \hookrightarrow \mathrm{U}_x$ *inside the Bierstone patches (according to Nota 2.2.32) are slices (Def. 1.1.20) in the total space of the bundle. That is, the induction/restriction-adjunct* (1.22) *of their inclusions are isomorphisms:*

$$
\begin{tikzcd}
\overset{G_x \circlearrowright}{\mathrm{P}|_{\mathrm{S}_x}} \arrow[r, hook, "P|_{\iota_x}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G_x \circlearrowright}{\mathrm{P}|_{G\cdot\mathrm{S}_x}} \arrow[d] \\
\underset{G_x \circlearrowright}{\mathrm{S}_x} \arrow[r, hook, "\iota_x"'] & \underset{G_x \circlearrowright}{G\cdot\mathrm{S}_x}
\end{tikzcd}
\quad \overset{\widetilde{(-)}}{\longleftrightarrow} \quad
\begin{tikzcd}
\overset{G \circlearrowright}{G\times_{G_x}\mathrm{P}|_{\mathrm{S}_x}} \arrow[r, "\widetilde{P|_{\iota_x}}", "\sim"'] \arrow[d] & \overset{G \circlearrowright}{\mathrm{P}|_{G\cdot\mathrm{S}_x}} \arrow[d] \\
\underset{G \circlearrowright}{G\times_{G_x}\mathrm{S}_x} \arrow[r, "\sim", "\widetilde{\iota_x}"'] & \underset{G \circlearrowright}{G\cdot\mathrm{S}_x}
\end{tikzcd}
\tag{2.49}
$$

Proof. First we claim that the underlying Γ-principal bundle (Cor. 2.1.10) of $G\times_{G_x}\mathrm{P}|_{\mathrm{S}_x}$ is locally trivializable:

By Lemma 2.2.22, there exists a tubular neighborhood $D\times\mathrm{S}_x \hookrightarrow G\cdot\mathrm{S}_x$ (using Assump. 1.1.2). Since this is contractible to S_x, and since the underlying Γ-principal bundle of $P|_{\mathrm{S}_x}$ is trivializable – being the restriction, along (2.48), of that of $\mathrm{P}|_{\mathrm{U}_x}$, which is trivializable by assumption (2.42) and by Lemma 2.2.27 – it follows that the underlying bundle of $\mathrm{P}|_{D\times\mathrm{S}_x}$ is trivializable. By G-equivariance the same is then true for all its G-translates; and since the G-translates of its base space $D\times\mathrm{S}_x \hookrightarrow G\cdot\mathrm{S}_x$ form a cover, this implies the claim.

Second, we claim that the adjunct square on the right of (2.49) is a morphism of Γ-principal bundles, hence that its top map is Γ-equivariant. A formal way to see this is to notice, with Lemma 2.2.25, that the top morphism is equivalently the induction/restriction-adjunct (1.22) along the lifted inclusion $\widehat{G}_x \hookrightarrow \widehat{G}$ (Ntn. 2.2.24).

These two claims together imply that the square on the right of (2.49) is a pullback square, by Lemma 1.1.19. Since its bottom morphism is an isomorphism (1.29) by the assumption that S_x is a slice, the top morphism is exhibited as the pullback of an iosmorphism and thus is itself an isomorphism (Example 1.0.7), which is the statement to be shown. □

Remark 2.2.34 (Comparison to the literature)**.** In [La81, Proof of Lem. 1.3] the adjunct (2.49) is written down in the special case when α is trivial (Rem. 2.2.28) but no reason is offered for it being an isomorphism. While it is clear by point-set analysis that the underlying function is a bijection, its homeomorphy is a little subtle. The above category-theoretic proof of Lem. 2.2.33 makes this transparent and immediately generalizes the argument to equivariant bundles internal to other ambient categories, such as to differentiable equivariant bundles.

Proposition 2.2.35 (Bierstone's local triviality implies Lashof-May's)**.** *An equivariant principal bundle over* X *(Def. 2.1.3, Cor. 2.1.6) that is locally trivial in the sense of Bierstone (Def. 2.2.26) is also locally trivial in the sense of Lashof & May (Def. 2.2.21).*

Proof. For $x\in\mathrm{X}$, the given Bierstone trivialization (2.42) over a Bierstone patch U_x further restricted to a slice S_x through x inside U_x (Ntn. 2.2.32) yields the following diagram, by Lemma 1.1.18

$$
\begin{tikzcd}
\mathrm{P}|_{\mathrm{S}_x} \arrow[r, "\sim"] \arrow[d] & \widehat{G}_x\times_{\widehat{G}_x}\mathrm{S}_x \arrow[d] \\
\mathrm{S}_x \arrow[r, equal] & \mathrm{S}_x
\end{tikzcd}
\tag{2.50}
$$

using that the underlying Γ-principal bundle (Cor. 2.1.10) of $\mathrm{P}|_{\mathrm{S}_x}$ is trivializable, by Lemma 2.2.27. Consider then the following pasting composite of the image of this diagram (2.50) under the induction functor $G\times_{G_x}(-)$ (1.21) with, on the left, the inverse of the identification (2.49) from Lemma 2.2.33, and, on the right, the equivalence induced via Lemma 2.2.25:

$$
\begin{tikzcd}
\mathrm{P}|_{G\cdot\mathrm{S}_x} \arrow[r, "\sim"] \arrow[d] & G\times_{G_x}\mathrm{P}|_{\mathrm{S}_x} \arrow[r, "\sim"] \arrow[d] & G\times_{G_x}\widehat{G}_x\times_{\widehat{G}_x}\mathrm{S}_x \arrow[r, "\sim"] \arrow[d] & \widehat{G}\times_{\widehat{G}_x}\mathrm{S}_X \arrow[d, "\mathrm{pr}_2\times_{\widehat{G}_x}\mathrm{id}"] \\
G\cdot\mathrm{S}_x \arrow[r, "\sim"] & G\times_{G_x}\mathrm{S}_x \arrow[r, equal] & G\times_{G_x}\mathrm{S}_x \arrow[r, equal] & G\times_{G_x}\mathrm{S}_x\,.
\end{tikzcd}
$$

The resulting composite rectangle manifestly exhibits a Lashof-May local trivialization (2.33), with open subsets indexed by the points of X. □

Proposition 2.2.36 (tom Dieck's local trivialization implies Bierstone's)**.** *An equivariant principal bundle over* X *(Def. 2.1.3, Cor. 2.1.6) that is locally trivial in the sense of tom Dieck (Def. 2.2.13) is also locally trivial in the sense of Bierstone (Def. 2.2.26).*

Proof. Consider a tom Dieck local trivialization (Def. 2.2.13), over patches U_i (2.26) given via Prop. 2.2.17 by $\widehat{H}_i$-coset space coprojections over H_i-cosets (Ntn. 2.2.10) for closed subgroups $H_i \subset G$.

In view of Ntn. 2.2.10 for the given tom Dieck local models and Ntn. 2.2.24 for the Bierstone local models that we have to reproduce, it is suggestive to write $\widehat{G}_i$ for the corresponding semidirect product subgroups:

$$\begin{array}{ccc} \widehat{G}_i & :\equiv & \Gamma \rtimes_\alpha H_i \\ \downarrow & & \downarrow \\ \widehat{G} & :\equiv & \Gamma \rtimes_\alpha H_i \end{array} \tag{2.51}$$

By Lemma 2.2.22 (for $\mathrm{S} = H_i$), we may find an open ball (2.34) around $\mathrm{e} \in G$ normal to $H_i \subset G$

$$D_i = D^\varepsilon N_{\mathrm{e}} H_i \subset G$$

which is an H-equivariant subspace under the conjugation action of H on itself

$$\overset{H_i \curvearrowright_{\mathrm{Ad}}}{D_i} \hookrightarrow \overset{H_i \curvearrowright_{\mathrm{Ad}}}{H_i} \tag{2.52}$$

and such that we have an isomorphism

$$D_i \times H_i \xrightarrow[\sim]{(d,h) \mapsto d \cdot h} D_i \cdot H_i \tag{2.53}$$

and hence an H_i-equivariant open neighborhood

$$\overset{H_i \curvearrowright_{\mathrm{Ad}}}{D_i} \xhookrightarrow[\text{open}]{} \overset{H_i \curvearrowright}{G/H_i}\,. \tag{2.54}$$

Using this isomorphism (2.53), observe the following $G \times \widehat{H}_i$-equivariant homeomorphism (where $h \mapsto \widehat{h}$ is the given lift (2.23)):

$$\begin{array}{ccc} \overset{G\times\widehat{H}_i \curvearrowright}{\Gamma \rtimes_\alpha (H_i \cdot D_i)} & \xrightarrow{\quad\sim\quad} & \overset{G\times\widehat{H}_i \curvearrowright}{(\Gamma \rtimes_\alpha H_i) \times D_i} \\ (\gamma, h' \cdot d) & \longmapsto & \big((\alpha(d^{-1})(\gamma), h'), d\big) \\ \Big\downarrow{\scriptstyle \widehat{h} \in \widehat{H}_i} & & \Big\downarrow{\scriptstyle \widehat{h} \in \widehat{H}_i} \\ \big(\gamma \cdot \alpha(d \cdot h')(\mathrm{pr}_1(\hat{h}^{-1})), h'h^{-1} \cdot hdh^{-1}\big) & \longmapsto & \Big(\big(\alpha(d^{-1})(\gamma) \cdot \alpha(h')(\mathrm{pr}_1(\hat{h}^{-1})), h'h^{-1}\big), hdh^{-1}\Big)\,. \end{array}$$

Here the vertical maps define the $\widehat{H}_i$-action, given by inverse right multiplication on $\Gamma \rtimes_\alpha G$ and by the conjugation action (2.52) (through pr_2), and where the remaining G-action is by left multiplication (through s). This is such that passage to $\widehat{H}_i$-quotients (Example 1.1.15) yields a G-equivariant homeomorphism of the following form, where on the right we use the adapted notation (2.51):

$$\big(\Gamma \rtimes_\alpha (H_i \cdot D_i)\big)/\widehat{H}_i \xrightarrow{\quad\sim\quad} \widehat{G}_i \times_{\widehat{H}_i} D_i\,. \tag{2.55}$$

It follows (with Lemma 1.1.18) that the following square is a pullback:

$$\begin{array}{ccc} \widehat{G}_i \times_{H_i} D_i & \longrightarrow & \widehat{G}/\widehat{H}_i \\ \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow \\ D_i & \hookrightarrow & G/H_i \end{array} \tag{2.56}$$

because the top morphism is manifestly Γ-equivariant, while the left vertical morphism is trivial as a Γ-principal bundle (by Lemma 2.2.27).

Now consider any point $x \in \mathrm{U}_i$. Without restriction of generality, we may assume that this point is taken by the classifying map of the given tom Dieck local trivialization to $[\mathrm{e}_G] \in G/H_i$, for if not then we may adjust the local trivialization, by Ex. 2.2.18.

Then consider the open neighborhood of x which is the preimage of D_i under the classifying map:

$$
\begin{tikzcd}
\overset{H_i \curvearrowright}{\mathrm{U}_x} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \overset{H_i \curvearrowright}{D_i} \arrow[d, hook] \\
\underset{H_i \curvearrowright}{\mathrm{U}_i} \arrow[r, "f_i"'] & \underset{G_x \curvearrowright}{G/H_i}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{G_x \curvearrowright}{\{x\}} \arrow[r, hook] \arrow[dr, hook] & \overset{G_x \curvearrowright}{\mathrm{U}_x} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \overset{G_x \curvearrowright}{D_i} \arrow[d, hook] \\
& \underset{G_x \curvearrowright}{\mathrm{U}_i} \arrow[r, "f_i"'] & \underset{H_i \curvearrowright}{G/H_i}
\end{tikzcd}
\tag{2.57}
$$

This inherits H_i-equivariance from D_i, but we may restrict this (Ex. 1.1.12) to the isotropy group $G_x \subset H_i \subset H$, shown on the right.

Finally, consider the following diagram:

$$
\begin{tikzcd}
& \mathrm{P}|_{\mathrm{U}_i} \arrow[rr] \arrow[dd] & & \widehat{G}/\widehat{H}_i \arrow[dd] \\
\widehat{G}_x \times_{\widehat{G}_x} \mathrm{U}_x \arrow[ur] \arrow[rr] \arrow[dd] & & \widehat{G}_i \times_{H_i} D \arrow[ur] \arrow[dd] & \\
& \mathrm{U}_i \arrow[rr, "f_i"] & & G/H_i \\
\mathrm{U}_x \arrow[ur, hook] \arrow[rr] & & D_i \arrow[ur, hook] &
\end{tikzcd}
\tag{2.58}
$$

Here the rear face is the given tom Dieck local trivialization and, as such, is a pullback. The bottom square is the above pullback (2.57) which defines the open neighborhood of any given point x in the given cover. To prove that we have a local trivialization in the sense of Bierstone, it is sufficient to prove it for this particular open neighborhood U_x, which means to produce a pullback square as shown on the left of (2.58).

But the right square of (2.58) is a pullback by (2.56), while the front square is similarly recognized as a pullback by Lemma 1.1.19, using that the underlying front left Γ-principal bundle is trivial, by Lemma 2.2.27. Therefore, the rear, right, and front squares are all pullbacks. Hence it follows that the left square is indeed a pullback, by the pasting law (Prop. 1.0.9). $\square$

This concludes the proofs of the propositions that make up Thm. 2.2.1.

2.3 Equivariant classifying spaces

We give a streamlined account of the *Murayama-Shimakawa construction* ([MS95][GMM17]) for would-be equivariant classifying spaces of equivariant principal topological bundles (their actual classifying property is discussed further below in §4.3). A transparent formulation is obtained by first establishing all relevant properties for equivariant principal bundles *internal* (Ntn. 1.0.23) to equivariant topological *groupoids* (discussed in §1.2), where the proofs amount to elementary verifications, and from where most of the desired statements are functorially induced (1.6) by the fact that topological realization preserves all finite limits (Lem. 1.2.31).

Ordinary classifying spaces. As a transparent template for the following equivariant discussion, we start by briefly discussing the classical construction of ordinary (i.e., not equivariant) universal principal bundles via topological realization of a universal formally principal bundle $\mathbf{E}\Gamma \xrightarrow{q_\Gamma} \mathbf{B}\Gamma$ *internal to topological groupoids* (Ex. 2.3.1 below). The elegance of this construction rests in the fact that the pertinent properties of the bundle $\mathbf{E}G \to \mathbf{B}G$ *internal to topological groupoids* are elementary and immediate to check explicitly (Lem. 2.3.3, Lem. 2.3.2), from where they are then simply inherited due to the good abstract properties of the topological realization functor that we have established in §1.2. The resulting topological realization

$$
E\Gamma \;:=\; |\mathbf{E}\Gamma| \xrightarrow{|q_\Gamma|} |\mathbf{B}\Gamma| \;=:\; BG
$$

is the classical universal Γ-principal bundle in its incarnation as the *Milgram bar-construction* [Mil67], which [ML70, §6] observed to be the realization coend (1.66) applied to the simplicial topological space which we recognize as the nerve (1.62) of the topological groupoid $\mathbf{E}\Gamma$.

Example 2.3.1 (Universal principal topological groupoid)**.** Consider any $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ with its combined left and right multiplication action:

$$\Gamma \circlearrowright \Gamma^{\mathrm{L,R}} \circlearrowleft \Gamma \quad \in \; (\Gamma \times \Gamma^{\mathrm{op}})\,\mathrm{Act}(\mathrm{kTopSpc})\,.$$

We denote the resulting left Γ-equivariant right Γ-action groupoid (Ex. 1.2.38) by:

$$\Gamma \circlearrowright \mathbf{E}\Gamma \;:=\; \Gamma \circlearrowright \Big(\Gamma \times \Gamma \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \Gamma\Big) \quad \in \; \Gamma\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big)\,. \tag{2.59}$$

Its left Γ-quotient is the delooping groupoid of Γ (Ex. 1.2.7):

$$\mathbf{E}\Gamma \xrightarrow{\;q\;} \Gamma\backslash(\mathbf{E}\Gamma) \;\simeq\; \mathbf{B}\Gamma \quad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})\,. \tag{2.60}$$

We discussed below (Rem. 4.1.4) that $\mathbf{E}\Gamma \to \mathbf{B}\Gamma$ (2.59) is the universal Γ-principal bundle not over the classifying space $B\Gamma$, but over the *moduli stack* $\mathbf{B}\Gamma$ (see Prop. 4.1.1).

Lemma 2.3.2 (Universal principal groupoid is formally principal)**.** *The quotient projection* (2.60) *exhibits the universal* Γ*-principal groupoid* (2.59) *as a formally principal bundle (Ntn. 1.0.25) internal to topological groupoids (Ntn. 1.2.1), in that the shear map is an isomorphism:*

$$\Gamma \times \mathbf{E}\Gamma \xrightarrow[\;\sim\;]{\big((-)\cdot(-),\,\mathrm{pr}_2\big)} (\mathbf{E}\Gamma) \underset{\mathbf{B}\Gamma}{\times} (\mathbf{E}\Gamma) \qquad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})\,.$$

Proof. Since limits of topological groupoids are computed degree-wise, this is equivalent to the following two horizontal component morphisms being isomorphisms of topological spaces (i.e., homeomorphisms):

$$\begin{tikzcd} \Gamma_L \times (\Gamma \times \Gamma_R) \arrow[r, "(\gamma_\ell,\,\gamma,\,\gamma_r) \mapsto (\gamma_\ell\cdot\gamma,\,\gamma,\,\gamma_r)"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma_1 \times \Gamma_2 \times \Gamma_R \arrow[d, shift left] \arrow[d, shift right] \\ \Gamma_L \times \Gamma \arrow[r, "(\gamma_\ell,\,\gamma) \mapsto (\gamma_\ell\cdot\gamma,\,\gamma)"] & \Gamma_1 \times \Gamma_2 \end{tikzcd} \quad \in \; \mathrm{kTopSpc}\,, \tag{2.61}$$

which is clearly the case. (We show subscripts only for readability, all these objects $\Gamma_{(-)}$ are copies of Γ.) ☐

Lemma 2.3.3 (Universal principal groupoid is equivalent to the point)**.** *There is an equivalence* (1.38) *of topological groupoids between the universal* Γ*-principal groupoid (Def. 2.3.1) and the point* $* := \mathrm{Cnst}(*)$.

$$\mathbf{E}\Gamma \underset{\mathrm{htpy}}{\simeq} * \quad \in \; \mathrm{Grpd}(\mathrm{kTopSpc})\,.$$

Proof. The canonical choice of equivalence (1.38) is

$$\Big(\Gamma \times \Gamma \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \Gamma\Big) \underset{\exists!\,R}{\overset{L:=\mathrm{e}}{\rightleftarrows}} (* \rightrightarrows *)\,, \qquad L \circ R \xRightarrow{(-)^{-1}} \mathrm{id}, \qquad \mathrm{id} \;=\; R \circ L\,. \tag{2.62}$$

☐

These lemmas readily imply the following classical fact:

Proposition 2.3.4 (Topological realization of universal principal groupoid is universal principal bundle)**.**
For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$*, the topological realization* (1.66)

$$E\Gamma \;:=\; |\mathbf{E}\Gamma| \quad \in \; \mathrm{kTopSpc} \tag{2.63}$$

of the universal Γ*-principal groupoid* $\mathbf{E}\Gamma$ *(Ex. 2.3.1)*

(i) *is contractible*

$$E\Gamma \underset{\mathrm{htpy}}{\simeq} *\,; \tag{2.64}$$

(ii) *inherits a* Γ*-action* $\Gamma \circlearrowright E\Gamma \quad \in \; \Gamma\,\mathrm{Act}(\mathrm{kTopSpc})$ *whose quotient coprojection coincides with the topological realization of the groupoid quotient* (2.60)

$$E\Gamma \longrightarrow \Gamma\backslash(E\Gamma) \;\simeq\; \overset{\text{Milgram classifying space}}{|\mathbf{B}\Gamma| \;=:\; B\Gamma}\,; \tag{2.65}$$

(iii) *and is principal, in that the shear map is a homeomorphism:*

$$\Gamma \times E\Gamma \xrightarrow[\;\sim\;]{((-)\cdot(-),\,\mathrm{pr}_2)} E\Gamma \underset{B\Gamma}{\times} E\Gamma \qquad \in \; \mathrm{kTopSpc}\,.$$

Proof. (i) That $E\Gamma$ is contractible follows from Lem. 1.2.32 applied to Lem. 2.3.3.
(ii) That $E\Gamma \to B\Gamma$ is the Γ-quotient projection follows from Lem. 1.2.40 applied to Ex. 2.3.1, which yields that the nerve preserves the quotient projection, followed by the fact that the remaining topological realization of simplicial spaces (1.65) preserves all colimits.
(iii) That the Γ-action on $E\Gamma$ is principal over $B\Gamma$ follows from the principal bundle structure on $\mathbf{E}\Gamma$ (from Lem. 2.3.2) since topological realization preserves finite limits, by Lem. 1.2.31, thus inducing a functor (1.6) of internal formally principal bundles:

$$\mathrm{Frm}\Gamma\mathrm{PrnBdl}\,(\mathrm{Grpd}(\mathrm{kTopSpc})) \xrightarrow{\mathrm{Frm}(|-|)\,\mathrm{PrnBdl}(|-|)} \mathrm{Frm}\Gamma\mathrm{PrnBdl}(\mathrm{kTopSpc})\,.$$ □

Example 2.3.5 (Topological groupoid refinement of the Borel construction)**.** For $\mathrm{X} \circlearrowleft \Gamma \in \Gamma^{\mathrm{op}}\,\mathrm{Act}(\mathrm{kTopSpc}))$, its right Γ-action groupoid (Ex. 1.2.38) is isomorphic to the quotient, in the 1-category Grpd(kTopSpc), of the product of the constant groupoid on X (Ex. 1.2.2), regarded with its induced inverse left action, with the universal principal groupoid on Γ (Ex. 2.3.1), regarded with its canonical left Γ-action (2.59), by the resulting product action:

$$(\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X}) \;\simeq\; \frac{\mathrm{Cnst}(\mathrm{X})\times\mathbf{E}\Gamma}{\Gamma} \;\in\; \mathrm{Grpd}(\mathrm{kTopSpc})\,. \tag{2.66}$$

Since topological realization (1.66) preserves all quotients, finite products (by Lem. 1.2.31) and constant groupoids (by Ex. 1.2.30), this means that the topological realization of the right action groupoid on X is its *Borel construction*:

$$|\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X}| \;\simeq\; \frac{|\mathrm{Cnst}(\mathrm{X})|\times|\mathbf{E}\Gamma|}{|\Gamma|} \;\simeq\; \frac{\mathrm{X}\times E\Gamma}{\Gamma} \;\in\; \mathrm{kTopSpc}\,, \tag{2.67}$$

whose path-connected components are those of the plain quotient, by (1.5) and (2.64):

$$\pi_0\left(\frac{\mathrm{X}\times E\Gamma}{\Gamma}\right) \;\simeq\; \pi_0\left(\frac{\mathrm{X}}{\Gamma}\right) \;\in\; \mathrm{Set} \tag{2.68}$$

Example 2.3.6 (Higher classifying spaces)**.** For $A \in \mathrm{AbGrp}(\mathrm{kTopSpc})$, its classifying space (2.65) itself carries the structure of a topological group, by Ex. 1.2.35. Therefore the classifying space construction (2.65) may be iterated, and we write, recursively:

$$B^0 A := A$$

and

$$B^{n+1}A \;:=\; B(B^n A) \;=\; B\,|(B^n A)\rightrightarrows *| \quad\in\; \mathrm{kTopSpc}\,. \tag{2.69}$$

The Murayama-Shimakawa construction. In equivariant generalization of Ex. 2.3.1, we make the following Def. 2.3.7, which, upon its topological realization (Prop. 2.3.16 below), is a streamlined rendering of the construction due to [MS95, p. 1293 (5 of 7)], specialized to discrete equivariance groups G according to the remark at the bottom of [MS95, p. 1294 (6 of 7)] (which seems to be the greatest validated generality, as observed in [GMM17, Scholium 3.12]):

Definition 2.3.7 (Universal equivariant principal groupoid via Murayama-Shimakawa construction)**.**
(i) Given a discrete group $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$, consider any $G \circlearrowright \Gamma \in \mathrm{Grp}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$ (2.1) with its combined left and right multiplication action

$$(G\circlearrowright\Gamma) \circlearrowright \left(G\circlearrowright\Gamma^{L,R}\right) \circlearrowleft (G\circlearrowright\Gamma) \;\;\in\; \left((G\circlearrowright\Gamma)\times(G\circlearrowright\Gamma^{\mathrm{op}})\right)\mathrm{Act}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))\,.$$

By Ex. 1.2.43, we obtain the resulting left $(G\circlearrowright\Gamma_L)$-equivariant right $(G\circlearrowright\Gamma_R)$-action G-groupoid

$$(G\circlearrowright\Gamma)\circlearrowright\mathbf{E}(G\circlearrowright\Gamma) := (G\circlearrowright\Gamma)\circlearrowright\Big((G\circlearrowright\Gamma)\times(G\circlearrowright\Gamma) \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} (G\circlearrowright\Gamma)\Big) \;\in\; (G\circlearrowright\Gamma)\,\mathrm{Act}\big(\mathrm{Grpd}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))\big)\,. \tag{2.70}$$

Notice that after forgetting the $(G\circlearrowright\Gamma)$-action on (2.70) there still remains the G-action inherited from the G-action on Γ:

$$G\circlearrowright\mathbf{E}\Gamma \;=\; G\circlearrowright\left(\Gamma\times\Gamma \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \Gamma\right) \;\in\; \mathrm{Grpd}\,(G\,\mathrm{Act}(\mathrm{kTopSpc}))\,. \tag{2.71}$$

The left $(G \circlearrowright \Gamma)$-quotient of (2.70) is the equivariant delooping groupoid (Ex. 1.2.44):

$$G \circlearrowright \mathbf{E}\Gamma \xrightarrow{G \circlearrowright q} (G \circlearrowright \Gamma)\backslash \mathbf{E}\Gamma \simeq G \circlearrowright \mathbf{B}\Gamma \quad \in G\,\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))\,. \tag{2.72}$$

(ii) We say that the *universal equivariant principal groupoid* over $(G \circlearrowright \Gamma)$ is the G-equivariant mapping groupoid (Ex. 1.2.39) of $G \circlearrowright \mathbf{E}G$ (2.59) into $G \circlearrowright \mathbf{E}\Gamma$ (2.71) with its $(G \circlearrowright \Gamma)$-action inherited from (2.70):

$$(G \circlearrowright \Gamma) \circlearrowright \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma) \quad \in (G \circlearrowright \Gamma)\,\mathrm{Act}(\mathrm{Grpd}(G\,\mathrm{Act}(\mathrm{kTopSpc})))\,. \tag{2.73}$$

Remark 2.3.8 (Conceptual nature of universal equivariant principal groupoid)**.** It may be surprising on first sight – and is the key innovation of [MS95] – that Def. 2.3.7 involves not just the evident G-groupoid $G \circlearrowright \mathbf{E}\Gamma$ (2.71), but the mapping groupoid $G \circlearrowright \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)$ (2.73). Of course, one recognizes the fixed locus $\mathrm{Maps}(EG, \mathrm{X})^G \simeq \mathrm{X}^{\mathrm{h}G}$ as a model for the *homotopy fixed locus* (going back to [Th83], see also [Vi21]), but this is more of a hint than an explanation. Instead, the construction had been justified (in [MS95][GMM17]) *a posteriori* by proof that its topological realization does happen to satisfy (under suitable conditions on G and Γ) abstract properties known to characterize universal equivariant principal bundles. Below in Thm. 4.3.19 we give a more conceptual derivation of Def. 2.3.7, showing that this construction arises naturally as the *equivariant shape* (4.116) of the proper-equivariant *moduli stack* of equivariant principal bundles.

Using our internalization technology, we prove in Prop. 2.3.16 that the topological realization of Def. 2.3.7 is an equivariantly contractible equivariant principal bundle, as a formal consequence of the following sequence of Lemmas, which establish elementary properties of the underlying equivariant principal groupoid.

Lemma 2.3.9 (Universal equivariant principal groupoid is right action groupoid)**.** *The universal equivariant principal groupoid* (2.73) *is a left* $(G \circlearrowright \Gamma)$*-equivariant right action* $(G \circlearrowright \Gamma)$*-groupoid of the form of Ex. 1.2.38:*

$$(G \circlearrowright \Gamma) \circlearrowright \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma) \;\simeq\; \Big(\mathrm{Fnctr}(\mathbf{E}G, \mathbf{E}\Gamma)^{L,R} \times \mathrm{Maps}(G, \Gamma) \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathrm{Fnctr}(\mathbf{E}G, \mathbf{E}\Gamma)\Big)\,, \tag{2.74}$$

where

$$\mathrm{Fnctr}(\mathbf{E}G, \mathbf{E}\Gamma) := \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)_0$$

is equipped with the argument-wise induced left and right $(G \circlearrowright \Gamma)$*-actions.*

Proof. This is a direct unwinding of the definitions: The point to observe is only that the space of natural transformations out of any fixed functor $\mathbf{E}G \to \mathbf{E}\Gamma$ is already isomorphic to that of all possible component functions, which is

$$\mathrm{Maps}((\mathbf{E}G)_0, (\mathbf{E}\Gamma)_1) \simeq \mathrm{Maps}(G, \Gamma)\,.$$
□

Lemma 2.3.10 (Mapping groupoid into universal principal groupoid preserves canonical quotient)**.**
Let $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and let* $(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0) \in \mathrm{Grpd}(\mathrm{kTopSpc})$ *be topologically discrete and connected as a groupoid. Then pushforward* q_* *along the quotient coprojection* $\mathbf{E}\Gamma \xrightarrow{q} \Gamma\backslash\mathbf{E}\Gamma = \mathbf{B}\Gamma$ (2.60) *of the mapping groupoid* (1.43) *into* $\Gamma \circlearrowright \mathbf{E}\Gamma$ (2.59) *exhibits the quotient of the latter by its induced pointwise left* Γ*-action:*

$$\begin{tikzcd} \mathrm{Maps}((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathbf{E}\Gamma) \arrow[d] \arrow[dr, "q_*"] & \\ \Gamma\backslash \mathrm{Maps}((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathbf{E}\Gamma) \arrow[r, "\sim"] & \mathrm{Maps}((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0), \mathbf{B}\Gamma)\,. \end{tikzcd}$$

This statement corresponds to the last part of [GMM17, Thm. 2.7]. We mean to give a transparent proof:

Proof. By the assumption that the domain groupoid is connected and discrete, we may choose a base object x_0 and a continuous assignment $f_{(-)}$ of morphisms connecting all objects to this one:

$$x_0 \in \mathrm{X}_0\,, \qquad \begin{array}{rcl} f_{(-)} : \mathrm{X}_0 & \longrightarrow & \mathrm{X}_1 \\ x & \longmapsto & (x_0 \xrightarrow{f_x} x) \end{array}$$

Using this, we obtain the following component functions, where $q : \mathbf{E}G \to \mathbf{B}G$ is the coprojection (2.60) from Ex. 2.3.1:

$$
\begin{array}{ccc}
F & \longmapsto & (F_0(x_0),\, q \circ F) \\
\mathrm{Maps}\big((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0),\, \mathbf{E}\Gamma\big) & \begin{array}{c}\xrightarrow{\qquad\qquad\qquad} \\ \xleftarrow{\qquad\qquad\qquad}\end{array} & (\Gamma \rightrightarrows \Gamma) \times \mathrm{Maps}\big((\mathrm{X}_1 \rightrightarrows \mathrm{X}_0),\, \mathbf{B}\Gamma\big)\,. \\
\Big(\big(x_1 \xrightarrow{f} x_2\big) \mapsto \big(\gamma \cdot \phi(f_{x_1}) \to \gamma \cdot \phi(f_{x_2})\big)\Big) & \longleftarrow\!\shortmid & (\gamma,\, \phi)\,.
\end{array}
\tag{2.75}
$$

Unwinding the definitions readily reveals that this is a pair of inverse continuous funcors. Moreover, the top morphism is evidently Γ-equivariant, with respect to the canonical left Γ-action on the $(\Gamma \rightrightarrows \Gamma)$-factor and the trivial action on the remaining factor on the right. Therefore, passing the quotient operation by Γ along the top isomorphism implies the claim. $\square$

Lemma 2.3.11 (Quotient coprojection of universal equivariant principal groupoid)**.** *Let* $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ *be a discrete group and* $(G \curvearrowright \Gamma) \in \mathrm{Grp}(G\mathrm{Act}(\mathrm{kTopSpc}))$*, then pushforward along the G-equivariant quotient coprojection* $G \curvearrowright \mathbf{E}\Gamma \xrightarrow{G \curvearrowright q} G \curvearrowright \mathbf{B}\Gamma$ (2.72) *of the equivariant mapping groupoid* (1.73) *out of* $G \curvearrowright \mathbf{E}G$ (2.59) *exhibits the equivariant quotient of the latter by its induced left* Γ*-action:*

$$
\begin{tikzcd}
\text{universal equivariant principal groupoid} & G \curvearrowright \mathrm{Maps}(\mathbf{E}G,\, \mathbf{E}\Gamma) \arrow[d] \arrow[dr, "G \curvearrowright q_*"] & \\
\text{universal equivariant base groupoid} & (G \curvearrowright \Gamma) \backslash \mathrm{Maps}(\mathbf{E}G,\, \mathbf{E}\Gamma) \arrow[r, "\sim"] & G \curvearrowright \mathrm{Maps}(\mathbf{E}G,\, \mathbf{B}\Gamma)\,.
\end{tikzcd}
\tag{2.76}
$$

Proof. Since the functor which forgets the G-action is a left adjoint (Ex. 1.1.9) and hence preserves colimits such as the quotient considered here, and since $\mathbf{E}G$ is connected (by definition) and topologically discrete (by assumption on G), the statement follows with Lem. 2.3.10. $\square$

In equivariant generalization of Lem. 2.3.3, we have:

Lemma 2.3.12 (Universal equivariant principal groupoid is equivalent to the point)**.** *There is an equivalence of G-equivariant topological groupoids between the universal equivariant principal groupoid* (2.73) *from (Def. 2.3.7) and the point:*

$$
G \curvearrowright \mathrm{Maps}(\mathbf{E}G,\, \mathbf{E}\Gamma) \underset{\text{htpy}}{\simeq} G \curvearrowright * \quad \in G\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))\,.
\tag{2.77}
$$

In particular, for all subgroups $H \subset G$ *the H-fixed subgroupoid of the mapping groupoid is equivalent to the point:*

$$
\underset{H \subset G}{\forall} \quad \big(G \curvearrowright \mathrm{Maps}(\mathbf{E}G,\, \mathbf{E}\Gamma)\big)^H \underset{\text{htpy}}{\simeq} * \quad \in G\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big)\,.
$$

Proof. Since the G-action on Γ is by group automorphisms (Lem. 2.1.4) the equivalence (2.62) contracting the universal principal groupoid $\mathbf{E}\Gamma$ is clearly G-equivariant, so that it is also contractible as a G-groupoid (2.71)

$$
G \curvearrowright \mathbf{E}\Gamma \underset{\text{htpy}}{\simeq} * \quad \in \mathrm{Grpd}(G\mathrm{Act}(\mathrm{kTopSpc}))\,.
\tag{2.78}
$$

The required equivalence (2.77) is obtained as the image of this equivalence (2.78) under the equivariant mapping groupoid 2-functor $G \curvearrowright \mathrm{Maps}(\mathbf{E}G, -)$ (1.73). $\square$

Lemma 2.3.13 (Equivariant mapping groupoid out of universal principal groupoid preserves constant groupoids)**.** *For* $(G \curvearrowright \Gamma) \in \mathrm{Grp}(G\mathrm{Act}(\mathrm{kTopSpc}))$*, we have an isomorphism of group objects*

$$
G \curvearrowright \mathrm{Maps}(\mathbf{E}G,\, \mathrm{Cnst}(\Gamma)) \underset{\sim}{\xrightarrow{\ \mathrm{ev}_e\ }} G \curvearrowright \Gamma \quad \in \mathrm{Grp}\big(G\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))\big)
$$

between Γ *and the equivariant mapping groupoid (Ex. 1.2.39) out of* $G \curvearrowright \mathbf{E}G$ (2.59) *into the constant groupoid (Ex. 1.2.2) on* Γ*.*

Proof.

$$
\begin{aligned}
\mathrm{Maps}\big(\mathbf{E}G, \mathrm{Cnst}(\Gamma)\big) & \simeq \mathrm{Maps}\big(\tau_0(\mathbf{E}G), \Gamma\big) \quad \text{by Prop. 1.2.5} \\
& \simeq \mathrm{Map}(*, \Gamma) \\
& \simeq \Gamma\,.
\end{aligned}
$$

$\square$

In equivariant generalization of Lem. 2.3.2, we have:

Lemma 2.3.14 (Universal equivariant principal groupoid is formally principal)**.** *The quotient coprojection* (2.76) *exhibits the universal equivariant principal groupoid* (2.73) *as a formally principal bundle (Ntn. 1.0.25) internal to G-equivariant topological groupoids (Def. 1.2.37),*

$$(G \curvearrowleft \Gamma) \curvearrowleft \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma) \xrightarrow{q_*} G \curvearrowleft \mathrm{Maps}(\mathbf{E}G, \mathbf{B}\Gamma) \;\;\in\; \mathrm{Frm}(G \curvearrowleft \Gamma)\,\mathrm{PrnBdl}\Big(G\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big)\Big)\,, \tag{2.79}$$

in that the shear map is an isomorphism:

$$(G \curvearrowleft \Gamma) \times \big(G \curvearrowleft \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)\big) \xrightarrow[\sim]{\big((-)\cdot(-),\,\mathrm{pr}_2\big)} \big(G \curvearrowleft \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)\big) \underset{G \curvearrowleft \mathrm{Maps}(\mathbf{E}G,\, \mathbf{B}\Gamma)}{\times} \big(G \curvearrowleft \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)\big)\,.$$

Proof. First, observe that the coprojection (2.72) is formally $(G \curvearrowleft \Gamma)$-principal:

$$(G \curvearrowleft \Gamma) \curvearrowleft \mathbf{E}\Gamma \xrightarrow{q} G \curvearrowleft \mathbf{B}\Gamma \;\;\in\; \mathrm{Frm}(G \curvearrowleft \Gamma)\,\mathrm{PrnBdl}\Big(G\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big)\Big)\,. \tag{2.80}$$

This follows by Lem. 2.3.2 combined with Lem. 1.1.10, observing that the isomorphisms (2.61) are clearly G-equivariant (Lem. 2.1.4).

Now, since the equivariant mapping groupoid functor $G \curvearrowleft \mathrm{Maps}(\mathbf{E}G, -)$ is a right adjoint (1.73) it preserves finite limits (Prop. 1.0.8) and hence induces (1.6) a functor of formally principal bundles, which by Lem. 2.3.13 is an endo-functor:

$$\mathrm{Frm}(G \curvearrowleft \Gamma)\,\mathrm{PrnBdl}\Big(\mathrm{Grpd}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\Big) \xrightarrow{\mathrm{Map}(\mathbf{E}G,\,-)} \mathrm{Frm}(G \curvearrowleft \Gamma)\,\mathrm{PrnBdl}\Big(\mathrm{Grpd}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\Big)\,.$$

The image of (2.80) under this functor gives the required structure (2.80). □

After this series of lemmas, we may finally conclude, in equivariant generalization of Prop. 2.3.4:

Notation 2.3.15 (Murayama-Shimakawa construction)**.**
For $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ and $\Gamma \in \mathrm{Grp}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$, we denote the topological realization (1.66) with its induced G-action (Prop. 1.2.41) of the G-equivariant universal $(G \curvearrowleft \Gamma)$-principal groupoid $\mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)$ (Def. 2.3.7) by

$$G \curvearrowleft E(G \curvearrowleft \Gamma) \;:=\; G \curvearrowleft |\mathrm{Map}(\mathbf{E}G, \mathbf{E}\Gamma)| \;\;\in\; G\,\mathrm{Act}(\mathrm{kTopSpc})\,. \tag{2.81}$$

Proposition 2.3.16 (Murayama-Shimakawa construction as equivariant principal bundle)**.**
Let $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $\Gamma \in \mathrm{Grp}(G\,\mathrm{Act}(\mathrm{kTopSpc}))$.

(i) *Then the Murayama-Shimakawa construction* (2.81) *is G-equivariantly contractible*

$$G \curvearrowleft E(G \curvearrowleft \Gamma) \underset{\mathrm{htpy}}{\simeq} G \curvearrowleft * \,. \tag{2.82}$$

(ii) *Hence, in particular, it has contractible fixed loci*

$$\underset{H \subset G}{\forall} \; \big(E(G \curvearrowleft \Gamma)\big)^H \underset{\mathrm{htpy}}{\simeq} *\,, \tag{2.83}$$

and inherits a $(G \curvearrowleft \Gamma)$*-action*

$$(G \curvearrowleft \Gamma) \curvearrowleft E(G \curvearrowleft \Gamma) \;\;\in\; G\,\mathrm{Act}(\mathrm{kTopSpc})\,.$$

(iii) *The corresponding quotient coprojection*

$$E(G \curvearrowleft \Gamma) \longrightarrow \Gamma \backslash E(G \curvearrowleft \Gamma) \;=:\; B(G \curvearrowleft \Gamma) \tag{2.84}$$

is a G-equivariant Γ*-principal bundle (Def. 2.1.3)*

$$\begin{array}{ccc} \underset{G}{\curvearrowright} \mathrm{Map}\big(\mathbf{E}G, \mathbf{E}\Gamma\big) & & \underset{G}{\curvearrowright} \overbrace{\big|\mathrm{Map}(\mathbf{E}G, \mathbf{E}\Gamma)\big|}^{E(G \curvearrowleft \Gamma)} \\ \Big\downarrow{\scriptstyle q_*} & \xmapsto{\;|-|\;} & \Big\downarrow{\scriptstyle \Gamma\backslash(-)} \\ \underset{G}{\curvearrowright} \mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma) & & \underset{G}{\curvearrowright} \underbrace{\big|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)\big|}_{B(G \curvearrowleft \Gamma)} \end{array} \;\;\in\; \mathrm{Frm}(G \curvearrowleft \Gamma)\,\mathrm{PrnBdl}\big(G\,\mathrm{Act}(\mathrm{kTopSpc})\big)\,.$$

Proof. **(i)** The equivariant contraction is the image under Lem. 1.2.32 and Prop. 1.2.41 of the contraction as an equivariant groupoid from Lem. 2.3.3.
(ii) The quotient coprojection is the image of the coprojection of equivariant groupoids from Lem. 2.3.11, using that this is preserved first by the nerve operation (1.62), due to Lem. 1.2.40 with Lem. 2.3.9, and then by topological realization (1.65), which preserves all colimits.
(iii) Formal principality follows from that that of Lem. 2.3.14 under the functor which is induced (1.6) from the fact that topological realization preserves finite limits (Prop. 1.2.31) and using that it sends the constant groupoid $\mathrm{Cnst}(\Gamma)$ to its underlying topological space Γ (Ex. 1.2.30)

$$\mathrm{Frm}\Gamma\mathrm{PrnBdl}\big(G\mathrm{Act}(\mathrm{Grpd}(\mathrm{kTopSpc}))\big) \xrightarrow{\mathrm{Frm}(|-|)\mathrm{PrnBdl}(|-|)} \mathrm{Frm}\Gamma\mathrm{PrnBdl}\big(G\mathrm{Act}(\mathrm{kTopSpc})\big)\,. \qquad \square$$

Fixed loci of equivariant classifying spaces. For the purposes of equivariant homotopy theory, the key property of the above equivariant universal bundles are (the homotopy types of) their fixed

Lemma 2.3.17 (Fixed loci of base of universal equivariant principal groupoid)**.** *For $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ a discrete group and $\Gamma \in \mathrm{Grp}(G\mathrm{Act}(\mathrm{kTopSpc}))$, we have for each subgroup $H \subset G$ an equivalence* (1.38) *of $W(H)$-equivariant (Ntn. 1.1.1) topological groupoids (Def. 1.2.37):*

$$\begin{aligned} &\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)^H && (2.85)\\ &\underset{\text{htpy}}{\simeq} \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}(\Gamma \rtimes H)\big)_{\mathbf{B}H} && (2.86)\\ &\simeq \big(\mathrm{CrsHom}(H, H \circlearrowright \Gamma)_{\mathrm{ad}} \times \Gamma \rightrightarrows \mathrm{CrsHom}(H, H \circlearrowright \Gamma)_{\mathrm{ad}}\big) \qquad \in W(H)\,\mathrm{Act}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big) && (2.87)\end{aligned}$$

between

(a) *the H-fixed sub-groupoid of the universal equivariant base groupoid* (2.76)*, and*
(b) *the groupoid of sections of $\mathbf{B}(\Gamma \rtimes H)$ (Def. 1.2.17), hence (by Prop. 1.2.18)*
(c) *the conjugation groupoid of crossed homomorphisms (Ntn. 1.2.16).*

This statement corresponds to that of [GMM17, Thm. 4.14, Cor. 4.15]. We mean to give a detailed proof. Its ingredients are needed below in the proof of Prop. 2.3.19.

Proof. First, observe that a G-equivariant function (1.14) on $G^L \times G^L$ (1.10) is equivalently a general function on G:

$$\begin{array}{ccccc} \mathrm{Maps}\big(G^L \times G^L, \Gamma\big)^G & \xleftrightarrow{\sim} & \mathrm{Maps}(G, \Gamma) & \xleftrightarrow{\sim} & \mathrm{Maps}_{/G}(G, \Gamma \times G) \\ \big((g_1, g_2) \mapsto F(g_1, g_2)\big) & \longmapsto & \big(g \mapsto F(e, g)\big) & \longmapsto & \big(g \mapsto (F(e,g), g)\big) \\ \big((g_1, g_2) \mapsto g_1 \cdot f(g_1^{-1} \cdot g_2)\big) & \longleftarrow & \big(g \mapsto f(g)\big) & \longleftarrow & \big(g \mapsto (f(g), g)\big) \end{array}$$

Using the isomorphism of $\mathbf{E}G$ with the pair groupoid on G (Ex. 1.2.8), this applies to the component functions of functors and natural transformations to yield the claim for the special case $H = G$:

$$\mathrm{Maps}\big((G^L \times G^L \rightrightarrows G^L), \Gamma^\alpha \rightrightarrows *\big)^G \xrightarrow{\quad\sim\quad} \mathrm{Maps}_{/(G \rightrightarrows *)}\big(G \rightrightarrows *, \Gamma \rtimes_\alpha G \rightrightarrows *\big) \tag{2.88}$$

$$\left(\begin{tikzcd} e \arrow[d] & & \bullet \arrow[r, "\eta(e)"] \arrow[d, "F(e,g_1)"'] & \bullet \arrow[d, "F'(e,g_1)"] \\ g_1 \arrow[d] & \mapsto & \bullet \arrow[r, "\eta(g_1)"] \arrow[d, "F(g_1,g_2)"'] & \bullet \arrow[d, "F(g_1,g_2)"] \\ g_2 & & \bullet \arrow[r, "\eta(g_2)"'] & \bullet \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} \bullet \arrow[d, "g_1"] & & \bullet \arrow[r, "(\eta(e){,}\, e)"] \arrow[d, "(F(e{,}g_1){,}g_1)"'] & \bullet \arrow[d, "(F'(e{,}g_1){,}g_1)"] \\ \bullet \arrow[d, "g' := g_1^{-1} \cdot g_2"] & \mapsto & \bullet \arrow[r, "(\eta(e){,}e)"] \arrow[d, "(F(e{,}g'){,}g')"'] & \bullet \arrow[d, "(F'(e{,}g'){,}g')"] \\ \bullet & & \bullet \arrow[r, "(\eta(e){,}e)"'] & \bullet \end{tikzcd}\right)$$

Notice that it is the semidirect product group structure that does make the assignment on the right be functorial, given the data on the left, e.g.:

$$\big(F(e, g_1), g_1\big) \cdot \big(\eta(e), e\big) = \big(F(e, g_1) \cdot \alpha(g_1)(\eta(e)),\, g_1\big) = \big(F(e, g_1) \cdot \eta(g_1),\, g_1\big)\,.$$

In order to generalize this proof to arbitrary $H \subset G$, choose a section of the coset space projection

$$
\begin{tikzcd}
 & G \arrow[d, two heads] \\
G/H \arrow[ur, dashed, "\sigma"] \arrow[r, equal] & G/H\,,
\end{tikzcd}
\qquad \text{such that } \sigma([\mathrm{e}]) = \mathrm{e}\,,
\tag{2.89}
$$

which exists and is continuous by the assumption that G is discrete. This induces a decomposition of the underlying set of G into H-orbits

$$
G \;\simeq\; \bigcup_{[g]\in G/H} H\cdot\sigma([g]) \qquad \in \mathrm{Set}
$$

and thus implies that the pair groupoid $(G^L \times G^L \rightrightarrows G^L)$ is generated, under (i) composition, (ii) taking inverses and (iii) acting with elements of H, by the following two classes of morphisms:

$$
\{(\mathrm{e}\to h)\mid h\in H\}\,,\quad \{(\mathrm{e}\to\sigma([g]))\mid [g]\in G/H\}\qquad \subset\; G\times G\,.
\tag{2.90}
$$

Using this, consider the following expression for a pair of continuous functors:

$$
\begin{array}{ccc}
\scriptstyle ((g_1,g_2)\,\mapsto\, F(g_1,g_1)) & \longmapsto & \scriptstyle (h\,\mapsto\,(F(\mathrm{e},h),\,h)) \\
\mathrm{Maps}\big((G\times G\rightrightarrows G),\,(\Gamma\rightrightarrows *)\big)^H & \underset{R}{\overset{L}{\rightleftarrows}} & \mathrm{Maps}_{/(H\rightrightarrows *)}\big((H\rightrightarrows *),\,(\Gamma\rtimes H\rightrightarrows *)\big) \\
\scriptstyle \left(\begin{smallmatrix} (\mathrm{e},h) & \mapsto\; \phi(h) \\ (\mathrm{e},\sigma([g])) & \mapsto\; \mathrm{e}\end{smallmatrix}\right) & \longleftarrow & \scriptstyle (h\,\mapsto\,(\phi(h),h))\,,
\end{array}
\tag{2.91}
$$

where L is restriction along $(H\times H\rightrightarrows H)\hookrightarrow(G\times G\rightrightarrows G)$ followed by the isomorphism (2.88) for $G=H$, while R is given on morphisms as follows:

$$
\left(
\begin{tikzcd}
\mathrm{e} \arrow[d] & \mapsto & \bullet \arrow[r, "\eta(\mathrm{e}) := \eta(\bullet)"] \arrow[d, "\phi(h)"'] & \bullet \arrow[d, "\phi'(h)"] \\
h & \mapsto & \bullet \arrow[r, "\eta(h) := \alpha(h)(\eta(\bullet))"'] & \bullet \\
\mathrm{e} \arrow[d] & \mapsto & \bullet \arrow[r, "\eta(\mathrm{e}) := \eta(\bullet)"'] \arrow[d, "\mathrm{e}"'] & \bullet \arrow[d, "\mathrm{e}"] \\
\sigma([g]) & \mapsto & \bullet \arrow[r, "\eta(\sigma([g])) := \eta(\bullet)"'] & \bullet
\end{tikzcd}
\right)
\quad \overset{R}{\longleftarrow\!\!\shortmid} \quad
\left(
\begin{tikzcd}
\bullet \arrow[d, "h"] & \mapsto & \bullet \arrow[r, "(\eta(\bullet)\text{,}\mathrm{e})"] \arrow[d, "(\phi(h)\text{,}h)"] & \bullet \arrow[d, "(\phi'(h)\text{,}h)"] \\
\bullet & \mapsto & \bullet \arrow[r, "(\eta(\bullet)\text{,}\mathrm{e})"'] & \bullet\,.
\end{tikzcd}
\right)
\tag{2.92}
$$

One readily checks that this is well-defined and that $L\circ R = \mathrm{id}$. Therefore, it now suffices to give a continuous natural transformation $\mathrm{id} \overset{\eta}{\Rightarrow} R\circ L$. This is obtained by choosing for any functor F in the groupoid on the left of (2.91) a natural transformation $\eta_F : F \Rightarrow R\circ L(F)$, given by the following component function:

$$
\begin{tikzcd}
\mathrm{e} \arrow[d] & \mapsto & \bullet \arrow[r, "\eta_F(\mathrm{e}) := \mathrm{e}"] \arrow[d, "F(\mathrm{e}\text{,}h)"'] & \bullet \arrow[d, "(R\circ L(F))(\mathrm{e}\text{,}h) = F(\mathrm{e}\text{,}h)"] \\
h & \mapsto & \bullet \arrow[r, "\eta_F(h) := \mathrm{e}"] & \bullet \\
\mathrm{e} \arrow[d] & \mapsto & \bullet \arrow[r, "\eta_F(\mathrm{e}) := \mathrm{e}"'] \arrow[d, "F(\mathrm{e}\text{,}\sigma({[h]}))"'] & \bullet \arrow[d, "(R\circ L(F))(\mathrm{e}\text{,}\sigma({[g]})) = \mathrm{e}"] \\
\sigma([g]) & \mapsto & \bullet \arrow[r, "\eta_F(\sigma({[g]})) := F(\mathrm{e}\text{,}\sigma({[g]}))^{-1}\text{.}"'] & \bullet
\end{tikzcd}
$$

It only remains to see that this is indeed natural in F, which amounts to checking that for any H-equivariant natural transformation β from F to F' the following two squares on the right commute (in $\mathbf{B}\Gamma$):

$$
\begin{tikzcd}
F \arrow[r, "\eta_F"] \arrow[d, "\beta"] & R\circ L(F) \arrow[d, "R\circ L(\beta)"] \\
F' \arrow[r, "\eta_{F'}"] & R\circ L(F')
\end{tikzcd}
\quad : \quad
\left\{
\begin{tikzcd}
 & & \bullet \arrow[r, "\eta_F(h) = \mathrm{e}"] \arrow[d, "\beta(h)"] & \bullet \arrow[d, "(R\circ L(\beta))(h) = \alpha(h)(\beta(e))"] \\
h & \mapsto & \bullet \arrow[r, "\eta_{F'}(h) = \mathrm{e}"'] & \bullet \\
 & & \bullet \arrow[r, "\eta_F(\sigma({[g]})) = F(e\text{,}\sigma({[g]}))^{-1}"] \arrow[d, "\beta(\sigma({[g]}))"] & \bullet \arrow[d, "(R\circ L(\beta))(\sigma({[g]})) = \beta(\mathrm{e})"] \\
\sigma([g]) & \mapsto & \bullet \arrow[r, "\eta_{F'}(\sigma({[g]})) = F'(e\text{,}\sigma({[g]}))^{-1}"'] & \bullet
\end{tikzcd}
\right.
$$

But the top square commutes by the H-equivariance of β (1.14), the bottom one by the naturality of β (1.37). $\square$

Proposition 2.3.18 (Connected components of fixed loci in equivariant classifying spaces)**.**
(i) *Let*
(a) $G \in \mathrm{Grp}(\mathrm{FinSet}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ *be a finite group;*
(b) $\Gamma \in \mathrm{Grp}\big(G\mathrm{Act}(\mathrm{kTopSpc})\big)$*;*
(c) *such that* $\mathrm{CrsHom}(G, G \circlearrowright \Gamma)/\sim_{\mathrm{ad}}$ (1.49) *is a discrete space (e.g. via Prop. 1.2.20).*
Then for each subgroup $H \subset G$, the connected components of the H-fixed locus of the base space $G \circlearrowright B(G \circlearrowright \Gamma)$ (2.84) *of the universal G-equivariant Γ-principal bundle; are the non-abelian group cohomology in degree 1* (1.49) *of H with coefficients in the restricted action $H \circlearrowright \Gamma$:*

$$\pi_0\Big(\big(B(G \circlearrowright \Gamma)\big)^H\Big) \;\simeq\; H^1_{\mathrm{Grp}}(H, H \circlearrowright \Gamma) \;\in\; \mathrm{Set}\,. \tag{2.93}$$

(ii) *If, moreover,*
(d) Γ *is a Lie group and* $\alpha : G \to \mathrm{Aut}_{\mathrm{Grp}}(\Gamma)$ *restricts to the identity on the center of G,*
then the underlying topological space of the fixed locus itself is homotopy equivalent to

$$\big(B(G \circlearrowright \Gamma)\big)^H \underset{\mathrm{htpy}}{\simeq} \coprod_{\substack{[\phi]\in \\ H^1_{\mathrm{Grp}}(H,\, H \circlearrowright \Gamma)}} \frac{\Gamma/C_\Gamma(\phi) \times E\Gamma}{\Gamma} \;\in\; \mathrm{kTopSpc}\,, \tag{2.94}$$

hence (when the $\Gamma \to \Gamma/C_\Gamma(\phi)$ admits local sections...) to a disjoint union, indexed by classes $[\phi]$ in the group 1-cohomology set, of ordinary classifying spaces of the stabilizer groups of the cocycles ϕ:

$$\big(B(G \circlearrowright \Gamma)\big)^H \;\simeq\; \coprod_{\substack{[\phi]\in \\ H^1_{\mathrm{Grp}}(H,\, H \circlearrowright \Gamma)}} B(C_\Gamma(\phi)) \;\in\; \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}})\,.$$

Proof. First, observe that we have a homotopy equivalence of the fixed locus with the Borel construction (2.67) of the space of crossed homomorphisms $H \to \Gamma$ (1.46) by the Γ-action of crossed conjugations (1.47):

$$\big(B(G \circlearrowright \Gamma)\big)^H \underset{\mathrm{htpy}}{\simeq} \frac{\mathrm{CrsHom}(G, G \circlearrowright \Gamma)}{\Gamma} \;\in\; \mathrm{kTopSpc}\,. \tag{2.95}$$

obtained as the following composite:

$$\begin{aligned}
\big(B(G \circlearrowright \Gamma)\big)^H &\simeq \big(\Gamma \backslash \big|\mathrm{Maps}(\mathbf{E}G, G \circlearrowright \Gamma)\big|\big)^H && \text{by (2.84)} \\
&\simeq \big|\Gamma \backslash \mathrm{Maps}(\mathbf{E}G, \mathbf{E}\Gamma)\big|^H && \text{by Lem. 1.2.40 \& Lem. 2.3.9,} \\
&\simeq \big|\mathrm{Maps}(\mathbf{E}G, \mathbf{B}\Gamma)\big|^H && \text{by Lem. 2.3.11} \\
&\simeq \Big|\big(\mathrm{Maps}(\mathbf{E}G, \mathbf{B}\Gamma)\big)^H\Big| && \text{by Lem. 1.2.31} \\
&\underset{\mathrm{htpy}}{\simeq} \big|\mathrm{CrsHom}(H, H \circlearrowright \Gamma) /\!\!/_{\mathrm{ad}} \Gamma\big| && \text{by Lem. 2.3.17} \\
&\simeq \tfrac{\mathrm{CrsHom}(H, H \circlearrowright \Gamma) \times E\Gamma}{\Gamma} && \text{by Ex. 2.3.5}\,.
\end{aligned}$$

With this, the first claim arises by the following sequence of bijections:

$$\begin{aligned}
\pi_0\Big(\big(B(G \circlearrowright \Gamma)\big)^H\Big) &\simeq \pi_0\left(\tfrac{\mathrm{CrsHom}(H, H \circlearrowright \Gamma) \times E\Gamma}{\Gamma}\right) && \text{by (2.95)} \\
&\simeq \pi_0\left(\tfrac{\mathrm{CrsHom}(H, H \circlearrowright \Gamma)}{\Gamma}\right) && \text{by (2.68)} \\
&\simeq \tfrac{\mathrm{CrsHom}(H, H \circlearrowright \Gamma)}{\Gamma} && \text{by (iii)} \\
&\simeq H^1_{\mathrm{Grp}}(H, H \circlearrowright \Gamma) && \text{by (1.49)}\,,
\end{aligned}$$

and the second by the following homeomorphisms:

$$\begin{aligned} \big(B(G \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma)\big)^H &\simeq \frac{\mathrm{CrsHom}(H, H \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma) \times E\Gamma}{\Gamma} && \text{by (2.95)} \\ &\simeq \coprod_{\substack{[\phi] \in \\ H^1_{\mathrm{Grp}}(H, H \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma)}} \frac{\Gamma / C_{/\Gamma}(\phi) \times E\Gamma}{\Gamma} && \text{by Prop. 1.2.20}\,. \end{aligned}$$

□

Proposition 2.3.19 (Weyl group action on connected components of fixed loci in equivariant classifying space)**.** *Under the equivalence of Prop. 2.3.18 the residual $W(H)$-action on $(B(G \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma))^H$ (Ex. 1.1.13) is, on the set of connected components, the Weyl group action on non-abelian first group cohomology from Prop. 1.2.21:*

$$W(H) \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \pi_0\Big(\big(B(G \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma)\big)^H\Big) \;\simeq\; W(H) \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} H^1_{\mathrm{Grp}}(H, H \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma) \;\;\in W(H)\,\mathrm{Act}(\mathrm{Set})\,.$$

Proof. We may transport the action along the explicit equivalence established inside the proof of Lem. 2.3.17: For $n \in N(H)$ and $\phi \in \mathrm{CrsHom}(H, H \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma)$ we need to check that

$$[L(n \cdot (R\phi))] \;=\; [\phi_n] \;\;\in H^1_{\mathrm{Grp}}(H, H \mathbin{\rotatebox[origin=c]{180}{$\circlearrowright$}} \Gamma)\,, \tag{2.96}$$

where L and R are from (2.91), ϕ_n is from (1.57), and $n \cdot (-)$ is the conjugation action by $n \in G$ on $\mathrm{Maps}(\mathbf{E}G, \mathbf{B}\Gamma)$.

Notice in the case $n \in H \subset N(H)$ that the n-action on the H-fixed locus is trivial by definition, while that on H^1_{Grp} is trivial by (1.58) in Prop. 1.2.21, so that there is nothing further to be proven.

Therefore, we may assume now that n is not in H. Since L does not depend on the choice of section σ in (2.89), the bijection of connected components induced by L and R is independent of this choice, so that *for any n* we may choose any such section convenient for the analysis. We now choose a σ (2.89) that takes n^{-1} to be the representative of its coset class:

$$\sigma([n^{-1}]) \;:=\; n^{-1}.$$

With this choice, the definition of R (2.92) says that $R\phi$ sends morphisms (in $\mathbf{E}G$) between n^{-1} and the neutral element to the neutral element:

$$(R\phi)(\mathrm{e}, n^{-1}) \;=\; \mathrm{e}\,, \quad (R\phi)(n^{-1}, \mathrm{e}) \;=\; \mathrm{e}\,. \tag{2.97}$$

Now for any $h \in H$, we compute as follows:

$$\begin{aligned} L\big(n \cdot (R\phi)\big)(h) &= \big(n \cdot (R\phi)\big)(\mathrm{e}, h) && \text{by (2.91)} \\ &= \alpha(n)\big((R\phi)(n^{-1}, n^{-1} \cdot h)\big) && \text{by (1.13)} \\ &= \alpha(n)\big((R\phi)(\mathrm{e}, n^{-1} \cdot h)\big) && \text{by (2.97)} \\ &= \alpha(n^{-1} \cdot h \cdot n \cdot n)\big((R\phi)(n^{-1} \cdot h^{-1} \cdot n, n^{-1})\big) && \text{by } H\text{-equivariance of } R\phi \\ &= \alpha(n^{-1} \cdot h \cdot n \cdot n)\big((R\phi)(n^{-1} \cdot h^{-1} \cdot n, \mathrm{e})\big) && \text{by (2.97)} \\ &= \alpha(n)\big((R\phi)(\mathrm{e}, n^{-1} \cdot h \cdot n)\big) && \text{by } H\text{-equivariance of } R\phi \\ &= \alpha(n)\big(\phi(n^{-1} \cdot h \cdot n)\big) && \text{by (2.92)} \\ &= \phi_n(h) && \text{by (1.57)}\,. \end{aligned} \tag{2.98}$$

This manifestly implies the desired (2.96). □

Remark 2.3.20 (Fixed loci of equivariant classifying spaces for Lie groups)**.** Applied to the special case when both G and Γ are compact Lie groups, Prop. 2.3.18 with Prop. 2.3.19 reproduces most of the content of [LM86, Thm. 10, Thm. 11]. The statement of Prop. 2.3.18 is essentially that of [GMM17, Thm. 4.23] (though it seems some topological conditions are missing there).

Remark 2.3.21 (Pseudo-principal nature of fixed loci in universal equivariant principal bundle)**.** Since the total space of the equivariant universal bundle has contractible fixed loci (2.83), while Prop. 2.3.18 says that its base space has, in general, fixed loci with several connected components, it follows that all except one of these base components carry *empty* fibers:

$$
\begin{array}{ccccc}
\big(E(G \curvearrowright \Gamma)\big)^H & \simeq & E(\Gamma^H) & \sqcup & \varnothing \\
\downarrow & & \downarrow & & \downarrow \\
\big(B(G \curvearrowright \Gamma)\big)^H & \simeq & B(\Gamma^H) & \sqcup & \coprod\limits_{\substack{[\phi \neq e] \in \\ H^1_{\mathrm{Grp}}(H, H \curvearrowright \Gamma)}} B\big(C_\Gamma(\phi)\big)
\end{array}
$$

While a bundle with empty fibers cannot be principal, these fixed loci of the universal equivariant principal bundle are still $W(H)$-equivariant *formally principal* Γ^H-principal bundles (Rem. 2.1.8), by Cor. 2.1.9.

Homotopy types of ordinary classifying spaces. We begin by establishing some homotopy theoretic properties of ordinary (not equivariant) classifying spaces (2.65).

Remark 2.3.22 (Local triviality)**.** Prop. 2.3.4 does not establish local trivializability of the Milgram Γ-princial bundle (compare Rem. 2.1.1), and this cannot be expected to hold for general Γ. Prop. 2.3.24 below is classical only in the case when $E\Gamma \to B\Gamma$ has been shown to be locally trivial (using that then it is guaranteed to be a Serre fibration, by Lem. 1.3.2). While Prop. 2.3.4 follows, by Rem. 1.3.26 and also by general facts of ∞-topos theory discussed in [NSS12a], the following is a more topological derivation (though it does crucially invoke Prop. 1.3.27) which is useful in the present context.

Lemma 2.3.23 (Homotopy fibration property of Borel construction)**.** *Let* $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *be well-pointed (Ntn. 1.3.17) then for any* $\mathrm{X} \curvearrowleft \Gamma \in \Gamma^{\mathrm{op}}\,\mathrm{Act}(\mathrm{kTopSpc})$ *the Borel construction (Ex. 2.3.5) sits in a homotopy fiber sequence (Ntn. 1.3.3) of the form*

$$
\begin{array}{ccc}
\mathrm{X} & \xhookrightarrow{\ \mathrm{hofib}(b)\ } & (\mathrm{X} \times E\Gamma)/G \\
& & \downarrow{\scriptstyle p} \\
& & B\Gamma
\end{array}
\tag{2.99}
$$

Proof. By Ex. 2.3.5, the sequence (2.99) is the image under topological realization (Ntn. 1.2.28) of the sequence of simplicial topological spaces which, in turn, is the image under taking nerves (1.62) of the evident sequence of topological action groupoids:

$$
N\Big((\mathrm{X} \rightrightarrows \mathrm{X}) \hookrightarrow (\mathrm{X} \times \Gamma^{\mathrm{op}} \rightrightarrows \mathrm{X}) \twoheadrightarrow (\Gamma^{\mathrm{op}} \rightrightarrows *) \Big) \;=\; \big(\mathrm{X} \hookrightarrow \mathrm{X} \times \Gamma^{\times \bullet} \twoheadrightarrow \Gamma^{\times \bullet} \big). \tag{2.100}
$$

Therefore, the claim follows by Prop. 1.3.27 as soon as we verify the following conditions:

(i) All simplicial spaces in (2.100) are good (Def. 1.3.16).

(ii) For each $n \in \mathbb{N}$ we have that $\mathrm{X} \hookrightarrow \mathrm{X} \times \Gamma^{\times n} \to \Gamma^{\times}_n$ is a homotopy fiber sequence (Ntn. 1.3.3).

(iii) The morphism on the right of (2.100) satisfies the homotopy Kan fibration property (1.60).

Condition (i) follows by Prop. 1.3.21. Condition (ii) is immediate from the fact that the projections $\mathrm{X} \times \Gamma^{\times n} \to \Gamma^{\times n}$ are Serre fibrations (by Lem. 1.3.2) with ordinary fiber X. Condition (iii) follows by Ex. 1.2.27. □

Proposition 2.3.24 (Milgram bundle is homotopy fiber sequence)**.** *If* $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}}$ *is a well-pointed topological group (Def. 1.3.17), then the sequence* (2.65)

$$
\Gamma \xrightarrow[\in\,\mathrm{hofib}(q)]{} E\Gamma \xrightarrow{\ q\ } B\Gamma
$$

is a homotopy fiber sequence (Ntn. 1.3.3).

Proof. This is Lem. 2.3.23 for the case when $\mathrm{X} = \Gamma$. □

Lemma 2.3.25 (Borel construction for free locally trivial actions)**.** *If* $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $\Gamma \curvearrowright \mathrm{X} \in G\,\mathrm{Act}(\mathrm{kTopSpc})$ *are such that*

(i) *both underlying spaces are connected,* $\pi_0 = *$,

(ii) *The coprojection* $\mathrm{X} \xrightarrow{q} \mathrm{X}/G$ *is a locally trivial G-fiber bundle.*

Then the Borel construction (2.67) *is weakly homotopy equivalent to the plain quotient space:*

$$
(\mathrm{X} \times E\Gamma)/\Gamma \xrightarrow{\ \in\,\mathrm{WHmtpEq}\ } \mathrm{X}/\Gamma \quad \in \ \mathrm{kTopSpc}_{\mathrm{Qu}}\,.
$$

Proof. Observe that any local trivialization of $\mathrm{X} \xrightarrow{q} \mathrm{X}/G$ lifts to a local trivialization of $\mathrm{X} \times EG \to (\mathrm{X} \times EG)/G$ through the morphism that projects out the EG-factor. This makes the resulting commuting diagram a morphism of Serre fiber sequences and hence of homotopy fiber sequences (Ntn. 1.3.3). The statement now follows from the five-lemma (in its generality for possibly non-abelian groups) applied to the resulting morphism of long exact sequences of homotopy groups. □

Lemma 2.3.26 (Classifying spaces of realizations of homotopy fibers of simplicial topological groups)**.** *If*

$$\mathscr{H}_\bullet \xrightarrow[\simeq \mathrm{hofib}(p_\bullet)]{i_\bullet} \widehat{\mathscr{G}}_\bullet \xrightarrow[\in \mathrm{hKanFib}]{p_\bullet} \mathscr{G}_\bullet \qquad \in \mathrm{Grp}(\Delta\mathrm{kTopSpc})_{\mathrm{wellpt}} \tag{2.101}$$

is a sequence of morphisms simplicial topological groups such that, as indicated:
(i) *all three simplicial groups are well-pointed (Ntn. 1.3.17);*
(ii) *the underlying sequence of simplicial spaces is a homotopy fiber sequence (Ntn. 1.3.3);*
(iii) *the underlying morphism $p_\bullet$ is a homotopy Kan fibration (Def. 1.2.23);*
and in addition
(iv) *under topological realization* (1.65)*, $|p_\bullet|$ is surjective on connected components;*
then the image of (2.101) *under passage to classifying spaces $B(-)$* (2.65) *of topologically realized groups $|-|$* (1.65) *is still a homotopy fiber sequence* (1.76)*, now of topological classifying spaces:*

$$B|\mathscr{H}_\bullet| \xrightarrow[\simeq \mathrm{hofib}(B|p_\bullet|)]{B|i_\bullet|} B|\widehat{\mathscr{G}}_\bullet| \xrightarrow{B|p_\bullet|} B|\mathscr{G}_\bullet| \qquad \in \mathrm{kTopSpc}\,. \tag{2.102}$$

Proof. First, with assumption (i), Lem. 1.3.23 implies that the underlying simplicial topological spaces in (2.101) are good (Def. 1.3.16). Together with assumptions (ii) and (iii), this implies, by Prop. 1.3.27, that the topological realization of (2.102) is a homotopy fiber sequence (Ntn. 1.3.3):

$$|\mathscr{H}_\bullet| \xrightarrow[\simeq \mathrm{hofib}(|p_\bullet|)]{|i_\bullet|} |\widehat{\mathscr{G}}_\bullet| \xrightarrow{|p_\bullet|} |\mathscr{G}_\bullet| \qquad \in \mathrm{Grp}(\mathrm{kTopSpc})_{\mathrm{wellpt}}\,, \tag{2.103}$$

all of whose entries are well-pointed simplicial topological groups, by Prop. 1.3.25 with assumption (i).

Next, by condition (iv) with Ex. 1.2.26, this implies that the image of (2.103) under forming nerves (1.62) of delooping groupoids (Ex. 1.2.7) is the homotopy fiber sequence of a homotopy Kan fibration (Def. 1.2.23):

$$N(|\mathscr{H}_\bullet| \rightrightarrows 1) \xrightarrow[\in \mathrm{hofib}(|p_\bullet|)]{N(|i_\bullet| \rightrightarrows 1)} N(|\widehat{\mathscr{G}}_\bullet| \rightrightarrows 1) \xrightarrow[\in \mathrm{hKanFib}]{N(|p_\bullet|)} N(|\mathscr{G}_\bullet| \rightrightarrows 1) \qquad \in \Delta\mathrm{kTopSpc}_{\mathrm{good}}\,, \tag{2.104}$$

all of whose entries are good simplicial spaces (Def. 1.3.16), by Prop. 1.3.21.

Therefore Prop. 1.3.27 applies also to (2.104) and thus yields the claim (2.102). □

Proposition 2.3.27 (2-Cocycle from topological central extension)**.** *Let $A \overset{i}{\hookrightarrow} \widehat{\Gamma} \overset{p}{\twoheadrightarrow} \Gamma$ be a central extension* (1.70) *of topological groups such that*
(i) *$\widehat{\Gamma} \to \Gamma$ is a locally trivial A-fiber bundle;*
(ii) *all groups are well-pointed (Ntn. 1.3.17);*
(iii) *A and $\widehat{\Gamma}$ are connected topological groups.*
Then there is a homotopy fiber sequence (Ntn. 1.3.3) of classifying spaces (2.65) *and higher classifying spaces (Eq. 2.3.6) of this form:*

$$B\widehat{\Gamma} \xrightarrow[\simeq \mathrm{hofib}(c)]{} B\Gamma \xrightarrow{c} B^2 A \quad \in \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}})\,.$$

Proof. Consider the following image under topological realization (1.66) of the diagram (1.71) of topological 2-groups from Ex. 1.2.36:

$$\begin{tikzcd}
|\widehat{\Gamma} \rightrightarrows \widehat{\Gamma}| \arrow[r, "|(\mathrm{id},\mathrm{e}) \rightrightarrows \mathrm{id}|", "\simeq \mathrm{hofib}(|\mathrm{pr}_2 \rightrightarrows 1|)"'] \arrow[d, equal] & |\widehat{\Gamma} \times A \rightrightarrows \widehat{\Gamma}| \arrow[r, "|\mathrm{pr}_2 \rightrightarrows 1|"] \arrow[d, "|p \circ \mathrm{pr}_1 \rightrightarrows p|"', "\in \mathrm{W}"] & |A \rightrightarrows *| \arrow[d, equal] \\
\widehat{\Gamma} \arrow[r] & \Gamma & BA
\end{tikzcd} \quad \in \mathrm{Grp}(\mathrm{kTopSpc})\,,$$

where in the bottom row we have used (1.67) and (2.65). Observe that, as indicated:

(i) the top sequence is a homotopy fiber sequence (Ntn. 1.3.3), by Lem. 2.3.23;

(ii) the middle morphism is a weak homotopy equivalence, by Lem. 2.3.25.

It follows by Lem. 2.3.26 that the same two properties are enjoyed by the further image of this diagram under passage to classifying spaces:

$$\begin{tikzcd}
B|\widehat{\Gamma} \rightrightarrows \widehat{\Gamma}| \arrow[r, "B|(\mathrm{id},\mathrm{e}) \rightrightarrows \mathrm{id}|", "\simeq \mathrm{hofib}(B|\mathrm{pr}_2 \rightrightarrows 1|)"'] \arrow[d, equal] & B\left|\widehat{\Gamma} \times A \rightrightarrows \widehat{\Gamma}\right| \arrow[r, "B|\mathrm{pr}_2 \rightrightarrows 1|"] \arrow[d, "B|p \circ \mathrm{pr}_1 \rightrightarrows p|"', "\in \mathrm{W}"] & B|A \rightrightarrows *| \arrow[d, equal] \\
B\widehat{\Gamma} \arrow[r] & B\Gamma \arrow[r, dashed, "c"] & B^2 A
\end{tikzcd} \quad \in \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}}),$$

where the dashed morphism is the unique morphism making the right square commute in the homotopy category. Now the commutativity of the left square implies the claim. □

Example 2.3.28 (Homotopy fiber sequence of the circle group)**.** Regarding the real line with its additive group structure and its Euclidean topology as a topological group

$$\mathbb{R} \quad \in \mathrm{Grp}(\mathrm{kTopSpc}),$$

consider the defining sequence of the circle group U_1

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathrm{U}_1 .$$

Under passage to higher classifying spaces (Ex. 2.3.6) and using that topological realization preserves finite limits (Lem. 1.2.31) and hence fibers, this induces for each $n \in \mathbb{N}$ a fiber sequence of topological groups

$$B^n\mathbb{Z} \hookrightarrow B^n\mathbb{R} \twoheadrightarrow B^n\mathrm{U}_1 . \tag{2.105}$$

Prop. 2.3.27 says that there is a continuous map $B^n\mathrm{U}_1 \xrightarrow{c} B^{n+1}\mathbb{Z}$ which extends this to a homotopy fiber sequence of the form

$$B^n\mathbb{Z} \longrightarrow B^n\mathbb{R} \xrightarrow{\mathrm{hofib}(c)} B^n\mathrm{U}_1 \underset{\sim}{\xrightarrow{c}} B^{n+1}\mathbb{Z} \quad \in \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}}). \tag{2.106}$$

where on the left we are using (12) in order to identify the homotopy fiber of the homotopy fiber of c. But $\mathbb{R} \underset{\mathrm{htpy}}{\simeq} *$ implies $B^n\mathbb{R} \underset{\mathrm{htpy}}{\simeq} B^n 1 \simeq *$ for all $n \in \mathbb{N}$, from which the long exact sequence of homotopy groups induced by (2.105) implies that the map c is a weak homotopy equivalence, as indicated.

Example 2.3.29 (Homotopy fiber sequence of the projective unitary group)**.** From the defining fiber sequence (1.89) of projective unitary group from Ex. 1.3.19, Prop. 2.3.27 implies a homotopy fiber sequence of classifying spaces of the form

$$B\mathrm{U}_\omega \longrightarrow B\mathrm{PU}_\omega \underset{\sim}{\xrightarrow{c}} B^2\mathrm{U}_1 \underset{\sim}{\longrightarrow} B^3\mathbb{Z} \quad \in \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}}),$$

where on the right we have appended the equivalence (2.106). But since $\mathrm{U}_\omega \underset{\mathrm{htpy}}{\simeq} *$ (1.87) also $B\mathrm{U}_\omega \underset{\mathrm{htpy}}{\simeq}$, whence the corresponding long exact sequence of homotopy groups implies that c is an equivalence, as indicated.

In summary, the combined 2-cocycle maps from Prop. 2.3.27 witness that the classifying space of the infinite complex projective group is an Eilenberg-MacLane space concentrated in degree 3:

$$B\mathrm{PU}_\omega \simeq B^3\mathbb{Z} = K(\mathbb{Z},3) \quad \in \mathrm{Ho}(\mathrm{kTopSpc}_{\mathrm{Qu}}). \tag{2.107}$$

We discuss a stacky refinement of this situation below in Ex. 3.3.30.

Part III

In cohesive ∞-stacks

Chapter 3

Equivariant ∞-topos theory

We recall and develop some background in modal and specifically in cohesive and globally equivariant ∞-topos theory that are needed below in chapter 4. Some of the following material is recalled from [Lur09][Sc13] [SS20-Orb] [FSS20-TCD], but several statements are new or at least not readily quotable from the literature. Nevertheless, the expert reader may want to skip this chapter and just refer back to it as need be.

– §3.1: Abstract homotopy theory (presentable ∞-categories).
– §3.2: Geometric homotopy theory (general ∞-toposes).
– §3.3: Cohesive homotopy theory (cohesive ∞-toposes).

3.1 Abstract homotopy theory

We record and develop some facts of the abstract homotopy theory of presentable ∞-categories in the guise of combinatorial simplicial model category theory. Then we highlight some aspects of the homotopy theory of simplicial transformation groups.

– §3.1.1: The homotopy 2-category of presentable ∞-categories.
– §3.1.2: Homotopy theory of simplicial transformation groups.

3.1.1 The homotopy 2-category of presentable ∞-categories

Besides the original [Qu67], standard textbook accounts of model category theory include [Ho99][Hi02][Lur09, A.2]. For review streamlined towards our context and applications see [FSS20-TCD, §A].

We briefly put this in perspective:

The idea of abstract homotopy theory. An *abstract homotopy theory* is meant to be a kind of category which adheres to what we may call the *higher gauge principle*, where instead of asking if any two parallel morphisms are equal one must ask whether there is a *gauge transformation* between them – here called a *homotopy* – and then for any two such homotopies one is to ask for *higher* gauge transformations, namely higher homotopies between these; and so forth. See diagrams (3) and (7) in §0.2.

Model categories. This informal idea had famously been formalized by the notion of *model categories* [Qu67], among which the "combinatorial" ones have come to be understood (see Rem. 3.1.6 below) as presenting exactly the "presentable" homotopy theories [Lur09], essentially meaning: those that have any relevance in practice. Since model categories *present* homotopy theories in terms of ordinary categories equipped with an ingenious construction principle for higher homotopies, they have proven to serve as useful toolboxes for operationally handling abstract homotopy in practice. In fact, those combinatorial model categories which are also locally cartesian closed (in an evident model category-theoretic sense) are so very well-adapted to this task that their presentation of homotopy theories can be entirely mechanized via a kind of programming language now known as *homotopy type theory* ([UFP13][Sh19]).

Homotopy theory of homotopy theories. But in the all-encompassing spirit of the gauge principle, there ought to be a *homotopy theory of homotopy theories*, necessary for making gauge-invariant sense of crucial notions such as that reflecting upon sub-homotopy theories of objects with special qualitative properties. By analogy (in fact: by de-categorification), ordinary categories form a ("very large") *2-category* Cat which provides the formal context ([Gry74]) for making invariant sense of notions such as that of reflective subcategories.

Accordingly, it eventually came to light that homotopy theories may be understood as homotopy-theoretic categories, now technically known as $(\infty,1)$*-categories* (or ∞*-categories*, for short, see [Lur09][Ci19][Rie19][Lan21] for introduction), whose *homotopy theory of homotopy theories* is a ("very large") homotopical 2-category Cat_∞, hence an "$(\infty,2)$-category". While it transpired that the resulting $(\infty,1)$-category theory formally behaves much along the lines familiar from ordinary category theory, thus providing the expert with a powerful way of effectively reasoning inside it, the sheer volume of technical detail needed to rigorously set up the $(\infty,2)$-category of $(\infty,1)$-categories, in any of its multitude of interdependent models, can be prohibitive.

The homotopy 2-category of homotopy theories. However, as highlighted in [RV15], this is often more than is necessary in practice, as much of the relevant formal theory of homotopy theories is reflected already within just the *homotopy 2-category* $\mathrm{Ho}_2(\mathrm{Cat}_\infty)$ that is obtained from the $(\infty,2)$-category by identifying gauge equivalent 2-morphisms [RV20][RV21]. In particular, adjunctions (10) between homotopy theories, such as the above-mentioned reflections onto sub-theories, may be fully understood already in $\mathrm{Ho}_2(\mathrm{Cat}_\infty)$ [RV15, §4.4.5].

The homotopy 2-category of combinatorial model categories. We observe here that this opens the door to short-cutting the rigorous construction of ∞-category theory in a dramatic way, if the homotopy 2-category of homotopy theories has a usefully direct construction from model category theory. This is indeed the case, at least for the practically relevant *presentable* homotopy theories (Prop. 3.1.6 below). The homotopy 2-category of presentable homotopy theories turns out to be simply the *2-localization* of the 2-category of combinatorial model categories at the Quillen equivalences, i.e. the 2-category obtained from that of combinatorial model categories by universally forcing its Quillen equivalences to become actual homotopy equivalences. This is a technical theorem (recently proven by [Pav21], inspired by [Ren06]) if one starts with existing definitions of $(\infty,1)$-categories. But we may thus take it to be the very definition of what might be called the *formal category theory of homotopy theories*:

$$\underset{\substack{\text{homotopy 2-category of}\\ \text{presentable homotopy theories}}}{\mathrm{Ho}_2(\mathrm{PresCat}_\infty)} \;:=\; \underset{\substack{\text{2-localization 2-category of}\\ \text{combinatorial model categories}}}{\mathrm{Loc}_2^{\mathrm{Qu}}(\mathrm{CombMdlCat})}\,.$$

This definition turns out to give a powerful but convenient handle on abstract homotopy theory.

Notation 3.1.1 (Category of simplicial sets)**.** We write $\Delta \in \mathrm{Cat}$ for the simplex category, and $\Delta\mathrm{Set} := \mathrm{PSh}(\Delta, \mathrm{Set})$ for the category of simplicial sets (see e.g. [Fr12][GJ09, §1]). For $n \in \mathbb{N}$ we write $[n] \in \Delta$ and then $\Delta[n] := y([n]) \in \Delta\mathrm{Set}$ for the objects of the simplex category, regarded as the standard simplicial simplices.

Remark 3.1.2 (Cartesian closed structure on simplicial sets)**.** $\Delta\mathrm{Set}$ being a category of presheaves means that the product of simplicial sets exists and is given for $X, Y \in \Delta\mathrm{Set}$ by

$$(X \times Y)_n \;\simeq\; X_n \times Y_n\,.$$

Notation 3.1.3 (Simplicial categories)**.**
(i) All categories in the following are simplicial categories ([Hi02, §9.1.1]), i.e., enriched ([Ke82][Bo94I, §6]) in the category of simplicial sets (Ntn. 3.1.1) with respect to its cartesian monoidal structure (Ntn. 3.1.2).

We write $\Delta\mathrm{Cat}$ for the large category of simplicial categories.

(ii) This means, in particular, that for any pair of objects of some $\mathscr{C} \in \Delta\mathrm{Cat}$, we have their *simplicial hom-complex*, to be denoted

$$X, A \in \mathscr{C} \;\;\vdash\;\; \mathscr{C}(X,A) \in \Delta\mathrm{Set}\,. \tag{3.1}$$

(iii) For $\mathscr{C} \in \Delta\mathrm{Cat}$, its *homotopy category* $\mathrm{Ho}(\mathscr{C})$ is the 1-category with the same objects and with hom-sets the connected components of the hom-complexes (3.1):

$$\mathrm{Ho}(\mathscr{C})(X,A) \;:=\; \pi_0\big(\mathscr{C}(X,A)\big)\,. \tag{3.2}$$

Notation 3.1.4 (2-Category of simplicial combinatorial model categories)**.**
(i) We write $\mathrm{CombMdlCat}_\Delta$ for the very large strict 2-category whose

(a) objects are are combinatorial model categories ([Du01b, Def. 2.1][Lur09, §A.2.6] review in [Ra14]) that are also simplicial model categories ([Qu67, §II.2][Hi02, §9.1.5][Lur09, Ex. A.3.1.4]);

(b) morphisms are left Quillen functors, to be denoted

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathscr{D} \arrow[l, shift left, "\exists R"] \end{tikzcd} ;$$

(iii) 2-morphisms are natural transformations between these left adjoints

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, bend right, "L'"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{D} \end{tikzcd} .$$

Definition 3.1.5 (Homotopy 2-category of presentable homotopy theories)**.** We write

$$\mathrm{CombMdlCat}_\Delta \xrightarrow{\mathrm{Loc}_\Delta^{\mathrm{WEqs}}} \mathrm{Loc}_2^{\mathrm{QuEq}}(\mathrm{CombMdlCat}_\Delta) \tag{3.3}$$

for the 2-localization [Pr96, §2][Ren06, §1.2] of the 2-category $\mathrm{CombMdlCat}_\Delta$ (Ntn. 3.1.4) at the class of left Quillen equivalences.

This localization exists by [Ren06, §2.3].[1]

Proposition 3.1.6 (Recovering the homotopy 2-category of presentable ∞-categories [Pav21])**.** *The 2-localization of the 2-category of simplicial combinatorial model categories in Def. 3.1.5 is equivalent to the homotopy 2-category of the* $(\infty,2)$*-category* $\mathrm{PresCat}_\infty$ *of presentable* ∞*-categories with left adjoint* ∞*-functors, according to [Lur09, §5]:*

$$\mathrm{Loc}_2^{\mathrm{QuEq}}(\mathrm{CombMdlCat}_\Delta) \simeq \mathrm{Ho}_2(\mathrm{PresCat}_\infty) . \tag{3.4}$$

Remark 3.1.7. Prop. 3.1.6 is the culmination of several well-known results:

(i) That presentable ∞-categories in the sense of [Lur09, §5.5], are, up to equivalence, the simplicial localizations of combinatorial model categories is [Lur09, Prop. A.3.7.6][Ci19, Thm. 7.11.16, Rem. 7.11.17].

(ii) That the image under simplicial localization of left Quillen functors between combinatorial model categories are left adjoint ∞-functors between these ∞-categories in the sense of [Lur09, §5.2], is [MG15]. More specifically, left Quillen functors between left proper combinatorial model categories functorially lift (by [BR14, Prop. A.3]) to simplicial Quillen adjunctions between Quillen-equivalent simplicial model categories, and these in turn lift functorially (by [Lur09, Prop. 5.2.4.6]) to adjunctions between the corresponding quasi-categories.

(iii) That every equivalence of presentable ∞-categories arises from a composite of (such simplicial localizations of) simplicial Quillen adjunctions between simplicial combinatorial model categories is claimed in [Lur09, Rem. A.3.7.7].

∞-Groupoids. In higher analogy to how Set is the archetypical category, the archetypical ∞-category is that of ∞-groupoids (e.g. [Lur09, §1.1.2]):

Notation 3.1.8 (Kan-Quillen model category of simplicial sets)**.** The classical Kan-Quillen model category of simplicial sets ([Qu67, §II.3][GJ09]), regarded as a simplicial combinatorial model category (Ntn. 3.1.4), we denote

$$\Delta\mathrm{Set}_{\mathrm{Qu}} \in \mathrm{CombMdlCat}_\Delta . \tag{3.5}$$

[1]Considered in [Ren06] is the 2-localization of the category of all combinatorial model categories, but by [Du01b, Thm 1.1] every combinatorial model category is Quillen equivalent to a (left proper) simplicial combinatorial model category, so that the above localization (3.3) exists equivalently.

Remark 3.1.9 (Simplicial tensoring Quillen adjunction)**.** For every $\mathscr{C} \in \mathrm{CombMdlCat}_\Delta$ (Ntn. 3.1.4) and $X \in \mathscr{C}^{\mathrm{co}}$, we have a morphism $\Delta\mathrm{Set}_{\mathrm{Qu}} \xrightarrow{X} \mathscr{C}$ in $\mathrm{CombMdlCat}_\Delta$, namely the left Quillen functor given by the tensoring (copowering) of the simplicial model category $\mathscr{C}$ over simplicial sets (Ntn. 3.1.8):

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift right=1ex, "\mathscr{C}(X,-)"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \Delta\mathrm{Set}_{\mathrm{Qu}} \arrow[l, shift right=1ex, "(-)\cdot X"'] \end{tikzcd} .$$

Notation 3.1.10 (∞-Groupoids)**.** We denote the homotopy theory (Def. 3.1.5) presented by the classical Kan-Quillen model category of simplicial sets (Ntn. 3.1.8) by

$$\mathrm{Grpd}_\infty \;:=\; \mathrm{Loc}_\Delta^{\mathrm{WEqs}}(\Delta\mathrm{Set}_{\mathrm{Qu}}) \;\in\; \mathrm{Ho}_2(\mathrm{PresCat}_\infty)\,. \tag{3.6}$$

Definition 3.1.11 (Hom ∞-groupoid)**.** For $\mathbf{C} \simeq \mathrm{Loc}_\Delta^{\mathrm{WEqs}}(\mathscr{C}) \in \mathrm{Ho}_2(\mathrm{PresCat}_\infty)$ (Def. 3.1.6) and for $X, A \in \mathscr{C}_{\mathrm{fib}}^{\mathrm{cof}}$ two bifibrant objects, we write

$$\mathbf{C}(X,A) \;:=\; \mathscr{C}(X,A) \quad \in \Delta\mathrm{Set}_{\mathrm{Qu}} \xrightarrow{\eta^{\mathrm{Loc}_\Delta^{\mathrm{W}}}} \mathrm{Grpd}_\infty$$

for the image of their simplicial hom-complex, regarded as an ∞-groupoid via Ntn. 3.1.10.

In fact, it suffices that the domain be cofibrant and the codomain fibrant to obtain the homotopy type of the hom-∞-groupoid, as shown by the following standard observation:

Lemma 3.1.12 (Hom ∞-groupoid from cofibrant domains into fibrant codomains)**.** *For* $\mathscr{C} \in \mathrm{MdlCat}_\Delta$*, let* $X \in \mathscr{C}^{\mathrm{cof}}$ *be a cofibrant object and* $A \in \mathscr{C}_{\mathrm{fib}}$ *be a fibrant object, and consider any fibrant replacement* $X \underset{\in \mathrm{Cof}\cap\mathrm{W}}{\xrightarrow{q_X}} PX$ *and cofibrant replacement* $QA \underset{\in \mathrm{Fib}\cap\mathrm{W}}{\xrightarrow{p_A}} A$*. Then the canonical comparison map*

$$\mathscr{C}(X,A) \underset{\in \mathrm{W}}{\xrightarrow{\mathscr{C}(q_X, p_A)}} \mathscr{C}(PX, QA) \;=\; \mathbf{C}(PX, QA) \;\in\; \mathrm{Grpd}_\infty$$

is an equivalence to the hom-∞-groupoid from Def. 3.1.11

Proof. Since p_X is an acylic cofibration and q_A is an acylic fibration, the axioms) (e.g. [Lur09, Rem. A.3.6.1 (2')]) of simplicially enriched model categories imply that both morphisms in the the following factorization are equivalences:

$$\mathscr{C}(p_X, q_A) \;:\; \mathscr{C}(X,A) \underset{\in \mathrm{W}\cap\mathrm{Fib}}{\xrightarrow{\mathscr{C}(\mathrm{id}_X, q_A)}} \mathscr{C}(PX, QA) \underset{\in \mathrm{W}\cap\mathrm{Fib}}{\xrightarrow{\mathscr{C}(q_X, \mathrm{id}_A)}} \mathscr{C}(PX, QA)\,.$$

Hence the composite is an equivalence. ☐

Proposition 3.1.13 (∞-Groupoids form a cartesian closed ∞-category)**.** *For* $S \in \mathrm{Grpd}_\infty$ *(Ntn. 3.1.10), there is a pair of adjoint ∞-functors* (10) *of the form*

$$\begin{tikzcd} \mathrm{Grpd}_\infty \arrow[r, shift right=1ex, "\mathrm{Grpd}_\infty(S,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Grpd}_\infty \arrow[l, shift right=1ex, "S\times(-)"'] \end{tikzcd} . \tag{3.7}$$

Notation 3.1.14 (Objects and equivalences in ∞-categories)**.** For $\mathbf{C} = \mathrm{Loc}_\Delta^{\mathrm{W}}(\mathscr{C})$ (Def. 3.1.5),

(i) we say that the homotopy 2-category of $\mathbf{C}$ is

$$\mathrm{Ho}(\mathbf{C}) \;:=\; \mathrm{Ho}(\mathrm{PresCat}_\infty)(\mathrm{Grpd}_\infty, \mathbf{C})\,;$$

(ii) we say that objects $X \in \mathscr{C}$ are *presentations* for their images, up to isomorphism, in the homotopy category of the model category $\mathscr{C}$ which we may identify with the homotopy category of $\mathscr{C}$

$$X \in \mathscr{C} \xrightarrow{\mathrm{Loc}^{\mathrm{WEqs}}} \mathrm{Ho}(\mathscr{C}) =: \mathrm{Ho}(\mathbf{C})\,. \tag{3.8}$$

Example 3.1.15 (Points of ∞-Groupoids)**.** All $S \in \mathrm{Grpd}_\infty$ (Ntn. 3.1.10) are equivalent to the hom-∞-groupoid (Def. 3.1.11) into themselves out of the point:

$$S \simeq \mathrm{Grpd}_\infty(*, S) \quad \in \mathrm{Grpd}_\infty\,. \tag{3.9}$$

Example 3.1.16 (Monomorphisms of ∞-groupoids)**.** A morphism of ∞-groupoids is a monomorphism (6) precisely if it is fully faithful (8), hence if it is, up to equivalence, the inclusion of a disjoint summand of connected components:

$$S \hookrightarrow S' \qquad \Leftrightarrow \qquad S' \simeq S \sqcup S' \setminus S \qquad \in \mathrm{Grpd}_\infty. \tag{3.10}$$

Moreover, by the homotopy-pullback definition (6) of monomorphisms, a morphism f in any ∞-category $\mathbf{C}$ is a monomorphism iff the induced map on hom-∞-groupoids out of any object is a monomorphism of ∞-groupoids (3.10):

$$A \overset{i}{\hookrightarrow} B \in \mathbf{C} \qquad \Leftrightarrow \qquad \underset{X \in \mathbf{C}}{\forall}\ \mathbf{C}(X, A) \xhookrightarrow{\mathbf{C}(X,i)} \mathbf{C}(X, B) \quad \in \mathrm{Grpd}_\infty\,.$$

Notation 3.1.17 (1-Groupoids)**.** Let $\mathscr{C}$ be a category with finite products. For $G \in \mathrm{Grp}(\mathscr{C})$, we denote

- its *delooping groupoid* by:
$$(G \rightrightarrows *) \ \in \ \mathrm{Grpd}(\mathscr{C})\,; \tag{3.11}$$
- its *action groupoid* of right multiplication action on itself by:
$$(G \times G \rightrightarrows G) \ :=\ \Big(G \times G \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} G\Big) \ \in\ \mathrm{Grpd}(\mathscr{C})\,. \tag{3.12}$$

More generally, for $G \circlearrowright X \ \in\ G\mathrm{Act}(\mathscr{C})$ a left action

$$\begin{array}{ccc} G \times X & \xrightarrow{\text{left action}} & X\,, \\ (g,x) & \longmapsto & g \cdot x \end{array} \qquad\qquad \begin{array}{ccc} X \times G & \xrightarrow{\text{induced right action}} & X \\ x, g & \longmapsto & x \cdot g \ :=\ g^{-1}\cdot x \end{array}$$

we denote

- its *right action groupoid* by
$$(X \times G \rightrightarrows X) \ :=\ \Big(X \times G \underset{(-)\cdot(-)}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\Big) \ \in\ \mathrm{Grpd}(\mathscr{C})\,, \tag{3.13}$$

  whose nerve hence has 2-cells of the following form:

$$N(X \times G \rightrightarrows X)_2 \quad = \quad \left\{ \begin{tikzcd} & x \cdot g_1 \arrow[dr, "{(x\cdot g_1, g_0)}"] \arrow[d, phantom, "{\overset{\|}{(x, g_1, g_0)}}\ \Downarrow"] & \\ x \arrow[ur, "{(x, g_1)}"] \arrow[rr, "{(x, g_1\cdot g_0)}"'] & & x \cdot g_1 \cdot g_0 \end{tikzcd} \ \middle|\ \begin{array}{l} x \in X, \\ g_1, g_0 \in G \end{array} \right\}.$$

For $\mathscr{C} = \mathrm{Set}$, we denote the images of these groupoids in Grpd_∞ (3.6) as follows:

$$\begin{array}{ccc} \mathrm{Grpd}(\mathrm{Set}) & \xrightarrow{\quad N \quad} & \mathrm{Grpd}_\infty \\ (X \times G \rightrightarrows G) & \longmapsto & X /\!\!/ G \\ (G \rightrightarrows *) & \longmapsto & * /\!\!/ G =: BG\,. \end{array}$$

Limits and colimits. We record some facts about homotopy (co-)limits (e.g. [DHKS04]) that we will need.

Example 3.1.18 (Homotopy pullback preserves projections out of Cartesian products)**.** We have the following diagram:

$$\begin{tikzcd} X \times A \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\scriptstyle(\mathrm{hbp})"] & X \times B \arrow[d, "\mathrm{pr}_2"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Since homotopy limits commute over each other, this homotopy Cartesian diagram is the objectwise product of the following two homotopy Cartesian diagrams:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\text{(hbp)}"] & X \arrow[d] \\ * \arrow[r] & *, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\text{(hbp)}"] & B \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

□

Example 3.1.19 (Colimits over simplicial diagrams of 0-truncated objects are coequalizers (e.g., [Rie14, Ex. 8.3.8])). The colimit of a simplicial diagram of 0-truncated objects

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathbf{H}_{\leq 0}$$

is the coequalizer of the first two face maps, hence the quotient of the set X_0 by the equivalence relation generated by X_1:

$$\underset{\longrightarrow}{\lim} X_\bullet \;\simeq\; \mathrm{coeq}\big(X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0\big) \;=:\; X_0/X_1 \,.$$

Free homotopy sets. We will need to rephrase some statements about homotopy groups, hence about homotopy classes of pointed maps of n-spheres, in terms of *free homotopy sets*, namely homotopy classes of unconstrained (unpointed) maps out of n-spheres.

Lemma 3.1.20 (Detecting weak homotopy equivalences via free homotopy sets [MMSu84, Thm. 2]). *A map $X \xrightarrow{f} Y \in (\mathrm{Grpd}_\infty)_{\geq 1}$ of connected ∞-groupoids is an equivalence, namely a weak homotopy equivalence*

$$\underset{n\in\mathbb{N}_+}{\forall} \quad \pi_n(X) \xrightarrow[\sim]{f_*} \pi_n(Y)\,,$$

if and only if it induces
(a) *isomorphisms on all free homotopy sets and*
(b) *a surjection on homotopy classes of maps out of the wedge of circles indexed by (the underlying set of) $\pi_1(Y)$:*

$$\underset{n\in\mathbb{N}_+}{\forall} \quad \tau_0\,\mathrm{Map}(S^n, X) \xrightarrow[\sim]{f_*} \tau_0\,\mathrm{Map}(S^n, Y) \qquad \text{and} \qquad \tau_0\,\mathrm{Map}\big(\underset{\pi_1(Y)}{\vee} S^1, X\big) \xrightarrow{f_*} \!\!\!\!\rightarrow \tau_0\,\mathrm{Map}\big(\underset{\pi_1(Y)}{\vee} S^1, Y\big)\,.$$

Lemma 3.1.21 (Truncation and n-fold free loop spaces). *For $X \in \mathrm{Grpd}_\infty$, and $n \in \mathbb{N}$, the following are equivalent:*

(i) *For all $d > n$ and all $x \in X$, we have $\pi_d(X,x) = *$, that is, X is n-truncated, $\tau_n(X) \simeq X$.*

(ii) *For all $d > n$ the point evaluation map is a weak equivalence:* $\mathrm{Map}(S^{d+1}, X) \xrightarrow[\in \mathrm{WHmtpEq}]{(\mathrm{pt}_d)^*} X\,.$

(iii) *For all $d > n$ the equatorial evaluation map is a weak equivalence:* $\mathrm{Map}(S^{d+1}, X) \xrightarrow[\in \mathrm{WHmtpEq}]{(i_d)^*} \mathrm{Map}(S^d, X)\,.$

This is an elementary argument using standard ingredients, but since the statement it is rarely made explicit in this form we spell out the proof:

Proof. First, to amplify the standard fact that the homotopy groups of a based loop space at *each* loop $\ell \in \Omega_x X$ (not necessarily the constant loop) are the shifted homotopy groups of the underlying space at the given basepoint $x \in X$, since the standard fiber sequence with the contractible based path space $P_x X$ yields a long exact sequence of homotopy groups of this form:

$$* \;\simeq\; \pi_{\bullet+1}(P_x X, \ell) \longrightarrow \pi_{\bullet+1}(X, x) \xrightarrow{\;\sim\;} \pi_\bullet(\Omega_x X, \ell) \longrightarrow \pi_\bullet(PX, \ell) \;\simeq\; *\,. \tag{3.14}$$

By induction, this implies that for all iterated loops $\sigma \in \Omega^d_x X$ we have

$$\pi_\bullet\big(\Omega^d_x X, \sigma\big) \;\simeq\; \pi_{\bullet+d}(X, x)\,. \tag{3.15}$$

Similarly, for any choice of $\sigma \in \mathrm{Map}(S^d, X)$ with basepoint $x := \mathrm{pt}^*_d(\sigma) \in X$, the d-fold loop space is the homotopy fiber of the evaluation map $(\mathrm{pt}_d)^* = \mathrm{Map}(* \xrightarrow{\mathrm{pt}_d} S^d, X)$ out of the mapping space from S^d into X:

$$\Omega^d_x X \xrightarrow{\mathrm{fib}_x((\mathrm{pt}_d)^*)} \mathrm{Map}(S^d, X) \xrightarrow{(\mathrm{pt}_d)^*} X\,.$$

With (3.15), this induces a long exact sequence of homotopy groups of the following form:

$$\cdots \longrightarrow \pi_{\bullet+1}(X,x) \longrightarrow \pi_{\bullet+d}(X,x) \longrightarrow \pi_\bullet\big(\mathrm{Map}(S^d, A), \sigma\big) \xrightarrow{((\mathrm{pt}_d)^*)_*} \pi_\bullet(X,x) \longrightarrow \pi_{\bullet+d-1}(X,x) \longrightarrow \cdots. \tag{3.16}$$

The implication (i) ⇒ (ii) follows immediately from the exactness of (3.16).

Next, since the equatorial embeddings $S^d \overset{i_d}{\hookrightarrow} S^{d+1}$ clearly respects the basepoint inclusion, we have for each $x \in X$ a commuting diagram of this form:

$$\begin{tikzcd} \Omega^{d+1}_x X \arrow[r, "\mathrm{fib}_x((\mathrm{pt}_d)^*)"] \arrow[d] & \mathrm{Map}(S^{d+1}, X) \arrow[r, "(\mathrm{pt}_d)^*"] \arrow[d, "(i_d)^*"'] & X \arrow[d, equal] \\ \Omega^d_x X \arrow[r, "\mathrm{fib}_x((\mathrm{pt}_d)^*)"] & \mathrm{Map}(S^d, X) \arrow[r, "(\mathrm{pt}_d)^*"] & X \end{tikzcd} \tag{3.17}$$

This gives the implication (ii) ⇒ (iii) by the 2-out-of-3 property for weak equivalences.

Moreover, for each $\sigma \in \mathrm{Map}(S^{d+1}, X)$ the morphism of long exact sequences of homotopy groups induced from (3.17) is of this form:

$$\begin{tikzcd} \pi_{\bullet+1}\big(\mathrm{Map}(S^{d+1}, X), \sigma\big) \arrow[r] \arrow[d, "((i_d)^*)_*"] & \pi_{\bullet+1}(X, x) \arrow[r] \arrow[d, equal] & \pi_{\bullet+d+1}(X, x) \arrow[r] \arrow[d, "0"] & \pi_\bullet\big(\mathrm{Map}(S^{d+1}, X), \sigma\big) \arrow[r] \arrow[d, "((i_d)^*)_*"] & \pi_\bullet(X, x) \arrow[d, equal] \\ \pi_\bullet\big(\mathrm{Map}(S^d, X), \sigma\big) \arrow[r] & \pi_{\bullet+1}(X, x) \arrow[r] & \pi_{\bullet+d}(X, x) \arrow[r] & \pi_\bullet\big(\mathrm{Map}(S^d, X), \sigma\big) \arrow[r] & \pi_\bullet(X, x)\,. \end{tikzcd}$$

Here the middle vertical morphism is the function constant on 0, as shown, since an equatorial extension of a based map from S^d to S^{d+1} is in particular an extension to D^{d+1} and hence a contracting homotopy.

So when $(i_d)^*$ is a weak homotopy equivalence, then the (non-abelian) five lemma for the above diagram implies that this 0-map is an isomorphism, which means that $\pi_{\bullet+d}(X,x) = 0$ for all x. This is the implication (iii) ⇒ (i) and hence concludes the proof. □

3.1.2 Homotopy theory of simplicial transformation groups

Simplicial Groups. In analogy to how plain simplicial sets model bare ∞-groupoids, so simplicial groups (e.g. [May67, §IV.17][Cu71, §3]) model bare ∞-groups:

Notation 3.1.22 (Homotopy theory of simplicial groups ([Qu67, §II 3.7][GJ09, §V]))**.** We write

$$\mathrm{Grp}(\Delta\mathrm{Set})_{\mathrm{proj}} \;\in\; \mathrm{MdlCat} \tag{3.18}$$

for the model category of simplicial groups ([May67, §17][Cu71, §3]), whose weak equivalences and fibrations are those of the underlying $\Delta\mathrm{Set}_{\mathrm{Qu}}$.

Lemma 3.1.23 (Simplicial groups are Kan complexes ([Moo54, Thm. 3][May67, Thm. 17.1][Cu71, Lem. 3.1]))**.** *The underling simplicial set of any simplicial group* (3.18) *is a Kan complex.*

The following model category theoretic version (Prop. 3.1.26) of the general looping/delooping equivalence (16) relates simplicial groups to reduced simplicial sets:

Notation 3.1.24 (Homotopy theory of reduced simplicial sets ([GJ09, §V Prop. 6.12]))**.** We write

$$\Delta\mathrm{Set}_{\geq 1,\mathrm{inj}} \;\in\; \mathrm{MdlCat}$$

for the model category of reduced simplicial sets (those $S \in \Delta\mathrm{Set}$ with a single vertex, $S_0 = *$) whose weak equivalences and cofibrations are those of the underlying $\Delta\mathrm{Set}_{\mathrm{Qu}}$ (Ntn. 3.1.8).

Lemma 3.1.25 (Fibrant reduced simplicial sets are Kan complexes [GJ09, §V, Lem. 6.6])**.** *While the forgetful functor*

$$(\Delta\mathrm{Set}_{\geq 1})_{\mathrm{inj}} \xrightarrow{\mathrm{undrlng}} \Delta\mathrm{Set}_{\mathrm{Qu}}$$

does not preserve all fibrations, it does preserve fibrant objects.

Proposition 3.1.26 (Quillen equivalence between simplicial groups and reduced simplicial sets [GJ09, §V, Prop. 6.3])**.** *The simplicial classifying space construction (Def. 3.1.27) is the right adjoint of a Quillen equivalence between the projective model structure on simplicial groups (Ntn. 3.1.22) and the injective model structure on reduced simplicial sets (Nota 3.1.24):*

$$\mathrm{Grp}(\Delta\mathrm{Set})_{\mathrm{proj}} \underset{\overline{W}(-)}{\overset{\simeq_{\mathrm{Qu}}}{\rightleftarrows}} (\Delta\mathrm{Set}_{\geq 1})_{\mathrm{inj}}\,.$$

Universal principal simplicial complex. With ∞-groups presented by simplicial groups $\mathscr{G}$, $\mathscr{G}$-principal ∞-bundles are presented by the construction traditionally known denoted "$W\mathscr{G}$":

Definition 3.1.27 (Universal principal simplicial complex [Kan58, Def. 10.3][May67, p. 87-788][GJ09, p. 269])**.** Let $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$.
(i) Its *standard universal principal complex* is the simplicial set

$$W\mathscr{G} \;\in\; \Delta\mathrm{Set}$$

whose

- component sets are
$$(W\mathscr{G})_n \;:=\; \mathscr{G}_n \times \mathscr{G}_{n-1} \times \cdots \times \mathscr{G}_0\,,$$
- face maps are given by
$$d_i\,(\gamma_n, \gamma_{n-1}, \cdots, \gamma_0) \;:=\; \begin{cases} \big(d_i(\gamma_n), d_{i-1}(\gamma_{n-1}), \cdots, d_0(\gamma_{n-i})\cdot\gamma_{n-i-1}, \gamma_{n-i-2}, \cdots, \gamma_0\big) & \text{for} \quad 0 < i < n \\ \big(d_n(\gamma_n), d_{n-1}(\gamma_{n-1}), \cdots, d_1(\gamma_1)\big) & \text{for} \quad i = n\,, \end{cases} \tag{3.19}$$
- degeneracy maps are given by
$$s_i\,(\gamma_n, \gamma_{n-1}, \cdots, \gamma_0) \;:=\; \big(s_i(\gamma_n), s_{i-1}(\gamma_{n-1}), \cdots, s_0(\gamma_{n-i}), e, \gamma_{n-i-1}, \cdots, \gamma_0\big)\,, \tag{3.20}$$
- and equipped with the left $\mathscr{G}$-action (Ex. 3.1.34) given by
$$\begin{array}{ccc} \mathscr{G} \times W\mathscr{G} & \longrightarrow & W\mathscr{G} \\ \big(h_n, (\gamma_n, \gamma_{n-1}, \cdots, \gamma_0)\big) & \longmapsto & \big(h_n \cdot \gamma_n, \gamma_{n-1}, \cdots, \gamma_0\big)\,. \end{array} \tag{3.21}$$

(ii) Its standard *simplicial delooping* or *simplicial classifying complex* $\overline{W}\mathscr{G}$ is the quotient by that action (3.21):

$$W\mathscr{G} \xrightarrow{q_{W\mathscr{G}}} \overline{W}\mathscr{G} \;:=\; (W\mathscr{G})/\mathscr{G}\,. \tag{3.22}$$

Example 3.1.28 (Low-dimensional cells of universal simplicial principal complex)**.** Unwinding the definition (3.19) of the face maps of $W\mathscr{G}$ (Def. 3.1.27) shows that its 1-simplices are of the form

$$(W\mathscr{G})_1 \;=\; \left\{ d_1(g_1) \xrightarrow{(g_1, g_0)} d_0(g_1)\cdot g_0 \;\middle|\; \begin{array}{l} g_0 \in \mathscr{G}_0 \\ g_1 \in \mathscr{G}_1 \end{array} \right\}$$

and its 2-simplices are of this form:

$$(W\mathscr{G})_2 \;=\; \left\{ \begin{tikzcd} & d_0 d_2(g_2)\cdot d_1(g_1) = d_1 d_0(g_2)\cdot d_1(g_1) \arrow[dr, "(d_0(g_2)\cdot g_1, g_0)"] \arrow[d, phantom, "\Downarrow (g_2, g_1, g_0)"] & \\ \begin{array}{c} d_1 d_2(g_2) \\ = d_1 d_1(g_2) \end{array} \arrow[ur, "(d_2(g_2), d_1(g_1))"] \arrow[rr, "(d_1(g_2), d_0(g_1)\cdot g_0)"'] & & \begin{array}{c} d_0 d_0(g_2)\cdot d_0(g_1)\cdot g_0 \\ = d_0 d_1(g_2)\cdot d_0(g_1)\cdot g_0 \end{array} \end{tikzcd} \;\middle|\; \begin{array}{l} g_0 \in \mathscr{G}_0, \\ g_1 \in \mathscr{G}_1, \\ g_2 \in \mathscr{G}_2 \end{array} \right\}.$$

Example 3.1.29 (Universal principal simplicial complex for ordinary group G)**.** If

$$G \in \mathrm{Grp} \hookrightarrow \mathrm{Grp}(\Delta\mathrm{Set})$$

is an ordinary discrete group, regarded as a simplicial group (hence the functor constant on G on the opposite simplex category), then the standard model of its universal principal complex (Def. 3.1.27) is isomorphic to the nerve of the action groupoid (3.12) of the right multiplication action of G on itself:

$$WG = N(G \times G \rightrightarrows G). \tag{3.23}$$

$$(WG)_2 = \left\{ \begin{tikzcd} & g_2 g_1 \arrow[dr, "(g_2 g_1, g_0)"] \arrow[d, phantom, "{\overset{\Vert}{(g_2,g_1,g_0)} \atop \Downarrow}"] & \\ g_2 \arrow[ur, "(g_2,g_1)"] \arrow[rr, "(g_2, g_1 g_0)"'] & {} & g_2 g_1 g_0 \end{tikzcd} \;\middle|\; g_0, g_1, g_2 \in G \right\}.$$

Accordingly, the standard simplicial delooping (3.22) of an ordinary group is isomorphic to the nerve of its delooping groupoid (3.11):

$$\overline{W}G \simeq N(G \rightrightarrows *).$$

Proposition 3.1.30 (Basic properties of standard simplicial principal complex [GJ09, §V, Lem. 4.1, 4.6, Cor. 6.8])**.** *For* $\mathscr{G} \in \mathrm{Grp}(\mathrm{SimplSets})$*, its standard universal principal complex (Def. 3.1.27) has the following properties:*
(i) $W\mathscr{G}$ *is contractible;*
(ii) $W\mathscr{G}$ *and* $\overline{W}\mathscr{G}$ *are Kan complexes.*

Proof. That $\overline{W}\mathscr{G}$ is Kan fibrant follows as the combination of Lem. 3.1.23, Prop. 3.1.26, and Lem. 3.1.25. This implies that $W\mathscr{G}$ is Kan fibrant since $W\mathscr{G} \xrightarrow{q} \overline{W}\mathscr{G}$ is a Kan fibration (3.34) (by Prop. 3.1.26, see Ex. 3.1.40). □

Proposition 3.1.31 ([GJ09, §V, Cor. 6.9])**.** *Let* $\mathscr{G}_1 \xrightarrow{\phi} \mathscr{G}_2$ *be a homomorphism of simplicial groups which is a Kan fibration of underlying simplicial sets. Then the induced morphism of simplicial classifying spaces* $\overline{W}\mathscr{G}_1 \xrightarrow{\overline{W}\phi} \overline{W}\mathscr{G}_2$ *(Def. 3.1.27) is a Kan fibration if and only if* ϕ *is a surjection on connected components.*

For plain groups $\mathscr{G} \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\Delta\mathrm{Set})$, the statement of Prop. 3.1.31 is also readily seen by direct inspection. Elementary as it is, it has some important consequences (see Lem. 3.2.86, Prop. 3.2.87 below).

Simplicial group actions. With bare ∞-groups presented by simplicial groups, their ∞-actions are presented by the homotopy theory of simplicial group actions (Ntn. 3.1.36 below), sometimes referred to as the **Borel model structure**.

Notation 3.1.32 (Simplicial group actions)**.** For $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$, we denote
(i) by

$$\mathbf{B}\mathscr{G} \in \Delta\mathrm{SetEnrCat}, \quad \mathbf{B}\mathscr{G}(*,*) := \mathscr{G}, \tag{3.24}$$

the simplicial groupoid with a single object $*$, with $\mathscr{G}$ as its unique hom-object and with composition "$\circ$" given by the *reverse* of the group product "$\cdot$"

$$\begin{array}{ccc} \mathscr{G} \times \mathscr{G} & \xrightarrow{\quad\quad \circ \quad\quad} & \mathscr{G} \\ (g_n, h_n) & \longmapsto & h_n \cdot g_n\,; \end{array} \tag{3.25}$$

(ii) the category of $\mathscr{G}$-actions on simplicial sets by:

$$\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set}) := \Delta\mathrm{Fnctr}(\mathbf{B}\mathscr{G}, \Delta\mathrm{Set}), \tag{3.26}$$

identified with the category of ΔSet-enriched functors from the delooping (3.24) of $\mathscr{G}$ to ΔSet (3.5).

Remark 3.1.33 (Simplicial group actions are from the left)**.** The convention (3.25) for the delooping $\mathbf{B}\mathscr{G}$ (3.24) implies that the simplicial $\mathscr{G}$-actions (3.26) are *left* actions:

$$
\begin{tikzcd}
\mathbf{B}\mathscr{G} \arrow[rr, "\mathscr{G} \circlearrowright X"] & & \Delta\mathrm{Set} \\
\bullet \arrow[d, "g_1"] \arrow[dd, bend right=60, swap, "g_1 \circ g_2 := g_2 \cdot g_1"] & & X \arrow[d, "g_1 \cdot (-)"] \arrow[dd, bend left=60, "(g_2 \cdot g_1)\cdot(-)"] \\
\bullet \arrow[d, "g_2"] & & X \arrow[d, "g_2 \cdot (-)"] \\
\bullet & & X
\end{tikzcd}
\qquad
\begin{array}{ccc}
\mathscr{G} \times X & \xrightarrow{\;\;(-)\cdot(-)\;\;} & X \\
(g_n, x_n) & \longmapsto & g_n \cdot x_n\,.
\end{array}
$$

Example 3.1.34 (Universal principal simplicial complex in $\mathscr{G}$-actions)**.** For $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$, the universal principal simplicial complex $W\mathscr{G}$ (Def. 3.1.27) becomes an object of (3.26) by the formula (3.21).

$$
\mathscr{G} \circlearrowright W\mathscr{G} \in \mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})\,. \tag{3.27}
$$

Making explicit the following elementary Ex. 3.1.35 is, serves to straighten out a web of conventions about (simplicial) group actions.

Example 3.1.35 (Simplicial group canonically acting on itself)**.** Any $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$ becomes an object of the category of simplicial $\mathscr{G}$-actions (3.26) in three canonical ways:

$$
\begin{array}{ccc}
\mathscr{G} \times \mathscr{G} & \xrightarrow{\substack{\text{left} \\ \text{multiplication action}}} & \mathscr{G}\,, \\
(g_n, h_n) & \longmapsto & g_n \cdot h_n
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{G} \times \mathscr{G} & \xrightarrow{\substack{\text{right inverse} \\ \text{multiplication action}}} & \mathscr{G}\,, \\
(g_n, h_n) & \longmapsto & h_n \cdot g_n^{-1}
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{G} \times \mathscr{G} & \xrightarrow{\substack{\text{adjoint/conjugation} \\ \text{action}}} & \mathscr{G}\,. \\
(g_n, h_n) & \longmapsto & g_n \cdot h_n \cdot g_n^{-1}
\end{array}
\tag{3.28}
$$

The first two are isomorphic in $\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})$ via the inversion operation:

$$
\begin{tikzcd}
(g_n, h_n) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & g_n \cdot h_n \arrow[ddd, mapsto] \\
& \mathscr{G} \times \mathscr{G} \arrow[r, "\substack{\text{left} \\ \text{multiplication}}"] \arrow[d, "\mathrm{id}\times(-)^{-1}"', "\wr"] & \mathscr{G} \arrow[d, "(-)^{-1}", "\wr"'] & \\
& \mathscr{G} \times \mathscr{G} \arrow[r, "\substack{\text{right inverse} \\ \text{multiplication}}"'] & \mathscr{G} & \\
(g_n, h_n^{-1}) \arrow[rrr, mapsto] & & & h_n^{-1} \cdot g_n^{-1}
\end{tikzcd}
\tag{3.29}
$$

Notation 3.1.36 (Model category of simplicial group actions ([DDK80, §2][Gu06, §5][GJ09, §V Thm. 2.3]))**.** For $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$, we have on the category of $\mathscr{G}$-actions (3.26) the projective model structure (the *coarse-* or *Borel-equivariant model structure*) whose fibrations and weak equivalences are those of the underlying $\Delta\mathrm{Set}_{\mathrm{Qu}}$ (Ntn. 3.1.8), which we denote as:

$$
\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}} \;:=\; \Delta\mathrm{Fnctr}(\mathbf{B}\mathscr{G}, \Delta\mathrm{Set})_{\mathrm{proj}}\,. \tag{3.30}
$$

Lemma 3.1.37 (Cofibrations of simplicial group actions [DDK80, Prop. 2.2 (ii)][Gu06, Prop. 5.3][GJ09, §V Lem. 2.4])**.** *The cofibrations of* $\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}}$ (3.30) *are the monomorphisms such that the* $\mathscr{G}$*-action on the simplices not in their image is free.*

Lemma 3.1.38 (Equivariant equivalence of simplicial universal principal complexes)**.** *For* $\mathscr{H} \overset{i}{\hookrightarrow} \mathscr{G}$ *a simplicial subgroup inclusion, the induced inclusion*

$$
W\mathscr{H} \xrightarrow[\in \mathrm{W}]{W(i)} W\mathscr{G} \quad \in \mathscr{H}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}}
$$

of their standard simplicial principal complexes (Def. 3.1.27) equipped with their canonical $\mathscr{H}$*-action* (3.21) *is a weak equivalence in the Borel-equivariant model structure* (3.30)*.*

Proof. The underlying simplicial sets of both are contractible, by Prop. 3.1.30), so that underlying any equivariant morphism between them is an simplicial weak homotopy equivalence. □

Proposition 3.1.39 (Quillen equivalence between Borel model structure and the slice over classifying complex)**.**
(i) *For any* $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$*, there is a simplicial adjunction*

$$\underset{\text{homotopy fiber}}{} \; (-) \times_{\overline{W}\mathscr{G}} W\mathscr{G} \quad \dashv \quad \big((-) \times W\mathscr{G}\big)/\mathscr{G} \; \underset{\text{Borel construction}}{} \tag{3.31}$$

between the Borel model structure (3.30) *and the slice model structure of* $\Delta\mathrm{Set}_{\mathrm{Qu}}$ (3.1.8) *over the simplicial classifying complex* $\overline{W}\mathscr{G}$ (3.22)*.*
(ii) *Hence there is a natural isomorphism of hom-complexes*

$$\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})\big((-) \times_{\overline{W}\mathscr{G}} W\mathscr{G}, (-)\big) \simeq \Delta\mathrm{Set}_{/\overline{W}\mathscr{G}}\big((-), ((-) \times W\mathscr{G})/\mathscr{G}\big) \in \Delta\mathrm{Set}, \tag{3.32}$$

which is a Quillen equivalence:

$$\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}} \underset{((-)\times W\mathscr{G})/\mathscr{G}}{\overset{(-)\times_{\overline{W}\mathscr{G}}W\mathscr{G}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Qu}}}}}} (\Delta\mathrm{Set}_{\mathrm{Qu}})_{/\overline{W}\mathscr{G}}\,. \tag{3.33}$$

Proof. The plain adjunction constituting a Quillen equivalence (3.33) is [DDK80, Prop. 2.3]. The simplicial enrichment (3.32), hence the natural bijections

$$\mathrm{Hom}\big(((-)\times_{\overline{W}\mathscr{G}}) \times \Delta[k], (-)\big) \simeq \mathrm{Hom}\big((-) \times \Delta[k], ((-) \times W\mathscr{G})/\mathscr{G}\big) \in \mathrm{Set},$$

are left somewhat implicit in [DDK80, Prop. 2.4], but it follows readily from the plain adjunction via the natural isomorphism

$$((-) \times_{\overline{W}\mathscr{G}} WG) \times \Delta[k] \simeq ((-) \times \Delta[k]) \times_{\overline{W}\mathscr{G}} W\mathscr{G},$$

which, in turn, follows from the pasting law (5):

$$\begin{tikzcd} (X \times_{\overline{W}G} WG) \times \Delta[k] \simeq (X \times \Delta[k]) \times_{\overline{W}G} WG \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X \times \Delta[k] \arrow[d, "\mathrm{pr}_1"] \\ X \times_{\overline{W}G} WG \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d] \\ WG \arrow[r] & \overline{W}G\,. \end{tikzcd}$$

□

Example 3.1.40 (Coprojections out of Borel construction are Kan fibrations)**.** For $\mathscr{G} \curvearrowright X \in \mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})$ such that the underlying simplicial set X being a Kan complex, hence such that

$$\mathscr{G} \curvearrowright X \xrightarrow{\in \mathrm{Fib}} * \quad \in \mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}},$$

the projection from the Borel construction (3.31) to the simplicial classifying space (Def. 3.1.27) is a Kan fibration, due to the right Quillen functor property (3.33):

$$\begin{tikzcd} X \arrow[r, "\mathrm{fib}(q)"] & (W\mathscr{G} \times X)/\mathscr{G} \arrow[d, "q"', "\in\mathrm{Fib}"] \\ & \overline{W}\mathscr{G} \end{tikzcd} \quad = \quad \left(W\mathscr{G} \times \left(\begin{smallmatrix} X \\ \downarrow \\ * \end{smallmatrix}\right)\right)/\mathscr{G}\,.$$

The fiber of this fibration, hence the *homotopy fiber* is clearly X.

In the special case where $\mathscr{G} \curvearrowright X = \mathscr{G} \curvearrowright \mathscr{G}_L$ is the multiplication action of the simplicial group on itself, this construction reduces to the *universal principal simplicial bundle* (3.22)

$$\begin{tikzcd} \mathscr{G} \arrow[r] & W\mathscr{G} \arrow[d, "q"', "\in\mathrm{Fib}"] \\ & \overline{W}\mathscr{G} \end{tikzcd} \quad = \quad \left(W\mathscr{G} \times \left(\begin{smallmatrix} \mathscr{G}_L \\ \downarrow \\ * \end{smallmatrix}\right)\right)/\mathscr{G}\,. \tag{3.34}$$

Notation 3.1.41 (Homotopy quotient of simplicial group actions)**.** For $\mathscr{G} \in \mathrm{Grp}(\Delta\mathrm{Set})$, we denote the right derived functor of the *Borel construction* right Quillen functor (3.33) by

$$(-)/\!\!/\mathscr{G} := \mathbb{R}\Big(\big((-) \times W\mathscr{G}\big)/\mathscr{G}\Big) \quad : \quad \mathrm{Ho}\big(\mathscr{G}\,\mathrm{Act}(\Delta\mathrm{Set})_{\mathrm{proj}}\big) \longrightarrow \mathrm{Ho}(\Delta\mathrm{Set}_{\mathrm{Qu}})\,.$$

Applied to the point $\mathscr{G} \curvearrowright *$, we also write

$$\mathbf{B}\mathscr{G} := */\!\!/\mathscr{G}\,. \tag{3.35}$$

As an example:

Proposition 3.1.42 (G-Sets in the homotopy theory over BG). *For $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\Delta\mathrm{Set})$, we have an equivalence between G-actions on sets and 0-truncated objects in the homotopy theory over BG (Ntn. 3.1.17):*

$$\begin{array}{ccccc} G\,\mathrm{Act}(\mathrm{Set}) & \simeq & \big((\mathrm{Grpd}_\infty)_{/BG}\big)_{\leq 0} & \hookrightarrow & (\mathrm{Grpd}_\infty)_{/BG} \\ {\scriptstyle G \circlearrowright X} & & \longmapsto & & {\scriptstyle (X /\!\!/ G \to BG)} \end{array}$$

Proof. Since $(X \times G \rightrightarrows X) \to (G \rightrightarrows *)$ is clearly a Kan fibration with fiber X, the latter is the homotopy fiber of $X /\!\!/ G \xrightarrow{B} G$. With this, the statement follows by Prop. 3.1.39. □

3.2 Geometric homotopy theory

We recall and develop basics of higher topos theory [Si00][Lur03][TV05][Jo08c][Lur09][Re10] with focus on the discussion of transformation groups in this context.

– §3.2.1: Simplicial sheaves and ∞-stacks.
– §3.2.2: General constructions in ∞-toposes.
– §3.2.3: Transformation groups in ∞-toposes.

∞-Toposes. As usual, we say *∞-topos* for ∞-categories of ∞-sheaves (of ∞-stacks), in contrast to the broader concept of *elementary ∞-toposes*.

For our purposes, the reader may take the following characterization of ∞-toposes to be their definition:

Proposition 3.2.1 (∞-Giraud theorem [Lur09, Prop. 6.1.0.6][Lur03, §2.2]). *A presentable ∞-category $\mathbf{H} \in \mathrm{Ho}_2(\mathrm{PresCat}_\infty)$ (Def. 3.1.5, Prop. 3.1.6) is an ∞-topos if and only if it satisfies the following three conditions:*
(a) *coproducts are disjoint,*
(b) *colimits are universal (see Prop. 3.2.6),*
(c) *groupoid objects are effective (see Prop. 3.2.15).*

Example 3.2.2 (The canonical base ∞-topos of ∞-groupoids). The ∞-category Grpd_∞ of ∞-groupoids (Ntn. 3.1.10) is an ∞-topos (Prop. 3.2.1), being the ∞-category ∞-sheaves on the point:

$$\mathrm{Grpd}_\infty \simeq \mathrm{Sh}_\infty(*)\,.$$

Proposition 3.2.3 (Terminal ∞-geometric morphism [Lur09, Prop. 6.3.4.1 with Def. 6.1.0.4]). *Given an ∞-topos $\mathbf{H}$ (Prop. 3.2.1) there is an essentially unique pair of adjoint ∞-functors* (10) *between $\mathbf{H}$ and Grpd_∞ (Ex. 3.2.2)*

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift right=1.5ex, "\Gamma"', "\mathrm{acc}" near start] \arrow[r, phantom, "\perp"] & \mathrm{Grpd}_\infty \arrow[l, shift right=1.5ex, "\mathrm{LCnst}"', "\mathrm{lex}" near start] \end{tikzcd}, \tag{3.36}$$

such that, in addition to the default (co-)limit preservation (11)*:*
(a) *the left adjoint* LCnst *preserves finite ∞-limits;*
(b) *the right adjoint Γ is accessible ([Lur09, Def. 5.4.2.5]).*

Remark 3.2.4 (Constructing locally constant ∞-stacks preserves finite products). That LCnst preserves finite limits means in particular that it preserves, via natural equivalences:
(a) the terminal object

$$\mathrm{LCnst}(*) \simeq *_{\mathbf{H}}\,; \tag{3.37}$$

(b) finite products

$$\mathrm{LCnst}(\mathscr{S} \times \cdots \times \mathscr{F}) \simeq \mathrm{LCnst}(\mathscr{S}) \times \cdots \times \mathrm{LCnst}(\mathscr{S})\,. \tag{3.38}$$

Example 3.2.5 (Global sections are co-represented by the terminal object)**.** Global sections (Prop. 3.2.3) are equivalently maps out of the terminal object:

$$\Gamma(-) \;\simeq\; \mathbf{H}(*, -)\,. \tag{3.39}$$

This is because:

$$\begin{aligned} \Gamma(X) &\simeq \mathrm{Grpd}_\infty\big(*, \Gamma(X)\big) && \text{by Ex. 3.1.15} \\ &\simeq \mathrm{Grpd}_\infty\big(*, \Gamma(X)\big) && \text{by (3.36) with (10)} \\ &\simeq \mathbf{H}(*, X) && \text{by (3.37).} \end{aligned}$$

Proposition 3.2.6 (Colimits are universal in an ∞-topos [Lur09, p. 532])**.** *In an ∞-topos* $\mathbf{H}$ *(Prop. 3.2.1), given a morphism* $f\,; A \to B$ *and a diagram* $X_\bullet \,:\, \mathscr{I} \to \mathbf{H}_{/B}$*,*
(i) *pullback along* f *preserves its ∞-colimit:*

$$\begin{tikzcd} f^*\big(\varinjlim X_\bullet\big) \simeq \varinjlim f^*(X_\bullet) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \varinjlim X_\bullet \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{3.40}$$

(ii) *In particular, products preserve colimits:*

$$\begin{tikzcd} A \times \big(\varinjlim X_\bullet\big) \simeq \varinjlim\,(A \times X_\bullet) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \varinjlim X_\bullet \arrow[d] \\ A \arrow[r] & * \end{tikzcd} \tag{3.41}$$

Groupoid objects in an ∞-topos.

Definition 3.2.7 (Groupoid objects in an ∞-topos [Lur09, Def. 6.1.2.7])**.** Given an ∞-topos $\mathbf{H}$ (Prop. 3.2.1),
(i) a simplicial object

$$\mathscr{X}_\bullet \;:\; \Delta^{\mathrm{op}} \longrightarrow \mathbf{H}$$

is called a *groupoid object* if it satifies the groupoidal Segal conditions, hence if for all $n \in \mathbb{N}$ and all partitions (not necessarily order-preserving)

$$\{0,1,\cdots,n\} \;\simeq\; \{i_0,\cdots,i_k\} \underset{\{i_k\}}{\sqcup} \{i_k,\cdots,i_n\}$$

the corresponding diagram of evaluations of $\mathscr{X}_\bullet$ is homotopy Cartesian

$$\begin{tikzcd} \mathscr{X}_{[n]} \arrow[r, "(i_0,\cdots,i_k)^*"] \arrow[d, "(i_k,\cdots,i_{n-k})^*"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathscr{X}_{[k]} \arrow[d, "(i_k)^*"] \\ \mathscr{X}_{[n-k]} \arrow[r, "(i_k)^*"'] & \mathscr{X}_{[0]}\,. \end{tikzcd} \tag{3.42}$$

Here $(i_0,\cdots,i_k) :\; \Delta[k] \to \Delta[n]$ denotes the unique unique order-preserving map of finite sets, whose image contains the vertices $i_0,\cdots,i_k$.
(ii) We write

$$\mathrm{Grpd}(\mathbf{H}) \hookrightarrow \mathrm{PSh}_\infty(\Delta, \mathbf{H})$$

for the full sub-∞-category of groupoid objects.

Example 3.2.8 (2-Horn fillers in groupoid objects)**.** For $n = 2$ the condition (3.42) says that

$$\begin{aligned} \mathscr{X}_{\{0<1<2\}} &\simeq \mathscr{X}_{\{0<1\}} \underset{\mathscr{X}_{\{1\}}}{\times} \mathscr{X}_{\{1<2\}}\,, \\ \mathscr{X}_{\{0<1<2\}} &\simeq \mathscr{X}_{\{0<2\}} \underset{\mathscr{X}_{\{0\}}}{\times} \mathscr{X}_{\{0<1\}}\,, \\ \text{and}\quad \mathscr{X}_{\{0<1<2\}} &\simeq \mathscr{X}_{\{1<2\}} \underset{\mathscr{X}_{\{2\}}}{\times} \mathscr{X}_{\{0<2\}}\,, \end{aligned}$$

expressing the fact that morphisms of the groupoid objects (namely the elements of $\mathscr{X}_{[1]}$) have
(a) essentially unique composites when composable,
(b) essentially unique left inverses, and
(c) essentially unique right inverses.

Example 3.2.9 (Plain objects as constant groupoid objects in an ∞-topos)**.** For an object $X \in \mathbf{H}$, the simplicial object $\mathrm{Cnst}(X)_\bullet$ which is constant on this value is a groupoid object (Def. 3.2.7). This construction extends to a full-sub-∞-category inclusion

$$\mathrm{Cnst}(-)_\bullet : \ \mathbf{H} \hookrightarrow \mathrm{Grpd}(\mathbf{H}) .$$

In particular, the groupoid constant on the terminal object $* \in \mathbf{H}$ is the terminal groupoid object

$$\mathrm{Cnst}(*)_\bullet \simeq * \quad \in \mathrm{Grpd}(\mathbf{H}) . \tag{3.43}$$

Example 3.2.10 (Čech groupoids)**.** Given a morphism $X \xrightarrow{p} \mathscr{X}$ in $\mathbf{H}$, its iterated fiber products with itself form a groupoid object (Def. 3.2.7):

$$C(p)_\bullet := X^{\times^\bullet_{\mathscr{X}}} \in \mathrm{Grpd}(\mathbf{H}) .$$

Effective epimorphisms and atlases.

Definition 3.2.11 (Effective epimorphism in an ∞-category [Lur09, §6.2.3])**.** A morphism in an ∞-category with finite limits and simplicial colimits is called an *effective epimorphism*, to be denoted in the form $X \xrightarrow{p} \mathrel{\mkern-14mu}\rightarrow \mathscr{X}$, if it is equivalently the ∞-colimit coprojection out of its Čech groupoid (Ex. 3.2.10)

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "p"] & X \arrow[d, "q_0"] \\
\varinjlim\limits_{[n]\in\Delta^{\mathrm{op}}} X^{\times^n_{\mathscr{X}}} \arrow[r, "\sim"] & \mathscr{X}
\end{tikzcd}$$

In view of Prop. 3.2.15, we also say that an effective epimorphism $X \twoheadrightarrow \mathscr{X}$ is an *atlas* for $\mathscr{X}$ (see [SS20-Orb, p. 27]) and we write

$$\mathrm{Atl}(\mathbf{H}) \hookrightarrow \mathrm{PSh}(\Delta^1, \mathbf{H})$$

for the sub-∞-category of the arrow category on the effective epimorphisms.

Notation 3.2.12 (Inhabited objects)**.** An object $X \in \mathbf{H}$ is called *inhabited* the morphism to the terminal object is an effective epimorphism: $X \twoheadrightarrow *$ (Def. 3.2.11).

Proposition 3.2.13 (Detecting effective epimorphisms on 0-truncations [Lur09, Prop. 7.2.1.14])**.** *A morphism $f : X \to \mathscr{X}$ in an ∞-topos $\mathbf{H}$, is an effective epimorphism (Def. 3.2.11), precisely if its 0-truncation $\tau_0(f) : \tau_0(X) \to \tau_0(\mathscr{X})$ is an effective epimorphism in $\mathbf{H}_0$.*

Example 3.2.14 (Pointed connected objects are those with point atlas)**.** In an ∞-topos $\mathbf{H}$, given a pointed object $* \xrightarrow{\mathrm{pt}} \mathscr{X}$, it is connected, in that $\tau_0(\mathscr{X}) \simeq *$ (Ntn. 3.2.54), precisely if the pointing is an effective epimorphism (Def. 3.2.11). Therefore, the pointed connected objects in $\mathbf{H}$ form a full sub-∞-category of the pointed objects in $\mathrm{Atl}(\mathbf{H})$:

$$\begin{array}{ccc}
\mathbf{H}^{*/}_{\geq 0} & \hookrightarrow & \mathrm{Atl}(\mathbf{H})^{(*\twoheadrightarrow *)/} \\
(* \xrightarrow{\mathrm{pt}} \mathscr{X}) & \longmapsto & \left(\begin{array}{ccc} * & \twoheadrightarrow & * \\ \downarrow & & \downarrow \mathrm{pt} \\ * & \overset{\mathrm{pt}}{\twoheadrightarrow} & \mathscr{X} \end{array}\right)
\end{array} \tag{3.44}$$

Proof. By Prop. 3.2.13 and using that τ_0 preserves the terminal object, it is sufficient to show that $\mathscr{X}$ is connected precisely if $* \xrightarrow{\tau_0(\mathrm{pt})} \tau_0(\mathscr{X})$ is an effective epimorphism in the 1-category $\mathbf{H}_0$. But in a 1-category any fiber product of the terminal object with itself is the terminal object itself. Therefore, the Čech nerve of $\tau_0(\mathrm{pt})$ in $\mathbf{H}_0$ is $\mathrm{Cnst}(*)_\bullet$, whose colimit coprojection is $* \to *$. $\square$

Proposition 3.2.15 (Groupoid objects in an ∞-topos are effective [Lur09, Cor. 6.2.3.5])**.**
Given an ∞-topos $\mathbf{H}$*, the operation of forming Čech groupoids (Ex. 3.2.10) constitutes an equivalence of ∞-categories*

$$\mathrm{Atl}(\mathbf{H}) \underset{C(-)_\bullet}{\overset{(-)_0 \to \varinjlim(-)}{\underset{\sim}{\rightleftarrows}}} \mathrm{Grpd}(\mathbf{H}) \tag{3.45}$$

between effective epimorphisms and groupoid objects (Def. 3.2.7)

Proposition 3.2.16 (The effective-epi/mono factorization system)**.** *In an ∞-topos* $\mathbf{H}$*, the classes of* (-1)*-truncated morphisms (hence of monomorphisms* (6)*) and of* (-1)*-connected morphisms (hence of effective epimorphisms (Def. 3.2.15)), form an orthogonal factorization system. In particular:*
(i) *Diagrams of the following form have essentially unique lifts*

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, two heads] & A \arrow[d, hook] \\ \mathscr{X} \arrow[r, "\forall"'] \arrow[ur, dashed, "\exists!"] & \mathscr{A} \end{tikzcd} \tag{3.46}$$

(ii) *An ∞-limit in the arrow category over* (-1)*-truncated morphisms is again* (-1)*-truncated*

$$\underset{\underset{i\in I}{\longleftarrow}}{\lim}\left(X_i \overset{f_i}{\hookrightarrow} Y_i\right) \;\simeq\; \left(\left(\underset{\underset{i\in I}{\longleftarrow}}{\lim} X_i\right) \xhookrightarrow{\underset{\underset{i\in I}{\longleftarrow}}{\lim} f_i} \left(\underset{\underset{i\in I}{\longleftarrow}}{\lim} Y_i\right)\right). \tag{3.47}$$

Shape theory of ∞-toposes.

Definition 3.2.17 (Pro-∞-groupoids [Lur09, Def. 7.1.6.1])**.** We write

$$\mathrm{ProGrpd}_\infty \hookrightarrow \mathrm{Fnctr}(\mathrm{Grpd}_\infty, \mathrm{Grpd}_\infty)^{\mathrm{op}} \tag{3.48}$$

for the opposite of the full sub-∞-category on those ∞-functors which preserve finite ∞-limits and κ-filtered ∞-colimits for some regular cardinal κ.

Lemma 3.2.18 (Factorization through pro-∞-groupoids)**.** *The ∞-Yoneda embedding* (3.54) *of* Grpd_∞ *factors through pro-∞-groupoids* (3.48)*:*

$$\begin{array}{ccc} \mathrm{Grpd}_\infty & \xhookrightarrow{\;y\;} & \mathrm{ProGrpd}_\infty\,. \\ S & \longmapsto & \mathrm{Grpd}_\infty(S,-) \end{array} \tag{3.49}$$

Proof. By cartesian closure, $\mathrm{Grpd}_\infty(S,-)\colon \mathrm{Grpd}_\infty \to \mathrm{Grpd}_\infty$ is a right adjoint and as such preserves finite ∞-limits (11) and κ-filtered colimits for some regular cardinal κ (by [Lur09, Prop. 5.4.7.7]). □

Definition 3.2.19 (Shape of an ∞-topos [Lur09, Def. 7.1.6.3])**.** The *shape* of an ∞-topos of the form (3.36) is the pro-∞-groupoid (3.48) given by

$$\mathrm{Shp}(\mathbf{H}) \;:=\; \Gamma(-)\circ \mathrm{LCnst} \;\in\; \mathrm{ProGrpd}_\infty\,. \tag{3.50}$$

Remark 3.2.20 (Shape of ∞-topos is well-defined)**.** The formula (3.50) is well-defined, in that $\Gamma \circ \mathrm{LCnst}$ preserves finite ∞-limits and κ-filtered ∞-colimits, by (3.36) and (11).

Proposition 3.2.21 (Shape of ∞-topos is étale pro-homotopy type of its terminal object [Hoy17, Def. 2.3])**.** *The shape of an ∞-topos in the sense of Def. 3.2.19 is equivalently the étale pro-homotopy type of its terminal object* $*_{\mathbf{H}} \in \mathbf{H}$*, namely the image of the terminal object under the pro-left adjoint to* LCnst*:*

$$\begin{array}{cccc} \mathrm{Shp}\;: & \mathbf{H} & \longrightarrow & \mathrm{Grpd}_\infty \\ & X & \longmapsto & \mathbf{H}(X, \mathrm{LCnst}(-)) \end{array} \tag{3.51}$$

in that:

$$\underset{\substack{\text{shape of}\\ \infty\text{-topos}}}{\mathrm{Shp}(\mathbf{H})} \;\simeq\; \underset{\substack{\text{étale pro-homotopy type}\\ \text{of terminal object}}}{\mathrm{Shp}(*_{\mathbf{H}})} \;\in\; \mathrm{ProGrpd}_\infty\,.$$

Proof. This is the composite of the following sequence of natural equivalences:

$$\begin{aligned} \mathrm{Shp}(\mathbf{H}) &:= \Gamma \circ \mathrm{LCnst}(-) && \text{by Def. 3.2.19} \\ &\simeq \mathrm{Grpd}_\infty(*, \Gamma \circ \mathrm{LCnst}(-)) && \text{by Prop. 3.1.13} \\ &\simeq \mathbf{H}(\mathrm{LCnst}(*), \mathrm{LCnst}(-)) && \text{by (3.36)} \\ &\simeq \mathbf{H}(*_{\mathbf{H}}, \mathrm{LCnst}(-)) && \text{by (3.37)} \\ &=: \mathrm{Shp}(*_{\mathbf{H}}) && \text{by (3.51)}\,. \end{aligned}$$

□

3.2.1 Simplicial Sheaves and ∞-stacks

With ∞-groupoids presented by simplicial sets, ∞-stacks are presented by simplicial presheaves, an observation that really goes back to [Bro73]. A textbook account with focus on the injective model structures on simplicial presheaves may be found in [Ja15]. Here we need to focus on their projective model structures whose role was first highlighted in [Du01a]. A more detailed review of the specific constructions of interest here may be found in [FSS20-TCD, pp. 105].

Simplicial sites and ∞-Sites. When discussing ∞-toposes we need some would-be ∞-categories which need not and typically are not presentable, and as such are outside the scope of our defining Prop. 3.1.6. These are the ∞-*sites* (Def. 3.2.23) over with ∞-(pre-)sheaves are defined and which hence serve to present ∞-toposes (see Prop. 3.2.27 below). But in fact, their presentation of ∞-toposes is practically the only purpose of ∞-sites, and for this it is sufficient to model the ∞-site as a simplicially enriched (preferably Kan complex-enriched) category (Ntn. 3.1.3). Last not least, the ∞-Yoneda lemma (Prop. 3.2.29) says that every ∞-site in this sense is a full sub-∞-category of its associated ∞-topos, hence of a presentable ∞-category (where either the ∞-site is small or else the ambient presentable ∞-category is "very large").

Therefore:

Definition 3.2.22 (Simplicial sites and ∞-sites [TV05, Def. 3.1.1])**.** A *simplicial site* $(\mathscr{S}, J)$ is a simplicial category $\mathscr{S} \in \Delta\mathrm{Cat}$ (Ntn. 3.1.3) equipped with a Grothendieck topology on its homotopy category (3.2).

Notation 3.2.23 (∞-Sites)**.** In order to notationally indicate that a construction depends only on the corresponding ∞-category of an ∞-site, we write $\mathbf{S}$ for it, as in (3.52) below.

Example 3.2.24 (∞-Sites from full sub-∞-categories of presentable ∞-categories)**.** For $\mathbf{C} \in \mathrm{PresCat}_\infty$, Prop. 3.1.6 provides a presenting simplicial combinatorial model category $\mathscr{C}$.

Hence for any class of objects of $\mathscr{C}$ we obtain an ∞-site (with trivial Grothendieck topology) given by the corresponding simplicial full subcategory

$$\mathscr{S} \hookrightarrow \mathscr{C}^{\mathrm{cof}}_{\mathrm{fib}} \hookrightarrow \mathscr{C}$$

of the full simplicial subcategory of fibrant- and cofibrant objects of $\mathscr{C}$ (i.e. with hom-∞-groupoids as in Def. 3.1.11). The equivalence class of the simplicial categories $\mathscr{S}$ arising this way as the presentation $\mathscr{C}$ varies up to equivalence in $\mathrm{Loc}_2^{\mathrm{QuEq}}(\mathrm{CombMdlCat}_\Delta)$ is labeled by the corresponding ∞-site $\mathbf{S}$ (Ntn. 3.2.23).

The main instance of this class examples (Exp. 3.2.24) that is of interest below is the *2-category of orbi-singularities* (or *global orbit category*), see Ntn. 3.3.57. A variation of this example is the orbit category of a topological group, which will appear for us (in Def. 4.3.14 below) as a special case of the following variant of the above construction:

Example 3.2.25 (∞-Sites from Cartesian closed presentable ∞-categories)**.** If $\mathbf{C} \in \mathrm{PresCat}_\infty$ is Cartesian closed (as in Prop. 3.2.55 below) with mapping object ∞-functor

$$\mathrm{Map}((-), (-)) \;:\; \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$$

and if

$$R \;:\; \mathbf{C} \to \mathrm{Grpd}_\infty$$

is any ∞-functor which preserves products, then we obtain an ∞-site $\mathbf{S}$ with objects any sub-class of that of $\mathbf{C}$ and mapping complex for for X, Y a pair of objects given by

$$\mathbf{S}(X,Y) \;:=\; R\big(\mathrm{Map}(X,Y)\big)$$

and with composition given by

$$\mathbf{S}(X,Y)\times\mathbf{S}(Y,Z) \;\simeq\; R\big(\mathrm{Map}(X,Y)\big)\times R\big(\mathrm{Map}(Y,Z)\big) \;\simeq\; R\big(\mathrm{Map}(X,Y)\times\mathrm{Map}(Y,Z)\big) \xrightarrow{R\left((-)\circ(-)\right)} \mathrm{Map}(X,Z)\,.$$

Simplicial presheaves and ∞-stacks.

Notation 3.2.26 (Model categories of simplicial presheaves)**.** Given a simplicial site $(\mathscr{S}, J)$ (Def. 3.2.22), we write:

(i) $\Delta\mathrm{PSh}(\mathscr{S}) \;\in\; \Delta\mathrm{Cat}$ for the simplicial category of simplicial presheaves on $\mathscr{S}$,

(ii) $\Delta\mathrm{PSh}(\mathscr{S})_{\mathrm{proj}} \;\in\; \mathrm{CombMdlCat}_{\Delta,\mathrm{prop}}$ for the global projective model category structure,

(iii) $\Delta\mathrm{PSh}(\mathscr{S},J)_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \;\in\; \mathrm{CombMdlCat}_{\Delta,\mathrm{prop}}$ for the J-local projective model category structure, whose weak equivalences are the J-stalk-wise weak equivalences in $\Delta\mathrm{Set}_{\mathrm{Qu}}$ (the "hypercomplete" local model structure).

For our purposes, the following may be taken to be the definition of ∞-toposes:

Proposition 3.2.27 (Presentation of ∞-stacks by simplicial presheaves ([TV05][Lur09], following [Du01a][Bro73]))**.** *Let $(\mathscr{S}, J)$ be a simplicial site.*

(i) *The Čech/stalk-local projective model category structure on simplicial presheaves over $\mathscr{S}$ presents the topological/hypercomplete ∞-topos over $\mathscr{S}$:*

$$\Delta\mathrm{PSh}(\mathscr{S})_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \xhookrightarrow{\mathrm{Loc}^{\mathrm{LclWEqs}}} \mathrm{Loc}^{\mathrm{LclWEqs}}_{\Delta}\big(\Delta\mathrm{PSh}(\mathscr{S},J)_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\big) \;\simeq\; \mathrm{Sh}_\infty(\mathbf{S},J) \quad\in\; \mathrm{Ho}_2(\mathrm{PresCat}_\infty)\,,. \tag{3.52}$$

(ii) *In particular, for all cofibrant X and fibrant A in $\Delta\mathrm{PSh}(\mathscr{C})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$ there is a natural equivalence of hom-∞-groupoids*

$$\Delta\mathrm{PSh}(\mathscr{S})(X,A) \;\simeq\; \mathrm{Sh}_\infty\left(\mathrm{Loc}^{\mathrm{LclWEqs}}(X),\mathrm{Loc}^{\mathrm{LclWEqs}}(A)\right)\,. \tag{3.53}$$

Notation 3.2.28 (Presentation of ∞-presheaves by simplicial presheaves)**.** When the Grothendieck topology J on $\mathscr{S}$ (in Prop. 3.2.27) is trivial, we write, as usual,

$$\mathrm{PSh}_\infty(\mathbf{S}) \;:=\; \mathrm{Sh}_\infty(\mathbf{S},\mathrm{triv}) \quad\in\; \mathrm{Ho}_2(\mathrm{PresCat}_\infty)$$

for the ∞-category of ∞-presheaves over the ∞-category $\mathbf{S}$.

Proposition 3.2.29 (∞-Yoneda lemma [LurYo, Prop. 8.2.1.3, Thm. 8.2.5.4][Lur09, Prop. 5.1.3.1, 5.5.2.1][RV09, Thm. 7.2.22])**.** *Let $\mathscr{S} \in \Delta\mathrm{Cat}$, not necessarily small.*

(i) *The assignment of representable ∞-presheaves is a fully faithful embedding* (8)

$$\begin{array}{ccc} \mathbf{S} & \xhookrightarrow{\;\;y\;\;} & \mathrm{PSh}_\infty(\mathbf{S})\,. \\ U & \longmapsto & \mathbf{S}(-,U) \end{array} \tag{3.54}$$

(ii) *For $U \in \mathbf{S}$ and $X \in \mathrm{PSh}_\infty(\mathbf{S})$,*

$$\mathrm{PSh}_\infty\left(y(U),X\right) \;\simeq\; X(U)\,. \tag{3.55}$$

Handling ∞-stacks via presenting simplicial presheaves. We discuss a few techniques for reasoning about ∞-stacks – on the right of (3.52) – by constructions in the 1-category of simplicial presheaves – on the left of (3.52).

Lemma 3.2.30 (Objectwise connected ∞-sheaves are connected)**.** *A sufficient (but not necessary) condition for an object $X \in \mathrm{Sh}_\infty(\mathbf{S})$ to be connected is that it is connected as an object of* $\mathrm{PSh}_\infty(\mathbf{S})$*, i.e., that for all $U \in \mathbf{S}$ we have that $X(U) \in \mathrm{Grpd}_\infty$ is connected.*

Proof. Consider a presentation of $\mathrm{Sh}_\infty(\mathbf{S})$ by a model category of simplicial presheaves. Then 0-truncation τ_0 is the derived functor of objectwise truncation of simplicial sets followed by ordinary sheafification. But under the given assumption, the objectwise truncation is the presheaf with constant value the singleton set. This is a sheaf, the terminal sheaf (by the fact that inclusion into presheaves preserves the empty limit), hence is already the derived truncation operation in question. □

Lemma 3.2.31 (Simplicial presheaves model homotopy colimit of their component sheaves)**.** *The localization* (3.52) *of any $X_\bullet \in \Delta\mathrm{PSh}(\mathscr{S})$ is equivalently the homotopy colimit of its components $X_n \in \mathrm{PSh}(\mathscr{S}) \xhookrightarrow{\mathrm{const}} \Delta\mathrm{PSh}(\mathscr{S})$:*

$$\mathrm{Loc}^{\mathrm{LclWEqs}}(X_\bullet) \;\simeq\; \varinjlim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Loc}^{\mathrm{LclWEqs}}(X_n) \;\;\in\; \mathrm{Loc}^{\mathrm{LclWEqs}}_{\Delta}\big(\Delta\mathrm{PSh}(\mathscr{S})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\big)\,.$$

Proof. The analogous statement for simplicial sets (i.e. for $\mathscr{S} = *$) is classical. This implies the statement for the global projective model structure since its homotopy colimits are computed objectwise. The full statement now follows since Bousfield localization to the local model structure is a (Quillen/derived/∞-) left adjoint. □

The following Lem 3.2.32 is clearly a special case of a much more general statement, but we highlight it in this form for definiteness, as we will have crucial use of this, for instance in the proof of Prop. 4.2.4 below:

Lemma 3.2.32 (Computing homotopy pullbacks of ∞-stacks)**.** *Given a simplicial site $(\mathscr{S}, J)$ and a diagram of simplicial presheaves (Ntn. 3.2.26) of the form*

$$\begin{tikzcd} & P \arrow[d, "\in \mathrm{PrjWEqs}"] \\ X \times_A \widehat{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{P} \arrow[d, "\in \mathrm{PrjFib}"] \\ X \arrow[r] & A \end{tikzcd} \quad \in\; \Delta\mathrm{PSh}(\mathscr{S}) \tag{3.56}$$

then the homotopy pullback (4) *of P with respect to the local projective model structure $\Delta\mathrm{PSh}(\mathscr{S},J)_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$ is presented already by the homotopy pullback in the global projective model structure, which in turn is presented by the ordinary pullback of any projective fibration resolution $\widehat{P}$:*

$$\mathrm{Loc}^{\mathrm{LclWEqs}}(X \times_A \widehat{P}) \;\simeq\; \mathrm{Loc}^{\mathrm{LclWEqs}}(X) \underset{\mathrm{Loc}^{\mathrm{LclWEqs}}(A)}{\times} \mathrm{Loc}^{\mathrm{LclWEqs}}(P) \quad \in\; \mathrm{Loc}^{\mathrm{WEqs}}_{\Delta}\big(\Delta\mathrm{PSh}(\mathscr{S},J)\big)\,.$$

Proof. The point is that the left Bousfield localization of the model categories has underlying identity functors but models the lex localization of presentable ∞-categories which preserves finite ∞-limits, in particular homotopy pullbacks (4):

$$\mathrm{Loc}^{\mathrm{WEqs}}_{\Delta}\Big(\Delta\mathrm{PSh}(\mathscr{S},J)_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \Delta\mathrm{PSh}(\mathscr{S},J)_{\mathrm{proj}}\Big) \;\simeq\; \Big(\mathrm{Sh}_\infty\big(\mathrm{Loc}^{\mathrm{WEqs}}_{\Delta}(\mathscr{S})\big) \overset{\mathrm{lex}}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}} \mathrm{PSh}_\infty\big(\mathrm{Loc}^{\mathrm{WEqs}}_{\Delta}(\mathscr{S})\big)\Big)\,.$$

This shows that the local homotopy pullback is given, by the image of the global homotopy pullback under the left derived functor of the identity, hence by a global cofibrant resolution of the global homotopy pullback. But left Bousfield localization does not change the class of cofibrations (just the class of acyclic cofibrations) so that global cofibrant resolution is implied by local cofibrant resolution, so that, under simplicial localization, the plain global homotopy pullback does present the local homotopy pullback.

Hence it just remains to see that for computing the global homotopy pullback of a cospan diagram of simplicial presheaves it is sufficient to resolve one of the two morphisms by a global fibration. Since the global projective model strcuture is evidently right proper (since $\Delta\mathrm{Set}_{\mathrm{Qu}}$ is so), this follows by a classical result (e.g. [Lur09, Prop. A.2.4.4]). □

This Lemma 3.2.32 is particularly useful since there is a convenient way to achieve the factorization in (3.56):

Notation 3.2.33 (Canonical projective path object for simplicial presheaves)**.** For $A \in \Delta\mathrm{PSh}(\mathscr{S})$ (Ntn. ref(Ntn. 3.2.26)), we write

$$A^I \,:\, s \mapsto \big(A(s)\big)^I := \mathrm{Map}\big(\Delta^1, A(s)\big) := \Delta\mathrm{Set}\big((-)\times\Delta^1, A(s)\big) \quad \in \Delta\mathrm{Set} \tag{3.57}$$

for the simplicial presheaf $A^I \in \Delta\mathrm{PSh}(\mathscr{S})$ which assigns to any object $s \in \mathscr{S}$ the simplicial function complex from Δ^1 into the value $A(s) \in \Delta\mathrm{Set}$ of the presheaf on that object.

This is evidently a *path space object* for A (in the sense of [Qu67, Def. I.4], review in [FSS20-TCD, Def. A.0.12]) in that with the evident face and degeneracy maps it sits in a diagram of simplicial presheaves of this form:

$$\begin{tikzcd} A \arrow[r, "\sigma", "\in \mathrm{PrjWEqs}"'] \arrow[rr, bend right, "\mathrm{diag}"'] & A^I \arrow[r, "(\mathrm{ev}_0{,}\mathrm{ev}_1)", "\in\mathrm{PrjFib}"'] & A \times A \end{tikzcd} \tag{3.58}$$

Lemma 3.2.34 (Factorization lemma for projective simplicial presheaves (specializing [Bro73, p. 421]))**.** *Given a morphism*

$$P \xrightarrow{f} A \quad \in \big(\Delta\mathrm{PSh}(\mathscr{S})_{\mathrm{proj}}\big)_{\mathrm{fib}}$$

of projectively fibrant simplicial presheaves (Ntn. 3.2.26), the following pullback (formed by using the path space object A^I from Ntn. 3.2.33) exhibits a factorization of this morphism through a weak equivalence followed by a fibration in $\Delta\mathrm{PSh}(\mathscr{S})_{\mathrm{proj}}$*:*

$$\begin{tikzcd} P \arrow[dr, dashed, "(\mathrm{id}_P{,}\sigma_A\circ f)", "\in\mathrm{PrjWEqs}"'] \arrow[drr, bend left, "\mathrm{id}_P"] \arrow[dddr, "f"'] & & \\ & P \underset{A}{\times} A^I \arrow[r] \arrow[d] \arrow[dd, bend right, "\in\mathrm{PrjFib}"'] \arrow[dr, phantom, "\text{(pb)}"] & P \arrow[d, "f"] \\ & A^I \arrow[r, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & A \\ & A\,. & \end{tikzcd} \tag{3.59}$$

Example 3.2.35 (Quotient stack of coset space is delooping of subgroup)**.** For $(\mathscr{S}, J)$ a (simplical) site, consider a presheaf G of ordinary groups, presenting a group stack in the ∞-topos over the site, and a subgroup-object H:

$$H \overset{i_H}{\hookrightarrow} G \;\in\; \mathrm{Grp}\big(\mathrm{PSh}(\mathscr{S})\big) \hookrightarrow \mathrm{Grp}\big(\Delta\mathrm{PSh}(\mathscr{S})\big) \xrightarrow{\mathrm{Grp}\left(\mathrm{Loc}^{\mathrm{LclWEqs}}\right)} \mathrm{Grp}\big(\mathrm{Sh}_\infty(\mathbf{S}, J)\big)\,.$$

Then the homotopy fiber of the ∞-stackification of the induced morphism of presheaves of nerves of delooping groupoids

$$N(H \rightrightarrows *) \xrightarrow{N(i_H \rightrightarrows *)} N(G \rightrightarrows *) \quad \in \Delta\mathrm{PSh}(\mathscr{S}) \xrightarrow{\mathrm{Loc}^{\mathrm{LclWEqs}}} \mathrm{Sh}_\infty(\mathscr{S}, J)$$

is computed as follows, where we are applying Lemma 3.2.32 to the projective fibration resolution of the above morphism which is provided by the factorization lemma 3.2.34:

$$\begin{tikzcd} N\big((G/H) \rightrightarrows (G/H)\big) & N(H \rightrightarrows *) \arrow[d, "N\left(\left(\mathrm{const}_e{,}(i_H{,}(-)^{-1})\right) \rightrightarrows \mathrm{const}_e\right) \in \mathrm{PrjWEqs}"] \\ N(G \times H^{\mathrm{op}} \rightrightarrows G) \arrow[u, "\in\mathrm{PrjWEqs}"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & N\big(G \times (G \times H^{\mathrm{op}}) \rightrightarrows G\big) \arrow[d, "N\left(\mathrm{pr}_2 \rightrightarrows \mathrm{const}_*\right) \in \mathrm{PrjFib}"] \\ N(* \rightrightarrows *) \arrow[r] & N(G \rightrightarrows *) \end{tikzcd} \quad \overset{\mathrm{Loc}^{\mathrm{LclWEqs}}}{\longmapsto} \quad \begin{tikzcd} G/H \arrow[r] \arrow[d, "\wr"] & \mathbf{B}H \arrow[d, "\wr"] \\ G /\!\!/ H \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (G/H) /\!\!/ G \arrow[d] \\ * \arrow[r] & \mathbf{B}G \end{tikzcd}$$

Comparison with Prop. 0.2.1 on the right shows first of all (for the case $H = 1$ the trivial subgroup), that the ∞-stackification of the nerve of the delooping groupoid of G is the ∞-topos theoretic delooping of the ∞-stackification of G:

$$\mathrm{Loc}^{\mathrm{LclWEqs}}\big(N(G \rightrightarrows *)\big) \;\simeq\; \mathbf{B}G\,; \tag{3.60}$$

and then, more generally, that the homotopy quotient of any $H \subset G$-coset object in the ∞-topos by its residual G-action is equivalently the delooping stack of H:

$$(G/H)/\!\!/G \;\simeq\; \mathbf{B}H \;\in\; \mathrm{Sh}_\infty(\mathbf{S},J)\,. \tag{3.61}$$

Example 3.2.36 (Double homotopy coset stacks)**.** In direct generalization of Exp. 3.2.35 the Factorization Lemma 3.2.34 combined with Lemma 3.2.32 shows that for a pair of presheaves of sub-groups

$$H_1, H_2 \in \mathrm{Grp}\big(\mathrm{PSh}(\mathscr{S})\big) \hookrightarrow \mathrm{Grp}\big(\Delta\mathrm{PSh}(\mathscr{S})\big) \xrightarrow{\;\mathrm{Grp}\left(\mathrm{Loc}^{\mathrm{LclWEqs}}\right)\;}$$

the homotopy fiber product of the delooping of their inclusion is the homotopy double coset object:

$$\begin{tikzcd} (G/H_1)/\!\!/H_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}H_1 \arrow[d, "\mathbf{B}(i_{H_1})"] \\ \mathbf{B}H_2 \arrow[r, "\mathbf{B}(i_{H_2})"'] & \mathbf{B}G \end{tikzcd} \quad \in \; \mathrm{Sh}_\infty(\mathbf{S},J)\,.$$

Here H_1 acts by multiplication from one side, H_2 by residual multiplication from the other side. Notice that the symmetry of the Cartesian diagram shows as once that

$$(G/H_1)/\!\!/H_2 \;\simeq\; (G/H_2)/\!\!/H_1\,.$$

This way, while the Factorization Lemma 3.2.34 serves to handle most our fibrancy needs (together with Prop. 3.3.27 below), the following Prop. 3.2.37 is our tool for ensuring cofibrancy (in Exp. 3.3.39 and Exp. 3.3.42 below):

Proposition 3.2.37 (Dugger's cofibrancy recognition [Du01a, Cor . 9.4])**.** *Let $\mathscr{S}$ be a 1-site. A sufficient condition for $X \in \mathrm{SimplPSh}(\mathscr{S})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$ (Ntn. 3.2.26) to be projectively cofibrant is that in each simplicial degree k, the component presheaf $X_k \in \mathrm{PSh}(\mathscr{S})$ is*

(i) *a coproduct $X_k \simeq \coprod_{i_k} U_{i_k}$ of representables $U_{i_k} \in \mathscr{S} \overset{y}{\hookrightarrow} \mathrm{PSh}(\mathscr{S})$;*

(ii) *whose degenerate cells split off as a disjoint summand: $X_k \simeq N_k \coprod \mathrm{im}(\sigma)$ for some N_k.*

Example 3.2.38 (Basic examples of projectively cofibrant simplicial presheaves)**.** Let $\mathscr{S}$ be a 1-site. Examples of projectively cofibrant simplicial presheaves over $\mathscr{S}$ include:

(i) every representable $U \in \mathscr{Y} \overset{y}{\hookrightarrow} \mathrm{SimplPSh}(\mathscr{S})$;

(ii) every constant simplicial presheaf $S \in \mathrm{SimpSets} \xhookrightarrow{\mathrm{const}} \mathrm{SimplPSh}(\mathscr{S})$;

and in joint generalization of these two cases:

(iii) every product $U \times S$ of a representable with a simplicial set.

In all cases the defining lifting property is readily checked. Alternatively, these follow with Prop. 3.2.37.

Kan extension to presheaves. A central operation in §3.3 below is the induction, by *Kan extension* of quadruples of adjoint ∞-functors between ∞-stacks from pairs of adjoint functors between simplicial presheaves. For completeness, we make explicit the elementary but crucial operations.

Lemma 3.2.39 (Kan extension of ∞-functor to ∞-presheaves)**.** *If $\mathbf{S}_1 \xrightarrow{f} \mathbf{S}_2$ is an ∞-functor between small ∞-sites (Ntn. 3.2.23), then precomposition $\phi^* := \mathrm{PSh}_\infty(\mathbf{S}_2) \xrightarrow{(-)\circ\phi} \mathrm{PSh}_\infty(\mathbf{S}_1)$ preserves both limits and colimits, and hence has left- and right adjoint ∞-functors*

$$
\begin{tikzcd}
\mathrm{PSh}_\infty(\mathbf{S}_1) \arrow[r, shift left=8, "f_!"] \arrow[r, phantom, shift left=4, "\perp"] & \mathrm{PSh}_\infty(\mathbf{S}_2) \arrow[l, "f^*" description] \arrow[l, phantom, shift left=4, "\perp"] \\
\end{tikzcd}
$$

$$\mathrm{PSh}_\infty(\mathbf{S}_1) \underset{f_*}{\overset{f_!}{\rightrightarrows}} \mathrm{PSh}_\infty(\mathbf{S}_2)\,, \quad f^* : \mathrm{PSh}_\infty(\mathbf{S}_2) \to \mathrm{PSh}_\infty(\mathbf{S}_1)\,, \quad f_! \dashv f^* \dashv f_* .$$

Lemma 3.2.40 (Left Kan extension on representables is original functor)**.** *On representables, the left Kan extension $f_!$ (Lem. 3.2.39) acts as the original functor f:*

$$f_!\big(y(s_1)\big) \;\simeq\; y\big(f(s_1)\big)\,.$$

Proof. For $s_1 \in \mathbf{S}_1$ and $X \in \mathrm{PSh}_\infty(\mathbf{S}_2)$ we have the following sequence of natural equivalences:

$$
\begin{aligned}
\mathrm{PSh}_\infty(\mathbf{S}_2)\,\big(f_!\big(y(s_1)\big), X\big) &\simeq \mathrm{PSh}_\infty(\mathbf{S}_1)\big(y(s_1), f^*(X)\big) \\
&\simeq X\big(f(s_1)\big) \\
&\simeq \mathrm{PSh}_\infty(\mathbf{S}_2)\Big(y\big(f(s_1)\big), X\Big)\,.
\end{aligned}
$$

Since these equivalences are natural in X, the ∞-Yoneda lemma (Prop. 3.2.29) for $\mathrm{PSh}_\infty(\mathbf{S}_2)^{\mathrm{op}}$, implies the claim. ∎

Lemma 3.2.41 (Left Kan extension preserves binary products if original functor does)**.** *If a pair of small ∞-categoris $\mathbf{S}_1, \mathbf{S}_2$ has binary products and an ∞-functor $\mathbf{S}_1 \xrightarrow{f} \mathbf{S}_2$ preserves these, in that for $s, s' \in \mathbf{S}_1$ there is a natural equivalence $f(s \times s') \simeq f(s) \times f(s')$, then its left Kan extension also preserves binary products. That is, for $X, X' \in \mathrm{PSh}_\infty(\mathbf{S}_1)$, there is a natural equivalence*

$$f_!(X \times X') \;\simeq\; f_!(X) \times f_1(X')\,.$$

Proof. We have the sequence of natural equivalences

$$
\begin{aligned}
f_!(X \times X') &\simeq f_!\Big(\big(\varinjlim_{s \to X} y(s)\big) \times \big(\varinjlim_{s' \to X} y(s')\big)\Big) \\
&\simeq f_!\Big(\varinjlim_{s \to X}\varinjlim_{s' \to X}\big(y(s) \times y(s')\big)\Big) \\
&\simeq \varinjlim_{s \to X}\varinjlim_{s' \to X} f_!\,\big(y(s) \times y(s')\big) \\
&\simeq \varinjlim_{s \to X}\varinjlim_{s' \to X} f_!\,(y(s)) \times f_!\,\big(y(s')\big) \\
&\simeq \Big(\varinjlim_{s \to X} f_!\,(y(s))\Big) \times \Big(\varinjlim_{s' \to X} f_!\,\big(y(s')\big)\Big).
\end{aligned}
$$

∎

Lemma 3.2.42 (Left Kan extension of fully faithful functor is fully faithful)**.** *The left Kan extension $f_!$ of a fully faithful functor f between small ∞-categories is itself fully faithful* (8)*:*

$$\mathbf{S}_1 \xhookrightarrow{\;i\;} \mathbf{S}_2 \qquad \Rightarrow \qquad \mathrm{PSh}_\infty(\mathbf{S}_1) \xhookrightarrow{\;i_!\;} \mathrm{PSh}_\infty(\mathbf{S}_2)\,.$$

Proof. For $X, X' \in \mathrm{PSh}_\infty(\mathbf{S}_2)$, we have the following sequence of natural equivalences:

$$
\begin{aligned}
\mathrm{PSh}_\infty(\mathbf{S}_1)\big(i_!(X),\, i_!(X')\big) &\simeq \mathrm{PSh}_\infty(\mathbf{S}_1)\Big(i_!\big(\varinjlim_{s\to X} y(s)\big),\, i_!\big(\varinjlim_{s'\to X'} y(s')\big)\Big) \\
&\simeq \mathrm{PSh}_\infty(\mathbf{S}_1)\Big(\varinjlim_{s\to X} i_!\,\big(y(s)\big)\,,\, \varinjlim_{s'\to X'} i_!\,\big(y(s')\big)\Big) \\
&\simeq \mathrm{PSh}_\infty(\mathbf{S}_1)\Big(\varinjlim_{s\to X} y\,\big(i(s)\big)\,,\, \varinjlim_{s'\to X'} y\,\big(i(s')\big)\Big) \\
&\simeq \varprojlim_{s\to X}\varinjlim_{s'\to X'} \mathrm{PSh}_\infty(\mathbf{S}_1)\,\big(y\,\big(i(s)\big)\,,\, y\,\big(i(s')\big)\big) \\
&\simeq \varprojlim_{s\to X}\varinjlim_{s'\to X'} \mathbf{S}_1\,\big(i(s),\, i(s')\big) \\
&\simeq \varprojlim_{s\to X}\varinjlim_{s'\to X'} \mathbf{S}_2(s,\, s') \\
&\simeq \varprojlim_{s\to X}\varinjlim_{s'\to X'} \mathrm{PSh}_\infty(\mathbf{S}_2)\,\big(y(s),\, y(s')\big) \\
&\simeq \mathrm{PSh}_\infty(\mathbf{S}_2)\Big(\varinjlim_{s\to X} y(s),\, \varinjlim_{s'\to X'} y(s')\Big) \\
&\simeq \mathrm{PSh}_\infty(\mathbf{S}_2)\,\big(X,\, X'\big)\,.
\end{aligned}
$$
□

Lemma 3.2.43 (Kan extensions of an adjoint pair yield an adjoint quadruple)**.** *Given a pair of adjoint ∞-functors between small ∞-categories*

$$\begin{tikzcd} \mathbf{S}_1 \arrow[r, shift left, "\ell"] \arrow[r, phantom, "\perp"] & \mathbf{S}_2 \arrow[l, shift left, "r"] \end{tikzcd}$$

the corresponding pullback functors on presheaves are adjoint to each other, $\ell^* \dashv r^*$, and their adjoint triples of Kan extensions (from Lem. 3.2.39) overlap:

$$\begin{tikzcd} \mathrm{PSh}_\infty(\mathbf{S}_1) \arrow[r, shift left=9, "\ell_!"] \arrow[r, shift left=3, "\ell_* \simeq r^*"] & \mathrm{PSh}_\infty(\mathbf{S}_2) \arrow[l, shift left=3, "\ell^* \simeq r_!"'] \arrow[l, shift left=9, "r_*"] \end{tikzcd}\,.$$

Proof. The characterizing hom-equivalence of the adjunction $\ell^* \dashv r^*$ is the composition of the following sequence of natural equivalences, for $X_i \in \mathrm{PSh}_\infty(\mathbf{S}_i)$ is checked along the lines of the proof of Lem. 3.2.40. From this, the overlap of the adjoint triples of Kan extensions follows by essential uniqueness of adjoints. □

Systems of local sections.

Notation 3.2.44 (∞-Yoneda embedding of slice sites)**.** For $\mathbf{S}$ a small ∞-category and $X \in \mathbf{S}$ an object, we have both the ∞-Yoneda embedding (3.54) of the slice $\mathbf{S}_{/X}$

$$
\begin{array}{ccc}
\mathbf{S}_{/X} & \xhookrightarrow{\;y_{(\mathbf{S}_{/X})}\;} & \mathrm{PSh}_\infty(\mathbf{S}_{/X}) \\
(U \xrightarrow{\phi} X) & \longmapsto & \mathbf{S}_{/X}\big(-,\,(U \xrightarrow{\phi} X)\big)
\end{array}
\tag{3.62}
$$

and also the slicing of the Yoneda embedding on $\mathbf{S}$:

$$
\begin{array}{ccc}
\mathbf{S}_{/X} & \xhookrightarrow{\;(y_{\mathbf{S}})_{/X}\;} & \mathrm{PSh}_\infty(\mathbf{S})_{/y_{\mathbf{S}}(X)} \\
(U \xrightarrow{\phi} X) & \longmapsto & \big(\mathbf{S}(-,U) \xrightarrow{\mathbf{S}(-,\phi)} \mathbf{S}(-,X)\big).
\end{array}
\tag{3.63}
$$

Proposition 3.2.45 (Systems of local sections of bundles internal to ∞-presheaves)**.** *For $\mathbf{B}$ an ∞-topos, $\mathbf{S}$ a small ∞-category and $X \in \mathbf{S}$ an object, the ∞-functor which assembles local sections of bundles of ∞-presheaves over (the image under the ∞-Yoneda embedding of) X is an equivalence of ∞-categories:*

$$
\mathrm{PSh}(\mathbf{S},\mathbf{B})_{/y_{\mathbf{S}}(X)} \xrightarrow[\sim]{\;\Gamma_{(-)}(-) := \mathrm{PSh}(\mathbf{S},\mathbf{B})_{/y_{\mathbf{S}}(X)}\big((y_{\mathbf{S}})_{/X}(-),\,-\big)\;} \mathrm{PSh}(\mathbf{S}_{/X},\mathbf{B})\,,
\tag{3.64}
$$

where $(y_{\mathbf{S}})_{/X}$ is from (3.62)*.*

Proof. For the analogous equivalence for 1-categories (which is classical, e.g. [KS06, Lem. 1.4.12]) one readily checks[2] that:

(1) the analogous functor is a right adjoint, and

(2) generalizes to a simplicial adjunction of simplicial presheaves over simplicial sites,

(3) where it is a Quillen equivalence for the projective model structure and its slice model structure.

This implies the claim, by Prop. 3.1.6. □

An alternative general argument for some equivalence of ∞-categories as in Prop. 3.2.45 is given in [Lur09, Cor. 5.1.6.12], in terms of quasi-categories. On the other hand, in our formulation the proof of Prop. 3.2.45 applies verbatim also in the case that the base presheaf is not necessarily representable (not necessarily in the image of the Yoneda embedding), if we use the following more general notion of slice ∞-site:

Definition 3.2.46 (Slice ∞-site)**.** Given an ∞-site (Ntn. 3.2.23) $\mathbf{S}$ and an ∞-presheaf X *in* $\mathrm{PSh}_\infty(\mathbf{S})$ (Ntn. 3.2.28) the *slice ∞-site* $\mathbf{S}_{/X}$ is the full sub-∞-category of the slice $\mathrm{PSh}_\infty(\mathbf{S})_{/X}$

$$\mathbf{S}_{/X} \hookrightarrow \mathrm{PSh}_\infty \mathbf{S}_{/X}\,. \tag{3.65}$$

on the morphisms with representable domain, i.e. those of the form $y_{\mathbf{S}}(U) \xrightarrow{\phi} X$ for some $U \in \mathbf{S}$.

By Prop. 3.2.63 this means that the hom-∞-groupoids in a slice ∞-site (3.65) are given by the following homotopy fiber product of ∞-groupoids:

$$\begin{tikzcd} \mathbf{S}_{/X}\Big(\big(y_{\mathbf{S}}(U) \xrightarrow{\phi} X\big), \big(y_{\mathbf{S}}(U') \xrightarrow{\phi'} X\big)\Big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{PSh}_\infty(\mathbf{S})\big(y_{\mathbf{S}}(U), y_{\mathbf{S}}(U')\big) \arrow[d, "\phi' \circ (-)"] \\ * \arrow[r, "\vdash \phi"'] & \mathrm{PSh}_\infty(\mathbf{S})\big(y_{\mathbf{S}}(U), X\big) \end{tikzcd} \tag{3.66}$$

With this, the proof of Prop. 3.2.45 generalizes to:

Proposition 3.2.47 (Fundamental theorem of ∞-presheaf ∞-topos theory)**.** *For* $\mathbf{S}$ *an ∞-site and any* $X \in \mathrm{PSh}_\infty(\mathbf{S})$ *there is an equivalence of ∞-categories between the slice of all ∞-presheaves over X and the ∞-presheaves over the slice ∞-site (Def. 3.2.46):*

$$\mathrm{PSh}_\infty(\mathbf{S})_{/X} \;\simeq\; \mathrm{PSh}_\infty\big(\mathbf{S}_{/X}\big)\,. \tag{3.67}$$

A proof for this statement in the special case that X is 1-truncated also essentially appears as [Ho08, Thm. 6.1(a)].

The equivalence (3.67) in Prop. 3.2.47 shows in particular that every slice of a presheaf ∞-topos (on the left) is again an ∞-topos (manifest on the right). This is the archetypical special case of the *fundamental theorem of ∞-topos theory* (e.g. [Ve19, Rem. 2.2.10], following the terminology for 1-toposes [Mc95]), which says that any slice of any ∞-topos is again an ∞-topos:

Proposition 3.2.48 (Fundamental theorem of ∞-topos theory [Lur09, Prop. 6.3.5.1])**.**
For $\mathbf{H}$ *an ∞-topos and* $B \in \mathbf{H}$ *an object, the slice ∞-category* $\mathbf{H}_{/B}$ (15) *is also an ∞-topos.*

Example 3.2.49 (Basic structures in slice infinity topos)**.** In $\mathbf{H}_{/B}$ the terminal object is (B, id_B).

Base change of ∞-toposes. In higher generalization of Lem. 1.1.7, much of the theory of (equivariant) ∞-toposes is driven by their base change adjoint triples:

[2] This is spelled out also at `ncatlab.org/nlab/show/slice+of+presheaves+is+presheaves+on+slice`

Proposition 3.2.50 (Base change). *For $\mathbf{H} \in \mathrm{Topos}_\infty$ and $B_1 \xrightarrow{f} B_2$ a morphism in $\mathbf{H}$, there is an adjoint triple of ∞-functors between the slice ∞-toposes (Prop. 3.2.48):*

$$
\begin{tikzcd}
\mathbf{H}_{/B_1}
\arrow[rr, shift left=14pt, "f_! = \Sigma_f"]
\arrow[rr, phantom, shift left=7pt, "\perp"]
\arrow[from=rr, "f^*" description]
\arrow[rr, phantom, shift right=7pt, "\perp"]
\arrow[rr, shift right=14pt, "f_* = \Pi_f"']
&& \mathbf{H}_{/B_2}
\end{tikzcd}
\qquad
\begin{array}{l}
\text{left base change}\\
\text{base change}\\
\text{right base change}
\end{array}
\tag{3.68}
$$

where f^* is given by pullback along f and $f_!$ is given by postcomposition with f.

Example 3.2.51 (Base change to absolute context). For $B \in \mathbf{H}$ any object the base change (Prop. 3.2.50) along its terminal morphism $B \xrightarrow{\exists !} *$ is the operation of taking the Cartesian product with B, and the corresponding adjoint triple (3.68) looks as follows:

$$
\begin{tikzcd}
\mathbf{H}_{/B}
\arrow[rr, shift left=14pt, "\mathrm{dom} = \Sigma_B"]
\arrow[rr, phantom, shift left=7pt, "\perp"]
\arrow[from=rr, "(-)\times B" description]
\arrow[rr, phantom, shift right=7pt, "\perp"]
\arrow[rr, shift right=14pt, "\Pi_B"']
&& \mathbf{H}_{/*} \simeq \mathbf{H}\,.
\end{tikzcd}
\tag{3.69}
$$

Lemma 3.2.52 (Frobenius reciprocity). *For $B_1 \xrightarrow{f} B_2$ a morphism in an ∞-topos $\mathbf{H}$, we have for $E_1 \in \mathbf{H}_{/B_1}$ and $E_2 \in \mathbf{H}_{/B_2}$ a natural equivalence*

$$
f_!\big(E_1 \times f^*(E_2)\big) \;\simeq\; f_!(E_1) \times_{B_2} E_2 \quad \in \mathbf{H}_{/B_2}
\tag{3.70}
$$

of left base changes (3.68).

Proof. Since, by Prop. 3.2.50, the left base change is given by postcomposition with f, this follows by the pasting law (5):

$$
\begin{tikzcd}
f_!(E_1) \times_{B_2} E_2 \arrow[r] \arrow[d] \arrow[rr, bend left, "p_{f_!(E_1 \times_{B_1} f^*(E_2))}"] \arrow[dr, phantom, "\text{(pb)}"] & f^*(E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & E_2 \arrow[d, "p_{E_2}"] \\
E_1 \arrow[r, "p_{E_1}"'] \arrow[rr, bend right, "p_{f_! E_1}"'] & B_1 \arrow[r, "f"'] & B_2
\end{tikzcd}
$$

□

3.2.2 General constructions in ∞-toposes

Truncation and connectivity.

Proposition 3.2.53 (n-Truncation modality [Lur09, 5.5.6.18, 6.5.1.2]). *For $n \in \{-2, -1, 0, 1, \cdots\}$ and $\mathbf{H}$ an ∞-topos, its full sub-∞-category of n-truncated objects is reflective, and the reflector τ_n preserves finite products:*

$$
\begin{tikzcd}
\mathbf{H}_n \arrow[r, hook, shift right=5pt, "i_n"'] \arrow[r, phantom, "\perp"] & \mathbf{H}\,. \arrow[l, shift right=5pt, "\tau_n"']
\end{tikzcd}
\tag{3.71}
$$

Notation 3.2.54 (Connected objects). An object $X \in \mathbf{H}$ is *connected* if $\tau_0 X \simeq *$. We write

$$
\mathbf{H}^{*/}_{\geq} \hookrightarrow \mathbf{H}_{\geq} \hookrightarrow \mathbf{H}
$$

for the full sub-∞-categories of (pointed and) connected objects.

Mapping stacks.

Proposition 3.2.55 (Mapping stacks)**.** *For* $\mathbf{H}$ *an* ∞*-topos and* $X \in \mathbf{H}$ *any object, the Cartesian product functor* $X \times (-)$ *has a right adjoint*

$$\mathbf{H} \underset{\mathrm{Map}(X,-)}{\overset{X\times(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{H}\,. \tag{3.72}$$

If $\mathbf{S}$ *is an* ∞*-site for* $\mathbf{H} \underset{\hookrightarrow}{\overset{\leftarrow \text{lex} -}{\perp}} \mathrm{PSh}_\infty(\mathbf{S})$, *then for* $U \in \mathbf{S}$ *the value of the mapping stack* $\mathrm{Map}(X,Y) \in \mathbf{H}$ *is naturally equivalent to*

$$\mathrm{Map}(X,Y)(U) \;\simeq\; \mathbf{H}\big(U, \mathrm{Map}(X,Y)\big) \;\simeq\; \mathbf{H}(U \times X, Y) \quad \in \mathrm{Grpd}_\infty\,. \tag{3.73}$$

Notation 3.2.56 (Evaluation map on mapping stacks)**.** The counit of the mapping stack adjunction (3.72) is the *evaluation map*:

$$X \times \mathrm{Map}(X,A) \xrightarrow{\mathrm{ev}_{X,A}} A\,. \tag{3.74}$$

Example 3.2.57 (Mapping space consists of global points of mapping stack)**.**

$$\mathbf{H}\big(*, \mathrm{Map}(X,A)\big) \;\simeq\; \mathbf{H}(X,A)\,. \tag{3.75}$$

Lemma 3.2.58 (∞-Stacks consist of their internal points)**.** *For* $X \in \mathbf{H}$ *there is a natural equivalence*

$$X \;\simeq\; \mathrm{Map}(*,X)\,.$$

Proof. For $U \in \mathbf{S}$, there is the natural equivalence (3.73)

$$\mathbf{H}(U,X) \;\simeq\; \mathbf{H}(U \times *, X) \;\simeq\; \mathbf{H}\big(U, \mathrm{Map}(*,X)\big)\,.$$

With this, the claim follows by the ∞-Yoneda lemma (Prop. 3.2.29). □

Lemma 3.2.59 (Looping of mapping stack is mapping stack into looping)**.** *We have a natural equivalence*

$$\Omega\mathrm{Map}(X,A) \;\simeq\; \mathrm{Map}(X, \Omega A)\,.$$

Proof. Since the looping operation is a limit, this follows from (3.72) by (11). □

Lemma 3.2.60 (Internal hom-adjointness)**.** *The kind of hom-equivalence that characterizes the right adjoint* (3.72) *also holds internally, in that for all* $X,Y,A \in \mathbf{H}$ *there is a natural equivalence*

$$\mathrm{Map}(X \times Y, A) \;\simeq\; \mathrm{Map}\big(X, \mathrm{Map}(Y,A)\big)\,.$$

Proof. For $U \in \mathbf{S}$, there is the natural equivalence (3.73)

$$\mathbf{H}\Big(U, \mathrm{Map}\big(X, \mathrm{Map}(Y,A)\big)\Big) \;\simeq\; \mathbf{H}\big(U \times X \times Y, A\big) \;\simeq\; \mathbf{H}\big(U, \mathrm{Map}(X \times Y, A)\big)\,.$$

With this, the claim follows by the ∞-Yoneda lemma (Prop. 3.2.29). □

Proposition 3.2.61 (Mapping stack construction preserves ∞-limits)**.** *The mapping stack construction (Prop. 3.2.55) preserves not only* ∞*-limits in its second argument, but also turns* ∞*-colimits in its first argument into* ∞*-limits: For* $A \in \mathbf{H}$ *and* $X_{(-)} : I \to \mathbf{H}$ *a small diagram, we have a natural equivalence*

$$\mathrm{Map}\big(\underset{\overrightarrow{i \in I}}{\lim} X_i, A\big) \;\simeq\; \underset{\overleftarrow{i \in I}}{\lim} \mathrm{Map}(X_i, A) \quad \in \mathbf{H}\,.$$

Proof. For $\mathbf{S}$ any ∞-site for $\mathbf{H}$ consider the following sequence of natural equivalences for $U \in \mathbf{S}$ any object:

$$\begin{aligned}
\mathbf{H}\Big(U, \mathrm{Map}\big(\underset{\overrightarrow{i \in I}}{\lim} X_i, A\big)\Big) &\simeq \mathbf{H}\Big(\big(\underset{\overrightarrow{i \in I}}{\lim} X_i\big) \times U, A\Big) && \text{by (3.73)} \\
&\simeq \mathbf{H}\Big(\underset{\overrightarrow{i \in I}}{\lim}\big(X_i \times U\big), A\Big) && \text{by (3.40)} \\
&\simeq \underset{\overleftarrow{i \in I}}{\lim} \mathbf{H}(X_i \times U, A) && \text{by (9)} \\
&\simeq \underset{\overleftarrow{i \in I}}{\lim} \mathbf{H}\big(U, \mathrm{Map}(X_i, A)\big) && \text{by (3.73)} \\
&\simeq \mathbf{H}\big(U, \underset{\overleftarrow{i \in I}}{\lim} \mathrm{Map}(X_i, A)\big) && \text{by (9)}
\end{aligned}$$

With this, the claim follows by the ∞-Yoneda lemma (Prop. 3.2.29). □

Proposition 3.2.62 (Base change preserves mapping stacks)**.** *For $B_1 \xrightarrow{f} B_2$ any morphism in an ∞-topos $\mathbf{H}$ we have for $X, Y \in \mathbf{H}_{/B_2}$ a natural equivalence*

$$f^*\mathrm{Map}(X,Y) \;\simeq\; \mathrm{Map}\big(f^*(X),\, f^*(Y)\big) \quad \in \mathbf{H}_{/B_!} \tag{3.76}$$

between the base change (Def. 3.2.50) of the mapping stack (3.72) *and the mapping stack of the base changes.*

Proof. For any $U \in \mathbf{H}_{/B_1}$, consider the following sequence of natural equivalences of Grpd_∞

$$\begin{aligned}
\mathbf{H}_{/B_1}\big(U, f^*\mathrm{Map}(X,Y)\big) &\simeq \mathbf{H}_{/B_2}\big(f_!U, \mathrm{Map}(X,Y)\big) && \text{by (3.68)}\\
&\simeq \mathbf{H}_{/B_2}\big((f_!U)\times_{B_2} X, Y\big) && \text{by (3.72)}\\
&\simeq \mathbf{H}_{/B_2}\big(f_!(U\times f^*(X)), Y\big) && \text{by (3.70)}\\
&\simeq \mathbf{H}_{/B_1}\big(U\times f^*(X), f^*(Y)\big) && \text{by (3.68)}\\
&\simeq \mathbf{H}_{/B_1}\big(U, \mathrm{Map}(f^*(X), f^*(Y))\big) && \text{by (3.72)}\,.
\end{aligned}$$

With this, the claim follows by the ∞-Yoneda lemma (Prop. 3.2.29). ☐

Slice mapping stacks.

Proposition 3.2.63 (Hom-∞-groupoids in slice toposes [Lur09, Prop. 5.5.5.12])**.** *Let $\mathbf{H}$ an ∞-topos over a base ∞-topos $\mathbf{B}$. Then for $B \in \mathbf{H}$ and $(E_1, p_1), (E_2, p_2) \in \mathbf{H}_{/B}$, the hom-$\infty$-groupoid* (7) *in the slice $\mathbf{H}_{/B}$ (Prop. 3.2.48) is naturally identified with the fiber of the plain hom-∞-groupoid:*

$$\mathbf{H}_{/B}(E_1,E_2) \;\simeq\; \mathbf{H}(E_1,E_2) \underset{\mathbf{H}(E_1,B)}{\times} \{p_1\} \qquad \text{i.e.,} \qquad
\begin{tikzcd}
\mathbf{H}_{/B}(E_1,E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{H}(E_1,E_2) \arrow[d, "\mathbf{H}(E_1,p_2)"] \\
* \arrow[r, "\vdash p_1"] & \mathbf{H}(E_1,B)
\end{tikzcd} \tag{3.77}$$

Proposition 3.2.64 (Slice mapping space is space of sections of pullback bundle)**.** *There is a natural equivalence*

$$\mathbf{H}_{/B}(E_1,E_2) \;\simeq\; \mathbf{H}_{/E_1}(E_1, p_1^*E_2)\,.$$

Proof. Factoring the bottom morphism $\vdash p_1$ in (3.77) as $p_1 \circ (\vdash \mathrm{id}_{E_1})$ and using the pasting law (5) yields the following identification:

$$\begin{tikzcd}
\mathbf{H}_{/B}(E_1,E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{H}(E_1,E_2) \arrow[d, dashed, "\mathbf{H}(E_1,p_2)"] \\
* \arrow[r, "\vdash p_1"] & \mathbf{H}(E_1,B)
\end{tikzcd}
\;\simeq\;
\begin{tikzcd}
\mathbf{H}_{/E_1}(E_1,p_1^*E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{H}(E_1,p_1^*E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{H}(E_1,E_2) \arrow[d, "\mathbf{H}(E_1,p_2)"] \\
* \arrow[r, "\vdash \mathrm{id}_{E_1}"'] & \mathbf{H}(E_1,E_1) \arrow[r, "\mathbf{H}(E_1,p_1)"'] & \mathbf{H}(E_1,B)
\end{tikzcd}$$

Here the pullback square on the far right is identified as shown by the fact that $\mathbf{H}(E_1,-)$ preserves all limits, hence in particular the pullback of p_2 along p_1. Hence the claim follows by recognizing the left square on the right as the claimined hom-space over E_1, by Prop. 3.2.63. ☐

Definition 3.2.65 (Slice mapping stack)**.** For $B \in \mathbf{H}$ and $(E_1,p_1),(E_2,p_2) \in \mathbf{H}_{/B}$ a pair of objects in the slice over B (Prop. 3.2.48), we say that the *slice mapping stack* between them is the object of $\mathbf{H}$ which is the fiber of the $\mathbf{H}$-internal hom between their total space objects:

$$\mathrm{Map}(E_1,E_2)_B \;:=\; \mathrm{Map}(E_1,E_2) \underset{\mathrm{Map}(E_1,B)}{\times} \{p_1\}, \quad \text{i.e.,} \quad
\begin{tikzcd}
\mathrm{Map}(E_1,E_2)_B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Map}(E_1,E_2) \arrow[d, "\mathrm{Map}(E_1,p_2)"] \\
* \arrow[r, "\vdash p_1"] & \mathrm{Map}(E_1,B)
\end{tikzcd} \;\in \mathbf{H}\,. \tag{3.78}$$

When $E_1 = B$, this may also be called the *stack of sections* of E_2:

$$\Gamma_B(E) \;:=\; \mathrm{Map}(B,E)_B\,. \tag{3.79}$$

Remark 3.2.66. The slice mapping stack (Def. 3.2.65) only depends on the connected component $\mathbf{B}\Omega_{p_1}\mathrm{Map}(E_1, B)$ of $\vdash p_1$ in $\mathrm{Map}(E_1, B)$ in that it is equivalently the top fiber product in the following diagram (obtained by applying the pasting law (5) to the homotopy image factorization as recalled e.g.in [SS20-Orb, Ex. 2.67])

$$
\begin{tikzcd}
\mathrm{Map}(E_1, E_2)_B \arrow[rr] \arrow[d, two heads] \arrow[drr, phantom, "\text{(pb)}"] & & \{p_1\} \arrow[d, two heads] \\
\mathrm{Map}(E_1, E_2)^{p_1} \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}\Omega_{p_1}\mathrm{Map}(E_1, B) \arrow[d, hook] \\
\mathrm{Map}(E_1, E_2) \arrow[rr, "\mathrm{Map}(E_1, p_2)"'] & & \mathrm{Map}(E_1, B)
\end{tikzcd}
\tag{3.80}
$$

Example 3.2.67 (Slice mapping stack into product projection)**.** In the case when $E_2 = X \times B$ with $p_2 = \mathrm{pr}_2$ the projection on the second factor, we have $\mathrm{Map}(E_1, E_2) \simeq \mathrm{Map}(E_1, X) \times \mathrm{Map}(E_1, B)$ with $\mathrm{Map}(E_1, p_2)$ being again the projection onto the second factor, so that the slice mapping stack in Def. 3.2.65 reduces to the plain mapping stack (by Ex. 3.1.18):

$$\mathrm{Map}(E_1, X \times B)_B \simeq \mathrm{Map}(E_1, X)\,.$$

Lemma 3.2.68 (Plots of slice mapping stack are slice homs)**.** *The slice mapping stack (Def. 3.2.65) is equivalently the right base change (Prop. 3.2.50) of the internal hom in the slice ∞-topos down to the base, in that*
(i) *for $B, U \in \mathbf{H}$ and $(E_1, p_1), (E_2, p_2) \in \mathbf{H}_{/B}$ (Prop. 3.2.48), we have a natural equivalence*

$$\mathrm{Map}(E_1, E_2)_B \simeq \prod_B \mathrm{Map}(p_1, p_2)\,. \tag{3.81}$$

(ii) *This means that there is a natural equivalence between the value of the slice mapping stack at U and the slice hom-space (Def. 3.2.63) between $(U \times E_1, \mathrm{pr}_2 \circ p_1)$ and (E_2, p_2):*

$$\mathbf{H}\big(U, \mathrm{Map}(E_1, E_2)_B\big) \simeq \mathbf{H}(U \times E_1, E_2)_B\,. \tag{3.82}$$

(iii) *In particular, the global points of the slice mapping stack constitute the slice mapping space* (3.77)*:*

$$\mathbf{H}\big(*, \mathrm{Map}(E_1, E_2)_B\big) \simeq \mathbf{H}(E_1, E_2)_B\,. \tag{3.83}$$

(iv) *and the space of sections* (3.79) *is equivalently the right base change to the point:*

$$\Gamma_B(E) = \mathrm{Map}(B, E)_B \simeq \prod_B (E, p)\,. \tag{3.84}$$

(v) *Generally, thinking of $E_1 = U \xrightarrow{i_U} B$ as a local patch of B, the slice mapping stack gives the U-local sections:*

$$\Gamma_U(E) := \Gamma_U\big(i_U^* E\big) \simeq \mathrm{Map}(U, E)_B \simeq \prod_B i_U^*(E, p)\,. \tag{3.85}$$

Proof. The first equivalence is the pre-image under the ∞-Yoneda lemma (Prop. 3.2.29) of the composite of the following sequence of natural equivalences for $U \in \mathbf{H}$:

$$
\begin{aligned}
\mathbf{H}\big(U, \textstyle\prod_B \mathrm{Maps}\,((E_1, p_1), (E_2, p_2))\big) &\simeq \mathbf{H}\big((U \times B, \mathrm{pr}_2), \mathrm{Maps}\,((E_1, p_1), (E_2, p_2))\big)_B && \text{by (10) with Prop. 3.2.50} \\
&\simeq \mathbf{H}\big((U \times E_1, p_1 \circ \mathrm{pr}_2), (E_2, p_2)\big)_B && \text{by (3.73) with Prop. 3.2.48} \\
&\simeq \mathbf{H}(U \times E_1, E_2) \underset{\mathbf{H}(U \times E_1, B)}{\times} \{p_1 \circ \mathrm{pr}_2\} && \text{by Prop. 3.2.63} \\
&\simeq \mathbf{H}\big(U, \mathrm{Map}(E_1, E_2)\big) \underset{\mathbf{H}(U, \mathrm{Map}(E_1, B))}{\times} \mathbf{H}(U, \{p_1\}) && \text{by (3.73)} \\
&\simeq \mathbf{H}\Big(U, \mathrm{Map}(E_1, E_2) \underset{\mathrm{Map}(E_1, B)}{\times} \{p_1\}\Big) && \text{by (11)} \\
&\simeq \mathbf{H}\big(U, \mathrm{Map}(E_1, E_2)_B\big) && \text{by Def. 3.2.65.}
\end{aligned}
$$

With this, the second statement is equivalently the first step in this sequence. □

3.2.3 Transformation groups in ∞-toposes

We discuss basics of *higher transformation group theory*, i.e., of higher groups and their group actions, internal to ∞-toposes.

Group objects in an ∞-topos.

Definition 3.2.69 (Group objects in an ∞-topos)**.** Given an ∞-topos $\mathbf{H}$,
(i) a *group object* $\mathscr{G}_\bullet$ is a groupoid object (Def. 3.2.7), to be denoted $(\mathbf{B}\mathscr{G})_\bullet$, which is equipped with a morphism $\mathrm{Cnst}(*)_\bullet \xrightarrow{\mathrm{pt}} (\mathbf{B}\mathscr{G})_\bullet$ from the terminal groupoid object (3.43), such that this is an equivalence in degree 0: $(\mathbf{B}\mathscr{G})_0 \simeq *$.
(ii) We write

$$\mathrm{Grp}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})^{*/} \tag{3.86}$$

for the full sub-∞-category of group objects. among the pointed groupoid objects.

Proposition 3.2.70 (Looping and delooping equivalence)**.** *Under the equivalence of groupoid objects with atlases (Prop. 3.2.15) group objects $\mathscr{G}$ (Def. 3.2.69) are identified as the loop objects $\mathscr{G} \simeq \Omega\mathbf{B}\mathscr{G}$ of pointed connected objects $* \twoheadrightarrow \mathbf{B}\mathscr{G}$,*

$$\begin{tikzcd}
\mathbf{H}^{*/}_{\geq 0} \arrow[d, hook] \arrow[rr, shift right, "\Omega(-)"'] & \sim & \mathrm{Grp}(\mathbf{H}) \arrow[ll, shift right, "\mathbf{B}(-)"'] \arrow[d, hook] \\
\mathrm{Atl}(\mathbf{H})^{(*\twoheadrightarrow *)/} \arrow[rr, shift right, "C(-)_\bullet^{(*\twoheadrightarrow *)/}"'] & \sim & \mathrm{Grpd}(\mathbf{H})^{\mathrm{Cnst}(*)_\bullet/}, \arrow[ll, shift right, "(-)_0 \to \underset{\longrightarrow}{\lim}(-)"']
\end{tikzcd} \tag{3.87}$$

where the delooping $\mathbf{B}\mathscr{G}$ is the homotopy colimit:

$$\mathbf{B}\mathscr{G} \;:=\; \underset{[n]\in\Delta^{\mathrm{op}}}{\underset{\longrightarrow}{\lim}} \mathscr{G}^{\times n} \xleftarrow{\;\;q\;\;} * \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow}\; \mathscr{G} \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathscr{G}\times\mathscr{G} \;\cdots\; . \tag{3.88}$$

Proof. By the equivalence at the bottom (Prop. 3.2.15) and by the definition of $\mathrm{Grp}(\mathbf{H})$ (Def. 3.2.69), the ∞-category in the top left is the full sub-∞-category of that on the bottom left on the point atlases. These are equivalently the pointed connected objects, by Prop. 3.2.14. Under this identification (3.44) the Čech nerve construction on a point atlas $* \twoheadrightarrow \mathbf{B}\mathscr{G}$ clearly is the loop space construction:

$$* \underset{\mathbf{B}\mathscr{G}}{\times} * \;\simeq\; \Omega\mathbf{B}\mathscr{G} \;\simeq\; \mathscr{G}\,.$$

□

Example 3.2.71 (Delooping preserves products)**.** The operation of delooping (Prop. 3.2.70) preserves products, in that the delooping of the direct product of $\mathscr{G}_1, \mathscr{G}_2 \in \mathrm{Grp}(\mathbf{H})$ is the naturally equivalent to the product of the separate deloopings:

$$\mathbf{B}(\mathscr{G}_1 \times \mathscr{G}_2) \;\simeq\; (\mathbf{B}\mathscr{G}_1) \times (\mathbf{B}\mathscr{G}_2)\,. \tag{3.89}$$

This follows from the inverse looping equivalence in Prop. 3.2.70, the fact that $\mathbf{B}\mathscr{G}_1 \times \mathbf{B}\mathscr{G}_2$ is connected, and using that looping, being itself a limit operation, commutes with products:

$$\Omega\big((\mathbf{B}\mathscr{G}_1) \times (\mathbf{B}\mathscr{G}_2)\big) \;\simeq\; (\Omega\mathbf{B}\mathscr{G}_1) \times (\Omega\mathbf{B}\mathscr{G}_2) \;\simeq\; \mathscr{G}_1 \times \mathscr{G}_1\,.$$

Lemma 3.2.72 (∞-Groups presented by presheaves of simplicial groups [NSS12b, Prop. 3.35, 3.73, Rem. 3.67])**.** *Let $(\mathscr{S}, J)$ be a 1-site with a terminal object and $\mathbf{H} := Sh_\infty((\mathscr{C}, J))$ its ∞-topos.*
(i) *Presheaves of simplicial groups represent ∞-groups*

$$\mathscr{G} \;\in\; \mathrm{Grp}\,(\Delta\mathrm{PSh}(\mathscr{S})) \xrightarrow{\mathrm{Loc}^J} \mathrm{Grp}(\mathbf{H})$$

with delooping equivalent to the objectwise $\overline{W}(-)$ (3.22)

$$\mathbf{B}(\mathscr{G}) \simeq \mathrm{Loc}^J\big(\overline{W}(\mathscr{G})\big).$$

(ii) *For $\mathscr{G} \circlearrowright X$, the homotopy quotient is represented by the right derived simplicial Borel constuction functor* (3.33) *given by the diagonal quotient of the Cartesian product with $W(-)$ (Def. 3.1.27):*

$$\mathrm{Loc}^J(X) /\!\!/ \mathrm{Loc}^J(\mathscr{G}) \simeq \mathrm{Loc}^J\big((X \times W\mathscr{G})/\mathscr{G}\big).$$

We consider examples of this construction in Ex. 3.3.30 below.

Proposition 3.2.73 (All bare ∞-groups are shapes of topological groups)**.** *For every bare ∞-group $\mathscr{G} \in \mathrm{Grp}(\mathrm{Grpd}_\infty)$, there exists a Hausdorff topological group $\Gamma \in \mathrm{Grp}(\mathrm{kHausSpc}) \xrightarrow{\mathrm{Cdfflg}} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$, such that*

$$\textstyle\int \Gamma \simeq \mathscr{G} \quad \in \mathrm{Grp}(\mathrm{Grpd}_\infty) \xhookrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty).$$

Proof. By Lem. 3.2.72, we have

$$\mathrm{Grp}(\mathrm{Grpd}_\infty) \simeq \mathrm{Loc}_\Delta^{\mathrm{WHmtpEq}}(\mathrm{Grp}(\Delta\mathrm{Set})).$$

Under this equivalence, we may think of the given ∞-group as a simplicial group $\mathscr{G}$.

Now since topological realization (1.65) preserves finite products (Lem. 1.2.31), the adjoint pair Quillen equivalence

$$\begin{tikzcd} \mathrm{TopSpc}_{\mathrm{Qu}} \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] & \Delta\mathrm{Set}_{\mathrm{Qu}} \arrow[l, bend right, "|-|"'] \end{tikzcd} \qquad \mathrm{TopSpc}_{\mathrm{Qu}} \underset{\mathrm{Pth}}{\overset{|-|}{\underset{\simeq_{\mathrm{Qu}}}{\rightleftarrows}}} \Delta\mathrm{Set}_{\mathrm{Qu}} \tag{3.90}$$

induces (1.6) an adjoint pair between group objects

$$\mathrm{Grp}(\mathrm{TopSpc}) \underset{\mathrm{Grp}(\mathrm{Pth})}{\overset{\mathrm{Grp}(|-|)}{\underset{\perp}{\rightleftarrows}}} \mathrm{Grp}(\Delta\mathrm{Set}).$$

Hence

$$\Gamma := |\mathscr{G}|,$$

is a Hausdorff topological group (since all CW complexes are Hausdorff, e.g. [Ha02, Porp. A.3]) such that

$$\begin{aligned} \textstyle\int\Gamma &\simeq \mathrm{Pth}\,|\mathscr{G}| && \text{by Prop. 3.3.46} \\ &\simeq \mathscr{G} \quad \in \mathrm{Loc}_\Delta^{\mathrm{WHmtpEq}}(\mathrm{Grp}(\Delta\mathrm{Set})) && \text{by (3.90)}. \end{aligned}$$

□

Group actions in an ∞-topos.

Definition 3.2.74 (Group actions internal to an ∞-topos)**.** Given an ∞-topos $\mathbf{H}$, for $X \in \mathbf{H}$ and $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$,
(i) we say that a *$\mathscr{G}$-action groupoid* on X is a groupoid object (Def. 3.2.7) which in degree k is equivalent to the cartesian product of X with k copies of $\mathscr{G}$:

$$(X /\!\!/ \mathscr{G})_\bullet \in \mathrm{Grpd}(\mathbf{H}), \qquad (X /\!\!/ \mathscr{G})_k \simeq X \times \mathscr{G}^{\times^k},$$

compatibly with the group structure on $\mathscr{G}$, hence for which the two simplicial objects form a diagram of homotopy Cartesian squares:

$$\begin{tikzcd} \vdots & \vdots \\ \mathscr{G}\times\mathscr{G}\times X \simeq (X/\!\!/\mathscr{G})_2 \arrow[r, "c_2"] \arrow[d, shift left] \arrow[d] \arrow[d, shift right] \arrow[dr, phantom, "\text{(pb)}"] & \mathscr{G}\times\mathscr{G} \arrow[d, shift left] \arrow[d] \arrow[d, shift right] \\ \mathscr{G}\times X \simeq (X/\!\!/\mathscr{G})_1 \arrow[r, "c_1"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "\text{(pb)}"] & \mathscr{G} \arrow[d, shift left] \arrow[d, shift right] \\ X \simeq (X/\!\!/\mathscr{G})_0 \arrow[r, "c_0"] & * \end{tikzcd} \tag{3.91}$$

(ii) We write

$$\begin{aligned} \mathscr{G}\,\mathrm{Act}(\mathbf{H}) &\hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(\mathbf{B}\mathscr{G})_\bullet} \\ G \circlearrowright X &\mapsto (X/\!\!/\mathscr{G})_\bullet \end{aligned} \tag{3.92}$$

for the full sub-∞-category of the slice of that of groupoid objects over the groupoid object underlying $\mathscr{G}$.

Proposition 3.2.75 (Homotopy quotients of ∞-actions)**.** *Under the equivalence of groupoid objects with atlases (Prop. 3.2.15), $\mathscr{G}$-actions (Def. 3.2.74) are identified with objects in the slice of* $\mathbf{H}$ *over* $\mathbf{B}\mathscr{G}$*:*

$$\begin{tikzcd} \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[d, hook] \arrow[r, shift right] & \mathscr{G}\,\mathrm{Act}(\mathbf{H}) \arrow[l, shift right, "(X\to X /\!\!/ \mathscr{G}) \,\leftarrow\!\shortmid\, G \circlearrowright X"', "\sim"] \arrow[d, hook] \\ \mathrm{Atl}(\mathbf{H})_{/(\ast\to\mathbf{B}\mathscr{G})} \arrow[r, shift right, "C(-)_\bullet^{/(\ast\to\mathbf{B}\mathscr{G})}"'] & \mathrm{Grpd}(\mathbf{H})_{/(\mathbf{B}\mathscr{G})_\bullet} \arrow[l, shift right, "(-)_0\to\underrightarrow{\lim}(-)"', "\sim"] \end{tikzcd},$$

where the base is the homotopy quotient

$$X /\!\!/ \mathscr{G} \;\simeq\; \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\, \mathscr{G}^{\times n}\times X \twoheadleftarrow\!\xleftarrow{\;\;q\;\;} X \; \substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow} \; \mathscr{G}\times X \; \substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathscr{G}\times\mathscr{G}\times X \;\cdots\cdots\; . \tag{3.93}$$

Proof. By the bottom equivalence (Prop. 3.2.15) and the definition of $\mathscr{G}\,\mathrm{Act}(\mathbf{H})$ (Def. 3.2.74), the ∞-category that appears in the top left must be the full sub-∞-category of $\mathrm{Atl}(\mathbf{H})_{/(\ast\,\to\,\mathbf{B}\mathscr{G})}$ whose objects are those squares

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d] & X /\!\!/ \mathscr{G} \arrow[d] \\ \ast \arrow[r, two heads] & \mathbf{B}\mathscr{G} \end{tikzcd},$$

which arise under the colimit operation from Cartesian morphisms of groupoid objects (3.91). But these are precisely the Cartesian such squares, by the theory of equifibered transformations ([Lur09, 6.1.3.9(4)][Re10, 6.5], see [SS20-Orb, Prop. 2.32]). (This is consistent, since effective epimorphisms are stable under pullback; so that for every such Cartesian square the top horizontal morphism is an effective epimorphism, since $\ast \twoheadrightarrow \mathbf{B}\mathscr{G}$ is.) But by the universal property of pullbacks, morphisms of pullback squares over $\ast \twoheadrightarrow \mathbf{B}\mathscr{G}$ are equivalent to morphsims of their right vertical component over $\mathbf{B}\mathscr{G}$, hence to morphisms in $\mathbf{H}_{/\mathbf{B}\mathscr{G}}$. □

Remark 3.2.76 (∞-Topos of ∞-Actions)**.** A key consequences of Prop. 3.2.75 is that $\mathscr{G}$-actions in an ∞-topos form themselves an ∞-topos:

$$\mathscr{G}\,\mathrm{Act}(\mathbf{H}) \;\simeq\; \mathbf{H}_{/\mathbf{B}\mathscr{G}} \;\in\; \mathrm{Topos}_\infty\,,$$

by the Fundamental Theorem of ∞-topos theory (Prop. 3.2.48).

In higher generalization of Exp. 1.1.8 we have:

Example 3.2.77 (Fixed loci of ∞-actions)**.** For $\mathscr{G}\in\mathrm{Grp}(\mathbf{H})$, the operations of forming

(i) the homotopy quotient of $\mathscr{G}$-actions (3.93)

(ii) the trivial $\mathscr{G}$-action

(iii) the fixed points of a $\mathscr{G}$-action [SS20-Orb, Def. 2.97]

constitute an adjoint triple of ∞-functors as shown on the left below, which under the equivalence of Prop. 3.2.75 is identified with the base change adjoint triple (3.68) along $\mathbf{B}\mathscr{G}\to\ast$:

$$\begin{tikzcd} \mathscr{G}\,\mathrm{Act}(\mathbf{H}) \arrow[r, shift left=3, "(-)/\!\!/\mathscr{G}"] \arrow[r, shift right=3, "(-)^{\mathscr{G}}"'] & \mathbf{H} \arrow[l, "\mathrm{trivial}"' description] \end{tikzcd} \qquad\simeq\qquad \begin{tikzcd} \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[r, shift left=3, "\Sigma_{\mathbf{B}\mathscr{G}}"] \arrow[r, shift right=3, "\Pi_{\mathbf{B}\mathscr{G}}"'] & \mathbf{H} \arrow[l, "\mathbf{B}\mathscr{G}\times(-)"' description] \end{tikzcd}. \tag{3.94}$$

(For the homotopy quotient this is part of the result of Prop. 3.2.75, for the trivial action it is evident, and for the fixed locus it is the very definition [SS20-Orb, Def. 2.97].)

Example 3.2.78 (Fixed loci as slice mapping stacks into homotopy quotients over deloopings)**.** Via (3.84), the fixed loci (3.94) of any $\mathscr{G}\circlearrowright X$ are equivalently given by the slice mapping stack (Def. 3.2.65) out of the base $\mathbf{B}\mathscr{G}$ into the homotopy quotient (cf. [SS20-Orb, Ex. 2.98]):

$$X^{\mathscr{G}} \;\simeq\; \mathrm{Map}\big(\mathbf{B}\mathscr{G}, X/\!\!/\mathscr{G}\big)_{\mathbf{B}\mathscr{G}}\,. \tag{3.95}$$

More generally, via (3.85), for $\mathscr{H} \xrightarrow{i} \mathscr{G}$ any homomorphism of ∞-groups, hence $\mathbf{B}i : \mathbf{B}\mathscr{H} \hookrightarrow \mathbf{B}\mathscr{G}$, the fixed loci of the induced action $\mathscr{H} \circlearrowright X$ are given by the slice mapping stack out of $\mathbf{B}\mathscr{H}$ over $\mathbf{B}\mathscr{G}$:

$$X^{\mathscr{H}} \simeq \mathrm{Map}\big(\mathbf{B}\mathscr{H}, X /\!\!/ \mathscr{G}\big)_{\mathbf{B}\mathscr{G}}. \tag{3.96}$$

Example 3.2.79 (∞-Stack of equivariant maps)**.** Given $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ and $\mathscr{G} \circlearrowright X, \mathscr{G} \circlearrowright Y \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ (Prop. 3.2.75), the $\mathscr{G}$-equivariant mapping stack between them is equivalently the slice mapping stack (Def. 3.2.65) between their homotopy quotients:

$$\begin{aligned} &\mathrm{Map}\big(\mathscr{G} \circlearrowright X, \mathscr{G} \circlearrowright Y\big)^{\mathscr{G}} \\ &\simeq \prod_{\mathbf{B}\mathscr{G}} \mathrm{Map}\big(X /\!\!/ \mathscr{G}, Y /\!\!/ \mathscr{G}\big) \quad \text{by (3.94)} \\ &\simeq \mathrm{Map}\big(X /\!\!/ \mathscr{G}, Y /\!\!/ \mathscr{G}\big)_{\mathbf{B}\mathscr{G}} \quad \text{by (3.81).} \end{aligned} \tag{3.97}$$

In higher generalization of Exp. 1.1.9 we have:

Proposition 3.2.80 (Free- and cofree $\mathscr{G}$-action)**.** *The forgetful functor from $\mathscr{G}$-actions (Def. 3.2.74) to underlying objects has a left adjoint and a right adjoint of the following form:*

$$\begin{array}{ccc} \mathscr{G}\,\mathrm{Act}(\mathbf{H}) & \xrightarrow{\mathrm{undrlng}} & \mathbf{H} \\ \mathscr{G} \circlearrowright P & \longmapsto & P \end{array} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, shift left=8, "\mathscr{G} \circlearrowright ((-)\times\mathscr{G})"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\mathscr{G} \circlearrowright \mathrm{Map}(\mathscr{G},-)"'] & \mathscr{G}\,\mathrm{Act}(\mathbf{H}) \arrow[l, "\mathrm{undrlng}"'] \end{tikzcd}.$$

Proof. Observe that, under Prop. 0.2.1, the underlying object functor is equivalently the pullback along $* \xrightarrow{\mathrm{pt}_{\mathbf{B}\mathscr{G}}} \mathbf{B}\mathscr{G}$:

$$P \simeq \mathrm{undrlng}(G \circlearrowright P) \simeq \mathrm{fib}\begin{pmatrix} P /\!\!/ \mathscr{G} \\ \downarrow \\ \mathbf{B}G \end{pmatrix} \simeq (\mathrm{pt}_{\mathbf{B}\mathscr{G}})^* \begin{pmatrix} P /\!\!/ \mathscr{G} \\ \downarrow \\ \mathbf{B}\mathscr{G} \end{pmatrix}. \tag{3.98}$$

Therefore, the existence of the adjoints is a special case of the base change adjoint triple (Prop. 3.2.50). By the discussion there, the left adjoint, seen on the slice topos, is given by postcomposition, which implies – by the pasting law (5) and using the looping equivalence (0.2.1) – that its underlying object is $(-) \times G$:

$$\begin{tikzcd} X \times G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d] \\ G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] \\ * \arrow[r, "\mathrm{pt}_{\mathbf{B}\mathscr{G}}"'] & \mathbf{B}\mathscr{G} \end{tikzcd} \qquad (\mathrm{pt}_{\mathbf{B}\mathscr{G}})^*(\mathrm{pt}_{\mathbf{B}\mathscr{G}})_!(X), \quad (\mathrm{pt}_{\mathbf{B}\mathscr{G}})_!(X)$$

Now observing the induced adjoint pair

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift right=4, "(\mathrm{pt}_{\mathbf{B}\mathscr{G}})^*\circ(\mathrm{pt}_{\mathbf{B}\mathscr{G}})_*"'] \arrow[r, phantom, "\perp"] & \mathbf{H} \arrow[l, shift right=4, "(\mathrm{pt}_{\mathbf{B}\mathscr{G}})^*\circ(\mathrm{pt}_{\mathbf{B}\mathscr{G}})_! \simeq (-)\times\mathscr{G}"'] \end{tikzcd}$$

implies the form of the right adjoint by the defining adjunction (3.72) and the essential uniqueness of adjoints. □

Some properties of ∞-group actions. We establish some properties of ∞-actions in ∞-toposes that we will need later on.

Lemma 3.2.81 (Homotopy quotient preserves connectedness)**.** *If $\mathscr{G} \circlearrowright X \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ is an action on a connected object $X \in \mathbf{H}_{\geq 1}$, then also the homotopy quotient $X /\!\!/ \mathscr{G}$ is connected.*

Proof. By [Lur09, Prop. 6.5.1.12], an object is connected precisely if its image under 0-truncation

$$\begin{tikzcd} \mathbf{H}_{\leq 0} \arrow[r, hook, shift right=2] \arrow[r, phantom, "\perp"] & \mathbf{H} \arrow[l, shift right=2, "\tau_0"'] \end{tikzcd}$$

is equivalent to the terminal object. But truncation is a left adjoint and hence sends homotopy quotients to homotopy quotients, $\tau_0\big(X /\!\!/ \mathscr{G}\big) \simeq \tau_0(X)/\tau_0(\mathscr{G})$. But homotopy quotient in the subcategory $H_{\leq 0}$ of 0-truncated objects are plain quotients, which send the point to the point. □

In higher generalization of Ex. 1.1.12, we have:

Proposition 3.2.82 (Restricted and left induced ∞-actions)**.** *For $G \in \mathrm{Grp}(\mathrm{Set})$ and $K \overset{i}{\hookrightarrow} G$ a subgroup inclusion, the induced left base change (via Prop. 3.2.75 and Prop. 3.2.50)*

$$G\,\mathrm{Act}(\mathbf{H}) \simeq \mathbf{H}_{/\mathbf{B}G} \xrightarrow{(Bi)_!} \mathbf{H}_{/\mathbf{B}G'} \simeq G'\,\mathrm{Act}(\mathbf{H})$$

satisfies

$$(Bi)_!(K \circlearrowright X) \simeq G \circlearrowright (X \times_K G) \quad \textit{and} \quad X /\!\!/ K \simeq (X \times_K G) /\!\!/ G. \tag{3.99}$$

Proof. Unwinding the definitions, and using Prop. 0.2.1 and the pasting law (5), this follows from the following pasting diagram

$$\begin{tikzcd}
X \times_K G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X /\!\!/ K \simeq (X \times_K G) /\!\!/ G \arrow[d] \\
G/K \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}K \arrow[d, "\mathbf{B}i"] \\
* \arrow[r] & \mathbf{B}G.
\end{tikzcd}$$

□

Lemma 3.2.83 (Presentation of homotopy fixed point spaces of equivariant moduli stacks)**.** *For $U \in \mathrm{CartSpc}$, $G \in \mathrm{Grp}(\Delta\mathrm{Set}) \to \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$, and $(\Gamma, \rho) \in G\,\mathrm{Act}(\mathrm{Grp}(\Delta\mathrm{PSh})) \longrightarrow G\,\mathrm{Act}(\mathrm{Grp}(\mathrm{SmthGrpd}_\infty))$, we have a natural equivalence of* Grpd_∞

$$(\mathrm{SmthGrpd}_\infty)_{/\mathbf{B}G}\big(U \times \mathbf{B}G,\, (\mathbf{B}\Gamma) /\!\!/ G\big) \simeq \Delta\mathrm{PSh}(\mathrm{CartSpc})_{/\overline{W}G}\big(U \times \overline{W}G,\, \big(\overline{W}\Gamma \times WG\big)/G\big).$$

Proof. By Lemma 3.2.72, we have that $\overline{W}(-)$ represents $\mathbf{B}(-)$ and that $\big((-) \times WG\big)/G$ represents $(-) /\!\!/ G$. It hence remains to see that the simplicial hom-set on the right has the correct homotopy type of the derived hom-set. For this it is sufficient, by Lemma 3.1.12, to show that:
(i) $U \times \overline{W}G$ is projectively cofibrant,
(ii) $\big(\overline{W}\mathscr{G} \times WG\big)/G$ is projectively fibrant.
The first statement is Example 3.2.38. The second statement follows since $\overline{W}\mathscr{G}$ is projectively fibrant as a simplicial presheaf (Prop. 3.1.30), hence is projectively fibrant as a G-action (3.30), and since $\big((-) \times WG\big)/G$ is a right Quillen functor (Prop. 3.1.39) and thus preserves fibrancy. □

Base change of ∞-actions along discrete group covers.

We discuss (Prop. 3.2.87 below) the base change of ∞-actions (Def. 3.2.74) along coverings of discrete groups, hence along deloopings of surjective homomorphisms

$$\widehat{G} \xrightarrow{p} G \quad \in \mathrm{Grp}(\mathrm{Set}) \xrightarrow{\mathrm{Grp}(\mathrm{LCnst})} \mathrm{Grp}(\mathbf{H}_{)}, \quad \text{delooping to} \quad B\widehat{G} \xrightarrow{Bp} BG \quad \in \mathrm{Grpd}_1 \xrightarrow{\mathrm{LCnst}_{\text{eq. (3.36)}}} \mathbf{H}_1. \tag{3.100}$$

The following Prop. 3.2.87 is one key aspect of "globalizing" G-equivariant homotopy theory (see also Rem. 4.1.28).

Lemma 3.2.84 (Terminal epimorphisms of delooping 1-groupoids)**.** *For all $G \in \mathrm{Grp}(\mathrm{Set})$ the terminal map out of its delooping groupoid*

$$BG \to * \quad \in \mathrm{Grpd}_1$$

is an epimorphism in Grpd_1, in that for all all $\mathscr{X} \in \mathrm{Grpd}_1 \hookrightarrow \mathrm{Grpd}_\infty$ the induced map

$$\mathrm{Map}(BG \to *, \mathscr{X}) \;:\; \mathscr{X} \to \mathrm{Map}(BG, \mathscr{X})$$

is a monomorphism (6)*, namely a fully faithful functor, i.e. the full inclusion of a connected component.*

Remark 3.2.85 (Epimorphisms of n-groupoids)**.** The proof of Lem. 3.2.84 is immediate by inspection, due to the fact that a natural transformation between functors out of $BG \simeq \mathrm{Loc}^{\mathrm{WHmtpEq}}(G \rightrightarrows *)$ has a single component corresponding to the point $* \to BG$.

The subtlety to notice here is just that this situation crucially relies on the ambient category being Grpd_1, as it fails in Grpd_2 and thus in Grpd_∞: If $\mathscr{X}$ is a 2-groupoid (or higher) then a (pseudo-)natural transformation between maps $BN \to \mathscr{X}$ has, in general, not just a component on the point, but also components over each element of G, which are not faithfully reflected on $* \to BG$. Indeed, for $BG \to *$ to be an epimorphism in Grpd_∞ the group G must be perfect, in that its abelianization is trivial ([Ra19][Hoy19, Lem. 3]).

Lemma 3.2.86 (∞-Action base change comonad along discrete group extension)**.** *If* $\widehat{G} \xrightarrow{\;p\;} G$ *is a surjective homomorphisms in* Grp(Set) (3.100)*, then the induced base change comonad (Prop. 3.2.50, via Prop. 0.2.1) of G-∞-actions in* $\mathbf{H}$ *along Bp*

$$\widehat{G}\mathrm{Act}(\mathbf{H}) \simeq \mathbf{H}_{/\mathbf{B}\widehat{G}} \underset{(Bp)^*}{\overset{(Bp)_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{H}_{/\mathbf{B}G} \simeq G\mathrm{Act}(\mathbf{H}) \tag{3.101}$$

is naturally equivalent to the cartesian product with the delooping groupoid BN of the kernel $N := \ker(p) \subset \widehat{G}$ equipped with its conjugation action:

$$(Bp)_!\,(Bp)^*\big(G \circlearrowright (-)\big) \;\simeq\; G \circlearrowright \big((-) \times BN\big)\,. \tag{3.102}$$

In particular, on 0-truncated objects the operation is the identity, in that for $G \circlearrowright W \in G\mathrm{Act}(\mathbf{H}_0)$ we have:

$$\tau_0\,(Bp)_!\,(Bp)^*(G \circlearrowright W) \;\simeq\; G \circlearrowright W\,. \tag{3.103}$$

Proof. Consider the following diagram in $\mathbf{H}$:

$$\begin{tikzcd}
W \times BN \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & ((Bp)^*W)/\!\!/\widehat{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & W/\!\!/G \arrow[d] \\
BN \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & B\widehat{G} \arrow[r, "Bp"] \arrow[d, "Bp"'] & BG \arrow[dl, equal] \\
* \arrow[r] & BG\,. &
\end{tikzcd}$$

Here the total middle vertical morphism is $(Bp)_!\,(Bp^*)\,(W)$, by Prop. 3.2.50, so that the total left Cartesian rectangle computes, in its top left corner, its underlying object. The bottom left rectangle is Cartesian by Prop. 3.2.72 with Lemma 3.1.31. Hence, by repeated use of the pasting law (5), the top left object is equivalently the pullback exhibited by the top left square and hence that by the total top rectangle. But by the factorization at the bottom, this total top pullback is the Cartesian product of the underlying object of $G \circlearrowright W$ with BN, as claimed. □

Proposition 3.2.87 (Base change of 2-actions along surjective group homomorphisms is faithful)**.** *If p is a surjective homomorphisms of discrete groups* (3.100)*, then the induced base change (Prop. 3.2.50, via Prop. 0.2.1) along Bp is*

(i) *fully faithful* (8) *on 0-truncated objects*

$$\widehat{G} \xrightarrow{\;p\;} G \qquad \vdash \qquad \widehat{G}\mathrm{Act}(\mathbf{H}_0) \simeq \Big(\mathbf{H}_{/B\widehat{G}}\Big)_0 \xleftarrow{\;(Bp)^*\;} \big(\mathbf{H}_{/BG}\big)_0 \simeq G\mathrm{Act}(\mathbf{H}_0)\,. \tag{3.104}$$

(ii) *faithful on 1-truncated objects, in the sense that $(Bp)^*$ induces a monomorphism, hence (Ex. 3.1.16) an injection of connected components of hom-∞-groupoids:*

$$\left.\begin{array}{r} \widehat{G} \xrightarrow{\;p\;} G \\ \text{and} \quad G \circlearrowright W, G \circlearrowright V \in G\mathrm{Act}(\mathbf{H}_1) \end{array}\right\} \;\vdash\; \mathrm{Hom}\big((Bp)^*(W), (Bp)^*(V)\big) \xleftarrow{\;(Bp)^*_{W,V}\;} \mathrm{Hom}(W, V)\,. \tag{3.105}$$

Proof. For the first statement, consider the following sequence of natural equivalences, for 0-truncated $G \circlearrowright V$, $G \circlearrowright W \in G\mathrm{Act}(\mathbf{H}_0)$:

$$
\begin{aligned}
\mathrm{Hom}\big((Bp)^*W,\,(Bp)^*V\big) &\simeq \mathrm{Hom}\big((Bp)_!(Bp)^*W,\,V\big) && \text{by (10)}\\
&\simeq \mathrm{Hom}\big(\tau_0(Bp)_!(Bp)^*W,\,V\big) && \text{Prop. 3.2.53 with (10)}\\
&\simeq \mathrm{Hom}(W,\,V) && \text{by (3.103).}
\end{aligned}
$$

By the ∞-Yoneda lemma (Prop. 3.2.29) the composite natural equivalence is a left inverse to $(Bp)^*_{(-,-)}$ and hence exhibits fully faithfulness on 0-truncated objects.

The second statement follows by analogous reasoning. For readability, we display the proof only for the special case when $\widehat{G} \xrightarrow{p} G$ is a *central* extension, so that the G-action on BN is trivial, hence the case when

$$
G \circlearrowright BN \;\simeq\; (Bp_G)^*(BN)\,, \qquad \text{for } p_G : G \to *. \tag{3.106}
$$

Then we have the following natural equivalences:

$$
\begin{aligned}
&\widehat{G}\mathrm{Act}(\mathbf{H}_1)\big((Bp)^*(G \circlearrowright W),\,(Bp)^*(G \circlearrowright V)\big)\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\big((Bp)_!(Bp)^*(G \circlearrowright W),\,G \circlearrowright V\big) && \text{by (10)}\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\,\big(G \circlearrowright W \times BN,\,G \circlearrowright V\big) && \text{by (3.102) in Lem. 3.2.86}\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\,\big(G \circlearrowright *,\,\mathrm{Maps}\,\big(G \circlearrowright W \times BN, G \circlearrowright V\big)\big) && \text{by (3.75)}\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\big(G \circlearrowright *,\,\mathrm{Maps}\,\big(G \circlearrowright BN, \mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big)\big) && \text{by Lem. 3.2.60}\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\big(G \circlearrowright BN,\,\mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by (3.72) with (10)}\\
&\simeq G\mathrm{Act}(\mathbf{H}_1)\big((Bp_G)^*BN,\,\mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by (3.106)}\\
&\simeq \mathbf{H}_1\big(BN,\,(Bp_G)_*\mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by Prop. 3.2.80}\\
&\simeq \mathbf{H}_1\big(\mathrm{LCnst}(BN),\,(Bp_G)_*\mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by (3.100)}\\
&\simeq \mathrm{Grpd}_1\big(BN,\,\Gamma\,(Bp_G)_*\,\mathrm{Map}(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by (3.36) with (10)}\\
&\simeq \mathrm{Grpd}_1\big(BN,\,G\mathrm{Act}(\mathbf{H}_1)(G \circlearrowright W,\,G \circlearrowright V)\big) && \text{by (3.39) with (3.75)}\\
&= \mathrm{Maps}\,(BN, \mathrm{Hom}(W,\,V))\;.
\end{aligned}
$$

With this, the claim follows by Lem. 3.2.84. $\square$

Higher shear maps. In preparation of the discussion of principal ∞-bundles (Def. 3.2.94 below), we discuss the higher analogs of the shear map (2.4):

Definition 3.2.88 (Higher shear maps)**.** Given $\mathscr{G} \circlearrowright X \in G\mathrm{Act}(\mathbf{H})$ (Def. 3.2.74),
(i) we say that its *shear map* is the universal morphism shear_1 in the following pasting diagram, whose front and rear faces are Cartesian, by construction:

$$
\begin{tikzcd}
\mathscr{G} \times X \arrow[rrr, "\rho"] \arrow[drr, dashed, "\mathrm{shear}_1"] \arrow[dd, "\mathrm{pr}_1"'] & & & X \arrow[dr, equal] \arrow[dd, two heads] & \\
& & X \times X \arrow[rr, "\mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\text{(pb)}"] & & X \arrow[dd] \\
X \arrow[rrr, two heads] \arrow[drr, equal] & & & X /\!\!/ \mathscr{G} \arrow[dr] & \\
& & X \arrow[rr] & & *
\end{tikzcd}
\tag{3.107}
$$

(ii) More generally, we say that the *higher shear maps*

$$\mathscr{G}^{\times_n} \times X \xrightarrow{\mathrm{shear}_n} X^{\times n+1}, \quad n \in \mathbb{N} \tag{3.108}$$

are the universal morphisms given by functorially forming the Čech nerves of id_X canonically mapping into $X/\!\!/G \to *$ (3.45):

$$\begin{array}{ccc}
\vdots & & \vdots \\
\mathscr{G} \times \mathscr{G} \times X & \xrightarrow{\mathrm{shear}_2} & X \times X \times X \\
\downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow \\
\mathscr{G} \times X & \xrightarrow{\mathrm{shear}_1} & X \times X \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
X & \overset{\mathrm{shear}_0}{=\!=\!=} & X \\
\downarrow & & \downarrow \\
X /\!\!/ \mathscr{G} & \longrightarrow & *
\end{array} \tag{3.109}$$

Example 3.2.89 (Recovering the traditional notion of shear map)**.** Let $\mathbf{H} = (\mathrm{SmthGrpd}_\infty)_{/X}$ be the slice ∞-topos over some $X \in \mathrm{DTopSpc} \hookrightarrow \mathrm{SmthGrpd}_\infty$, and let $\Gamma \in \mathrm{Grp}(\mathrm{DTopSpc}) \xhookrightarrow{(-)\times X} \mathrm{Grp}(\mathbf{H})$ and $\Gamma \circlearrowright P \in G\,\mathrm{Act}(\mathrm{DTopSpc}_{/X}) \hookrightarrow G\,\mathrm{Act}(\mathbf{H})$ be a fiberwise topological group action on a topological bundle P over X.

Then its shear map (3.107) according to Def. 3.2.88 is the universal dashed morphism in the following diagram:

$$\begin{tikzcd}
\Gamma \times P \arrow[rr, "\rho"] \arrow[dr, dashed, "\mathrm{shear}_1"'] \arrow[dd, "\mathrm{pr}_1"'] & & P \arrow[dr, equals] \arrow[dd, two heads] & \\
& P \times_X P \arrow[rr, "\mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_2"'] \arrow[drr, phantom, "\scriptstyle (\mathrm{pb})"] & & P \arrow[dd] \\
P \arrow[rr, two heads] \arrow[dr, equals] & & P /\!\!/_X \Gamma \arrow[dr] & \\
& P \arrow[rr] & & X
\end{tikzcd}$$

This manifestly coincides with the traditional shear map (2.4).

Free ∞-actions.

Definition 3.2.90 (Free, transitive and regular ∞-actions)**.** We say that $\mathscr{G} \circlearrowright X \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ (Def. 3.2.74) is

- *free* if its shear map (3.107) is (-1)-truncated:

$$\mathscr{G} \circlearrowright X \text{ is free} \quad \Leftrightarrow \quad \mathscr{G} \times X \xhookrightarrow{\mathrm{shear}_1} X \times X\,; \tag{3.110}$$

- *transitive* if its shear map is (-1)-connected:

$$\mathscr{G} \circlearrowright X \text{ is transitive} \quad \Leftrightarrow \quad \mathscr{G} \times X \xrightarrow{\mathrm{shear}_1} \!\!\!\!\to X \times X\,; \tag{3.111}$$

- *regular* if its shear map is an equivalence:

$$\mathscr{G} \circlearrowright X \text{ is regular} \quad \Leftrightarrow \quad \mathscr{G} \times X \xrightarrow[\sim]{\mathrm{shear}_1} X \times X\,. \tag{3.112}$$

Lemma 3.2.91 (Higher shear maps are (-1)-truncated if first shear map is)**.** *Given $\mathscr{G} \circlearrowright X \in G\,\mathrm{Act}(\mathbf{H})$, if its shear map* shear_1 *is (-1)-truncated or an equivalence, then so are all higher shear maps (Def. 3.2.88), respectively:*

$$\mathscr{G} \times X \xhookrightarrow{\mathrm{shear}_1} \mathscr{X} \times \mathscr{X} \quad \Rightarrow \quad \underset{n \in \mathbb{N}}{\forall}\; \mathscr{G} \times X \xhookrightarrow{\mathrm{shear}_n} \mathscr{X} \times \mathscr{X}\,,$$

$$\mathscr{G} \times X \xrightarrow[\sim]{\mathrm{shear}_1} \mathscr{X} \times \mathscr{X} \quad \Rightarrow \quad \underset{n \in \mathbb{N}}{\forall}\; \mathscr{G} \times X \xrightarrow[\sim]{\mathrm{shear}_n} \mathscr{X} \times \mathscr{X}\,.$$

Proof. Observe that for all effective epimorphisms $X \twoheadrightarrow \mathscr{X}$ we have, for all $n \in \mathbb{N}$, homotopy-cartesian squares as follows:

$$
\begin{tikzcd}
 & & X^{\times^{n+2}_{\mathscr{X}}} \arrow[dl] \arrow[dr] & & \\
 & X^{\times^{n+1}_{\mathscr{X}}} \arrow[dl] \arrow[dr] & \text{(pb)} & X^{\times^{2}_{\mathscr{X}}} \arrow[dl] \arrow[dr] & \\
X^{\times^{n}_{\mathscr{X}}} \arrow[dr] & \text{(pb)} & X \arrow[dl] \arrow[dr] & \text{(pb)} & X \arrow[dl] \\
 & \mathscr{X} & & \mathscr{X} &
\end{tikzcd}
$$

Namely, the two bottom squares are cartesian by construction of Čech nerves, while the top square is cartesian by the groupoidal Segal conditions (3.42) satisfied by groupoid objects.

Applied to the morphism of effective epimorphisms

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, two heads] & X \arrow[d] \\
X /\!\!/ \mathscr{G} \arrow[r] & *
\end{tikzcd}
$$

this implies that the $n+1$st shear map is the following homotopy pullback in the arrow category of $\mathbf{H}$:

$$
\begin{tikzcd}
X \times G^{\times^{n+1}} \arrow[rr] \arrow[dr, dashed, "\mathrm{shear}_{n+1}"'] \arrow[dd] & & X \times G \arrow[dr, "\mathrm{shear}_1"] \arrow[dd, "\mathrm{pr}_1"'] & \\
 & X^{\times^{n+2}} \arrow[rr] \arrow[dd] & & X \times X \arrow[dd, "\mathrm{pr}_1"] \\
X \times G^{\times^{n}} \arrow[rr] \arrow[dr, "\mathrm{shear}_n"'] & & X \arrow[dr, equal] & \\
 & X^{\times^{n+1}} \arrow[rr] & & X
\end{tikzcd}
$$

Since the classes of (-1)-truncated morphisms and of equivalences (include all identity morphisms) and both are stable (3.47) under ∞-limits in the arrow category (by Prop. 3.2.16), the claim follows by induction. $\square$

Proposition 3.2.92 (Inhabited homotopy quotients of regular ∞-actions are terminal)**.** *If $X \in \mathbf{H}$ is inhabited (Ntn. 3.2.12) then an ∞-action $\mathscr{G} \circlearrowright X \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ (Def. 3.2.74) is regular (Def. 3.2.90) iff its homotopy quotient (Prop. 3.2.75) is terminal:*

$$
(X \twoheadrightarrow *) \qquad \Rightarrow \qquad \big(\mathscr{G} \circlearrowright X \text{ is regular} \quad \Leftrightarrow \quad X /\!\!/ \mathscr{G} \simeq * \big)\,.
$$

Proof. This follows by inspection of the defining diagram (3.109) of the higher shear maps, noticing that, under the given assumption on X, not only its left but also its right bottom morphism is an effective epimorphism, so that the bottom rectangle is the image under forming homotopy quotients of the morphisms of simplicial objects. Therefore: If the action is regular in that shear_1 is an equivalence (3.112), then all top horizontal morphisms in (3.109) are equivalences, by Lem. 3.2.91, whence all top squares are Cartesian, so that the bottom morphism is the image of an equivalence under the ∞-colimit functor, and hence itself an equivalence.

Conversely, if the homotopy quotient is terminal, then the bottom morphism is an equivalence, and hence so is the shear map, by its defining diagram (3.107). $\square$

Proposition 3.2.93 (Homotopy quotient of free actions is ordinary quotient of 0-truncation)**.** *For $\mathbf{H}$ an ∞-topos, with $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ if $\mathscr{G} \circlearrowright X \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ is free (Def. 3.2.90) and X is inhabited (Ntn. 3.2.12), then the homotopy quotient* (3.93) *coincides with the ordinary quotient of the 0-truncation:*

$$
X /\!\!/ \mathscr{G} \simeq i_0\big(\tau_0(X)/\tau_0(\mathscr{G})\big) := i_0\Big(\mathrm{coeq}\big(\tau_0(\mathscr{G}) \times \tau_0(X) \underset{\rho}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \tau_0(X)\big)\Big) \quad \in \mathbf{H}_0 \xhookrightarrow{i_0} \mathbf{H}\,.
$$

Proof. First we show that $X /\!\!/ \mathscr{G}$ is 0-truncated. For this it is sufficient to demonstrate that, for all $U \in \mathbf{H}$ and $\mathscr{K} \in \mathrm{Grp}(\mathrm{Grpd}_\infty) \xrightarrow{\mathrm{Grp(LCnst)}} \mathrm{Grp}\mathbf{H}$, every morphism out of $U \times \mathbf{B}\mathscr{K}$ into it factors through the projection onto U:

$$\begin{tikzcd}
U \times \mathbf{B}\mathscr{K} \arrow[r, "\forall"] \arrow[d, "\mathrm{pr}_1"'] & X /\!\!/ \mathscr{G} \arrow[d] \\
U \arrow[r] \arrow[ur, dashed, "\exists"] & \ast\,.
\end{tikzcd}$$

To see this, consider the following square diagram of augmented simplicial objects, obtained by choosing atlases as shown in the bottom row and then forming the four Čech groupoids running vertically (Ex. 3.2.10):

$$\begin{tikzcd}[column sep=small, row sep=small]
\vdots & & \vdots & \\
\mathscr{K}\times\mathscr{K}\times U \arrow[rr] \arrow[dr, two heads] \arrow[dd] & \vdots & \mathscr{G}\times\mathscr{G}\times X \arrow[dr, hook] \arrow[dd] & \vdots \\
& U \arrow[rr, hook] \arrow[ur, dashed] \arrow[dd] & & X\times X\times X \arrow[dd] \\
\mathscr{K}\times U \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & \mathscr{G}\times X \arrow[dr, hook] \arrow[dd] & \\
& U \arrow[rr, hook] \arrow[ur, dashed] \arrow[dd] & & X\times X \arrow[dd] \\
U \arrow[rr, two heads] \arrow[dr, equal] \arrow[dd, two heads] & & X \arrow[dr, equal] \arrow[dd, two heads] & \\
& U \arrow[rr, hook] \arrow[ur, dashed] \arrow[dd, two heads] & & X \arrow[dd, two heads] \\
U\times\mathbf{B}\mathscr{K} \arrow[rr] \arrow[dr] & & X/\!\!/\mathscr{G} \arrow[dr] & \\
& U \arrow[rr] \arrow[ur, dashed] & & \ast
\end{tikzcd} \tag{3.113}$$

Here in all of the upper horizontal squares

- the left morphism is (-1)-connected $\twoheadrightarrow$, since every ∞-group $\mathscr{K} \in \mathrm{Grp}(\mathrm{Grpd}_\infty)$ is inhabited (Ntn. 3.2.12) and LCnst, being lex left adjoint, preserves this property;
- the right morphism is (-1)-truncated $\hookrightarrow$, by Lem. 3.2.91.

Since n-connected/truncated morphisms in ∞-presheaf categories such as $\mathrm{PSh}_\infty(\Delta, \mathbf{H})$ are detected objectwise, this means that the total square diagram of simplicial objects is a (-1)-connected/truncated-lifting problem, and since $\mathrm{PSh}_\infty(\Delta, \mathbf{H})$ is again an ∞-topos this lifting problem has an essentially unique solution (3.46), hence a *compatible* set of dashed lifts as shown above. Therefore, forming again the ∞-colimit over the vertical simplicial diagrams recovers the bottom square (Prop. 3.2.15) but now itself equipped with a lift. This is the required lift which proves that $X /\!\!/ \mathscr{G}$ is 0-truncated:

$$i_0\, \tau_0\, (X /\!\!/ \mathscr{G}) \;\simeq\; X /\!\!/ \mathscr{G}\,, \qquad \mathbf{H}_0 \underset{i_0}{\overset{\tau_0}{\underset{\sim}{\leftrightarrows}}} \mathbf{H}\,. \tag{3.114}$$

With this, we may conclude as follows:

$$\begin{aligned}
X /\!\!/ \mathscr{G} &\simeq i_0\, (\tau_0\, (X /\!\!/ \mathscr{G})) && \text{by (3.114)} \\
&\simeq i_0 \Big(\tau_0 \big(\underset{[n]\in\Delta^{\mathrm{op}}}{\varinjlim} \mathscr{G}^{\times^n} \times X \big) \Big) && \text{by (3.93)} \\
&\simeq i_0 \Big(\underset{[n]\in\Delta^{\mathrm{op}}}{\varinjlim} \big(\tau_0(\mathscr{G})\big)^{\times^n} \times \tau_0(X) \Big) && \text{by (3.71)} \\
&\simeq i_0 \Big(\mathrm{coeq}\big(\tau_0(\mathscr{G}) \times \tau_0(X) \rightrightarrows \tau_0(X) \big) \Big) && \text{by Ex. 3.1.19} \\
&= i_0\, (\tau_0(X)/\tau_0(\mathscr{G}))\,.
\end{aligned}$$

□

Definition 3.2.94 (Principal ∞-bundles)**.** Given an ∞-topos $\mathbf{H}$, with $X \in \mathbf{H}$ and $\mathscr{G} \in \mathrm{Grp}(\mathbf{H}) \xrightarrow{\mathrm{Grp}(X\times(-))} \mathrm{Grp}(\mathbf{H}_{/X})$,

(i) we say that a regular ∞-action $\mathscr{G} \circlearrowright P$ (Def. 3.2.90) of $\mathscr{G}$ internal to the slice topos $\mathbf{H}_{/X}$ (Prop. 3.2.48) is a *formally $\mathscr{G}$-principal ∞-bundle* over X and an actual *$\mathscr{G}$-principal* ∞ over X if it is inhabited, $P \twoheadrightarrow \ast$ (Nota. 3.2.12).

(ii) We write

$$\mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H})_X \hookrightarrow \mathrm{Frm}\mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H})_X \hookrightarrow G\,\mathrm{Act}(\mathbf{H}_{/X}) \tag{3.115}$$

for the full sub-∞-categories of the ∞-category of all $\mathscr{G}$-actions (3.92) on the (formally) principal ones.

Remark 3.2.95 (Local triviality)**.** In contrast to the the 1-category theoretic analog Def. 2.2.2 (Rem. 2.1.1), the above Def. 3.2.94 does not include an explicit local triviality clause, but just formulates the internal notion of (formally) principal bundles (Ntn. 1.0.25). It is instead the nature of the ambient ∞-toposes which implies the local triviality of internal principal bundles (Thm. 4.1.2, Thm. 4.2.7 below).

Proposition 3.2.96 (Every ∞-action is principal over its homotopy quotient)**.** *In an ∞-topos $\mathbf{H}$, every ∞-action $\mathscr{G} \circlearrowright P \in \mathscr{G}\,\mathrm{Act}(\mathbf{H})$ (Def. 3.2.74) is principal (Def. 3.2.94) relative to the coprojection into its homotopy quotient $X := P /\!\!/ \mathscr{G}$ (3.93).*

Proof. Since the left base change functor $\sum\limits_X : \mathbf{H}_{/X} \to \mathbf{H}$ (Prop. 3.2.50)

– preserves ∞-colimits and hence homotopy quotients, being a left adjoint,

– sends products to fiber products,

the defining diagram on the left is sent by $\sum_X$ to the following diagram on the right:

$$\mathbf{H}_{/X} \ni \begin{tikzcd}[column sep=small] (X\times\mathscr{G})\times P \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & P \arrow[dr, equal] \arrow[dd] & \\ & P\times P \arrow[rr] \arrow[dd] & & P \arrow[dd] \\ P \arrow[rr] \arrow[dr, equal] & & P/\!\!/(X\times\mathscr{G}) \arrow[dr] & \\ & P \arrow[rr] & & \ast \end{tikzcd} \quad \overset{\Sigma_X}{\longmapsto} \quad \begin{tikzcd}[column sep=small] \mathscr{G}\times P \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & P \arrow[dr, equal] \arrow[dd] & \\ & P\underset{X}{\times} P \arrow[rr] \arrow[dd] & & P \arrow[dd] \\ P \arrow[rr] \arrow[dr, equal] & & P/\!\!/\mathscr{G} \arrow[dr, equal] & \\ & P \arrow[rr] & & X \end{tikzcd} \in \mathbf{H}$$

Here the dashed morphism on the right is manifestly an equivalence. Since left base change also reflects equivalences, so is the dashed morphism on the left, which is the claim to be proven. ☐

Theorem 3.2.97 (Delooping groupoids are moduli stacks for principal ∞-bundles [NSS12a, Thm. 3.17])**.** *In an ∞-topos $\mathbf{H}$, for every ∞-group $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ (Def. 3.2.69) there is for each $X \in \mathbf{H}$ a natural equivalence of ∞-categories*

$$\begin{tikzcd} \mathbf{H}(X, \mathbf{B}\mathscr{G}) \arrow[r, "\sim"'] \arrow[rr, bend right, hook, "\mathrm{hofib}"'] & \mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H})_X \arrow[r, hook] & \mathscr{G}\,\mathrm{Act}(\mathbf{H}_{/X}) \end{tikzcd}$$

$$(X \xrightarrow{c} \mathbf{B}\mathscr{G}) \longmapsto \left(\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \ast \arrow[d, two heads] \\ X \arrow[r, "c"'] & \mathbf{B}\mathscr{G} \end{tikzcd} \right)$$

between $\mathscr{G}$-principal ∞-bundles over X (Def. 3.2.94) and morphisms from X to the delooping $\mathbf{B}\mathscr{G}$ (3.88).

Proof. The homotopy fiber functor is fully faithful by Prop. 3.2.15, factors as shown by Prop. 3.2.96, and its factorization is essentially surjective by Prop. 3.2.92. ☐

3.3 Cohesive homotopy theory

Where general ∞-toposes reflect geometry (geometric homotopy theory, §3.2) in a broad sense, encompassing exotic notions of space such as encountered, say, in arithmetic geometry, some of their properties may be too exotic compared to tamer notions of space as expected in classical differential topology ([Mi64][Be21]); for instance in that their shape (Def. 3.2.19) may be a pro-∞-groupoid that is not represented by a plain ∞-groupoid.

The notion of *cohesive ∞-toposes* (Def. 3.3.1 below) narrows in on those *gros* ∞-toposes that share more of the abstract properties of categories of spaces expected in differential topology; for instance in that
(1) all their slices have genuine ∞-groupoidal shape (Prop. 3.3.5 below) and
(2) they contain a good supply of *concrete spaces*, namely of sets equipped with cohesive (e.g. topological or smooth) structure (see Prop. 3.3.37 below).

We first discuss the general concept of cohesive ∞-groupoids and then consider standard models that encapsulate traditional differential and equivariant topology:

– §3.3.1: smooth cohesion,
– §3.3.2: orbi-singular cohesion.

Definition 3.3.1 (Cohesive ∞-topos [SSS12, §3.1][Sc13][SS20-Orb, §3.1.1], following [Law07])**.**
(i) An ∞-topos $\mathbf{H}_{\cup}$ over some base ∞-topos $\mathbf{B}$ is called *cohesive* if its global section geometric morphism $\mathbf{H}_{\cup} \xrightarrow{\mathrm{Pnt}:=\Gamma} \mathbf{B}$ extends to an adjoint quadruple of the following form:

$$
\begin{array}{ccccccc}
 & \times\!\!\longleftarrow & \mathrm{Shp} & \longrightarrow & & & \textbf{shape} \\
 & & \bot & & & & \\
 & \longleftarrow & \mathrm{Dsc} & \hookrightarrow & & & \textbf{discrete} \\
 & & \bot & & & & \\
\mathbf{H}_{\cup} & \longrightarrow & \mathrm{Pnt} & \longrightarrow & \mathbf{B} & & \textbf{points} \\
 & & \bot & & & & \\
 & \longleftarrow & \mathrm{Cht} & \hookrightarrow & & & \textbf{chaotic}
\end{array}
\tag{3.116}
$$

meaning that:
(a) $\mathrm{Dsc} :\equiv \mathrm{LCnst}$ is fully faithful (8) and has a left adjoint (10), denoted Shp, which preserves finite products.
(b) $\mathrm{Pnt} :\equiv \Gamma$ has a right adjoint, denoted Cht.
(ii) We denote the induced adjoint triple of endo-∞-functors on $\mathbf{H}$ by

$$
\begin{array}{ccc}
\textbf{pure shape aspect} & \int & := \mathrm{Dsc}\circ\mathrm{Shp} \\
 & \bot & \\
\textbf{purely discrete aspect} & \flat & := \mathrm{Dsc}\circ\mathrm{Pnt} \\
 & \bot & \\
\textbf{purely continuous aspect} & \sharp & := \mathrm{Cht}\circ\mathrm{Pnt}
\end{array}
\tag{3.117}
$$

Remark 3.3.2 (Idempotency of cohesive adjoints)**.** The fact that the adjoints Dsc and Cht in (3.116) are fully faithful is equivalent to the following (co)unit transformations being equivalences

$$
\mathrm{Shp}\circ\mathrm{Dsc} \xrightarrow[\sim]{\varepsilon^{\int}} \mathrm{id}\,, \qquad \mathrm{id} \xrightarrow[\sim]{\eta^{\flat}} \mathrm{Pnt}\circ\mathrm{Dsc}\,, \qquad \mathrm{Pnt}\circ\mathrm{Cht} \xrightarrow[\sim]{\varepsilon^{\sharp}} \mathrm{id}\,,
\tag{3.118}
$$

and equivalent to the (co)monads (3.117) being idempotent, in that their (co)multiplications are natural equivalences:

$$
\int\circ\int \xrightarrow[\sim]{\mathrm{Dsc}\left(\varepsilon^{\int}_{\mathrm{Shp}(-)}\right)} \int\,, \qquad \flat \xrightarrow[\sim]{\mathrm{Dsc}\left(\eta^{\flat}_{\mathrm{Pnt}(-)}\right)} \flat\circ\flat\,, \qquad \sharp\circ\sharp \xrightarrow[\sim]{\mathrm{Cht}\left(\varepsilon^{\sharp}_{\mathrm{Pnt}(-)}\right)} \sharp\,,
\tag{3.119}
$$

as well as

$$
\int\circ\flat \xrightarrow[\sim]{\mathrm{Dsc}\left(\varepsilon^{\int}_{\mathrm{Pnt}(-)}\right)} \flat\,, \qquad \int \xrightarrow[\sim]{\mathrm{Dsc}\left(\eta^{\flat}_{\mathrm{Shp}(-)}\right)} \flat\circ\int.
\tag{3.120}
$$

It is this idempotency which makes these (co)monads act as *projecting out* certain qualities or *modes of being* of objects, here: *being pure shape*, *being purely discrete* and *being chaotic*[3], while the adjointness $\int \dashv \flat \dashv \sharp$ makes them project out *dual* such qualities, with the (co)unit transformations exhibiting every object $X \in \mathbf{H}$ as having quality in between the given opposing extremes, e.g.:

$$
\underset{\textbf{purely discrete aspect}}{\flat X} \xrightarrow{\varepsilon^{\flat}_X} X \xrightarrow{\eta^{\int}_X} \underset{\textbf{pure shape aspect}}{\int X}.
\qquad
\underbrace{\flat X \longrightarrow \int X}_{\textbf{points-to-pieces transform}}
\tag{3.121}
$$

[3]This follows the traditional terminology for "chaotic topology"; see Ex. 3.3.20.

The following simple class of examples of cohesive ∞-toposes (Ex. 3.3.3) is already interesting in itself (as in our key Def. 3.3.62 below); and other important examples (such as the pivotal $\mathrm{SmthGrpd}_\infty$, Ntn. 3.3.26 below) are obtained by (co-)restricting an example in this class from ∞-presheaves to ∞-sheaves (when possible):

Example 3.3.3 (Discrete cohesion)**.** Let $\mathbf{S}$ be a small ∞-category regarded as an ∞-site with trivial Grothendieck topology. If $\mathbf{S}$ has a terminal object, then its ∞-(pre-)sheaves $\mathbf{H} = \mathrm{Sh}_\infty(\mathbf{S})$ form a cohesive ∞-topos (Def. 3.3.1). A transparent abstract way to see this is to note that the existence of a terminal object means equivalently that its inclusion i is right adjoint to the unique functor p to the terminal category. This implies the adjoint quadruple by Lem. 3.2.43 with Lem. 3.2.42:

$$
\begin{tikzcd}
\mathbf{S} \arrow[r, shift left=1.2ex, "p"] \arrow[r, phantom, "\perp"] & \ast \arrow[l, hook', shift left=1.2ex, "i"]
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathrm{PSh}_\infty(\mathbf{S}) \arrow[r, shift left=4ex, "p_!"] \arrow[r, hookleftarrow, shift left=2ex, "p^* \simeq i_!"' description] \arrow[r, "p_* \simeq i^*" description] \arrow[r, hookleftarrow, shift right=2ex, "i_*" description] & \mathrm{Sh}_\infty(\ast) \simeq \mathrm{Grpd}_\infty\,.
\end{tikzcd}
$$

Moreover:
(1) the left Kan extension $i_!$ is fully-faithful since i is (by Lem. 3.2.42) and it preserves finite products since since p does (by Lem. 3.2.40, Lem. 3.2.41)
(2) a right adjoint of a right adjoint of a fully faithful functor is itself fully faithful.

Remark 3.3.4 (The axioms on the shape modality)**.** That the cohesive shape operation $\int$ (3.117) preserves finite products means that
(a) it preserves binary products, in that for all $X, Y \in \mathbf{H}$ we have natural equivalences

$$\textstyle\int(X \times Y) \simeq (\int X) \times (\int Y), \qquad \text{equivalently} \qquad \mathrm{Shp}(X \times Y) \simeq \mathrm{Shp}(X) \times \mathrm{Shp}(Y)\,. \tag{3.122}$$

(b) it preserves the terminal object, in that

$$\textstyle\int \ast \simeq \ast, \quad \text{equivalently}: \quad \mathrm{Shp}(\ast) \simeq \ast. \tag{3.123}$$

The following are some key implications of the fact/requirement that the shape modality preserves finite products:
(i) together with idempotency of modalities (Rem. 3.118), it implies the *projection formula*:

$$\textstyle\int(X \times \int Y) \simeq (\int X) \times (\int Y), \qquad \text{equivalently}: \qquad \mathrm{Shp}(X \times \mathrm{Dsc}(S)) \simeq \mathrm{Shp}(X) \times S. \tag{3.124}$$

(ii) together with the preservation (11) of simplicial colimits and the effectiveness of group objects (Prop. 3.2.15), it implies that passage to shape preserves group objects (Def. 3.2.69) and their deloopings (3.88) and more generally action groupoids (Def. 3.2.74) and their homotopy quotients (3.93):

$$\textstyle\int \mathbf{B}\mathcal{G} \simeq \mathbf{B}\int\mathcal{G} \overset{(4.12)}{=:} B\mathcal{G}, \quad \int(X /\!\!/ \mathcal{G}) \simeq (\int X) /\!\!/ (\int \mathcal{G}) \tag{3.125}$$

The analogous statement holds for $\flat$, for the same reasons:

$$\flat \mathbf{B}\mathcal{G} \simeq \mathbf{B}\flat\mathcal{G} \overset{(4.12)}{=:} B\flat\mathcal{G}, \quad \flat(X /\!\!/ \mathcal{G}) \simeq (\flat X) /\!\!/ (\flat \mathcal{G}), \tag{3.126}$$

but $\flat$ preserves also all fiber products (in particular) and hence preserves all groupoid objects (Def. 3.2.7, not just the group action objects);
(iii) concretely, (3.123) implies that $\flat$ computes the underlying (geometrically discrete) ∞-*groupoid of points*

$$
\begin{aligned}
\flat X &\simeq \mathrm{Grpd}_\infty(\ast, \flat X) && \text{by (3.9)} \\
&\simeq \mathbf{H}(\textstyle\int \ast, X) && \text{by (3.116) with (11)} \\
&\simeq \mathbf{H}(\ast, X) && \text{by (3.123);}
\end{aligned}
\tag{3.127}
$$

(iv) together with the mapping-stack adjunction (3.72), it implies a canonical natural transformation for $X, A \in \mathbf{H}$

$$\textstyle\int \mathrm{Map}(X, A) \xrightarrow{\widetilde{\int \mathrm{ev}}} \mathrm{Map}\big(\int X, \int A\big), \tag{3.128}$$

this being the adjunct (10) of the shape of the evaluation map (3.74):

$$\textstyle(\int X) \times (\int \mathrm{Map}(X, A)) \xrightarrow{\sim} \int\,(X \times \mathrm{Map}(X, A)) \xrightarrow{\int \mathrm{ev}} \int A\,.$$

Less immediate is that the shape operation also preserves certain classes of homotopy fiber products: this is the content of Prop. 3.3.8 and Prop. 3.3.10 below.

Some basic properties of cohesive ∞-toposes.

Proposition 3.3.5 (Shape of slice of cohesive ∞-topos is cohesive shape)**.** *Let* $\mathbf{H}$ *be a cohesive ∞-topos (Def. 3.3.1) with shape modality* (3.117)

$$\int : \; \mathbf{H} \xrightarrow{\mathrm{Shp}} \mathrm{Grpd}_\infty \xrightarrow{\mathrm{Dsc}} \mathbf{H}\,.$$

Then for any $X \in \mathbf{H}$ *the shape of the slice ∞-topos over X, in the sense of Def. 3.2.19, is equivalently the cohesive shape of X, under the embedding* (3.49) *of ∞-groupoids into pro-∞-groupoids:*

$$\mathrm{Shp}(\mathbf{H}_{/X}) \;\simeq\; \mathrm{Shp}(X) \;\;\in\; \mathrm{Grpd}_\infty \hookrightarrow \mathrm{ProGrpd}_\infty\,.$$

Proof. The base change adjoint triple of the slice (Prop. 3.2.50) combines with part (1) of cohesion (Def. 3.3.1)

$$\Gamma_X \;:\; \begin{tikzcd}[column sep=huge]
\mathbf{H}_{/X}
\arrow[r, shift left=16pt, "\Sigma_X"{description}]
\arrow[r, phantom, shift left=8pt, "\perp"]
\arrow[from=r, "X\times(-)"{description}]
\arrow[r, phantom, shift right=8pt, "\perp"]
\arrow[r, shift right=16pt, "\Pi_X"{description}]
& \mathbf{H}
\arrow[r, shift left=16pt, "\mathrm{Shp}"{description}]
\arrow[r, phantom, shift left=8pt, "\perp"]
\arrow[from=r, hook', "\mathrm{Disc}"{description}]
\arrow[r, phantom, shift right=8pt, "\perp"]
\arrow[r, shift right=16pt, "\mathrm{Pnts}"{description}]
& \mathrm{Grp}_\infty
\end{tikzcd} \;:\; \mathrm{LCnst}_X \tag{3.129}$$

and by essential uniqueness of adjoints and of the terminal geometric morphism, the composite is equivalently the global section geometric morphism $(\Gamma_X \dashv)$ of the slice, as shown above.

Now consider, for $S_1, S_2 \in \mathrm{Grpd}_\infty$ the following sequence of natural equivalences:

$$\begin{aligned}
\mathrm{Grp}_\infty\big(S_1, \Gamma_X \circ \mathrm{LCnst}_X(S_2)\big) &\simeq \mathbf{H}\big(\mathrm{LCnst}_X(S_1), \mathrm{LCnst}_X(S_2)\big)_X && \text{by the composite adjunction in (3.129)}\\
&\simeq \mathbf{H}\big(X \times \mathrm{Disc}(S_1), X \times \mathrm{Disc}(S_2)\big)_X && \text{by the factorization in (3.129)}\\
&\simeq \mathbf{H}\big(X \times \mathrm{Disc}(S_1), \mathrm{Disc}(S_2)\big) && \text{by the left adjunction in (3.129)}\\
&\simeq \mathrm{Grp}_\infty\big(\mathrm{Shp}\,(X \times \mathrm{Disc}(S_1))\,, S_2\big) && \text{by the right adjunction in (3.129)}\\
&\simeq \mathrm{Grp}_\infty\big(\mathrm{Shp}(X) \times S_1, S_2\big) && \text{by (3.124)}\\
&\simeq \mathrm{Grp}_\infty\big(S_1, \mathrm{Grp}_\infty(\mathrm{Shp}(X), S_2)\big) && \text{by (3.7)}\,.
\end{aligned}$$

With this, the ∞-Yoneda lemma (Prop. 3.2.29) implies the natural equivalence that is to be proven. □

Lemma 3.3.6 (Cohesive points are intrinsic points)**.** *If* $\mathbf{H}$ *is cohesive over* Grpd_∞*, then for all* $X \in \mathbf{H}$ *we have a natural equivalence*

$$\mathrm{Pnts}(X) \;\simeq\; \mathbf{H}(*, X)\,. \tag{3.130}$$

Proof.

$$\begin{aligned}
\mathrm{Pnts}(X) &\simeq \mathrm{Grpd}_\infty\big(*, \mathrm{Pnts}(X)\big) && \text{by (3.9)}\\
&\simeq \mathbf{H}\big(\mathrm{Dsc}(*), X\big) && \text{by cohesion}\\
&\simeq \mathbf{H}(*, X) && \text{by cohesion}\,.
\end{aligned}$$

□

Lemma 3.3.7 (Mapping space consists of cohesive points of mapping stack)**.** *If* $\mathbf{H}$ *is cohesive over* $\mathbf{B}$*, then for all* $X, A \in \mathbf{H}$ *we have a natural equivalence between their hom-space* (7) *and the cohesive points of their mapping stack* (3.72)*:*

$$\mathbf{H}(X, A) \;\simeq\; \mathrm{Pnt}\,\mathrm{Map}(X, A)\,. \tag{3.131}$$

More generally, for $B \in \mathbf{H}$ *and* $E_1, E_2 \in \mathbf{H}_{/B}$*, the cohesive points of the slice mapping stack (Def. 3.2.65) yield the mapping space of the slice:*

$$\mathbf{H}(E_1, E_2)_B \;\simeq\; \mathrm{Pnt}\,\mathrm{Map}(E_1, E_2)_B\,. \tag{3.132}$$

Proof. For the first case:

$$\begin{aligned}
\mathbf{H}(X, A) &\simeq \mathbf{H}\big(*, \mathrm{Map}(X, A)\big) && \text{by Ex. 3.2.57}\\
&\simeq \mathrm{Pnt}\,\mathrm{Map}(X, A) && \text{by Ex. 3.3.6}\,.
\end{aligned}$$

More generally, for the second case:

$$\begin{aligned}
\mathbf{H}(E_1, E_2)_B &\simeq \mathrm{Map}\big(*, \mathrm{Map}(E_1, E_2)_B\big) && \text{by Lem. 3.2.68}\\
&\simeq \mathrm{Pnt}\,\mathrm{Map}(E_1, E_2)_B && \text{by Ex. 3.3.6.}
\end{aligned}$$

□

Proposition 3.3.8 (Shape preserves fiber products over discrete objects [Sc13, Thm. 3.8.19][SS20-Orb, Lem. 3.5])**.**
For $\mathbf{H}$ *a cohesive* ∞*-topos (Def. 3.3.1) over* Grpd_∞*, for* $B \in \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \mathbf{H}$ *a discrete object and* $X, Y \in \mathbf{H}_{/B}$ *two cohesive objects fibered over B, there is a natural equivalence*

$$\textstyle\int \big(X \underset{B}{\times} Y\big) \;\simeq\; (\int X) \underset{B}{\times} (\int Y) \tag{3.133}$$

between the shape (3.117) *of their homotopy fiber product over S and the homotopy fiber product of their shapes.*

Proof. The Shp $\dashv$ Dsc-adjunction passes to slices ([Lur09, Prop. 5.2.5.1]) and, by a generalization of the Grothendieck construction, slices of ∞-toposes over inverse images of ∞-groupoids B are equivalent to the ∞-categories of functors from B to the them ([BP22, Lem. 3.10]):

$$\begin{tikzcd}
\mathrm{Fnctr}\big(B, \mathbf{H}\big) \arrow[rrrr, bend left, "\mathrm{Fnctr}(B,\mathrm{Shp})"] & \simeq & \mathbf{H}_{/\mathrm{Dsc}(B)} \arrow[r, shift left, "\mathrm{Shp}_{/B}"] \arrow[r, phantom, "\perp"] & \mathrm{Grpd}_{\infty/B} \arrow[l, shift left, "\mathrm{Dsc}_{/B}"] & \simeq \; \mathrm{Fnctr}\big(B, \mathrm{Grpd}_\infty\big) . \arrow[llll, bend left, "\mathrm{Fnctr}(B,\mathrm{Dsc})"]
\end{tikzcd}$$

Under these equivalences, the right adjoint is readily seen to be given by objectwise application, as shown at the bottom, from which it follows by uniqueness of ∞-adjoints that the left adjoint is similarly given by objectwise application of the shape operation, as shown at the top. But since limits of ∞-presheaves (here: over B) are computed objectwise, and since both Shp and Dsc preserve finite products, it follows that so do their slicings over S. Now the claim follows since the fiber product in question is equivalently the plain product in the slice $\mathbf{H}_{/S}$. ☐

Lemma 3.3.9 (Reverse pasting law for ∞-groupoids)**.** *Given a homotopy pasting diagram in* Grpd_∞ *of the form*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
A \arrow[r, two heads, "p"'] & B \arrow[r] & C ,
\end{tikzcd}$$

such that
- *the total rectangle and the left square are homotopy Cartesian* (4)*,*
- *the bottom left morphism is an effective epimorphism (Def. 3.2.11),*

then also the right square is homotopy Cartesian.

Proof. Using the presentation of the situation in the classical model structure on simplicial sets (Ntn. 3.1.10), we may consider the analogous pasting diagram in ΔSet, such that the two bottom morphisms, say, are Kan fibrations. Since ΔSet is a presheaf category and hence regular (by Exp. 1.0.12) we are thus reduced to showing that the 1-category theoretic reverse pasting law from Lem. 1.0.14 applies, which is the case if the bottom left Kan fibration (representing) p is a regular epimorphism. Since ΔSet is in fact a topos, this is equivalent to showing that this Kan fibration is an epimorphism (again by Exp. 1.0.12), hence a degreewise surjection of simplicial sets. But the assumption that p is an effective epimorphism of Grpd_∞ means that it is a surjection on connected components (by Prop. 3.2.13), and every Kan fibration which is a surjection on connected components is already surjective in degree 0 (seen by lifting against the acyclic cofibrations $\Delta^0 \hookrightarrow \Delta^1$) and hence in each degree n (seen by lifting against $\Delta^0 \hookrightarrow \Delta^n$). ☐

Proposition 3.3.10 (Shape preserves homotopy fibers of deloopings out of discrete domains)**.** *For* $\mathbf{H}$ *a cohesive* ∞*-topos (Def. 3.3.1) over* Grpd_∞*, the shape operation preserves homotopy fibers of morphisms out of cohesively discrete* ∞*-groupoids,* $X \simeq \flat X$*, into deloopings of cohesive* ∞*-groups* $\mathscr{G}$*:*

$$\mathrm{fib}(f) \longrightarrow X \xrightarrow{\;f\;} \mathbf{B}\mathscr{G} \qquad\qquad \vdash \qquad\qquad \textstyle\int \mathrm{fib}(f) \simeq \mathrm{fib}\big(\int f\big) \longrightarrow X \xrightarrow{\;\int f\;} B\mathscr{G}$$

Proof. Since every $X \in \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \mathrm{SmthGrpd}_\infty$ is a coproduct of its connected components, and since coproducts are preserved by homotopy pullback (Prop. 3.2.6) and by $\int$ (11) we may assume without restriction of generality that X is connected, hence (due to Prop. 3.2.70) that $X \simeq B\mathscr{K}$ for some ∞-group $\mathscr{K}$, so that the point inclusion $* \twoheadrightarrow X$ is an effective epimorphism of (smooth) ∞-groupoids (by Prop. 3.2.13). Consider then the pasting diagram shown on the left below, where we are using the pasting law (5) and the looping/delooping equivalence (16) in order to identify $\mathscr{G}$ as the fiber appearing in the top left:

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & * \arrow[d] \\ * \arrow[r, two heads] & B\mathscr{K} \arrow[r, "f"'] & \mathbf{B}\mathscr{G} \end{tikzcd} \quad \vdash \quad \begin{tikzcd} \int\mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \int\mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & * \arrow[d] \\ * \arrow[r, two heads] & B\mathscr{K} \arrow[r, "\int f"'] & B\mathscr{G}\,. \end{tikzcd}$$

Under applying the shape modality to this diagram, shown on the right, the total rectangle remains a homotopy pullback square with the bottom left morphism remaining an effective epimorphism (both by Rem. 3.3.4), and the left square remains a pullback by Prop. 3.3.8. Therefore the reverse pasting law for homotopy pullbacks (Prop. 3.3.9) implies that also the right square remains a homotopy pullback, which is equivalently the claim to be shown. □

Cohesive charts.

Notation 3.3.11 (Cohesive charts [SS20-Orb, Def. 3.9])**.** Given a cohesive ∞-topos $\mathbf{H}$ we say that an ∞-site Chrt for $\mathbf{H}$ is an *∞-category of cohesive charts* if, under the ∞-Yoneda embedding (3.54), all its objects have contractible shape:

$$\forall \mathrm{U} \in \mathrm{Chrt} \xhookrightarrow{y} \mathbf{H} \;:\; \int \mathrm{U} \simeq *; \tag{3.134}$$

equivalently:

$$\forall \mathrm{U} \in \mathrm{Chrt} \xhookrightarrow{y} \mathbf{H} \;:\; \mathrm{Shp}(\mathrm{U}) \simeq *. \tag{3.135}$$

Example 3.3.12 (Flat modality over a site of cohesive charts)**.** If a cohesive ∞-topos $\mathbf{H}$ over Grpd_∞ has cohesive Chrt (Ntn. 3.3.11) then for all $S \in \mathrm{Grpd}_\infty$ the corresponding geometrically discrete cohesive object is constant as an ∞-presheaf on the charts, in that for all $U \in \mathrm{Chrt}$ there is a natural equivalence:

$$\big(\mathrm{Disc}(S)\big)(\mathrm{U}) \simeq S \quad \in \mathrm{Grpd}_\infty\,. \tag{3.136}$$

Moreover, for $A \in$ there is a natural equivalence

$$(\flat A)(\mathrm{U}) \simeq A(*)\,.$$

Proof. The first statement is the composite of the following sequence of natural equivalences:

$$\begin{aligned} \big(\mathrm{Disc}(S)\big)(\mathrm{U}) &\simeq \mathbf{H}\big(\mathrm{U}, \mathrm{Disc}(S)\big) && \text{by (3.55)} \\ &\simeq \mathbf{H}\big(\mathrm{Shp}(*)S\big) && \text{by (3.117)} \\ &\simeq \mathbf{H}(*, S) && \text{by (3.135)} \\ &\simeq S && \text{by (3.9)}\,. \end{aligned}$$

From this the second statement follows by Lem. 3.3.6. □

Lemma 3.3.13 (Mapping stack into discrete object is discrete)**.** *Let* $\mathbf{H}$ *be a cohesive ∞-topos with a category* Chrt *of cohesive charts (Ntn. 3.3.11). Then:*
(i) *the hom-equivalence of the* $\int \dashv \flat$*-adjunction also holds internally,*
(ii) *a mapping stack* (3.72) *into a discrete object is itself discrete:*

$$\flat A \simeq A \quad \Rightarrow \quad \underset{X \in \mathbf{H}}{\forall} \; \mathrm{Map}(X, A) \simeq \mathrm{Map}(\textstyle\int X, A) \simeq \flat\mathrm{Map}(X, A) \overset{(3.120)}{\simeq} \textstyle\int \mathrm{Map}(X, A)\,. \tag{3.137}$$

Proof. By the ∞-Yoneda lemma (Prop. 3.2.29), we may check the statement on $\mathrm{U} \in \mathrm{Chrt}$:

$$
\begin{aligned}
\mathrm{Map}(X,A)(\mathrm{U}) &\simeq \mathbf{H}(\mathrm{U}\times X, A) && \text{by (3.73)}\\
&\simeq \mathbf{H}(\mathrm{U}\times X, \flat A) && \text{by assumption}\\
&\simeq \mathbf{H}(\textstyle\int(\mathrm{U}\times X), A) && \text{by (3.117)}\\
&\simeq \mathbf{H}((\textstyle\int \mathrm{U})\times(\textstyle\int X), A) && \text{by (3.122)}\\
&\simeq \mathbf{H}(\textstyle\int X, A) && \text{by (3.134)}\\
&\simeq \mathbf{H}(X, \flat A) && \text{by (3.117)}\\
&\simeq \mathbf{H}(X, A) && \text{by assumption}\\
&\simeq \mathbf{H}(*\times X, A) && \\
&\simeq \mathrm{Pts}\,\mathrm{Map}(X,A) && \text{by (3.73)}\\
&\simeq (\mathrm{Dsc}\,\mathrm{Pnt}\,\mathrm{Map}(X,A))(\mathrm{U}) && \text{by (3.136)}\\
&= (\flat\mathrm{Map}(X,A))(\mathrm{U}) && \text{by (3.117)}\,.
\end{aligned}
$$

□

3.3.1 Smooth Cohesion

We turn attention now to the archetypical example of those cohesive ∞-toposes (Def. 3.3.1) given by generalized spaces that are built form (are ∞-colimits of) *Cartesian spaces* $\mathbb{R}^n$ glued by smooth functions between them, i.e. built from the elementary *charts* of differential topology. The *concrete* 0-truncated objects among these *smooth ∞-groupoids* are known as *diffeological spaces* (Ntn. 3.3.15 below) and we begin by recalling and developing general topology as seen inside the category of diffeological spaces.

Diffeological spaces.

Definition 3.3.14 (Good open covers)**.** For $\mathrm{X} \in \mathrm{TopSpc}$ a topological manifold,
(i) an open cover $\{\mathrm{U}_i \hookrightarrow \mathrm{X}\}_{i\in I}$ is *good* if all finite intersections of patches are (empty or) homeomorphic to an open ball:

$$
\underset{n\in\mathbb{N}}{\forall}\ \underset{i_0,\cdots,i_n\in I}{\forall}\quad \mathrm{U}_{i_0}\cap\cdots\cap\mathrm{U}_{i_n} \simeq \begin{cases}\mathbb{R}^n & \text{or}\\ \varnothing\,. \end{cases} \tag{3.138}
$$

(ii) If X is a smooth manifold, we say a good cover is *differentiably good* ([FStS12, Def. 6.3.9]) if the identifications in (3.138) can be taken to be diffeomorphisms instead of just homeomorphisms.

Notation 3.3.15 (Cartesian spaces and Diffeological spaces)**.** **(i)** We write

$$
\mathrm{CartSpc} \hookrightarrow \mathrm{SmthMfd} \tag{3.139}
$$

for the category whose objects are the $\mathbb{R}^n$ for $n \in \mathbb{N}$ and whose morphisms are the smooth functions between these.
(ii) We regard this as a site with respect to the coverage (Grothendieck pre-topology) whose covers of $\mathbb{R}^n$ are the *differentiably good* open covers ([FStS12, Def. 6.3.9]) hence the open covers $\{U_i \hookrightarrow \mathbb{R}^n\}_{i\in I}$, such that all non-empty finite intersections $U_{i_0}\cap\cdots\cap U_{i_k}$, $k \in \mathbb{N}$, of patches are *diffeomorphic* to an open ball, and hence to $\mathbb{R}^n$.
(iii) We say that a sheaf on this site is a *smooth set* and is a *diffeological space* ([So80][So84][IZ85], see [BH08] [IZ13]) if it is a *concrete sheaf* ([Du79b]):

$$
\mathrm{DfflgSpc} \xhookrightarrow{i_{\sharp 1}} \mathrm{SmthGrpd}_0 \;:=\; \mathrm{Sh}(\mathrm{CartSpc})\,. \tag{3.140}
$$

This means that a functor

$$
\begin{array}{ccccl}
X: & \mathrm{CartSpc}^{\mathrm{op}} & \longrightarrow & \mathrm{Set} & \\
& \mathbb{R}^n & \longmapsto & X(\mathbb{R}^n) & \text{set of plots}
\end{array} \tag{3.141}
$$

encodes a diffeological space if and only if it is a sheaf and for all $n \in \mathbb{N}$ the following natural morphism is a monomorphism, hence, for each n, an injection of the set of plots into the set of all functions of underlying point sets:

$$\begin{array}{ccccc} X(\mathbb{R}^n) \;\simeq\; \mathrm{SmthGrpd}_0(\mathbb{R}^n, X) & \hookrightarrow & \mathrm{Set}\big(\mathrm{SmthGrpd}_0(*, \mathbb{R}^n), \mathrm{SmthGrpd}_0(*, X)\big) & =: & (\sharp X)(\mathbb{R}^n)\,. \\ (\mathbb{R}^n \xrightarrow{\phi} X) & \longmapsto & \big((* \xrightarrow{r} \mathbb{R}^n) \mapsto (* \xrightarrow{r} \mathbb{R}^n \xrightarrow{\phi} X)\big) & & \end{array}$$

(iv) The category of diffeological spaces contains the category of smooth manifolds as a full subcategory, via the extended Yoneda embedding:

$$\mathrm{CartSpc} \hookrightarrow \mathrm{SmthMfd} \overset{y}{\hookrightarrow} \mathrm{DfflgSpc}\,. \tag{3.142}$$

Remark 3.3.16 (Diffeological mapping spaces)**.** On general grounds, the category of diffeological spaces (Ntn. 3.3.15) is Cartesian closed, in that for any $X \in \mathrm{DfflgSpc}$ the cartesian product operation with X, given by

$$X \times Y \,:\, \mathbb{R}^n \longmapsto X(\mathbb{R}^n) \times Y(\mathbb{R}^n)\,, \tag{3.143}$$

has a right adjoint

$$\begin{tikzcd} \mathrm{DfflgSpc} \arrow[r, phantom, "\perp"] & \mathrm{DfflgSpc} \arrow[l, bend right, "X \times (-)"'] \arrow[l, leftarrow, bend left, "\mathrm{Map}(X,-)"] \end{tikzcd}$$

given by

$$\mathrm{Map}(X, Y) \,:\, \mathbb{R}^n \longmapsto \mathrm{DfflgSpc}\,(X \times \mathbb{R}^n, Y)\,. \tag{3.144}$$

Example 3.3.17 (Continuous diffeology and D-topology)**.** For any $X \in \mathrm{DfflgSpc}$ (3.140) the underlying set becomes a topological space $\mathrm{Dtplg}(X) \in \mathrm{TopSpc}$ via the *D-topology* ([IZ85, Def. 1.2.3][IZ13, §2.8][CSW14, §3]), which is the final (i.e. finest) topology such that all plots (3.141) become continuous functions:

$$\big(\mathbb{R}^n \xrightarrow{\phi} X\big) \;\in\; \mathrm{DfflgSpc}(\mathbb{R}^n, X) \;\simeq\; X(\mathbb{R}^n) \qquad \vdash \qquad \mathbb{R}^n \xrightarrow[\text{continuous}]{\phi} \mathrm{Dtplg}(X)\,.$$

Conversely, for any $X \in \mathrm{TopSpc}$ the underlying set becomes a diffeological space (3.140) via the *continuous diffeology* $\mathrm{Cdfflg}(X) \in \mathrm{DfflgSpc}$ whose plots (3.141) are the continuous functions

$$\mathrm{Cdfflg}(X) \;:\; \mathbb{R}^n \longmapsto \mathrm{TopSpc}(\mathbb{R}^n, X)\,.$$

Notation 3.3.18 (D-topological spaces)**.** We write

$$\begin{tikzcd} \mathrm{TopSpc} \arrow[r, bend right, "\mathrm{Cdfflg}"'] \arrow[r, phantom, "\perp"] & \mathrm{DTopSpc} \arrow[l, hook', bend right] \end{tikzcd}$$

for the coreflective subcategory on the *Delta-generated spaces* [Dug03] also called *numerically generated spaces* [SYH10] (see also [CSW14, §3.2]), hence the topological spaces whose open subsets $U \subset X$ are precisely those whose pre-images $\phi^{-1}(U)$ are open under all continuous functions of the form $\Delta^n \xrightarrow{\phi} X$, or, equivalently, under all those of the form $\mathbb{R}^n \xrightarrow{\phi} X$, for all $n \in \mathbb{N}$.

The following statement justifies abbreviating "Delta-generated topological spaces" to "D-topological spaces":

Proposition 3.3.19 (Adjunction between topological and diffeological spaces fixes Delta-generated spaces)**.** *The constructions in Ex. 3.3.17 constitute a pair of adjoint functors between tological and diffeological spaces (Ntn. 3.3.15) which factors through compactly generated spaces (Ntn. 1.0.16) and further through D-topological spaces (Ntn. 3.3.18) as follows:*

$$\begin{tikzcd} \overset{\text{topological spaces}}{\mathrm{TopSpc}} \arrow[r, bend right, "k"'] \arrow[r, phantom, "\perp"] & \overset{\text{topological k-spaces}}{\mathrm{kTopSpc}} \arrow[l, hook', bend right] \arrow[r, bend right, "\underset{\text{continuous diffeology}}{\mathrm{Cdfflg}}"'] \arrow[r, phantom, "\perp"] & \overset{\substack{\text{Delta-generated}\\ \text{topological spaces}}}{\mathrm{DTopSpc}} \arrow[l, hook', bend right] \arrow[r, phantom, "\perp"] & \overset{\text{diffeological spaces}}{\mathrm{DfflgSpc}} \arrow[l, bend right, "\overset{\text{D-topology}}{\mathrm{Dtplg}}"'] \arrow[r, hook, "(3.162)"'] & \mathrm{SmthGrpd}_\infty \end{tikzcd} \tag{3.145}$$

Proof. This is essentially the observation of [SYH10, §3], see also [CSW14, §3.2]: One checks that $\mathrm{Dtplg} \dashv \mathrm{Cdfflg}$ is an adjunction ([SYH10, Lem. 3.1]) whose associated comonad is idempotent ([SYH10, Lem. 3.3])

$$\mathrm{Dtplg} \circ \mathrm{Cdfflg} \circ \mathrm{Dtplg} \circ \mathrm{Cdfflg} \xrightarrow[\sim]{\mathrm{Dtplg}(\varepsilon_{\mathrm{Cdfflg}(-)})} \mathrm{Dtplg} \circ \mathrm{Cdfflg}$$

and whose fixed objects $\mathrm{Dtplg} \circ \mathrm{Cdfflg}(X) \xrightarrow[\sim]{\varepsilon_X} X$ are precisely the Delta-generated spaces ([SYH10, Prop. 3.2]). This implies the claimed factorization relative to all topological spaces by the factorization theorem for idempotent adjunctions (e.g. [Grd21, Thm. 3.8.8]). Moreover, Delta-generated spaces are also coreflective among k-spaces (by [Vo71, Prop. 1.5], see also [Gau09, p. 7]), whereby the full factorization (3.145) is implied by uniqueness of adjoints. □

Example 3.3.20 (Chaotic topology)**.** For $S \in \mathrm{Set} \hookrightarrow \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \mathrm{SmthGrpd}_\infty$ its *sharp* or *chaotic* aspect, in the notation of Def. 3.3.1, is the D-topological space obtained by equipping S with the *chaotic topology*, i.e., the *coarsest* or *initial* topology

$$\sharp\mathrm{Dsc}(S) \;\simeq\; \mathrm{Cdfflg}\big(S, \mathrm{Op}_S := \{\varnothing, S\}\big)\,.$$

Definition 3.3.21 (Smooth extended simplices)**.** Write

$$\Delta^\bullet_{\mathrm{smth}} \;:\; \Delta \longrightarrow \mathrm{CartSpc} \hookrightarrow \mathrm{SmthMfd} \xhookrightarrow{y} \mathrm{DfflgSpc}$$

for the cosimplicial diffeological space (Ntn. 3.3.15) given by the extended smooth simplices:

$$\Delta^n_{\mathrm{smth}} \;:=\; \Big\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \,\Big\vert\, \sum_{k=0}^{n} x_k \;=\; 1\Big\}$$

co-degeneracy and co-face maps given by addition of consecutive variables and by insertion of zeros, respectively.

Notation 3.3.22 (Diffeological singular complex)**.** We write

$$\begin{array}{rcl} \mathrm{Pth}\colon \mathrm{DfflgSpc} & \longrightarrow & \Delta\mathrm{Set} \\ X & \longmapsto & \mathrm{DfflgSpc}\,(\Delta^\bullet_{\mathrm{smth}}, X) \end{array} \tag{3.146}$$

for the nerve operation induced by Def. 3.3.21.

Proposition 3.3.23 (Singular simplicial complex of continuous-diffeology represents weak homotopy type)**.** *For* $\mathrm{X} \in \mathrm{TopSpc}$ *there is a natural transformation to the ordinary singular simplicial complex of a topological space from the diffeological singular simplial complex* (3.146) *of its continuous-diffeological space* (3.145) *which takes values in weak homotopy equivalences:*

$$\mathrm{Pth}\,(\mathrm{Cdfflg}(\mathrm{X})) \xrightarrow[\in \mathrm{WHmtpEq}]{} \mathrm{Pth}(\mathrm{X}) \quad \in\; \Delta\mathrm{Set} \to \mathrm{Grpd}_\infty\,.$$

Proof. The existence of these weak homotopy equivalences is the content of [CW14, Prop. 4.14]. Inspection of the proof given there shows that these are indeed natural transformations. □

Proposition 3.3.24 (Diffeological mapping spaces have correct underlying homotopy type)**.** *The ordinary singular simplicial complex of the mapping space between a pair of k-topological spaces* (1.3) *is naturally weakly homotopy equivalent to the diffeological singular simplicial complex* (3.146) *of the diffeological mapping space* (3.144) *of their continuous-diffeological incarnations* (3.145)*:*

$$X, Y \,\in\, \mathrm{kTopSpc} \quad \vdash \quad \mathrm{Pth}\,\mathrm{Cdfflg}\,(\mathrm{Map}(X, Y)) \xrightarrow[\in \mathrm{WHmtpEq}]{} \mathrm{Pth}\;\mathrm{Map}\big(\mathrm{Cdfflg}(X), \mathrm{Cdfflg}(Y)\big)\,.$$

Proof. We deduce this as a corollary of results proven in [SYH10]. This gives the existence of a weak homotopy equivalence $\phi \,\in\, \mathrm{WHmtpEq}$, whose image under Cdfflg is of this form:

$$\mathrm{Cdfflg}\,(\mathrm{Map}(X, Y)) \xrightarrow[\in \mathrm{Cdfflg}(\mathrm{WHmtpEq})]{\mathrm{Cdfflg}(\phi)} \mathrm{Cdfflg}\,(\mathbf{smap}(X, A)) \simeq \mathrm{Map}\big(\mathrm{Cdfflg}(X), \mathrm{Cdfflg}(Y)\big)\,, \tag{3.147}$$

where **smap** is some topologization of the set of maps (defined on [SYH10, p. 6]) of which we only need to know that:

- its continuous-diffeological incarnation is the diffeological mapping space (by [SYH10, Prop. 4.7]) as shown on the right of (3.147),
- it is weakly homotopy equivalent to the topological mapping space with its compact-open topology (by [SYH10, Prop. 5.4]) hence to the k-fication of that (by [Vo71, Prop. 1.2 (h)]), as shown on the left of (3.147).

Now consider the naturality square of ϕ (3.147) under the natural weak homotopy equivalence from Prop. 3.3.23:

$$\begin{tikzcd} \mathrm{Pth}\,\mathrm{Cdfflg}\,(\mathrm{Map}(X, Y)) \arrow[r, "\mathrm{Pth}\,\mathrm{Cdfflg}(\phi)"] \arrow[d, "\in \mathrm{WHmtpEq}"] & \mathrm{Pth}\,\mathrm{Cdfflg}\,(\mathbf{smap}(X, Y)) \simeq \mathrm{Pth}\,\mathrm{Map}\big(\mathrm{Cdfflg}(X), \mathrm{Cdfflg}(Y)\big). \arrow[d, "\in \mathrm{WHmtpEq}"] \\ \mathrm{Pth}\,(\mathrm{Map}(X, Y)) \arrow[r, "\mathrm{Pth}(\phi)", "\in \mathrm{WHmtpEq}"'] & \mathrm{Pth}\,(\mathbf{smap}(X\,Y)) \end{tikzcd}$$

With the bottom morphism and the vertical morphisms being weak homotopy equivalences, it follows that also the top morphism is a weak homotopy equivalence (by the 2-out-of-3 property), which is the claim to be shown. □

For maps out of discrete spaces, we have stronger statements, such as the following:

Lemma 3.3.25 (Diffeological mapping groupoid out of discrete into continuous-diffeological groupoid**).**
Let $(S_1 \rightrightarrows S_0) \in \mathrm{Grpd}(\mathrm{Set}) \hookrightarrow \mathrm{Grpd}(\mathrm{DTopSpc})$ *be a topologically discrete groupoid and* $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *be any topological group.*
(i) *We have an isomorphism of diffeological groupoids:*

$$\mathrm{Map}\big((S_1 \rightrightarrows S_0),\, (\mathrm{Cdfflg}(\Gamma) \rightrightarrows *)\big) \;\simeq\; \mathrm{Cdfflg}\,\mathrm{Map}\big((S_1 \rightrightarrows S_0),\, (\Gamma \rightrightarrows *)\big) \;\in\; \mathrm{Grpd}(\mathrm{DfflgSpc})\,,$$

where on the left the mapping groupoid is formed in DfflgSpc*, while on the right it is formed in* kTopSpc*.*
(ii) *In particular, for* $G \in \mathrm{Grp}(\mathrm{Set})$ *we have an isomorphism*

$$\mathrm{Map}\big(\mathbf{E}G, \mathbf{B}\mathrm{Cdfflg}(\Gamma)\big) \;\simeq\; \mathrm{Cdfflg}\,\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)\,.$$

Proof. The mapping groupoids are equalizers of maps between product spaces, e.g.

$$\mathrm{Map}\big((S_1 \rightrightarrows S_0),\, (\Gamma \rightrightarrows *)\big) \xhookrightarrow{\;\;\mathrm{eq}\;\;} \prod_{f \in S_1} \Gamma \underset{F \mapsto F(f)\cdot F(f')}{\overset{F \mapsto F(f\circ f')}{\rightrightarrows}} \prod_{(f,f') \in S_1 \underset{S_0}{\times} S_1} \Gamma\,.$$

Since Cdfflg is a right adjoint, it preserves all these limits. □

Smooth ∞-groupoids. Much like diffeological spaces (Ntn. 3.3.15) are "geometric sets probeable by smooth manifolds" so smooth ∞-groupoids (Ntn. 3.3.26 below) are "geometric ∞-groupoids probeable by smooth manifolds" ([SSS12, Def. 3.1][Sc13, §4.4][SS20-Orb, Ex. 3.18][FSS20-TCD, Def. A.57]). In fact, diffeological spaces are equivalently the concrete 0-truncated smooth ∞-groupoids (recalled as Prop. 3.3.37 below). The generalization from concrete sheaves on CartSpc to all sheaves embeds diffeological spaces into a cohesive topos, and discarding the truncation condition makes this a cohesive ∞-topos (Prop. 3.3.36 below).

Notation 3.3.26 (Smooth ∞-groupoids**).** We write

$$\mathrm{SmthGrpd}_\infty \;:=\; \mathrm{Sh}_\infty(\mathrm{CartSpc})$$

for the hypercomplete ∞-topos over CartSpc (Ntn. 3.3.15), presented by the local projective moel structure (Ntn. 3.2.26) on simplicial presheaves over CartSpc:

$$\begin{aligned} \mathrm{SmthGrpd}_\infty \;&:=\; \mathrm{Loc}^{\mathrm{LclWEqs}}\big(\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\underset{\mathrm{loc}}{\mathrm{proj}}}\big) \\ &\simeq\; \mathrm{Loc}^{\mathrm{LclWEqs}}\big(\Delta\mathrm{PSh}(\mathrm{SmthMfd})_{\underset{\mathrm{loc}}{\mathrm{proj}}}\big) \;\in\; \mathrm{Ho}_2(\mathrm{Topos}_\infty)\,. \end{aligned} \tag{3.148}$$

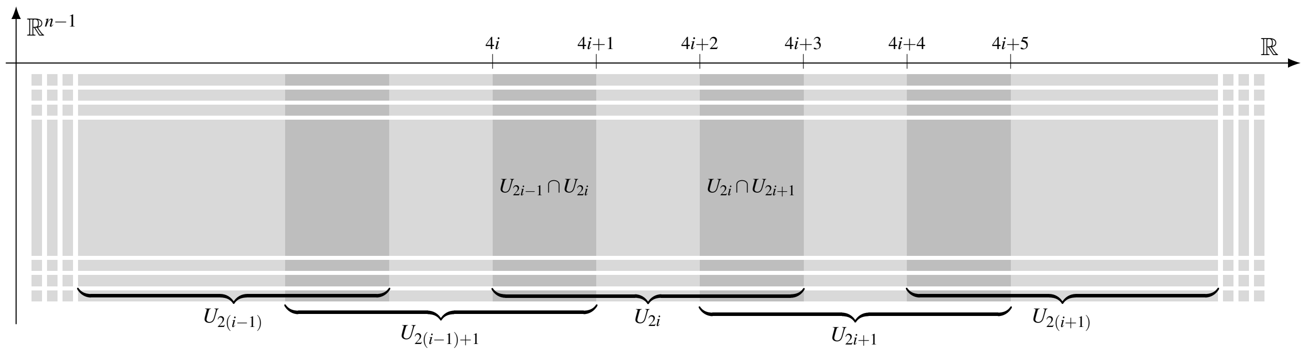
In addition to the general recognition of projectively co-fibrant simplicial presheaves (from Prop. 3.2.37), we have the following convenient recognition principle for fibrant simplicial presheaves presenting smooth ∞-groupoids (Ntn. 3.3.26), based on results in [Pav22]:

Proposition 3.3.27 (Recognition of fibrant simplicial presheaves over CartSpc**).** *An object* $\mathscr{X} \in \Delta\mathrm{PSh}(\mathrm{CartSpc})_{\underset{\mathrm{loc}}{\mathrm{proj}}}$ (3.148) *is fibrant precisely if it satisfies the following conditions for all* $\mathbb{R}^n \in \mathrm{CartSpc}$*:*

1. *the value* $\mathscr{X}(\mathbb{R}^n) \in \Delta\mathrm{Set}_{\mathrm{fib}}$ *is a Kan complex, i.e.* $\mathscr{X}$ *is globally projectively fibrant;*
2. *with*

$$\Big(U_{2i} := (4i, 4i+3) \times \mathbb{R}^{n-1} \xhookrightarrow{\;\;\iota_{2i}\;\;} \mathbb{R}^n\Big)_{i \in \mathbb{Z}}, \qquad \Big(U_{2i+1} \,:=\, (4i+2, 4i+5) \times \mathbb{R}^{n-1} \xhookrightarrow{\;\;\iota_{2i+1}\;\;} \mathbb{R}^n\Big)_{i \in \mathbb{Z}}$$

a pair of disjoint collections of parallel rectangular strips, which jointly cover $\mathbb{R}^n$*,*



the resulting square

$$
\begin{tikzcd}
\mathscr{X}(\mathbb{R}^n) \arrow[r, "\big(\mathscr{X}(\iota_{2i})\big)_{i\in\mathbb{Z}}"] \arrow[d, "\big(\mathscr{X}(\iota_{2i+1})\big)_{i\in\mathbb{Z}}"'] \arrow[dr, phantom, "\text{(hpb)}"] & \prod_{i\in\mathbb{Z}} \mathscr{X}(U_{2i}) \arrow[d] \\
\prod_{i\in\mathbb{Z}} \mathscr{X}(U_{2i+1}) \arrow[r] & \prod_{j\in\mathbb{Z}} \mathscr{X}(U_j \cap U_{j+1})
\end{tikzcd}
\tag{3.149}
$$

is a homotopy pullback square.

Proof. On general grounds the projectively locally fibrant simplicial are those which are objectwise Kan complexes and which satisfy homotopy descent. That over the site of smooth manifolds homotopy descent is equivalent to the above *Brown-Gersten type condition* (3.149) follows from [Pav22, Prop. 4.11]. □

This has the following remarkable consequences:

Lemma 3.3.28 (Local fibrancy of delooping groupoids over Cartesian spaces)**.** *For* $\Gamma \in \mathrm{Grp}\big(\mathrm{Sh}(\mathrm{CartSpc})\big)$ *any sheaf of groups over the site* CartSpc (3.139)*, the simplicial nerve of its delooping groupoid is locally projectively fibrant:*

$$
N(\Gamma \rightrightarrows *) \;\in\; \Big(\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\Big)_{\mathrm{fib}}.
$$

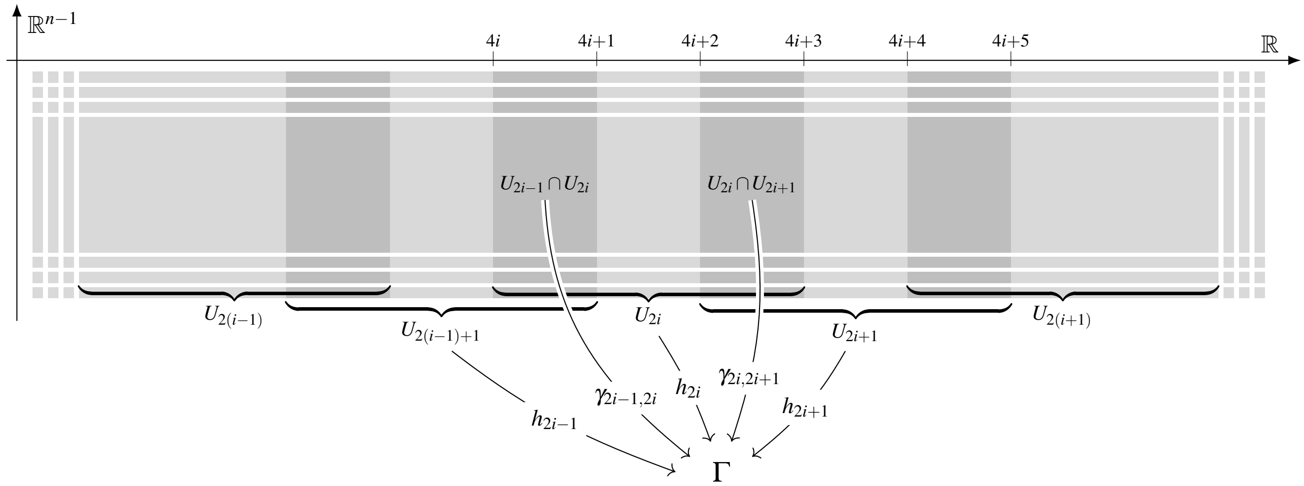
We thank Dmitri Pavlov for discussion of the following proof.

Proof. In the given case, the homotopy pullback in (3.3.27) is readily seen to be the groupoid whose objects are tuples of *transition functions*

$$
\Big(\gamma_{j,j+1} \in \Gamma\big((2j+2, 2j+3) \times \mathbb{R}^{n-1}\big)\Big)_{j\in\mathbb{Z}}
\tag{3.150}
$$

and whose morphisms are *gauge transformations* between these, of the following form, with composition given by the group operations in the groups $\Gamma(2j, 2j+3)$:

$$
\Big(h_j \in \Gamma\big((2j, 2j+3) \times \mathbb{R}^{n-1}\big)\Big)_{j\in\mathbb{Z}} \;:\; (\gamma_{j,j+1})_{j\in\mathbb{Z}} \longrightarrow \Big(h_j^{-1}\big|_{(2j+2,2j+3)} \cdot \gamma_{j,j+1} \cdot h_{j+1}\big|_{(2j+2,2j+3)}\Big)_{j\in\mathbb{Z}}.
\tag{3.151}
$$



We need to show that this groupoid (3.151) is equivalent to $\Gamma(\mathbb{R}^n) \rightrightarrows *$, which means to show that (i) it is connected and (ii) that the automorphisms of the element with trivial transition functions $(\mathrm{e})_{j\in\mathbb{Z}}$ are in bijection to $\Gamma(\mathbb{R}^n)$.

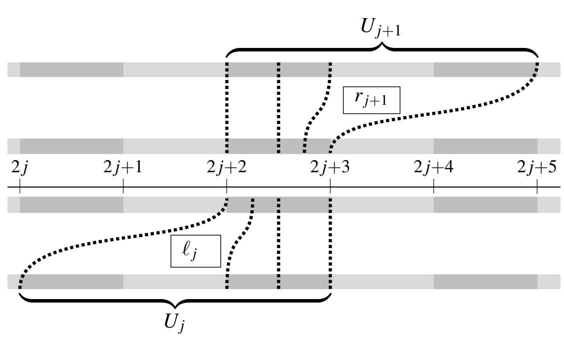
The second statement (ii) is clear: For a gauge transformation (3.151) to preserve the trivial transition functions $(\gamma_{j,j+1} = \mathrm{e})_{j\in\mathbb{Z}}$ its components $(h_j)_{j\in\mathbb{Z}}$ must agree on intersections, hence they glue by the assumed sheaf property of Γ.

For the first statement (i), the strategy is to use gauge transformations "on the left" to gauge away transition functions "on the right of their domain", and vice versa. The subtlety is to do this gauging "on the left" without destroying the trivial gauge already achieved "on the right".

We close by spelling out one way to do this in detail. So let $(\gamma_{j,j+1})_{j\in\mathbb{Z}}$ be any tuple of transition functions (3.150). We will produce a morphism (3.151) of the form $(\gamma_{j,j+1})_{j\in\mathbb{Z}} \to (\mathrm{e})_{j\in\mathbb{Z}}$.

To this end, choose diffeomorphisms (by the usual methods for bump functions, e.g. as in [Le12, Prop. 2.25]) of this form:

$$
\begin{tikzcd}
\big(2j,2j+2\big) \arrow[r, squiggly, "\sim"'] \arrow[d, hook] & \big(2j+2,2j+2\tfrac{1}{4}\big) \arrow[dr, hook] & \big(2j+2,2j+2\tfrac{1}{2}\big) \arrow[d, hook] \arrow[rr, equal] & & \big(2j+2,2j+2\tfrac{1}{2}\big) \arrow[d, hook] \\
U_j = \big(2j,2j+3\big) \arrow[rr, "\ell_j", "\sim"'] & & \big(2j+2,2j+3\big) & & \big(2j+2,2j+5\big) = U_{j+1} \arrow[ll, "r_{j+1}"', "\sim"] \\
\big(2j+2\tfrac{1}{2},2j+3\big) \arrow[u, hook] \arrow[rr, equal] & & \big(2j+2\tfrac{1}{2},2j+3\big) \arrow[u, hook] & \big(2j+2\tfrac{3}{4},2j+3\big) \arrow[ul, hook] & \big(2j+3,2j+5\big) \arrow[l, squiggly, "\sim"] \arrow[u, hook]
\end{tikzcd}
$$



Now, to start with, consider the morphism

$$
\left(h_j := \left\{ \begin{array}{ccc} \ell_0^*(\gamma_{01}) & | & j = 0 \\ \mathrm{e} & | & \text{otherwise} \end{array} \right. \right)_{j\in\mathbb{Z}} \;\;:\;\; (\gamma_{j,j+1})_{j\in\mathbb{Z}} \longrightarrow \big(\gamma'_{j,j+1}\big)_{j\in\mathbb{Z}}
$$

to new transition functions which satisfy

$$\gamma'_{01}\big|_{\left(2\frac{1}{2},3\right)} = \mathrm{e}\,;$$

and from there the further morphism

$$\left(h_j := \left\{\begin{array}{ccc} r_1^*(\gamma'_{01})^{-1} & | & j=1 \\ \mathrm{e} & | & \text{otherwise} \end{array}\right.\right)_{j\in\mathbb{Z}} \;:\; (\gamma'_{j,j+1})_{j\in\mathbb{Z}} \longrightarrow \big(\gamma''_{j,j+1}\big)_{j\in\mathbb{Z}}$$

to new transition functions which satisfy

$$\gamma''_{01} = \mathrm{e}\,.$$

Next, consider the following morphism, whose components to the right are all trivial, while those to the left are defined inductively:

$$\left(h_j := \left\{\begin{array}{ccc} \mathrm{e} & | & j\geq 0 \\ \ell_j^*\Big(\gamma''_{j,j+1}\cdot\big(h_{j+1}\big)\big|_{(2j+2,2j+3)}\Big) & | & j<0 \end{array}\right.\right)_{j\in\mathbb{Z}} \;:\; (\gamma''_{j,j+1})_{j\in\mathbb{Z}} \longrightarrow \big(\gamma'''_{j,j+1}\big)_{j\in\mathbb{Z}}\,.$$

This yields transition functions with the property

$$\underset{\substack{j\in\mathbb{Z}\\ j\leq 0}}{\forall} \quad \gamma'''_{j,j+1}\big|_{\left(2j+2\frac{1}{2},2j+3\right)} = \mathrm{e}\,;$$

and from there we have a morphism

$$\left(h_j := \left\{\begin{array}{ccc} \mathrm{e} & | & j\geq 0 \\ r_{j+1}^*\big(\gamma'''_{j,j+1}\big)^{-1} & | & j<0 \end{array}\right.\right)_{j\in\mathbb{Z}} \;:\; (\gamma'''_{j,j+1})_{j\in\mathbb{Z}} \longrightarrow \big(\gamma''''_{j,j+1}\big)_{j\in\mathbb{Z}}$$

to transition functions with the property

$$\underset{\substack{j\in\mathbb{N}\\ j\leq 0}}{\forall} \quad \gamma''''_{j,j+1} = \mathrm{e}\,.$$

Finally, applying the analogous inductive construction also mirror-symmetrically for $j\to+\infty$ serves to trivialize also the remaining transition functions to the right, giving a morphism $(\gamma''''_{j,j+1})_{j\in\mathbb{Z}} \to (\mathrm{e})_{j\in\mathbb{Z}}$. The composite morphism $(\gamma_{j,j+1})_{j\in\mathbb{Z}} \to (\gamma'_{j,j+1})_{j\in\mathbb{Z}} \to (\gamma''_{j,j+1})_{j\in\mathbb{Z}} \to (\gamma'''_{j,j+1})_{j\in\mathbb{Z}} \to (\gamma''''_{j,j+1})_{j\in\mathbb{Z}} \to (\mathrm{e})_{j\in\mathbb{Z}}$ is the desired one, exhibiting the connectedness of the descent groupoid. □

Topological transformation groups seen in SmthGrpd$_\infty$**.** Before discussing the cohesion of smoooth ∞-groupoids (Prop. 3.3.36 below), we showcase some aspects of topological transformation groups seen under the embedding of D-topological spaces into smooth ∞-groupoids:

Example 3.3.29 (Stabilizer groups as the loop objects of homotopy quotient stacks)**.** Given a D-topological group action

- $\Gamma \in \mathrm{Grp}(\mathrm{DTopSpc}) \hookrightarrow \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$
- $\Gamma \circlearrowright \mathrm{X} \in \Gamma\mathrm{Act}\big(\mathrm{DTopSpc}\big) \hookrightarrow \Gamma\mathrm{Act}\big(\mathrm{SmthGrpd}_\infty\big)$

seen in SmthGrpd$_\infty$ under (3.145), then the simplicial nerve of their topological action groupoid (Exp. 1.2.6)

$N\big((\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X})\big) \in \Delta\mathrm{DTopSpc} \hookrightarrow \Delta\mathrm{DfflgSpc} \hookrightarrow \Delta\mathrm{PSh}(\mathrm{CartSpc})$

represents the ∞-topos theoretic homotopy quotient (according to Prop. 3.2.75)

$$\mathrm{Loc}_\Delta^{\mathrm{PrjWEqs}}\Big(N\big((\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X})\big)\Big) \simeq \mathrm{X}/\!\!/\Gamma \quad \in \mathrm{SmthGrpd}_\infty\,.$$

Moreover, for any $x \in \mathrm{X}$ its topological stabilizer group is canonically identified with the looping (13) of the homotopy quotient at that point:

$$\begin{tikzcd} \mathrm{Stab}_\Gamma(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "\vdash x"] \\ * \arrow[r, "\vdash x"'] & \mathrm{X}/\!\!/\Gamma \end{tikzcd} \quad \in \ \mathrm{SmthGrpd}_\infty\,. \tag{3.152}$$

Proof. For the first statement, it is immediate to see that we have a pullback diagram of simplicial presheaves as shown on the left below, which hence gives the required homotopy pullback of ∞-stacks shown on the right, due to Lem. 3.2.32:

$$\begin{tikzcd} N(\mathrm{X} \rightrightarrows \mathrm{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & N(\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X}) \arrow[d, "\in \mathrm{PrjFib}"] \\ N(* \rightrightarrows *) \arrow[r] & N(\Gamma \rightrightarrows *) \end{tikzcd} \quad \xmapsto{\mathrm{Loc}_\Delta^{\mathrm{PrjWEqs}}} \quad \begin{tikzcd} \mathrm{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{X}/\!\!/\Gamma \arrow[d] \\ * \arrow[r] & \mathbf{B}\Gamma \end{tikzcd}$$

Similarly, for computing the looping, the same Lemma 3.2.32 (together with direct inspection of the right morphism, or else appeal to the factorization lemma, 3.2.34) gives:

$$\begin{tikzcd} & N(* \rightrightarrows *) \arrow[d, "\vdash x"', "\in \mathrm{PrjWEqs}"] \\ N\big(\mathrm{Stab}_\Gamma(x) \rightrightarrows \mathrm{Stab}_\Gamma(x)\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & N\big((\{x\}\times\Gamma)\times\Gamma \rightrightarrows (\{x\}\times\Gamma)\big) \arrow[d, "\in \mathrm{PrjFib}"] \\ N(* \rightrightarrows *) \arrow[r, "\vdash x"'] & (\mathrm{X}\times\Gamma \rightrightarrows \mathrm{X}) \end{tikzcd} \quad \xmapsto{\mathrm{Loc}_\Delta^{\mathrm{PrjWEqs}}} \quad \begin{tikzcd} \mathrm{Stab}_\Gamma(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "\vdash x"] \\ * \arrow[r, "\vdash x"'] & \mathrm{X}/\!\!/\Gamma \end{tikzcd}$$

□

Example 3.3.30 (Projective representations of finite groups seen via simplicial presheaves)**.** The topological action groupoid (Ex. 1.2.6) of the circle group U_1 acting by multiplication on the unitary group U_ω (1.86) carries an evident structure of a group object (by the degreewise group operations in U_ω and in $\mathrm{U}_1 \times \mathrm{U}_\omega$), hence of a topological strict 2-group (Ntn. 1.2.33), so that its simplicial nerve (Ntn. 1.2.24) is a simplicial topological group:

$$(\mathrm{U}_\omega \times \mathrm{U}_1 \rightrightarrows \mathrm{U}_\omega) \ \in \ \mathrm{Grp}\big(\mathrm{Grpd}(\mathrm{kTopSpc})\big)\,, \quad N\big((\mathrm{U}_\omega \times \mathrm{U}_1 \rightrightarrows \mathrm{U}_\omega)\big) \ \in \ \mathrm{Grp}\big(\Delta\mathrm{kTopSpc}\big)\,.$$

Under the inclusion from Prop. 3.3.19

$$\mathrm{kTopSpc} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DTopSpc} \hookrightarrow \mathrm{DfflgSpc} \hookrightarrow \mathrm{SmthGrpd}_\infty$$

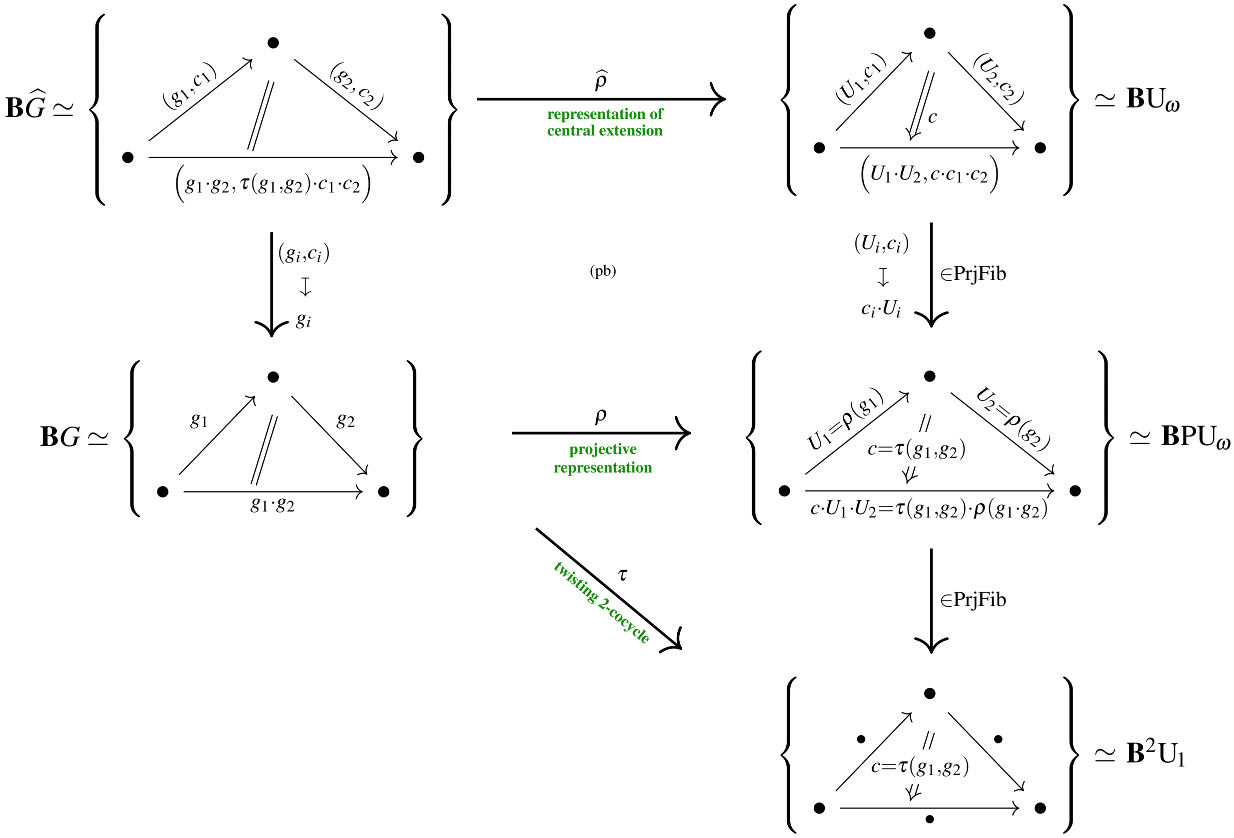
and using Lemma 3.3.29, this becomes a smooth 2-group, represented by a presheaf of simplicial groups on CartSpc:

$$\mathrm{Cdfflg}\Big(N\big((\mathrm{U}_\omega \times \mathrm{U}_1 \rightrightarrows \mathrm{U}_\omega)\big)\Big) \ : \ \mathbb{R}^n \ \mapsto \ N\big((\mathrm{Map}(\mathbb{R}^n, \mathrm{U}_\omega) \times \mathrm{Map}(\mathbb{R}^n, \mathrm{U}_1) \rightrightarrows \mathrm{Map}(\mathbb{R}^n, \mathrm{U}_\omega))\big)\,. \tag{3.153}$$

Since the U_1-quotient projection $\mathrm{U}_\omega \to \mathrm{PU}_\omega$ is locally trivial (1.89), so that every map from an $\mathbb{R}^n$ to PU_ω lifts to a map to U_ω, it follows that the canonical map from the simplicial classifying space (3.22) of (3.153) to that of the group object represented by PU_ω (1.88) is a weak equivalence in the projective model structure over CartSpc (Ntn. 3.2.26 which presents $\mathrm{SmthGrpd}_\infty$ (Ntn. 3.3.26) of models for the delooping $\mathbf{B}\mathrm{PU}_\omega$ (Lemma 3.2.72):

$$\mathrm{Cdfflg}\Big(\overline{W}N\big((\mathrm{U}_\omega \times \mathrm{U}_1 \rightrightarrows \mathrm{U}_\omega)\big)\Big) \xrightarrow[\in \mathrm{PrjWEqs}]{} \mathrm{Cdfflg}\big(\overline{W}(\mathrm{PU}_\omega)\big) \ \simeq \ \mathbf{B}\mathrm{PU}_\omega$$

Inspection of the formula (3.19) for $\overline{W}$ shows that this simplicial presheaf presentation of $\mathbf{B}\mathrm{PU}_\omega \in \mathrm{SmthGrpd}_\infty$ looks as shown in the middle right of the following diagram, where the top right shows an analogous presentation of $\mathbf{B}\mathrm{U}_\omega$ which is such as to make the delooped quotient $\mathbf{B}\mathrm{U}_\omega \to \mathbf{B}\mathrm{PU}_\omega$ be presented by a fibration in $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\mathrm{proj}}$:



(3.154)

The bottom left of this diagram shows $\overline{W}G \simeq \mathbf{B}G$ (still by Lemma 3.2.72) for a finite group G, which, since G is discrete, is a projectively cofibrant simplicial presheaf (by Prop. 3.2.37). It follows (by Lem. 3.1.12) that the elements of the hom-groupoid $\mathrm{SmthGrpd}_\infty\big(\mathbf{B}G, \mathbf{B}\mathrm{PU}_\omega\big)$ are all representable by morphisms of simplicial presheaves denoted "ρ" in this diagram, which, again by Prop. 3.3.19, are equivalently morphisms of simplicial D-topological spaces (in fact of topological 2-functors between D-topological 2-groupoids).

The upshot is that the presentation for $\mathbf{B}\mathrm{PU}_\omega$ chosen on the right makes transparently manifest how such *projective unitary representations* $G \xrightarrow{\rho} \mathrm{PU}_\omega$ (e. g. [Ta77][Co09][EU14, §5], review in [Me17]) encode:
(i) a U_1-valued 2-cocycle τ, by postcomposition with $\mathbf{B}\mathrm{PU}_\omega \to \mathbf{B}^2\mathrm{U}_1$ (this construction is the stacky refinement of Ex. 2.3.29);
(ii) a genuine unitary representation $\widehat{\rho}$ of the U_1-extension $\widehat{G}$ of G which is classified by τ, this being simply the pullback of ρ along the top right projective fibration (compare [EU14, §5]).

Abstracting away from the presentations by simplicial presheaves, the bottom right fibration in implies (by Lem. 3.2.32) that we have a homotopy fiber sequence $\mathbf{B}\mathrm{U}_\omega \to \mathbf{B}\mathrm{PU}_\omega \to \mathbf{B}^2\mathrm{U}_1$ in $\mathrm{SmthGrpd}_\infty$, which, by Prop. 0.2.1 gives an identification

$$\mathbf{B}\mathrm{PU}_\omega \;\simeq\; \mathbf{B}\mathrm{U}_\omega /\!\!/ \mathbf{B}\mathrm{U}_1 \quad \in \; \mathrm{SmthGrpd}_\infty\,.$$

It follows that for $\tau \in H^2_{\mathrm{Grp}}(G;\mathrm{U}_1)$, the isomorphism classes of $[\tau]$-projective unitary representations of G are in natural bijection with the connected components of the slice hom-groupoid (3.77)

$$\Big(\mathrm{Rep}^{[\tau]}(G)\Big)_{/\sim_{\mathrm{iso}}} \;\simeq\; \tau_0\,\mathbf{H}\big((\mathbf{B}G,\tau),\, \mathbf{B}\mathrm{U}_\omega /\!\!/ \mathbf{B}\mathrm{U}_1\big)_{\mathbf{B}^2\mathrm{U}_1}\,. \tag{3.155}$$

We use the occasion of Ex. 3.3.30 to record that key facts of standard character theory generalize to projective representations and to establish a fact (Prop. 3.3.35 below) which we need below in Lem. 4.1.45:

Lemma 3.3.31 (Characters of projective representations of finite groups [Che15, Prop. 2.2 (1)])**.** *For G a finite group and $[\tau] \in H^2_{\mathrm{Grp}}(G, \mathrm{U}_1)$ a 2-cocycle, the characters*

$$\chi_\rho \;\; \in \; \mathrm{Map}(G, \mathrm{U}_1)$$

of $[\tau]$-projective unitary representations (Ex. 3.3.30)

$$\begin{tikzcd} \mathbf{B}G \arrow[r, "\rho"] \arrow[dr, "\tau"'] & \mathbf{B}\mathrm{U}_d /\!\!/ \mathbf{B}\mathrm{U}_1 \arrow[r] & \mathbf{B}\mathrm{U}_\omega /\!\!/ \mathbf{B}\mathrm{U}_1 \simeq \mathbf{B}\mathrm{PU}_\omega \arrow[dl] \\ & \mathbf{B}^2\mathrm{U}_1 & \end{tikzcd}$$

of any finite dimension $d \in \mathbb{N}$, which are still given by the trace of the unitary operators $\rho(g) \in \mathrm{U}_d$ (3.154)

$$\chi_\rho(g) \;:=\; \mathrm{Tr}\big(\rho(g)\big)\,, \tag{3.156}$$

still detect isomorphy of projective representations (3.155)*:*

$$\chi_{\rho_1} \;=\; \chi_{\rho_2} \qquad \Leftrightarrow \qquad [\rho_1] = [\rho_2] \;\; \in \; \Big(\mathrm{Rep}^{[\tau]}(G)\Big)_{/\sim_{\mathrm{iso}}}. \tag{3.157}$$

This means that projective character theory looks much like plain character theory; for instance:

Example 3.3.32 (Ordinary characters act on projective characters)**.** The tensoring (see also (4.60) below) of an ordinary unitary G-representation $\mathbf{B}G \xrightarrow{\rho} \mathbf{B}\mathrm{U}_d$ with a $[\tau]$-projective representation $\mathbf{B}G \xrightarrow{\rho} \mathbf{B}\mathrm{U}_{d'}$ (Ex. 3.3.30) is again a $[\tau]$-projective representation

$$\begin{array}{ccc} \mathrm{Rep}(G) \times \mathrm{Rep}^{[\tau]}(G) & \xrightarrow{\quad\quad \otimes \quad\quad} & \mathrm{Rep}^{[\tau]}(G) \\ (\rho, \rho') & \longmapsto & \rho \otimes \rho'\,, \end{array} \tag{3.158}$$

and under this operation the characters (3.156) multiply, as usual:

$$\chi_{\rho\otimes\rho'} \;=\; \chi_\rho \cdot \chi_{\rho'}\,. \tag{3.159}$$

Example 3.3.33 (Projective regular representation)**.** For G a finite group and $[\tau] \in H^2_{\mathrm{Grp}}(G; \mathrm{U}_1)$,
(i) the *regular* $[\tau]$-projective representation (Ex. 3.3.30) is the complex linear span of the underlying set of G

$$\mathbb{C}[G]^{[\tau]} \in \mathrm{Rep}^{[\tau]}(G)$$

with action defined on the canonical basis elements $\{v_h | g \in G\} \subset \mathbb{C}[G]$ by the formula

$$\rho(g)(v_k) \;:=\; \tau(g,k) \cdot v_{g\cdot k}\,. \tag{3.160}$$

That this definition satisfies the action property is equivalently the cocycle condition on τ:

$$\begin{array}{rcll} \rho(g_1)\big(\rho(g_2)(v_{g_3})\big) & = & \rho(g_1)\big(\tau(g_2,g_3)\cdot v_{g_2\cdot g_3}\big) & \text{by (3.160)} \\ & = & \tau(g_1, g_2\cdot g_3)\cdot\tau(g_2,g_3)\cdot v_{g_1\cdot g_2\cdot g_3} & \text{by (3.160)} \\ & = & \tau(g_1,g_2)\cdot\tau(g_1\cdot g_2, g_3)\cdot v_{g_1\cdot g_2\cdot g_3} & \text{by cocycle property} \\ & = & \tau(g_1,g_2)\cdot\rho(g_1\cdot g_2)(v_{g_3}) & \text{by (3.160)}\,. \end{array}$$

(ii) The character (3.3.31) of the regular projective representation evidently is, independently of τ, the same as that of the ordinary regular representation:

$$\chi_{\mathbb{C}[G]^\tau} \;:\; g \;\longmapsto\; \left\{ \begin{array}{ccl} \mathrm{ord}(G) & | & g = \mathrm{e} \\ 0 & | & \text{otherwise.} \end{array} \right. \tag{3.161}$$

Proposition 3.3.34 (Regular projective representation decomposes into all projective irreps [Che15, Prop. 2.3]). *For G a finite group and $[\tau] \in H^2_{\mathrm{Grp}}(G;\mathrm{U}_1)$, the $[\tau]$-projective regular representation (Ex. 3.3.33) decomposes as the direct sum of all irreducible $[\tau]$-projective representations, each appearing with multiplicity equal to the complex dimension of its representation space:*

$$\mathbb{C}[G]^{[\tau]} \;\simeq\; \bigoplus_{\substack{[\mu]\in \\ \left(\mathrm{Rep}^{[\tau]}(G)_{\mathrm{irr}}\right)_{/\sim_{\mathrm{iso}}}}} \dim_{\mathbb{C}}(\mu)\cdot\mu \;\;\in\; \mathrm{Rep}^{[\tau]}(G)\,.$$

We deduce from this the following fact (which will be needed in Lem. 4.1.45 below):

Proposition 3.3.35 (Tensoring projective representations with the plain regular representation). *For G a finite group, and $[\tau] \in H^2_{\mathrm{Grp}}(G;\mathrm{U}_1)$, the tensoring* (3.158) *of any $[\tau]$-projective representation ρ (Ex. 3.3.30) with the plain regular representation is the direct sum of* $\dim_{\mathbb{C}}(\rho)$ *copies of the $[\tau]$-projective regular representation (Ex. 3.3.33), and hence a direct sum of all irreducible $[\tau]$-projective representations:*

$$\rho \in \mathrm{Rep}^{[\tau]}(G) \quad\vdash\quad \begin{cases} \mathbb{C}[G]\otimes\rho & \simeq\; \dim_{\mathbb{C}}(\rho)\cdot\mathbb{C}[g]^{\tau} \\ & \simeq \displaystyle\bigoplus_{\substack{[\mu]\in \\ \left(\mathrm{Rep}^{[\tau]}(G)_{\mathrm{irr}}\right)_{/\sim_{\mathrm{iso}}}}} \dim_{\mathbb{C}}(\rho)\cdot\dim_{\mathbb{C}}(\mu)\cdot\mu \quad \textit{by Prop. 3.3.34.} \end{cases}$$

Proof. The character (3.156) of the tensor product representation is

$$\begin{aligned} \chi\big(\mathbb{C}[G]\otimes\rho\big) &= \chi\big(\mathbb{C}[G]\big)\cdot\chi(\rho) && \text{by (3.159)} \\ &= \big(|G|,0,0,\cdots,\big)\cdot\big(\dim_{\mathbb{C}}(\rho),\mathrm{tr}(\rho(g_1)),\cdots\big) && \text{by (3.161)} \\ &= \big(|G|\cdot\dim_{\mathbb{C}}(\rho),0,0,\cdots\big) \\ &= \dim_{\mathbb{C}}(\rho)\cdot\big(|G|,0,0,\cdots\big) \\ &= \dim_{\mathbb{C}}(\rho)\cdot\chi\big(\mathbb{C}[G]^{\tau}\big) && \text{by (3.161)}\,. \end{aligned}$$

Hence the claim follows by (3.157). □

Smooth cohesion. Smooth ∞-groupoids constitute the main example of cohesive ∞-toposes that we consider here (see also [SS20-Orb, §3.1] for this case and further variants):

Proposition 3.3.36 (Smooth cohesion [SSS12, §3.1][Sc13, Prop. 4.4.8]). *The ∞-topos* $\mathrm{SmthGrpd}_\infty$ *of smooth ∞-groupoids (Ntn. 3.3.26) is cohesive in the sense of Def. 3.3.1: The adjoint quadruple arises just as in Ex. 3.3.3 from the inclusion* $* \simeq \{\mathbb{R}^0\} \hookrightarrow \mathrm{CartSpc}$ *of the terminal object into the site* (3.139) *of smooth Cartesian spaces, and is then seen to factor through the full inclusion* $\mathrm{SmthGrpd}_\infty \simeq \mathrm{Sh}_\infty(\mathrm{CartSpc}) \hookrightarrow \mathrm{PSh}_\infty$.

Properties of smooth cohesion are derived in [Sc13, §4.4] and key facts are summarized in [SS20-Orb, Ex. 3.18][FSS20-TCD, Ex. A.58] and above in §0.1. We proceed to highlight some of these in more detail.

To start with, contact with the differential topology discussed above, is made by observing that diffeological spaces, and with them the D-topological spaces (Prop. 3.3.19), are most naturally included among smooth ∞-groupoids:

Proposition 3.3.37 (Diffeological spaces are the concrete 0-truncated smooth ∞-groupoids [Sc13, Prop. 4.4.15]). *The canonical inclusion of diffeological spaces (Ntn. 3.3.15) into smooth ∞-groupoids (Ntn. 3.3.26) is equivalently the inclusion of the concrete 0-truncated smooth ∞-groupoids (cf. [SS20-Orb, Ex. 3.18(ii)]):*

$$\mathrm{DfflgSpc} \;\simeq\; \mathrm{SmthGrpd}_{0,\sharp_1} \longrightarrow \mathrm{SmthGrpd}_\infty\,. \tag{3.162}$$

Cohesive shape of smooth manifolds.

Proposition 3.3.38 (Quillen functor for shape modality on smooth ∞-groupoids)**.** *Consider the shape-adjunction on smooth ∞-groupoids (Def. 3.3.26)*

$$\mathrm{SmthGrpd}_\infty \underset{\mathrm{Dsc}}{\overset{\mathrm{Shp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Grpd}_\infty .$$

(i) *This adjunction is, equivalently, the left derived functor of the colimit operation on simplicial presheaves over Cartesian spaces, regarded as functors* $\mathrm{CartSpc}^{\mathrm{op}} \to S\Delta\mathrm{Set}$*, in that the following is a Quillen adjunction:*

$$\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \underset{\mathrm{const}}{\overset{\underset{\longrightarrow}{\lim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot_{\mathrm{Qu}}}}}} \Delta\mathrm{Set}_{\mathrm{Qu}} . \tag{3.163}$$

(ii) *Moreover, on a simplicial presheaf satisfying Dugger's cofibrancy condition (Prop. 3.2.37)*

$$\varnothing \xrightarrow{\in\mathrm{Cof}} \coprod_{i_\bullet \in I_\bullet} \mathbb{R}^{n_{i_\bullet}} \xrightarrow{\in\mathrm{W}} X \quad \in \Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proh}\\ \mathrm{loc}}} , \tag{3.164}$$

the shape is given by the simplicial set obtained by contracting all copies of Cartesian spaces to the point:

$$\int X \;\simeq\; \coprod_{i_\bullet \in I_\bullet} * \;\; \in \; \Delta\mathrm{Set}_{\mathrm{Qu}} .$$

Proof. First, observe that the colimit over a representable functor is the point (e.g. [SS20-Orb, Lem. 2.40])

$$\underset{\longrightarrow}{\lim} \mathbb{R}^n := \underset{\longrightarrow}{\lim}\, y(\mathbb{R}^n) \simeq * \;\; \in \mathrm{Set} \hookrightarrow \Delta\mathrm{Set} , \tag{3.165}$$

so that the colimit of a simplicial presheaf of the form (3.164) is the simplicial set obtained by replacing all copies of Cartesian spaces by a point:

$$\begin{aligned} \underset{\longrightarrow}{\lim}\Big(\coprod_{i_\bullet \in I_\bullet} \mathbb{R}^{n_{i_\bullet}}\Big) &\simeq \Big(\underset{\longrightarrow}{\lim} \mathbb{R}^{n_{i_\bullet}}\Big) && \text{since colimits commute with coproducts} \\ &\simeq \coprod_{i_\bullet \in I_\bullet} * && \text{by (3.165)} . \end{aligned} \tag{3.166}$$

Next, it is clear that (3.163) is a simplicial Quillen adjunction for the *global* projective model structure. To show that it is also Quillen for the local model structure it is hence sufficient, by [Lur09, Cor. A.3.7.2], to see that the right adjoint preserves fibrant objects. By adjunction, this is equivalent to the statement that for $\{U_i \hookrightarrow X\}$ a differentiably good open cover, with $U := \coprod_i U_i$, we have a simplicial weak homotopy equivalence

$$\underset{\longrightarrow}{\lim}\, y(U^{\times_X^\bullet}) \xrightarrow{\in\mathrm{W}} * . \tag{3.167}$$

But, by (3.166), the left hand side of (3.167) is the simplicial set obtained by contracting summands of the Čech nerve of the good cover to the point. Therefore, since any Cartesian space is contractible, the *nerve theorem* ([MC67, Thm. 2], review in [Ha02, Prop. 4G.3]) implies (3.167). With this, the last statement follows from the fact that left derived functors may be computed on any cofibrant resolution:

$$\begin{aligned} \int X &\simeq \big(\mathbb{L}\underset{\longrightarrow}{\lim}\big)(X) && \text{by (3.163)} \\ &\simeq \underset{\longrightarrow}{\lim}\Big(\coprod_{i_\bullet \in I_\bullet} \mathbb{R}^n\Big) && \text{by Prop. 3.2.37} \\ &\simeq \coprod_{i_\bullet \in I_\bullet} * && \text{by (3.166)} . \end{aligned}$$

□

Example 3.3.39 (Good open covers are projectively cofibrant resolutions of smooth manifolds)**.**
Every $\mathrm{X} \in \mathrm{SmthMfd} \xrightarrow{y} \Delta\mathrm{PSh}(\mathrm{CartSpc})$ admits a *differentiably good open cover* (Def. 3.3.14), namely an open cover such that all non-empty finite intersections of patches are *diffeomorphic* to an open ball, and hence to $\mathbb{R}^{\dim(X)}$:

$$\big\{\mathrm{U}_i \simeq \mathbb{R}^{\dim(\mathrm{X})} \hookrightarrow \mathrm{X}\big\}_{i \in I} , \quad \text{s.t.} \quad \underset{\substack{k \in \mathbb{N} \\ i_0, i_1, \cdots, i_k \in I}}{\forall} \quad \mathrm{U}_{i_0} \cap \mathrm{U}_{i_1} \cap \cdots \cap \mathrm{U}_{i_k} \underset{\mathrm{diff}}{\simeq} \mathbb{R}^{\dim(\mathrm{X})} \;\; \text{if non-empty} . \tag{3.168}$$

By Dugger's recognition (Prop. 3.2.37), this means that the corresponding Čech nerve is projectively cofibrant (hence also locally projectively cofibrant); moreover, its canonical morphism to X is clearly a stalkwise weak equivalence, so that it provides a cofibrant resolution of X in the local model structure (Def. 3.3.26):

$$\varnothing \xrightarrow[\in\mathrm{LclPrjCof}]{} \widehat{\mathrm{X}}^{\times_{\mathrm{X}}^\bullet} \xrightarrow[\in\mathrm{LclWEqs}]{} X \;\; \in \Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}} , \quad \text{for } \widehat{\mathrm{X}} := \sqcup_i \mathrm{U}_i .$$

In conclusion:

Proposition 3.3.40 (Cohesive shape of smooth manifolds is their homotopy type)**.** *For* $X \in \mathrm{SmthMfd} \xhookrightarrow{y} \mathrm{SmthGrpd}_\infty$ *(Def. 3.3.26), their cohesive shape* (3.117) *is their standard underlying homotopy type.*

Proof. By Ex. 3.3.39, the smooth manifold admits a differentiably good open cover whose Čech nerve constitutes a cofibrant resolution of the manifold in the local projective model structure on simplicial presheaves. By Prop. 3.3.38, this implies that the shape of the manifold is the homotopy type of the simplicial set obtained by replacing each direct summand in this Čech nerve by a point. That this simplicial set represents the standard homotopy type of the manifold is the classical *nerve theorem* ([MC67, Thm. 2], review in [Ha02, Prop. 4G.3]). □

Example 3.3.41 (Standard cofibrant resolution of the smooth circle)**.** Considering the smooth circle as the quotient of the real numbers by the integers

$$\mathbb{Z} \xhookrightarrow{\;\;i\;\;} \mathbb{R} \xrightarrow{\;\;p\;\;}\!\!\!\!\rightarrow S^1 \;\;\in\;\; \mathrm{SmthMfd} \xhookrightarrow{y} \Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\,,$$

Dugger's recognition (Prop. 3.2.37) shows that the Čech nerve of p consistitutes a cofibrant resolution of the circle in the projective local model structure (Def. 3.3.26)

$$\begin{array}{ccccccc} \varnothing & \xrightarrow{\;\in\mathrm{Cof}\;} & \mathbb{R} \times \mathbb{Z}^{\times^\bullet} & \xrightarrow{\qquad\sim\qquad} & \mathbb{R}^{\times^\bullet_{S^1}} & \xrightarrow{\;\in\mathrm{W}\;} & S^1 \\ & & (r, \vec{n}) & \longmapsto & (r, (r+n_1), (r+n_1+n_2), \cdots). & & \end{array}$$

Equivariant Čech cocycles from projectively cofibrant resolution of action groupoids. In immediate generalization of Example 3.3.39 we have the following further consequence of Dugger's cofibrancy condition (Prop. 3.2.37):

Example 3.3.42 (Čech-action groupoid of equivariant good open cover is local cofibrant resolution)**.** For $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}\big(\Delta\mathrm{PSh}(\mathrm{CartSpc})\big)$. let $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$, with properly equivariant good open cover $G \circlearrowright \widehat{\mathrm{X}} := G \circlearrowright \big(\underset{i\in I}{\sqcup} \mathrm{U}_i\big) \twoheadrightarrow G \circlearrowright \mathrm{X}$ (Def. 1.1.24, which exists by Prop. 1.1.25). Then the diagonal of the Čech nerve of its action groupoid is a cofibrant replacement of the simplicial nerve of the action groupoid of X in $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$:

$$\varnothing \xrightarrow[\in\mathrm{LclPrjCof}]{} N\Big(\, \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \underset{\mathrm{pr}_3 \cdot \mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \widehat{\mathrm{X}}\,\Big) \xrightarrow[\in\mathrm{LclWEqs}]{} N\big(\mathrm{X} \times G^{\mathrm{op}} \rightrightarrows \mathrm{X}\big) \tag{3.169}$$

$$\left\{ \begin{tikzcd} (x,i) \arrow[r, "{((x,i,i),g)}"] \arrow[d, "{((x,i,j),e)}"'] \arrow[dr, "{((x,i,j),g)}"] & (g\cdot x,\, g\cdot i) \arrow[d, "{((g\cdot x, g\cdot i, g\cdot j),e)}"] \\ (x,j) \arrow[r, "{((x,j,j),g)}"'] & (g\cdot x,\, g\cdot j) \end{tikzcd} \right\}$$

Remark 3.3.43 (Equivariant Čech cocycles)**.** Example 3.3.42 implies that, under the conditions stated there, we have for $\mathscr{G} \in \mathrm{Grp}\big(\Delta\mathrm{PSh}(\mathrm{CartSpc})\big)$ an equivalence of ∞-groupoids

$$\mathrm{SmthGrpd}_\infty\big(\mathrm{X}/\!\!/G, \mathbf{B}\mathscr{G}\big) \;\simeq\; \mathrm{Loc}^{\mathrm{WHmtpEq}}\Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(\big(\, \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \underset{\mathrm{pr}_3 \cdot \mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \widehat{\mathrm{X}}\,\big), \overline{W}\mathscr{G}\Big)\,, \tag{3.170}$$

where in the second argument we used Lemma 3.2.72. The elements on the right may be thought of as *G-equivariant Čech $\mathscr{G}$-cocycles on* X. These will serve to represent equivariant principal ∞-bundles in §4.1 and

§4.2 below.

$$
\mathrm{X}/\!\!/G \xrightarrow{\;\;\text{modulating map}\;\;} \mathbf{B}\Gamma \qquad \in \mathrm{SmthGrpd}_\infty
$$

$$
N\big(\widehat{\mathrm{X}}\times_{\mathrm{X}}\widehat{\mathrm{X}}\times G \rightrightarrows \widehat{\mathrm{X}}\big) \xrightarrow{\;\;\text{equivariant Čech cocycle}\;\;} N\big(\Gamma \rightrightarrows \ast\big) \qquad \in \Delta\mathrm{PSh}(\mathrm{CartSpc})
$$

$$
\begin{tikzcd}
& (x,j) \arrow[rr] \arrow[dr] & & g\cdot(x,j) \arrow[dr] & & & & \bullet \arrow[rr, "\rho_g(x,j)"] \arrow[dr, "\gamma_{jk}(x)"] & & \bullet \arrow[dr] & \\
& & (x,k) \arrow[rr] & & g\cdot(x,k) & \longmapsto & & & \bullet \arrow[rr] & & \bullet \\
(x,i) \arrow[uur, "\big((x,i,j),e\big)"] \arrow[urr] \arrow[rr, "\big((x,i,i),g\big)"'] & & g\cdot(x,i) \arrow[uur] \arrow[urr] & & & & \bullet \arrow[uur, "\gamma_{ij}(x)"] \arrow[urr, "\gamma_{ik}(x)"'] \arrow[rr, "\rho_g(x,i)"'] & & \bullet \arrow[uur] \arrow[urr] & &
\end{tikzcd}
\tag{3.171}
$$

Smooth path ∞-groupoids. It is a familiar fact that the singular simplicial complex[4] $\mathrm{Pth}(\mathrm{X})$ of a topological space X, being a Kan complex and as such representing an ∞-groupoid (3.6) may be thought of as the *∞-groupoid of paths* in X, whose objects are the points of X and whose $n+1$-morphisms are the continuous paths between n-dimensional paths in X. Here we discuss how this phenomenon is subsumed by the concept of ∞-groupoids of smooth paths not just in D-topological spaces, but more generally in smooth ∞-groupoids (Def. 3.3.44 below).

Below, Prop. 3.3.46 [BBP19] (following [Pv14], see also [Bu20, §3] with precursor arguments in [BNV16, Lem. 7.5]) says that these ∞-groupoids of smooth paths in smooth ∞-groupoids still reflect their smooth shape. This a cornerstone fact for analysis of the cohesion of smooth ∞-groupoids. In particular, it implies the *smooth Oka principle* (Thm. 3.3.51 below) which in turn implies the fundamental classification theorems for smooth principal ∞-bundles (Prop. 4.1.12 below) and for suitable equivariant principal ∞-bundles (Thm. 4.1.56 below) over smooth manifolds, as we show below in §4.1.

Conceptually, this result (Prop. 3.3.46) says that morphisms out of the shape of a smooth ∞-groupoid X encode *parallel transport of flat ∞-connections* ([SSS12, p. 22], see also [ScSh14]) or equivalently *local systems of constant ∞-coefficients* ([FSS20-TCD, §2.2])

$$
\underset{\substack{\text{smooth }\infty\text{-groupoid of}\\ \text{smooth higher paths in }X}}{\mathbf{Pth}(X)} \;\simeq\; \underset{\text{shape of }X}{\int X} \xrightarrow{\substack{\text{parallel transport of}\\ \text{flat }\infty\text{-connection}}} \mathbf{B}\mathscr{G} \qquad \overset{(10)}{\longleftrightarrow} \qquad X \xrightarrow{\substack{\text{local system of}\\ \text{constant }\infty\text{-coefficients}}} \flat\mathbf{B}\mathscr{G}\,.
$$

Conversely, this means that in any cohesive ∞-topos the shape modality encodes, in particular, flat ∞-connections on principal ∞-bundles. This was the key motivation for axiomatic cohesion on ∞-toposes in [SSS12][Sc13].[5]

Definition 3.3.44 (Smooth path ∞-groupoid)**.** For $A \in \mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CartSpc})$, we write

$$
\mathrm{Pth}(A) \;:=\; \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\, A(\Delta^n_{\mathrm{smth}}) \;\;\in \mathrm{Grpd}_\infty \qquad \substack{\infty\text{-groupoid of smooth paths}\\ \text{and their gauge transformations.}}
\tag{3.172}
$$

for the ∞-groupoid whose n-morphisms are k-gauge transformations of $(n-k)$-dimensional smooth paths in X, in the sense of plots from the smooth extended n-simplices from Def. 3.3.21.

[4] This is the standard singular simplicial set often denoted $\mathrm{Sing}(-)$ or similar. Even though this notation is classical and standard, it clashes badly with any appropriate notation for "singularities" in the sense of orbifolds (as well as of schemes, stacks, etc.) that we need below (in Def. 3.3.57, Def. 3.3.62 and Prop. 3.3.77). Moreover, in the generalization of this classical construction from topological spaces to diffeological spaces (Ntn. 3.3.22 below) and further to smooth ∞-groupoids (Def. 3.3.44 below) it becomes increasingly manifest that the actual *conceptual* nature of the singular simplicial complex is as a model for the *path ∞-groupoid* of a space. For these reasons we choose the non-standard but more suggestive notation $\mathrm{Pth}(-)$ instead of $\mathrm{Sing}(-)$.

[5] Notice that this relation of cohesive ∞-toposes to higher gauge theory is invisible in the cohesive 1-toposes considered in [Law07]: The shape modality of a cohesive $(n,1)$-topos reflects parallel transport for flat $(n-1)$-connections. Here 1-connections are the traditional Cartan-Ehresmann connections, while 0-connections are just functions, and flat 0-connections are just locally constant functions. Hence it takes at least a cohesive $(2,1)$-topos (such as $\mathrm{SmthGrpd}_1$) for its cohesion to axiomatically reflect ordinary connections/gauge fields.

Example 3.3.45 (Smooth shape of diffeological space given by diffeological singular simplicial complex)**.** For $X \in \mathrm{DfflgSpc} \hookrightarrow \mathrm{SmthGrpd}_\infty$, the smooth path ∞-groupoid (3.174) coincides with the diffeological singular simplicial complex (3.146) up to weak homotopy equivalence:

$$\mathrm{Pth}(X) \in \Delta\mathrm{Set} \xrightarrow{\mathrm{Loc}_\Delta^{\mathrm{WHmtpEq}}} \mathrm{Grpd}_\infty\,. \tag{3.173}$$

Proposition 3.3.46 (Smooth cohesive shape given by smooth path ∞-groupoid [Pv14][BBP19])**.**
(i) *For $A \in \mathrm{SmthGrpd}_\infty$, the smooth path ∞-groupoid of A, namely the presheaf*

$$\mathrm{Pth}(A) \;:\; U \;\longmapsto\; \mathrm{Pth}\big([U,A]\big) \;=\; \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\;\big(A(\Delta^n_{\mathrm{smth}}\times U)\big) \;\in\; \mathrm{Grpd}_\infty \tag{3.174}$$

of smoothly parameterized paths (Def. 3.3.44) in A, is in fact an ∞-sheaf

$$\mathbf{Pth}(A) \;\in\; \mathrm{Sh}_\infty(\mathrm{CartSpc}) \;\simeq\; \mathrm{SmthGrpd}_\infty\,.$$

(ii) *As such, this is equivalent to the cohesive shape* (3.117) *of A:*

$$\int A \;\simeq\; \mathbf{Pth}(A)\,.$$

(iii) *In Particular, this means, by evaluation on $\mathbb{R}^0 \in \mathrm{CartSpc}$, that*

$$\mathrm{Shp}(A) \;\simeq\; \mathrm{Pth}(A) \qquad \textit{and} \qquad \mathbf{Pth}(A) \;\simeq\; \mathrm{Dsc}\,\mathrm{Pth}(A)\,. \tag{3.175}$$

Example 3.3.47 (Connected components of smooth shape are concordance classes)**.** For $A \in \mathrm{SmthGrpd}_\infty$, there is a natural isomorphism

$$\begin{aligned} \tau_0 \int A &\simeq \tau_0 \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\; A && \text{by Prop. 3.3.46} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\; \tau_0 A && \text{by (11)} \\ &\simeq \big(\tau_0 A(*)\big)/\big(\tau_0 A(\Delta^1_{\mathrm{smth}})\big) && \text{by Lem. 3.1.19} \end{aligned} \tag{3.176}$$

between the connected components of the shape of A and its *concordance classes*

$$\tau_0 A(*) \;=:\; (A)_{/\sim_{\mathrm{iso}}} \longrightarrow\!\!\!\!\!\rightarrow (A)_{/\sim_{\mathrm{conc}}} \;:=\; \big(\tau_0 A(*)\big)/\big(\tau_0 A(\Delta^1_{\mathrm{smth}})\big)\,. \tag{3.177}$$

Notice that concordance classes are a coarsening, in general, of isomorphism classes on the point by isomorphism classes on the interval. In particular, if two smooth ∞-groupoids are equivalent after 0-truncation, then they already have the same concordance classes:

$$\tau_0 A \;\simeq\; \tau_0 A' \;\;\in\; \mathrm{SmthGrpd}_\infty \qquad \Rightarrow \qquad (A)_{/\sim_{\mathrm{conc}}} \;\simeq\; \big(A'\big)_{/\sim_{\mathrm{conc}}} \;\;\in\; \mathrm{Set}\,. \tag{3.178}$$

Proposition 3.3.48 (Smooth cohesive shape of topological spaces is their weak homotopy type)**.** *The weak homotopy type of any topological space is naturally identified with the cohesive shape* (3.117) *seen in* $\mathrm{SmthGrpd}_\infty$ *(Ntn. 3.3.26) of its continuous-diffeological incarnation* (3.145)*:*

$$\begin{tikzcd} \mathrm{kTopSpc} \arrow[rr, "\mathrm{Cdfflg}"] \arrow[dr, "\mathrm{Pth}"'] & \arrow[d, phantom, "\Swarrow\sim"] & \mathrm{SmthGrpd}_\infty \arrow[dl, "\mathrm{Shp}"] \\ & \mathrm{Grpd}_\infty & \end{tikzcd}$$

Proof. For $X \in \mathrm{TopSpc}$, we need to exhibit a natural equivalence

$$\mathrm{Shp}\circ\mathrm{Cdfflg}(X) \xrightarrow[\sim]{} \mathrm{Pth}(X)\,.$$

But we have:

$$\begin{aligned} \mathrm{Shp}\circ\mathrm{Cdfflg}(X) &\simeq \mathrm{Pth}\circ\mathrm{Cdfflg}(X) && \text{by Prop. 3.3.46} \\ &\simeq \mathrm{Pth}(X) && \text{by Prop. 3.3.23 with Ex. 3.3.45.} \end{aligned}$$

$\square$

More generally:

Proposition 3.3.49 (Smooth shape of good simplicial spaces is weak homotopy type of their realization)**.** *The weak homotopy type of the topological realization* $|\mathrm{X}_\bullet|$ *(Ntn. 1.2.28) of any good simplicial topological space* $\mathrm{X}_\bullet$ *(Def. 1.3.16) is naturally identified with the cohesive shape* (3.117) *seen in* $\mathrm{SmthGrpd}_\infty$ *(Ntn. 3.3.26) of its degreewise continuous-diffeological incarnation* (3.145)*:*

$$\begin{tikzcd} \Delta\mathrm{kTopSpc}_{\mathrm{good}} \arrow[rr, "\mathrm{Cdfflg}((-)_\bullet)"] \arrow[dr, "\mathrm{Pth}\,|-|"'] \arrow[rr, phantom, shift right=4ex, "\Swarrow\sim"] & & \mathrm{SmthGrpd}_\infty \arrow[dl, "\mathrm{Shp}"] \\ & \mathrm{Grpd}_\infty & \end{tikzcd}$$

Proof. We have the following sequence of natural equivalences in Grpd_∞:

$$\begin{aligned} \mathrm{Shp}\,(\mathrm{Cdfflg}(\mathrm{X})_\bullet) &\simeq \mathrm{Shp}\,\underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\;\mathrm{Cdfflg}(\mathrm{X}_n) && \text{by Lem. 3.2.31} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\;\mathrm{Shp}\,\mathrm{Cdfflg}(\mathrm{X})_n && \text{by (11)} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\;\mathrm{Pth}(\mathrm{X}_n) && \text{by Prop. 3.3.48} \\ &\simeq \mathrm{Pth}|\mathrm{X}_\bullet| && \text{by Rem. 1.3.26}. \end{aligned}$$

□

Proposition 3.3.50 (Shape of smooth ∞-topos over continuous-diffeology is weak homotopy type)**.** *For* $\mathrm{X} \in \mathrm{TopSpc}$*, its weak homotopy type coincides with the shape (in the sense of Def. 3.2.19) of the slice ∞-topos (Ntn. 3.2.48) over its continuous-diffeological incarnation* $\mathrm{Cdfflg}(\mathrm{X}) \in \mathrm{SmthGrpd}_\infty$ (3.145) *in that*

$$\mathrm{Shp}\big((\mathrm{SmthGrpd}_\infty)_{/\mathrm{Cdfflg}(\mathrm{X})}\big) \;\simeq\; \mathrm{Pth}(\mathrm{X}) \;\in\; \mathrm{Grpd}_\infty \hookrightarrow \mathrm{Pro}(\mathrm{Grpd}_\infty)\,.$$

Proof.

$$\begin{aligned} \mathrm{Shp}\big((\mathrm{SmthGrpd}_\infty)_{/\mathrm{Cdfflg}(\mathrm{X})}\big) &\simeq \mathrm{Shp}\,(\mathrm{Cdfflg}(\mathrm{X})) && \text{by Prop. 3.3.5} \\ &\simeq \mathrm{Pth}(\mathrm{X}) && \text{by Prop. 3.3.48}. \end{aligned}$$

□

Theorem 3.3.51 (Smooth Oka principle for maps out of smooth manifolds)**.**
Let $A \in \mathrm{SmthGrpd}_\infty$ *and* $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthGrpd}_\infty$.
(i) *The shape of the mapping stack* (3.72) *from X to A is naturally equivalent the mapping stack from X to the shape of A:*

$$\textstyle\int \mathrm{Map}(X,A) \;\simeq\; \mathrm{Map}(\int X, \int A) \;\in\; \mathrm{SmthGrpd}_\infty\,.$$

(ii) *This means, equivalently, that*

$$\mathrm{Shp}\,(\mathrm{Map}(X,A)) \;\simeq\; \mathrm{Grpd}_\infty\,(\mathrm{Shp}(X), \mathrm{Shp}(A))\,.$$

Proof. Noticing that also SmthMfd is a site of definition for $\mathrm{SmthGrpd}_\infty$, observe the following sequence of natural equivalences of ∞-groupoids, for $U \in \mathrm{SmthMfd}$:

$$\begin{aligned} \mathrm{Map}(X,\textstyle\int A)(U) &\simeq \mathrm{Sh}_\infty(\mathrm{SmthMfd})\,(X\times U, \textstyle\int A) && \text{by Prop. 3.2.55} \\ &\simeq \mathrm{PSh}_\infty(\mathrm{SmthMfd})\Big(X\times U,\, \big(U' \mapsto \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\, A\,\big(U'\times\Delta^n_{\mathrm{smth}}\big)\big)\Big) && \text{by Prop. 3.3.46} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\,\mathrm{PSh}_\infty(\mathrm{SmthMfd})\Big(X\times U,\, \big(U' \mapsto A\,\big(U'\times\Delta^n_{\mathrm{smth}}\big)\big)\Big) && \text{by [Lur09, Cor. 5.1.2.3]} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\,\mathrm{PSh}_\infty(\mathrm{SmthMfd})\big(X\times U, \mathrm{Map}(\Delta^n_{\mathrm{smth}}, A)\big) && \text{by Prop. 3.2.55} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\,\mathrm{PSh}_\infty(\mathrm{SmthMfd})\big(\Delta^n_{\mathrm{smth}}\times X\times U, A\big) && \text{by Prop. 3.2.55} \\ &\simeq \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}}\,\big(\mathrm{Map}(X,A)\,\big(\Delta^n_{\mathrm{smth}}\times U\big)\big) && \text{by Prop. 3.2.55} \\ &\simeq \big(\textstyle\int \mathrm{Map}(X,A)\big)\,(U) && \text{by Prop. 3.3.46}. \end{aligned}$$

By the ∞-Yoneda lemma (Prop. 3.2.29), the composite of these natural equivalences yields the first claim. From this the second claim follows by cohesion (the definition $\int \simeq \mathrm{Dsc}\circ\mathrm{Shp}$, the adjunction $\mathrm{Shp} \dashv \mathrm{Dsc}$ and the fully faithfulness of Dsc). □

Resolvable orbi-singularities. The smooth Oka principle (Thm. 3.3.51) applies, a priori, in the generality of domains that are smooth manifolds X with maps $\mathrm{X} \to A$ into any coefficient objects $A \in \mathrm{SmthGrpd}_\infty$. However, for particular coefficients it may happen that maps into them do not distinguish a given domain ∞-stacks from its approximation by some smooth manifold, in which case the smooth Oka principle will generalize to that particular situation as well.

A first class of examples of this generalization occurs when the domain is the moduli stack $\mathbf{B}G$ of a finite group G, hence the orbifold $* /\!\!/ G$ consisting of a single point which is a G-singularity. We may observe that a canonical choice of approximation in this case is given by the "blow-up" of this point (e.g. [Tu20, §29.2], Ntn. 3.3.54 below) to a *smooth spherical space form* (Def. 3.3.52 below) namely to the quotient manifold S^{n+2}_{sm}/G of a smooth n-sphere (possibly exotic) by a smooth and free action of the group G: For any given coefficients A, the distinction between $\mathbf{B}G$ and S^{n+2}_{sm}/G is carried by morphisms $S^{n+2}_{\mathrm{sm}} \to A$, and hence disappears whenever A, or at least its shape, is suitably *n-truncated* (see Ntn. 4.1.30 below).

Therefore, if G-singularities are resolvable in this sense, and the coefficients are suitably truncated, we obtain an *orbi-smooth Oka principle*, which is Thm. 4.1.55 below. Here we first recall relevant facts about free actions of finite groups on spheres and the resulting smooth spherical space forms.

Definition 3.3.52 (Smooth spherical space forms[6])**.** If $G \in \mathrm{Grp}(\mathrm{FinSet})$ and $G \curvearrowright S^{n+2}_{\mathrm{sm}} \in G\mathrm{Act}(\mathrm{SmthMfd})$ is a free action on a smooth $n+2$-sphere (possibly exotic), then the quotient smooth manifold

$$S^{n+2}_{\mathrm{sm}} /\!\!/ G \;\simeq\; S^{n+2}_{\mathrm{sm}}/G \;\; \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthGrpd}_\infty \,,$$

is a *smooth spherical space form*.

Proposition 3.3.53 (Existence of smooth spherical space forms)**.** *For G a finite group, the following conditions are equivalent:*

- **(i)** *For each prime number p, if $H \subset G$ is a subgroup of order p^2 or $2p$, then H is a cyclic group.*
- **(ii)** *G has a continuous free action on a topological d-sphere S^d_{top} for some $d \in \mathbb{N}$.*
- **(iii)** *For each $n \in \mathbb{N}$, there exists $d \geq n+2$ and a smooth manifold structure S^d_{sm} on the d-sphere (possibly exotic) such that G has a smooth free action on S^d_{sm}, hence such that S^d_{sm}/G is a smooth spherical space form (Def. 3.3.52).*

Proof. The equivalence of the first two statements is [MTW76, Thm. 0.5-0.5]. The implication of the third from the second statement is [MTW83, Thm. 5], which asserts, in more detail, that G has a smooth action (at least) on smooth spheres of dimension $2^{2k \cdot e(G)-1}$, for all $k \in \mathbb{N}$, where $e(G) \in \mathbb{N}_+$ is the *Artin-Lam exponent* of G [Lam68][Ya70], the only feature of which that matters here is that it is positive. Both of these statements are reviewed in [Ham14, Thm. 6.1]. Finally, the implication of the second from the third statement is trivial. □

Notation 3.3.54 (Resolvable orbi-singularities)**. (i)** If a finite group G satisfies the equivalent conditions of Prop. 3.3.53, we say that there *exist resolutions of G-singularities* or that *G-singularities are resolvable*.
(ii) We write

$$\begin{tikzcd} \overset{\substack{\text{finite groups that have} \\ \text{resolvable singularities}}}{\mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}} \arrow[r, hook] \arrow[d] & \overset{\substack{\text{finite groups with covers that} \\ \text{have resolvable singularities}}}{\mathrm{Grp}(\mathrm{FinSet})_{\mathrm{covrslvbl}}} \arrow[r, hook] \arrow[d] & \mathrm{Grp}(\mathrm{FinSet}) \arrow[d] \\ \mathrm{rslvblSnglrt} \arrow[r, hook] & \mathrm{covrslvblSnglrt} \arrow[r, hook] & \mathrm{Snglrt} \end{tikzcd} \tag{3.179}$$

for the full subcategories on finite groups G and on orbi-singularities $\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}$ (Ntn. 3.3.57) on the resolvable ones, and, respectively, for those which admit a cover, hence a surjective homomorphism

[6] Historically and by default, the term "spherical space form" refers to Riemannian geometry, where the G-action on a round S^{n+2}_{rm} is required to be (free and) by isometries (e.g. [Wo11][Al18]). In generality, if the action is just required to be (free and) continuous, one speaks of "topological spherical space forms" [MTW76], see review in [Ham14]. The terminology "smooth spherical space form" for the case of concern here, where the action is (free and) by diffeomorphisms seems to originate with [Mad78] following the construction of interesting examples in [MTW76], see Prop. 3.3.53.

$$
\widehat{G} \overset{p}{\twoheadrightarrow} G\,,
$$

such that $\widehat{G}$ has resolvable singularities.

(iii) Given $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ we say that the canonical morphism

$$
S^d_{\mathrm{sm}}/G \xrightarrow{\;(S^d_{\mathrm{sm}} \to *)\,/\!\!/\,G\;} *\,/\!\!/\,G = \mathbf{B}G\,,
$$

$$
S^d_{\mathrm{sm}}/G \xrightarrow{\;\curlyvee(S^d_{\mathrm{sm}} \to *)\,/\!\!/\,G\;} \underset{\curlyvee}{G}
$$

is a *blow-up* of the $\widehat{G}$-orbi-singularity.

(iv) Notice that the homotopy fiber of this morphism is the smooth sphere:

$$
\begin{tikzcd}
S^d_{\mathrm{sm}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & * \arrow[d] \\
S^d_{\mathrm{sm}}/G \arrow[rr, "(S^d_{\mathrm{sm}} \to *)\,/\!\!/\,G"] & & *\,/\!\!/\,G\,.
\end{tikzcd}
$$

Example 3.3.55 (ADE-groups have (cover-)resolvable singularities)**.** For $G \subset \mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ a finite subgroup, its left quaternion multiplication action on the unit sphere $S^{4k+3}_{\mathrm{sm}} \simeq S(\mathbb{H}^{k+1})$ in $k+1$-Quaternion space is evidently smooth and free, for each $k \in \mathbb{N}$.

(i) For example, for $k=0$ this is just the restriction of left multiplication action of $\mathrm{Sp}(1) \simeq S^3_{\mathrm{sm}}$ on itself; while for $k=1$ this is the restriction of the $\mathrm{Sp}(1)$-action which exhibits the 7-sphere as a $\mathrm{Sp}(1)$-principal bundle over the 4-sphere via the quaternionic Hopf fibration $t_{\mathbb{H}}$ (e.g. [GWZ86]):

$$
\begin{tikzcd}
S^3_{\mathrm{sm}} \simeq \mathrm{Sp}(1) \arrow[r, hook] & S^7_{\mathrm{sm}} \arrow[loop above, "G"] \arrow[d, "t_{\mathbb{H}}", "\text{quaternionic Hopf fibration}"'] \\
& S^4_{\mathrm{sm}}
\end{tikzcd}
\tag{3.180}
$$

(ii) Indeed, by the famous *ADE-classification* of finite subgroups of $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ (for pointers to modern proofs see [HSS18, Rem. A.9]) these must be, up to isomorphism, among the following list:

Label	**Finite subgroup of SU(2)**	**Order**	**Name of group**
$\mathbb{A}_r$	$\mathbb{Z}_{r+1}$	n	Cyclic
$\mathbb{D}_{r+4}$	$2\mathrm{D}_{r+2}$	$4(r+2)$	Binary dihedral
$\mathbb{E}_6$	2T	24	Binary tetrahedral
$\mathbb{E}_7$	2O	48	Binary octahedral
$\mathbb{E}_8$	2I	120	Binary icosahedral

Incidentally, for all these finite $G \subset \mathrm{Sp}(1)$ their integral group cohomology is (e.g. [EG17, p. 12]):

$$
H^n_{\mathrm{Grp}}(G;\mathbb{Z}) \simeq \left\{
\begin{array}{ll}
\mathbb{Z} & \mid\; n = 0 \\
G^{\mathrm{ab}} = G/[G,G] & \mid\; n = 2 \bmod 4 \\
\mathbb{Z}_{/\mathrm{ord}(G)} & \mid\; n \text{ a positive multiple of } 4 \\
0 & \mid\; \text{otherwise.}
\end{array}
\right.
\tag{3.181}
$$

and this means (by the exact sequence $\mathbb{R} \to \mathbb{Z} \to \mathrm{U}_1$ and using that $H^{\geq 1}_{\mathrm{Grp}}(G;\mathbb{R}) = 0$ for all finite groups) in particular that the abelianization of G is identified with its group of multiplicative characters:

$$
\mathrm{Hom}(G,\mathrm{U}_1) \simeq H^1_{\mathrm{Grp}}(G;\mathrm{U}_1) \simeq H^2_{\mathrm{Grp}}(G;\mathbb{Z}) \simeq G^{\mathrm{ab}}\,.
\tag{3.182}
$$

By the above ADE-classification, the subgroups of $G \subset \mathrm{Sp}(1)$ which are not cyclic can only be of order

- $2^2 \cdot (r+2)$ (for $2D_{r+2}$, $r \in \mathbb{N}$)
- $24 = 2^3 \cdot 3$ (for $2T$)
- $48 = 2^4 \cdot 3$ (for $2O$)
- $120 = 2^3 \cdot 3 \cdot 5$ (for $2I$),

and none of these orders are of the form p^2 or $2p$, in accord with Prop. 3.3.53.
(ii) This is in contrast to the plain dihedral groups D_{r+2} whose order $2(r+2)$ can violate the condition in Prop. 3.3.53, showing that not all finite subgroups of $\mathrm{SO}(3)$ act freely (and continuously) on any sphere.

However, every subgroup $G \subset \mathrm{SO}(3)$ is (double-)covered by a finite subgroup $\widehat{G} \in \mathrm{SU}(2)$:

$$\begin{tikzcd} \widehat{G} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{SU}(2) \arrow[d, two heads] \\ G \arrow[r, hook] & \mathrm{SO}(3)\,. \end{tikzcd}$$

(iii) In conclusion, the finite subgroups of $\mathrm{SU}(2)$ have resolvable singularities in the sense of Ntn. 3.3.54, and the finite subgroups of $\mathrm{SO}(3)$ have covers by groups that have resolvable singularities (3.179):

$$\begin{aligned} \mathrm{FinSub}\,(\mathrm{SU}(2)) &\subset \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}, \\ \mathrm{FinSub}\,(\mathrm{SO}(3)) &\subset \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{covrslvbl}}\,. \end{aligned}$$

These ADE equivariance groups already subsume key examples of general and recent interest, e.g. all those considered in: [GSV83][HHR16][BG10, §8][SS19-Tad][KL20][Lu21].

Remark 3.3.56 (Further cover-resolvable singularities)**.** A necessary condition for a finite to group have cover-resolvable singularities in the sense of Ntn. 3.3.54 is (see [Saw21]) that it be a *$\mathscr{P}'$-group* in the sense of [Wa13], where the classification of groups with this necessary property is discussed.

3.3.2 Singular cohesion

Here we review and develop basics of *cohesive global homotopy theory* or *singular cohesion* from [SS20-Orb, §3.2], combining the cohesive homotopy theory from [Sc13] with the cohesive perspective on global homotopy theory from [Re14a].

The orbit category. The following Ntn. 3.3.57 is known under a variety of different names and notations in the literature on global homotopy theory (see [SS20-Orb, Rem. 3.47]). Since none of the symbols in use seem particularly illuminating and no common convention has been established, we follow the notation in [SS20-Orb], which has the advantage of transparently expressing the conceptual role of this definition in cohesive global homotopy theory (Def. 3.3.62 below):

Notation 3.3.57 (2-Category of orbi-singularities [SS20-Orb, Def. 3.64])**.**
(i) We write

$$\begin{array}{ccccc} \mathrm{Snglrt} & := & \mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1} & \hookrightarrow & \mathrm{Grpd}_\infty \\ \overset{G}{\prec} & & & \longmapsto & BG \end{array} \tag{3.183}$$

for the ∞-site (Ntn. 3.2.23) of finite connected groupoids, namely the full sub-category (via Exp. 3.2.24) of Grpd_∞ (Ntn. 3.1.10) on the delooping groupoids BG (3.11) of finite groups $G \in \mathrm{Grp}(\mathrm{FinSet})$.
(ii) When regarded as objects in Snglrt we denote these by the symbol "$\overset{G}{\prec}$".
(iii) This means that for $\overset{K}{\prec}, \overset{G}{\prec} \in \mathrm{Snglrt}$ corresponding to $K, G \in \mathrm{Grp}(\mathrm{FinSet})$ we have

$$\mathrm{Snglrt}\big(\overset{K}{\prec}, \overset{G}{\prec}\big) \;\simeq\; \mathrm{Grp}(K, G)/\!\!/G\,, \tag{3.184}$$

where on the right we have the conjugation-action groupoid of G acting by conjugation on group homomorphisms $K \to G$.

Notation 3.3.58 (Category of G-orbits [Br67, §I.3])**.** For $G \in \mathrm{Grp}(\mathrm{Set})$, the *category of G-orbits* or *G-orbit category* is the full subcategory of G-actions (on sets) on the connected transitive G-sets (the types of G-orbits), hence on the quotient sets G/H for any subgroup $H \subset G$ (the stabilizer subgroup of any one point in the orbit):

$$G/H \;\in\; G\mathrm{Orb} \longhookrightarrow G\,\mathrm{Act}(\mathrm{Set})\,.$$

(We consider the generalization of the orbit category, from finite groups to topological groups, below in Def. 4.3.14.)

Lemma 3.3.59 (0-Truncated slice of singularities)**.** *For $G \in \mathrm{Grp}$, the full sub-(2,1)-category of the slice of* Snglrt *(Def. 3.3.57) over* $\overset{G}{\curlyvee}$ (3.183) *on the 0-truncated objects (those whose homotopy fiber is a set) is*
(a) *reflective and*
(b) *given by the subgroup inclusions $H \overset{i_H}{\hookrightarrow} G$:*

$$\mathrm{Snglrt}_{/\overset{G}{\curlyvee}} \underset{\hookleftarrow}{\overset{\tau_0}{\underset{\perp}{\longrightarrow}}} \big(\mathrm{Snglrt}_{/\overset{G}{\curlyvee}}\big)_{\leq 0} \;\simeq\; \left\{ \begin{tikzcd} \overset{H}{\curlyvee} \arrow[rr, dashed] \arrow[dr, "\overset{i_H}{\curlyvee}"'] & \arrow[d, phantom, "\Swarrow"] & \overset{K}{\curlyvee} \arrow[dl, "\overset{i_K}{\curlyvee}"] \\ & \overset{G}{\curlyvee} & \end{tikzcd} \right\} \tag{3.185}$$

for the full sub-(2,1)-category of the slice of Snglrt *(Def. 3.3.57) over* $\overset{G}{\curlyvee}$ (3.183) *on the 0-truncated objects, hence on the morphisms $\overset{H}{\curlyvee} \overset{\overset{i}{\curlyvee}}{\to} \overset{G}{\curlyvee}$ corresponding to subgroup inclusions $H \overset{i}{\hookrightarrow} G$.*

Lemma 3.3.60 (G-orbits are the 0-truncated objects in slice over G-orbi-singularity)**.** *For $G \in \mathrm{Grp}$ the singularities in the 0-truncated slice over* $\overset{G}{\curlyvee}$ (3.185) *are equivalently the G-orbits (Ntn. 3.3.58) in that we have an equivalence in* Cat_∞

$$\begin{array}{ccccc} \big((\mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1})_{/BG}\big)_{\leq 0} & \simeq & \big(\mathrm{Snglrt}_{/\overset{G}{\curlyvee}}\big)_{\leq 0} & \xrightarrow{\sim} & G\mathrm{Orb} \\ (BH \xrightarrow{Bi_H} BG) & \mapsto & (\overset{H}{\curlyvee} \overset{\overset{i}{\curlyvee}}{\to} \overset{G}{\curlyvee}) & \longmapsto & G/H\,, \end{array}$$

where $i_H : H \hookrightarrow G$ are subgroup inclusions.

Proof. By Prop. 3.1.42, we have an equivalence

$$\big((\mathrm{Grpd}_\infty)_{/BG}\big)_{\leq 0} \xrightarrow[\sim]{\mathrm{fib}} G\,\mathrm{Act}(\mathrm{Set})\,.$$

This implies the statement by the observation that the homotopy fiber of $Bi_H : BH \to BG$ is G/H. ☐

In summary:

Proposition 3.3.61 (G-Orbit category is reflective subcategory of slice over G-singularity)**.** *For $G \in \mathrm{Grp}(\mathrm{FinSet})$, the G-orbit category (Ntn. 3.3.58) is a reflective subcategory of the slice of* Snglrt *(Ntn. 3.3.57) over* $\overset{G}{\curlyvee}$*:*

$$\begin{array}{ccc} \mathrm{Snglrt}_{/\overset{G}{\curlyvee}} & \underset{\hookleftarrow}{\overset{\tau_0}{\underset{\perp}{\longrightarrow}}} & G\mathrm{Orb} \\ \overset{K}{\curlyvee} \\ \downarrow{\scriptstyle \overset{\phi}{\curlyvee}} & \longmapsto & G/K = G/\mathrm{im}(\phi)\,. \\ \overset{G}{\curlyvee} \end{array} \tag{3.186}$$

Proof. This follows by Lem. 3.3.59 combined with Lem. 3.3.60. ☐

Equivariant homotopy theories.

Definition 3.3.62 (G-Equivariant and globally equivariant homotopy theories)**.** For $\mathbf{H}_\cup$ an ∞-topos and $G \in \mathrm{Grp}(\mathrm{Set})$,
(i) we write

$$\mathrm{Snglr}(\mathbf{H}_\cup) \;:=\; \mathrm{Sh}_\infty\big(\mathrm{Snglrt}, \mathbf{H}_\cup\big) \qquad \text{and} \qquad G(\mathbf{H}_\cup) \;:=\; \mathrm{Sh}_\infty\big(G\mathrm{Orb}, \mathbf{H}_\cup\big) \tag{3.187}$$

for the ∞-toposes of ∞-(pre-)sheaves on the 2-site of orbi-singularities (Def. 3.3.57) and the site of G-orbits (Ntn. 3.3.58), respectively.

(ii) Since their ∞-sites have trivial Grothendieck topology and a terminal object, these ∞-toposes (3.187) are cohesive over $\mathbf{H}_{\cup}$ (by Ex. 3.3.3), witnessed by adjoint quadruples which we denote like this:

$$
\begin{array}{ccccc}
 & \xleftarrow{\quad} \mathrm{Cncl} \xrightarrow{\quad} & & \textbf{conical} \\
 & \perp & & \\
 & \xleftarrow{\quad} \mathrm{Spc} \xhookrightarrow{\quad} & & \textbf{spatial} \\
\mathrm{Snglr}\mathbf{H}_{\cup} & \perp & \mathbf{H}_{\cup} & \\
 & \xleftarrow{\quad} \mathrm{Smth} \xrightarrow{\quad} & & \textbf{smooth} \\
 & \perp & & \\
 & \xleftarrow{\quad} \mathrm{Snglt} \xhookrightarrow{\quad} & & \textbf{singular}
\end{array}
\tag{3.188}
$$

(iii) We denote the resulting modalities by:

$$
\begin{array}{lcll}
\textbf{purely conical aspect} & \vee & := & \mathrm{Spc} \circ \mathrm{Cncl} \\
 & \perp & & \\
\textbf{purely smooth aspect} & \cup & := & \mathrm{Spc} \circ \mathrm{Smth} \\
 & \perp & & \\
\textbf{purely orbi-singular aspect} & \prec & := & \mathrm{Snglt} \circ \mathrm{Smth}\,.
\end{array}
\tag{3.189}
$$

Example 3.3.63 (Classical equivariant homotopy theory)**.** In the base case $\mathbf{H}_{\cup} = \mathrm{Grpd}_{\infty}$, the ∞-toposes from Def. 3.3.62 are:

(i) the classical equivariant homotopy theory

$$
G\mathrm{Grpd}_{\infty} \simeq \mathrm{PSh}_{\infty}\big(\mathrm{Orb}(G)\big)
$$

from Elmendorf's theorem in its homotopically enhanced version (2) due to [DK84] (see also [CP96], review in [Blu17, Thm. 1.3.8]),

(ii) the global equivariant homotopy theory $\mathrm{Snglr}\mathrm{Grpd}_{\infty}$ in its unstable form first highlighted in [Re14a].

More general instances of Def. 3.3.62 enhance this classical equivariant homotopy theory by cohesive geometric structure:

Singular-cohesive ∞-toposes.

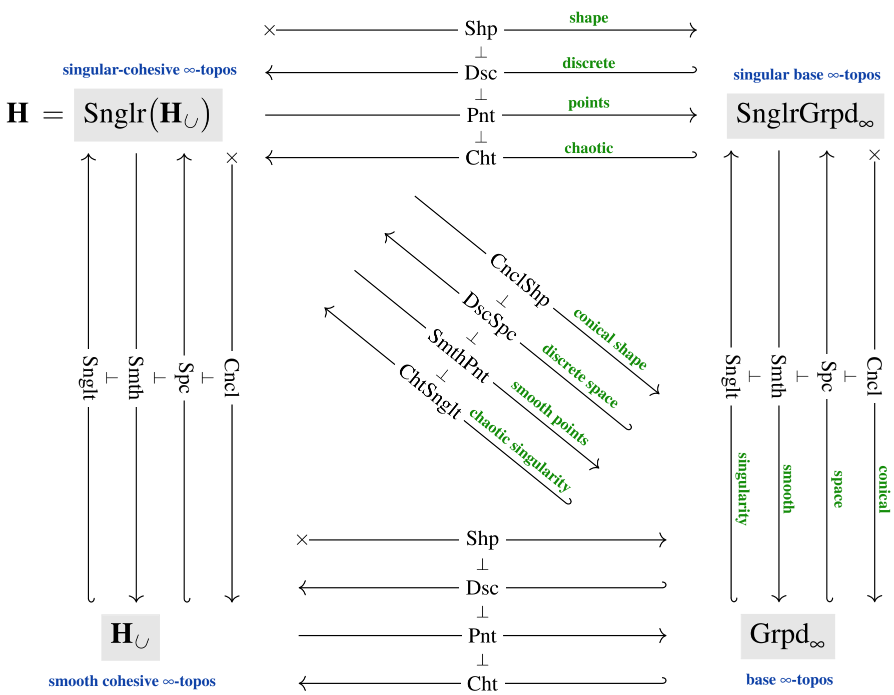
Notation 3.3.64 (Smooth charts)**.** Given a cohesive ∞-topos $\mathbf{H}_{\cup}$, we say that a *category of smooth charts* is an ∞-category Chrt of charts for $\mathbf{H}_{\cup}$ according to Ntn. 3.3.11, subject to the additional condition that all $U \in \mathrm{Chrt}$ are 0-truncated, hence that Chrt is in fact a 1-category.

Definition 3.3.65 (Singular-cohesive ∞-toposes)**.** We say that an ∞-topos $\mathbf{H}$ is *singular-cohesive* over Grpd_{∞} if it is equivalently the global homotopy theory (Def. 3.3.62) of a cohesive ∞-topos $\mathbf{H}_{\cup}$ (Def. 3.3.1) which admits a site Chrt of smooth charts (Def. 3.3.64):

$$
\mathbf{H} \simeq \mathrm{Sh}_{\infty}\left(\mathrm{Snglrt}, \mathbf{H}_{\cup}\right) \simeq \mathrm{Sh}_{\infty}\left(\mathrm{Snglrt} \times \mathrm{Chrt}, \mathrm{Grpd}_{\infty}\right)\,.
\tag{3.190}
$$

This implies[7] that it carries a system of adjoint ∞-functors of the following form:

[7] It would be desireable to state a converse implication, hence to *characterize* singular-cohesive ∞-toposes by specification of the system of adjoint functors they carry, instead of resorting to a presentation over a site. While desireable, this is not necessary for the present purpose, and we leave this question for elsewhere.



(3.191)

We are going to focus on the special case relevant to the differential topology of orbifolds:

Notation 3.3.66 (Singular-smooth ∞-groupoids [SS20-Orb, Ex. 3.56])**.** We write

$$\mathrm{SnglrSmthGrpd}_{\infty} \;:=\; \mathrm{Sh}\left(\mathrm{Snglrt}, \mathrm{SmthGrpd}_{\infty}\right) \;\simeq\; \mathrm{Sh}_{\infty}\left(\mathrm{Snglrt} \times \mathrm{CartSpc}\right)$$

for the singular-cohesive ∞-topos (Def. 3.3.65) over that of smooth ∞-groupoids (Ntn. 3.3.26).

Example 3.3.67 (Singular-cohesive aspects of orbi-singularities [SS20-Orb, Lem. 3.61, Prop. 3.62])**.** For $\mathbf{H}$ a singular-cohesive ∞-topos (Def. 3.3.65) and $G \in \mathrm{Grp}(\mathrm{FinSet}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathrm{Grp}(\mathbf{H})$, the canonical G-orbisingularity (Ntn. 3.3.57)

$$\prec^{G} \in \mathrm{Snglrt} \xhookrightarrow[\;]{\overset{(3.54)}{y}} \mathrm{PSh}(\mathrm{Snglrt}, \mathrm{Grpd}_{\infty}) \;=\; \mathrm{SnglrGrpd}_{\infty} \xhookrightarrow{\overset{(3.191)}{\mathrm{Dsc}}} \mathbf{H}$$

has the following modal aspects (3.189):

(1) its *purely conical aspect* is the point:

$$\vee \prec^{G} \;\simeq\; * \,. \tag{3.192}$$

(2) Its *purely smooth aspect* is the delooping (3.88):

$$\cup \prec^{G} \;\simeq\; \mathbf{B}G \,. \tag{3.193}$$

(3) It is the *orbi-singularization* (3.189) of the delooping of G (3.88):

$$\prec^{G} \;\simeq\; \prec \mathbf{B}G \,. \tag{3.194}$$

Proof. The first statement (3.192) follows via Lem. 3.2.40, as $\mathrm{Cncl} \simeq p_!$ is the left Kan extension of (see Ex. 3.3.3)

$$\mathrm{Snglrt} \underset{\longleftarrow}{\overset{p}{\underset{\perp}{\longrightarrow}}} * \,.$$

This implies, for $\mathrm{U} \times \prec^{K} \in \mathrm{Chrt} \times \mathrm{Snglrt}$, the following natural equivalences

$$
\begin{aligned}
\mathbf{H}\big(\mathrm{U}\times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \cup \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) &\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$<$}(\mathrm{U}\times \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}),\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) && \text{by (3.189)}\\
&\simeq \mathbf{H}\big(\underbrace{(\rotatebox[origin=c]{70}{$<$}\mathrm{U})}_{\substack{\simeq \rotatebox[origin=c]{70}{$<$}\cup U\\ \simeq \mathrm{U}}}\times \underbrace{(\rotatebox[origin=c]{70}{$<$}\overset{G}{\rotatebox[origin=c]{70}{$\prec$}})}_{\simeq \overset{1}{\rotatebox[origin=c]{70}{$\prec$}}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) && \text{by (3.188) \& (3.192)}\\
&\simeq \mathbf{H}\big(\mathrm{U}\times \overset{1}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathbf{H}\big(\mathrm{U}\times \overset{1}{\rotatebox[origin=c]{70}{$\prec$}}, \flat \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathbf{H}\big(\textstyle\int(\mathrm{U}\times \overset{1}{\rotatebox[origin=c]{70}{$\prec$}}),\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathbf{H}\big(\overset{1}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathrm{Snglrt}\big(\overset{1}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) && \text{by (3.54)}\\
&\simeq \mathrm{Grpd}(B1, BG)\\
&\simeq BG\\
&\simeq \mathbf{H}\big(\mathrm{U}\times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \mathbf{B}G\big)\,.
\end{aligned}
$$

This implies the statement (3.88) by the ∞-Yoneda lemma (Prop. 3.2.29).

Similarly, for the last statement (3.194):

$$
\begin{aligned}
\mathbf{H}\big(\mathrm{U}\times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}},\ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}G\big) &\simeq \mathbf{H}\big(\mathrm{U}\times(\rotatebox[origin=c]{70}{$\prec$}\mathbf{B}K),\ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}G\big)\\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(\mathrm{U}\times \mathbf{B}K),\ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}G\big)\\
&\simeq \mathbf{H}(\mathrm{U}\times \mathbf{B}K, \mathbf{B}G)\\
&\simeq \mathbf{H}\big(\mathrm{Dsc}(BK), \mathrm{Dsc}(BG)\big)\\
&\simeq \mathrm{Grpd}_\infty(BK, BG)\\
&\simeq \mathrm{Snglrt}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathbf{H}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\\
&\simeq \mathbf{H}\big(\mathrm{U}\times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}},\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\,.
\end{aligned}
$$

□

Example 3.3.68 (Cohesive loci of orbi-singularities)**.** Given a singular-cohesive ∞-topos $\mathbf{H}$, and $G \in \mathrm{Grp}(\mathrm{FinSet})$, then

$$
\mathscr{X}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \overset{(3.2.29)}{\simeq} \mathbf{H}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big) \overset{(3.75)}{\simeq} \mathrm{Smth}\Big(\mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big)\Big)\,. \tag{3.195}
$$

is the cohesive mapping space of G-orbi-singularities into $\mathscr{X}$.

In particular, if $\mathscr{X} \in \mathbf{H}$ is smooth $\cup\mathscr{X} \simeq \mathscr{X}$, then it has no orbi-singularities and hence these mapping spaces are equivalently those out of the point:

$$
\cup(\mathscr{X}) \simeq \mathscr{X} \quad \Rightarrow \quad \cup\mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big) \simeq \mathscr{X}\,. \tag{3.196}
$$

More generally, given any other smooth object U, the cohesive U-parameterized loci of orbi-singularities in a smooth object form the smooth mapping space out of U:

$$
\begin{matrix} \cup(\mathscr{X}) \simeq \mathscr{X} \\ \cup(U) \simeq U \end{matrix} \quad \Rightarrow \quad \cup\mathrm{Map}\big(U\times \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big) \simeq \cup\mathrm{Map}(U, \mathscr{X})\,. \tag{3.197}
$$

Proof. The equivalence (3.197) is, under the ∞-Yoneda lemma, the composite of the following sequence of natural equivalences:

$$
\begin{aligned}
\mathbf{H}\big(U \times \overset{G}{\prec}, \mathscr{X}\big) &\simeq \mathbf{H}\big(U \times \overset{G}{\prec}, \cup\mathscr{X}\big) && \text{by assumption} \\
&\simeq \mathbf{H}\big(\curlyvee(U \times \overset{G}{\prec}), \mathscr{X}\big) && \text{by (3.189)} \\
&\simeq \mathbf{H}\big(\underbrace{(\curlyvee U)}_{\substack{\simeq \curlyvee\cup U \\ \simeq U}} \times \underbrace{(\curlyvee \overset{G}{\prec})}_{\simeq *}, \mathscr{X}\big) && \text{by (3.188) \& (3.192)} \\
&\simeq \mathbf{H}(U, \mathscr{X}) \,.
\end{aligned}
$$

The statement (3.196) is the special case of (3.197) for $U = *$. □

Example 3.3.69 (Orbi-singularization does not preserve deloopings)**.** Beware that the rightmost adjoint modality $\prec$ (3.189) has no reason to preserve ∞-colimits, hence no reason to preserve homotopy quotients (3.93) such as deloopings $\mathbf{B}(-)$ (3.88); and indeed generically it does not:

(i) For $G \in \mathrm{Grp}(\mathrm{FinSet})$ and $\Gamma \in \mathrm{Grp}(\mathrm{Set}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathbf{H}$, the value of the orbi-singularization of the delooping $\mathbf{B}\Gamma$ on the G-orbi-singularity $\overset{G}{\prec}$ is the Γ-conjugation action groupoid of group homomorphisms $G \to \Gamma$:

$$
(\prec\mathbf{B}\Gamma)(\overset{G}{\prec}) \underset{(3.199)}{\simeq} \mathrm{Grpd}(\mathbf{B}G, \mathbf{B}\Gamma) \simeq \big(\mathrm{Grp}(G,\Gamma) \times \Gamma \rightrightarrows \mathrm{Grp}(G,\Gamma)\big) \quad \in \mathrm{Grpd} \hookrightarrow \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \mathbf{H}\,, \tag{3.198}
$$

because:

$$
\begin{aligned}
(\prec\mathbf{B}\Gamma)(\overset{G}{\prec}) &\simeq \mathbf{H}(\overset{G}{\prec}, \prec\mathbf{B}\Gamma) && \text{by Prop. 3.2.29} \\
&\simeq \mathbf{H}(\cup \overset{G}{\prec}, \mathbf{B}\mathscr{G}) && \text{by (3.189)} \\
&\simeq \mathbf{H}(\mathbf{B}G, \mathbf{B}\mathscr{G}) && \text{by Ex. 3.3.67} \\
&\simeq \mathbf{H}(\flat\mathbf{B}G, \flat\mathbf{B}\Gamma) && \text{by (3.126)} \\
&\simeq \mathrm{Grpd}_\infty(\mathbf{B}G, \mathbf{B}\Gamma) && \text{by Def. 3.3.1.}
\end{aligned}
\tag{3.199}
$$

So as soon as there is more than one conjugacy class of group homomorphisms $G \to \Gamma$, (3.198) says that $\prec\mathbf{B}\Gamma$ has non-trivial π_0, hence is not connected, hence cannot be a delooping, by Prop. 3.2.70.

(ii) This failure of the rightmost singular modality $\prec$ to preserve deloopings is ultimately (we discuss this in §4.3 below) the source of the rich set of connected components of equivariant classifying spaces that we had seen in Prop. 2.3.18, Rem. 2.3.21.

(iii) This state of affairs is what makes twisted equivariant cohomology rich and subtle: While plain twisted cohomology is about sections of higher fiber bundles, twisted equivariant cohomology is about sections of the image under $\prec$ of such higher fiber bundles.

Remark 3.3.70 (Singularity-wise evaluation of cohesive modalities)**.** Def. 3.3.65 means that under the identification

$$
\mathbf{H} = \mathrm{Snglr}\mathbf{H}_{\cup} = \mathrm{Sh}(\mathrm{Snglrt}, \mathbf{H}_{\cup})
$$

the cohesive modalities on $\mathbf{H}$ act functorially over each $\overset{G}{\prec} \in \mathrm{Snglrt}$ as the cohesive modalities on $\mathbf{H}_{\cup}$, in that for $\mathscr{X} \in \mathbf{H}$ regarded as $\mathscr{X} : \overset{G}{\prec} \mapsto \mathscr{X}(\overset{G}{\prec})$, we have the following natural equivalences:

$$
\begin{aligned}
(\int\mathscr{X})(\overset{G}{\prec}) &\simeq \int\Big(\mathscr{X}(\overset{G}{\prec})\Big)\,, \\
(\flat\mathscr{X})(\overset{G}{\prec}) &\simeq \flat\Big(\mathscr{X}(\overset{G}{\prec})\Big)\,, \\
(\sharp\mathscr{X})(\overset{G}{\prec}) &\simeq \sharp\Big(\mathscr{X}(\overset{G}{\prec})\Big)\,.
\end{aligned}
\tag{3.200}
$$

Proposition 3.3.71 (Some singular modalities commute with some cohesive modalities)**.** *In a singular-cohesive ∞-topos* $\mathbf{H}$ *(Def. 3.3.65) such that* $\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}$ *admits cohesive charts (Ntn. 3.3.11)*

– *all cohesive modalities* (3.117) *commute with* $\rotatebox[origin=c]{70}{$\subset$}$ (3.189)*,*

– *all singular modalities* (3.189) *commute with* $\flat$ (3.117)*,*

in that there are natural equivalences of the following form:

$$
\begin{array}{|c|c|c|}
\cline{2-2}
\multicolumn{1}{c|}{} & \flat \circ \rotatebox[origin=c]{70}{$<$} & \multicolumn{1}{c}{} \\
\multicolumn{1}{c|}{} & \simeq \rotatebox[origin=c]{70}{$<$} \circ \flat & \multicolumn{1}{c}{} \\
\hline
\int \circ \rotatebox[origin=c]{70}{$\subset$} & \flat \circ \rotatebox[origin=c]{70}{$\subset$} & \sharp \circ \rotatebox[origin=c]{70}{$\subset$} \\
\simeq \rotatebox[origin=c]{70}{$\subset$} \circ \int & \simeq \rotatebox[origin=c]{70}{$\subset$} \circ \flat & \simeq \rotatebox[origin=c]{70}{$\subset$} \circ \sharp \\
\hline
\multicolumn{1}{c|}{} & \flat \circ \rotatebox[origin=c]{70}{$\prec$} & \multicolumn{1}{c}{} \\
\multicolumn{1}{c|}{} & \simeq \rotatebox[origin=c]{70}{$\prec$} \circ \flat & \multicolumn{1}{c}{} \\
\cline{2-2}
\end{array}
\tag{3.201}
$$

Proof. For Chrt denoting any site (of charts) for $\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}$ we have, by Def. 3.3.62, equivalences of ∞-categories of the following form:

$$
\begin{array}{c}
\mathbf{H} \\
\wr\!\| \\
\mathrm{Sh}\big(\mathrm{Snglrt}, \mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}\big) \;\simeq\; \mathrm{Sh}\big(\mathrm{Snglrt} \times \mathrm{Chrt}, \mathrm{Grpd}_\infty\big) \;\simeq\; \mathrm{Sh}\big(\mathrm{Chrt}, \mathrm{SnglrGrpd}_\infty\big)\,.
\end{array}
\tag{3.202}
$$

Here we may assume without restriction, by (3.123), that Chrt contains the terminal object

$$
*_{\rotatebox[origin=c]{70}{$\subset$}} \;\in\; \mathrm{Chrt} \overset{y}{\hookrightarrow} \mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}\,.
$$

But this means that, under the equivalence on the right of (3.202), $\flat$ is the operation of evaluating on $*_{\rotatebox[origin=c]{70}{$\subset$}}$ and then extending as a constant sheaf, by Ex. 3.3.12, in that there are natural equivalences:

$$
\underset{U \,\in\, \mathrm{Chrt}}{\forall} \quad (\flat \mathscr{X})(U) \simeq \mathscr{X}(*_{\rotatebox[origin=c]{70}{$\subset$}}) \quad \in\; \mathrm{SnglrGrpd}_\infty\,.
\tag{3.203}
$$

The analogous statement holds for $\rotatebox[origin=c]{70}{$\subset$}$ under the equivalence on the left of (3.202), by the definition (3.188) via Ex. 3.3.3:

$$
\underset{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \,\in\, \mathrm{Snglrt}}{\forall} \quad (\rotatebox[origin=c]{70}{$\subset$}\mathscr{X})\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \;\simeq\; \mathscr{X}\big(\overset{1}{\rotatebox[origin=c]{70}{$\prec$}}\big) \quad \in\; \mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}\,.
\tag{3.204}
$$

Moreover, under the equivalence on the left of (3.202) the cohesive modalities act objectwise over singularities (by Rem. 3.3.70), e.g. for $\int$ we have natural equivalences

$$
(\textstyle\int \mathscr{X})\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \simeq \textstyle\int \Big(\mathscr{X}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big) \quad \in\; \mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}\,;
\tag{3.205}
$$

while under the equivalence on the right the singular modalities act objectwise over sharts, e.g. for $\rotatebox[origin=c]{70}{$\prec$}$ we have natural equivalences

$$
(\rotatebox[origin=c]{70}{$\prec$}\mathscr{X})(U) \;\simeq\; \rotatebox[origin=c]{70}{$\prec$}\big(\mathscr{X}(U)\big) \quad \in\; \mathrm{SnglrGrpd}_\infty\,.
\tag{3.206}
$$

Using this, the claim follows with the ∞-Yoneda lemma (Prop. 3.2.29). For example, for the cases of $(\int, \rotatebox[origin=c]{70}{$\subset$})$ and $(\flat, \rotatebox[origin=c]{70}{$\prec$})$ we have the following sequences of natural equivalences in $\overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \in \mathrm{Snglrt}$ and in $U \in \mathrm{Chrt}$, respectively:

$$
\begin{aligned}
(\rotatebox[origin=c]{70}{$\subset$}\textstyle\int \mathscr{X})\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \;&\simeq\; (\textstyle\int \mathscr{X})\big(\overset{1}{\rotatebox[origin=c]{70}{$\prec$}}\big) && \text{by (3.204)} \\
&\simeq\; \textstyle\int \Big(\mathscr{X}\big(\overset{1}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big) && \text{by (3.205)} \\
&\simeq\; \textstyle\int \Big((\rotatebox[origin=c]{70}{$\subset$}\mathscr{X})\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big) && \text{by (3.204)} \\
&\simeq\; (\textstyle\int \rotatebox[origin=c]{70}{$\subset$}\mathscr{X})\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) && \text{by (3.205),}
\end{aligned}
$$

and analogously:

$$
\begin{aligned}
(\flat \curlyvee \mathscr{X})(U) &\simeq (\curlyvee \mathscr{X})(*_{\cup}) && \text{by (3.203)}\\
&\simeq \curlyvee\big(\mathscr{X}(*_{\cup})\big) && \text{by (3.206)}\\
&\simeq \curlyvee((\flat \mathscr{X})(U)) && \text{by (3.203)}\\
&\simeq (\curlyvee \flat \mathscr{X})(U) && \text{by (3.206).}
\end{aligned}
$$

The proof of the remaining cases is obtained by substitution. □

Remark 3.3.72 (Noncommuting composite modalities)**.** The mixed pairs of singular/cohesive modalities remaining besides those in Prop. 3.3.71 do *not* commute, in general

$$
\boxed{\begin{aligned} &\textstyle\int \circ \vee \\ \not\simeq\; &\vee \circ \textstyle\int \end{aligned}}
\qquad
\boxed{\begin{aligned} &\sharp \circ \vee \\ \not\simeq\; &\vee \circ \sharp \end{aligned}}
$$

$$
\boxed{\begin{aligned} &\textstyle\int \circ \curlyvee \\ \not\simeq\; &\curlyvee \circ \textstyle\int \end{aligned}}
\qquad
\boxed{\begin{aligned} &\sharp \circ \curlyvee \\ \not\simeq\; &\curlyvee \circ \sharp \end{aligned}}
$$

The failure of these equivalences reflects the key information contained in singular cohesion. For example:
- $\curlyvee \circ \int (\mathrm{X} /\!\!/ G)$ extracts homotopy fixed points of a topological G-space $G \circlearrowright \mathrm{X}$,
- $\int \circ \curlyvee (\mathrm{X} /\!\!/ G)$ extracts the geometric fixed points (Def. 3.3.85).

The fact that the latter are a finer invariant than the former is the hallmark of proper-equivariant homotopy theory, here brought out as an aspect of cohesive homotopy theory.

Example 3.3.73 (Smooth charts are smooth and orbisingular)**.** In a singular-cohesive ∞-topos **H** (Def. 3.3.65) all smooth charts $U \in \mathrm{Chrt}$ (Def. 3.3.64) are both smooth as well as orbisingular ([SS20-Orb, Lem. 3.65], because they are 0-truncated)

$$
U \in \mathrm{Chrt} \xhookrightarrow{\;y\;} \mathbf{H} \qquad \vdash \qquad \cup(U) \simeq U\,, \quad \text{and} \quad \curlyvee(U) \simeq U\,. \tag{3.207}
$$

Remark 3.3.74. Ex. 3.207 shows that being $\curlyvee$-modal is to be read as "all singularities *that appear* are orbisingularities", subsuming the trivial case where no singularities are present in the first place.

Example 3.3.75 (Morphisms between orbi-singularizations of smooth objects)**.** For $X, A \in \mathbf{H}_{\cup} \xhookrightarrow{\mathrm{Spc}} \mathbf{H}$, we have that the hom-space (7) between their orbi-singularization is naturally equivalent to their hom-space as smooth objects:

$$
\begin{aligned}
\mathbf{H}(\curlyvee X,\, \curlyvee A) &= \mathbf{H}(\mathrm{Snglt} \circ \mathrm{Smth}(X),\, \mathrm{Snglt} \circ \mathrm{Smth}(A))\\
&\simeq \mathbf{H}(\mathrm{Snglt}(X),\, \mathrm{Snglt}(A))\\
&\simeq \mathbf{H}_{\cup}(X, A)
\end{aligned}
$$

by the fact that $\mathrm{Snglt} : \mathbf{H}_{\cup} \hookrightarrow \mathbf{H}$ is fully faithful. More generally, for $G \in \mathrm{Grp}(\mathbf{H}_{\cup})$ and $G \circlearrowright X,\, G \circlearrowright A \in G\mathrm{Act}(\mathbf{H}_{\cup}) \xhookrightarrow{\mathrm{Spc}} G\mathrm{Act}(\mathbf{H})$ the slice hom-spaces (3.77) of (the orbisingularizations of) the homotopy quotients are equivalent as follows:

$$
\mathbf{H}_{/\curlyvee \mathbf{B}G}\big(\curlyvee(X /\!\!/ G),\, \curlyvee(A /\!\!/ G)\big) \;\simeq\; \mathbf{H}_{/\mathbf{B}G}(X /\!\!/ G, A /\!\!/ G)\,.
$$

Remark 3.3.76. This means that effects of proper equivariance appear only for coefficients that are neither smooth nor just orbisingularizations of smooth objects. The canonical example of interesting coefficients for proper equivariance is the shape of orbi-singularizations of smooth objects.

***G*-Singular cohesive ∞-toposes.** Finally we connect the general singular-cohesive homotopy theory to traditional proper G-equivariant homotopy theory by observing, with [Re14a], that the latter serves as a base of cohesion for the slice of the former over the generic G-orbi-singularity.

Proposition 3.3.77 (G-Singular-cohesive ∞-topos)**.** *For $G \in \mathrm{Grp}(\mathrm{Set})$ a discrete group, the slice ∞-topos (Prop. 3.2.48) of any singular-cohesive ∞-topos $\mathbf{H}$ (Def. 3.3.65) over the orbi-singularity $\rotatebox[origin=c]{70}{$\prec^{G}* (3.183) *is cohesive over the proper G-equivariant homotopy theory of $\mathbf{H}_{\cup}$ (Def. 3.3.62) in that there exists an adjoint quadruple of this form:*

$$
\mathbf{H}_{/\rotatebox[origin=c]{70}{$\prec$}^{G}} \;=\; \mathrm{Snglr}(\mathbf{H}_{\cup})_{/\rotatebox[origin=c]{70}{$\prec$}^{G}}
\begin{array}{c}
\xrightarrow{\quad G\mathrm{OrbCncl}\quad} \\
\perp \\
\xhookleftarrow{\quad G\mathrm{OrbSpc}\quad} \\
\perp \\
\xrightarrow{\quad G\mathrm{OrbSmth}\quad} \\
\perp \\
\xhookleftarrow{\quad G\mathrm{OrbSnglt}\quad}
\end{array}
G(\mathbf{H}_{\cup})\,,
\qquad
\begin{array}{c}
\textbf{G-orbi-conical} \\
\textbf{G-orbi-space} \\
\textbf{G-orbi-smooth} \\
\textbf{G-orbi-singularity}
\end{array}
\tag{3.208}
$$

where

$$
G\mathrm{OrbSmth}(\mathscr{X}) \;:\; G/H \;\mapsto\; \mathrm{Smth}\Big(\mathrm{Map}\big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\big)_{\rotatebox[origin=c]{70}{$\prec$}^{G}}\Big)
\tag{3.209}
$$

assigns the smooth aspect (3.188) *of the slice mapping stack (Def. 3.2.65) (the geometric fixed loci, Def. 3.3.85).*

Essentially this was observed in [Re14a], there for the base case $\mathbf{H} = \mathrm{Grpd}_{\infty}$ and for compact (hence finite if discrete) Lie groups G. For the discrete equivariance groups considered here the statement, for any $\mathbf{H}$, is a formal consequence of the reflection from Prop. 3.186 (as in [Re14a, §7.1, 7.2]), as spelled out in the following proof.

Proof. With the equivalence from Prop. 3.2.45, the adjoint quadruple follows as the Kan extension from Lem. 3.2.43 of the adjunction of sites in Prop. 3.3.61:

$$
\begin{array}{c}
\mathrm{Snglrt}_{/\rotatebox[origin=c]{70}{$\prec$}^{G}}
\begin{array}{c}
\xrightarrow{\;\;\tau\;\;} \\
\perp \\
\xhookleftarrow[\;\;i\;\;]{}
\end{array}
G\mathrm{Orb}
\\[1em]
\Rightarrow \quad \mathrm{Snglr}(\mathbf{H}_{\cup})_{/\rotatebox[origin=c]{70}{$\prec$}^{G}} \xrightarrow{\;\sim\;} \mathrm{PSh}_{\infty}\Big(\mathrm{Snglrt}_{/\rotatebox[origin=c]{70}{$\prec$}^{G}}, \mathbf{H}_{\cup}\Big)
\begin{array}{c}
\xrightarrow{\quad \tau_{!}\quad} \\
\perp \\
\xhookleftarrow{\quad \tau^{*} \simeq i_{!}\quad} \\
\perp \\
\xrightarrow{\quad \tau_{*} \simeq i^{*}\quad} \\
\perp \\
\xhookleftarrow{\quad i_{*}\quad}
\end{array}
\mathrm{Sh}_{\infty}\big(G\mathrm{Orb}, \mathbf{H}_{\cup}\big) \;=\; G\mathbf{H}_{\cup}\,.
\end{array}
\tag{3.210}
$$

Moreover, that the left adjoint $\tau_{!}$ preserves finite products follows (by Lem. 3.2.40, Lem. 3.2.41) since τ preserves finite products after extension to free cocompletions.

The second statement (3.209) follows with the explicit realization of the equivalence in Prop. 3.2.45 and using the definition of i^{*}:

$$
\begin{aligned}
G\mathrm{OrbSmth}(\mathscr{X}) \;&\simeq\; \Big(G/H \mapsto \mathrm{PSh}\Big(\mathrm{Snglrt}_{\rotatebox[origin=c]{70}{$\prec$}^{G}}, \mathbf{H}_{\cup}\Big)_{\rotatebox[origin=c]{70}{$\prec$}^{G}}\Big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\Big)\Big) && \text{by Prop. 3.2.45} \\
&\simeq\; \Big(G/H \mapsto \mathrm{PSh}\Big(\mathrm{Snglrt}_{\rotatebox[origin=c]{70}{$\prec$}^{G}}, \mathbf{H}_{\cup}\Big)\Big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\Big) \underset{\mathrm{PSh}_{\infty}\big(\mathrm{Snglrt}_{\rotatebox[origin=c]{70}{$\prec$}^{G}}, \mathbf{H}_{\cup}\big)\big(\rotatebox[origin=c]{70}{$\prec$}^{H},\, \rotatebox[origin=c]{70}{$\prec$}^{G}\big)}{\times} \Big\{ \rotatebox[origin=c]{70}{$\prec$}^{i_H} \Big\}\Big) && \text{by Prop. 3.2.63} \\
&\simeq\; \Big(G/H \mapsto \mathscr{X}\big(\rotatebox[origin=c]{70}{$\prec$}^{H}\big) \underset{\rotatebox[origin=c]{70}{$\prec$}^{G}(\rotatebox[origin=c]{70}{$\prec$}^{H})}{\times} \big\{ \rotatebox[origin=c]{70}{$\prec$}^{i_H} \big\}\Big) && \text{by Prop. 3.2.29} \\
&\simeq\; \Big(G/H \mapsto \mathrm{Smth}\Big(\mathrm{Map}\big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\big)\Big) \underset{\mathrm{Smth}\big(\mathrm{Map}(\rotatebox[origin=c]{70}{$\prec$}^{H},\, \rotatebox[origin=c]{70}{$\prec$}^{G})\big)}{\times} \Big\{ \rotatebox[origin=c]{70}{$\prec$}^{i_H} \Big\}\Big) && \text{by (3.195)} \\
&\simeq\; \Big(G/H \mapsto \mathrm{Smth}\Big(\mathrm{Map}\big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\big) \underset{\mathrm{Map}(\rotatebox[origin=c]{70}{$\prec$}^{H},\, \rotatebox[origin=c]{70}{$\prec$}^{G})}{\times} \big\{ \rotatebox[origin=c]{70}{$\prec$}^{i_H} \big\}\Big)\Big) && \text{by (11)} \\
&\simeq\; \Big(G/H \mapsto \mathrm{Smth}\Big(\mathrm{Map}\big(\rotatebox[origin=c]{70}{$\prec$}^{H}, \mathscr{X}\big)_{\rotatebox[origin=c]{70}{$\prec$}^{G}}\Big)\Big) && \text{by Def. 3.2.65.}
\end{aligned}
$$
□

Definition 3.3.78 (G-orbi-singular modalities)**.** Given a singular-cohesive ∞-topos (Def. 3.3.65) and $G \in \mathrm{Grp}(\mathrm{Set})$, we write, in generalization of (3.189):

$$
\begin{array}{rccl}
\text{singular aspect relative to } G\text{-orbi-singularities} & \vee_{\overset{G}{\curlyvee}} & := & G\mathrm{OrbSpc} \circ G\mathrm{OrbCncl} \\
 & \perp & & \\
\text{smooth aspect relative to } G\text{-orbi-singularities} & \cup_{\overset{G}{\curlyvee}} & := & G\mathrm{OrbSpc} \circ G\mathrm{OrbSmth} \\
 & \perp & & \\
\text{orbisingular aspect relative to } G\text{-orbi-singularities} & \curlyvee_{\overset{G}{\curlyvee}} & := & G\mathrm{Snglt} \circ G\mathrm{OrbSmth}\,.
\end{array}
\tag{3.211}
$$

Example 3.3.79 (Idempotency of G-orbi-singular modalities)**.** By the general idempotency of cohesive modalities (Rem. 3.118) we have, for instance, that G-orbi-spaces (3.3.77) are G-orbi-smooth (Def. 3.3.78) and hence coincide with their G-orbi-conical aspect (3.211):

$$
G\mathrm{OrbSpc} \;\simeq\; \cup_{\overset{G}{\curlyvee}} \circ G\mathrm{OrbSpc}\,, \qquad \vee_{\overset{G}{\curlyvee}} \circ G\mathrm{OrbSpc} \;\simeq\; G\mathrm{OrbSpc}\,.
$$

Example 3.3.80 (Orbi-singularization of homotopy quotient)**.** For $\mathbf{H}$ a singular-cohesive ∞-topos, given $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_{\cup}) \hookrightarrow G\mathrm{Act}(\mathbf{H})$, we may naturally regard the orbi-singularization (3.189) of its homotopy quotient (Ntn. 3.1.41) as an object in the slice of $\mathbf{H}$ over the G-orbisingularity (3.208):

$$
\curlyvee \left(\begin{array}{c} X /\!\!/ G \\ \big\downarrow {\scriptstyle (X \to *)/\!\!/ G} \\ * /\!\!/ G \end{array} \right) \;=\; \left(\begin{array}{c} \curlyvee(X /\!\!/ G) \\ \big\downarrow \\ \overset{G}{\curlyvee} \end{array} \right) \;\in\; \mathbf{H}_{/\overset{G}{\curlyvee}}\,.
$$

If X here is 0-truncated, then this construction lands in $\cup_{\overset{G}{\curlyvee}}$-modal objects (by Prop. 3.3.92).

Lemma 3.3.81 (Conical quotient of orbi-singularity relative to a G-orbi-singularity)**.** *Let $K, G \in \mathrm{Grp}(\mathrm{FinSet})$ and $K \xrightarrow{\phi} G$ any group homomorphism, regarded as an object $\overset{K}{\curlyvee} \in \mathrm{Snglrt}_{/\overset{G}{\curlyvee}} \overset{y}{\hookrightarrow} \mathbf{H}_{/\overset{G}{\curlyvee}}$. Then the G-relative conical aspect* (3.211) *of ϕ is given by the image factorization $K \twoheadrightarrow \mathrm{im}(\phi) \hookrightarrow G$, in that:*

$$
\vee_{\overset{G}{\curlyvee}}\big(\overset{K}{\curlyvee}\big) \;\simeq\; \overset{\mathrm{im}(\phi)}{\curlyvee} \quad \in \mathbf{H}_{/\overset{G}{\curlyvee}}\,.
$$

Proof. By (3.210) with Lemma 3.2.40 and Prop. 3.3.61. □

Cohesive G-Orbispaces. The following terminology (Ntn. 3.3.82) is the direct generalization of that of [Re14a] (following [HG07]) from discrete to cohesive global homotopy theory, as discussed in [SS20-Orb, §4.1].

Notation 3.3.82 (Cohesive G-orbispace)**.** For $\mathbf{H}$ a singular-cohesive ∞-topos and $G \in \mathrm{Grp}(\mathrm{Set})$, we say that those objects $\mathscr{X} \in \mathbf{H}_{/\overset{G}{\curlyvee}}$, which are $\cup_{\overset{G}{\curlyvee}}$-modal (Def. 3.3.78), hence in the image of the full inclusion (3.208)

$$
G(\mathbf{H}_{\cup}) \xhookrightarrow{\;G\mathrm{OrbSpc}\;} \mathrm{Snglr}(\mathbf{H}_{/\overset{G}{\curlyvee}})
$$

are the *cohesive G-orbi-spaces*.

Remark 3.3.83 (Interpretation of G-orbispaces)**.** The cohesive G-orbispaces (Ntn. 3.3.82) may be thought of as those singular-cohesive spaces which are *smooth away from G-singularities* or *smooth except possibly for G-singularities*.

Example 3.3.84 (The terminal G-orbispace)**.** For $G \in \mathrm{Grp}(\mathrm{Set})$ with $\mathbf{B}G \in \mathbf{H}_{/\mathbf{B}G}$ and $\overset{G}{\curlyvee} \in \mathbf{H}_{/\overset{G}{\curlyvee}}$ regarded as the identity maps on themselves, hence as the respective terminal objects (by Ex. 3.2.49), we have that the G-smooth aspect (3.211) of the shape (3.117) of the orbi-singularization (3.189) of the former terminal object $\mathbf{B}G$ is the latter terminal object $\overset{G}{\curlyvee}$, hence the terminal G-orbispace (Ntn. 3.3.82):

$$
\cup_{\overset{G}{\curlyvee}} \textstyle\int \curlyvee \mathbf{B}G \;\simeq\; \overset{G}{\curlyvee} \;\simeq\; *_{\overset{G}{\curlyvee}} \quad \in G\mathbf{H}_{\cup} \xhookrightarrow{\;G\mathrm{Spc}\;} \mathbf{H}_{\overset{G}{\curlyvee}}\,.
$$

This follows immediately since $\curlyvee$ and $\cup_{\overset{G}{\curlyvee}}$ are right adjoints and hence preserve all limits (11) and since $\int$ preserves all finite products (Rem. 3.3.4).

Geometric fixed loci. We expand on the construction of geometric fixed loci from (3.208).

Definition 3.3.85 (Geometric fixed loci [SS20-Orb, Def. 3.69])**.** For
 – $G \in \mathrm{Grp}(\mathrm{FinSet})$,
 – $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_{\cup}) \hookrightarrow G\mathrm{Act}(\mathbf{H})$:

(i) We say, for $H \subset G$ a subgroup, that the *geometric H-fixed locus* of $G \circlearrowright X$ is the smooth aspect (3.189) of the stack of sections (3.78) of the orbi-singularization (3.189) of its homotopy quotient (3.93):

$$\begin{aligned} H\mathrm{FxdLoc}(G \circlearrowright X) &:= \big(\mathrm{FxdLoc}(G \circlearrowright X)\big)(G/H) \\ &:= \cup\mathrm{Map}\big(\prec^{H},\ \prec(X /\!\!/ G)\big)_{\prec^{G}} \quad \in \mathbf{H}_{\cup} \hookrightarrow \mathbf{H}\,. \end{aligned} \tag{3.212}$$

Since orbi-singularization $\prec$ is fully faithful on smooth ($\cup$-modal) objects, this is equivalently the stacky fixed locus of Ex. 3.2.77, as it should be:

$$\begin{aligned} H\mathrm{FxdLoc}(G \circlearrowright X) &\simeq \cup\mathrm{Map}\big(\prec^{H},\ \prec(X /\!\!/ G)\big)_{\prec^{G}} \\ &\simeq \mathrm{Map}\big(\mathbf{B}H, X /\!\!/ G\big)_{\mathbf{B}G} \\ &\simeq X^{H} \qquad \text{by (3.96)} \end{aligned}$$

(ii) We say that the *shape of the geometric H-fixed locus* in X is the ∞-groupoid of sections of the shape of the orbi-singularization of the homotopy quotient, regarded as a discrete spatial object:

$$\int H\mathrm{FxdLoc}(G \circlearrowright X) \overset{\text{Prop. 3.3.91}}{\simeq} \mathrm{Spc}\,\mathrm{Dsc}\,\mathbf{H}\big(\prec^{H}, \int \prec(X /\!\!/ G)\big)_{\prec^{G}} \ \in \mathrm{Grpd}_{\infty} \hookrightarrow \mathbf{H}\,. \tag{3.213}$$

Remark 3.3.86 (Orbi-spaces)**.**
(i) By functoriality of the mapping stack construction, as $H \subset G$ varies the fixed loci (3.212) arrange into an object of the G-equivariant homotopy theory of $\mathbf{H}_{\cup}$ (Def. 3.3.62), which, according to Prop. 3.3.77, coincides with the smooth aspect relative to G-singularities (3.209) of the orbi-singularization of the homotopy quotient:

$$\mathrm{FxdLoc}(G \circlearrowright X) \;=\; G\mathrm{OrbSmth}\big(\prec(X /\!\!/ G)\big) \;=\; \mathrm{Smth}\,\mathrm{Map}\big(\prec^{(-)},\ \prec(X /\!\!/ G)\big)_{\prec^{G}} \ \in G\mathbf{H}_{\cup}\,. \tag{3.214}$$

(ii) Now, as $H \subset G$ varies, the fixed loci (3.213) arrange into an object of the G-equivariant homotopy theory of $\mathrm{Grpd}_{\infty} \overset{\mathrm{Dsc}}{\hookrightarrow} \mathbf{H}_{\cup}$ (Def. 3.3.62), which, according to Prop. 3.3.77, coincides with the smooth aspect relative to G-singularities (3.209) of *the shape of* the orbi-singularization of the homotopy quotient:

$$\int \mathrm{FxdLoc}(G \circlearrowright X) \;\simeq\; G\mathrm{OrbSmth}\big(\int \prec(X /\!\!/ G)\big) \;\in\; \begin{tikzcd}[column sep=small] & G\mathbf{H}_{\cup} \arrow[dr, hook, "G\mathrm{OrbSpc}"] & \\ G\mathrm{Grpd}_{\infty} \arrow[ur, hook, "\mathrm{Dsc}"] \arrow[dr, hook, "G\mathrm{OrbSpc}"'] & & \mathrm{Snglr}(\mathbf{H}_{\cup})_{/\prec^{G}} \\ & \mathrm{SnglrGrpd}_{\infty/\prec^{G}} \arrow[ur, hook, "\mathrm{Dsc}"'] & \end{tikzcd} . \tag{3.215}$$

Regarded under this embedding, we have

$$\int \mathrm{FxdLoc}(G \circlearrowright X) \;\simeq\; \cup_{\prec^{G}} \int \mathrm{FxdLoc}(G \circlearrowright X) \;\simeq\; \cup_{\prec^{G}} \int \prec(X /\!\!/ G)\,, \tag{3.216}$$

reflecting the fact that any such *G-orbi-space* (see [SS20-Orb, §4.1] for more discussion and further pointers to the literature) is smooth (non-singular) except for orbi-singularities corresponding to subgroups of G.

Remark 3.3.87 (Cocycle ∞-groupoids on cohesive orbi-spaces)**.** In summary, the above implies that, for $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_{\cup})$ and $\mathscr{A} \in G\mathbf{H}_{\cup}$, we have a natural equivalence

$$\mathbf{H}\big(\prec(X /\!\!/ G), G\mathrm{OrbSnglt}(\mathscr{A})\big)_{\prec^{G}} \;\simeq\; G\mathbf{H}_{\cup}\big(\mathrm{FxdLoc}(X), \mathscr{A}\big) \tag{3.217}$$

obtained as the following composite:

$$\begin{aligned} \mathbf{H}\big(\prec(X /\!\!/ G), G\mathrm{OrbSnglt}(\mathscr{A})\big)_{\prec^{G}} &\simeq G\mathbf{H}_{\cup}\big(G\mathrm{OrbSmth}\,(\prec(X /\!\!/ G))\,, \mathscr{A}\big) && \text{by (3.208)} \\ &\simeq G\mathbf{H}_{\cup}\Big(\mathrm{Smth}\,\mathrm{Map}\big(\prec^{(-)},\ \prec(X /\!\!/ G)\big)_{\prec^{G}}, \mathscr{A}\Big) && \text{by (3.209)} \\ &= G\mathbf{H}_{\cup}\,(\mathrm{FxdLoc}(X), \mathscr{A}) && \text{by (3.214).} \end{aligned}$$

This is the basis for the definition of proper-equivariant cohomology in cohesive ∞-toposes (Def. 4.3.20 below).

Remark 3.3.88 (Subsuming generalized geometric fixed points)**.**
(i) If $X \in \mathbf{H}_{\cup,0}$ is smooth and 0-truncated, then the geometric fixed loci of $X /\!\!/ G$, according to Def. 3.3.85, consist of the expected fixed loci of the given G-∞-action on X.
(ii) However, if X is not 0-truncated, then its isotropy groups, as far as they receive non-trivial homomorphisms from the subgroups $H \subset G$, do contribute to the geometric fixed loci in the sense of Def. 3.3.85.
(iii) The archetypical example of this effect occurs when $X = \mathbf{B}\Gamma$ is a delooping (3.2.70), hence a single point with isotropy. This is exactly the case of equivariant moduli stacks, discussed in §4.3 below. Their generalized geometric fixed loci is the source of all the interesting and characteristic structure of equivariant classifying spaces embodied by the Murayama-Shimakawa construction (Thm. 4.3.19 below).

We proceed to demonstrate the equivalence (3.215) in Prop. 3.3.91 below:

Lemma 3.3.89. *The shape of the geometric fixed locus is the smooth aspect* (3.189) *of the internal fixed locus*

$$\cup \circ \mathrm{Map}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}} \simeq \mathrm{Spc} \circ \mathrm{Dsc} \circ \mathbf{H}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}}.$$

Proof. For $\orbisingular^{K} \times U \in \mathrm{Snglrt} \times \mathrm{Chrt}$, we have the following sequence of natural equivalences:

$$\begin{aligned}
\Big(\cup \mathrm{Map}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}}\Big)\Big(\orbisingular^{K} \times U\Big) &\simeq \Big(\mathrm{Map}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}}\Big)(U) && \text{by [SS20-Orb, Def. 3.52]}\\
&\simeq \mathbf{H}\big(\orbisingular^{G} \times U, \int \mathscr{X}\big)_{\orbisingular^{G}} && \text{by Lem. 3.2.68}\\
&\simeq \mathbf{H}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}} && \text{by [SS20-Orb, Def. 3.1]}\\
&\simeq \Big(\mathrm{Spc} \circ \mathrm{Dsc} \circ \mathbf{H}\big(\orbisingular^{G}, \int \mathscr{X}\big)_{\orbisingular^{G}}\Big)\Big(\orbisingular^{K} \times U\Big) && \text{by [SS20-Orb, Def. 3.50].}
\end{aligned}$$

Therefore, the claim follows by the ∞-Yoneda lemma (Lem. 3.2.29). □

The following simple version (Prop. 3.3.90) of an "orbi-smooth Oka principle" (1) will serve as a workhorse lemma in following proofs. Notice here that by our definitions the groups H and G are discrete, but that there is absolutely no condition on the geometry embodied in $\mathscr{X}$.

Proposition 3.3.90 (Orbi-smooth Oka principle for maps out of relative orbi-singularities)**.** *Let* $\mathbf{H}$ *be a singular-cohesive ∞-topos (Def. 3.3.65). Then for*

- $\orbisingular^{G} \in \mathrm{Snglrt}$ (Ntn. 3.3.57),
- $\orbisingular^{H} \in \mathrm{Snglrt}_{/\orbisingular^{G}}$,
- $\mathscr{X} \in \mathbf{H}_{/\orbisingular^{G}}$,

we have a natural equivalence

$$\int \mathrm{Map}\big(\orbisingular^{H}, \mathscr{X}\big)_{\orbisingular^{G}} \simeq \mathrm{Map}\big(\int \orbisingular^{H}, \int \mathscr{X}\big)_{\orbisingular^{G}} \simeq \mathrm{Map}\big(\orbisingular^{H}, \int \mathscr{X}\big)_{\orbisingular^{G}} \tag{3.218}$$

exhibiting the shape modality (3.117) *as commuting over the slice mapping stack construction (Def. 3.2.65) out of orbi-singularities.*

Proof. For the special case when $\orbisingular^{G} = *$, i.e. $G = 1$ the trivial group, hence the case when $\orbisingular^{H} \in \mathrm{Snglrt}$, $X \in \mathbf{H}$ and $\mathrm{Map}(\orbisingular^{H}, X)$ the ordinary mapping stack (3.72), consider the following sequence of natural equivalences:

$$
\begin{aligned}
\int \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big) & = \int\Big(\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, U\big) \mapsto \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big)\big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big) \\
& \simeq \int\Big(\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, U\big) \mapsto \mathbf{H}\big(U \times \overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \mathscr{X}\big)\Big) && \text{by (3.73)} \\
& \simeq \int\Big(\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, U\big) \mapsto \Big(\mathscr{X}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \times U\big)\Big)\Big) && \text{by (3.55)} \\
& \simeq \int\Big(\Big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, U\Big) \mapsto \Big(\mathscr{X}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big)(U)\Big) && \text{by (3.190)} \\
& \simeq \Big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \mapsto \Big(\int\Big(\mathscr{X}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big)\Big)\Big) && \text{by (3.200)} \\
& \simeq \Big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \mapsto \Big(\big(\int\mathscr{X}\big)\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big)\Big) && \text{by (3.200)} \\
& \simeq \Big(\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, U\big) \mapsto \Big(\big(\int\mathscr{X}\big)\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \times U\big)\Big)\Big) && \text{by (3.190)} \\
& \simeq \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, \int\mathscr{X}\big) && \text{by (3.73)}.
\end{aligned}
\tag{3.219}
$$

The composite of these equivalence yields (3.218) for the special case $\overset{G}{\rotatebox[origin=c]{70}{$\prec$}} = *$.

To obtain from this the statement for general $\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}$, consider the following sequence of natural equivalences:

$$
\begin{aligned}
\int \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, X\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} & = \int\Big(\mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, X\big) \underset{\mathrm{Map}(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}},\, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}})}{\times} *\Big) && \text{by Def. 3.2.65} \\
& \simeq \Big(\int \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, X\big)\Big) \underset{\mathrm{Map}(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}},\, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}})}{\times} \big(\int *\big) && \text{by Prop. 3.3.8 with Lem. 3.3.13} \\
& \simeq \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, \int X\big) \underset{\mathrm{Map}(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}},\, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}})}{\times} * && \text{by (3.219) and (3.123)} \\
& = \mathrm{Map}\big(\overset{H}{\rotatebox[origin=c]{70}{$\prec$}}, \int X\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Def. 3.2.65}.
\end{aligned}
$$

The composite of these equivalences yields the desired (3.218). □

Proposition 3.3.91 (Shape of geometric fixed locus)**.** *The shape of the geometric fixed locus in the sense of* (3.213) *in Def. 3.3.85 is indeed the image under* $\int$ *of the geometric fixed locus* (3.212)

$$
\int \cup \mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \;\simeq\; \mathrm{Spc} \circ \mathrm{Dsc} \circ \mathbf{H}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}\,.
$$

Proof.

$$
\begin{aligned}
\int \cup \mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} & \simeq \cup \int \mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by [SS20-Orb, 3.67]} \\
& \simeq \cup \mathrm{Map}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Prop. 3.3.90} \\
& \simeq \mathrm{Spc} \circ \mathrm{Dsc} \circ \mathbf{H}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}, \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Lem. 3.3.89}.
\end{aligned}
$$

□

Proposition 3.3.92 (Orbi-space incarnation of G-space is orbi-singularization of homotopy quotient)**.**
For $\mathbf{H}$ *a singular-cohesive* ∞*-topos (Def. 3.3.65), consider* $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_{\cup,0})$

(i) *Then there are natural equivalence*

$$
G\mathrm{OrbSpc}\big(\ \mathrm{FxdLoc}(G \circlearrowright X)\big) \;\simeq\; \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \;\in\; \mathbf{H}_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \tag{3.220}
$$

$$
G\mathrm{OrbSpc}\big(\int \mathrm{FxdLoc}(G \circlearrowright X)\big) \;\simeq\; \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \;\in\; (\mathbf{H}_\flat)_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \hookrightarrow \mathbf{H}_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \tag{3.221}
$$

between

(a) *the* G*-orbispace* (3.208) *associated with the image* (3.214) *of* $G \circlearrowright X$ *in cohesive* G*-equivariant homotopy theory, and*

(b) *the orbisingularization of its cohesive homotopy quotient (Ex. 3.3.80).*

(ii) *In particular, the orbi-singularization of the homotopy quotient of a 0-truncated cohesive space is smooth away from G-singularities (Def. 3.3.78):*

$$\cup_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}\begin{pmatrix} \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \\ \downarrow \\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \end{pmatrix} \simeq \begin{pmatrix} \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \\ \downarrow \\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \end{pmatrix} \in \mathbf{H}_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}},$$

$$\cup_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}\begin{pmatrix} \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \\ \downarrow \\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \end{pmatrix} \simeq \begin{pmatrix} \int \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G) \\ \downarrow \\ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \end{pmatrix} \in (\mathbf{H}_\flat)_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}.$$

Proof. We show the proof of (3.221); the proof of (3.220) follows by the exact same steps, just with all occurences of $\int$ removed. For $\overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \in \mathrm{Snglrt}_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}$, we have the following sequence of natural equivalences in $\mathbf{H}_\cup$:

$$\begin{aligned}
\mathrm{Snglr}\mathbf{H}_\cup\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, G\mathrm{OrbSpc}\,(\textstyle\int \mathrm{FxdLoc}(\mathrm{X}))\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} &\simeq G\mathbf{H}_\cup\big(G\mathrm{OrbCncl}(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}), \textstyle\int \mathrm{FxdLoc}(\mathrm{X})\big) && \text{by (3.208)}\\
&\simeq G\mathbf{H}_\cup\big(\overset{\mathrm{im}(K)}{\rotatebox[origin=c]{70}{$\prec$}}, \textstyle\int \mathrm{FxdLoc}(\mathrm{X})\big) && \text{by Lem. 3.3.81}\\
&\simeq \textstyle\int G\mathbf{H}_\cup\big(\overset{\mathrm{im}(K)}{\rotatebox[origin=c]{70}{$\prec$}}, \mathrm{FxdLoc}(\mathrm{X})\big) && \text{by (3.200)}\\
&\simeq \textstyle\int \mathrm{Smth}\,\mathrm{Map}\big(\overset{\mathrm{im}(K)}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(\mathrm{X} /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Prop. 3.2.29 \& Def. 3.3.85}\\
&\simeq \textstyle\int \mathrm{Smth}\,\mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(\mathrm{X} /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Lem. 3.3.93}\\
&\simeq \mathrm{Smth}\textstyle\int \mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(\mathrm{X} /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Prop. 3.3.71}\\
&\simeq \mathrm{Smth}\,\mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \textstyle\int \rotatebox[origin=c]{70}{$\prec$}(\mathrm{X} /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by Prop. 3.3.90}\\
&\simeq \mathrm{Snglr}\mathbf{H}_\cup\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \textstyle\int \rotatebox[origin=c]{70}{$\prec$}(\mathrm{X} /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} && \text{by (3.131)}.
\end{aligned}$$

Therefore, the claim follows by the ∞-Yoneda lemma (Prop. 3.2.29). □

Lemma 3.3.93 (Orbi-singularizations of homotopy G-quotients of 0-truncated objects have only G-singularities)**.** *Given a singular-cohesive ∞-topos* $\mathbf{H}$ *let* $G \in \mathrm{Grp}(\mathrm{FinSet})$ *and* $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_0)$. *Then for* $\overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \in \mathrm{Snglrt}_{/\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}}$ *there is a natural equivalence*

$$\mathrm{Smth}\,\mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}} \simeq \mathrm{Smth}\,\mathrm{Map}\big(\overset{\mathrm{im}(K)}{\rotatebox[origin=c]{70}{$\prec$}}, \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G)\big)_{\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}},$$

where

$$K \twoheadrightarrow \mathrm{im}(K) \hookrightarrow G$$

is the image factorization of the group homomorphism (3.184) *that underlies* $\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}$.

Proof. The assumption that $X \in \mathbf{H}_\cup$ is 0-truncated implies that for $U \in \mathrm{Chrt}$ also $\mathbf{H}(U, X) = X(U) \in \mathrm{Grpd}_0 \hookrightarrow \mathrm{Grpd}_\infty$ is 0-truncated, hence a set. This implies that the homotopy qouotient of this set of U-plots (see [SS20-Orb, Lem. 3.12]) is a 1-groupoid and as such the disjoint union of deloopings of its isotropy groups, which are the stabilizer subgroups $\mathrm{Stab}_G(\phi) \subset G$ of plots $U \xrightarrow{\phi} X$:

$$X(U) /\!\!/ G \simeq \coprod_{[x] \in \pi_0(X(U)/\!\!/G)} B\mathrm{Stab}_G(x) \in \mathrm{Grpd}_1\,. \tag{3.222}$$

With this, we first have the following sequence of natural equivalences for $U \in \mathrm{Chrt}$:

$$
\begin{aligned}
\mathrm{Smth}\,\mathrm{Map}\big(\Upsilon^{K},\, \Upsilon(X /\!\!/ G)\big)_{\Upsilon^{G}}(U) \;&\simeq\; \mathbf{H}\big(U \times \Upsilon^{K},\, \Upsilon(X /\!\!/ G)\big)_{\Upsilon^{G}} && \text{by Lem. 3.2.68}\\
&\simeq\; \mathbf{H}\big((\Upsilon U) \times (\Upsilon BK),\, \Upsilon(X /\!\!/ G)\big)_{\Upsilon^{G}} && \text{by (3.207)}\\
&\simeq\; \mathbf{H}\big(\Upsilon(U \times BK),\, \Upsilon(X /\!\!/ G)\big)_{\Upsilon^{G}} && \text{by (11)}\\
&\simeq\; \mathbf{H}(U \times BK, X /\!\!/ G))_{BG} && \text{by Ex. 3.3.75}\\
&\simeq\; \mathrm{Grpd}_\infty\big(BK\,, X(U) /\!\!/ G)\big)_{BG} && \text{by [SS20-Orb, Lem. 3.12]}\\
&\simeq\; \mathrm{Grpd}\Bigg(BK\,, \coprod_{[\phi]\in \pi_0(X(U)/\!\!/G)} B\mathrm{Stab}_G(x)\Bigg)_{BG} && \text{by (3.222)}\\
&\simeq \coprod_{[\phi]\in \pi_0(X(U)/\!\!/G)} \mathrm{Grpd}\big(BK\,, B\mathrm{Stab}_G(\phi)\big)_{BG} && \text{since } BK \text{ is connected}\\
&\simeq \coprod_{[\phi]\in \pi_0(X(U)/\!\!/G)} \mathrm{Grpd}\big(B\mathrm{im}(\phi)\,, B\mathrm{Stab}_G(x)\big)_{BG} && \text{by Prop. 3.3.61 .}
\end{aligned}
$$

Now, running backwards through this chain of equivalences, but with K replaced by $\mathrm{im}(K)$ throughout, implies the claim, by the ∞-Yoneda lemma (Prop. 3.2.29). □

Base change of proper equivariant homotopy theories.

Lemma 3.3.94 (Base change in equivariant homotopy theory along coverings of the equivariance group)**.**

(i) *A surjective homomorphism of finite groups,* $p : \widehat{G} \twoheadrightarrow G$ (3.100) *induces a reflective subcategory inclusion of orbit categories (Ntn. 3.3.58)*

$$
\begin{tikzcd}
\widehat{G}\mathrm{Orb} \arrow[r, shift left=1ex, "p"] \arrow[r, phantom, "\scriptscriptstyle{(\mathrm{pb})}"] & G\mathrm{Orb}\,. \arrow[l, hook, shift left=1ex]
\end{tikzcd}
$$

(ii) *Moreover, by Kan extension (Lem. 3.2.39) this induces, for* $\mathbf{H}_{\cup}$ *any* ∞*-topos, an adjoint quadruple between the proper equivariant homotopy theories: (Def. 3.3.62) of this form:*

$$
\begin{tikzcd}
\widehat{G}\mathrm{Grpd}_\infty
\arrow[r, shift left=3ex, "p_!"]
\arrow[r, phantom, shift left=2ex, "\perp"]
\arrow[r, phantom, shift left=0ex, "\perp"]
\arrow[r, shift right=1ex, "p_*"']
\arrow[r, phantom, shift right=2ex, "\perp"]
& G\mathrm{Grpd}_\infty\,.
\arrow[l, hook', shift right=1ex, "p^*"']
\arrow[l, hook', shift left=3ex, "p^!"]
\end{tikzcd}
\tag{3.223}
$$

Proof. The second statement follows from the first by Lem. 3.2.43 with Lem. 3.2.42.

A transparent way to see the adjunction in the first statement is to use the identification from Lem. 3.3.60, in terms of which the adjunction is the composite

$$
\begin{tikzcd}
\widehat{G}\mathrm{Orb} \simeq \big((\mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1})_{/B\widehat{G}}\big)_0 \arrow[r, hook] & (\mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1})_{/B\widehat{G}} \arrow[r, shift left=1ex, "(Bp)_!"] \arrow[r, phantom, "\perp"] & (\mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1})_{/BG} \arrow[l, hook', shift left=1ex, "(Bp)^*"] \arrow[r, shift left=1ex, "\tau_0"] \arrow[r, phantom, "\perp"] & \big((\mathrm{Grpd}^{\mathrm{fin}}_{1,\geq 1})_{/BG}\big)_0 \simeq G\mathrm{Orb} \arrow[l, hook', shift left=1ex]
\end{tikzcd}
$$

of the left base change along Bp (Prop. 3.2.50) with the 0-truncation reflection (Prop. 3.2.53), observing that the right adjoint preserves 0-truncation and hence factors.

Under the equivalence of Lem. 3.3.60, the right adjoint is of course given by regarding a G-set as a $\widehat{G}$-set by acting through p. It is immediate to see that this operation is fully faithful when p is surjective. □

Chapter 4

Equivariant principal ∞-bundles

In this last and main chapter we introduce the notion of equivariant principal ∞-bundles, show how these subsume the traditional equivariant bundles (from chapter 2) and use smooth cohesive homotopy theory (from §3.3.1) to prove a new classification theorem for the case of truncated structure groups. Finally we use singular-cohesive homotopy theory (from §3.3.2) to show that the equivariant moduli stacks of equivariant bundles are generally given by the Murayama-Shimakawa construction and to re-formulate the classification theorem in terms of proper-equivariant homotopy theory.

– §4.1: introducing G-equivariant principal ∞-bundles as principal bundles internal to $\mathbf{B}G$-slices of ∞-toposes.
– §4.2: recovering the classical notion of equivariant bundles and proof of the classification theorem.
– §4.3: equivariant moduli stacks and the proper-equivariant formulation of the classification theorem.

Notation 4.0.1 (Singular cohesion with cohesive charts)**.** In all of the following, $\mathbf{H}$ denotes a singular-cohesive ∞-topos (Def. 3.3.65) hence such that the cohesive sub-topos $\mathbf{H}_{\cup} \hookrightarrow \mathbf{H}$ has cohesive 1-charts (Def. 3.3.64), hence a 1-site Chrt of definition, such that $\int U \simeq *$ for all $U \in$ Chrt. For all classification results we consider the case $\mathbf{H}_{\cup} = \mathrm{SmthGrpd}_\infty$ from §3.3.1.

Assumption 4.0.2 (Proper equivariant cohesive homotopy theory)**.** In all of the following we strengthen the Assumption 1.1.2, used in chapter 1, and demand that

– *equivariance groups G are discrete groups (not necessariliy finite)*;

and for all classification results discussed in the following we also assume that

– *domain spaces* X *carry the structure of smooth manifolds*.

The first of these assumptions

$$G \in \mathrm{Grp}(\mathrm{Set}) \xhookrightarrow{\mathrm{Dsc}} \mathrm{Grp}(\mathbf{H}) \tag{4.1}$$

implies (by Rem 3.3.4) that also the delooping (16) of the equivariance group is discrete and hence pure shape:

$$\flat \mathbf{B}G \xrightarrow[\sim]{\epsilon^\flat_{\mathbf{B}G}} \mathbf{B}G \xrightarrow[\sim]{\eta^{\int}_{\mathbf{B}G}} \int \mathbf{B}G \underset{\mathrm{Def\,.4.1.9}}{:=} BG\,,$$

and the same holds (by $\flat \circ \rotatebox[origin=c]{70}{$\prec$} \simeq \rotatebox[origin=c]{70}{$\prec$} \circ \flat$, Prop. 3.3.71) for its orbi-singularization (Ex. 3.3.67):

$$\flat \underset{\rotatebox[origin=c]{70}{$\prec$}}{G} \xrightarrow[\sim]{\epsilon^\flat_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}}} \underset{\rotatebox[origin=c]{70}{$\prec$}}{G} \xrightarrow[\sim]{\eta^{\int}_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}}} \int \underset{\rotatebox[origin=c]{70}{$\prec$}}{G}\,.$$

Remark 4.0.3 (Discrete equivariance ∞-groups in cohesive global homotopy theory)**.** Assumption 4.0.2 (4.1) is used in our proofs (in Thm. 4.1.55 below, just as in Prop. 3.3.90 above) to guarantee, via Prop. 3.3.8, that $\int$ preserves homotopy fibers over $\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}$.

(i) At this point it is worth recalling ([SS20-Orb, Rem. 3.64]) that in *cohesive* global homotopy theory according to Def. 3.3.65 it is not useful to promote the 2-category Snglrt (Ntn. 3.3.57) from discrete to compact Lie groups with

shapes of mapping stacks of their delooping between them, as commonly done in non-cohesive global homotopy theory. The reason is, conceptually, that this secretly introduces a notion of cohesion into the site, which does not properly interplay with the cohesion that is seen inside the cohesive ∞-topos over this site. For example, the crucial relations (3.193) and (3.194) would fail, in general.
(ii) On the other hand, in contrast to traditional equivariant homotopy theory (and aside from the particular classification Theorems 4.2.7 and 4.3.24 below), cohesive global homotopy theory works just as well for *higher* equivariance groups in $\mathrm{Grp}(\mathrm{Grpd}_\infty) \xrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathrm{Grp}(\mathbf{H})$, as long as they are geometrically discrete, hence for plain equivariance ∞-groups. But if G_i are topological 1-groups, then their shapes $\int G_i$ are such geometrically discrete ∞-groups, and there is the canonical comparison morphism (3.128) between their usual hom-∞-groupoids:

$$\int \mathrm{Map}(\mathbf{B}G_1, \mathbf{B}G_2) \xrightarrow{\widetilde{\int \mathrm{ev}}} \mathrm{Map}(\mathbf{B}\int G_1, \mathbf{B}\int G_2)\,. \tag{4.2}$$

(iii) The cohesive global homotopy theory as set up here captures all those aspects of non-discrete equivariance groups that are still reflected on the right of (4.2). For these, no further condition on the topologies is required, for example the G_i could even be non-compact topological groups such as loop groups.

4.1 As bundles internal to G-∞-actions

In higher analogy with the discussion in (2.1) we introduce equivariant principal ∞-bundles as principal bundles internal to ∞-toposes of G-∞-actions (Def. 4.1.22 below). We show that ordinary topological equivariant bundles (according to chapter 2) faithfully embed into this higher geometric theory, and we use this to prove their classification theorem for truncated structure groups (Thm. 4.2.7).

In order to warm up and to establish some preliminaries, we first present an analogous re-proof of the classical Milgram-classification of ordinary topological principal bundles (Thm. 4.1.2, Thm. 4.1.13):

– §4.1.1: Smooth principal ∞-bundles.
– §4.1.2: Smooth equivariant principal ∞-bundles.

4.1.1 Smooth principal ∞-bundles

Principal ∞-bundles. We begin by recalling and developing some facts about plain (i.e. not equivariant) principal ∞-bundles ([NSS12a][NSS12b]) with focus on their incarnation in $\mathrm{SmthGrpd}_\infty$ (Ntn. 3.3.26).

Recovering principal bundles among principal ∞-bundles. We discuss first how ordinary principal bundles embed into the theory of principal ∞-bundles in the sense of Prop. 0.2.1. The following Prop. 4.1.2 is the blueprint for the analogous embedding of equivariant principal bundles into the theory of equivariant principal ∞-bundles which we prove in Thm. 4.2.7 below.

Proposition 4.1.1 (Universal Γ-principal bundle over the Γ-moduli stack)**.** *For a diffeological group*

$$\Gamma \in \mathrm{Grp}\big(\mathrm{DfflgSpc}\big) \hookrightarrow \mathrm{Grp}\big(\mathrm{PSh}(\mathrm{CartSpc})\big)\,,$$

such as a D-topological group or a Lie group
(i) *the simplicial nerve (Ntn. 1.2.24) of the delooping groupoid of* Γ *(Ex. 1.2.7)*

$$\mathbf{B}\Gamma = (\Gamma \rightrightarrows *) \in \mathrm{Grpd}(\mathrm{kTopSpc}) \xhookrightarrow{N\mathrm{Grpd}(\mathrm{Cdfflg})} \Delta\mathrm{PSh}(\mathrm{CartSpc})$$

is fibrant in the local projective model structure of simplicial presheaves over the site of Cartesian spaces (Ntn. 3.3.26), and hence a fibrant representative of the delooping of Γ *in the ∞-topos according to Prop. 0.2.1:*

$$\mathbf{B}\Gamma \in \big(\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\big)_{\mathrm{fib}} \xrightarrow{\mathrm{Loc}_{\Delta}^{\mathrm{LclWEqs}}} \mathrm{SmthGrpd}_\infty\,.$$

(ii) *Moreover, pullback of the universal* Γ*-principal groupoid* (2.59)

$$\mathbf{E}\Gamma = (\Gamma \times \Gamma \rightrightarrows \Gamma) \in \mathrm{Grpd}(\mathrm{kTopSpc}) \xhookrightarrow{N\mathrm{Grpd}(\mathrm{Cdfflg})} \Delta\mathrm{PSh}(\mathrm{CartSpc})$$

along morphisms of simplicial presheaves $X \xrightarrow{c} \mathbf{B}\Gamma$ *represents their homotopy fibers in* $\mathrm{SmthGrpd}_\infty$*:*

$$\begin{tikzcd} X \underset{\mathbf{B}\Gamma}{\times} \mathbf{E}\Gamma \arrow[r] \arrow[d, ->>] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathbf{E}\Gamma \arrow[d] \\ X \arrow[r] & \mathbf{B}\Gamma \end{tikzcd} \in \Delta\mathrm{PSh}(\mathrm{CartSpc}) \quad \Rightarrow \quad \mathrm{Loc}^{\mathrm{LclWEqs}}\big(X \underset{\mathbf{B}\Gamma}{\times} \mathbf{E}\Gamma\big) \simeq \mathrm{Loc}^{\mathrm{LclWEqs}}(X) \underset{\mathbf{B}\Gamma}{\times} * \in \mathrm{SmthGrpd}_\infty\,.$$

Proof. The local fibrancy is Lemma 3.3.28 and the delooping property is Lemma 3.2.72in view of Ex. 3.1.27. The second statement (as well as this delooping property) follows by Lem. 3.2.32 since

$$* \xrightarrow[\in \mathrm{PrjWEqs}]{} \mathbf{E}\Gamma \xrightarrow[\in \mathrm{PrjFib}]{} \mathbf{B}\Gamma$$

is evidently a fibration resolution of the point inclusion in the global projective model structure. □

Theorem 4.1.2 (Topological/smooth principal bundles embed among smooth principal ∞-bundles)**.** *Let*

– $\Gamma \in \mathrm{Grp}(\mathrm{DHausSpc}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Cdfflg})} \mathrm{Grp}(\mathrm{DfflgSpc}) \hookrightarrow \mathrm{Grp}\left(\mathrm{PSh}(\mathrm{CartSpc})\right)$,

or

– $\Gamma \in \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{DfflgSpc}) \hookrightarrow \mathrm{Grp}\left(\mathrm{PSh}(\mathrm{CartSpc})\right)$.

Then the groupoid of traditional Γ*-principal fiber bundles over* X *(Rem. 2.1.1), respectively topological or smooth, is naturally equivalent to the groupoid of* Γ*-principal ∞-bundles over* SmthMfd *internal to* $\mathrm{SmthGrpd}_\infty$*, in the sense of Prop. 0.2.1:*

$$\underset{\text{ordinary topological/smooth-principal bundles}}{\Gamma\mathrm{PrnFibBdl}(\mathrm{DfflgSpc})_X} \simeq \underset{\text{among smooth principal ∞-bundles}}{\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_X} \in \mathrm{Grpd} \hookrightarrow \mathrm{Grpd}_\infty\,. \tag{4.3}$$

Proof. Since X is assumed to be a smooth manifold, it admits a differentiably good open cover ([FStS12, Prop. A.1]), namely an open cover $\widehat{\mathrm{X}} := \underset{i\in I}{\sqcup} \mathrm{U}_i \twoheadrightarrow \mathrm{X}$, such that all non-empty finite intersections of patches are *diffeomorphic* to an open ball, and hence to $\mathbb{R}^{\dim(X)}$:

$$\left\{\mathrm{U}_i \simeq \mathbb{R}^{\dim(X)} \hookrightarrow \mathrm{X}\right\}_{i\in I}, \quad \text{s.t.} \quad \underset{\substack{k\in\mathbb{N} \\ i_0, i_1, \cdots, i_k \in I}}{\forall} \quad \mathrm{U}_{i_0} \cap \mathrm{U}_{i_1} \cap \cdots \cap \mathrm{U}_{i_k} \underset{\mathrm{diff}}{\simeq} \mathbb{R}^{\dim(\mathrm{X})} \quad \text{if non-empty}\,. \tag{4.4}$$

Now we consider three consecutive equivalences of groupoids:

(i) Since the cover is good, Lem. **??** implies that every Γ-principal bundle has a trivialization over the cover which gives a diffeological Čech 1-cocycle with coefficients in Γ:

$$\begin{array}{c} \mathrm{P} \\ \downarrow \\ \mathrm{X} \end{array} \quad \rightsquigarrow \quad \left\{\mathrm{U}_i \cap \mathrm{U}_j \xrightarrow[\mathrm{dfflg}]{\gamma_{ij}} \Gamma\right\}_{i,j\in I} \quad \text{s.t} \quad \underset{i\in I}{\forall}\left(\gamma_{ii} = \mathrm{const}_{\mathrm{e}}\right) \text{ and } \underset{i,j,k\in I}{\forall}\left(\gamma_{ij}\cdot\gamma_{jk} = \gamma_{ik}\right), \tag{4.5}$$

and that morphisms of Γ-principal bundles over X bijectively correspind to diffeological Čech coboundaries between these cocycles:

$$\mathrm{P} \xrightarrow{h} \mathrm{P}' \quad \leftrightarrow \quad \left\{ \mathrm{U}_i \xrightarrow[\mathrm{dfflg}]{h_i} \Gamma \;\middle|\; \underset{i,j\in I}{\forall}\, h_i \cdot \gamma'_{ij} = \gamma_{ij}\cdot h_j \right\}, \tag{4.6}$$

so that this construction constitutes a natural equivalence between the groupoid of principal fiber bundles and the action groupoid of Čech coboundaries acting on Čech cocycles:

$$\Gamma\mathrm{PrnFibBdl}(\mathrm{DfflgSpc})_{\mathrm{X}} \simeq \left(\hat{Z}^1(\widehat{\mathrm{X}};\Gamma) \times \hat{B}^0(\widehat{\mathrm{X}};\Gamma) \rightrightarrows \hat{Z}^1(\widehat{\mathrm{X}};\Gamma)\right). \tag{4.7}$$

(ii) By direct inspection (see Rem. 3.3.43) and using the defining fully-faithful embedding $\mathrm{DfflgSpc} \hookrightarrow \mathrm{PSh}(\mathrm{CartSpc})$, the nerve of this latter groupoid is manifestly isomorphic to the hom-complex (3.1) of simplicial presheaves from the Čech nerve of the open cover

$$N\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \rightrightarrows \widehat{\mathrm{X}}\big) \simeq \widehat{\mathrm{X}}^{\times_{\mathrm{X}}\bullet} \in \Delta\mathrm{PSh}(\mathrm{CartSpc}) \tag{4.8}$$

to the nerve of the delooping groupoid of Γ:

$$N\Big(\hat{Z}^1(\widehat{\mathrm{X}};\Gamma)\times\hat{B}^0(\widehat{\mathrm{X}};\Gamma)\rightrightarrows\hat{Z}^1(\widehat{\mathrm{X}};\Gamma)\Big) \;\simeq\; \Delta\mathrm{PSh}(\mathrm{CartSpc})\big(\widehat{\mathrm{X}}^{\times^\bullet_{\mathrm{X}}}, N(\Gamma\rightrightarrows *)\big)\,. \tag{4.9}$$

(iii) This has the homotopy type of the correct hom-∞-groupoid (Def. 3.1.11)

$$\Delta\mathrm{PSh}(\mathrm{CartSpc})\big(\widehat{\mathrm{X}}^{\times^\bullet_{\mathrm{X}}}, N(\Gamma\rightrightarrows *)\big) \;\simeq\; \mathrm{SmthGrpd}_\infty(\mathrm{X}, \mathbf{B}G) \;\in\; \mathrm{Grpd}_\infty\,, \tag{4.10}$$

because:

- $N(\Gamma\rightrightarrows *)$, which represents (3.8) the delooping (0.2.1) of Γ (by Prop. 3.2.72), is fibrant in $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$, by Prop. 4.1.1,
- $\widehat{\mathrm{X}}^{\times^\bullet_{\mathrm{X}}}$ is a local projective cofibrant resolution of X, by Example 3.3.39.

Therefore, (4.10) follows by Prop. 3.1.12. In conclusion, the composite of the equivalences (4.7), (4.9), and (4.10) yields the desired equivalence (4.3). □

Remark 4.1.3 (Local triviality of principal bundles internal to ∞-topos is implied)**.** Thm. 4.1.2 gives an equivalence to *locally trivial* topological principal bundles (Rem. 2.1.1), even though the Definition 3.2.94 of smooth principal ∞-bundles, as principal bundles internal to the ∞-topos $\mathrm{SmthGrpd}_\infty$, does not explicitly state a local triviality clause. The equivariant generalization of this phenomenon is the content of Thm. 4.2.7 below; see also the discussion at the beginning of §4.2.

Remark 4.1.4 (Universality of the universal principal bundle over the moduli stack)**.**
(i) While the above proof of Thm. 4.1.2 handles principal bundles entirely in terms of their Čech cocycles, it entails a systematic way of reconstructing the actual bundles from these Čech cocycles, as follows: By Prop. 4.1.1 and Prop. 0.2.1, the following pullback of simplicial presheaves

$$\begin{tikzcd}
\mathrm{P} \arrow[d] & \big(\Gamma\times\widehat{\mathrm{X}}\times_{\mathrm{X}}\widehat{\mathrm{X}}\rightrightarrows\Gamma\times\widehat{\mathrm{X}}\big) \arrow[l, two heads, "\in\mathrm{LclWEqs}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\Gamma\times\Gamma\rightrightarrows\Gamma) \arrow[d] & = & \mathbf{E}G \arrow[d] \\
\mathrm{X} & \big(\widehat{\mathrm{X}}\times_{\mathrm{X}}\widehat{\mathrm{X}}\rightrightarrows\widehat{\mathrm{X}}\big) \arrow[l, two heads, "\in\mathrm{LclWEqs}"'] \arrow[r, "c"'] & (\Gamma\rightrightarrows *) & \simeq & \mathbf{B}\Gamma
\end{tikzcd}$$

computes the Γ-principal ∞-bundle classified by a Čech cocycle c, according to Thm. 3.2.97. But inspection of the pullback diagram shows that this bundle $\mathrm{P}\to\mathrm{X}$ is precisely the principal bundle reconstructed from a Čech 1-cocycle in the traditional way.
(ii) Therefore, the *groupoid* $\mathbf{E}\Gamma$ from (2.59) (whose topological realization $EG = |\mathbf{E}\Gamma|$ (2.63) is the universal Γ-principal bundle over the classifying space BG) plays the role of the universal Γ-principal bundle over the *moduli stack* $\mathbf{B}\Gamma$.

Concordance of principal ∞-bundles. The notion of concordance of principal bundles (Def. 2.2.6) has an evident generalization to principal ∞-bundles (and, of course, yet more generally to any contravariant structure over spaces with cylinder objects).

Definition 4.1.5 (Concordance of smooth principal ∞-bundles)**.** Let $\mathscr{G}\in\mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$.
(i) We say that a *concordance* between $P_1, P_2 \in \mathscr{G}\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_X$ is a principal ∞-bundle on the cylinder $X\times\mathbb{R}$

$$\widehat{P} \;\in\; \mathscr{G}\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{X\times\mathbb{R}}$$

such that over the two endpoints it restricts to the two bundles, respectively, up to equivalence:

$$\widehat{P}|_{X\times\{0\}} \;\simeq\; P_0 \quad\text{ and }\quad \widehat{P}|_{X\times\{1\}} \;\simeq\; P_1, \qquad P_1, P_2 \in \mathscr{G}\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_X\,.$$

(ii) We write

$$\big(\mathscr{G}\mathrm{PrnBdl}_X\big)_{/\sim_{\mathrm{conc}}} \;\in\; \mathrm{Set}$$

for the set of concordance classes of $\mathscr{G}$-principal bundles in $\mathrm{SmthGrpd}_\infty$.

The equivalence between isomorphism classes and concordance classes of topological principal bundles (Thm. 2.2.8) still holds after including them among smooth principal ∞-bundles:

Proposition 4.1.6 (Concordant topological principal bundles are isomorphic as smooth ∞-bundles)**.** *Given*

– $\mathrm{X} \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthGrpd}_\infty$,

– $\Gamma \in \mathrm{Grp}(\mathrm{DHausSpc}) \xhookrightarrow{\mathrm{Grp(Cdfflg)}} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$,

the canonical comparison morphism (4.11) *from isomorphism classes to concordance classes (Def. 4.1.5) of* Γ*-principal ∞-bundles over* X *is a bijection*

$$\big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;\sim\;} \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}}.$$

Proof. This is the following composite of natural bijections:

$$\begin{aligned} & \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \\ \simeq\;& \big(\Gamma\mathrm{PrnFibBdl}(\mathrm{DfflgSpc})_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} && \text{by Prop. 4.1.2} \\ \simeq\;& \big(\Gamma\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} && \text{by Prop. 3.3.19} \\ \simeq\;& \big(\Gamma\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}} && \text{by Thm. 2.2.8} \\ \simeq\;& \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}} && \text{by Props. 3.3.19, 4.1.2}\,. \end{aligned}$$

□

Proposition 4.1.7 (Shape of mapping stack into moduli stack is ∞-groupoid of concordances)**.** *Let* $\mathbf{H} = \mathrm{SmthGrpd}_\infty$. *Then, for any* $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ *and* $X \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H}$*, the 0-truncation of the shape of the mapping stack from* X *to the* $\mathscr{G}$*-moduli stack is in natural bijection to the set of concordance classes (Def. 4.1.5) of* $\mathscr{G}$*-principal ∞-bundles on* X*:*

$$\tau_0\,\mathrm{Shp}\,\mathrm{Map}(X, \mathbf{B}\mathscr{G}) \;\simeq\; \big(\mathscr{G}\,\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_X\big)_{/\sim_{\mathrm{conc}}}.$$

Proof. In the following sequence of natural bijections, the key step is the "smooth Oka principle" (Thm. 3.3.51) which allows to take the shape-operation out of the mapping stack construction:

$$\begin{aligned} \tau_0\,\mathrm{Shp}\,\mathrm{Map}(X, \mathbf{B}\mathscr{G}) \;&\simeq\; \tau_0\,\mathrm{Pth}\,\mathrm{Map}(X, \mathbf{B}\mathscr{G}) && \text{by Prop. 3.3.46} \\ &\simeq\; \tau_0 \varinjlim \big(\mathrm{Map}(X, \mathbf{B}\mathscr{G})(\Delta^\bullet_{\mathrm{smth}})\big) && \text{by Def. 3.3.44} \\ &\simeq\; \tau_0 \varinjlim \Big(\mathbf{H}\big(X \times \Delta^\bullet_{\mathrm{smth}}, \mathbf{B}\mathscr{G}\big)\Big) && \text{by Prop. 3.2.55} \\ &\simeq\; \tau_0 \varinjlim \big(\mathscr{G}\,\mathrm{PrnBdl}_{X \times \Delta^\bullet_{\mathrm{smth}}}\big) && \text{by Prop. 0.2.1} \\ &\simeq\; \varinjlim \tau_0 \big(\mathscr{G}\,\mathrm{PrnBdl}_{X \times \Delta^\bullet_{\mathrm{smth}}}\big) && \text{by Prop. 3.2.53} \\ &\simeq\; \big(\tau_0\,(\mathscr{G}\,\mathrm{PrnBdl}_X)\big) \big/ \Big(\tau_0\,\Big(\mathscr{G}\,\mathrm{PrnBdl}_{X \times \Delta^1_{\mathrm{smth}}}\Big)\Big) && \text{by Lem. 3.1.19} \\ &\simeq\; \big(\mathscr{G}\,\mathrm{PrnBdl}_X\big)_{\sim_{\mathrm{conc}}} && \text{by Def. 4.1.5.} \end{aligned}$$

□

Remark 4.1.8 (The ∞-groupoid of concordances)**.** Without applying 0-truncation, the argument in Prop. 4.1.7 shows that the full shape of the mapping stack into a moduli stack is the ∞-groupoid of concordances and higher order concordances between principal ∞-bundles. Here the points-to-pieces transformation $\flat \to \int$ (3.121) on these mapping stacks regards higher morphisms of principal ∞-bundles inside higher concordances:

$$\mathscr{G}\,\mathrm{PrnBdl}_X \;\simeq \mathbf{H}(X, \mathbf{B}\mathscr{G}) \;\simeq\; \underbrace{\flat\,\mathrm{Map}(X, \mathbf{B}\mathscr{G})}^{\substack{\infty\text{-groupoid of morphisms of} \\ \text{principal } \mathscr{G}\text{-bundles on } X}} \xrightarrow{\;\eta^{\int} \circ\, \varepsilon^{\flat}\;} \overbrace{\textstyle\int \mathrm{Map}(X, \mathbf{B}\mathscr{G})}^{\substack{\infty\text{-groupoid of concordances of} \\ \text{principal } \mathscr{G}\text{-bundles on } X}} \xrightarrow{\;\tau_0\;} \big(\mathscr{G}\,\mathrm{PrnBdl}_X\big)_{/\sim_{\mathrm{conc}}}. \tag{4.11}$$

Classification of principal ∞-bundles. We demonstrate the existence of classifying spaces for concordance classes of principal bundles over smooth manifolds in the full generality of principal ∞-bundles internal to $\mathrm{SmthGrpd}_\infty$.

Definition 4.1.9 (Classifying shapes for smooth principal ∞-bundles)**.** Let $\mathbf{H}$ a cohesive ∞-topos (Def. 3.3.1). Then for $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ we write (recalling (3.125))

$$B\mathscr{G} := \int(\mathbf{B}\mathscr{G}) \quad \in \mathbf{H}_\flat \hookrightarrow \mathbf{H} \tag{4.12}$$

for the shape (3.117) of the universal moduli stack (according to Prop. 0.2.1) of $\mathscr{G}$-principal bundles.

Example 4.1.10 (Classifying shapes for discrete structure ∞-groups)**.** If $G \in \mathrm{Grp}(\mathrm{Grpd}_\infty) \xhookrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathrm{Grp}(\mathbf{H})$ so that $\flat G \simeq G \simeq \int G$ then the object BG from Def. 4.1.9 coincides with the delooping $\mathbf{B}G$:

$$\flat\mathbf{B}G \simeq \mathbf{B}G \simeq \int\mathbf{B}G = BG \simeq \flat BG, \tag{4.13}$$

by (3.125) and (3.126). In particular this means that for all $\Gamma \in \mathrm{Grp}(\mathbf{H})$

$$B\Gamma \simeq B\textstyle\int\Gamma.$$

For instance if $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathbf{H})$ and $n \in \mathbb{N}$, then

$$BG \simeq K(G,1) \quad \in \mathrm{Grpd} \xhookrightarrow{\mathrm{Dsc}} \mathbf{H}$$

is the homotopy type of a traditional Eilenberg-MacLane space, regarded among cohesive homotopy types.

Proposition 4.1.11 (Milgram classifying spaces models classifying shape)**.**
If $\Gamma \in \mathrm{Grp}(\mathrm{DTopSpc}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Cdfflg})} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$ *is well-pointed (Ntn. 1.3.17), then the homotopy type of its Milgram classifying space* $B\Gamma$ (2.65) *coincides with the classifying shape* $B(\mathrm{Cdfflg}(\Gamma))$ (4.12)*:*

$$\mathrm{Shp}(\mathrm{Cdfflg}(B\Gamma)) \simeq B(\mathrm{Cdfflg}(\Gamma)) \quad \in \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \mathrm{SmthGrpd}_\infty .$$

Proof.

$$\begin{aligned}
B(\mathrm{Cdfflg}(\Gamma)) &= \mathrm{Shp}(\mathbf{B}(\mathrm{Cdfflg}(\Gamma))) && \text{by Def. 4.1.9}\\
&\simeq \mathrm{Shp}\Big(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Cdfflg}(\Gamma)^{\times_n}\Big) && \text{by (3.88)}\\
&\simeq \varinjlim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Shp}(\mathrm{Cdfflg}(\Gamma)^{\times_n}) && \text{by (11)}\\
&\simeq \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (\mathrm{Shp}(\mathrm{Cdfflg}(\Gamma)))^{\times_n} && \text{by Rem. 3.3.4}\\
&\simeq \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (\mathrm{Pth}(\mathrm{Cdfflg}(\Gamma)))^{\times_n} && \text{by Prop. 3.3.46}\\
&\simeq \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (\mathrm{Pth}(\Gamma))^{\times_n} && \text{by Prop. 3.3.23}\\
&\simeq \mathrm{Pth}\,|\Gamma^{\times_\bullet}| && \text{by (1.98)}\\
&\simeq \mathrm{Pth}(B\Gamma) && \text{by (2.65)}\\
&\simeq \mathrm{Pth}\,\mathrm{Cdfflg}(B\Gamma) && \text{by (3.3.23)}\\
&\simeq \mathrm{Shp}\,\mathrm{Cdfflg}(B\Gamma) && \text{by Prop. 3.3.46.}
\end{aligned}$$

□

Theorem 4.1.12 (Concordance classification of smooth principal ∞-bundles)**.**
For any $\mathscr{G} \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$, the space $B\mathscr{G}$ (Def. 4.1.9) the concordance classes (Def. 4.1.5) of $\mathscr{G}$-principal ∞-bundles over any $\mathrm{X} \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthGrpd}_\infty$ in that there is a natural bijection

$$\begin{aligned}(\mathscr{G}\,\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}})_{/\sim_{\mathrm{conc}}} &= \tau_0 \textstyle\int \mathrm{Map}(\mathrm{X}, \mathbf{B}\mathscr{G}) \\ &\simeq \tau_0\, \mathrm{Map}\big(\textstyle\int \mathrm{X}, B\mathscr{G}\big) \\ &= H^1(X; \textstyle\int \mathscr{G}) \quad \in \mathrm{Set}.\end{aligned}$$

Proof.

$$\begin{aligned}\tau_0\, \mathbf{H}(X, B\mathscr{G})) &\simeq \tau_0\, \mathrm{Pnt}\,\mathrm{Map}(X, B\mathscr{G}) && \text{by Lem. 3.3.7} \\ &\simeq \tau_0\, \mathrm{Pnt}\,\mathrm{Map}(X, \textstyle\int \mathbf{B}\mathscr{G}) && \text{by Def. 4.1.9} \\ &\simeq \tau_0\, \mathrm{Pnt} \textstyle\int \mathrm{Map}(X, \mathbf{B}\mathscr{G}) && \text{by Thm. 3.3.51} \\ &\simeq \tau_0\, \mathrm{Shp}\,\mathrm{Map}(X, \mathbf{B}\mathscr{G}) && \text{by (3.118)} \\ &\simeq (\mathscr{G}\,\mathrm{PrnBdl}_X)_{/\sim_{\mathrm{conc}}} && \text{by Prop. 4.1.7.}\end{aligned}$$

□

Theorem 4.1.13 (Isomorphism classification of topological principal bundles)**.** *In $\mathbf{H} = \mathrm{SmthGrpd}_\infty$, consider*

– $\Gamma \in \mathrm{Grp}(\mathrm{DHausSpc}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Cdfflg})} \mathrm{Grp}(\mathbf{H})$ *being well-pointed (Ntn. 1.3.17);*
– $X \in \mathrm{SmthMfd} \xhookrightarrow{\mathrm{Cdfflg}} \mathbf{H}$.

Then there is a natural bijection

$$\big(\Gamma\,\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_X\big)_{/\sim_{\mathrm{iso}}} \simeq \tau_0\, \mathrm{Map}(\textstyle\int X, B\Gamma)$$

between isomorphism classes of Γ-principal fiber bundles over X (Ntn. 2.2.2) and homotopy classes of maps from (the shape of) X to the classifying shape of Γ (Def. 4.1.9, Prop. 4.1.11).

Proof.

$$\begin{aligned}\big(\Gamma\,\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_X\big)_{/\sim_{\mathrm{iso}}} &\simeq \big(\Gamma\,\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_X\big)_{/\sim_{\mathrm{conc}}} && \text{by Thm. 2.2.8} \\ &\simeq \big(\Gamma\,\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_X\big)_{/\sim_{\mathrm{conc}}} && \text{by Prop. 4.1.2} \\ &\simeq \tau_0\, \mathrm{Map}(\textstyle\int \mathrm{X}, B\Gamma) && \text{by Prop. 4.1.12.}\end{aligned}$$

□

Remark 4.1.14 (Recovering the classification theorem for topological principal bundles via cohesion)**.**
(i) The statement of Thm. 4.1.13 reproduces the time-honored classification result for topological principal bundles for Milgram- ([Mil67], review in [RS17, Thm. 3.5.1]) for the special case when the domain is a smooth manifold.

(ii) While the statement of Thm. 4.1.13 is classical, the new proof via cohesive homotopy theory has the advantage that it generalizes to equivariant principal bundles, where it provides a new classification result for the case of truncated structure groups (Thm. 4.1.56 below).

This is what we turn to next.

4.1.2 Smooth equivariant principal ∞-bundles

The general theory of principal ∞-bundles in any ∞-topos, discussed in §4.1.1 *immediately* generalizes to the equivariant case, by internalizing it into the corresponding ∞-category of G-∞-actions, in direct analogy with the discussion in §2.1, and using that G-∞-actions in any ∞-topos form themselves again an ∞-topos. Moreover, the classifying theory of smooth principal ∞-bundles (from Prop. 4.1.12) generalizes to a general classification result for smooth equivariant principal ∞-bundles, by the "singular-smooth Oka principle".

Equivariant ∞-groups. We generalize (Def. 4.1.16 below) the notion of G-equivariant topological groups (Def. 2.1.2) to ∞-group stacks in ∞-toposes, and (in higher analogy with Lem. 2.1.4) identify them with semidirect product ∞-groups (Prop. 4.1.19). We currently fail to show that all ∞-groups internal to G-∞-actions are of this form (Rem. 4.1.20).

Recall from [SS20-Orb, Def. 2.101] that for $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ (Def. 4.1.16) we have the group-automorphism group $\mathrm{Aut}_{\mathrm{Grp}}(\mathbf{H})$ equipped with canonical actions on $\mathscr{G}$ and on $\mathbf{B}\mathscr{G}$ preserving their basepoints:

$$
\underset{\text{group-automorphism group}}{\mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G}) \;:=\; \mathrm{Aut}_{*}(\mathbf{B}\mathscr{G})}
\qquad
\underset{\text{pointed action on delooping}}{
\begin{tikzcd}
\ast \arrow[r] \arrow[d, "\mathrm{pt}_{\mathbf{B}G}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G}) \arrow[d, "(\mathrm{pt}_{\mathbf{B}\mathscr{G}}) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G})"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\mathbf{B}\mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\mathbf{B}\mathscr{G}) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G}) \arrow[d] \\
\ast \arrow[r] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G})
\end{tikzcd}}
\qquad
\underset{\text{canonical action on group}}{
\begin{tikzcd}
\ast \arrow[r] \arrow[d, "e"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G}) \arrow[d, "e /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G})"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Gamma /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G}) \arrow[d] \\
\ast \arrow[r] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\mathscr{G})
\end{tikzcd}}
\tag{4.14}
$$

Proposition 4.1.15 (Quotient groups by group automorphisms [SS20-Orb, Prop. 2.102]). *Consider* $G, \Gamma \in \mathrm{Grp}(\mathbf{H})$ (16) *and an action of* G *on* Γ (18) *by group automorphisms, i.e., a pasting diagram in* $\mathbf{H}$ *of this form:*

$$
\begin{tikzcd}
\Gamma \arrow[r, "G\text{-action on }\Gamma\text{ (18)}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Gamma /\!\!/ G \arrow[r, "\text{by group automorphisms (4.14)}"] \arrow[d, "\rho"] \arrow[dr, phantom, "\text{(pb)}"] & \Gamma /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(\Gamma) \arrow[d, "\rho_{\mathrm{Aut}}"] \\
\ast \arrow[r, "\mathrm{pt}_{\mathbf{B}G}"'] & \mathbf{B}G \arrow[r, "\vdash \rho"'] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\Gamma)
\end{tikzcd}
\tag{4.15}
$$

Then there is group structure on the homotopy quotient in the slice

$$
\Gamma /\!\!/ G \;\in\; \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}G})
\tag{4.16}
$$

such that

$$
\mathbf{B}\,(\Gamma /\!\!/ G) \;\simeq\; (\mathbf{B}\Gamma) /\!\!/ G \quad \in \; \mathbf{H}_{/\mathbf{B}G}\,.
\tag{4.17}
$$

Definition 4.1.16 (Equivariant ∞-groups). We write

$$
G\mathrm{EquvGrp}(\mathbf{H}) \hookrightarrow \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}G}) \;\simeq\; \mathrm{Grp}\,(G\,\mathrm{Act}(\mathbf{H}))
\tag{4.18}
$$

for the ∞-category of group objects in the slice over $\mathbf{B}G$ that arise via Prop. 4.1.15 from group objects equipped with G-actions by group automorphisms.

Example 4.1.17 (Equivariant ∞-group for trivial G-action). A G-action on Γ is trivial precisely if $\Gamma /\!\!/ G \simeq \Gamma \times \mathbf{B}G$, which by (4.17) is the case (using Ex. 3.1.18) precisely if

$$
\mathbf{B}(\Gamma /\!\!/ G) \;\;\simeq\;\; (\mathbf{B}\Gamma) \times (\mathbf{B}G) \quad \in \mathbf{H}_{/\mathbf{B}G}\,.
\tag{4.19}
$$

Definition 4.1.18 (Split ∞-group extensions). Given $G \in \mathrm{Grp}(\mathbf{H})$,
(i) a *split group extension* of G is a diagram of the form

$$
\begin{tikzcd}
 & G \arrow[d, "i"'] \arrow[dr, equal] & \\
\Gamma \arrow[r, "\mathrm{fib}(p)"] & \Gamma \rtimes G \arrow[r, "p"] & G
\end{tikzcd}
\quad \in \; G\mathrm{EquvGrp}(\mathbf{H})\,,
\tag{4.20}
$$

where the horizontal sequence is a fiber sequence.
(ii) We write

$$
\mathrm{Splt}\,G\,\mathrm{Extnss} \hookrightarrow \mathrm{Grp}(\mathbf{H})^{G/}_{/G}
$$

for the full subcategory on the split extensions.

Proposition 4.1.19 (Equivariant groups are equivalent to split extensions of equivariance group)**.** *The operation of delooping of split extensions $\Gamma \xrightarrow{i} \Gamma \rtimes G \xrightarrow{p}$ (Def. 4.1.18). exhibits their equivalence with G-equivariant groups (Def. 4.1.16):*

$$\mathrm{Splt}\,G\mathrm{Extnss}(\mathbf{H}) \xrightarrow{\ \sim\ } G\mathrm{EquvGrp}(\mathbf{H}),$$

such that (4.17) *is further identified with*

$$(\mathbf{B}\Gamma) /\!\!/ G \simeq \mathbf{B}(\Gamma \rtimes G)\,. \tag{4.21}$$

Proof. First, to see that we have a functor, consider for any split extension the following pasting diagram:

$$\begin{tikzcd} * \arrow[rr, "\mathrm{pt}_{\mathbf{B}G}"] \arrow[d, "\mathrm{pt}_{\mathbf{B}\Gamma}"'] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}G \arrow[d, "\mathbf{B}i"] \arrow[dd, bend left=60, "\mathrm{id}"] \\ \mathbf{B}\Gamma \arrow[rr, "\mathbf{B}\mathrm{fib}(p)"] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}(\Gamma \rtimes G) \arrow[d, "\mathbf{B}p"] \\ * \arrow[rr, "\mathrm{pt}_{\mathbf{B}G}"'] & & \mathbf{B}G \end{tikzcd} \tag{4.22}$$

Here the bottom Cartesian square is the delooping of the given fiber sequence (4.20) (using (16) in Prop. 0.2.1), from which the top Cartesian square follows by the pasting law (5) and using that the pullback of an equivalence is an equivalence. Hence the bottom square exhibits a G-action on $\mathbf{B}\Gamma$ (by (18) in Prop. 0.2.1) with homotopy quotient as claimed in (4.21); and the top square exhibits $\mathrm{pt}_{\mathbf{B}\Gamma}$ as a homotopy fixed point of this action ([SS20-Orb, Def. 2.97]), and hence a G-action on Γ by group automorphisms ([SS20-Orb, Def. 2.101]).

Similarly, n-morphism of split extensions are n-morphisms of the diagrams (4.22) which fix the top and bottom copies of $\mathbf{B}G$ and hence the top and bottom copies of the point. These are equivalently the pointed n-morphisms between G actions on the $\mathbf{B}\Gamma$ and hence are equivalently n-morphisms between the G-actions by group automorphisms on Γ. This means that we have a fully faithful functor.

It just remains to see that this is essentially surjective. Hence for a given G-action on Γ by group automorphisms consider the following pasting diagram:

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}G \arrow[r, "\vdash\rho"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\Gamma) \arrow[d, "(\mathrm{pt}_{\mathbf{B}\Gamma})/\!\!/\mathrm{Aut}_{\mathrm{Grp}}(\Gamma)"] \arrow[dd, bend left=60, "\mathrm{id}"] \\ \mathbf{B}\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\mathbf{B}\Gamma)/\!\!/G \arrow[r] \arrow[d, "\rho"] \arrow[dr, phantom, "\text{(pb)}"] & (\mathbf{B}\Gamma)/\!\!/\mathrm{Aut}_{\mathrm{Grp}}(\Gamma) \arrow[d, "\rho_{\mathrm{Aut}}"] \\ * \arrow[r, "\mathrm{pt}_{\mathbf{B}G}"'] & \mathbf{B}G \arrow[r, "\vdash\rho"'] & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(\Gamma) \end{tikzcd}$$

Here the bottom square is that which defines the given automorphism action (4.15) and all the other squares follow with the pasting law (5) from (4.14). Now the two squares on the left exhibit the delooping of a split group extension since $(\mathbf{B}\Gamma)/\!\!/G$ is connected (by Lemma 3.2.81), and hence of the form (4.21). □

Remark 4.1.20 (Are all groups in G-Actions equivalent to semidirect product groups?)**.** One might expect that any group object in the slice $\Gamma/\!\!/G \in \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}G})$ is an equivariant group in the sense of Def. 4.1.16, hence that the full inclusion (4.18) is in fact essentially surjective and hence an equivalence.

(i) We currently do not have a proof that this is the case, but the following pasting diagram shows something close, at least:

$$\begin{tikzcd} & G \arrow[r] \arrow[d, "(\mathrm{e},\mathrm{id})"'] \arrow[dr, phantom, "\text{(d)}"] & * \arrow[d] & \\ G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(d)}"] & \Gamma \times G \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\text{(c)}"] & \Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(b)}"] & * \arrow[d] \\ * \arrow[r] & \Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(b)}"] & \Gamma/\!\!/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(a)}"] & \mathbf{B}G \simeq \Gamma/\!\!/(\Gamma \rtimes G) \arrow[d] \\ & * \arrow[r] & \mathbf{B}G \arrow[r] & \sum_{\mathbf{B}G} \mathbf{B}(\Gamma/\!\!/G) \simeq \mathbf{B}(\Gamma \rtimes G)\,. \end{tikzcd}$$

Here:
(a) is the homotopy pullback that exhibits $\Gamma /\!\!/ G$ as a group object in the slice over $\mathbf{B}G$;
(b) is the homotopy pullback that exhibits $\Gamma /\!\!/ G$ as the homotopy quotient of a G-action on Γ;
(c) is the homotopy fiber product which exhibits the shear map equivalence of Γ as a principal G-bundle over $\Gamma /\!\!/ G$;
(d) is a homotopy pullback implied from this by the pasting law (5).
We see from this that $\Gamma \times G$ carries a group structure, to be denoted $\Gamma \rtimes G$, whose delooping is the bottom right object:

$$\sum_{\mathbf{B}G} \mathbf{B}(\Gamma /\!\!/ G) \;\simeq\; \mathbf{B}(\Gamma \rtimes G)\,. \tag{4.23}$$

Moreover, $G \xrightarrow{(\mathrm{e},\mathrm{id})} \Gamma \times G$ is exhibited as a homomorphism of group objects. Also, we see that $G \xrightarrow{(\mathrm{e},\mathrm{id})} \Gamma \times G \xrightarrow{\rho} \Gamma$ are G-equivariant maps for G acting by right multiplication on itself. This implies that ρ is the given G-action.
(ii) In conclusion, this shows almost all the structure required of a split extension according to Prop. 4.1.19, except that we have yet to conclude group structure on Γ itself, such that the morphism $\Gamma \to \Gamma /\!\!/ G \simeq \Gamma \rtimes G$ is in fact a *homo*morphism.

Lemma 4.1.21 (Homotopy quotients by equivariant actions of equivariant groups)**.** *Given*
- $\Gamma /\!\!/ G \,\in\, G\mathrm{EquvGrp}(\mathbf{H}) \hookrightarrow \mathrm{Grp}\big(\mathbf{H}_{/\mathbf{B}G}\big)$ *(Def. 4.1.16)*
- $(\Gamma /\!\!/ G) \circlearrowright (A /\!\!/ G) \,\in\, (\Gamma /\!\!/ G)\,\mathrm{Act}\big(\mathbf{H}_{/\mathbf{B}G}\big)$

then the underlying object of the homotopy quotient of this equivariant action formed in $\mathbf{H}_{/\mathbf{B}G} \simeq G\mathrm{Act}\mathbf{H}$ is equivalent to the homotopy quotient of A by $\Gamma \rtimes G$ in $\mathbf{H}$, as exhibited by the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & A /\!\!/ G \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & A /\!\!/ (\Gamma \rtimes G) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \big(A /\!\!/ G\big) /\!\!/ \big(\Gamma /\!\!/ G\big) \arrow[d] \\
\ast \arrow[r] & \mathbf{B}G \arrow[r] & \mathbf{B}(\Gamma \rtimes G) \arrow[r, "\sim", "\text{Prop. 4.1.19}"'] \arrow[dr] & \mathbf{B}\Gamma /\!\!/ G . \arrow[d] \\
 & & & \mathbf{B}G
\end{tikzcd} \tag{4.24}$$

Proof. This uses Prop. 4.1.19 for the identification in the bottom right and then Prop. 3.2.75 with Prop. 3.2.48, and the pasting law (5) to identify the pullbacks. ☐

Equivariant principal ∞-bundles. In view of Prop. 0.2.1 and in direct analogy with Def. 2.1.3, we consider the following evident definition of equivariant principal ∞-bundles in any ∞-topos:

Definition 4.1.22 (Equivariant principal ∞-bundles)**.** For
– $G \in \mathrm{Grp}(\mathbf{H})$,
– $\Gamma /\!\!/ G \,\in\, G\mathrm{EquvGrp}(\mathbf{H})$ (Def. 4.1.16),

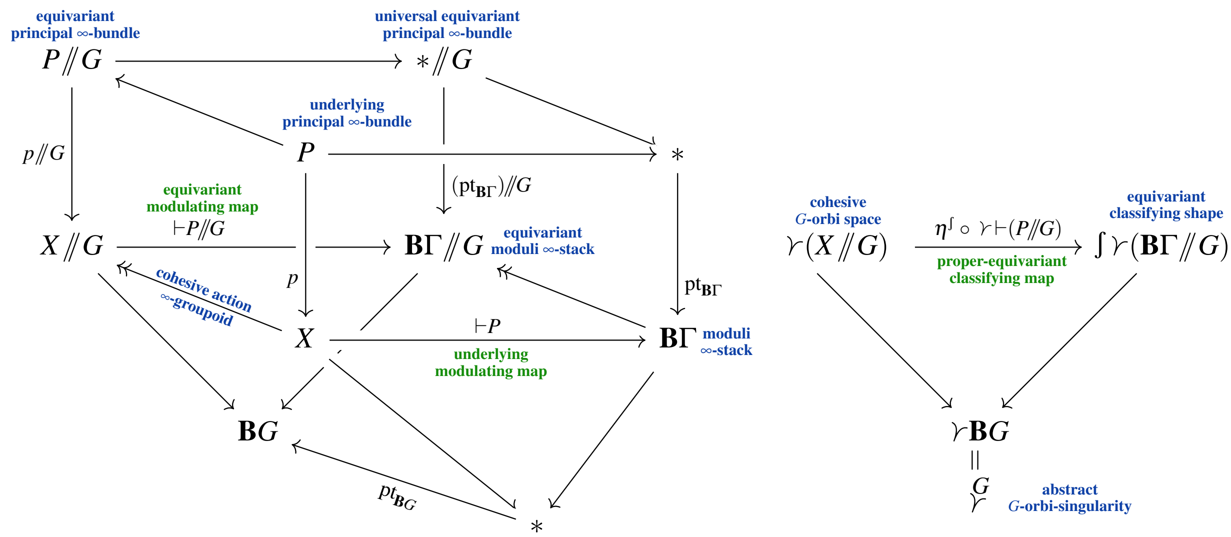
the *G-equivariant Γ-principal bundles* on any $G \circlearrowright X \,\in\, G\mathrm{Act}(\mathbf{H}) \simeq \mathbf{H}_{/\mathbf{B}G}$ are the $\mathscr{G}$-principal bundles (18) for $\mathscr{G} := \Gamma /\!\!/ G$ from (4.16), hence the objects of the following equivalent ∞-groupoids:

$$\begin{array}{rcll}
G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & := & (\Gamma /\!\!/ G)\,\mathrm{PrnBdl}(\mathbf{H}_{/\mathbf{B}G})_{(X/\!\!/G)} & \text{internalization into slice} \\
 & \simeq & \mathbf{H}\big(X /\!\!/ G, \mathbf{B}(\Gamma /\!\!/ G)\big)_{\mathbf{B}G} & \text{by Prop. 0.2.1} \\
 & \simeq & G\mathrm{Act}(\mathbf{H})\,\big(G \circlearrowright X,\, G \circlearrowright (\mathbf{B}\Gamma)\big) & \text{by Prop. 0.2.1}\,.
\end{array} \tag{4.25}$$

Remark 4.1.23 (Equivariant principal ∞-bundles and their underlying principal ∞-bundles)**.** Let $\Gamma \in G\mathrm{Grp}(\mathbf{H})$. The base change (3.68) along $\ast \xrightarrow{\mathrm{pt}_{\mathbf{B}G}} \mathbf{B}G$ exhibits plain Γ-principal bundles (18) underlying (Rem. 3.2.80) G-equivariant Γ-principal bundles on any $G \circlearrowright X$ (Def. 4.1.22):

$$\begin{array}{ccc}
G\mathrm{Equiv}\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & \xrightarrow{\quad\mathrm{undrlng}\quad} & \Gamma\mathrm{PrnBdl}(\mathbf{H})_X \\
\wr\!\mid & & \wr\!\mid \\
\mathbf{H}_{/\mathbf{B}G}\,(X /\!\!/ G,\, (\mathbf{B}\Gamma) /\!\!/ G) & \xrightarrow{\quad(\mathrm{pt}_{\mathbf{B}G})^\ast = \mathrm{fib}\quad} & \mathbf{H}(X, \mathbf{B}\Gamma) \\
(\vdash P /\!\!/ G) \,\simeq\, (\vdash P) /\!\!/ G & \longmapsto & \vdash P
\end{array} \tag{4.26}$$

This situation is shown in the following diagram on the left, all whose squares are Cartesian:



The diagram on the right shows the corresponding classifying map to the proper-equivariant classifying shape, discussed in §4.3 below (Def. 4.3.1).

Example 4.1.24 (Semidirect product-group homotopy quotient of underlying equivariant principal ∞-bundles)**.** Let $G \circlearrowright \Gamma \in G\mathrm{EquvGrp}(\mathbf{H})$ (Def. 4.1.16) with semidirect product ∞-group $\Gamma \rtimes G$ (4.21) then the total space P of the underlying Γ-principal bundle P (4.26) of a G-equivariant Γ-principal bundle satisfies

$$
\begin{tikzcd}
P /\!\!/ (\Gamma \rtimes G) \arrow[rr, "\sim"] \arrow[dr] & & X /\!\!/ G \arrow[dl, "\vdash P /\!\!/ G"] \\
& \mathbf{B}(\Gamma \rtimes G) \arrow[d, "\mathbf{B}\mathrm{pr}_2"] & \\
& \mathbf{B}G\,. &
\end{tikzcd}
\tag{4.27}
$$

This follows via Prop. 0.2.1 by applying the pasting law (5) to this diagram:

$$
\begin{tikzcd}
P \arrow[rr] \arrow[d] \arrow[drr, phantom, "\scriptstyle (\mathrm{pb})"] & & * \arrow[d] \\
P /\!\!/ G \arrow[rr] \arrow[d] \arrow[drr, phantom, "\scriptstyle (\mathrm{pb})"] & & \mathbf{B}G \arrow[d] \\
X /\!\!/ G \arrow[rr, "\vdash P /\!\!/ G"] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G\,. &
\end{tikzcd}
$$

In particular, (4.27) implies a natural identification of the fixed loci of G-equivariant Γ-principal bundles for (sub-)group $\eta \to \Gamma \rtimes G$ with the local sections of its modulating morphism:

$$
\begin{aligned}
P^{\eta} & \simeq \mathrm{Map}\big(\mathbf{B}\eta, P /\!\!/ (\Gamma \rtimes G)\big)_{\mathbf{B}(\Gamma \rtimes G)} && (3.96) \\
& \simeq \mathrm{Map}\big(\mathbf{B}\eta, X /\!\!/ G\big)_{\mathbf{B}(\Gamma \rtimes G)} && \text{by (4.1.24).}
\end{aligned}
\tag{4.28}
$$

Proposition 4.1.25 (Equivariant principal ∞-bundles for trivial action on structure ∞-groups)**.** *When the G-action on the structure ∞-group Γ is trivial, so that $(\mathbf{B}\Gamma) /\!\!/ G \simeq (\mathbf{B}\Gamma) \times (\mathbf{B}G)$ (Ex. 4.1.17), then G-equivariant Γ-principal ∞-bundles (Def. 4.1.22) are equivalently:*

(i) *Γ-principal bundles on the G-quotient stack:*

$$\begin{array}{ccc} G\,\mathrm{Equiv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & \xrightarrow{\quad\sim\quad} & \Gamma\mathrm{PrnBdl}(\mathbf{H})_{X/\!\!/G} \\ (G \circlearrowright X \xrightarrow{c} \mathbf{B}\Gamma) & \longmapsto & (X /\!\!/ G \xrightarrow{c/\!\!/G} \mathbf{B}\Gamma) \end{array} \tag{4.29}$$

(ii) *G-Actions on Γ-principal bundles*

$$\begin{tikzcd} G\,\mathrm{Equiv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X \arrow[rr, "\sim"] \arrow[dr, "\underset{(4.26)}{\mathrm{undrlng}}"'] && G\,\mathrm{Act}\big(\Gamma\mathrm{PrnBdl}(\mathbf{H})\big)_{G \circlearrowright X} \arrow[dl, "\underset{(3.2.80)}{\mathrm{undrlng}}"] \\ & \Gamma\mathrm{PrnBdl}(\mathbf{H})_X\,, & \end{tikzcd} \tag{4.30}$$

where on the right we consider G base-changed (3.69) *to a group object in Γ-principal bundles* (18)*:*

$$G \,\in\, \mathrm{Grp}(\mathbf{H}) \xrightarrow{\mathrm{Grp}((-)\times\mathbf{B}\mathscr{G})} \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}\mathscr{G}}) \xrightarrow{\simeq} \mathrm{Grp}\big(\mathscr{G}\,\mathrm{PrnBdl}(\mathbf{H})\big)\,.$$

Proof. The first equivalence in the composite of the following sequence of natural equivalences:

$$\begin{array}{rll} G\,\mathrm{Equiv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & \simeq \mathbf{H}_{/\mathbf{B}G}\big(X/\!\!/G,\, \mathbf{B}(\Gamma/\!\!/G)\big) & \text{by (4.25)} \\ & \simeq \mathbf{H}_{/\mathbf{B}G}\big(X/\!\!/G,\, (\mathbf{B}\Gamma)/\!\!/G\big) & \text{by (4.17)} \\ & \simeq \mathbf{H}_{/\mathbf{B}G}\big(X/\!\!/G,\, (\mathbf{B}\Gamma)\times(\mathbf{B}G)\big) & \text{by (4.19)} \\ & \simeq \mathbf{H}\big(X/\!\!/G,\, \mathbf{B}\Gamma\big) & \text{by (3.69)} \\ & \simeq \Gamma\mathrm{PrnBdl}(\mathbf{H})_{X/\!\!/G} & \text{by (18)}\,. \end{array}$$

This already implies also the second equivalence, since now both sides are identified with maps of the form $X/\!\!/G \xrightarrow{c/\!\!/G} \mathbf{B}\Gamma$. It just remains to see that the underlying objects coincide, in that the diagram in (4.30) commutes. To find the underlying object of a G-action in Γ-principal bundles, we need to compute (by (3.98)) the base change

$$\big(\mathbf{H}_{/\mathbf{B}\Gamma}\big)_{/(\mathbf{B}\Gamma)\times(\mathbf{B}G)} \xrightarrow{\mathrm{pt}^*_{(\mathbf{B}\Gamma)\times(\mathbf{B}G)}} \mathbf{H}_{/\mathbf{B}\Gamma}\,.$$

Since fiber products in a slice are computed in the underlying ∞-topos (e.g. [SS20-Orb, Prop. 2.53]) we are thus reduced to checking the pullback square as shown on the top of the following diagram on the left:

$$\begin{tikzcd} X \arrow[rr] \arrow[d, "c"'] \arrow[drr, phantom, "\text{(pb)}"] && X/\!\!/G \arrow[d, "(c/\!\!/G,\,\rho_G)"] \\ \mathbf{B}\Gamma \arrow[rr, "(\mathrm{id}_{\mathbf{B}\Gamma},\,\mathrm{pt}_{\mathbf{B}G})"] \arrow[dr, "\mathrm{id}"'] && (\mathbf{B}\Gamma)\times(\mathbf{B}G) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathbf{B}\Gamma & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "c"'] \arrow[dr, phantom, "\text{(pb)}"] & X/\!\!/G \arrow[d, "(c/\!\!/G,\,\rho_G)"] \\ \mathbf{B}\Gamma \arrow[r, "(\mathrm{id}_{\mathbf{B}\Gamma},\,\mathrm{pt}_{\mathbf{B}G})"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (\mathbf{B}\Gamma)\times(\mathbf{B}G) \arrow[d, "\mathrm{pr}_2"] \\ * \arrow[r] & \mathbf{B}G \end{tikzcd}$$

By forming the pasting composite shown on the right, this follows by the pasting law (5). □

Example 4.1.26 (Equivariant bundle gerbes)**.** Let $\mathbf{H} = \mathrm{SmthGrpd}_\infty$ (Ntn. 3.3.26).

(i) If we regard the circle Lie group $\mathrm{U}_1 \in \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathbf{H})$ as equipped with trivial equivariance action, and consider its delooping 2-group $\Gamma = \mathbf{B}\mathrm{U}_1 \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.70), then the equivariant $\mathbf{B}\mathrm{U}_1$-principal ∞-bundles – in the sense of Def. 4.1.22, Prop. 4.1.25 – are equivalently "equivariant bundle gerbes" [NiSc11][MRSV17] (with several equivalent precursors, e.g. [GT09, §5], and generalizing earlier discussion in [MSt03][Me03]). This follows from an analysis over action-Čech groupoids (Ex. 3.3.42 below) analogous to that in the proof of Prop. 4.2.4 below, see around [Sc13, Obs. 1.2.71] for more.

(ii) If instead we consider as structure 2-group the *discrete* underlying 2-group $\Gamma = \flat\mathbf{B}\mathrm{U}_1$ (3.127), then equivariant $\flat\mathbf{B}\mathrm{U}_1$-principal ∞-bundles are equivalent to *flat* equivariant bundle gerbes, classified by "discrete torsion" [Sh03a][GT09, §5].

(iii) If we regard $\mathbb{Z}_{/2} \circlearrowright \mathrm{U}_1$ as a $\mathbb{Z}_{/2}$-equivariant group with respect to the complex conjugation action, then $G = \mathbb{Z}_{/2}$-equivariant $\mathbb{Z}_{/2} \circlearrowright \mathbf{B}\mathrm{U}_1$-principal ∞-bundles are equivalently "Jandl gerbes" [SSW07][GSW11] or "real bundle gerbes" [HSMV16].

(iv) Generally, if the equivariance group is given as an extension of $\mathbb{Z}_{/2}$

$$1 \longrightarrow N \hookrightarrow G \overset{p}{\twoheadrightarrow} \mathbb{Z}_{/2} \longrightarrow 1$$

and we let G act on $\mathbf{B}\mathrm{U}_1$ through p (this is the situation of "orbi-orientifolds", see [SS19-Tad, (67)]), then G-equivariant $\mathbb{Z}_{/2} \circlearrowright \mathbf{B}\mathrm{U}_1$-principal ∞-bundles are the equivariant generalization of Jandl gerbes.
(iv) Notice that the cohesive shape of this structure 2-group coincides with that of the infinite projective unitary 1-group (Ex. 1.3.19):

$$\mathbf{B}\mathrm{U}_1 \xrightarrow{\eta^{\int}_{\mathbf{B}\mathrm{U}_1(1)}} B^2\mathbb{Z} \xleftarrow{\eta^{\int}_{\mathrm{PU}_\omega}} \mathrm{PU}_\omega\,.$$

It is in this second guise that equivariant bundle gerbes (Jandl gerbes) model the general 3-twists of equivariant K-theory (KR-theory); see Ex. 4.4.2 and Rem. 4.5.6 below.

Example 4.1.27 (Higher Jandl gerbes)**.** Proceeding as in Ex. 4.1.26 for any $p \in \mathbb{N}$,
(i) the $\mathbb{Z}_{/2}$-equivariant $(p+1)$-groups which are the higher deloopings $\mathbb{Z}_{/2} \circlearrowright \Gamma = \mathbb{Z}_{/2} \circlearrowright \mathbf{B}^p\mathrm{U}_1$ of the circle group are the structure groups of $\mathbb{Z}_{/2}$-equivariant $\mathbf{B}^p\mathrm{U}_1(1)$-principal bundles which are, equivalently, "higher Jandl gerbes" according to [FSS15, §4.4]; see also [FSS20-TCD, Ex. 2.12] for further pointers on higher bundle gerbes;
(ii) for geometrically discrete equivariant structure 2-group $\mathbb{Z}_{/2} \circlearrowright \Gamma = \mathbb{Z}_{/2} \circlearrowright \flat\mathbf{B}^2\mathrm{U}_1$ (3.127), these equivariant 2-bundles are classified by higher "discrete torsion", in the sense of [Sh03b].

Equivariant bundles under base change of the equivariance group. We discuss the behaviour equivariant bundles under base change of the equivariance group

– along an inclusion: Rem. 4.1.28;
– along a surjection: Prop. 4.1.29.

Remark 4.1.28 (Globally equivariant nature of equivariant principal bundles)**.** Given $G \in \mathrm{Grp}(\mathrm{Set})$ and a subgroup inclusion $K \overset{i}{\hookrightarrow} G$, the induced left base change adjunction $(Bi)_! \dashv (Bi)^*$ (Prop. 3.2.82) says, for G-equivariant structure groups $G \circlearrowright \Gamma$ (Def. 4.1.16), that
(i) K-equivariant Γ-principal bundles (Def. 4.1.22) on some $K \circlearrowright X$ are equivalently G-equivariant Γ-principal bundles on $X \times_K G$:

$$\begin{array}{ccc}
K\,\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & \simeq & G\,\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_{X\times_K G} \\
\begin{array}{c}(Bi)_!(X/\!\!/K) \simeq (X\times_K G)/\!\!/G \longrightarrow \mathbf{B}\Gamma/\!\!/G \\ \searrow \quad \swarrow \\ BG\end{array} & \leftrightarrow & \begin{array}{c}X/\!\!/K \longrightarrow \mathbf{B}\Gamma/\!\!/K \simeq (Bi)^*(\mathbf{B}\Gamma/\!\!/G)\,. \\ \searrow \quad \swarrow \\ BK\end{array}
\end{array}$$

(ii) In particular, when the equivariance group action on the structure group Γ is trivial, so that (by Ex. 4.1.17)

$$\begin{aligned}
(Bi)^*\mathbf{B}\Gamma/\!\!/G &\simeq (Bi)^*(\mathbf{B}\Gamma \times \mathbf{B}G) \\
&\simeq \mathbf{B}\Gamma \times \mathbf{B}K \\
&\simeq \mathbf{B}\Gamma/\!\!/K
\end{aligned}$$

holds for *every* pair of groups and monomorphism $K \overset{i}{\hookrightarrow} G$, then

$$\begin{array}{ccc}
K\,\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X & \simeq & G\,\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_{X\times_K G} \\
\begin{array}{c}(Bi)_!(X/\!\!/K) \simeq (X\times_K G)/\!\!/G \longrightarrow \mathbf{B}\Gamma\times\mathbf{B}G \\ \searrow \quad \swarrow_{\mathrm{pr}_2} \\ BG\end{array} & \leftrightarrow & \begin{array}{c}X/\!\!/K \longrightarrow \mathbf{B}\Gamma\times\mathbf{B}K \simeq (Bi)^*(\mathbf{B}\Gamma\times\mathbf{B}G)\,. \\ \searrow \quad \swarrow_{\mathrm{pr}_2} \\ BK\end{array} \\
(X\times_K G)/\!\!/G \longrightarrow \mathbf{B}\Gamma & \leftrightarrow & X/\!\!/K \longrightarrow \mathbf{B}\Gamma
\end{array}$$

reflects the fact (Prop. 4.1.25) that equivariant Γ-principal bundles only depend on the homotopy quotient stack of their domain, and that $(X \times_K G)/\!\!/G \simeq X/\!\!/K$ (3.99).
(iii) This state of affairs is (as previously highlighted in [Re14a, §1.3][Schw18, p. xi]) the avatar of the incarnation of equivariant structures in *globally equivariant* homotopy theory, which we discuss as such in Prop. 4.3.5 below.

Proposition 4.1.29 (Equivariant bundles under extensions of the equivariance group)**.** *For* $p : \widehat{G} \xrightarrow{p} G$ *a surjection of equivariance groups* (3.100)*, there is for*

– *1-truncated* $G \curvearrowright X \in G\mathrm{Act}(\mathbf{H}_1)$ *and*

– *0-truncated* $G \curvearrowright \Gamma\; G\mathrm{Act}\big(\mathrm{Grp}(\mathbf{H}_0)\big)$,

a natural full inclusion

$$G\mathrm{Equiv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X \xhookrightarrow{(Bp)^*} \widehat{G}\mathrm{Equiv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H})_X$$

of the 1-groupoid G-equivariant into that of $\widehat{G}$*-equivariant* Γ*-principal bundles on X, where* $\widehat{G}$ *acts on* Γ *and on X through p.*

Proof. Under the given assumption, both $G \curvearrowright X$ as well as $G \curvearrowright \mathbf{B}\Gamma$ are 1-truncated. Therefore, the statement follows (recalling Def. 4.1.22) as an immediate consequence of (3.105) in Prop. 3.2.87. □

Conditions for classification of equivariant principal ∞-bundles. We proceed to generalize the cohesive classification result for principal ∞-bundles (Thm. 4.1.13) to equivariant principal ∞-bundles. The crucial step in the plain proof was the smooth Oka principle (Thm. 3.3.51) and our proof strategy here is to reduce the equivariant case to this plain case by "blowing-up" all orbifold singularities to smooth spherical space forms (Def. 3.3.52) after stabilizing the dimension. This requires assuming that

(1) G-orbisingularities are resolvable (via Ntn. 3.3.54) and

(2) the equivariant bundle do not detect the difference between an orbi-singularity and its resolution, which is guaranteed by

  (i) a truncation condition on the structure group, Ntn. 4.1.30 below,

  (ii) a "stability" condition on the bundles, Ntn. 4.1.33 below.

The first condition is satisfied, for instance, by all finite subgroups of SU(2), but not by all finite subgroups of SO(3) (by Ex. 3.3.55). The second is mainly a truncation condition which, while rarely satisfied by compact Lie groups, is the default assumption for twists in twisted equivariant Whitehead-generalized cohomology theory, where it just means that the twist has a bounded degree. The archetypical example of this are the degree-3 twists of equivariant K-theory, whose classification result is recovered by the following result (discussed in Ex. 4.4.2).

Notation 4.1.30 (Cohesive groups with truncated classifying shape)**.** We say that a 0-truncated smooth group $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)_{\leq 0}$, $\tau_0\Gamma \simeq \Gamma$, has *truncated classifying shape* if

(i) its shape is truncated:

$$\underset{n\in\mathbb{N}}{\exists}\;\; \tau_n \textstyle\int \Gamma \simeq \int \Gamma; \tag{4.31}$$

(ii) isomorphism classes of Γ-principal bundles over smooth manifolds are already concordance classes (3.177):

$$\underset{\substack{\mathrm{X}\in\\ \mathrm{SmthMfd}}}{\forall}\quad \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;\sim\;} \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}}.$$

This second condition is equivalent, by Prop. 4.1.12, to $B\Gamma$ (4.12) being a classifying space for isomorphism classes of Γ-principal bundles over smooth manifolds:

$$\underset{\substack{\mathrm{X}\in\\ \mathrm{SmthMfd}}}{\forall}\quad \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;\sim\;} \tau_0\,\mathrm{Map}(\textstyle\int \mathrm{X}, B\Gamma) \simeq H^1(X; \int\Gamma). \tag{4.32}$$

Notice that, by possibly increasing any truncation degree n in (4.31) a little, Prop. 3.3.53 guarantees that we may always assume, just for notational definiteness, that G has a smooth free action on some smooth $n+2$-sphere S^{n+2}_{sm} (possibly exotic):

$$\underset{n\in\mathbb{N}}{\exists}\quad G \curvearrowright S^{n+2}_{\mathrm{sm}} \in G\mathrm{Act}(\mathrm{SmthMfd})_{\mathrm{free}} \;\;\text{and}\;\; \tau_n \textstyle\int\Gamma \simeq \int\Gamma. \tag{4.33}$$

We shall say for short that Γ has *n-truncated classifying shape* to indicate such a compatible choice of truncation degree.

Lemma 4.1.31 (Principal bundles on blow-ups for structure group with truncated classifying shape)**.** *Let* $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54) and* $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)_{\leq 0}$ *with n-truncated classifying shape (Ntn. 4.1.30). Then, for* $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$*, pullback along the projection*

$$\mathrm{X} /\!\!/ G \xleftarrow{\;p_{\mathrm{X}} := \mathrm{pr}_1 /\!\!/ G\;} (\mathrm{X} \times S^{n+2}_{\mathrm{sm}}) /\!\!/ G \;\simeq\; (\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G$$

from the product with a compatible G-sphere (4.33) *is a natural surjection*

$$\big(G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;p_{\mathrm{X}}^*\;}\!\!\!\!\to \Big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{(\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G}\Big)_{/\sim_{\mathrm{iso}}} \tag{4.34}$$

from isomorphism classes of G-equivariant Γ*-principal bundles on* X (4.29) *onto the set of isomorphism classes of plain* Γ*-principal bundles on* $(\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G$*.*

Proof. In terms of cocycles we need to show that $\tau_0 \mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma) \xrightarrow{\;p_{\mathrm{X}}^*\;}\!\!\!\!\to \tau_0 \mathbf{H}\big((\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \mathbf{B}\Gamma\big)$. To see this, consider the homotopy colimit diagram that exhibits the G-quotients:

$$\begin{tikzcd}
(\mathrm{X} \times S^{n+2}_{\mathrm{sm}}) \times G \times G \arrow[r, "p"] \arrow[d, shift left=2, "\partial_0"] \arrow[d, "\partial_1"] \arrow[d, shift right=2, "\partial_2"'] & \mathrm{X} \times G \times G \arrow[d, shift left=2, "\partial_0"] \arrow[d, "\partial_1"] \arrow[d, shift right=2, "\partial_2"'] & \\
(X \times S^{n+2}_{\mathrm{sm}}) \times G \arrow[r, "p"] \arrow[d, shift left, "\partial_0"] \arrow[d, shift right, "\partial_1"'] & \mathrm{X} \times G \arrow[d, shift left, "\partial_0"] \arrow[d, shift right, "\partial_1"'] \arrow[dddr, bend right=10, "\partial_1^* c_0'"' description] \arrow[dddr, bend left=30, "\partial_0^* c_0'" description] & \\
\mathrm{X} \times S^d_{\mathrm{sm}} \arrow[r, "p"] \arrow[d, "q"] & \mathrm{X} \arrow[d, "q"] \arrow[ddr, "c_0'" description] & \gamma_1' : \partial_1^* c_0' \overset{\sim}{\Rightarrow} \partial_0^* c_0' \\
(\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G \arrow[r, "p"] \arrow[dr, bend right=15, "c"'] & \mathrm{X} /\!\!/ G \arrow[dr, "c'" description] & \\
& & \mathbf{B}\Gamma
\end{tikzcd} \tag{4.35}$$

Hence given any cocycle $c : (\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G \to \mathbf{B}\Gamma$ we need to show that it is isomorphic to one in the image of p^*. But by the universal property of the middle vertical colimit in (4.35), such cocycles c on the quotient are equivalent to cocycles $c_0 : \mathrm{X} \times S^{n+2}_{\mathrm{sm}} \to \mathbf{B}\mathscr{G}$ on the covering space which are equipped with *descent data* in the form of an isomorphism $\partial_1^* c_0 \xrightarrow[\sim]{\gamma_1} \partial_0^* c_0$ such that[1]

$$\begin{tikzcd}
& \partial_2^*\partial_0^* c_0 \arrow[r, equal] & \partial_0^*\partial_1^* c_0 \arrow[dr, "\partial_0^*\gamma_1", "\sim"'] & \\
\partial_2^*\partial_1^* c_0 \arrow[ur, "\partial_2^*\gamma_1", "\sim"'] \arrow[dr, equal] & & & \partial_0^*\partial_0^* c_0 \arrow[dl, equal] \\
& \partial_1^*\partial_1^* c_0 \arrow[r, "\partial_1^*\gamma_1", "\sim"'] & \partial_1^*\partial_0^* c_0 &
\end{tikzcd} \tag{4.36}$$

[1] We are displaying the standard form of the descent condition. If one unwinds the universal property of the homotopy colimit in (4.35) one finds the diagram (4.36) with another cocycle $c_1 : \mathrm{X} \times S^{n+2}_{\mathrm{sm}} \times G \to \mathbf{B}\Gamma$ at its center and equipped with isomorphisms to all three corners. These three isomorphisms (which are not shown) clearly carry the same information as the three that are shown in (4.36), up to gauge equivalence of descent data.

Now, by the assumption that Γ has truncated classifying shape, the cocycles on $\mathrm{X}\times S^{n+2}_{\mathrm{sm}}$ must be trivial along the sphere, in that we we have the following natural bijections:

$$
\begin{aligned}
\Big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}\times S^{n+2}_{\mathrm{sm}}}\Big)_{/\sim_{\mathrm{iso}}} &\simeq \tau_0\,\mathrm{Map}\big(\mathrm{Shp}(\mathrm{X}\times S^{n+2}_{\mathrm{sm}}), B\Gamma\big) && \text{by (4.32)}\\
&\simeq \tau_0\,\mathrm{Map}\big(S^{n+2}\times \mathrm{Shp}\,\mathrm{X}, B\Gamma\big) && \text{by (3.122)}\\
&\simeq \tau_0\,\mathrm{Map}\big(\mathrm{Shp}\,\mathrm{X}, \mathrm{Map}(S^{n+2}, B\Gamma)\big) && \text{by Lem. 3.2.60}\\
&\simeq \tau_0\,\mathrm{Map}(\mathrm{Shp}\mathrm{X}, B\Gamma) && \text{by Lem. 3.1.21 using (4.33)}\\
&\simeq \big(\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} && \text{by (4.32).}
\end{aligned}
$$

This implies that the required isomorphism to a cocycle in the image of p^* exists:

$$
c_0' : X \to \mathbf{B}\mathscr{G}, \qquad c_0 \xrightarrow[\sim]{t_1} p^* c_0'.
$$

Via this isomorphism, the above equivariant structure γ on c_0 is transported to an equivariant structure γ' on c_0':

$$
\begin{tikzcd}
i^*\partial_0^* c_1 \arrow[r, "i^*\partial_0^* t_1", "\sim"'] \arrow[d, "i^*\gamma", "\wr"'] & i^*\partial_0^* p^* c_1' \arrow[r, equal] & i^* p^* \partial_0^* c_1' \arrow[r, equal] & \partial_0^* c_1' \arrow[d, dashed, "\gamma_1'"] \\
i^*\partial_1^* c_1 \arrow[r, "i^*\partial_1^* t_1", "\sim"'] & i^*\partial_1^* p^* c_1' \arrow[r, equal] & i^* p^* \partial_1^* c_1' \arrow[r, equal] & \partial_1^* c_1' .
\end{tikzcd}
\tag{4.37}
$$

By its conjugation nature, (4.37) clearly inherits the descent condition (4.36), hence constitutes a co-cone under the right vertical homotopy colimiting diagram in (4.35), as shown there. Thus, by the universal property of homotopy colimits in the bottom row of (4.35) , this gives the required isomorphism

$$
c \xrightarrow[\sim]{t} p^* c' .
$$

As this holds for every given c, p^* is surjective. □

The point of the pullback (4.34) is that it gives a handle on the classification of G-equivariant Γ-principal bundles on X through that of ordinary Γ-principal bundles on $(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})/G$:

Lemma 4.1.32 (Classifying maps on blowups for structure group of truncated shape)**.** *Let $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ (Ntn. 3.3.54) and $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)_{\leq 0}$ with n-truncated classifying shape (Ntn. 4.1.30). Then*

$$
\mathrm{Map}\big(\textstyle\int(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})/G, B\Gamma\big) \simeq \mathrm{Map}\big(\textstyle\int \mathrm{X}/\!\!/G, B\Gamma\big) .
$$

Proof. We compute as follows:

$$
\begin{aligned}
\mathrm{Map}\big(\textstyle\int(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})/G, B\Gamma\big) &\simeq \mathrm{Map}\big(\textstyle\int(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})/\!\!/G, B\Gamma\big) && \text{by Prop. 3.2.93}\\
&\simeq \mathrm{Map}\Big(\textstyle\int \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (\mathrm{X}\times S^{n+2}_{\mathrm{sm}})\times G^{\times_n}, B\Gamma\Big) && \text{by (3.93)}\\
&\simeq \mathrm{Map}\Big(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \textstyle\int(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})\times G^{\times_n}, B\Gamma\Big) && \text{by (11)}\\
&\simeq \varprojlim_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Map}\big(\textstyle\int(\mathrm{X}\times S^{n+2}_{\mathrm{sm}})\times G^{\times_k}, B\Gamma\big) && \text{by Prop. 3.2.61}\\
&\simeq \varprojlim_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Map}\big((\textstyle\int\mathrm{X}\times G^{\times_k})\times(\textstyle\int S^{n+2}_{\mathrm{sm}}), B\Gamma\big) && \text{by (3.122)}\\
&\simeq \varprojlim_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Map}\big(\textstyle\int\mathrm{X}\times G^{\times_k}, \mathrm{Map}(S^{n+2}, B\Gamma)\big) && \text{by Lem. 3.2.60}\\
&\simeq \varprojlim_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Map}\big(\textstyle\int\mathrm{X}\times G^{\times_k}, B\Gamma\big) && \text{by Lem. 3.1.21 using (4.33)}\\
&\simeq \mathrm{Map}\Big(\varinjlim_{[k]\in\Delta^{\mathrm{op}}} \textstyle\int\mathrm{X}\times G^{\times_k}, B\Gamma\Big) && \text{by Prop. 3.2.61}\\
&\simeq \mathrm{Map}(\textstyle\int\mathrm{X}/\!\!/G, B\Gamma) && \text{by (3.93).}
\end{aligned}
$$

□

While the pullback operation (4.34) is always a surjection, it is often far from being injective. The following terminology captures the idea of choosing a natural sub-class of equivariant bundles that makes this pullback a bijection:

Notation 4.1.33 (Blowup-stable equivariant principal bundles)**.** Consider $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$ of truncated classifying shape (Ntn. 4.1.30).
(i) We say that a natural system of sub-groupoids of G-equivariant Γ-principal bundles on smooth G-manifolds X with resolvable singularities (Ntn. 3.3.54)

$$\mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \hookrightarrow \mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma) \tag{4.38}$$

is a system of *blowup-stable* equivariant bundles if restriction along this inclusion makes the surjections (4.34) into bijections:

$$\underset{\substack{G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}} \\ G \circlearrowright \mathrm{X} \,\in\, G\mathrm{Act}(\mathrm{SmthMfd})}}{\forall} \quad \begin{tikzcd} \tau_0\,\mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \tau_0\,\mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma) \arrow[r, two heads, "p_{\mathrm{X}}^*"] & \tau_0\,\mathbf{H}\big((\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \mathbf{B}\Gamma\big) \end{tikzcd}. \tag{4.39}$$

(ii) By naturality of the inclusion (4.38) in the domain manifold, we have this inclusion also over products of the form $U \times \mathrm{X}/\!\!/G$ with $U \in \mathrm{CartSpc}$ (Ntn. 3.3.15) (regarded as equipped with the trivial G-action), so that a choice of stable equivariant bundles (4.39) induces a monomorphism of their moduli stacks:

$$\mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \hookrightarrow \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)\,,$$

where, in view of (3.73), we write

$$\mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \;\in\; \mathrm{SmthGrpd}_\infty\,, \qquad \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}}(U) \;:=\; \mathbf{H}(U \times \mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}}\,. \tag{4.40}$$

(iii) If Γ above is equipped with a G-action, and there is a notion of stable equivariant bundles for structure group $\Gamma \rtimes G$-bundles then we have also the corresponding subgroupoid of stable objects

$$\begin{array}{l} \mathbf{H}\big(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G\big)^{\mathrm{stbl}}_{\mathbf{B}G} \\ := \mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}(\Gamma \rtimes G))^{\mathrm{stbl}} \underset{\mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}G)}{\times} * \end{array} \quad\hookrightarrow\quad \begin{array}{l} \mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}(\Gamma \rtimes G)) \underset{\mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}G)}{\times} * \\ \overset{(3.77)}{\simeq} \mathbf{H}\big(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G\big)_{\mathbf{B}G}\,. \end{array}$$

among all G-equivariant $G \circlearrowright \Gamma$-principal bundles according to (4.25).
(iv) Finally, in the case (iii) we obtain also the corresponding monomorphism of moduli stacks of equivariant bundles formed as slice mapping stacks (Def. 3.2.65):

$$\begin{array}{l} \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G)_{\mathbf{B}G} \\ := \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G)^{\mathrm{stbl}} \underset{\mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}G)}{\times} * \end{array} \quad\hookrightarrow\quad \begin{array}{l} \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G) \underset{\mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}G)}{\times} * \\ \overset{(3.78)}{=} \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G)_{\mathbf{B}G}\,. \end{array}$$

Remark 4.1.34. While blowup-stability in Ntn. 4.1.33 may look like a strong assumption to make, the condition can in fact be solved nicely explicitly by "averaging coboundaries of equivariant Čech cocycles over blowup spheres", at least in the following classes of examples:
(i) equivariant bundles with truncated compact Lie structure groups (Thm. 4.1.35 below).
(ii) ADE-equivariant stable projective bundles are blowup-stable (Thm. 4.1.47, Thm. 4.1.52 below).

Theorem 4.1.35 (Equivariant bundles with truncated compact Lie structure are blowup-stable)**.** *Let $\Gamma = \mathbb{T}^r \rtimes K$ be the semidirect product of a connected compact abelian Lie group (an r-torus $\mathbb{T}^r \simeq (\mathrm{U}_1)^{\times^r}$) with a discrete group K. If the equivariance group $G \in \mathrm{Grp}(\mathrm{Set})$ has a free action on some S^{n+2}_{sm} with $n \geq 1$, then for all $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$ the pullbacks (4.34) are injections:*

$$\tau_0\,\mathbf{H}\big(\mathrm{X}/\!\!/G, \mathbf{B}(\mathbb{T}^r \rtimes K)\big) \xrightarrow{\;\;p^*\;\;} \tau_0\,\mathbf{H}\big((\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \mathbf{B}(\mathbb{T}^r \rtimes K)\big)\,.$$

With Lem. 4.1.31 this means that the full class of equivariant bundles with truncated compact Lie structure is blowup-stable in the sense of Ntn. 4.1.33.

Proof. We need to show the implication

$$
\exists \quad \big(\mathrm{X} \times S^{n+2}_{\mathrm{sm}}\big)/G
\begin{tikzcd}[column sep=huge]
{} \arrow[r, bend left, "p_{\mathrm{X}}^*(\vdash P)"] \arrow[r, bend right, "p_{\mathrm{X}}^*(\vdash P')"'] \arrow[r, phantom, "\sim\Downarrow f"] & {}
\end{tikzcd}
\mathbf{B}(\mathbb{T}^r \rtimes K)
\qquad \Rightarrow \qquad
\exists \quad \mathrm{X}/\!\!/G
\begin{tikzcd}[column sep=huge]
{} \arrow[r, bend left, "\vdash P"] \arrow[r, bend right, "\vdash P'"'] \arrow[r, phantom, "\sim\Downarrow f"] & {}
\end{tikzcd}
\mathbf{B}(\mathbb{T}^r \rtimes K)\,.
$$

Choosing, by Prop. 1.1.25, any proper equivariant good open cover $\widehat{\mathrm{X}}$ of X, we may equivalently express, by Ex. 3.3.42, the isomorphism f as a continuous natural transformation between continuous functors on the product of the G-action Čech groupoid (3.169) with S^{n+2}_{sm}:

$$
\begin{tikzcd}[column sep=huge]
N\Big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times S^{n+2}_{\mathrm{sm}} \times G^{\mathrm{op}} \rightrightarrows \widehat{X} \times S^{n+2}_{\mathrm{sm}}\Big) \arrow[r, bend left, "p_{\mathrm{X}}^*(\vdash P)"] \arrow[r, bend right, "p_{\mathrm{X}}^*(\vdash P')"'] \arrow[r, phantom, "\Downarrow f"] & N\big((\mathbb{T}^r \rtimes K \rightrightarrows *)\big)
\end{tikzcd}
\tag{4.41}
$$

$$
\begin{tikzcd}
& g_1 \cdot (\widehat{x},\, p) \arrow[dr] & \\
& & g_2 \cdot (\widehat{x},\, p) \\
(\widehat{x}, p) \arrow[uur] \arrow[urr] & &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
& \bullet \arrow[rrr, "(g\cdot f)(\widehat{x},p) = f(g_1\cdot\widehat{x}, g_1\cdot p)"] \arrow[dr] & & & \bullet \arrow[dr] & \\
& & \bullet \arrow[rrr] & & & \bullet \\
\bullet \arrow[uur, "{(\rho(\widehat{x},g_1),\alpha(\widehat{x},g_1))}"] \arrow[urr] \arrow[rrr, "f(\widehat{x},p)"'] & & & \bullet \arrow[uur, "{(\rho'(\widehat{x},g_1),\alpha'(\widehat{x},g_1))}"] \arrow[urr] & &
\end{tikzcd}
\tag{4.42}
$$

$$
\begin{tikzcd}
& ((x,j),\, p) \arrow[dr] & \\
& & ((x,k),\, p) \\
((x,i),p) \arrow[uur] \arrow[urr] & &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
& \bullet \arrow[rrr, "f_j(x,p)"] \arrow[dr] & & & \bullet \arrow[dr] & \\
& & \bullet \arrow[rrr] & & & \bullet \\
\bullet \arrow[uur, "g_{ij}(x)"] \arrow[urr] \arrow[rrr, "f_i(x,p)"'] & & & \bullet \arrow[uur, "g'_{ij}(x)"] \arrow[urr] & &
\end{tikzcd}
\tag{4.43}
$$

We need to show that f may compatibly be replaced by an equivariant functions which is constant on S^{n+2}_{sm}. Our strategy is to replace the function on the $n+2$-sphere by its average $\overline{f}$ as seen by integration against the unit volume form on the sphere. Since this is a continuous operation, it turns Čech cocycles into Čech cocycles.

The key point to achieve this is that $\mathbb{T}^r \rtimes K$ is 1-truncated while the dimension of S^{n+2}_{sm} is larger than 1, by assumption. This implies that every map from the $(n+2)$-sphere lifts to a map through the universal covering space:

$$
\begin{tikzcd}
& \mathbb{R}^r \rtimes K \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] \\
S^{n+2}_{\mathrm{sm}} \arrow[ur, dashed, "{(\vec{f}(\widehat{x},-),\phi(\widehat{x}))}"] \arrow[r, "f(\widehat{x},-)"] & \mathbb{T}^r \rtimes K \arrow[r] & \mathbf{B}(\mathbb{Z}^r \rtimes K)\,.
\end{tikzcd}
$$

Therefore, it makes sense to assign

$$
f \;\longmapsto\; \overline{f}(\widehat{x}) \;:=\; \bigg(\int\limits_{S^{n+2}_{\mathrm{sm}}} \vec{f}(\widehat{x},p)\,\mathrm{dvol}(p),\, \phi(\widehat{x}) \bigg) \;\in\; \mathbb{R}^r \rtimes K \twoheadrightarrow \mathbb{T}^r \rtimes K\,,
\tag{4.44}
$$

which is evidently well-defined as a map to $\mathbb{T}^n \rtimes K$, in that it is independent of the choice of lift. Here $\phi(\widehat{x}) \in K$ is the K-component of the original function, which is already constant along the sphere, since K is assumed to be discrete and S^{n+2}_{sm} is connected.

It remains to check that the averaged component function $\overline{f_i} : U_i \to \Gamma \rtimes K$ is again a natural transformation of the form (4.41). The key point here is that the automorphism group of the r-torus acts linearly on component vectors by matrix multiplication with invertible integer-valued matrices, which implies that it is compatible with the integration operation:

$$\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{T}^r) \;\simeq\; \mathrm{GL}(r,\mathbb{Z}) \;\subset\; \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^r)\,. \tag{4.45}$$

Using this we find:

$$\begin{aligned}
&\overline{f}(g\cdot\widehat{x}) \\
&= \left(\int_{S^{n+2}_{\mathrm{sm}}} \vec{f}(g\cdot\widehat{x}, g\cdot p)\,\mathrm{dvol}(p),\, \phi(g\cdot\widehat{x})\right) \;=\; \left(\int_{S^{n+2}_{\mathrm{sm}}} \big(g\cdot\vec{f}(\widehat{x},p)\big)\,\mathrm{dvol}(p),\, \phi(g\cdot\widehat{x})\right) && \text{by (4.44)} \\
&= \left(\int_{S^{n+2}_{\mathrm{sm}}} \left(\alpha(\widehat{x},g)^{-1}\Big(\vec{f}(\widehat{x},p) + \phi(\widehat{x})\big(\vec{\rho}\,'(\widehat{x},g)\big) - \vec{\rho}(\widehat{x},g)\Big)\right)\mathrm{dvol}(p),\, \alpha(\widehat{x},g)^{-1}\cdot\phi(\widehat{x})\cdot\alpha'(\widehat{x},g)\right) && \text{by (4.42)} \\
&= \left(\alpha(\widehat{x},g)^{-1}\Big(\int_{S^{n+2}_{\mathrm{sm}}} f(\widehat{x},p)\,\mathrm{dvol}(p) + \phi(\widehat{x})\big(\vec{\rho}\,'(\widehat{x},g)\big) - \vec{\rho}(\widehat{x},g)\Big),\, \alpha(\widehat{x},g)^{-1}\cdot\phi(\widehat{x})\cdot\alpha(\widehat{x},g)\right) && \text{by (4.45)} \\
&= \left(\alpha(\widehat{x},g)^{-1}\Big(\overline{f}(\widehat{x}) + \phi(\widehat{x})\big(\vec{\rho}\,'(\widehat{x},g)\big) - \vec{\rho}(\widehat{x},g)\Big),\, \alpha(\widehat{x},g)^{-1}\cdot\phi(\widehat{x})\cdot\alpha(\widehat{x},g)\right) && \text{by (4.44)} \\
&= (g\cdot\overline{f})(u) && \text{as in (4.42)}\,.
\end{aligned}$$

This demonstrates that $\overline{f}$ still makes the equivariance naturality diagrams (4.42) commute. Verbatim the same computation, just with equivariant transitions replaced by transition functions, shows that it also makes the Čech naturality squares (4.43) commute. Therefore $\overline{f}$ is an isomorphism between $p_{\mathrm{X}}^*(\vdash P)$ and $p_{\mathrm{X}}^*(\vdash P')$ which is itself the pullback to S^{n+2}_{sm} of an isomorphism between $\vdash P$ and $\vdash P'$ themselves. □

Blow-up stability of ADE-equivariant projective bundles. We now solve the blowup-stability condition (Ntn. 4.1.33) for G-equivariant projective bundles in the case of ADE equivariance groups $G \subset \mathrm{Sp}(1)$ (Ex. 3.3.55) and show that this recovers the traditional stability condition (4.46) due to [AS04, §6]. This is Thm. 4.1.47 below. We prove this by an explicit construction on the level of equivariant Čech cocycles. This proof immediately generalizes to the garded and semidirect product structure group $\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_{/2}$ (Thm. 4.1.52 below). In this generality the result eventually implies the classification theorem for stable equivariant $\mathbb{Z}_{/2} \circlearrowleft \mathrm{PU}^{\mathrm{gr}}_{\omega}$-principal bundles twisting equivariant KR-theory, see Ex. 4.4.2 below.

Traditionally, for $\mathscr{G}$ a compact Lie group, a projective $\mathscr{G}$-representation $\mathscr{G} \circlearrowleft \mathrm{P}(\mathscr{H})$ is called *stable* if it is stable under tensoring with the regular Hilbert space representation $L^2(\mathscr{G})$ of square-integrable functions on the group G (with respect to the Haar measure), equipped with the canonical $\mathscr{G}$-action given by pullback of such functions along the left multiplication action of G on itself ([AS04, p. 28]):

$$\mathscr{G} \circlearrowleft \mathscr{H} \in \mathscr{G}\,\mathrm{Act}(\mathrm{Hilb}) \text{ is stable} \qquad \Leftrightarrow \qquad \mathscr{G} \circlearrowleft \big(\mathscr{H} \otimes L^2(\mathscr{G})\big) \;\simeq\; \mathscr{G} \circlearrowleft \mathscr{H}\,. \tag{4.46}$$

This means equivalently that all projective irreps of the given projective twist appear as direct summands with infinite multiplicity (see Lem. 4.1.45 below). Likewise, an equivariant $\mathrm{PU}(\mathscr{H})$-bundle V is called *stable* if its total space is stable against tensoring with $L^2\mathscr{G}$, hence if all its isotropy actions are stable in the above sense (see [BEJU14][LU14, §15][EU16, §5]).

This condition is traditionally motivated as ensuring that the space of Fredholm operators on $\mathscr{G} \circlearrowleft \mathscr{H}$ is a classifying $\mathscr{G}$-space for equivariant KU-theory (see Ex. 4.4.2 below).

Since our blow-up stability condition (Ntn. 4.1.33) is a purely bundle-theoretic notion, it is interesting to observe how it recovers the traditional operator-algebraic notion: In our situation $\mathscr{G} = \mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ is the compact Lie group that contains our finite equivariance groups G, and find that under the identification $L^2(\mathrm{Sp}(1)) \simeq L^2(S^3_{\mathrm{rm}})$, the stable $\mathscr{G}$-Hilbert space of equivariant K-theory at a G-orbi-singularity is identified with that of quantum states on the blow-up $S^3 \to *$ of the singularity: The proof of Thm. 4.1.47 below crucially proceeds by "quantizing" cocycle data on the blowup S^3 of an orbifold singularity, identifying it with components of the "wavefunction" that the cocycle takes values in, and vice versa.

Notation 4.1.36 (Infinite tensor product of L^2-Hilbert spaces)**.** We write Hilb $\in$ Cat for the category of complex countably-dimensional Hilbert spaces.

(i) For $(\mathrm{X}, \mathrm{dvol})$ a Riemannian manifold, regarded as a measure space, of unit volume

$$\int_{\mathrm{X}} 1\,\mathrm{dvol} \;=\; 1$$

(these are already the most general measure spaces that we need here) we write

$$L^2(\mathrm{X}, \mathrm{dvol}) \quad \in \; \mathrm{Hilb}$$

for the usual Hilbert space of its square-integrable complex-valued functions modulo those supported on subsets of vanishing measure.

(ii) Recalling that the tensor product of L^2-Hilbert spaces reflects the Cartesian product of underlying measure spaces

$$L^2(\mathrm{X}, \mathrm{dvol}) \otimes L^2(\mathrm{X}', \mathrm{dvol}') \;\simeq\; L^2\big(\mathrm{X} \times \mathrm{X}', \mathrm{dvol} \wedge \mathrm{dvol}'\big),$$

consider the (co)limiting case of an infinite number of (tensor) products:

$$(\mathrm{X}, \mathrm{dvol})^{\times^\infty} \;:=\; \underset{\substack{\longleftarrow \\ S \in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} \mathrm{X}^S \qquad \longmapsto \qquad \underset{\substack{\longrightarrow \\ S \in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} L^2(\mathrm{X}, \mathrm{dvol})^{\otimes S} \quad =: \; L^2(\mathrm{X}, \mathrm{dvol})^{\otimes^\infty}, \tag{4.47}$$

where the (co)limit is over the directed system of inclusions of finite subsets of $\mathbb{N}$:

$$\begin{tikzcd} S' \arrow[r, hook, "i"] & S \\ \mathrm{X}^{S'} & \mathrm{X}^S \arrow[l, two heads, "X^i"'] \\ L^2(\mathrm{X})^{\otimes S'} \arrow[r, "(\mathrm{X}^i)^*"] & L^2(\mathrm{X})^{\otimes S}. \end{tikzcd} \tag{4.48}$$

(iii) This means that the colimiting Hilbert space $L^2(\mathrm{X}, \mathrm{dvol})^{\otimes^\infty}$ (4.47) is that generated by those square-integrable functions f on $(\mathrm{X}, \mathrm{dvol})^{\times^\infty}$ that depend on any finite number of variables (hence those which factor as $(\mathrm{X}, \mathrm{dvol})^{\times^\infty} \to \mathrm{X}^{S_f} \to \mathbb{C}$ for some finite subset $S_f \subset \mathbb{N}$) with the inner product on these generating elements being

$$\langle f, g \rangle \;=\; \prod_{s \in F_f \cup F_g} \langle f_s, g_s \rangle \;=\; \prod_{s \in F_f \cup F_g} \int_{\mathrm{X}} \overline{f}_s(x^s) \cdot g_s(x^s)\, \mathrm{dvol}(x^s)\,. \tag{4.49}$$

Remark 4.1.37 (Interpretations and properties of infinite tensor products of L^2-Hilbert spaces)**.**

(i) The space (4.49) may equivalently be identified with:

(a) the Hilbert space of square integrable functions on the infinite product measure space $(\mathrm{X}^{\times^\infty}, \mathrm{dvol}^{\times^\infty})$ (e.g., by [Par05, Ex. 6.3.11]),

(b) the ("incomplete" [vN39, §4.1], or "Guichardet" [Gui69], or "grounded" [BSZ92, p. 126]) infinite tensor product of Hilbert spaces (review in [Wea01, Def. 2.5.1]), if we regard each $L^2(\mathrm{X}, \mathrm{dvol})$ as equipped with the vacuum state given by

$$\begin{tikzcd} * & \mathrm{X} \arrow[l, "p_{\mathrm{X}}"'] \\ \mathbb{C} \simeq L^2(*) \arrow[r, "(p_{\mathrm{X}})^*"] & L^2(\mathrm{X}, \mathrm{dvol}) \\ 1 \arrow[r, mapsto] & \mathrm{const}_1 =: |\mathrm{vac}\rangle\,. \end{tikzcd} \tag{4.50}$$

(ii) In this notation, the inclusions of Hilbert spaces in (4.48) operate by "adding more copies of the system in its vacuum state", where necessary. For example:

$$\begin{tikzcd} \{1,3\} \arrow[r, hook, "i"] & \{1,2,3\} \arrow[r, hook] & \mathbb{N} \\ \mathrm{X}^2 & \mathrm{X}^3 \arrow[l, two heads, "(\mathrm{pr}_1, \mathrm{pr}_3)"'] & \mathrm{X}^\infty \arrow[l, two heads, "(\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3)"'] \\ L^2(\mathrm{X})^{\otimes 2} \arrow[r, "(\mathrm{X}^i)^*"] & L^2(\mathrm{X})^{\otimes 3} \arrow[r] & L^2(\mathrm{X})^{\otimes^\infty} \\ |\psi_1\rangle \otimes |\psi_3\rangle \arrow[r, mapsto] & |\psi_1\rangle \otimes |\mathrm{vac}\rangle \otimes |\psi_3\rangle \arrow[r, mapsto] & |\psi_1\rangle \otimes |\mathrm{vac}\rangle \otimes |\psi_3\rangle \otimes \big(|\mathrm{vac}\rangle\big)^{\otimes^\infty}. \end{tikzcd} \tag{4.51}$$

(iii) This construction is functorial; in particular, for $U \in \mathrm{U}\big(L^2(\mathrm{X})\big)$ a unitary operator acting on a single copy of the Hilbert space which fixes the vaccum state (4.50), it extends to a diagonal unitary action on the infinite Guichardet tensor product:

$$\left.\begin{array}{l} U \in \mathrm{U}\big(L^2(\mathrm{X})\big), \\ U|\mathrm{vac}\rangle = |\mathrm{vac}\rangle \end{array}\right\} \quad \vdash \quad U^{\otimes^\infty} \in \mathrm{U}\big(L^2(\mathrm{X})^{\otimes^\infty}\big). \tag{4.52}$$

For example, a diagonal unitary action on the state (4.51) is of this form:

$$U^{\otimes^\infty}\Big(|\psi_1\rangle \otimes |\mathrm{vac}\rangle \otimes |\psi_3\rangle \otimes \big(|\mathrm{vac}\rangle\big)^{\otimes^\infty}\Big) \;=\; |U(\psi_1)\rangle \otimes |\mathrm{vac}\rangle \otimes |U(\psi_3)\rangle \otimes \big(|\mathrm{vac}\rangle\big)^{\otimes^\infty}.$$

While all countably infinite-dimensional Hilbert spaces are abstractly isomorphic, functorial families of them – such as the combined tensor products and representation spaces that we encounter in a moment – have a more individual structure. We next turn attention specifically to the following Hilbert space:

Example 4.1.38 (Hilbert space of finite higher spin chains)**.** The infinite Guichardet tensor product (4.47)

$$\mathscr{H} \;:=\; L^2\big(S^3_{\mathrm{rm}}\big)^{\otimes^\infty} \;=\; L^2\big(\mathrm{Sp}(1)\big)^{\otimes^\infty} \;\in\; \mathrm{Hilb} \tag{4.53}$$

of copies of the Hilbert space of square-integrable functions on the round unit 3-sphere:

$$\mathscr{H}_{\mathrm{sngl}} \;:=\; L^2\big(S^3_{\mathrm{rm}}\big) \;\in\; \mathrm{Hilb} \tag{4.54}$$

is generated by the (necessarily finite-dimensional) complex irreducible representations (irreps) of the $\mathrm{Spin}(3)$ group. Indeed, under the isomorphism

$$S^3_{\mathrm{rm}} \simeq \mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq \mathrm{Spin}(3) \;\in\; \mathrm{RiemMfd}$$

(for these compact Lie groups equipped with their normalized Haar measure), the Peter-Weyl theorem implies that under the canonical action (4.62) by pullback along the (inverse) left multiplication action of these groups on themselves we have the following direct sum decomposition of G-Hilbert spaces (e.g. [HM20, Thm. 3.28]):

$$\mathrm{Spin}(3) \circlearrowright L^2(S^3_{\mathrm{rm}}) \;\simeq \bigoplus_{\substack{[\mathbf{n}] \,\in\, (\mathrm{Rep}(\mathrm{Spin}(3))_{\mathrm{irr}})_{/\sim_{\mathrm{iso}}} \\ \dim_{\mathbb{C}}(\mathbf{n}) = n \,\in\, \mathbb{N}_+}} n \cdot \mathbf{n} \;\in\; \mathrm{Rep}\big(\mathrm{Spin}(3)\big). \tag{4.55}$$

It is suggestive to observe that the G-subspaces

$$\mathrm{Spin}(3) \circlearrowright \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \cdots \otimes \mathbf{2} \hookrightarrow \mathrm{Sp}(1) \circlearrowright L^2\big(S^3_{\mathrm{rm}}\big)^{\otimes^\infty} \;\in\; \mathrm{Spin}(3)\,\mathrm{Act}(\mathrm{Hilb})$$

which are finite tensor products of the fundamental representation (the defining representation of $\mathrm{SU}(2)$)

$$\mathbf{2} \simeq \mathrm{Span}\big(|\downarrow\rangle, |\uparrow\rangle\big)$$

are known as the Hilbert spaces of finite *Heisenberg spin chain*-models (e.g. [Sab18, §4][Sto20, §2]), and the subspaces of finite tensor products of any given higher dimensional representation

$$\mathrm{Spin}(3) \circlearrowright \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \cdots \otimes \mathbf{n} \hookrightarrow \mathrm{Spin}(2) \circlearrowright L^2\big(S^3_{\mathrm{rm}}\big)^{\otimes^\infty} \;\in\; \mathrm{Spin}(3)\,\mathrm{Act}(\mathrm{Hilb})$$

are those of *higher spin chain* models. Since the Hilbert space (4.53) naturally includes these finite higher spin chain Hilbert spaces, together with all finite "mixed higher spin" chains,

$$\mathrm{Spin}(3) \circlearrowright \mathbf{n}_1 \otimes \mathbf{n_2} \otimes \mathbf{n}_3 \otimes \cdots \otimes \mathbf{n}_\ell \hookrightarrow \mathrm{Spin}(3) \circlearrowright L^2\big(S^3_{\mathrm{rm}}\big)$$

we will loosely refer to it as the *higher spin chain Hilbert space*.[2]

[2] Some authors also consider the infinite Heisenberg spin chain given by the Guichardet tensor product $\mathbf{2}^{\otimes^\infty}$ with respect to regarding $|\downarrow\rangle \in \mathbf{2}$ as the vacuum state. Beware that this space is *not* naturally a subspace of our Hilbert spaces, whose selected vacuum state is instead $|\mathrm{vac}\rangle \in \mathbf{1}$.

Definition 4.1.39 (Projective-unitary group on higher mixed spin chains)**.**
(i) We now write

$$\mathrm{U}_{\omega} = \mathrm{U}_{\omega'} := \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big) \quad \in \mathrm{Grp}(\mathrm{kTopSpc}) \xrightarrow{\mathrm{Grp}(\mathrm{Cdfflg})} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty) \tag{4.56}$$

for the unitary group (1.86) specifically of the Hilbert space (4.53) of "higher spin chains", so that the projective unitary group PU_ω (1.88) is now specifically the quotient of (4.56) by the (topological) circle group

$$1 \longrightarrow \mathrm{U}_1 \hookrightarrow L^2(S^3_{\mathrm{rm}})^{\otimes^\infty} \twoheadrightarrow \mathrm{PU}_\omega \longrightarrow 1\,.$$

Of course, this is isomorphic to any other construction of the bundle $\mathrm{U}_1 \to \mathrm{U}_\omega \to \mathrm{PU}_\omega$, but in this incarnation we have a manifest inclusion of the following non-isomorphic but weakly-equivalent version of the circle group.
(ii) We write

$$\mathrm{U}_{1'} := \underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} \mathrm{Map}\big((S^3_{\mathrm{sm}})^F, \mathrm{U}_1\big) \quad \in \mathrm{Grp}(\mathrm{DTopSpc}) \tag{4.57}$$

for the colimit of D-topological groups of U_1-valued continuous functions on the 3-sphere over the system (4.48). (This is the space of U_1-valued functions on $(S^3_{\mathrm{sm}})^{\times^\infty}$ (4.47) which depend on any finite number of the factor spaces.)
(iii) The system of evident multiplication actions

$$F, F' \in \mathrm{FinSub}(\mathbb{N}) \;\vdash\; \mathrm{Map}\big((S^3_{\mathrm{top}})^F, \mathrm{U}_1\big) \times L^2\big((S^3_{\mathrm{rm}})^{F'}\big) \longrightarrow L^2\big((S^3_{\mathrm{rm}})^{(F\cup F')}\big) \longrightarrow L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}$$

passes to the double colimit to yield an action of $\mathrm{U}_{1'}$ (4.57) on $\mathrm{U}_{\omega'}$ (4.56)

$$\mathrm{U}_{1'} \times L^2(S^3_{\mathrm{rm}})^{\otimes^\infty} \longrightarrow L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}$$

which is manifestly unitary and hence, adjointly, an inclusion into the unitary group (4.56)

$$\mathrm{U}_{1'} \hookrightarrow \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big) = \mathrm{U}_{\omega'}\,. \tag{4.58}$$

(iv) The quotient of the unitary group $\mathrm{U}_{\omega'}$ (4.56) by its subgroup $\mathrm{U}_{1'}$ (4.58) we denote $\mathrm{PU}_{\omega'}$:

$$\begin{tikzcd}
\mathrm{U}_{1'} \arrow[r, hook] & \mathrm{U}_{\omega'} \arrow[r, two heads] & \mathrm{PU}_{\omega'} := \mathrm{U}_{\omega'}/\mathrm{U}_{1'} \\
\underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} \mathrm{Map}\big((S^3_{\mathrm{rm}})^F, \mathrm{U}_1\big) \arrow[u, phantom, "\rotatebox{90}{$=:$}"] \arrow[r, hook] & \mathrm{U}\left(\underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} L^2\big(S^3_{\mathrm{rm}}\big)^{\otimes^F} \right) \arrow[u, phantom, "\rotatebox{90}{$=:$}"]
\end{tikzcd} \tag{4.59}$$

Remark 4.1.40 (Adjoining a spinor system to the chain)**.** The notions in Def. 4.1.39 are such as to seamlessley allow "adjoining one more copy" of the elementary system (4.54) to the infinite tower (4.53):
(i) For example, the tensor product of the Hilbert space (4.53) with one elementary system (4.54) from the left is canonically re-included into the former by shifting all other copies up:

$$\begin{tikzcd}
L^2(S^3_{\mathrm{rm}}) \otimes L^2(S^3_{\mathrm{rm}})^{\otimes^\infty} \arrow[r, hook, "\otimes"] & L^2(S^3_{\mathrm{rm}})^{\otimes^\infty} \\
L^2(S^3_{\mathrm{rm}}) \otimes L^2(S^3_{\mathrm{rm}})^{\otimes^{\{s_1,\cdots,s_k\}}} \arrow[u, hook] \arrow[r, hook] & L^2(S^3_{\mathrm{rm}})^{\otimes^{\{1,s_1+1,\cdots,s_k+1\}}} \arrow[u, hook] \\
|\psi\rangle \otimes \big(|\psi_{s_1}\rangle \otimes \cdots \otimes |\psi_{s_k}\rangle\big) \arrow[r, mapsto] & \big(|\psi\rangle \otimes |\psi_{s_1}\rangle \otimes \cdots \otimes |\psi_{s_k}\rangle\big)\,,
\end{tikzcd}$$

and this tensoring induces a corresponding inclusion of unitary and projective unitary groups from Def. 4.1.39:

$$\begin{tikzcd}
1 \times \mathrm{U}_1 \arrow[r] \arrow[d] & \mathrm{U}_1 \arrow[d] \\
\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big) \times \mathrm{U}_{\omega'} \arrow[r, hook, "\otimes"] \arrow[d] & \mathrm{U}_{\omega'} \arrow[d] \\
\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big) \times \mathrm{PU}_{\omega} \arrow[r] & \mathrm{PU}_{\omega}\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times \mathrm{U}_{1'} \arrow[r] \arrow[d] & \mathrm{U}_{1'} \arrow[d] \\
\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big) \times \mathrm{U}_{\omega'} \arrow[r, hook, "\otimes"] \arrow[d] & \mathrm{U}_{\omega'} \arrow[d] \\
\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big) \times \mathrm{PU}_{\omega'} \arrow[r] & \mathrm{PU}_{\omega'}\,.
\end{tikzcd} \tag{4.60}$$

(ii) Similarly, since S^3_{sm} is compact (certainly in the traditional sense, but also as an object of the ∞-topos $\mathrm{SmthGrpd}_\infty = \mathrm{Sh}_\infty(\mathrm{SmthMfd})$, still in the sense that all of its covers have a finite sub-cover, see [Sc13, §3.6.4]) and since the colimit (4.57) is over monomorphisms, it follows ([Sc13, Prop. 3.6.61]) that every map from S^3_{sm} to $\mathrm{U}_{1'}$ (4.57) factors through one of its finite stages, so that $\mathrm{Map}(S^3_{\mathrm{sm}}, \mathrm{U}_{1'})$ is naturally re-included into $\mathrm{U}_{1'}$:

$$\mathrm{Map}\big(S^3_{\mathrm{sm}}, \mathrm{U}_{1'}\big) \overset{\sim}{\hookrightarrow} \mathrm{U}_{1'}\,, \tag{4.61}$$

namely as follows:

$$\begin{aligned}
\mathrm{Map}\big(S^3_{\mathrm{sm}}, \mathrm{U}_{1'}\big) \;&=\; \mathrm{Map}\Bigg(S^3_{\mathrm{sm}},\, \underset{\substack{F\in\\ \mathrm{FinSub}(\mathbb{N})}}{\varinjlim}\mathrm{Map}\Big((S^3_{\mathrm{sm}})^F, \mathrm{U}_1\Big)\Bigg) && \text{by (4.57)}\\
&\simeq\; \underset{\substack{F\in\\ \mathrm{FinSub}(\mathbb{N})}}{\varinjlim}\mathrm{Map}\Big(S^3_{\mathrm{sm}}, \mathrm{Map}\big((S^3_{\mathrm{top}})^F, \mathrm{U}_1\big)\Big) && \text{by [Sc13, Prop. 3.6.61]}\\
&\simeq\; \underset{\substack{F\in\\ \mathrm{FinSub}(\mathbb{N})}}{\varinjlim}\mathrm{Map}\Big((S^3_{\mathrm{top}})^{*\sqcup F}, \mathrm{U}_1\Big) \;=\; \underset{\substack{F\in\\ \mathrm{FinSub}(\mathbb{N}_+)}}{\varinjlim}\mathrm{Map}\Big((S^3_{\mathrm{top}})^{F}, \mathrm{U}_1\Big) && \text{by (3.72)}\\
&\hookrightarrow\; \underset{\substack{F\in\\ \mathrm{FinSub}(\mathbb{N})}}{\varinjlim}\mathrm{Map}\Big((S^3_{\mathrm{top}})^{F}, \mathrm{U}_1\Big) && \text{by inspection}\\
&=\; \mathrm{U}_{1'} && \text{by (4.57).}
\end{aligned}$$

Example 4.1.41 (G-Action on space of higher spin chains)**.** For $G \subset \mathrm{Sp}(1)$ any subgroup (in particular a finite subgroup, Ex. 3.3.55), its left multiplication action on $S^3_{\mathrm{sm}} = \mathrm{Sp}(1)$ induces, by isometric pullback of square-integrable functions, a unitary representation of G on $L^2\big(S^3_{\mathrm{rm}}\big)$ (4.54):

$$\begin{array}{ccc}
G & \longrightarrow & \mathrm{U}\big(L^2\big(S^3_{\mathrm{rm}}\big)\big)\\
g & \longmapsto & (g^{-1})^* =: \displaystyle\int_{S^3_{\mathrm{rm}}} |g^{-1}\cdot p\rangle\langle p|
\end{array} \tag{4.62}$$

which diagonally extends (4.52) to an action by $\mathrm{U}_{\omega'}$ (4.56) on the infinite tensor product Hilbert space (4.53):

$$\begin{array}{ccc}
G & \longrightarrow & \mathrm{U}_{\omega'} = \mathrm{U}\Big(L^2\big(S^3_{\mathrm{rm}}\big)^{\otimes^\infty}\Big)\\
g & \longmapsto & (g^{-1})^* =: \displaystyle\bigotimes_{\mathbb{N}}\int_{S^3_{\mathrm{rm}}} |g^{-1}\cdot p\rangle\langle p|\,.
\end{array} \tag{4.63}$$

Lemma 4.1.42 (Shape of projective unitary group on higher spin chains)**.**
(i) *The shape of the modified circle group* (4.57) *is still that of a* $K(\mathbb{Z},1)$*:*

$$\int \mathrm{U}_{1'} \;\simeq\; B\mathbb{Z}\,.$$

(ii) *The shape of the modified projective unitary group* (4.59) *is still that of a* $K(\mathbb{Z},2)$*:*

$$\int \mathrm{PU}_{\omega'} \;\simeq\; B^2\mathbb{Z}\,.$$

Proof. Since $\pi_{\geq 2}(S^1) = *$, the long exact sequence of homotopy groups induced by the homotopy fiber sequence

$$\Omega^3\mathrm{U}_1 \xrightarrow{\;\mathrm{fib}(\mathrm{ev}_*)\;} \mathrm{Map}(S^3, S^1) \xrightarrow[\in \mathrm{WHmtpEq}]{\;\mathrm{ev}_*\;} S^1 \tag{4.64}$$

shows that the map on the right (evaluating at any chosen basepoint) is a weak homotopy equivalence. Therefore:

$$\begin{aligned}
\int\mathrm{Map}(S^3_{\mathrm{sm}}, \mathrm{U}_1) \;&\simeq\; \mathrm{Map}\big(\textstyle\int S^3_{\mathrm{sm}}, \int\mathrm{U}_1\big) && \text{by Thm. 3.3.51}\\
&\simeq\; \mathrm{Map}\big(S^3, S^1\big) && \text{by Prop. 3.3.40 and Prop. 3.3.46}\\
&\simeq\; S^1 \;\simeq\; B\mathbb{Z} && \text{by (4.64).}
\end{aligned} \tag{4.65}$$

By induction on the number $k \in \mathbb{N}$ of 3-sphere factors, it follows more generally that:

$$\begin{aligned}
\int \mathrm{Map}\Big((S^3_{\mathrm{sm}})^{\times^{k+1}}, \mathrm{U}_1\Big) \;&\simeq\; \int \mathrm{Map}\Big(S^3_{\mathrm{sm}}, \mathrm{Map}\big((S^3_{\mathrm{sm}})^k, \mathrm{U}_1\big)\Big) && \text{by Lem. 3.2.60}\\
&\simeq\; \mathrm{Map}\Big(\int S^3_{\mathrm{sm}}, \int \mathrm{Map}\big((S^3_{\mathrm{sm}})^k, \mathrm{U}_1\big)\Big) && \text{by Thm. 3.3.51}\\
&\simeq\; \mathrm{Map}\big(S^3, S^1\big) && \text{by induction from (4.65)}\\
&\simeq\; S^1 \simeq B\mathbb{Z} && \text{by (4.64).}
\end{aligned} \tag{4.66}$$

Finally, since passing to shape commutes over colimits, we find that the large circle group $\mathrm{U}_{1'}$ still has the shape of the plain circle, as claimed:

$$\begin{aligned}
\int \mathrm{U}_{1'} \;&=\; \int \underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} \mathrm{Map}\big((S^3_{\mathrm{sm}})^F, \mathrm{U}_1\big) && \text{by (4.57)}\\
&\simeq\; \underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} \int \mathrm{Map}\big((S^3_{\mathrm{sm}})^F, \mathrm{U}_1\big) && \text{by (11)}\\
&\simeq\; \underset{\substack{\longrightarrow \\ F\in \\ \mathrm{FinSub}(\mathbb{N})}}{\lim} S^1 && \text{by (4.66)}\\
&\simeq\; S^1 \simeq B\mathbb{Z} && \text{colimit over constant diagram .}
\end{aligned} \tag{4.67}$$

With this, the second claim follows as for the ordinary PU_ω:

$$\begin{aligned}
\int \mathrm{PU}_{\omega'} \;&=\; \int \big(\mathrm{U}_{\omega'}/\mathrm{U}_{1'}\big) && \text{by (4.59)}\\
&\simeq\; \int \big(\mathrm{U}_{\omega'} /\!\!/ \mathrm{U}_{1'}\big) && \text{by Prop. 3.2.93}\\
&\simeq\; \big(\int \mathrm{U}_{\omega'}\big) /\!\!/ \big(\int \mathrm{U}_{1'}\big) && \text{by (11)}\\
&\simeq\; * /\!\!/ B\mathbb{Z} && \text{by (1.87) \& (4.67)}\\
&\simeq\; B^2\mathbb{Z} && \text{by (3.88) .}
\end{aligned} \tag{4.68}$$

□

Example 4.1.43 (Inclusion of standard into modified projective unitary group is weak equivalence)**.** Arguing as Lem. 4.1.42, one sees that

(a) the canonical inclusions of the ordinary projective unitary group PU_ω into $\mathrm{PU}_{\omega'}$;

(b) the self-inclusion of the latter under the infinite tensoring (4.60)

are both weak homotopy equivalences:

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[r, hook] \arrow[d, hook] & \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big) \arrow[r, two heads] \arrow[d, equal] & \mathrm{PU}_\omega \arrow[d, "\in \mathrm{WHmtpEq}"] \\
\mathrm{U}_{1'} \arrow[r, hook] \arrow[d, hook] & \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big) \arrow[r, two heads] \arrow[d, hook, "\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big)\otimes(-)"'] & \mathrm{PU}_{\omega'} \arrow[d, "\in \mathrm{WHmtpEq}"] \\
\mathrm{U}_{1'} \arrow[r, hook] & \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\big) \arrow[r, two heads] & \mathrm{PU}_{\omega'}
\end{tikzcd} \tag{4.69}$$

Definition 4.1.44 (Blowup-stable ADE-equivariant projective bundles)**.** For a finite subgroup $G \subset \mathrm{Sp}(1)$ (Ex. 3.3.55) and $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$, consider any equivariant good open cover $G \circlearrowright \widehat{\mathrm{X}} = G \circlearrowright \big(\underset{i\in I}{\sqcup} U_i\big)$ (by Prop. 1.1.25).

(i) On its Čech action groupoid (Ex. 3.3.42), we have the following canonical equivariant Čech cocycle (Rem. 3.3.43) with values in $\mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big)$ (4.62), acting via the canonical unitary G-action from Ex. 4.1.41:

$$c_{\mathrm{reg}}: \quad \Big(\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}}\big) \times G \rightrightarrows \widehat{\mathrm{X}}\Big) \xrightarrow{\;p_{\widehat{\mathrm{X}}}\;}\!\!\!\!\rightarrow \big((G \rightrightarrows *)\big) \xrightarrow{\big(((-)^{-1})^* \rightrightarrows *\big)} \Big(\big(\mathrm{U}\big(L^2(S^3)\big) \rightrightarrows *\big)\Big). \tag{4.70}$$

(ii) We say that a G-equivariant principal bundle $\mathrm{X}/\!\!/X \xrightarrow{[c]} \mathbf{B}\mathrm{PU}_{\omega'}$ (Def. 4.1.22) with structure group PU_{ω} or $\mathrm{PU}_{\omega'}$ (Def. 4.1.39) is *stable* if the operation of tensoring (4.60) any of its representing equivariant Čech cocycles (3.170)

$$c: \quad \Big(\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}}\big) \times G \rightrightarrows \widehat{\mathrm{X}} \Big) \longrightarrow (\mathrm{PU}_{\omega'} \rightrightarrows *)$$

with the canonical cocycle (4.70)

$$\begin{array}{ccccc} & c & \longmapsto & c_{\mathrm{reg}} \otimes c & \\ \mathrm{PU}\Big(L^2(S^3_{\mathrm{rm}})^{\otimes\infty}\Big) \ni & c\big((x,i,j),g\big) & \longmapsto & \int\limits_{S^3_{\mathrm{rm}}} |g^{-1}\cdot p\rangle\langle p| \otimes c\big((x,i,j),g\big) & \in \mathrm{U}\Big(L^2(S^3_{\mathrm{rm}}) \otimes L^2(S^3_{\mathrm{rm}})^{\otimes\infty}\Big) \end{array} \tag{4.71}$$

is stable, in that it fixes the cocycle up to isomorphism:

$$[\mathrm{c}] \text{ is stable} \quad \Leftrightarrow \quad [c] = \big[c_{\mathrm{reg}} \otimes c\big]\,. \tag{4.72}$$

Lemma 4.1.45 (Stable projective isotropy representations; cf. [BEJU14][LU14, §15][EU16, §5])**.** *For $G \subset \mathrm{Sp}(1)$ a finite subgroup and $[\tau] \in H^2_{\mathrm{Grp}}(G;\mathrm{U}_1)$, a $[\tau]$-projective unitary G-representation (Ex. 3.3.30)*

$$\begin{tikzcd} \mathbf{B}G \arrow[rr, "\rho^\tau"] \arrow[dr, "\tau"'] & & \mathbf{B}\mathrm{U}_\omega /\!\!/ \mathbf{B}\mathrm{U}_1 \arrow[dl] \\ & \mathbf{B}^2\mathrm{U}_1 & \end{tikzcd}$$

is stable in the sense of Def. 4.1.44, in that it is stable under tensoring with $G \circlearrowright L^2(\mathrm{Sp}(1))$, iff it is is isomorphic to the direct sum of all irreducible $[\tau]$-projective representations each appearing with countably infinite multiplicity.

$$\rho^\tau \text{ stable} \quad \Leftrightarrow \quad \rho^\tau \simeq \bigoplus_{\substack{[\mu]\in \\ \left(\mathrm{Rep}^{[\tau]}(G)_{\mathrm{irr}}\right)_{/\sim_{\mathrm{iso}}}}} \mu^{\otimes\infty}\,.$$

Proof. In view of (4.55) and by Schur's Lemma, the restriction map

$$\begin{array}{ccc} \mathrm{Sp}(1) & \xleftarrow{\;\; i \;\;} & G \\ G \circlearrowright L^2\big(\mathrm{Sp}(1)\big) & \xrightarrow{\;\; i^* \;\;} & L^2\big(G\big) \simeq G \circlearrowright \mathbb{C}[G] \end{array}$$

exhibits the regular G-representation $\mathbb{C}[G]$ as a direct summand of $L^2(\mathrm{Sp}(1))$ (which, of course, itself is the regular $\mathrm{Sp}(1)$-representation).

But tensoring a $[\tau]$-projective representation with the regular representation adjoins a positive finite number of copies of all $[\tau]$-projective irreps, by Prop. 3.3.35. This operation is thus stable if and only if an infinite number of direct summands of each $[\tau]$-projective irrep is already present in $\rho^{[\tau]}$. □

Example 4.1.46 (Blowup-stabilization of ADE-equivariant $\mathrm{PU}_{\omega'}$-principal bundles)**.** If we equip the infinite tensor Hilbert space (4.53) with the diagonal canonical G-action (4.63), then it is stable under tensoring with one copy of its G-tensor factors, regarded as equipped with its canonical G-action (4.62):

$$\overset{\circlearrowright G}{L^2(S^3_{\mathrm{rm}})} \otimes \overset{\circlearrowright G}{L^2(S^3_{\mathrm{rm}})^{\otimes\infty}} \simeq \overset{\circlearrowright G}{L^2(S^3_{\mathrm{rm}})^{\otimes\infty}} \quad \in \; G\mathrm{Act}(\mathrm{Hilb})\,.$$

More generally, for $G \circlearrowright \mathscr{H}$ any G-Hilbert space, its tensor product with $G \circlearrowright L^2(S^3_{\mathrm{rm}})^{\otimes\infty}$ is stable under tensoring with a single copy of the G-Hilbert space $L^2(S^3_{\mathrm{rm}})$. In this manner, if we write, in analogy with (4.70),

$$c_{\mathrm{reg}}^{\otimes\infty}: \quad \Big(\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}}\big) \times G \rightrightarrows \widehat{\mathrm{X}} \Big) \xrightarrow{\;p_{\widehat{\mathrm{X}}}\;} \big((G \rightrightarrows *) \big) \xrightarrow{\;\left(\left(((-)^{-1})^*\right)^{\otimes\infty} \rightrightarrows * \right)\;} \Bigg(\Big(\overbrace{\mathrm{U}\big(L^2(S^3)^{\otimes\infty}\big)}^{\mathrm{U}_{\omega'}} \Big) \Bigg) \tag{4.73}$$

then tensoring with this cocycle serves to stabilize G-equivariant PU_{ω}- and $\mathrm{PU}_{\omega'}$-principal bundles $[c]$ according to Def. 4.1.44) in that:

$$\big[c_{\mathrm{reg}} \otimes (c_{\mathrm{reg}}^{\otimes\infty} \otimes c)\big] = \big[(c_{\mathrm{reg}}^{\otimes\infty} \otimes c)\big] \quad \in \big(G\mathrm{Equv}\,\mathrm{PU}_{\omega'}\,\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)\big)_{/\sim_{\mathrm{iso}}}\,. \tag{4.74}$$

Theorem 4.1.47 (Blowup-stability of ADE-equivariant $\mathrm{PU}_{\omega'}$-principal bundles)**.** *For $G \subset \mathrm{Sp}(1)$ a finite subgroup (Ex. 3.3.55):*
(i) *The stable G-equivariant $\mathrm{PU}_{\omega'}$-principal bundles according to Def. 4.1.44 do form a blow-up stable class in the sense of Ntn. 4.1.33 with respect to blowing up by the Hopf action $G \circlearrowright S^7_{\mathrm{sm}}$ (3.180).*
(ii) *Every stable ADE-equivariant $\mathrm{PU}_{\omega'}$-principal bundle reduces to a stable equivariant $\mathrm{PU}^{\mathrm{gr}}_{\omega}$-principal bundle along the comparison map $\mathrm{PU}^{\mathrm{gr}}_{\omega} \to \mathrm{PU}^{\mathrm{gr}}_{\omega'}$ (4.69).*

Proof. With any equivariant good open cover chosen as in Def. 4.1.44, we need to show that the pullback operation on equivariant Čech cocycles (Rem. 3.3.43)

$$\Big(\mathrm{Fnctr}\big((\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G \rightrightarrows \widehat{\mathrm{X}}),\, (\mathrm{PU}_{\omega} \rightrightarrows *)\big)\Big)_{/\sim_{\mathrm{iso}}} \xrightarrow{\;p^*_{S^7_{\mathrm{sm}}}\;} \Big(\mathrm{Fnctr}\Big(\big((\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times S^7_{\mathrm{sm}}) \times G \rightrightarrows \widehat{\mathrm{X}} \times S^7_{\mathrm{sm}}\big),\, (\mathrm{PU}_{\omega'} \rightrightarrows *)\Big)\Big)_{/\sim_{\mathrm{iso}}}$$

restricts on stable equivariant $\mathrm{PU}_{\omega'}$-principal bundles to a bijection and on stable equivariant PU_{ω}-principal bundles at least to a surjection. (Here "Fnctr" is short for the groupoid of morphisms of simplicial presheaves, hence, in the present case, of functors of D-topological groupoids, Ntn. 1.2.1).

(1) *Injectivity:* Given a coboundary f between the pullback of a pair of cocycles

$$\begin{tikzcd}[column sep=huge]
\Big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times S^7_{\mathrm{sm}} \times G \rightrightarrows \widehat{\mathrm{X}} \times S^7_{\mathrm{sm}}\Big) \arrow[r, bend left, "p^*_{S^7_{\mathrm{sm}}} c"] \arrow[r, bend right, "p^*_{S^7_{\mathrm{sm}}} c'"'] \arrow[r, phantom, "\sim \Downarrow f"] & (\mathrm{PU}_{\omega'} \rightrightarrows *)
\end{tikzcd} \tag{4.75}$$

$$\begin{tikzcd}
(\widehat{x},\, p) \arrow[d] & & \bullet \arrow[r, "f(\widehat{x}{,}p)"] \arrow[d, "\rho(\widehat{x}{,}g)"'] & \bullet \arrow[d, "\rho'(\widehat{x}{,}g)"] \\
g \cdot (\widehat{x},\, p) & \mapsto & \bullet \arrow[r, "f(g\cdot\widehat{x}{,}g\cdot p)"] & \bullet
\end{tikzcd} \;\in\; \mathrm{PU}_{\omega'} \tag{4.76}$$

$$\begin{tikzcd}
(\widehat{x},\, p) \arrow[d] & & \bullet \arrow[r, "f(\widehat{x}{,}p)"] \arrow[d, "c(\widehat{x}{,}\widehat{x}')"'] & \bullet \arrow[d, "c'(\widehat{x}{,}\widehat{x}')"] \\
(\widehat{x}',\, p) & \mapsto & \bullet \arrow[r, "f(\widehat{x}'{,}\,p)"] & \bullet
\end{tikzcd} \;\in\; \mathrm{PU}_{\omega'} \tag{4.77}$$

we will construct a coboundary $\overline{f}$ between the tensoring of the original cocycles with c_{reg} (4.71):

$$\begin{tikzcd}[column sep=huge]
\Big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G \rightrightarrows \widehat{\mathrm{X}}\Big) \arrow[r, bend left, "c_{\mathrm{reg}} \otimes c"] \arrow[r, bend right, "c_{\mathrm{reg}} \otimes c'"'] \arrow[r, phantom, "\sim \Downarrow \overline{f}"] & (\mathrm{PU}_{\omega'} \rightrightarrows *)
\end{tikzcd} \tag{4.78}$$

$$\begin{tikzcd}
\widehat{x} \arrow[d] & & \bullet \arrow[r, "\overline{f}(\widehat{x})"] \arrow[d, "\tilde{\rho}(\widehat{x}{,}g)"'] & \bullet \arrow[d, "\tilde{\rho}'(\widehat{x}{,}g)"] \\
g \cdot \widehat{x} & \mapsto & \bullet \arrow[r, "\overline{f}(g\cdot\widehat{x})"] & \bullet
\end{tikzcd} \;\in\; \mathrm{PU}_{\omega'} \tag{4.79}$$

$$\begin{tikzcd}
\widehat{x} \arrow[d] & & \bullet \arrow[r, "\overline{f}(\widehat{x})"] \arrow[d, "\tilde{c}(\widehat{x}{,}\widehat{x}')"'] & \bullet \arrow[d, "\tilde{c}'(\widehat{x}{,}\widehat{x}')"] \\
\widehat{x}' & \mapsto & \bullet \arrow[r, "\overline{f}(g\cdot\widehat{x})"] & \bullet
\end{tikzcd} \;\in\; \mathrm{PU}_{\omega'}\,. \tag{4.80}$$

By the assumption that both c and c' are stable (4.72), this will imply that they were already isomorphic before pullback.

To motivate the following construction of an isomorphism $\overline{f}$ (4.78), notice that *if* the G-action on S^7_{sm} *had* a fixed point $p_0 \hookrightarrow S^7_{\mathrm{sm}}$, then we could simply restrict both cocycles along its inclusion to immediately obtain the desired isomorphism. Now, instead of a fixed point, the left G-action in question has for each point $p_0 \in S^4_{\mathrm{sm}}$ its full 3-spherical $\mathrm{Sp}(1)$-orbit $S^3_{p_0} \hookrightarrow S^7_{\mathrm{sm}}$, this being the fiber of the quaternionic Hopf fibration (3.180) over this point:

$$
\begin{tikzcd}
S^3_{p_0} \arrow[r, "i_{p_0}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7_{\mathrm{sm}} \arrow[d] \\
* \arrow[r, "p_0"'] & S^4_{\mathrm{sm}} \,.
\end{tikzcd}
\tag{4.81}
$$

Therefore the strategy is to "evaluate" at some $p_0 \in S^4_{\mathrm{sm}}$ and "average" the value of f over the remaining $S^3_{p_0} \hookrightarrow S^7_{\mathrm{sm}}$ over this point, by absorbing its operator values on this 3-sphere fiber on the left of the higher spin chain:

$$
\overline{f}(\widehat{x}) \;:=\; \int\limits_{S^3_{p_0}} |p\rangle\langle p| \otimes f(\widehat{x},p) \quad \in\; \mathrm{PU}\Big(L^2(S^3_{\mathrm{rm}}) \otimes L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}\Big) \;\simeq\; \mathrm{PU}_{\omega'}\,.
\tag{4.82}
$$

In order to check that this does yield a coboundary as desired, notice that we may lift the component function f from a projective-unitary to a unitary map $\widehat{f}$:

$$
\begin{tikzcd}
 & \mathrm{U}_{\omega'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d] \\
\widehat{\mathrm{X}} \times S^7_{\mathrm{sm}} \arrow[ur, dashed, "\widehat{f}"] \arrow[r, "f"] & \mathrm{PU}_{\omega'} \arrow[r] & \mathbf{B}\mathrm{U}_{1'}\,.
\end{tikzcd}
$$

Namely, under the universal property of the pullback on the right, such a lift is induced by a trivialization of the bottom composite map shown above, and this exists by the classification theory for $\mathrm{PU}_{\omega'}$-principal bundles (Thm. 4.1.13) and since $\int \mathrm{PU}_{\omega'}$ is 2-truncated (by Lem. 4.1.42), while $\int(\widehat{\mathrm{X}} \times S^7_{\mathrm{sm}})$ is is a disjoint union of copies of S^7.

Using an analogously constructed lift for $\rho(-,-)$, the equivariance condition (4.76) says that the two products of $\widehat{f}$ and $\widehat{\rho}$ make two lifts to $\mathrm{U}_{\omega'}$ of the same map $\mathrm{PU}_{\omega'}$, which hence differ by a map to $\mathrm{U}_{1'}$. This, of course, remains the case after restriction to $S^3_{p_0} \hookrightarrow S^7$, where we hence have:

$$
\widehat{f}\big(\widehat{x}, (-)\big) \cdot \widehat{\rho}'(\widehat{x}, g) \;=\; \widehat{\rho}(\widehat{x}, g) \cdot \widehat{f}\big(g \cdot \widehat{x}, g \cdot (-)\big) \;\; \mathrm{mod} \;\; \mathrm{Map}(S^3_{p_0}, \mathrm{U}_{1'}) \xrightarrow{(4.61)} \mathrm{U}_{1'}\,.
\tag{4.83}
$$

In fact, the projectivizing maps on the right exist in smooth dependence on $\widehat{x}$.

Using this, we may verify that $\overline{f}$ (4.82) satisfies condition (4.79):

$$
\begin{aligned}
\overline{f}(\widehat{x}) \cdot \widetilde{\rho}'(\widehat{x},g) \quad & = \; \overline{f}(\widehat{x}) \cdot \int\limits_{S^3_{p_0}} |g^{-1} \cdot p\rangle\langle p| \otimes \rho'(\widehat{x},g) && \text{by (4.71)} \\
& = \; \Bigg[\int\limits_{S^3_{p_0}} |g^{-1} \cdot p\rangle\langle p| \otimes \widehat{f}(\widehat{x}, g^{-1} \cdot p) \cdot \widehat{\rho}'(\widehat{x},g) \Bigg] && \text{by (4.82)} \\
& = \; \Bigg[\int\limits_{S^3_{p_0}} |g^{-1} \cdot p\rangle\langle p| \otimes \widehat{\rho}(\widehat{x},g) \cdot \widehat{f}\big(g \cdot \widehat{x}, p\big) \Bigg] && \text{by (4.83)} \\
& = \; \widetilde{\rho}(\widehat{x},g) \cdot \overline{f}\big(g \cdot \widehat{x}\big) && \text{by (4.71) \& (4.82).}
\end{aligned}
\tag{4.84}
$$

Here square brackets denote $\mathrm{U}_{1'}$-equivalence classes of $\mathrm{U}_{\omega'}$-operators.

The remaining condition (4.80) is verified analogously:

$$\begin{aligned} \overline{f}(\widehat{x}) \cdot \widetilde{c}\,'(\widehat{x}, \widehat{x}\,') &= \left[\int\limits_{S^3_{p_0}} |p\rangle\langle p| \otimes \widehat{f}(\widehat{x}, p) \cdot \widehat{c}'(\widehat{x}, \widehat{x}\,') \right] && \text{by (4.71) \& (4.82)} \\ &= \left[\int\limits_{S^3_{p_0}} |p\rangle\langle p| \otimes \widehat{c}'(\widehat{x}, \widehat{x}\,') \cdot \widehat{f}(\widehat{x}\,', p) \right] && \text{by (4.77)} \\ &= \widetilde{c}\,'(\widehat{x}, \widehat{x}\,') \cdot \overline{f}(\widehat{x}\,') && \text{by (4.71) \& (4.82)}\,. \end{aligned} \tag{4.85}$$

(2) *Surjectivity:* First, consider the further pullback of the equivariant bundles to the blowup by just the 3-spherical fiber of the Hopf fibration over any chose basepoint $p_0 \in S^4_{\mathrm{sm}}$, as in (4.81), hence all the way to the last item of the following chain of morphisms:

$$\mathrm{X}/\!\!/G \xleftarrow{\;\mathrm{pr}_1\;} \mathrm{X}/\!\!/G \times S^4_{\mathrm{sm}} \xleftarrow{(\mathrm{X}\times t_{\mathbb{H}})/G} (\mathrm{X} \times S^7_{\mathrm{sm}})/G \xleftarrow{(\mathrm{X}\times i_{p_0})/G} \big(\mathrm{X} \times (S^3_{\mathrm{sm}})_{p_0}\big)/G \;:\, p_{S^3_{\mathrm{sm}}}\,. \tag{4.86}$$

Observe that on this 3-spherical blowup, the regular unitary Čech cocycle c_{reg} (4.70) and its infinite tensor product $c_{\mathrm{reg}}^{\otimes^\infty}$ (4.73) both have a canonical trivialization after identifying $S^3_{\mathrm{sm}} \simeq \mathrm{Sp}(1)$:

$$\begin{tikzcd} \big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times \mathrm{Sp}(1) \rightrightarrows \widehat{\mathrm{X}} \times \mathrm{Sp}(1)\big) \arrow[rr, bend left, "c_{\mathrm{reg}}"] \arrow[rr, bend right, "\mathrm{const}"'] \arrow[rr, phantom, "\wr\Downarrow"] && \mathrm{Hilb} \end{tikzcd}$$

$$\begin{tikzcd} p \arrow[d] & & L^2(\mathrm{Sp}(1))^{\otimes^\infty} \arrow[r, "(p^{-1})^*"] \arrow[d, "(g^{-1})^*"'] & L^2(\mathrm{Sp}(1))^{\otimes^\infty} \arrow[d, equal] \\ g \cdot p & \longmapsto & L^2(\mathrm{Sp}(1))^{\otimes^\infty} \arrow[r, "\left((g\cdot p)^{-1}\right)^*"] & L^2(\mathrm{Sp}(1))^{\otimes^\infty} \end{tikzcd}$$

By tensoring (4.60) with this trivialization and using that tensoring $\mathrm{PU}_{\omega'}$ with the constant $L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}$ is a weak homotopy equivalence (4.69), it follows that for every isomorphism class $[c]$ of a G-equivariant $\mathrm{PU}_{\omega'}$-principal bundle over X, its pullback to $\mathrm{X} \times S^3_{p_0}$ coincides with the pullback of its stabilization $[c_{\mathrm{reg}}^{\otimes^\infty} \otimes c]$ (4.74):

$$p^*_{S^3_{\mathrm{sm}}}\big[c_{\mathrm{reg}}^{\otimes^\infty} \otimes c\big] \;=\; p^*_{S^3_{\mathrm{sm}}}[c] \quad \in \tau_0\mathbf{H}\big((\mathrm{X} \times S^3_{\mathrm{sm}})/G, \mathrm{PU}_{\omega'}\big)\,.$$

But since the intermediate pullback along i_{p_0} (4.86) is an injection on isomorphism classes of $\mathrm{PU}_{\omega'}$-principal bundles (Lem. 4.1.48)

$$\tau_0\mathbf{H}\big((\mathrm{X} \times S^7_{\mathrm{sm}})/G, \mathrm{PU}_{\omega'}\big) \xhookrightarrow{\big((\mathrm{X}\times i_{p_0})/G\big)^*} \tau_0\mathbf{H}\big((\mathrm{X} \times S^3_{\mathrm{sm}})/G, \mathrm{PU}_{\omega'}\big)\,, \tag{4.87}$$

we find that this equality between classes of equivariant bundles and their stabilization must hold already after pullback to the blowup by S^7:

$$p^*_{S^7_{\mathrm{sm}}}\big[c_{\mathrm{reg}}^{\otimes^\infty} \otimes c\big] \;=\; p^*_{S^7_{\mathrm{sm}}}[c] \quad \in \tau_0\mathbf{H}\big((\mathrm{X} \times S^3_{\mathrm{sm}})/G, \mathrm{PU}_{\omega'}\big)\,.$$

Since Lem. 4.1.31 says that $p^*_{S^7_{\mathrm{sm}}}[-]$ is surjective, we conclude that already $p^*_{S^7_{\mathrm{sm}}}\big[c_{\mathrm{reg}}^{\otimes^\infty} \otimes (-)\big]$ is surjective. Since $c^*_{\mathrm{reg}} \otimes (-)$ produces stable bundles (Ex. 4.1.46) this is the surjectivity statement to be proven, for structure group $\mathrm{PU}_{\omega'}$. Moreover, this surjectivity argument applies verbatim also to the structure group PU_{ω}. □

Lemma 4.1.48 (Pullback of PU_ω- and $\mathrm{PU}_{\omega'}$-bundles from 7-spherical to 3-spherical blowup is injection)**.** *Pullback of equivariant bundles with structure group* PU_ω *or* $\mathrm{PU}_{\omega'}$ *along* i_{p_0} (4.86) *is an injection on isomorphism classes:*

$$\begin{tikzcd}
\tau_0 \mathbf{H}\big((\mathrm{X}\times S^7)/G, \mathbf{B}\mathrm{PU}_{\omega^{(\prime)}}\big) \arrow[r, hook, "(i_{p_0})^*"] \arrow[d, equal] & \tau_0 \mathbf{H}\big((\mathrm{X}\times S^3)/G, \mathbf{B}\mathrm{PU}_{\omega^{(\prime)}}\big) \arrow[d, equal] \\
H^3\big(\int \mathrm{X}/\!\!/G; \mathbb{Z}\big) \arrow[r, hook, "(\mathrm{id},0)"] & H^0\big(\int \mathrm{X}/\!\!/G; \mathbb{Z}\big) \oplus H^3\big(\int \mathrm{X}/\!\!/G; \mathbb{Z}\big).
\end{tikzcd}$$

Proof. Since the G-action both on S^7 as well as on S^3 are free, the standard classification of principal bundles over smooth manifolds (Thm. 4.1.13) applies, and since $B\mathrm{PU}_{\omega'} \simeq B\int \mathrm{PU}_{\omega'} \simeq B^3\mathbb{Z}$ (Lem. 4.1.42) this says that the classification for PU'_ω-structure is the same as that of PU_ω-bundles, given by integral 3-cohomology of the shape of the domain manifold.

First observe, from the long exact sequences of homotopy groups induced by the fiber sequences

$$\begin{tikzcd}
0 \simeq \Omega^7 B^3\mathbb{Z} \arrow[r] & \mathrm{Map}(S^7, B^3\mathbb{Z}) \arrow[d, "\mathrm{ev}_*"] \\
& B^3\mathbb{Z}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbb{Z} \simeq \Omega^3 B^3\mathbb{Z} \arrow[r] & \mathrm{Map}(S^3, B^3\mathbb{Z}) \arrow[d, "\mathrm{ev}_*"] \\
& B^3\mathbb{Z}.
\end{tikzcd}$$

that we have equivalences

$$G \circlearrowright \mathrm{Map}(S^7, B^3\mathbb{Z}) \simeq B^3\mathbb{Z} =: A_7, \qquad G \circlearrowright \mathrm{Map}(S^3, B^3\mathbb{Z}) \simeq \mathbb{Z}\times B^3\mathbb{Z} =: A_3 \;\in\; G\mathrm{Act}(\mathrm{Grpd}_\infty), \tag{4.88}$$

which identify the induced G-action (via the given G-action on the sphere) as trivial: In both cases the factor $B^3\mathbb{Z}$ comes from the maps that factor through the point, while the factor $\mathbb{Z}$ is the winding of S^3 over itself under any factorization through a generator $S^3 \to B^3\mathbb{Z}$ of $\pi_3(B^3\mathbb{Z}) \simeq \mathbb{Z}$. Both of these are manifestly G-invariant. Moreover, this shows that the pullback map between these mapping spaces is the inclusion of the 0-winding sector:

$$\begin{tikzcd}
\mathrm{Map}(S^7, B^3\mathbb{Z}) \arrow[r, "(i_{p_0})^*"] \arrow[d, "\wr"] & \mathrm{Map}(S^3, B^3\mathbb{Z}) \arrow[d, "\wr"] \\
A_7 = B^3\mathbb{Z} \arrow[r, hook, "a\mapsto(0,a)"] & \mathbb{Z}\times B^3\mathbb{Z} =: A_3
\end{tikzcd} \tag{4.89}$$

Now, for $d \in \{3,7\}$, we compute much as in Lem. 4.1.32:

$$\begin{aligned}
\mathrm{Map}\Big(\int\big(\mathrm{X}\times S^d_{\mathrm{sm}}\big)/G, B^3\mathbb{Z}\Big) &\simeq \mathrm{Map}\big((\textstyle\int \mathrm{X}\times S^d)/\!\!/G, B^3\mathbb{Z}\big) \\
&\simeq \mathrm{Map}\Big(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} G^{\times^n}\times \textstyle\int \mathrm{X}\times S^d, B^3\mathbb{Z}\Big) \\
&\simeq \mathrm{Map}\Big(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \coprod_{G^n} \textstyle\int \mathrm{X}\times S^d, B^3\mathbb{Z}\Big) && \text{by discreteness of } G \\
&\simeq \varprojlim_{[n]\in\Delta^{\mathrm{op}}} \prod_{G^n} \mathrm{Map}\big(\textstyle\int \mathrm{X}, B^3\mathbb{Z}\big) && \text{by (9)} \\
&\simeq \varprojlim_{[n]\in\Delta^{\mathrm{op}}} \prod_{G^n} \mathrm{Map}\big(\textstyle\int \mathrm{X}, \mathrm{Map}(S^d, B^3\mathbb{Z})\big) \\
&\simeq \varprojlim_{[n]\in\Delta^{\mathrm{op}}} \prod_{G^n} \mathrm{Map}\big(\textstyle\int \mathrm{X}, A_d\big) && \text{by (4.88)} \\
&\simeq \mathrm{Map}\Big(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} \coprod_{G^n} \textstyle\int \mathrm{X}, A_d\Big) && \text{by (9) \& (4.88)} \\
&\simeq \mathrm{Map}\big(\textstyle\int \mathrm{X}/\!\!/G, A_d\big).
\end{aligned}$$

The claim follows by appying this natural equivalence to (4.89). ☐

With this concrete cocycle model for stability of equivariant $\mathrm{PU}_{\omega'}$-principal bundles in hand, we may now easily generalize to projective unitary operators on graded Hilbert spaces and/or to their semidirect product with the action of complex conjugation.

Notation 4.1.49 (Grading and Complex conjugation action on $\mathrm{PU}_{\omega'}$).
(i) In direct generalization of (1.90), the $\mathbb{Z}_{/2}$-action by complex conjugation on all the groups in Ex. 4.1.43 is clearly compatible with all the morphisms there. In particular, we have $\mathbb{Z}_{/2}$-equivariant weak homotopy equivalences

$$\overset{\mathbb{Z}_{/2}}{\circlearrowright}\mathrm{PU}_{\omega} \xrightarrow[\in \mathrm{WHmtpEq}]{} \overset{\mathbb{Z}_{/2}}{\circlearrowright}\mathrm{PU}_{\omega'} \xrightarrow[\in \mathrm{WHmtpEq}]{\mathrm{U}\left(L^2(S^3_{\mathrm{rm}})^{\otimes\infty}\right)\otimes(-)} \overset{\mathbb{Z}_{/2}}{\circlearrowright}\mathrm{PU}_{\omega'} \quad \in \mathbb{Z}_{/2}\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big)$$

and the corresponding semidirect product groups

$$\mathrm{U}_{1'} \rtimes \mathbb{Z}_{/2} \hookrightarrow \mathrm{U}_{\omega'} \rtimes \mathbb{Z}_{/2} \twoheadrightarrow \mathrm{PU}_{\omega'} \rtimes \mathbb{Z}_{/2} \;\in\; \mathrm{Grp}(\mathrm{DTopSpc})\,.$$

(ii) Moreover, all the above constructions on the Hilbert space $L^2(S^3_{\mathrm{rm}})^{\otimes\infty}$ evidently generalize to constructions on its tensor product with any other Hilbert space, in particular to

$$L^2(S^3_{\mathrm{rm}})^{\otimes\infty} \otimes \mathbb{C}^2 \;\simeq\; L^2(S^3_{\mathrm{rm}})^{\otimes\infty}_{+} \oplus L^2(S^3_{\mathrm{rm}})^{\otimes\infty}_{-}\,. \tag{4.90}$$

(iii) Under this identification we have, in analogous variation of $\mathrm{PU}^{\mathrm{gr}}_{\omega}$ (1.94), the modified graded projective unitary group and its semidirect product with complex conjugation:

$$\mathrm{U}_{1'} \rtimes \mathbb{Z}_{/2} \hookrightarrow \mathrm{U}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2} \twoheadrightarrow \mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2} \;\in\; \mathrm{Grp}(\mathrm{DTopSpc})\,. \tag{4.91}$$

(iv) Finally, the tensoring operation (4.60) evidently lifts to these semidirect products

$$\begin{aligned} \mathrm{U}\big(L^2(S^3_{\mathrm{rm}})\big) \times \big(\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}\big) &\xrightarrow{\;\otimes\;} \mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2} \\ \mathrm{U}^{\mathrm{gr}}_{\omega'} \times \big(\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}\big) &\xrightarrow{\;\otimes\;} \mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2} \end{aligned} \tag{4.92}$$

Lemma 4.1.50 (Shape of graded projective groups). *The shape of the graded projective groups* $\mathrm{PU}^{\mathrm{gr}}_{\omega}$ (1.94) *and* $\mathrm{PU}^{\mathrm{gr}}_{\omega'}$ (4.91) *is:*

$$\int \mathrm{PU}^{\mathrm{gr}}_{\omega} \;\simeq\; \int \mathrm{PU}^{\mathrm{gr}}_{\omega'} \;\simeq\; \mathbb{Z}_{/2} \times B^2\mathbb{Z} \;\in\; \mathrm{Grp}(\mathrm{Grpd}_\infty)\,;$$

and under this equivalence the complex conjugation action (1.90) *becomes the sign involution* $\mathbb{Z}_{/2} \circlearrowright \mathbb{Z}_{\mathrm{cjg}}$*:*

$$\int \overset{\mathbb{Z}_{/2}}{\circlearrowright}\mathrm{PU}^{\mathrm{gr}}_{\omega} \;\simeq\; \mathbb{Z}_{/2} \times \overset{\mathbb{Z}_{/2}}{\circlearrowright}B^2\mathbb{Z}_{\mathrm{cjg}} \;\in\; \mathbb{Z}_{/2}\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{Grpd}_\infty)\big)\,.$$

Proof. This is almost the same computation as in (4.68), the only difference being that $\int \mathrm{U}_{\omega} \simeq *$ gets replaced by $\int \mathrm{U}^{\mathrm{gr}}_{\omega} \simeq \mathbb{Z}_{/2}$ (on which the circle quotient action is trivial, by definition):

$$\begin{aligned} \int \mathrm{PU}^{\mathrm{gr}}_{\omega'} &= \int\big(\mathrm{U}^{\mathrm{gr}}_{\omega'}/\mathrm{U}_{1'}\big) && \text{by (4.91)} \\ &\simeq \int\big(\mathrm{U}^{\mathrm{gr}}_{\omega'}/\!\!/\mathrm{U}_{1'}\big) && \text{by Prop. 3.2.93} \\ &\simeq \big(\int \mathrm{U}^{\mathrm{gr}}_{\omega'}\big)/\!\!/\big(\int \mathrm{U}_{1'}\big) && \text{by (11)} \\ &\simeq \big((\int \mathbb{Z}_{/2}) \times (\int \mathrm{U}_{\omega}) \times (\int \mathrm{U}_{\omega})\big)/\!\!/\big(\int \mathrm{U}_{1'}\big) && \text{by (3.122)} \\ &\simeq \mathbb{Z}_{/2}/\!\!/B\mathbb{Z} && \text{by (1.87) \& (4.67)} \\ &\simeq \mathbb{Z}_{/2} \times B^2\mathbb{Z} && \text{by (3.88)}\;. \end{aligned} \tag{4.93}$$

The second statement follows by observing that under the equivalence

$$\mathrm{U}_1 \simeq \mathbb{R}/\!\!/\mathbb{Z}$$

the conjugation action on the left becomes sign inversion on real (hence on integer) numbers on the right. ☐

Now in direct generalization of Def. 4.1.44:

Definition 4.1.51 (Blowup-stable ADE-equivariant $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$-principal bundles)**.** We say that an ADE-equivariant principal bundle with structure group $\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_{/2}$ or $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$ (Ntn. 4.1.49) is *stable* if its class is invariant under the tensoring (4.92) with the regular cocycle (4.70).

In direct generalization of Thm. 4.1.47, we have:

Theorem 4.1.52 (Blowup-stablility of ADE-equivariant $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$-principal bundles)**.** *For $G \subset \mathrm{Sp}(1)$ a finite subgroup (Ex. 3.3.55),*
(i) *The G-equivariant principal bundles with structure group $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$ (Ntn. 4.1.49) which are stable according to Def. 4.1.51 do form a blow-up stable class in the sense of Ntn. 4.1.33 with respect to blowing up by the Hopf action $G \circlearrowright S^7_{\mathrm{sm}}$* (3.180)*.*
(ii) *Every stable ADE-equivariant $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$-principal bundle reduces to a stable equivariant $\mathrm{PU}_{\omega} \rtimes \mathbb{Z}_{/2}$-principal bundle along the comparison map $\mathrm{PU}_{\omega} \to \mathrm{PU}_{\omega'}$* (4.69)*.*

Proof. We just need to observe that the evident generalization of the proof of Thm. 4.1.47 still goes through. First, since the argument there is manifestly natural in forming tensor products with the underlying Hilbert space, it evidently generalizes to the graded structure groups $\mathrm{PU}^{\mathrm{gr}}_{\omega}$ and $\mathrm{PU}^{\mathrm{gr}}_{\omega'}$. Moreover, since $\mathbb{Z}_{/2}$ is discrete and S^7_{sm} is connected, the equivariant cocycles and the coboundary (4.75) now have components which are pairs

$$\big(\rho(\widehat{x},p), \sigma(\widehat{x})\big)\,, \quad \big(f(\widehat{x},p), \kappa(\widehat{x})\big) \quad \in\; \mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$$

with (not only σ but also) κ not depending on $p \in S^7_{\mathrm{sm}}$. This implies that their conjugation actions are compatible with the integrals in the computations (4.84) and (4.85), so that the proof given there generalizes straightforwardly. □

The orbi-smooth Oka principle. We may now prove a constrained generalization of the "smooth Oka principle" (Thm. 3.3.51) from smooth manifolds to smooth orbifolds, subject to the conditions discussed above (truncated structure group and blowup-stability over resolvable singularities).

Lemma 4.1.53 (Concordance of stable equivariant bundles on blowups)**.**
Let $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ (Ntn. 3.3.54) and $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_{\infty})_{\leq 0}$ with truncated classifying shape (Ntn. 4.1.30), so that a notion of stable G-equivariant Γ-principal bundles exists (Ntn. 4.1.33) then

$$\tau_0 \textstyle\int \mathrm{Map}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \;\simeq\; \tau_0 \textstyle\int \mathrm{Map}\big((\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \mathbf{B}\Gamma\big)\,.$$

Proof. By stability (4.39) we have such a bijection already on isomorphism classes, both over X and over $\mathrm{X} \times \Delta^1_{\mathrm{smth}}$. This implies the claimed bijection on concordance classes by (3.178) in Ex. 3.3.47. □

Remark 4.1.54 (Cardinality bound on homotopy classes of maps to structure group)**.** In addition to the substantial condition of blowup-stability for runcated structure group over resolvable orbi-singularities, the following proof of the orbi-smooth Oka principle (Thm. 4.1.55) requires a mild technical condition (in the step (4.98) below): The cardinality of the fundamental group

$$\pi_1 \,\mathrm{Map}\big(\textstyle\int \mathrm{X}/\!\!/G, B\Gamma \rtimes G\big)$$

must be countable, at every basepoint. At the canonical basepoint this means that

$$H^0_G\big(\mathrm{X}; \textstyle\int \Gamma \rtimes G\big) \;=\; \pi_0 \,\mathrm{Map}\big(\textstyle\int \mathrm{X}/\!\!/G, \textstyle\int \Gamma \times G\big)$$

must be countable, and due to the assumption that $\int \Gamma$ is truncated, this essentially just means that the Borel-equivariant cohomology of X in every degree is a countable set.

Theorem 4.1.55 (Orbi-smooth Oka principle for stable truncated maps out of good orbifolds with resolvable singularities)**.** *Given*

– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*

– $G \circlearrowright \Gamma \in G\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big)$

such that

– $\Gamma \rtimes G$ *is of truncated classifying shape (Ntn. 4.1.30),*

– *there is a notion of blowup-stability of* G*-equivariant* $\Gamma \rtimes G$*-principal bundles (Ntn. 4.1.33),*

(i) *then, for all*

– $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$ *such that* $\pi_1 \mathrm{Map}\big(\mathrm{X} /\!\!/ G, B(\Gamma \rtimes G)\big)$ *is countable (Rem. 4.1.54);*

the canonical comparison morphism (3.128) *restricted to stable equivariant bundles is an equivalence in* Grpd_∞*:*

$$\int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G)^{\mathrm{stbl}}_{\mathbf{B}G} \xrightarrow[\sim]{\widetilde{\int_{\mathrm{ev}}}} \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G}\,. \tag{4.94}$$

(ii) *When the* G*-action on* Γ *is trivial, this reduces to*

$$\int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma)^{\mathrm{stbl}} \xrightarrow[\sim]{\widetilde{\int_{\mathrm{ev}}}} \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma)\,. \tag{4.95}$$

Proof. First, we may use the above lemmas to obtain the bijections (4.99) in the unsliced case, for any Γ' of truncated classifying shape, notably for $\Gamma' \in \{\Gamma, \Gamma \rtimes G\}$:

$$\begin{aligned} \tau_0 \int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma')^{\mathrm{stbl}} &\simeq \tau_0 \int \mathrm{Map}\big((\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \mathbf{B}\Gamma'\big) && \text{by Lem. 4.1.53} \\ &\simeq \tau_0\, \mathrm{Map}\big(\int (\mathrm{X} \times S^{n+2}_{\mathrm{sm}})/G, \int \mathbf{B}\Gamma'\big) && \text{by Thm. 3.3.51} \\ &\simeq \tau_0\, \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma') && \text{by Lem. 4.1.32}\,. \end{aligned} \tag{4.96}$$

This implies, for $\mathrm{K} \in \mathrm{SmthMfd} \hookrightarrow G\mathrm{Act}(\mathrm{SmthMfd})$ a smooth manifold regarded as equipped with the trivial G-action, that:

$$\begin{aligned} \tau_0\, \mathrm{Map}\Big(\int \mathrm{K}, \int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma')^{\mathrm{stbl}}\Big) &\simeq \tau_0 \int \mathrm{Map}\Big(\mathrm{K}, \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma')^{\mathrm{stbl}}\Big) && \text{by Thm. 3.3.51} \\ &\simeq \tau_0 \int \mathrm{Map}\big(\mathrm{K} \times \mathrm{X} /\!\!/ G, \mathbf{B}\Gamma'\big)^{\mathrm{stbl}} && \text{by Lem. 3.2.60} \\ &\simeq \tau_0\, \mathrm{Map}\big(\int (\mathrm{K} \times \mathrm{X} /\!\!/ G), \int \mathbf{B}\Gamma'\big) && \text{by (4.96)} \\ &\simeq \tau_0\, \mathrm{Map}\big((\int \mathrm{K}) \times (\int \mathrm{X} /\!\!/ G)), \int \mathbf{B}\Gamma'\big) && \text{by (3.122)} \\ &\simeq \tau_0\, \mathrm{Map}\big(\int \mathrm{K}, \mathrm{Map}(\int \mathrm{X} /\!\!/ G), \int \mathbf{B}\Gamma')\big) && \text{by Lem. 3.2.60}\,, \end{aligned}$$

hence that

$$\underset{\mathrm{K} \in \mathrm{SmthMfd}}{\forall} \quad \tau_0\, \mathrm{Map}\Big(\int \mathrm{K}\ \ , \int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma')^{\mathrm{stbl}} \to \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma')\Big) \ \text{ is a bijection.} \tag{4.97}$$

Considering this for the cases

$$\mathrm{K} = *\,, \qquad \mathrm{K} = S^k_{\mathrm{sm}},\ k \in \mathbb{N}_+\,, \qquad \mathrm{K} = \mathbb{R}^2 \setminus \Big(\coprod_{\pi_1 \mathrm{Map}(\mathrm{X} /\!\!/ G, B\Gamma')} *\Big), \tag{4.98}$$

(here the last item uses the assumption that $\pi_1 \mathrm{Map}\big(\mathrm{X} /\!\!/ G, B(\Gamma \rtimes G)\big)$ is a countable set, Rem. 4.1.54, in order to embed that many points into $\mathbb{R}^2$) and observing that, by Prop. 3.3.40,

$$\int * \simeq *\,, \qquad \int S^k_{\mathrm{sm}} \simeq S^k, \qquad \int \bigg(\mathbb{R}^2 \setminus \Big(\coprod_{H^0(\mathrm{X} /\!\!/ G; \int \Gamma')} *\Big)\bigg) \simeq \bigvee_{H^1(\mathrm{X} /\!\!/ G; \int \Gamma')} S^1\,,$$

implies the second claim (4.95), by Lem. 3.1.20 applied to (4.97).

From this follows the first claim (4.94) by the fact that the shape modality preserves fiber products over discrete objects (Prop. 3.3.8):

$$
\begin{aligned}
\int \mathrm{Map}\big(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G)\big)^{\mathrm{stbl}}_{BG}
& \simeq \int\Big(\mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G))^{\mathrm{stbl}} \underset{\mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}G)}{\times} \{(X \to *) /\!\!/ G\}\Big) && \text{by Def. 3.2.65}
\\
& \simeq \int\Big(\mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G))^{\mathrm{stbl}} \underset{\mathrm{Map}(\int \mathrm{X} /\!\!/ G, BG)}{\times} \{(\int X \to *) /\!\!/ G\}\Big) && \text{by Lem. 3.3.13}
\\
& \simeq \int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G)) \underset{\mathrm{Map}(\int \mathrm{X} /\!\!/ G, BG)}{\times} \{(\int X \to *) /\!\!/ G\} && \text{by Prop. 3.3.8}
\\
& \simeq \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma /\!\!/ G) \underset{\mathrm{Map}(\int \mathrm{X} /\!\!/ G, BG)}{\times} \{(\int X \to *) /\!\!/ G\} && \text{by (4.95)}
\\
& \simeq \mathrm{Map}(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma /\!\!/ G)_{BG} && \text{by Def. 3.2.65.}
\end{aligned}
$$

□

From this, we immediately obtain, in equivariant generalization of Thm. 4.1.12:

Theorem 4.1.56 (Concordance classification of smooth equivariant principal bundles)**.** *Given*

– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*

– $\Gamma \rtimes G \in \mathrm{Grp}(\mathrm{SmthGrpd}_0)$ *of truncated classifying shape (Ntn. 4.1.30),*

– *a notion of blowup-stability of G-equivariant* $\Gamma \rtimes G$*-principal bundles (Ntn. 4.1.33);*

– $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$ *such that* $\pi_1 \mathrm{Map}\big(\mathrm{X} /\!\!/ G, B(\Gamma \rtimes G)\big)$ *is countable (Rem. 4.1.54);*

there is a natural identification of concordance classes (Def. 2.2.6) of stable G-equivariant Γ*-principal bundles* (4.29) *with first non-abelian Borel-equivariant cohomology:*

$$
\begin{array}{ccc}
\big(G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}_{\mathrm{X}}\big)_{/\sim_{\mathrm{conc}}} & \xrightarrow[\sim]{\quad\quad} & H^1_G(X; \int\Gamma)
\\
\| & & \|
\\
\tau_0 \int \mathrm{Map}\big(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G\big)^{\mathrm{stbl}}_{BG} & \xrightarrow[\sim]{\widetilde{\int \mathrm{ev}}} & \tau_0 \, \mathrm{Map}\big(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\Gamma /\!\!/ G\big)_{BG} .
\end{array}
\tag{4.99}
$$

Proof. This is the restriction to connected components of (4.94) in Thm. 4.1.55. □

Below in Thm. 4.2.7 we refine this classification statement from concordance classes to isomorphism classes of stable equivariant bundles.

Example 4.1.57 (Orbi-smooth Oka principle for PU_ω-coefficients over ADE-singularities)**.** Let $G \subset \mathrm{Sp}(1)$ be a finite subgroup (Ex. 3.3.55) and consider the structure group PU_ω (Ex. 1.3.19) with the corresponding stability condition (Thm. 4.1.47). We describe the shape of the topological space of conjugations of stable morphisms $\mathbf{B}G \to \mathbf{B}\mathrm{PU}_\omega$, hence of isomorphisms of stable G-equivariant PU_ω-principal bundles over the point, according to the orbi-smooth Oka principle (Thm. 4.1.55). This may be summarized by the following diagram:

$$
\begin{tikzcd}
\mathrm{Map}(BG, B^3\mathbb{Z}) \arrow[r, "\text{Ex. 2.3.29}", "\sim"'] \arrow[d, "(4.101)"', "\sim"] & \mathrm{Map}(\int \mathbf{B}G, \int \mathbf{B}\mathrm{PU}_\omega) \arrow[r, "\text{Thm. 4.1.55}", "\sim"'] & \int \mathrm{Map}(\mathbf{B}G, \mathbf{B}\mathrm{PU}_\omega)^{\mathrm{stbl}}
\\
B\mathrm{Hom}(G, \mathrm{U}_1) \times B^3\mathbb{Z} \;\simeq\; B\Big(\mathrm{Hom}(G, \mathrm{U}_1) \times \big(\prod_{\mathrm{Irr}(G)} \mathrm{U}_\omega\big)/\mathrm{U}_1\Big) \arrow[rr, "B\mathrm{Ad}((-)\otimes(-))\ (4.104)"'] & & \arrow[u]
\end{tikzcd}
\tag{4.100}
$$

(Here $\mathrm{Irr}(G)$ denotes the set of isomorphism classes of irredible unitary G-representations.)

The interesting point to highlight is what concretely the morphisms in $\mathrm{Map}(\mathbf{B}G, \mathbf{B}\mathrm{PU}_\omega)^{\mathrm{stbl}}$ (on the top right) are like and how exactly these have the shape predicted by the orbi-smooth Oka principle (on the top left). After some re-casting of the mapping shape on the bottom left, this is brought out by the assignment on the bottom right, which we describe in (4.104) below.

First to see the equivalence on the left of (4.100), we use that the integral group cohomology of G is concentrated in even degrees (3.181) and that the second integral cohomology gives the group characters (3.182): The first fact gives $\tau_0\,\mathrm{Map}(BG, B^3\mathbb{Z}) \simeq *$ which implies (with Prop. 3.2.70) that $\mathrm{Map}(BG, B^3\mathbb{Z}) \simeq B\Omega\mathrm{Map}(BG, B^3\mathbb{Z}) \simeq B\mathrm{Map}(BG, \Omega B^3\mathbb{Z}) \simeq B\mathrm{Map}(BG, B^2\mathbb{Z})$; and then the second fact gives $\tau_0\,\mathrm{Map}(BG, B^2\mathbb{Z}) \simeq \mathrm{Hom}(G, \mathrm{U}_1)$. Proceeding in this manner shows that each of these connected components has homotopy groups concentrated in degree 2 on $\mathrm{Map}(BG, \Omega^3 B^3\mathbb{Z}) \simeq \mathrm{Map}(BG, \mathbb{Z}) \simeq \mathbb{Z}$, hence that $\mathrm{Map}(BG, B^2\mathbb{Z}) \simeq \mathrm{Hom}(G, \mathrm{U}_1) \times B^2\mathbb{Z}$. In conclusion:

$$\mathrm{Map}(BG, B^3\mathbb{Z}) \simeq B\big(H^2_{\mathrm{Grp}}(G;\mathbb{Z}) \times B^2 H^0_{\mathrm{Grp}}(G;\mathbb{Z})\big) \simeq B\big(\mathrm{Hom}(G, \mathrm{U}_1) \times B^2\mathbb{Z}\big) \simeq B\mathrm{Hom}(G, \mathrm{U}_1) \times B^3\mathbb{Z}\,. \tag{4.101}$$

The fact that the space (4.101) is connected means that there is an essentially unique stable morphism $\mathbf{B}G \to \mathbf{B}\mathrm{PU}_\omega$. Indeed, by Lem. 4.1.45 these stable projective representations must be the direct sums of infinite copies of all $[\tau]$-projective G-representations for some $[\tau] \in H^2_{\mathrm{Grp}}(G;\mathrm{U}_1) \simeq H^3_{\mathrm{Grp}}(G;\mathbb{Z})$; but since the third integral group cohomology in the present case is trivial (3.181), there is only the case $[\tau] = [0]$, and hence the only stable projective representation is, up to isomorphism, the projectivization of

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{U}\Big(\bigoplus\limits_{\mu \in \mathrm{Irr}(G)} \mu \otimes \mathscr{H}\Big) \\ g & \longmapsto & \bigoplus\limits_{\mu} \mu(g) \otimes \mathrm{id} \end{array} \tag{4.102}$$

for any countably infinite-dimensional Hilbert space $\mathscr{H}$.

Therefore the question is: *How can the group characters $\kappa \in \mathrm{Hom}(G, \mathrm{U}_1)$ encode projective endo-intertwiners of the stable representation* (4.102)*?* Direct inspection shows that on $\bigoplus\limits_{\mu} \mu$ this is accomplished by the linear map which direct summand-wise is the linear identification

$$\begin{array}{ccc} \mu & \longrightarrow & \mu \otimes \big(\kappa \times_{\mathrm{U}_1} \mathbb{C}\big) \\ v & \longmapsto & v \otimes 1 \end{array} \tag{4.103}$$

of the irrep μ with its tensor product with κ, the latter regarded as a complex linear representation on $\mathbb{C}$ with canonical basis vector $1 \in \mathbb{C}$. This operation fails, in general, to be a genuine unitary homomorphism, but the failure is exactly the phases $\kappa(g) \in \mathrm{U}_1$, so that this does yield a projective intertwiner, as shown by the following commuting diagram (using the 2-groupoidal notation for the projective group from Ex. 3.3.30):

$$\begin{tikzcd} \mathbf{B}G \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \big(\kappa,\, [U_\mu]_\mu\big)"] & & \mathbf{B}\mathrm{U}_\omega /\!\!/ \mathbf{B}\mathrm{U}_1 \simeq \mathbf{B}\mathrm{PU}_\omega \\ \bullet \arrow[dd, "g"] & \bigoplus\limits_{\mu \in \mathrm{Irr}(G)} \mu \otimes \mathscr{H} \arrow[rr, "\bigoplus\limits_{\mu}\Big(\big(\mu \xrightarrow{v \mapsto v\otimes 1} \mu\otimes\kappa\big)\otimes U_\mu\Big)"] \arrow[dd, "\bigoplus\limits_{\mu} \mu(g)\otimes \mathrm{id}"] & & \bigoplus\limits_{\mu \in \mathrm{Irr}(G)} \mu \otimes \mathscr{H} \arrow[dd, "\bigoplus\limits_{\mu} \mu(g)\otimes \mathrm{id}"] \\ \longmapsto & & \kappa(g) \Nearrow & \\ \bullet & \bigoplus\limits_{\mu \in \mathrm{Irr}(G)} \mu \otimes \mathscr{H} \arrow[rr, "\bigoplus\limits_{\mu}\Big(\big(\mu \xrightarrow{v \mapsto v\otimes 1} \mu\otimes\kappa\big)\otimes U_\mu\Big)"'] & & \bigoplus\limits_{\mu \in \mathrm{Irr}(G)} \mu \otimes \mathscr{H} \end{tikzcd} \tag{4.104}$$

Here the $U_\mu \in \mathrm{U}_\omega$ are any unitary operators acting on the Hilbert space of infinite copies of the irrep μ.

Example 4.1.58 (Orbi-smooth Oka principle for PU_ω-coefficients over A-singularities)**.** It is instructive to further spell out the construction of the projective endo-intertwiners from Ex. 4.1.57 for the case that $G = \mathbb{Z}_{/n}$ is a cyclic group. Here $\mathrm{Irr}(G) = \mathrm{Irr}(\mathbb{Z}_{/n}) \simeq \mathbb{Z}_{/n}$ and the standard $n \times n$ matrix presentation of the regular representation $\bigoplus\limits_{\mu} \mu$ is

$$1 \leq q, q' \leq n \qquad \vdash \qquad \big(U_{\mathrm{reg}}([k])\big)_{q,q'} \;=\; \exp\big(2\pi \mathrm{i} \cdot k \tfrac{q-1}{n}\big) \cdot \delta_{q,q'}\,.$$

Moreover, the tensoring with $\kappa \in \mathrm{Hom}(\mathbb{Z}_{/n}, \mathrm{U}_1) \simeq \mathbb{Z}_{/n}$ acts on these irreps by cyclically permuting them, so that the matrix representation for the projective intertwiner (4.103) is

$$1 \leq q, q' \leq n \qquad \vdash \qquad \big(U_\kappa\big)_{q,q'} \;=\; \delta_{q,q'-\kappa}\,.$$

That this gives the required diagram (4.104)

$$\begin{tikzcd}
\bullet \arrow[d, "{[k]}"] & \bullet \arrow[r, "U_{([\kappa],[\vec{c}])}"] \arrow[d, "U_{\mathrm{reg}}\left([k]\right)"'] \arrow[dr, phantom, "\theta_{[\kappa]}([k])"] & \bullet \arrow[d, "U_{\mathrm{reg}}\left([k]\right)"] \\
\bullet & \bullet \arrow[r, "U_{([\kappa],[\vec{c}])}"'] & \bullet
\end{tikzcd} \tag{4.105}$$

follows from a straightforward computation:

$$\begin{aligned}
\big(U_{\mathrm{reg}}([k]) \cdot U_{[\kappa],[\vec{c}]}\big)_{q,q'} &\;=\; \exp\big(2\pi \mathrm{i} \cdot k \tfrac{q-1}{n}\big) \cdot \delta_{q,q'-\kappa} \\
\big(U_{[\kappa],[\vec{c}]} \cdot U_{\mathrm{reg}}([1])\big)_{q,q'} &\;=\; \exp\big(2\pi \mathrm{i} \cdot k \tfrac{\kappa+q-1}{n}\big) \cdot \delta_{q,q'-\kappa}
\end{aligned}$$

whence

$$\theta_{[\kappa]}([k]) \;=\; \exp\big(2\pi \mathrm{i} \cdot k \tfrac{\kappa}{n}\big)\,.$$

Curiously, the projective self-intertwining relation (4.105) of the [0]-projective regular $\mathbb{Z}_{/n}$-representation happens to coincide, in the case $\kappa = 1$, with the defining equation of the finite matrix approximation of the "non-commutative torus" as in [LLS01, (2.29)].

4.2 Local local triviality is implied

We have seen in §2.1 that the definition of principal bundles, in Cartan's original sense and as formalized by Grothendieck's internal notion of pseudo-torsors, namely without requiring local triviality (Rem. 2.1.1), has excellent abstract mathematical properties that make their correct generalization to equivariant principal bundles (or any other generalization) a purely formal matter. While a form of local triviality of bundles may also be formulated internally as soon as the ambient category is regular, we have seen in §2.2 that, internal to 1-categories of group actions, the resulting notion is overly restrictive, and that some ingenuity and labor is required to establish a notion of equivariant local triviality that seems more appropriate.

However, in §4.1, we observed that the analogous internalization procedure that yields equivariant bundles internal to the category of topological spaces applies to yield equivariant ∞-bundles internal to any ∞-topos. Here we prove (Thm. 4.3.24 below) that the latter, when seen externally, are *automatically* locally trivial in the proper equivariant sense of §2.2. We may think of this as saying that Cartan's original definition of principal bundles is the *correct* one, when regarded, with Grothedieck, as a formal theory waiting to be internalized (interpreted) in an ambient category with finite limits; and that the constraint of local triviality is not part of the formal theory of principal bundles, but is to be provided by the semantic context in which the theory is interpreted.

Thm. 4.3.24 below shows that contexts which do provide this feature in a useful way are the Grothendieck(-Simpson-Toën-Vezzosi-Lurie-Rezk) ∞-toposes. In fact, their underlying $(2,1)$-toposes are sufficient for capturing equivariant local triviality with respect to equivariance 1-groups, as $(n+1,1)$-toposes will be sufficient for capturing equivariance n-groups, but traditional 1-toposes are sufficient only for equivariance 0-groups, which includes only the trivial group. Specifically, the model $\mathbf{H} = \mathrm{SmthGrpd}_\infty$ is such that interpreting equivariant principal bundles inside it, in Cartan-Grothendieck's sense, and then restricting the resulting notion along the full inclusion $\mathrm{DTopSpc} \hookrightarrow \mathrm{SmthGrpd}_\infty$, recovers the traditional notion of equivariant principal bundles, *including* now the proper equivariant local triviality constraint.

In this sense, the traditional theory of equivariant bundles is completed by embedding it into the more general theory of equivariant ∞-bundles in singular-cohesive ∞-toposes.

Notation 4.2.1 (Equivariance groups)**.** Throughout this section,

(i) $G \in \mathrm{Grp}(\mathrm{Set}) \xhookrightarrow{\mathrm{Grp}(\mathrm{Dsc})} \mathrm{Grp}\,(\Delta\mathrm{PSh}(\mathrm{CartSpc}))$
denotes a discrete equivariance group regarded as a constant simplicial preaheaf of groups,

(ii) $G \circlearrowright \Gamma \in \mathrm{Grp}\,(G\mathrm{Act}(\mathrm{DHausSpc})) \longhookrightarrow \mathrm{Grp}\,(G\mathrm{Act}\,(\Delta\mathrm{PSh}(\mathrm{CartSpc})))$
denotes a G-equivariant D-topological Hausdorff group Γ, regarded as a group in G-actions on simplicially constant simplicial presheaves.

The universal equivariant principal bundle over the moduli stack. First, we generalize the construction in Rem. 4.1.4 of universal principal bundles over the moduli stack to the equivariant context.

Definition 4.2.2 (Crossed adjoint action through a crossed homomorphism)**.** Given a crossed homomorphism $\phi : G \hookrightarrow \Gamma$, hence a homomorphic lift $\widehat{(-)}; G \to \Gamma \rtimes G$, we denote by $G \circlearrowright \Gamma_{\widehat{\mathrm{adj}}} \in G\mathrm{Act}(\mathrm{kTopSpc})$ the adjoint action which on the right is twisted by $\widehat{(-)}$:

$$
\begin{array}{ccc}
G \times \Gamma & \longrightarrow & \Gamma \\
(g,\gamma) & \longmapsto & \alpha(g)(\gamma\cdot\phi(\gamma)^{-1}) = (\mathrm{e},g)\cdot(\gamma,\mathrm{e})\cdot\widehat{g}^{-1}
\end{array}
$$

Lemma 4.2.3 (The universal G-equivariant Γ-principal bundle)**.** *The action groupoid of the crossed adjoint action $\Gamma_{\widehat{\mathrm{adj}}}$ (Def. 4.2.2) provides a fibrant replacement* $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\mathrm{proj}}$ *of the delooping of $G \xrightarrow{g\mapsto(\mathrm{e},g)} \Gamma \rtimes G$:*

$$
\begin{tikzcd}
N(G \rightrightarrows *) \arrow[rr, "N(g\mapsto(\mathrm{e},(\mathrm{e},g)))", "\in\mathrm{WEqs}"'] \arrow[dr, "N(g\mapsto(\mathrm{e},g))"'] & & N\big(\Gamma_{\widehat{\mathrm{adj}}} \times (\Gamma\rtimes G) \rightrightarrows \Gamma\big). \arrow[dl, "N((\gamma',g)\mapsfrom(\gamma,(\gamma',g)))", "\in\mathrm{PrjFib}"'] \\
& N((\Gamma\rtimes G) \rightrightarrows *) &
\end{tikzcd}
$$

Proof. It is clear that the diagram commutes and that the right morphism is a projective fibration. To see that the top morphism is a weak equivalence it is sufficient to observe that we have in fact a homotopy equivalence between the representing groupoids:

$$
\begin{array}{ccc}
g & \longmapsfrom & (\gamma,(\gamma',g)) \\
(* \times G \rightrightarrows *) & \rightleftarrows & (\Gamma \times (\Gamma\rtimes G) \rightrightarrows \Gamma) \\
g & \longmapsto & (\mathrm{e},(\mathrm{e},g))\,.
\end{array}
$$

In one direction, the composite of these two functors equals the identity, in the other we have the following natural transformation to the identity:

$$
\begin{tikzcd}
\gamma \arrow[r, "(\gamma{,}\mathrm{e})^{-1}"] \arrow[d, "(\gamma'{,}g)"'] & \mathrm{e} \arrow[d, "(\mathrm{e}{,}g)"] \\
\alpha(g^{-1})(\gamma\cdot\gamma') \arrow[r, "(\alpha(g^{-1})(\gamma\cdot\gamma'){,}\mathrm{e})^{-1}"'] & \mathrm{e}
\end{tikzcd}
$$

□

Proposition 4.2.4 (Equivariant principal bundles from Čech cocycles)**.** *For*
– $G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$,
– $G \circlearrowright \Gamma \in \mathrm{Grp}\,(G\mathrm{Act}(\mathrm{DTopSpc})) \longhookrightarrow \mathrm{Grp}\,(G\mathrm{Act}(\mathrm{SmthGrpd}_\infty))$,
– $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$,
the operation of forming the homotopy fiber of modulating morphisms lands in G-action groupoids of bundles in $\mathrm{DTopSpc} \hookrightarrow \mathrm{SmthGrpd}_\infty$ *and constitutes a natural fully faithful embedding*

$$
\begin{tikzcd}
\mathbf{H}\big(\mathrm{X}/\!\!/G, \mathbf{B}(\Gamma\rtimes G)\big)_{BG} \arrow[r, phantom, "\simeq"] \arrow[rr, bend right, "\mathrm{hofib}"'] & G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)_{\mathrm{X}} \arrow[r, hook] & G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathrm{DTopSpc})_{\mathrm{X}}
\end{tikzcd}
\tag{4.106}
$$

of the groupoids of G-equivariant Γ*-principal bundles internal to* $\mathrm{SmthGrpd}_\infty$ *and internal to* $\mathrm{DTopSpc}$*, respectively.*

Proof. Abbreviating $\mathbf{H} := \mathrm{SmthGrpd}_\infty$, we want to compute the operation that sends a cocycle c to its homotopy fiber in $\mathbf{H}_{/\mathbf{B}G}$, which is equivalently given by the following homotopy pullback in $\mathbf{H}$:

$$
\begin{tikzcd}
\mathrm{P} /\!\!/ G \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}G \arrow[d] \\
\mathrm{X} /\!\!/ G \arrow[rr, "c"] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
\tag{4.107}
$$

We will compute this in model category $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$ by using the following representatives:

(1) By Prop. 1.1.25), we may find a properly equivariant good open cover $G \circlearrowright \widehat{\mathrm{X}} := G \circlearrowright (\underset{i \in I}{\sqcup} \mathrm{U}_i) \twoheadrightarrow G \circlearrowright \mathrm{X}$ (Def. 1.1.24) and, by Ex. 3.3.42, this yields a local projective cofibrant representative $N(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}})$ of $\mathrm{X} /\!\!/ G$.

(2) By Prop. 4.1.1, we have a local projective fibrant representative for $\mathbf{B}(\Gamma \rtimes G)$ given by $N(\Gamma \rtimes G \rightrightarrows *) \in \Delta\mathrm{PSh}(\mathrm{CartSpc})_{\substack{\mathrm{proj}\\ \mathrm{loc}}}$.

(3) We have the evident projective fibration

$$
\begin{array}{ccc}
N(\Gamma \rtimes G \rightrightarrows *) & & N(\mathrm{Map}(\mathbb{R}^n, \Gamma \rtimes G) \rightrightarrows *) \\
\Big\downarrow{\scriptstyle \in \mathrm{PrjFib}} & : \quad \mathbb{R}^n \longmapsto & \Big\downarrow{\scriptstyle \in \mathrm{KanFib}} \\
N(G \rightrightarrows *) & & N(\mathrm{Map}(\mathbb{R}^n, \Gamma \rtimes G) \rightrightarrows *) \,.
\end{array}
$$

(4) By the simplicial model enrichment of $\Delta\mathrm{PSh}(\mathrm{CartSpc})_{\mathrm{proj}}$, this implies the following Kan fibration model:

$$
\begin{array}{ccc}
\mathbf{H}\big(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G)\big) & \simeq & \Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(N(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}), N(\Gamma \rtimes G \rightrightarrows *)\Big) \\
\Big\downarrow & & \downarrow{\scriptstyle \in \mathrm{KanFib}} \\
\mathbf{H}\big(\mathrm{X} /\!\!/ G, \mathbf{B}G\big) & \simeq & \Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(N(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}), N(G \rightrightarrows *)\Big).
\end{array}
$$

(5) Hence we have a presentation of the equivariant bundles by equivariant Čech cocycles (Rem. 3.3.43):

$$
\begin{aligned}
\mathbf{H}\big(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} & \simeq \mathrm{fib}\left(\begin{array}{c} \mathbf{H}(\mathrm{X} /\!\!/ G, \mathbf{B}(\Gamma \rtimes G)) \\ \downarrow{\scriptstyle \mathbf{B}\mathrm{pr}_2} \\ \mathbf{H}(\mathrm{X} /\!\!/ G, \mathbf{B}G) \end{array}\right) \qquad \text{by Prop. 3.2.63} \\
& \simeq \mathrm{fib}\left(\begin{array}{c} \Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(N\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}\big), N(\Gamma \rtimes G \rightrightarrows *)\Big) \\ \downarrow \\ \Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(N\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}\big), N(G \rightrightarrows *)\big)\Big) \end{array}\right) \\
& \simeq \Delta\mathrm{PSh}(\mathrm{CartSpc})\Big(N\big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}\big), N(\Gamma \rtimes G \rightrightarrows *)\Big)_{N(G \rightrightarrows *)}
\end{aligned}
\tag{4.108}
$$

(6) Therefore, and using Lem. 3.2.32, the homotopy pullback (4.107) is equivalently computed by the following pullback of simplicial presheaves (in equivariant generalization of Rem. 4.1.4):

$$
\begin{tikzcd}
\overset{\curvearrowright G}{\mathrm{P}} \arrow[d] & \big(\mathrm{P} \times G^{\mathrm{op}} \rightrightarrows \mathrm{P}\big) \arrow[d] & \big(\Gamma \times \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \Gamma \times \widehat{\mathrm{X}}\big) \arrow[l, two heads, "\in \mathrm{LclWEqs}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Big(\Gamma_{\widehat{\mathrm{adj}}} \times (\Gamma \rtimes G) \rightrightarrows \Gamma\Big) \arrow[d, "\in \mathrm{PrjFib}"] \\
\underset{\curvearrowleft G}{\mathrm{X}} & \big(\mathrm{X} \times G^{\mathrm{op}} \rightrightarrows \mathrm{X}\big) & \big(\widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times G^{\mathrm{op}} \rightrightarrows \widehat{\mathrm{X}}\big) \arrow[l, two heads, "\in \mathrm{LclWEqs}"'] \arrow[r] \arrow[dr] & ((\Gamma \rtimes G) \rightrightarrows *) \arrow[d] \\
& & & N(G \rightrightarrows *)
\end{tikzcd}
\tag{4.109}
$$

The equivariant principal bundles obtained this way are by construction all locally trivializable over $\widehat{\mathrm{X}}$, which means that the morphisms between them are in natural bijection to the morphsims between their local trivialization data, hence between the Čech cocycles on the right. □

Lemma 4.2.5 (Equivariant principal bundles constructed from Čech cocycles are proper equivariant fibrations)**.** *The G-equivariant Γ-principal ∞-bundles obtained as in* (4.109) *are proper equivariant Serre fibrations (as in Prop. 1.3.4).*

Proof. By Ex. 3.3.42, we may compute cocycles on Čech-action groupoids of equivariant good open covers. The underlying bundle is evidently locally trivial while only the G-action may vary. Hence fixed-point-wise we have locally trivial bundles, hence Serre fibrations. □

Lemma 4.2.6 (Concordant equivariant principal bundles constructed from Čech cocycles are isomorphic)**.** *Equivariant principal bundles in the image of* (4.106) *are isomorphic as soon as they are concordant.*

Proof. With Lem. 4.2.5, this follows by the same proof as Thm. 2.2.8. □

In equivariant generalization of Thm. 4.1.2 and Thm. 4.1.13, we now have:

Theorem 4.2.7 (Isomorphism classification of stable equivariant bundles with truncated structure over good orbifolds with resolvable singularities)**.** *For*

– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*

– $\Gamma \in G\mathrm{Act}\big(\mathrm{Grp}(\mathrm{DTopSpc})\big)$ *well-pointed (Ntn. 1.3.17), of truncated classifying shape (Ntn. 4.1.30), with countably many connected components, and with a notion of blowup-stability of G-equivariant $\Gamma \rtimes G$-principal bundles (Ntn. 4.1.33)*

then:

(i) *the construction* (4.106) *from Prop. 4.2.4 on stable equivariant bundles is an equivalence onto the full subgroupoid of equivariantly locally trivial bundles (Def. 2.2.2):*

$$\underbrace{\mathbf{H}\big(\mathrm{X}/\!\!/G,\, \mathbf{B}(\Gamma \rtimes G)\big)^{\mathrm{stbl}}_{\mathbf{B}G} \simeq G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}_{\mathrm{X}} \xrightarrow{\;\sim\;} G\mathrm{Equv}\,\Gamma\mathrm{PrnFibBdl}(\mathrm{DTopSpc})^{\mathrm{stbl}}_{\mathrm{X}}}_{\underset{\sim}{\mathrm{hofib}}} . \tag{4.110}$$

(ii) *Isomorphism classes of these equivariant topological bundles are classified by $\int\mathbf{B}(\Gamma \rtimes G) \simeq B\int\Gamma \rtimes G \in (\mathrm{Grpd}_\infty)_{/\mathbf{B}G}$ (see Ex. 4.1.10), hence coincide with equivalence classes of of G-equivariant $\int\Gamma$-principal ∞-bundles (Def. 4.1.22) in that:*

$$\begin{aligned}
\big(G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} &\simeq \tau_0\,\mathrm{Map}\big(\textstyle\int\mathrm{X}/\!\!/G,\, B\Gamma/\!\!/G\big)^{\mathrm{stbl}}_{\mathbf{B}G} \\
&\simeq \big(G\mathrm{Equv}\textstyle\int\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} .
\end{aligned} \tag{4.111}$$

Proof. The second statement is the following composite bijection:

$$\begin{aligned}
\big(G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}\big)_{/\sim_{\mathrm{iso}}} &\simeq \big(G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathrm{SmthGrpd}_\infty)^{\mathrm{stbl}}\big)_{/\sim_{\mathrm{conc}}} && \text{by Lem. 4.2.6} \\
&\simeq \tau_0\textstyle\int\mathrm{Map}\big(\mathrm{X}/\!\!/G,\, \mathbf{B}\Gamma/\!\!/G\big)^{\mathrm{stbl}}_{\mathbf{B}G} && \text{by Ex. 3.3.47} \\
&\simeq \tau_0\,\mathrm{Map}\big(\textstyle\int\mathrm{X}/\!\!/G,\, \textstyle\int\mathbf{B}\Gamma/\!\!/G\big)^{\mathrm{stbl}}_{\mathbf{B}G} && \text{by Thm. 4.1.55.}
\end{aligned}$$

Using this we now obtain the first statement: Consider an equivariant principal bundle $G \circlearrowright \mathrm{P} \to G \circlearrowright \mathrm{X}$ obtained as in (4.109) relative to a choice of proper equivariant regular good open cover (Def. 1.1.24) $G \circlearrowright (\underset{i \in I}{\sqcup} \mathrm{U}_i) \twoheadrightarrow G \circlearrowright \mathrm{X}$. By Thm. 2.2.1 it is sufficient to exhibit, for every $x \in \mathrm{X}$ with stabilizer subgroup $G_x := \mathrm{Stab}_G(x) \subset G$, a G_x-equivariant neighborhood $G_x \circlearrowright \mathrm{U}_x$ of x over which $G_x \circlearrowright \mathrm{P}$ restricts to a Bierstone local model (2.43).

Now, by the fact that we have an open cover, there exists $i \in I$ with $x \in \mathrm{U}_i$; and by Rem. 1.1.26 this is already a G_x-neighborhood $\mathrm{U}_x := \mathrm{U}_i$. Restricting the construction (4.109) to this patch yields the following diagram:

$$
\begin{tikzcd}
N\left((\Gamma\times \mathrm{U}_x)_{\rho_x}\times G_x^{\mathrm{op}} \rightrightarrows (\Gamma\times\mathrm{U}_x)_{\rho_x}\right) \arrow[rrr] \arrow[d] & & & N\left(\Gamma_{\widehat{\mathrm{adj}}}\times(\Gamma\rtimes G)\rightrightarrows\Gamma\right) \arrow[d] \\
N\left(\mathrm{U}_x\times G_x^{\mathrm{op}}\rightrightarrows\mathrm{U}_x\right) \arrow[r, "N\left(\mathrm{U}_i\to *\times(-)^{-1}\rightrightarrows \mathrm{U}_i\to *\right)"] \arrow[rrr, bend right, "c_{|\mathrm{U}_i}"'] & N\left(G_x\rightrightarrows *\right) \arrow[rr, dashed, "\left(\mathrm{pr}\times\widehat{(-)}^{-1}\rightrightarrows\mathrm{pr}\right)"] \arrow[urr, phantom, "\mathrm{(pb)}"] & & N\left((\Gamma\rtimes G)\rightrightarrows *\right).
\end{tikzcd}
$$

Here the factorization, up to some simplicial homotopy shown at the bottom, of the restricted cocycle $c_{|\mathrm{U}_i}$ through a dashed morphism (as shown) follows by the classification statement (4.111) which we have already proven, using now that U_x is a patch in a *good* open cover and hence contractible, so that any cocycle for G_x-equivariant Γ-principal bundles on U_x factors through

$$
\int(\mathrm{U}_x /\!\!/ G_x) \;\simeq\; * /\!\!/ G_x \;\simeq\; \mathrm{Loc}^{\mathrm{LclWEqs}}\left(N\left(G_x\rightrightarrows *\right)\right).
$$

But this dashed morphism of simplicial presheaves is manifestly a crossed homomorphism in the guise (1.2.13), hence a lift $\widehat{G_x}$ as in Ntn. 2.2.10:

$$
\begin{tikzcd}
N\left(G_x\rightrightarrows *\right) \arrow[rr, "\Delta\mathrm{PSh}\left(\widehat{(-)}\rightrightarrows *\right)"] \arrow[dr, "N\left((-)^{-1}\rightrightarrows *\right)"'] & & N\left((\Gamma\rtimes G)\rightrightarrows *\right). \arrow[dl, "(\mathrm{pr}_2\rightrightarrows *)"] \\
& N\left(G\rightrightarrows *\right) &
\end{tikzcd}
$$

With $\Gamma_{\widehat{\mathrm{adj}}}$ from Def. 4.2.2, this identifies the above pullback bundle as the trivial Γ-bundle over U_x equipped with the following action:

$$
\begin{array}{ccc}
(\Gamma\times\mathrm{U}_x)_{\rho_x}\times G_x^{\mathrm{op}} & \longrightarrow & \Gamma\times\mathrm{U}_x \\
((\gamma,u),g) & \longmapsto & \mathrm{pr}_1\left((\mathrm{e},g)\cdot(\gamma,\mathrm{e})\cdot\widehat{g}^{-1}\right)
\end{array}
$$

This is precisely the action on local Bierstone models found in (2.45). Hence the above pullback exhibits the claimed equivariant local trivialization. □

Example 4.2.8 (G-Equivariant principal bundles over G-coset spaces from basic geometric homotopy theory)**.** For $G \circlearrowright \Gamma$ as in Thm. 4.1.13, we find, with Thm. 4.2.7, the form of locally trivial G-equivariant Γ-principal bundles on the coset spaces G/H as originally consider by tom Dieck in 1969 (Lem. 2.2.15) by a glance at the following homotopy Cartesian diagram in $\mathrm{SmthGrpd}_\infty$ (using just the basic facts from Ex. 3.2.35. Ex. 3.2.36 and Prop. 1.2.18):

$$
\begin{tikzcd}
\overbrace{(\Gamma\rtimes G)}^{\widehat{G}}/\widehat{H} \arrow[d] & \left((\Gamma\rtimes G)/\widehat{H}\right)/\!\!/G \arrow[rr] \arrow[d] & & \mathbf{B}G \arrow[d] \\
G/H & (G/H)/\!\!/G \arrow[r, "\sim", "\text{Ex. 3.2.35}"'] \arrow[dr] & \mathbf{B}H \arrow[r, "\mathbf{B}(i_{\widehat{H}})", "\text{Prop. 1.2.18}"'] \arrow[d, "\mathbf{B}(i_H)"] \arrow[ur, phantom, "\substack{\text{Ex. 3.2.36}\\ \text{(pb)}}"] & \mathbf{B}(\Gamma\rtimes G) \arrow[dl, "\mathbf{B}\mathrm{pr}_2"] \\
& & \mathbf{B}G. &
\end{tikzcd}
\tag{4.112}
$$

Lemma 4.2.9 ($(\Gamma\rtimes G)$-CW-Complex structure on equivariant bundles over G-CW complexes)**.** *Given $G \circlearrowright \Gamma$ as in Thm. 4.2.7, with Γ connected, and given a G-equivariant Γ-principal fiber bundle* P *over a G-CW complex* X

- $G \circlearrowright \mathrm{X} \in G\mathrm{CWCplx}$ *(Ex. 1.3.6)*
- $(\Gamma\rtimes G)\circlearrowright \mathrm{P} \in G\mathrm{Equiv}\,\Gamma\,\mathrm{PrnFibBdl}_{\mathrm{X}}$

then P *is a proper $(\Gamma\rtimes G)$-CW complex, namely a $(\Gamma\rtimes G)$-equivariant cell complex with respect to cell attachments of the form*

$$
(\Gamma\rtimes G)/\times\partial\Delta^n_{\mathrm{top}} \hookrightarrow (\Gamma\rtimes G)/\times\Delta^n_{\mathrm{top}}
$$

Proof. By the classification result of Thm. 4.2.7 we have a homotopy Cartesian square of the following form

$$\begin{tikzcd}
\mathrm{P}/\!\!/G \arrow[rr] \arrow[d, "p/\!\!/G"] \arrow[drr, phantom, "\text{(pb)}"] && \mathbf{B}G \arrow[d] \\
\mathrm{X}/\!\!/G \arrow[rr, "c"] \arrow[dr] && \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}$$

Denote the assumed G-CW-complex structure of X as follows:

$$\mathrm{X}_0 \;=\; \varnothing\,, \qquad \mathrm{X}_{\bullet+1} \;=\; \mathrm{X}_\bullet \coprod_{G/H_\bullet \times \partial\Delta^{n_\bullet}_{\mathrm{top}}} G/H_\bullet \times \Delta^{n_\bullet}_{\mathrm{top}}\,, \qquad \mathrm{X} \;=\; \varinjlim_{k\in\mathbb{N}} \mathrm{X}_k$$

Due to colimits in $\mathrm{SmthGrpd}_\infty$ being universal (Prop. 3.2.6) we find from this the above homotopy pullback that

$$\mathrm{P}_\bullet \;=\; \mathrm{X}_\bullet \underset{\mathbf{B}\Gamma}{\times} *\,, \qquad \mathrm{P} \;=\; \varinjlim_{k\in\mathbb{N}} \mathrm{P}_k\,.$$

Hence it is now sufficient to show that

$$\mathrm{P}_{\bullet+1} \;\simeq\; \mathrm{P}_\bullet \coprod_{(\Gamma\rtimes G)/\widehat{H_\bullet} \times \partial\Delta^{n_\bullet}_{\mathrm{top}}} (\Gamma \rtimes G)/\widehat{H_\bullet} \times \Delta^{n_\bullet}_{\mathrm{top}}$$

namely that for each $k \in \mathbb{N}$ we have homotopy Cartesian squares of this form:

$$\begin{tikzcd}
\mathrm{P}_{k+1}/\!\!/G \arrow[d, "p_{k+1}/\!\!/G"] & \simeq & \Big(\mathrm{P}_k \coprod_{(\Gamma\rtimes G)/\widehat{H_{k+1}} \times \partial\Delta^{n_{k+1}}_{\mathrm{top}}} (\Gamma\rtimes G)/\widehat{H_{k+1}} \times \Delta^{n}_{\mathrm{top}} \Big)/\!\!/G \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] && \mathbf{B}G \arrow[d] \\
\mathrm{X}_{k+1}/\!\!/G & \simeq & \Big(\mathrm{X}_k \coprod_{G/H_{k+1} \times \partial\Delta^{n_{k+1}}_{\mathrm{top}}} G/H_{k+1} \times \Delta^{n_{k+1}}_{\mathrm{top}} \Big)/\!\!/G \arrow[rr, "{\left(c_{k+1},\, \mathbf{B}\left(i_{\widehat{H}_{k+1}}\right)\right)}"] \arrow[dr] && \mathbf{B}(\Gamma\rtimes G) \arrow[dl] \\
&&& \mathbf{B}G\,. &
\end{tikzcd}$$

But (again by universal colimits and using the classification result Thm. 4.2.7 and the connectivity assumption to deduce that the cocycle must be trivializable on the simplex), this is the case as soon as we have homotopy Cartesian squares of this form:

$$\begin{tikzcd}
\big((\Gamma\rtimes G)/\widehat{H_{k+1}}\big)/\!\!/G \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] && \mathbf{B}G \arrow[d] \\
(G/H_{k+1})/\!\!/G \arrow[r, "\sim"] \arrow[dr] & \mathbf{B}H_{k+1} \arrow[r, "\mathbf{B}\left(i_{\widehat{H_{k+1}}}\right)"] \arrow[d] & \mathbf{B}(\Gamma\rtimes G) \arrow[dl] \\
& \mathbf{B}G\,, &
\end{tikzcd}$$

and this holds by Ex. 4.2.8. □

Equivariant fiber bundles. Specializing the general notion of associated fiber ∞-bundles ([NSS12a, Prop. 4.10] [SS20-Orb, Prop. 2.92]) to the equivariant context of Def. 4.1.22, we obtain the following notion of equivariant fiber ∞-bundles, which by Thm. 4.2.7 are immediately guaranteed to satisfy the traditional equivariant local triviality conditions.

Definition 4.2.10 (Equivariant associated ∞-bundles)**.** For
- $G \circlearrowright \Gamma \in G\mathrm{EquvGrp}(\mathbf{H})$ (Def. 4.1.16)
- $(\Gamma /\!\!/ G) \circlearrowright A /\!\!/ G \in (\Gamma /\!\!/ G)\,\mathrm{Act}\big(\mathbf{H}_{/\mathbf{B}G}\big)$ (Prop. 3.2.75)

we say that
(i) the projection (recall the vertical equivalences from Prop. 4.1.19 and Lem. 4.1.21)

$$
\begin{tikzcd}
(A /\!\!/ G) /\!\!/ (\Gamma /\!\!/ G) \arrow[rr] \arrow[d, phantom, "\wr"] & & \mathbf{B}\Gamma /\!\!/ G \arrow[d, phantom, "\wr"] \\
A /\!\!/ (\Gamma \rtimes G) \arrow[rr] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
\tag{4.113}
$$

is the *universal G-equivariant Γ-associated A-fiber ∞-bundle*;
(ii) given a G-equivariant Γ-principal ∞-bundle P (Def. 4.1.22) its *associated G-equivariant A-fiber ∞-bundle* is the pullback of the universal such (4.113) along the modulating map of P:

$$
\begin{tikzcd}
\overset{\substack{\text{equivariant}\\ \text{principal }\infty\text{-bundle}}}{P /\!\!/ G} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\substack{\Gamma /\!\!/ (\Gamma \rtimes G)\\ \wr}}{\mathbf{B}G} \arrow[d] \\
X /\!\!/ G \arrow[rr, "\vdash P /\!\!/ G"] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
\quad \vdash \quad
\begin{tikzcd}
\overset{\substack{\text{equivariant associated}\\ A\text{-fiber }\infty\text{-bundle}}}{E /\!\!/ G} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\substack{\text{universal } G\text{-equivariant}\\ \Gamma\text{-associated}\\ A\text{-fiber }\infty\text{-bundle}}}{A /\!\!/ (\Gamma \rtimes G)} \arrow[d] \\
X /\!\!/ G \arrow[rr, "\vdash P /\!\!/ G"] \arrow[dr] & & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}
\tag{4.114}
$$

Conversely, given an A-fiber ∞-bundle arising as on the right, we say that its equivariant structure ∞-group is Γ.

It is a general fact that sections of Γ-associated A-fiber ∞-bundles are equivalently Γ-equivariant maps into A out of the underlying Γ-principal ∞-bundle [NSS12a, Cor. 4.18]. The following lemma spells this out for equivariant ∞-bundles.

Proposition 4.2.11 (G-Equivariant sections of Γ-associated equivariant bundles as $(\Gamma \rtimes G)$-equivariant maps)**.** *Given $G \circlearrowright \Gamma \in G\mathrm{EquvGrp}(\mathbf{H})$ (Def. 4.1.16) with semidirect product* (4.21) *to be abbreviated as $\widehat{G} := \Gamma \rtimes G$, and P a G-equivariant Γ-principal ∞-bundle over $X \in \mathbf{H}$, modulated (according to Def. 4.1.22) by $\mathrm{tw} : X /\!\!/ G \xrightarrow{\vdash (P /\!\!/ G)} \mathbf{B}(\Gamma \rtimes G)$ with E an associated A-fiber ∞-bundle (according to Def. 4.2.10), the following sliced mapping stacks (Def. 3.2.65) are naturally equivalent:*

$$
\overset{\substack{\infty\text{-stack of equivariant sections of}\\ \text{associated equivariant } A\text{-fiber }\infty\text{-bundle}}}{\mathrm{Map}\big(\mathrm{X} /\!\!/ G, E /\!\!/ G\big)_{X /\!\!/ G}}
\simeq
\overset{\substack{\infty\text{-stack of lifts of modulating map through}\\ \text{universal equivariant } \Gamma\text{-associated } A\text{-fiber }\infty\text{-bundle}}}{\mathrm{Map}\big((\mathrm{X},\mathrm{tw}) /\!\!/ G, \mathrm{A} /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}}}
\simeq
\overset{\substack{\infty\text{-stack of maps of homotopy quotients in slice}\\ \text{from equivariant principal bundle to typical fiber}}}{\mathrm{Map}\big(\mathrm{P} /\!\!/ \widehat{G}, \mathrm{A} /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}}}
\simeq
\overset{\substack{\infty\text{-stack of } \Gamma \rtimes G\text{-equivariant maps from}\\ \text{equivariant principal bundle to typical fiber}}}{\mathrm{Map}\Big(\widehat{G} \circlearrowright \mathrm{P}, \widehat{G} \circlearrowright \mathrm{A}\Big)^{\widehat{G}}}
$$

$$
\left\{
\begin{tikzcd}[sep=small]
& E /\!\!/ G \arrow[d] \\
X /\!\!/ G \arrow[r, equal] \arrow[ur, dashed] \arrow[dr] & X /\!\!/ G \arrow[d] \\
& \mathbf{B}G
\end{tikzcd}
\right\}
\qquad
\left\{
\begin{tikzcd}[sep=small]
& A /\!\!/ (\Gamma \rtimes G) \arrow[d] \\
X /\!\!/ G \arrow[r, "\mathrm{tw}"] \arrow[ur, dashed] \arrow[dr] & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
\mathbf{B}G &
\end{tikzcd}
\right\}
\qquad
\left\{
\begin{tikzcd}[sep=small]
& A /\!\!/ (\Gamma \rtimes G) \arrow[d] \\
P /\!\!/ \widehat{G} \arrow[r] \arrow[ur, dashed] \arrow[dr] & \mathbf{B}(\Gamma \rtimes G) \arrow[dl] \\
\mathbf{B}G &
\end{tikzcd}
\right\}
\qquad
\left\{
\begin{tikzcd}
\overset{\Gamma \rtimes G}{\circlearrowright} P \arrow[r, dashed] & \overset{\Gamma \rtimes G}{\circlearrowright} A
\end{tikzcd}
\right\}
$$

Proof. (i) Under the identification (3.82) of plots of the slice mapping stack with slice homs, and using the ∞-Yoneda lemma (Prop. 3.2.29), the first equivalence is the universal property of the defining pullback (4.114). (ii) The second equivalence follows immediately from the identification $P /\!\!/ (\Gamma \rtimes G) \simeq X /\!\!/ G \in \mathbf{H}_{/\mathbf{B}G}$ (4.27). (iii) The third equivalence is a special case of Ex. 3.2.79. □

Base change of concordances of equivariant bundles along coverings of the equivariance group. We use the classification Theorem 4.2.7 to describe the base change of concordances of equivariant principal bundles along coverings of the equivariance group (Lem. 4.2.13 below).

Notation 4.2.12 (Equivariance group base change on mapping stacks)**.** For
– $\mathbf{H}$ a cohesive ∞-topos with a 1-site Chrt of charts (Ntn. 3.3.11),
– $p : \widehat{G} \to G \;\; \in \mathrm{Grp}(\mathrm{Set})$,
– $G \circlearrowright \mathscr{X}, G \circlearrowright \mathscr{A} \;\; \in G\mathrm{Act}(\mathbf{H})$,
(i) we write

$$(Bp)^*_{\mathscr{X},\mathscr{A}} : \; \mathrm{Map}\big(\mathscr{X} /\!\!/ G, \mathscr{A} /\!\!/ G\big)_{\mathbf{B}G} \longrightarrow \mathrm{Map}\big(\mathscr{X} /\!\!/ \widehat{G}, \mathscr{A} /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}}$$

for the component morphism of the base change of slice mapping stacks (Def. 3.2.65) along the delooping (Prop. 3.2.70) of p. Namely:
(ii) Under Lem. 3.2.68 and the ∞-Yoneda lemma (Prop. 3.2.29), this is given at stage $U \in \mathrm{Chrt} \overset{y}{\hookrightarrow} \mathbf{H}$ by the $(U \times \mathscr{X}, \mathscr{Y})$-component of the base change functor (Prop. 3.2.50):

$$(Bp)^*_{\mathscr{X},\mathscr{A}}(U) \;=\; (Bp)^*_{U\times\mathscr{X},\mathscr{A}} \;:\; \mathbf{H}\big(U \times \mathscr{X} /\!\!/ G, \mathscr{A} /\!\!/ G\big)_{\mathbf{B}G} \longrightarrow \mathbf{H}\big(U \times \mathscr{X} /\!\!/ \widehat{G}, \mathscr{A} /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}}.$$

Lemma 4.2.13 (Base change of concordances of equivariant bundles along covering of equivariance group)**.** *For*
– $\widehat{G} \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*
– $p : \;\widehat{G} \twoheadrightarrow G$ *a surjective homomorphism* (3.100)*,*
– $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd}) \hookrightarrow G\mathrm{Act}(\mathrm{SmthGrpd}_\infty)$*,*
– $G \circlearrowright \Gamma \in G\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kHausSpc})\big) \xhookrightarrow{\mathrm{Cdfflg}} G\mathrm{Act}\big(\mathrm{Grp}(\mathrm{SmthGrpd}_\infty)\big)$*, well-pointed (Ntn. 1.3.17) and of truncated classifying shape (Ntn. 4.1.30) with a notion of stable equivariant bundles (Ntn. 4.1.33),*
the shape (3.116) *of the base change* $(Bp)^*_{\mathrm{X}/\!\!/G,\mathbf{B}\Gamma}$ *on slice mapping stacks (Ntn. 4.2.12) – hence the induced morphism* (3.128) *of concordance ∞-groupoids (Rem. 4.1.8) of equivariant Γ-principal bundles – is a monomorphism (Ex. 3.1.16) for stable equivariant bundles:*

$$\int \mathrm{Map}\big(\mathrm{X} /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G\big)^{\mathrm{stbl}}_{\mathbf{B}G} \xhookrightarrow{\;\int (Bp)^*_{\mathscr{X},\mathbf{B}\Gamma}\;} \int \mathrm{Map}\big(\mathrm{X} /\!\!/ \widehat{G}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)^{\mathrm{stbl}}_{\mathbf{B}\widehat{G}}.$$

Proof. For $\Delta^\bullet_{\mathrm{smth}} \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthGrpd}_\infty$ (Def. 3.3.21), Prop. 3.2.87 gives that (3.105)

$$(Bp)^*_{\Delta^\bullet_{\mathrm{smth}}\times\mathrm{X},\mathbf{B}\Gamma} \;:\; \mathbf{H}\big(\Delta^\bullet_{\mathrm{smth}} \times \mathrm{X} /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G\big)^{\mathrm{stbl}}_{\mathbf{B}G} \hookrightarrow \mathbf{H}\big(\Delta^\bullet_{\mathrm{smth}} \times \mathrm{X} /\!\!/ \widehat{G}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)^{\mathrm{stbl}}_{\mathbf{B}\widehat{G}}$$

is a degreewise monomorphism of simplicial ∞-groupoids. With Prop. 3.3.46 and Lem. 3.2.68, it follows that the morphism in question is equivalently the image under the simplicial ∞-colimit operation of a monomorphism of simplicial ∞-groupoids:

$$\int (Bp)^*_{\mathrm{X},\mathbf{B}\Gamma} \;\simeq\; \underset{[n]\in\Delta^{\mathrm{op}}}{\underrightarrow{\lim}} \Big(\mathbf{H}\big(\Delta^n_{\mathrm{smth}} \times \mathrm{X} /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G\big)^{\mathrm{stbl}}_{\mathbf{B}G} \hookrightarrow \mathbf{H}\big(\Delta^n_{\mathrm{smth}} \times \mathrm{X} /\!\!/ \widehat{G}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)^{\mathrm{stbl}}_{\mathbf{B}\widehat{G}}\Big).$$

Now, since monomorphisms are characterized by a pullback property (6), it is sufficient to show that this homotopy pullback is preserved by the colimit operation. Prop. 1.3.27 says that a sufficient condition for this to be the case is that

$$\tau_0 \mathbf{H}\big(\Delta^n_{\mathrm{smth}} \times \mathrm{X} /\!\!/ \widehat{G}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)^{\mathrm{stbl}}_{\mathbf{B}\widehat{G}} \;\simeq\; \tau_0 \mathbf{H}\big(\mathrm{X} /\!\!/ \widehat{G}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)^{\mathrm{stbl}}_{\mathbf{B}\widehat{G}}$$

is independent of n. But, since

$$\begin{aligned} \int \big(\Delta^n_{\mathrm{smth}} \times (-)\big) &\simeq \int \Delta^n_{\mathrm{smth}} \times \int(-) && \text{by (3.122)} \\ &\simeq \int \mathbb{R}^n \times \int(-) && \text{by Def. 3.3.21} \\ &\simeq \int(-) && \text{by Prop. 3.3.40,} \end{aligned}$$

this follows by the classification result for $\widehat{G}$-equivariant Γ-principal bundles (Thm. 4.2.7, using here the assumptions on $\widehat{G}$ and on Γ and the stability condition). □

4.3 Equivariant moduli stacks

The discussion in §4.1, culminating in Thm. 4.2.7, may be summarized as saying that the stacky delooping

$$\begin{pmatrix} \mathbf{B}(\Gamma \rtimes G) \\ \downarrow \\ \mathbf{B}G \end{pmatrix} \simeq \begin{pmatrix} \mathbf{B}\Gamma /\!\!/ G \\ \downarrow \\ * /\!\!/ G \end{pmatrix} \in (\mathbf{H}_{\cup})_{/\mathbf{B}G}$$

is the *moduli stack* (as in Prop. 3.2.97) for (equivariantly locally trivial) G-equivariant Γ-principal bundles. Remarkably, this is always a Borel-equivariant object (though with higher geometric structure). But for these to serve as twists in a good twisted equivariant generalized cohomology theory (this is discussed in §4.5 below) one needs to equivalently regard these moduli stacks as orbi-singularized objects in the slice $\mathbf{H}_{/\underset{\prec}{G}}$ of a singular-cohesive ∞-topos (§3.3.2).

This requires embedding the Borel-equivariant moduli stacks into proper-equivariant homotopy theory, which we achieve (in Def. 4.3.1 below) simply by forming their orbi-singularization:

$$\prec \begin{pmatrix} \mathbf{B}(\Gamma \rtimes G) \\ \downarrow \\ \mathbf{B}G \end{pmatrix} \simeq \begin{pmatrix} \prec(\mathbf{B}\Gamma /\!\!/ G) \\ \downarrow \\ \underset{\prec}{G} \end{pmatrix} \in \mathbf{H}_{/\underset{\prec}{G}}. \tag{4.115}$$

Notice that these proper equivariant moduli stacks (4.115) still modulate equivariant principal bundles, now regarded over cohesive orbi-spaces $\prec(X /\!\!/ G)$ (recall Prop. 3.3.92 and [SS20-Orb, §4]), in that:

$$\begin{aligned} \mathbf{H}\big(\prec(X /\!\!/ G),\, \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\underset{\prec}{G}} &\simeq \mathbf{H}\big(X /\!\!/ G, \mathbf{B}\Gamma /\!\!/ G\big)_{\mathbf{B}G} && \text{by Ex. 3.3.75} \\ &\simeq G\mathrm{Equv}\,\Gamma\mathrm{PrnBdl}(\mathbf{H}_{\cup}) && \text{by Def. 4.1.22}. \end{aligned}$$

The underlying equivariant homotopy type of these orbi-singular objects (4.115) is their pure shape (3.117) hence the *equivariant shape* of the corresponding homotopy quotients:

$$\int \prec \begin{pmatrix} \mathbf{B}(\Gamma \rtimes G) \\ \downarrow \\ \mathbf{B}G \end{pmatrix} \simeq \begin{pmatrix} \int \prec(\mathbf{B}\Gamma /\!\!/ G) \\ \downarrow \\ \underset{\prec}{G} \end{pmatrix} \in \mathbf{H}_{/\underset{\prec}{G}}. \tag{4.116}$$

We show (Thm. 4.3.19 below) that their G-(orbi-)spatial aspect $\cup_{\underset{\prec}{G}} \int \prec(\mathbf{B}\Gamma /\!\!/ G)$ (Def. 4.3.1 below) is given equivalently by the Murayama-Shimakawa construction from §2.3, which serves to give that construction general abstract meaning.

For example, this shows that the abstract reason why equivariant classifying spaces are not connected (see Rem. 2.3.21) even though they arise as the images of moduli stacks which *are* connected (according to Thm. 3.2.97), is that they arise so under orbi-singularization $\prec$, which is the (only) one of the three singular modalities (3.189), that does *not* preserve deloopings (Ex. 3.3.69). This is the conceptual origin (by Prop. 2.3.18 with Thm. 4.3.19 and Thm. 4.3.7 below) of the rich sets of connected components labeled by non-abelian group cohomology classes of crossed homomorphisms which is so characteristic for equivariant principal bundle theory (as witnessed by the ubiquitous Ntn. 2.2.10).

Finally, we show (Thm. 4.3.24 below) how the Borel-equivariant classification of equivariant principal bundles from Thm. 4.2.7 embeds into proper-equivariant theory (Thm. 4.3.24) and we conclude with a list of examples and applications of the classification result in this form (Ex. 4.4.1, Ex. 4.4.2).

Proper equivariant classifying spaces. In direct proper-equivariant generalization of the notation in Def. 4.1.9, we set:

Definition 4.3.1 (Proper equivariant classifying shape)**.** For $\mathbf{H}$ a singular-cohesive ∞-topos (Def. 3.3.65),

– $G \in \mathrm{Grp}(\mathbf{H})$,

– $\Gamma \rtimes G \in \mathrm{Grp}(\mathbf{H}_{/\mathbf{B}G})$,

(i) we write

$$B_G\Gamma \;:=\; \cup_{\overset{G}{\curlyvee}} \textstyle\int \curlyvee \mathbf{B}(\Gamma \rtimes G) \quad \in \; G\mathrm{Grpd}_\infty \xhookrightarrow{\;\mathrm{Dsc}\,G\mathrm{OrbSpc}\;} \mathbf{H}_{/\overset{G}{\curlyvee}} \tag{4.117}$$

for the G-smooth aspect (Def. 3.3.78) of the shape (Def. 3.3.1) of the orbi-singular aspect (Def. 3.3.62) of the equivariant moduli stack (from Prop. 3.2.97, Thm. 4.2.7):

$$\begin{array}{ccccccccccccc}
 & \text{delooping} & & \text{orbi-singularization} & & \text{shape} & & G\text{-orbi-spatial} & & & & & \\
\mathrm{Grp}\Big((\mathbf{H}_\cup)_{/\mathbf{B}G}\Big) & \xrightarrow{\mathbf{B}} & (\mathbf{H}_\cup)_{/\mathbf{B}G} & \xrightarrow{\curlyvee} & \mathbf{H}_{/\overset{G}{\curlyvee}} & \xrightarrow{\mathrm{Shp}} & \mathrm{SnglrGrpd}_{\infty/\overset{G}{\curlyvee}} & \xrightarrow{G\mathrm{OrbSmth}} & G\mathrm{Grpd}_\infty & \xhookrightarrow{\mathrm{Dsc}} & G\mathbf{H}_\cup & \xhookrightarrow{G\mathrm{OrbSpc}} & \mathbf{H}_{/\overset{G}{\curlyvee}} \\
\Gamma /\!\!/ G & \mapsto & \mathbf{B}(\Gamma \rtimes G) & \mapsto & \curlyvee\mathbf{B}(\Gamma \rtimes G) & \mapsto & \mathrm{Shp}\curlyvee\mathbf{B}(\Gamma \rtimes G) & \longmapsto & B_G\Gamma & & & & \\
\begin{array}{c}\text{cohesive}\\ G\text{-equivariant}\\ \infty\text{-group}\end{array} & & \begin{array}{c}\text{moduli stack of}\\ G\text{-equivariant}\\ \Gamma\text{-principal}\\ \infty\text{-bundles}\end{array} & & \begin{array}{c}\text{proper-equivariant}\\ \text{moduli stack}\end{array} & & \begin{array}{c}\text{global-equivariant}\\ \text{homotopy type of}\\ \text{proper-equivariant}\\ \text{moduli stack}\end{array} & & \begin{array}{c}G\text{-orbi-space}\\ \text{underlying}\\ \text{proper-equivariant}\\ \text{moduli stack}\end{array} & & & &
\end{array}$$

(ii) For $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_\cup)$ with $X \in \mathbf{H}_{\cup,0} \hookrightarrow \mathbf{H}$, we denote by

$$\chi \;:\; \big(G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathbf{H}_\cup)_X\big)_{/\sim_{\mathrm{iso}}} \longrightarrow H^0_{\overset{G}{\curlyvee}}(X; B_G\Gamma) \tag{4.118}$$

the natural map from equivalence classes of G-equivariant Γ-principal ∞-bundles over X to the non-abelian proper-equivariant cohomology of the G-orbi-space underlying $G \circlearrowright X$ with coefficients in $B_G\Gamma$ (4.117).

Remark 4.3.2 (The equivariant characteristic class map)**.** The map χ (4.118) is the image under 0-truncation of the following composite:

$$\begin{aligned}
& G\mathrm{Equv}\Gamma\mathrm{PrnBdl}(\mathbf{H}_\cup)_X \\
& \simeq \mathbf{H}\big(X/\!\!/G, \mathbf{B}\Gamma/\!\!/G\big)_{\mathbf{B}G} \xrightarrow{\;\int\curlyvee\;} \mathbf{H}\big(\textstyle\int\curlyvee(X/\!\!/G), \int\curlyvee(\mathbf{B}\Gamma/\!\!/G)\big)_{\overset{G}{\curlyvee}} \\
& \qquad\qquad \simeq \mathbf{H}\Big(G\mathrm{OrbSpc}\big(\mathrm{Shp}\,\mathrm{FxdLoc}(X)\big), \textstyle\int\curlyvee(\mathbf{B}\Gamma/\!\!/G)\Big)_{\overset{G}{\curlyvee}} && \text{by Prop. 3.3.92} \\
& \qquad\qquad \simeq \mathbf{H}\Big(\vee_{\overset{G}{\curlyvee}}\, G\mathrm{OrbSpc}\big(\mathrm{Shp}\,\mathrm{FxdLoc}(X)\big), \textstyle\int\curlyvee(\mathbf{B}\Gamma/\!\!/G)\Big)_{\overset{G}{\curlyvee}} && \text{by Ex. 3.3.79} \\
& \qquad\qquad \simeq \mathbf{H}\Big(G\mathrm{OrbSpc}\big(\mathrm{Shp}\,\mathrm{FxdLoc}(X)\big), \cup_{\overset{G}{\curlyvee}} \textstyle\int\curlyvee(\mathbf{B}\Gamma/\!\!/G)\Big)_{\overset{G}{\curlyvee}} && \text{by (3.211)} \\
& \qquad\qquad \simeq G\mathrm{Grpd}_\infty\big(\mathrm{Shp}\,\mathrm{FxdLoc}(X), B_G\Gamma\big) && \text{by (3.208).}
\end{aligned}$$

We proceed to compute (in Thm. 4.3.19 below) the geometric fixed loci of the proper-equivariant classifying spaces $B_G\Gamma$ (4.117).

Lemma 4.3.3 (Geometric fixed loci of equivariant classifying spaces)**.** *In a singular-cohesive ∞-topos $\mathbf{H}$ (Def. 3.3.65), given*

– $G \in \mathrm{Grp}(\mathrm{Set})$,

– $G \circlearrowright \Gamma \in \mathrm{Grp}\big(G\mathrm{Act}(\mathbf{H})\big)$,

(i) *the geometric H-fixed loci (Def. 3.3.85) of the equivariant moduli stack* (4.115) *are*

$$\cup\mathrm{Map}\big(\overset{H}{\curlyvee}, \curlyvee(\mathbf{B}\Gamma/\!\!/G)\big)_{\overset{G}{\curlyvee}} \;\simeq\; \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma/\!\!/G\big)_{\mathbf{B}G}; \tag{4.119}$$

(ii) *their shape is, equivalently, the H-fixed locus in the equivariant classifying shape* (4.117)*:*

$$B_G\Gamma \;:\; G/H \;\mapsto\; \cup\mathrm{Map}\big(\overset{H}{\curlyvee}, \textstyle\int\curlyvee\mathbf{B}\Gamma/\!\!/G\big)_{\overset{G}{\curlyvee}} \;\simeq\; \int\mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma/\!\!/G\big)_{\mathbf{B}G}. \tag{4.120}$$

Proof. For $U \times \overset{G}{\curlyvee} \in \mathrm{Chrt} \times \mathrm{Snglrt}$, we have the following sequence of natural equivalences:

$$
\begin{aligned}
&\Big(\cup\Big(\mathrm{Map}\big(\prec^{H},\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec^{G}}\Big)\Big)\Big(U\times \prec^{K}\Big) \\
&\qquad\simeq\ \Big(\mathrm{Map}\big(\prec^{H},\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec^{G}}\Big)(U) && \text{by Def. 3.3.62} \\
&\qquad\simeq\ \mathbf{H}\big(\prec^{H}\times U,\ \prec\mathbf{B}\Gamma /\!\!/ G\big) \underset{\mathbf{H}(\prec^{H}\times U,\ \prec^{G})}{\times} \{*\} && \text{by Prop. 3.2.63, Lem. 3.2.68} \\
&\qquad\simeq\ \mathbf{H}\big(\prec^{H}\times \prec U,\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big) \underset{\mathbf{H}(\prec^{H}\times \prec U,\ \prec^{G})}{\times} \{*\} && \text{by (3.207)} \\
&\qquad\simeq\ \mathbf{H}\big(\prec(\mathbf{B}H\times U),\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big) \underset{\mathbf{H}(\prec(\mathbf{B}H\times U),\ \prec\mathbf{B}G)}{\times} \{*\} && \text{by (11) \& Ex. 3.3.67} \\
&\qquad\simeq\ \mathbf{H}\big(\mathbf{B}H\times U,\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big) \underset{\mathbf{H}(\mathbf{B}H\times U,\ \mathbf{B}G)}{\times} \{*\} && \text{by Ex. 3.3.75} \\
&\qquad\simeq\ \mathbf{H}\big(\mathbf{B}H\times U\times \prec^{K},\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big) \underset{\mathbf{H}(\mathbf{B}H\times U\times \prec^{K},\ \mathbf{B}G)}{\times} \{*\} && \text{by Ex. 3.3.68} \\
&\qquad\simeq\ \Big(\mathrm{Map}\big(\mathbf{B}H,\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\mathbf{B}G}\Big)(U\times \prec^{K}) && \text{by Props. 3.2.63, 3.2.68.}
\end{aligned}
$$

Since this is natural in $U\times \prec^{K} \in \mathrm{CartSpc}\times\mathrm{Snglrt} \xhookrightarrow{y} \mathbf{H}$, the first statement (4.119) follows by the ∞-Yoneda lemma (Prop. 3.2.29). With this, the second statement (4.120) is implied as follows:

$$
\begin{aligned}
\cup\Big(\mathrm{Map}\big(\prec^{H}, \int\prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec\mathbf{B}G}\Big) &\ \simeq\ \cup\int\Big(\mathrm{Map}\big(\prec^{H},\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec\mathbf{B}G}\Big) && \text{by Prop. 3.3.90} \\
&\ \simeq\ \int\cup\Big(\mathrm{Map}\big(\prec^{H},\ \prec\mathbf{B}\Gamma /\!\!/ G\big)_{\prec\mathbf{B}G}\Big) && \text{by Prop. 3.3.71} \\
&\ \simeq\ \int\mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G\big)_{\mathbf{B}G} && \text{by (4.119).}
\end{aligned}
$$

□

Example 4.3.4 (The case of trivial G-action on the structure ∞-group)**.** In the special case when the action of G on Γ is trivial, in that $\mathbf{B}\Gamma /\!\!/ G \simeq (\mathbf{B}\Gamma)\times(\mathbf{B}G)$ (Ex. 4.1.17), we have (by Ex. 3.2.67)

$$
\mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G\big)_{\mathbf{B}G} \ \simeq\ \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma)\,.
$$

Consequently, in the case the statement of Lem. 4.3.3 reduces to the following two equivalences

$$
\cup\mathrm{Map}\big(\prec^{H},\ \prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec^{G}} \ \simeq\ \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma)\,, \tag{4.121}
$$

$$
B_G\Gamma\ :\ G/H\ \mapsto \cup\mathrm{Map}\big(\prec^{H}, \int\prec(\mathbf{B}\Gamma /\!\!/ G)\big)_{\prec^{G}} \ \simeq\ \int\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma)\,. \tag{4.122}
$$

In fact, in this case (Ex. 4.3.4) the equivariant classifying shapes have a “globally equivariant” incarnation (in the terminology of [Schw18]), in proper-equivariant refinement of Rem. 4.1.28 and along the lines of [Re14a, §1.3-1.4]:

Proposition 4.3.5 (Globally equivariant classifying shape)**.** *If the action of the equivariance group G on the structure group Γ is trivial, so that their semi-direct product is their direct product*

$$
\Gamma \rtimes G \simeq \Gamma\times G\,, \tag{4.123}
$$

then the equivariant shape (4.116) *of the delooping of the structure group*

$$
\int\prec\mathbf{B}\Gamma\quad\in\ \mathbf{H} \tag{4.124}
$$

is the globally equivariant classifying shape, in that, for all $G\in$ Grp(Set) *and $G \circlearrowright X\ \in\ G\,$*Act$(\mathbf{H}_{\cup,0})$*, we have a natural equivalence*

$$
\mathbf{H}\big(\prec(X /\!\!/ G), \int\prec\mathbf{B}\Gamma\big)\ \simeq\ \mathbf{H}\big(\prec(X /\!\!/ G), B_G\Gamma\big)_{\prec^{G}}
$$

between the global hom-∞-groupoid into (4.124) *and the slice-hom-∞-groupoid into* (4.117).

Proof. This is the composite of the following sequence of natural equivalences,

$$
\begin{aligned}
\mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), B_G\Gamma\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} &= \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), \cup_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}(\Gamma \rtimes G)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by Def. 4.3.1} \\
&\simeq \mathbf{H}\big(\vee_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} \rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}(\Gamma \rtimes G)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by (3.211), (10)} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}(\Gamma \rtimes G)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by Prop. 3.3.92 with Ex. 3.3.79} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}(\Gamma \times G)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by (4.123)} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), ʃ \rotatebox[origin=c]{70}{$\prec$}\big((\mathbf{B}\Gamma) \times (\mathbf{B}G)\big)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by Ex. 4.1.17 or (3.89)} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), (ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\Gamma) \times (ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}G)\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by (11) and (3.122)} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), (ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\Gamma) \times \underset{\rotatebox[origin=c]{70}{$\prec$}}{G}\big)_{\underset{\rotatebox[origin=c]{70}{$\prec$}}{G}} && \text{by (3.194)} \\
&\simeq \mathbf{H}\big(\rotatebox[origin=c]{70}{$\prec$}(X /\!\!/ G), ʃ \rotatebox[origin=c]{70}{$\prec$}\mathbf{B}\Gamma\big) && \text{by Ex. 3.2.51.}
\end{aligned}
$$

□

Definition 4.3.6 (Stable proper equivariant classifying shape)**.** Given $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ (Ntn. 3.3.54), and $G \circlearrowright \Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$ of truncated classifying shape (Ntn. 4.1.30) and with a notion of stable equivariant bundles (Ntn. 4.1.33), we write

$$
\begin{array}{ccccc}
 & & (B_G\Gamma)^{\mathrm{stbl}} & \hookrightarrow & B_G\Gamma \\
G/H & \mapsto & ʃ\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)^{\mathrm{stbl}}_{\mathbf{B}G} & \hookrightarrow & \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G}
\end{array}
$$

for the subobject of the proper-equivariant classifying shape (Def. 4.3.1) which is given on its values (4.120) by the stable component in the sense of Ntn. 4.1.33.

Theorem 4.3.7 (Equivariant homotopy groups of equivariant classifying shapes)**.** *Given*

– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*

– $G \circlearrowright \Gamma \in G\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big)$ *of truncated classifying shape (Ntn. 4.1.30) and with a notion of of blowup-stable equivariant* $\Gamma \rtimes G$*-principal bundles (Ntn. 4.1.33);*

– *such that* $\pi_1 \mathrm{Map}\big(BG, B(\Gamma \rtimes G)\big)$ *is countable (Rem. 4.1.54);*

then the equivariant homotopy groups of the blowup-stable component of the equivariant classifying shape $B_G\Gamma$ (4.3.1) *at stage* $H \subset G$ *are given by the non-abelian group cohomology of* H *with coefficients in* $H \circlearrowright ʃ\Gamma$*, in that we have natural isomorphisms as follows:*

$$
\underset{\substack{H \subset G \\ n \in \mathbb{N}}}{\forall} \qquad \pi^H_n\big((B_G\Gamma)^{\mathrm{stbl}}\big) \;\simeq\; H^{1-n}_{\mathrm{Grp}}(H; ʃ\Gamma) \;=\; H^1_{\mathrm{Grp}}(H; \Omega^n ʃ\Gamma)\,.
$$

Proof.

$$
\begin{aligned}
\pi^H_n\big((B_G\Gamma)^{\mathrm{stbl}}\big) &= \pi_n\Big((B_G\Gamma)^{\mathrm{stbl}}(G/H)\Big) \\
&\simeq \pi_n\big(ʃ\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ H)^{\mathrm{stbl}}_{BH}\big) && \text{by Lem. 4.3.3} \\
&\simeq \pi_n\big(\mathrm{Map}(ʃ\mathbf{B}H, ʃ\mathbf{B}\Gamma)_{\mathbf{B}H}\big) && \text{by Thm. 4.1.55} \\
&\simeq \pi_n\big(\mathrm{Map}(BH, Bʃ\Gamma /\!\!/ G)_{BH}\big) && \text{by Def. 4.1.9, Ex. 4.1.10} \\
&\simeq \left\{ \begin{array}{ll} \tau_0 \mathrm{Map}(BH, Bʃ\Gamma)_{BH} & \mid n = 0 \\ \tau_0 \mathrm{Map}(BH, B\Omega^n ʃ\Gamma) & \mid n > 0 \end{array} \right. && \text{by Lem. 3.2.59} \\
&= H^1_{\mathrm{Grp}}(H; \Omega^n ʃ\Gamma) && \text{e.g. [FSS20-TCD, Ex. 2.4]}\,.
\end{aligned}
$$

□

Remark 4.3.8 (Proper equivariant classifying shapes may depend on structure group beyond its shape)**.** While plain classifying shapes (Def. 4.1.9) are sensitive only to the shape of the structure group, in that $B\Gamma \simeq B\int\Gamma$ (Ex. 4.1.10), for proper equivariant classifying shapes (Def. 4.3.1) this generally fails, in as far as the geometric fixed points of $\mathbf{B}\Gamma$ differ from that of its shape (the "homotopy fixed points"):

$$B_G\Gamma \underset{\text{i.g.}}{\not\simeq} B_G(\textstyle\int\Gamma)\,.$$

Concretely, Lem. 4.3.3 shows that the two are related by taking the shape operation into the mapping stack along the comparison morphism (3.128):

$$\begin{array}{ccccc} & & B_G\Gamma & \longrightarrow & B_G(\int\Gamma) \\ G/H & \mapsto & \int\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G} & \xrightarrow{\widetilde{\int\mathrm{ev}}} & \mathrm{Map}(\mathbf{B}H, B\Gamma /\!\!/ G)_{\mathbf{B}G}\,. \end{array} \tag{4.125}$$

In general, there is no reason for (4.125) to be an equivalence. However, it is so under suitable conditions where a smooth Oka principle applies (such as in Prop. 4.1.55):

Proposition 4.3.9 (Stable equivariant classifying shape of truncated groups coincides with that of their shape)**.** *Given*

– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{rslvbl}}$ *(Ntn. 3.3.54),*

– $\Gamma \in \mathrm{Grp}(\mathrm{SmthGrpd}_0)$ *of truncated classifying shape (Ntn. 4.1.30),*

then the stable proper G-equivariant classifying shape of Γ *(Def. 4.3.6) is equivalent to the plain proper equivariant classifying shape (Def. 4.3.1) of* $\int\Gamma$*:*

$$B_G(\Gamma)^{\mathrm{stbl}} \;\simeq\; B_G(\textstyle\int\Gamma) \quad \in G\mathrm{Grpd}_\infty\,.$$

Proof. For $H \subset G$ a subgroup, we have the following sequence of natural equivalences in $\mathrm{Grpd}_\infty \overset{\mathrm{Dsc}}{\hookrightarrow} \mathbf{H}_\cup$:

$$\begin{aligned} (B_G\Gamma)^{\mathrm{stbl}}(G/H) &\simeq \textstyle\int\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)^{\mathrm{stbl}}_{\mathbf{B}G} && \text{by Def. 4.3.6} \\ &\simeq \mathrm{Map}(\textstyle\int\mathbf{B}H, \int\mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G} && \text{by Thm. 4.1.55} \\ &\simeq \mathrm{Map}(\mathbf{B}H, \mathbf{B}\textstyle\int\Gamma /\!\!/ G)_{BG} && \text{by (3.125)} \\ &\simeq \textstyle\int\mathrm{Map}(\mathbf{B}H, \mathbf{B}\int\Gamma /\!\!/ G)_{BG} && \text{by (3.137)} \\ &\simeq \Big(B_G(\textstyle\int\Gamma)\Big)(G/H) && \text{by Lem. 4.3.3.} \end{aligned}$$

By naturality in $G/H \in G\mathrm{Orb}$ (Ntn. 3.3.58), this establishes the claimed equivalence of ∞-presheaves. □

Lemma 4.3.10 (Base change of equivariant classifying shapes along a covering of the equivariance group)**.** *For* $G \circlearrowright \Gamma \in G\mathrm{Act}(\mathrm{Grp}(\mathbf{H}_0))$ *and a surjective homomorphism of discrete groups* $p : \widehat{G} \twoheadrightarrow G$ (3.100)*, there is a natural monomorphism* (6)

$$B_G\Gamma \xhookrightarrow{(Bp)^*} p_* B_{\widehat{G}}\Gamma \quad \in G\mathrm{Grpd}_\infty$$

from the equivariant classifying shape (Def. 4.3.1) of $G \circlearrowright \Gamma$ *to the direct image* p_* (3.223) *of that of the cover* $\widehat{G} \circlearrowright \Gamma := (Bp)^*(G \circlearrowright \Gamma)$*; see* (3.101)*.*

Proof. First, observe that for $G/H \in G\mathrm{Orb} \overset{y}{\hookrightarrow} G\mathrm{Grpd}_\infty$ there is the following sequence of natural equivalences

$$\begin{aligned} G\mathrm{Grpd}_\infty(G/H, p_* B_{\widehat{G}}\Gamma) &\simeq G\mathrm{Grpd}_\infty(p^*(G/H), B_{\widehat{G}}\Gamma) && \text{by (3.223) with (10)} \\ &\simeq G\mathrm{Grpd}_\infty\Big(\widehat{G}/\widehat{H}, B_{\widehat{G}}\Gamma\Big) && \text{by Lem. 3.3.94 with Lem. 3.2.40} \\ &\simeq (B_{\widehat{G}}\Gamma)\Big(\widehat{G}/\widehat{H}\Big) && \text{by Prop. 3.2.29} \\ &\simeq \textstyle\int\mathrm{Map}\big(\mathbf{B}\widehat{H}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}} && \text{by Lem. 4.3.3;} \end{aligned}$$

hence, by the ∞-Yoneda lemma (Prop. 3.2.29) an equivalence

$$p_* B_{\widehat{G}}\Gamma \simeq \int \mathrm{Map}\big(\mathbf{B}\widehat{(-)}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}} \simeq \int \mathrm{Map}\big((Bp)^*\mathbf{B}(-), (Bp)^*\mathbf{B}\Gamma /\!\!/ G\big)_{\mathbf{B}\widehat{G}}. \tag{4.126}$$

Under this identification, the natural monomorphism in question is that from Prop. 3.2.87:

$$\begin{aligned}
(B_G\Gamma)\,(G/(-)) &\simeq \int \mathrm{Map}\big(\mathbf{B}(-), \mathbf{B}\Gamma /\!\!/ G\big)_{\mathbf{B}G} && \text{by Lem. 4.3.3}\\
&\overset{(Bp)^*}{\hookrightarrow} \int \mathrm{Map}\big(\mathbf{B}\widehat{(-)}, \mathbf{B}\Gamma /\!\!/ \widehat{G}\big)_{\mathbf{B}\widehat{G}} && \text{by Prop. 3.2.87}\\
&\simeq (p_* B_{\widehat{G}}\Gamma)\,(G/(-)) && \text{by (4.126).}
\end{aligned}$$

□

Proper topological orbits. We use the above characterization of equivariant classifying shapes (Lem. 4.3.3) to show (Prop. 4.3.15 below) that the slice of the category of G-orbits over the G-equivariant Γ-classifying shape is fully faithfully embedding into the ∞-site of proper Γ-orbits (Def. 4.3.14 below) – under a common assumption on the topological group Γ (Def. 4.3.11 and Prop. 4.3.13 below). Further below, this embedding under this admissibility conditionimplies the *twisted Elmendorf theorem* (Thm. 4.5.3).

Definition 4.3.11 (Admissible G-equivariant twisting)**.** For $G \in \mathrm{Grp}(\mathrm{FinSet})$ a finite group we say that a G-equivariant topological group $G \circlearrowright \Gamma \in \mathrm{Grp}\big(G\mathrm{Act}(\mathrm{DTopSpc})\big)$ – or rather its delooping stack – is *admissible G-equivariant twisting* if for all (necessarily finite) subgroups $H \subset G$ and all $H \subset G$-crossed homomorphisms $\rho \in \mathrm{CrsHom}(H, H \circlearrowright \Gamma)$ (1.2.11) the operations of

(i) passing to shape $\int$ (3.117)

(ii) forming the looping Ω_ρ (13) at ρ (via the identification of Prop. 1.2.18)

commute on the slice mapping stack (Def. 3.2.65) from $\mathbf{B}H$ to $\mathbf{B}(\Gamma \rtimes G)$ over $\mathbf{B}G$, up to equivalence:

$$\left.\begin{array}{l}\mathbf{B}(\Gamma \rtimes G) \text{ is admissible}\\ \text{as } G\text{-equivariant twisting}\end{array}\right\} \Leftrightarrow \underset{\substack{H \subset G\\ \rho \in \mathrm{CrsHom}(H,\, H \circlearrowright \Gamma)}}{\forall} \Omega_\rho \int \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} \simeq \int \Omega_\rho \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G}. \tag{4.127}$$

In the special case that the G-action on Γ is trivial, this condition reduces to:

$$\left.\begin{array}{l}\mathbf{B}(\Gamma \times G) \text{ is admissible}\\ \text{as } G\text{-equivariant twisting}\end{array}\right\} \Leftrightarrow \underset{\substack{H \subset G\\ \rho \in \mathrm{Hom}(H,\, \Gamma)}}{\forall} \Omega_\rho \int \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma\big) \simeq \int \Omega_\rho \mathrm{Map}\big(\mathbf{B}H, \mathbf{B}\Gamma\big). \tag{4.128}$$

Remark 4.3.12 (Alternative formulations of admissible G-equivariant twists)**.** Noticing that

$$\begin{aligned}
&\Omega_\rho \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma)_{\mathbf{B}G}\\
&\simeq \Omega_\rho \big(\mathrm{Hom}(H, \Gamma) /\!\!/ \Gamma\big) && \text{by (1.54)}\\
&\simeq \mathrm{Stab}_\Gamma(\rho) && \text{by (3.152)}
\end{aligned}$$

is the topological stabilizer group of ρ under the conjugation action by Γ, and recalling

$$\begin{aligned}
&B\,\mathrm{Stab}_\Gamma(\rho)\\
&\simeq \int \mathbf{B}\,\mathrm{Stab}_\Gamma(\rho) && \text{by Prop. 4.1.11}\\
&\simeq B \int \mathrm{Stab}_\Gamma(\rho) && \text{by (3.125)}
\end{aligned}$$

we may, under the looping/delooping equivalence (Prop. 3.2.70), equivalently rewrite (4.127) as follows, which proves the claim:

$$\left.\begin{array}{l}\mathbf{B}(\Gamma \rtimes G) \text{ is admissible} \\ \text{as } G\text{-equivariant twisting}\end{array}\right\} \quad \Leftrightarrow \quad \underset{\substack{H \subset G \\ \rho \in \mathrm{CrsHom}(H,\, H \circlearrowright \Gamma)}}{\forall} \quad \Big(\textstyle\int \mathrm{Map}\big(\mathbf{B}H,\, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G}\Big)_{\rho} \;\simeq\; B\,\mathrm{Stab}_{\Gamma}(\rho)\,, \tag{4.129}$$

where on the right $(-)_\rho$ denotes the connected component of ρ. Therefore, a sufficient condition for admissible G-equivariant twists is that the Γ-conjugacy classes of crossed homomorphisms bijectively label these connected components:

$$\left.\begin{array}{l}\mathbf{B}(\Gamma \rtimes G) \text{ is admissible} \\ \text{as } G\text{-equivariant twisting}\end{array}\right\} \quad \Leftarrow \quad \underset{H \subset G}{\forall} \quad \textstyle\int \mathrm{Map}\big(\mathbf{B}H,\, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} \;\simeq\; \coprod_{\substack{[\rho] \in \\ \mathrm{CrsHom}(H, H \circlearrowright \Gamma)/\Gamma}} B\,\mathrm{Stab}_{\Gamma}(\rho)\,, \tag{4.130}$$

Moreover, noticing that

$$\begin{aligned} & \coprod_{[\rho]} B\mathrm{Stab}_{\Gamma}(\rho) \\ & \simeq \textstyle\int \coprod_{[\rho]} \mathbf{B}\,\mathrm{Stab}_{\Gamma}(\rho) && \text{by Prop. 4.1.11} \\ & \simeq \textstyle\int \coprod_{[\rho]} \big(\Gamma/\mathrm{Stab}_{\Gamma}(\rho)\big) /\!\!/ \Gamma && \text{by Exp. 3.2.35} \end{aligned}$$

we find that a yet stronger (i.e. further from necessary but still) sufficient condition for admissibility is that the conjugacy classes already label the connected components of the action groupoid of crossed homomorphisms (i.e. even before passing to its shape):

$$\left.\begin{array}{l}\mathbf{B}(\Gamma \rtimes G) \text{ is admissible} \\ \text{as } G\text{-equivariant twisting}\end{array}\right\} \quad \Leftarrow \quad \underset{H \subset G}{\forall} \quad \mathrm{Map}\big(\mathbf{B}H,\, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G} \;\simeq\; \coprod_{\substack{[\rho] \in \\ \mathrm{CrsHom}(H, H \circlearrowright \Gamma)/\Gamma}} \big(\Gamma/\mathrm{Stab}_{\Gamma}(\rho)\big) /\!\!/ \Gamma\,. \tag{4.131}$$

For trivial G-action on Γ these specializations all simplify as in (4.128).

Proposition 4.3.13 (Examples of admissible G-equivariant twistings)**.** *The condition of Def. 4.3.11 holds at least for:*

(1) Γ *any Lie group (not necessarily compact) with trivial G-action,*
(2) Γ *a compact Lie group with any G-action,*
(3) $\Gamma = \mathrm{PU}_\omega$ (1.88) *with trivial G-action.*

Proof. (1) For Γ a Lie group with trivial G-action, the sufficient condition (4.131) follows readily from the classical result [CF64, Lem. 31.8], as noticed in [Re14a, p. 5]. In the special case that Γ is almost connected (i.e. the quotient space $\Gamma/\Gamma_{\mathrm{e}}$ by the connected component of the neutral element is compact), the sufficient condition (4.130) is verified in [LU14, Thm. 6.3 with §13].
(2) For Γ a compact Lie group with any G-action, the sufficient condition (4.130) follows from the combination of [LM86, Thm. 10][GMM17, Thm. 4.24] with [MS95][GMM17, Thm. 3.11].
(3) For $\Gamma = \mathrm{PU}_\omega$ with trivial G-action, the sufficient condition (4.130) is verified in [LU14, §15 with §13]. □

We now consider the generalization of the orbit category (Ntn. 3.3.58) from finite groups G to topological groups Γ. The following Def. 4.3.14 is the usual definition of the orbit category of a topological group as in [DK84, §2.1 and Thm. 3.1], only that we"

(i) consider that topological group to be decomposed as the semidirect product $\Gamma \rtimes G$ of a G-equivariant topological group Γ with a finite group G (since this is the case of interest in the following);

(ii) restrict attention to orbits given by finite subgroups (indicated by the adjective "proper");

(ii) understand (as in [Re14a]) the ΔSet-enrichment considered in [DK84] as making an ∞-category (an ∞-site in the sense of Ntn. 3.2.23, by appeal to our Prop. 3.3.48);

(iv) immediately reformulate (4.132) the equivariant mapping space between the orbits as a slice mapping stack between the corresponding delooped subgroups, in direct generalization of Lemma 3.3.60 (to which the following reduces when $\Gamma = 1$ is the trivial group).

Definition 4.3.14 (Proper topological orbit ∞-category)**.** For any G-equivariant D-topological group (not necessarily compact)
- $G \in \mathrm{Grp}(\mathrm{FinSet})$
- $G \circlearrowright \Gamma \in \mathrm{Grp}\big(G\,\mathrm{Act}(\mathrm{DTopSpc})\big)$

we say that the proper *orbit ∞-category* of its semidirect product group $\Gamma \rtimes G$ is the ∞-site (Ntn. 3.2.23) $\mathrm{Orb}(\Gamma \rtimes G)$ whose objects are finite subgroups

$$\eta \in \mathrm{Grp}(\mathrm{FinSet})\,, \qquad \eta \xhookrightarrow{i_\eta} \Gamma \rtimes G$$

and whose hom-∞-groupoids are the shapes of the equivariant mapping spaces between the corresponding coset spaces $(\Gamma \rtimes G)/\eta$, hence equivalently the shapes of the slice mapping stacks (3.78) between the deloopings of the subgroups, relative to $\mathbf{B}(\Gamma \rtimes G)$:

$$\begin{aligned}
&\mathrm{Orb}(\Gamma \rtimes G)\Big((\Gamma \rtimes G)/\eta_1,\, (\Gamma \rtimes G)/\eta_2\Big) \\
&:= \int\!\Big(\mathrm{Map}\Big((\Gamma \rtimes G) \circlearrowright \big((\Gamma \rtimes G)/\eta_1\big),\, (\Gamma \rtimes G) \circlearrowright \big((\Gamma \rtimes G)/\eta_2\big)\Big)^{(\Gamma \rtimes G)}\Big) \\
&\simeq \int \prod_{\mathbf{B}(\Gamma \rtimes G)} \mathrm{Map}\big(\mathbf{B}(i_{\eta_1}),\, \mathbf{B}(i_{\eta_2})\big) && \text{by Exp. 3.2.77} \\
&\simeq \int\!\Big(\mathrm{Map}\big(\mathbf{B}\eta_1,\, \mathbf{B}\eta_2\big)_{\mathbf{B}(\Gamma \rtimes G)}\Big) && \text{by Lem. 3.2.68.}
\end{aligned} \tag{4.132}$$

Proposition 4.3.15 (Slice of finite orbit category over equivariant classifying shape inside proper topological orbit category)**.** *For $G \in \mathrm{Grp}(\mathrm{FinSet})$ and $G \circlearrowright \Gamma \in G\,\mathrm{Act}(\mathrm{DTopSpc})$ such that $\mathbf{B}(\Gamma \rtimes G)$ is admissible G-equivariant twisting (Def. 4.3.11), then the slice ∞-site (Def. 3.2.46) of the G-orbit category (Ntn. 3.3.58) over the G-equivariant classifying shape $B_G\Gamma$ (Def. 4.3.1) is equivelently the full sub-∞-site of the topological orbit ∞-category of $\Gamma \rtimes G$ (Def. 4.3.14)*

$$\mathrm{Orb}(G)_{/B_G\Gamma} \;\simeq\; \mathrm{Orb}(\Gamma \rtimes G)_{\mathrm{crs}} \xhookrightarrow{i_{\mathrm{crs}}} \mathrm{Orb}(\Gamma \rtimes G) \qquad \in\; \mathrm{Site}_\infty\,. \tag{4.133}$$

on orbits of the form $\Gamma/\widehat{H}$ for subgroups $\widehat{H} \subset \Gamma \rtimes G$ (Ntn. 2.2.10) which are lifts of subgroups $H \subset G$:

$$\begin{tikzcd}
\widehat{H} \arrow[r, hook, "i_{\widehat{H}}"] & \Gamma \rtimes G \arrow[d, "\mathrm{pr}_2"] \\
H \arrow[u, "\wr"] \arrow[r, hook, "i_H"] & G\,.
\end{tikzcd} \tag{4.134}$$

Proof. Recall from Lem. 4.3.3 that, as an ∞presheaf over $\mathrm{Orb}(G)$, the equivariant classifying shape is equivalently given by the assignment

$$B_G\Gamma \;:\; G/H \;\longmapsto\; \int \mathrm{Map}\big(\mathbf{B}H,\, \mathbf{B}(\Gamma \rtimes G)\big)_{\mathbf{B}G}\,.$$

By the ∞-Yoneda lemma (Prop. 3.2.29) this implies that the objects of the slice $\mathrm{Orb}(G)_{/B_G\Gamma}$ are lifts $\widehat{H}$ of subgroups $H \subset G$ to $\Gamma \rtimes G$ as in (4.134); while the hom-∞-groupoid in the slice between a pair of such objects is naturally equivalent to that between the corresponding $(\Gamma \rtimes G)$-orbits, as follows:

$$
\begin{aligned}
& \mathrm{Orb}(G)_{/B_G\Gamma}\Big(\big(G/H_1, \mathbf{B}i_{\widehat{H}_1}\big),\, \big(G/H_2, \mathbf{B}i_{\widehat{H}_2}\big) \Big) \\
\simeq\; & \mathrm{Orb}(G)(G/H_1, G/H_2) \underset{G\mathrm{Grpd}_\infty\left(\mathrm{FxdLoc}(G/H_1),\, B_G\Gamma\right)}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} && \text{by (3.66)} \\
\simeq\; & \mathrm{Map}\big(\mathbf{B}H_1, \mathbf{B}H_2\big)_{\mathbf{B}G} \underset{\int \mathrm{Map}\left(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G)\right)_{\mathbf{B}G}}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} && \text{by Lem. 3.3.60 \& Lem. 4.3.3} \\
\simeq\; & \mathrm{Map}\big(\mathbf{B}H_1, \mathbf{B}H_2\big)_{\mathbf{B}G}^{\widehat{H}_1} \underset{B\Omega_{\widehat{H}_1} \int \mathrm{Map}\left(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G)\right)_{\mathbf{B}G}}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} && \text{by (3.80)} \\
\simeq\; & \mathrm{Map}\big(\mathbf{B}H_1, \mathbf{B}H_2\big)_{\mathbf{B}G}^{\widehat{H}_1} \underset{\int \mathbf{B}\Omega_{\widehat{H}_1} \mathrm{Map}\left(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G)\right)_{\mathbf{B}G}}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} && \text{by (4.127)} \\
\simeq\; & \int \Bigg(\mathrm{Map}\big(\mathbf{B}H_1, \mathbf{B}H_2\big)_{\mathbf{B}G}^{\widehat{H}_1} \underset{\mathbf{B}\Omega_{\widehat{H}_1} \mathrm{Map}\left(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G)\right)_{\mathbf{B}G}}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} \Bigg) && \text{by Lem. 3.3.10} \\
\simeq\; & \int \Bigg(\mathrm{Map}\big(\mathbf{B}H_1, \mathbf{B}H_2\big)_{\mathbf{B}G} \underset{\mathrm{Map}\left(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G)\right)_{\mathbf{B}G}}{\times} \Big\{ \mathbf{B}\big(i_{\widehat{H}_1}\big) \Big\} \Bigg) && \text{by (3.80)} \\
\simeq\; & \int \mathrm{Map}\Big(\big(\mathbf{B}H_1, \mathbf{B}(i_{\widehat{H}_1})\big),\, \big(\mathbf{B}H_2, \mathbf{B}(i_{\widehat{H}_2})\big) \Big)_{\mathbf{B}(\Gamma \rtimes G)} && \text{by Lem. 4.135} \\
\simeq\; & \mathrm{Orb}(\Gamma \rtimes G)\Big((\Gamma \rtimes G)/\widehat{H_1},\, (\Gamma \rtimes G)/\widehat{H_2} \Big) && \text{by (4.132).}
\end{aligned}
$$

Here in the second but last step we used the following observation (which could be stated much more generally):

$$
\mathrm{Map}(\mathbf{B}H_1, \mathbf{B}H_2)_{\mathbf{B}G} \underset{\mathrm{Map}(\mathbf{B}H_1,\, \mathbf{B}(\Gamma \rtimes G))_{\mathbf{B}G}}{\times} \big\{ \mathbf{B}i_{\widehat{H}_2 s} \big\} \;\;\simeq\;\; \mathrm{Map}\Big(\big(\mathbf{B}H_1, \mathbf{B}(i_{\widehat{H}_1})\big),\, \big(\mathbf{B}H_2, \mathbf{B}(i_{\widehat{H}_2})\big) \Big)_{\mathbf{B}(\Gamma \rtimes G)}. \tag{4.135}
$$

To see this, considering the following diagram in $\mathrm{SmthGrpd}_\infty$, which commutes by (4.134):

$$
\begin{tikzcd}
\mathrm{Map}(\mathbf{B}H_1, \mathbf{B}H_2) \arrow[r, "\left(\mathbf{B}(i_{H_2})\right)\circ(-)"] \arrow[d, "\mathbf{B}\left(i_{\widehat{H}_2}\right)\circ(-)"'] & \mathrm{Map}(\mathbf{B}H_1, \mathbf{B}G) \arrow[d, equal] & \ast \arrow[l, "\vdash \mathbf{B}(i_{H_2})"'] \arrow[d, equal] & \mathrm{Map}(\mathbf{B}H_1, \mathbf{B}H_2)_{\mathbf{B}G} \arrow[d] \\
\mathrm{Map}(\mathbf{B}H_1, \mathbf{B}(\Gamma \rtimes G)) \arrow[r, "\left(\mathbf{B}(\mathrm{pr}_2)\right)\circ(-)"] & \mathrm{Map}(\mathbf{B}H_1, \mathbf{B}G) & \ast \arrow[l, "\vdash \mathbf{B}(i_{H_2})"'] & \mathrm{Map}(\mathbf{B}H_1, \mathbf{B}(\Gamma \rtimes G))_{\mathbf{B}G} \\
\ast \arrow[u, "\vdash \mathbf{B}(i_{\widehat{H}_1})"] \arrow[r] & \ast \arrow[u, "\vdash \mathbf{B}(i_{H_1})"'] & \ast \arrow[l] \arrow[u, equal] & \ast \arrow[u] \\
\mathrm{Map}\Big(\big(\mathbf{B}H_1, \mathbf{B}(i_{\widehat{H}_1})\big),\, \big(\mathbf{B}H_2, \mathbf{B}(i_{\widehat{H}_2})\big) \Big)_{\mathbf{B}(\Gamma \rtimes G)} \arrow[r] & \ast & \ast \arrow[l] &
\end{tikzcd}
$$

By Def. 3.2.65, forming limits horizontally over the rows or vertically over the columns yields the diagrams of slice mapping stacks shown on the right and on the bottom, respectively. The further limits over these two diagrams manifestly yield the two sides of the equivalence (4.135), which hence follows by the general fact that limits commute over each other. □

Definition 4.3.16 (Systems of shapes of fixed loci in topological Γ-spaces)**.** For $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ and $\Gamma \circlearrowright \mathrm{X} \in \Gamma\mathrm{Act}(\mathrm{kTopSpc})$, we write

$$
\int \mathrm{FxdLoc}(\mathrm{X}) \in \mathrm{PSh}_\infty\big(\mathrm{Orb}(\Gamma)\big)
$$

for the ∞-presheaf (Ntn. 3.2.26) over the proper orbit category of Γ (Def. 4.3.14) which assigns the shapes (presented by the classical singular simplicial complexes, via Prop. 3.3.24) of the fixed loci (Ntn. 1.1.1) of X:

$$\mathrm{FxdLoc} \;:\; \Gamma/\eta \;\longmapsto\; \int\Big(\mathrm{Map}\big(\Gamma \circlearrowleft (\Gamma/\eta),\, \Gamma \circlearrowleft \mathrm{X}\big)^{\Gamma}\Big)\,.$$

In the special case that $\Gamma = G$ is a discrete group, this reduces equivalently to (3.213).

Example 4.3.17 (System of $(\Gamma \rtimes G)$-fixed loci of a G-equivariant Γ-principal topological bundle)**.** Let
- $G \in \mathrm{Grp}(Sets)$,
- $G \circlearrowleft \Gamma \in \mathrm{Grp}\big(G\mathrm{Act}(\mathrm{kTopSpc})\big)$,
- $G \circlearrowleft \mathrm{X} \in G\mathrm{Act}(\mathrm{kTopSpc})$,
- $G \circlearrowleft \mathrm{P} \in G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}}$.

Then the induced $(\Gamma \rtimes G)$-action on the total space P has non-empty fixed loci only for finite subgroups which are lifts $\widehat{H}$ (2.29) of subgroups $H \subset G$, and for these the fixed locus is isomorphic to that of the base space:

$$\mathrm{P} \in G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl} \quad\Rightarrow\quad \mathrm{FxdLoc}(\mathrm{P}) \,:\, \Gamma/\eta \mapsto \left\{ \begin{array}{ll} \mathrm{P}^{\widehat{H}} & \Bigg|\;\; \eta \simeq \begin{tikzcd} \widehat{H} \arrow[r, hook] & \Gamma \rtimes G \arrow[d, "\mathrm{pr}_2"] \\ H \arrow[u, "\sim"] \arrow[r, hook] & G \end{tikzcd} \\ \varnothing & \big|\;\; \text{otherwise.} \end{array} \right. \tag{4.136}$$

(To see this, recall the proof of Prop. 2.2.17: The Γ-action is free, so that for each $h \in H \subset G$ which fixes a point $x \in \mathrm{X}$ in the base, there is a *unique* element $\gamma_x(h) \in \Gamma$ which compensates the h-action on the fiber P_x.)

But this means that a morphism of presheaves out of $\int \mathrm{FxdLoc}(\mathrm{P})$ is uniquely fixed by its restriction to the full sub-∞-site $\mathrm{Orb}(\Gamma \rtimes G)_{\mathrm{crs}}$ (4.133) on orbits corresponding to these subgroups, in that for all $\mathscr{A} \in \mathrm{PSh}_\infty\big(\mathrm{Orb}(\Gamma \rtimes G)\big)$ we have the following natural equivalences:

$$\mathrm{P} \in G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}} \quad\Rightarrow\quad \begin{array}{ll} \mathrm{PSh}_\infty\big(\mathrm{Orb}(\Gamma \rtimes G)\big)\Big(\int \mathrm{FxdLoc}(\mathrm{P}),\, \mathscr{A}\Big) & \\ \underset{\sim}{\xrightarrow{i^*_{\mathrm{crs}}}} \mathrm{PSh}_\infty\big(\mathrm{Orb}(\Gamma \rtimes G)_{\mathrm{crs}}\big)\Big(\int \mathrm{FxdLoc}(\mathrm{P}),\, \mathscr{A}\Big) & \text{by (4.136)}^3 \\ \simeq \mathrm{PSh}_\infty\big(\mathrm{Orb}(G)_{/B_G\Gamma}\big)\Big(\int\big(\mathrm{FxdLoc}(\mathrm{X}), c\big),\, \mathscr{A}\Big) & \begin{array}{l}\text{by Prop. 4.3.15} \\ \text{and (4.28).}\end{array} \end{array} \tag{4.137}$$

Here in the last line $c :\; \mathrm{X} \to \mathbf{B}\Gamma$ is the (G-equivariant) modulating morphism of the given equivariant bundle.

Recovering the Murayama-Shimakawa construction. We show (Thm. 4.3.19 below) that the Murayama-Shimakawa construction of equivariant classifying spaces (reviewed in §2.3) may be understood as a natural topological groupoid model for the more abstractly defined proper-equivariant moduli stacks (4.115) and hence for their proper-equivariant classifying shapes (Def. 4.3.1).

Lemma 4.3.18 (Geometric fixed loci of equivariant classifying spaces by Murayama-Shimakawa construction)**.** *In* $\mathbf{H} = \mathrm{SmthGrpd}_\infty$*,*

– $G \circlearrowleft \Gamma \in \mathrm{Grp}\,(G\mathrm{Act}(\mathrm{kTopSpc})) \xrightarrow{\mathrm{Grp}(G\mathrm{Act}(\mathrm{Cdfflg}))} \mathrm{Grp}\,(G\mathrm{Act}(\mathrm{SmthGrpd}_\infty))$,

– *$H \subset G$ a subgroup,*

there is a natural equivalence

$$\int \mathrm{Map}\big(\mathbf{B}H,\, (\mathbf{B}\Gamma)/\!\!/G\big)_{\mathbf{B}G} \;\simeq\; \mathrm{Pth}\Big(\big|\mathrm{Map}(\mathbf{E}G,\, \mathbf{B}\Gamma)\big|^{H}\Big) \;\in\; \mathrm{Grpd}_\infty \overset{\mathrm{Dsc}}{\hookrightarrow} \mathrm{SmthGrpd}_\infty \tag{4.138}$$

between the shapes of the slice mapping stack (Def. 3.2.65) and that of the H-fixed locus of the base of the Murayama-Shimakawa construction (2.84).

Proof. First, for $\mathrm{U} \in \mathrm{CartSpc}$ we have the following sequence of natural equivalences in Grpd_∞, where in the last lines we are using Lem. 3.3.25 to notationally conflate $\mathrm{Cdfflg}(\Gamma)$ with Γ:

$$
\begin{aligned}
\mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G}(\mathrm{U}) & \simeq (\mathrm{SmthGrpd}_\infty)_{/\mathbf{B}G}\,(\mathrm{U} \times \mathbf{B}H,\, (\mathbf{B}\Gamma) /\!\!/ G) && \text{by Lem. 3.2.68} \\
& \simeq (\mathrm{SmthGrpd}_\infty)_{/\mathbf{B}H}\,(\mathrm{U} \times \mathbf{B}H,\, (\mathbf{B}\Gamma) /\!\!/ H) && \text{by Prop. 3.2.64} \\
& \simeq \Delta\mathrm{PSh}(\mathrm{CartSpc})_{/\overline{W}H}\,\big(\mathrm{U} \times \overline{W}H,\, \big(\overline{W}\Gamma \times WH\big)/H\big) && \text{by Lem. 3.2.83} \\
& \simeq H\,\mathrm{Act}\big(\Delta\mathrm{PSh}(\mathrm{CartSpc})\big)\big(\mathrm{U} \times WH, \overline{W}\Gamma\big) && \text{by Prop. 3.1.39} \\
& \simeq H\,\mathrm{Act}\,(\Delta\mathrm{PSh}(\mathrm{CartSpc}))\,\big(\mathrm{U} \times WG, \overline{W}\Gamma\big) && \text{by Ex. 3.1.38} \\
& \simeq H\,\mathrm{Act}\big(\Delta\mathrm{PSh}(\mathrm{CartSpc})\big)\big(\mathrm{U} \times (G \times G \rightrightarrows G),\, (\Gamma \rightrightarrows *)\big) && \text{by Ex. 3.1.29} \\
& \simeq \Big(\mathrm{Cdfflg}\,\mathrm{Map}\big((G \times G \rightrightarrows G),\, (\Gamma \rightrightarrows *)\big)^H\Big)(\mathrm{U}) && \text{as in [SS20-Orb, Lem. A.5].}
\end{aligned}
$$

By naturality (in U) of the composite equivalence, the Yoneda lemma (Prop. 3.2.29) implies that we have an equivalence

$$
\mathrm{Map}(\mathbf{B}G, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G} \;\simeq\; \mathrm{Cdfflg}\,\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)^H \;\in\; \mathrm{Grpd}(\mathrm{DfflgSpc}) \hookrightarrow \mathrm{SmthGrpd}_\infty\,. \tag{4.139}
$$

Now the claim is implied from the following composite:

$$
\begin{aligned}
\textstyle\int \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G} & \simeq \textstyle\int \mathrm{Cdfflg}\big(\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)^H\big) && \text{by (4.139)} \\
& \simeq \textstyle\int \mathrm{Cdfflg}\Big(\big(\mathrm{CrsHom}(H, H \circlearrowright \Gamma) \times \Gamma \rightrightarrows \mathrm{CrsHom}(H, H \circlearrowright \Gamma)\big)\Big) && \text{by (2.87)} \\
& \simeq \mathrm{Pth}\,\big|\,\big(\mathrm{CrsHom}(H, H \circlearrowright \Gamma) \times \Gamma \rightrightarrows \mathrm{CrsHom}(H, H \circlearrowright \Gamma)\big)\,\big| && \text{by Prop. 3.3.49 \& Prop. 1.3.21} \\
& \simeq \mathrm{Pth}\Big|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)^H\Big| && \text{by (2.87)} \\
& \simeq \mathrm{Pth}\Big(\big|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)\big|^H\Big) && \text{by Lem. 1.2.31.}
\end{aligned}
$$

Here the passage through the action groupoid of crossed homomorphisms just serves to produce a homotopy equivalent topological groupoid whose nerve is manifestly good, by Prop. 1.3.21, so that its plain topological realization is seen to have the correct weak homotopy type. □

Theorem 4.3.19 (Murayama-Shimakawa construction gives shape of equivariant moduli stacks)**.** *For $G \in \mathrm{Grp}(\mathrm{Set})$ and $G \circlearrowright \Gamma \in G\,\mathrm{Act}\,(\mathrm{Grp}(\mathrm{DTopSpc}))$ with Γ well-pointed (Ntn. 1.3.17), the proper-equivariant classifying space from Def. 4.3.1 is equivalent, under Elmendorf's theorem* (2)*, to the Muryama-Shimakawa construction (Ntn. 2.3.15)*

$$
B_G\Gamma := \cup_{\underset{\curlyvee}{G}} \textstyle\int \curlyvee \mathbf{B}(\Gamma \rtimes G) \;\simeq\; \mathrm{Pth}\,|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)|^{(-)} \quad \in\; G\mathrm{Grpd}_\infty \xhookrightarrow{G\mathrm{Dsc}} G\mathbf{H}_{\cup} \xhookrightarrow{G\mathrm{OrbSpc}} \mathrm{Snglr}G(\mathbf{H}_{\cup})_{/\underset{\curlyvee}{G}} \simeq \mathbf{H}_{/\underset{\curlyvee}{G}}\,.
$$

Proof. For $H \subset G$, we have the following natural equivalences in Grpd_∞:

$$
\begin{aligned}
B_G\Gamma \,:\, G/H \longmapsto \;& \big(\cup_{\underset{\curlyvee}{G}} \textstyle\int \curlyvee(\mathbf{B}\Gamma /\!\!/ G)\big)(G/H) && \text{by Def. 4.3.1} \\
\simeq \;& \cup \mathrm{Map}\big(\underset{\curlyvee}{H}, \textstyle\int \curlyvee (\mathbf{B}\Gamma /\!\!/ G)\big)_{\mathbf{B}G} && \text{by (3.209)} \\
\simeq \;& \textstyle\int \mathrm{Map}(\mathbf{B}H, \mathbf{B}\Gamma /\!\!/ G)_{\mathbf{B}G} && \text{by Lem. 4.3.3} \\
\simeq \;& \mathrm{Pth}\,|\mathrm{Map}(\mathbf{E}G, \mathbf{B}\Gamma)|^H && \text{by Lem. 4.3.18.}
\end{aligned}
$$

□

Classification in Borel-equivariant cohomology and in proper-equivariant cohomology. We now formulate the classification result for equivariant principal bundles in terms of equivariant cohomology.

Definition 4.3.20 (Borel-equivariant and proper-equivariant cohomology in a cohesive ∞-topos)**.** Given a singular-cohesive ∞-topos $\mathbf{H}$ (Def. 3.3.65),

(i) for $G \curvearrowright A \in G\mathrm{Act}(\mathbf{H}_{\cup})$, we have that

$$H^0_G(X;A) \;\coloneqq\; \tau_0\, \mathbf{H}(X /\!\!/ G, A /\!\!/ G)_{\mathbf{B}G} \tag{4.140}$$

is the *Borel-equivariant cohomology* of X with coefficients in A;

(ii) for $\mathscr{A} \in G\mathbf{H}_{\cup}$, we have that

$$H_{\underset{\curlyvee}{G}}(X;\mathscr{A}) \;\coloneqq\; \tau_0\, \mathbf{H}\big(\curlyvee(X /\!\!/ G), G\mathrm{OrbSnglt}(\mathscr{A})\big)_{\underset{\curlyvee}{G}} \overset{(3.217)}{\simeq} \tau_0\, G\mathbf{H}_{/\cup}(\mathrm{FxdLoc}X, \mathscr{A})\,, \tag{4.141}$$

is the *proper equivariant cohomology* of X with coefficients in $\mathscr{A}$.

The following Prop. 4.3.21 generalizes the corresponding classical statement for bare global homotopy theory to general singular-cohesive homotopy theory:

Proposition 4.3.21 (Proper-equivariant cohomology subsumes Borel-equivariant cohomology)**.** *For*
– $G \curvearrowright X \in G\mathrm{Act}(\mathbf{H}_{\cup,0})$,
– $G \curvearrowright A \in G\mathrm{Act}(\mathbf{H}_{\cup})$,
Borel-equivariant cohomology with coefficients in A is in natural bijection with the proper-equivariant cohomology (Def. 4.3.20) with coefficients in $\mathrm{FxdLoc}(A)$ *(Def. 3.3.85):*

$$H^0_G(X;A) \;\simeq\; H^0_{\underset{\curlyvee}{G}}(X;\mathrm{FxdLoc}(A))\,.$$

Proof. We have the following sequence of natural equivalences:

$$\begin{aligned}
\mathbf{H}(X /\!\!/ G, A /\!\!/ G)_{\mathbf{B}G} \;&\simeq\; \mathbf{H}\big(\curlyvee(X /\!\!/ G),\, \curlyvee(A /\!\!/ G)\big)_{\underset{\curlyvee}{G}} && \text{by Ex. 3.3.75} \\
&\simeq\; \mathbf{H}\big(G\mathrm{OrbSpc}(\mathrm{FxdLoc}(X)),\, \curlyvee(A /\!\!/ G)\big)_{\underset{\curlyvee}{G}} && \text{by Prop. 3.3.92} \\
&\simeq\; G\mathbf{H}_{\cup}\big(\mathrm{FxdLoc}(X), G\mathrm{OrbSmth}(\curlyvee(A /\!\!/ G))\big) && \text{by (3.208)} \\
&\simeq\; G\mathbf{H}_{\cup}\big(\mathrm{FxdLoc}(X), \mathrm{FxdLoc}(A)\big) && \text{by Def. 3.3.85.}
\end{aligned}$$

This implies the claim by passage to sets of connected components. □

Remark 4.3.22 (Interpretation)**.** Specialized to pure shape coefficients (which is essentially the only case considered in traditional literature on equivariant cohomology), Prop. 4.3.21 means that Borel-equivariant cohomology only captures those proper-equivariant coefficients which are systems of homotopy fixed loci $\mathrm{FxdLoc}(\int A)$ but not coefficients $\int \mathrm{FxdLoc}(A)$ which are systems of shapes of geometric fixed loci.

Example 4.3.23 (First non-abelian Borel-equivariant cohomology as coarse proper-eqjuivariant cohomology)**.** Specialized to delooped pure shape coefficients $A \coloneqq \mathbf{B}\mathscr{G} \simeq B\int\mathscr{G}$ for any $G \curvearrowright \mathscr{G} \in G\mathrm{Act}(\mathbf{H}_{\cup})$, Prop. 4.3.21 says that first non-abelian Borel-equivariant cohomology is equivalently proper equivariant cohomology with coefficients in the coarse equivariant classifying spaces from Rem. 4.3.8:

$$H^1_G(X;\textstyle\int\Gamma) \;=\; H^0_G(X;B\Gamma) \;\simeq\; H^1_{\underset{\curlyvee}{G}}(X;B_G(\textstyle\int\Gamma))\,.$$

Finally, we obtain the classification of stable equivariant principal bundles (Thm. 4.2.7) in terms of proper equivariant cohomology and generalized from resolvable to cover-resolvable singularities:

Theorem 4.3.24 (Classification of equivariant bundles with truncated structure over orbifolds with cover-resolvable singularities)**.** *For*
– $G \in \mathrm{Grp}(\mathrm{FinSet})_{\mathrm{covrslvbl}}$ *(Ntn. 3.3.54),*
– $\Gamma \in \mathrm{Grp}(\mathrm{DHausSpc})$ *of truncated classifying shape (Ntn. 4.1.30) and with a notion of stable equivariant bundles (Ntn. 4.1.33),*
isomorphism classes of stable topological G-equivariant Γ*-principal bundles over any* $G \curvearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$ *are classified equivalently by*

(i) *proper-equivariant cohomology (Def. 4.3.20) with coefficients in the stable proper-equivariant classifying shape $B_G\Gamma$ (Def. 4.3.6);*

(ii) *proper-equivariant cohomology (Def. 4.3.20) with coefficients in the coarse equivariant classifying shape $B_G(\int\Gamma)$ (Rem. 4.3.8);*

(iii) *first Borel-equivariant cohomology with coefficients in $\int\Gamma$:*

$$\underbrace{\big(G\mathrm{Equiv}\Gamma\mathrm{PrnFibBdl}(\mathrm{DTopSpc})_{\mathrm{X}}^{\mathrm{stbl}}\big)_{/\sim_{\mathrm{iso}}}}_{\substack{\text{isomorphism classes of}\\ \text{stable equivariant principal topological bundles}}} \simeq \underbrace{H^1_G(X;\textstyle\int\Gamma) \;=\; H^0_G(X;B\Gamma)}_{\text{Borel-equivariant cohomology}} \;\simeq\; \underbrace{H^0_{\underset{\curlyvee}{G}}\big(X;B_G(\textstyle\int\Gamma)\big) \;\simeq\; H^0_{\underset{\curlyvee}{G}}\big(X;(B_G\Gamma)^{\mathrm{stbl}}\big)}_{\text{proper-equivariant cohomology}}\,.$$

Proof. First, consider the case where G-singularities are resolvable. Then the first equivalence is Thm. 4.2.7 in view of Def. 4.3.20, the second equivalence follows by Ex. 4.3.23 of Prop. 4.3.21, and then the third by Prop. 4.3.9.

It remains to generalize this statement to the case that G-singularities may not be resolvable, but there exists a surjective homomorphism $p: \widehat{G} \twoheadrightarrow G$ (3.100) such that $\widehat{G}$ has resolvable singularities. To this end, notice that, for $G \circlearrowright X \in G\mathrm{Act}(\mathbf{H}_0)$, we have the natural monomorphism

$$\begin{aligned} G\mathrm{Grpd}_\infty\big(X^{(-)}, B_G\Gamma\big) &\hookrightarrow G\mathrm{Grpd}_\infty\big(X^{(-)}, p_* B_{\widehat{G}}\Gamma\big) && \text{by Lem. 4.3.10 with Ex. 3.1.16} \\ &\simeq \widehat{G}\mathrm{Grpd}_\infty\big(p^* X^{(-)}, B_{\widehat{G}}\Gamma\big) && \text{by Lem. 3.3.94 with (10).} \end{aligned} \tag{4.142}$$

By naturality of the base change $(Bp)^*$, this monomorphism is compatible with that from Prop. 4.1.29, making a commuting square

$$\begin{tikzcd} \big(G\mathrm{Equiv}\Gamma\mathrm{PrnBdl}_X\big)_{/\sim_{\mathrm{iso}}} \arrow[r] \arrow[d, hook, "\text{Prop. 4.1.29}"] & \tau_0 G\mathrm{Grpd}_\infty\big(X^{(-)}, B_G\Gamma\big) \arrow[r, equal] \arrow[d, hook, "\text{Lem. 4.3.10 , (4.142)}"] & H^0_{\underset{\curlyvee}{G}}(X, B_G\Gamma) \\ \Big(\widehat{G}\mathrm{Equiv}\Gamma\mathrm{PrnBdl}_X\Big)_{/\sim_{\mathrm{iso}}} \arrow[r, "\sim"] & \tau_0\widehat{G}\mathrm{Grpd}_\infty\big(p^*X^{(-)}, B_{\widehat{G}}\Gamma\big) \arrow[r, equal] & H^0_{\underset{\curlyvee}{\widehat{G}}}(X, B_{\widehat{G}}\Gamma)\,. \end{tikzcd}$$

Here the bottom map is a bijection by the previous argument, which applies to $\widehat{G}$ by assumption; therefore the top map is also a bijection, being its restriction to the G-equivariant bundles among the $\widehat{G}$-equivariant bundles along $(Bp)^*$. □

Part IV

Examples and applications

We close by briefly indicating examples and applications of the machinery of singular-cohesive homotopy theory in general and in view of the classification result, Thm. 4.3.24, in particular. Further discussion is relegated to [SS22-TEC][SS22-TED].

4.4 Classification results

Example 4.4.1 (Equivariant bundles with truncated compact Lie structure group)**.** Via Thm. 4.1.35, Thm. 4.3.24 subsumes the following special cases known in the literature (here over smooth G-manifolds $G \circlearrowright \mathrm{X}$ with finite equivariance group G satisfying the conditions of Ntn. 3.3.54):

(i) a **discrete group** $\Gamma \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$.

In this case $\Gamma \simeq \int\Gamma \simeq K(\Gamma, 0)$ is an Eilenberg-MacLane space, hence truncated, and $B\Gamma$ is classifying, by Thm. 4.1.13. Hence Γ has truncated classifying shape, according to Ntn. 4.1.30.

The statement of Thm. 4.3.24 in this case recovers [May90, Thm. 5].

Notice that Γ-principal bundles for Γ a discrete group serve as twists for ordinary cohomology ("local coefficient bundles", e.g. [BFGM03, Def. 3.1, Lem. 4.2]), see [FSS20-TCD, Ex. 2.32]. In particular, in the case $\Gamma = \mathbb{Z}_{/2}$ these are the twists of integral cohomology [GS17a], reflecting the fact that $\mathbb{Z}_{/2}$ is the group of units (see [FSS20-TCD, Ex. 2.37]) of the integral Eilenberg-MacLane spectrum: $\mathrm{GL}_1(H\mathbb{Z}) \simeq \{\pm 1\} \simeq \mathbb{Z}_{/2}$.

Therefore, equivariant Γ-principal bundles in this case constitute twists of equivariant ordinary cohomology. Despite the relative simplicity of this example, the classification of these equivariant twists is already quite rich, given by the non-abelian group cohomology of crossed homomorphisms with coefficients in Γ (Thm. 4.3.7).

(ii) a **compact abelian Lie group** $\Gamma \in \mathrm{AbGrp}(\mathrm{CptSmthMfd})$.

Since compact abelian Lie groups are, up to isomorphism, direct products $\Gamma = \mathbb{T}^r \times$ of finite abelian groups A with r-dimensional tori ($r \in \mathbb{N}$) hence with k-fold direct products of the circle group with itself, their shape is an Eilenberg-MacLane space

$$\mathrm{Shp}\,(\mathbb{T}^r \times A) \;\simeq\; K(A,0) \times K(\mathbb{Z}^r, 1)\,,$$

and hence truncated. Moreover, $B\Gamma$ is classifying, by Thm. 4.1.13.

The statement of Thm. 4.3.24 in this case recovers [LMSe83, Lem. 1, Thm. 2][May90, Thm. 3, Thm. 10].

(iii) an **extension of a discrete group K by a torus** $\mathbb{T}^r \hookrightarrow \Gamma \twoheadrightarrow K$.

This is the evident joint generalization of the above cases: The shape of $B\Gamma$ in this case is a homotopy 2-type, now possibly with a non-trivial k-invariant.

The statement of Thm. 4.3.24 in this case, restricted to the point and for trivial G-action on Γ, recovers [Re18, Thm. 1.2].

Namely, and in more generality:

Let Γ be a truncated compact Lie group, hence a semidirect product $\Gamma \simeq \mathbb{T}^r \rtimes K$ of a torus group $\mathbb{T}^r = (\mathrm{U}_1)^{\times r}$ with a finite group K; and G a finite equivariance group with resolvable singularities (Ntn. 3.3.54), assume that its action on Γ is such that

$$(\mathbb{T}^r \rtimes K) \rtimes G \;\simeq\; \mathbb{T}^r \rtimes (K \rtimes G) \quad \in \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)\,.$$

(The special case of this assumption which is considered in all the above references is that the G-action on Γ is trivial. On the opposite extreme, the assumption is satisfied for $K = 1$ and general G-action on $\mathbb{T}^r$.)

With this assumption, Thm. 4.1.35 says that all G-equivariant bundles with structure group $\Gamma \rtimes G$ are blowup-stable. Therefore the orbi-smooth Oka principal (Thm. 4.1.55) says, for all $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$, that we have an equivalence

$$\int \mathrm{Map}\big(\mathrm{X} /\!\!/ G, \mathbf{B}(\mathbb{T}^r \rtimes K) /\!\!/ G\big)_{\mathbf{B}G} \;\simeq\; \mathrm{Map}\big(\textstyle\int \mathrm{X} /\!\!/ G, B(\mathbb{T}^{\rtimes} K) /\!\!/ G\big)_{BG} \quad \in \mathrm{Grpd}_\infty$$

and Thm. 4.2.7 says that this restricts on connected components to the classification theorem for equivariantly locally trivial G-equivariant $G \circlearrowright \Gamma$-principal bundles:

$$\big(G\mathrm{Equv}\,(G \circlearrowright \Gamma)\,\mathrm{PrnFibBdl}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}} \;\simeq\; \tau_0 \mathrm{Map}\big(\textstyle\int \mathrm{X} /\!\!/ G,\, B\Gamma /\!\!/ G\big)_{BG} \;=:\; H^1_G\big(\mathrm{X}, \Gamma_G\big)\,.$$

Example 4.4.2 (Classification of equivariant PU_ω-bundles)**.** Equivariant bundles with structure group the infinite projective groups, from Ex. 1.3.19, are meant to serve as geometric twists

- for equivariant KU-theory (going back to [AR03][AS04, §6]),
- for KO-theory (going back to [DD63], the modern perspective is in [AGG14], considered via a variety of explicit models in [Ro89][MMS03][HJ13][GS18][GS19b][GY21]),
- and for their unification in KR-theory [At66] (considered via a variety of explicit models in [Mou11][Mou12] [Mou14][FM13, §7][HSMV16][Go17]).

So consider the above classification result, in turn, for:

(i) the **projective unitary group on a Hilbert space** $\Gamma = \mathrm{PU}_\omega$ (1.88).

This has the truncated shape of an Eilenberg-MacLane space $\int \mathrm{PU}_\omega \simeq K(\mathbb{Z}, 2)$ (by Ex. 2.3.29), which is classifying, by Thm. 4.1.13.

Via the characterization of blowup-stable projective bundles from Thm. 4.1.47 for ADE-equivariance groups, the statement of Thm. 4.3.24 in this case recovers [AS04, Prop. 6.3][BEJU14, Thm. 3.8][LU14, 15.17] (see also [TXLG04, Cor. 2.41]), and the statement of Thm. 4.3.7 in this case recovers [BEJU14, Thm. 1.10].

(ii) the **projective unitary group on a graded Hilbert space** $\Gamma = \mathrm{PU}^{\mathrm{gr}}_\omega$ (1.94).

Via the characterization of blowup-stable projective graded bundles from Thm. 4.1.52 for ADE-equivariance groups, this special case of Thm. 4.3.24 gives the classification of twists of equivariant K-theory in degrees 1 and 3 combined.

(iii) the **projective unitary group with complex conjugation action** $\mathbb{Z}_{/2} \circlearrowright \Gamma = \mathbb{Z}_{/2} \circlearrowright \mathrm{PU}_\omega$ or $= \mathbb{Z}_{/2} \circlearrowright \mathrm{PU}^{\mathrm{gr}}_\omega$.

Via Thm. 4.1.52, the statement of Thm. 4.3.24 in this case generalizes the previous two cases to Atiyah's "Real" K-theory (KR-theory) [At66]; see also Ex. 4.1.26, whence we may refer to equivariant $\mathbb{Z}_{/2} \circlearrowright \mathrm{PU}_\omega$-principal bundles as *Real-equivariant* projective bundles. Their classification result in this generality seems to be new.

Namely:

For $G \subset \mathrm{Sp}(1)$ (Ex. 3.3.55) equipped with a homomorphism $G \xrightarrow{ct} \mathbb{Z}_{/2}$ and for $G \circlearrowright \mathrm{X} \in G\mathrm{Act}(\mathrm{SmthMfd})$, the isomorphism classes of blowup-stable (Def. 4.1.51, Thm. 4.1.52) G-equivariant $\mathbb{Z}_{/2} \circlearrowright \mathrm{PU}^{\mathrm{gr}}_{\omega'}$-principal bundles on X (4.1.52) are in natural bijection with the Borel-equivariant integral cohomology of X with coefficients in degrees 1 and conjugation-local coefficients in degree 3:

$$\begin{tikzcd}
\overset{\substack{\text{isomorphism classes of stable}\\ \text{Real-equivariant projective graded bundles}}}{\big(G\mathrm{Equv}\,\mathrm{PU}^{\mathrm{gr}}_{\omega'}\,\mathrm{PrnBdl}^{\mathrm{stbl}}_{\mathrm{X}}\big)_{/\sim_{\mathrm{iso}}}} \arrow[r, "\sim"] \arrow[d, "\wr"', "\substack{\text{Thm. 4.3.24}\\ \text{via}\\ \text{Thm. 4.1.52}}"] & \overset{\substack{\text{Borel-equivariant integral cohomology}\\ \text{in degree 1 and conjugation-local degree3}}}{H^1_G\big(\mathrm{X}, \mathbb{Z}\big) \times H^3_G\big(\mathrm{X}, \mathbb{Z}_{ct}\big)} \arrow[d, equal] \\
\tau_0\,\mathrm{Map}\Big(\int \mathrm{X} /\!\!/ G, \int \mathbf{B}\mathrm{PU}^{\mathrm{gr}}_{\omega'} /\!\!/ \mathbb{Z}_{/2}\Big)_{B\mathbb{Z}_{/2}} \arrow[r, "\sim", "\text{Lem. 4.1.50}"'] & \tau_0\,\mathrm{Map}\Big(\int \mathrm{X} /\!\!/ G, B\mathbb{Z}_{/2} \times B^3\mathbb{Z} /\!\!/ \mathbb{Z}_{/2}\Big)_{B\mathbb{Z}_{/2}}\,.
\end{tikzcd} \tag{4.143}$$

(For trivial equivariance group $G = 1$ this subsumes the classification of [MP88, Thm. 3.6][AS04, Prop. 2.3].)

Moreover, Thm. 4.3.7 implies that the equivariant homotopy groups of the equivariant classifying shape are given by the integral group cohomology of the equivariance group with conjugation-local coefficients:

$$\pi^H_k\big(B_G(\mathrm{PU}^{\mathrm{gr}}_\omega)^{\mathrm{stbl}}\big) \;\simeq\; \tau_0\,\mathrm{Map}\big(BH, B^{3-k}\mathbb{Z}_{\mathrm{cjg}}\big)_{BH} \;\simeq\; H^{3-k}_{\mathrm{Grp}}\big(H, \mathbb{Z}_{\mathrm{cjg}}\big)\,. \tag{4.144}$$

See also Ex. 4.1.57 for how exactly this encodes the space of projective intertwiners of stable projective G-representations.

4.5 Twisted equivariant cohomology

We use the above singular-cohesive homotopy theory to obtain (in Thm. 4.5.3 below) a twisted (parameterized/sliced) generalization of the classical Elmendorf-Dwyer-Kan theorem (reviewed as Prop. 4.5.1 below). Before proving the twisted generalization of the EDK theorem, we now explain how this serves as the foundation for a general notion of *twisted & equivariant* generalized differential cohomology theory (subsuming the fairly well-known example of twisted equivariant K-theory. Ex. 4.5.4 below) which we will lay out in [SS22-TEC][SS22-TED]:

The notion of twisted equivariant cohomology. In asking for the proper conceptual nature for twisted & equivariant generalized cohomology, the first step must be understanding the notion of "generalized cohomology" as such. The concpet that traditionally goes by this ambitious name – which here we rather refer to as *Whitehead-generalized cohomology* theories (i.e. those represented by bare spectra, for review and pointers see [FSS20-TCD, Ex. 1.0.13]) – is an important example but not nearly general enough, as it fails to subsume even common generalizations such as sheaf cohomology or non-abelian cohomology, let alone the twisted equivariant differential non-abelian cohomology that is needed in many applications envisioned notably in theoretical physics.

But some reflection shows (see [SS20-Orb, p. 6]) that a notion of generalized cohomology with a good chance to live up to its name is: Hom-sets in the homotopy categories of ∞-toposes (as in §3.2). Concretely, for $\mathbf{H}$ an ∞-topos and $\mathscr{A} \in \mathbf{H}$ any object, the generalized cohomology with coefficients in $\mathscr{A}$ is simply the homotopy-classes of maps into $\mathscr{A}$:

$$\overset{\text{truly generalized cohomology}}{H^0(-;\mathscr{A})} := \pi_0 \mathbf{H}(-,\mathscr{A}) : \mathrm{Ho}(\mathbf{H})^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{4.145}$$

The Giraud-Rezk-Lurie axioms which characterize ∞-toposes (Prop. 3.2.1) guarantee that these hom-sets have the main properties expected of (non-abelian) cohomology sets. In particular, if $\mathscr{A}$ is pointed-connected, it follows (Prop. 3.2.70) that $\mathscr{A} \simeq \mathbf{B}\mathscr{G}$ for an ∞-group object $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ and that

$$\overset{\text{first non-abelian generalized cohomology}}{H^1(-;\mathscr{G})} := H^0(-;\mathbf{B}\mathscr{G}) : \mathrm{Ho}(\mathbf{H})^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{4.146}$$

classifies $\mathscr{G}$-principal ∞-bundles in $\mathbf{H}$ (Prop. 0.2.1), in grand generalization of the classical statement from Chern-Weil theory, for G a (compact) Lie group (review in [FSS20-TCD, §6]).

Remarkably, both *twisting* as well as *equivariance* are native to this generalized notion of generalized cohomology (4.145), and in a unified manner: Indeed, by the *fundamental theorem of ∞-topos theory* (Prop. 3.2.48) the *slice ∞-category* $\mathbf{H}_{/\mathscr{B}}$ over any $\mathscr{B} \in \mathbf{H}$ is again an ∞-topos and its intrinsic cohomology (4.145) on any $(\mathscr{X},\tau) \in \mathbf{H}_{/\mathscr{B}}$ with *local coefficients* $(\mathscr{A}, p_{\mathscr{A}}) \in \mathbf{H}_{/\mathscr{B}}$ is the *τ-twisted* cohomology of $\mathscr{X}$:

$$H^\tau(\mathscr{X}; p_{\mathscr{A}}) = H^0\big((\mathscr{X},\tau), (\mathscr{A}, p_{\mathscr{A}})\big)\,,$$

with its correct functoriality in pairs consisting of a domain object and a twist:

$$\overset{\text{twisted generalized cohomology}}{H^\bullet(-;p_{\mathscr{A}})} := H^0\big((-,\bullet);(\mathscr{A},p_{\mathscr{A}})\big) : \mathrm{Ho}\big(\mathbf{H}_{/\mathscr{B}}\big)^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{4.147}$$

The traditional "abelian" namely Whitehead-generalized twisted cohomology theories, i.e., those represented by bundles of spectra, fit into this more general generalized notion of twisted cohomology (4.147) most neatly, due to the remarkable fact that the ∞-category of bundles of spectra in any ∞-topos $\mathbf{H}$ is itself again an ∞-topos, the *tangent ∞-topos* $T\mathbf{H}$ ([Jo08c, §35.5][Sc13, §4.1.2][Lur17, §7.3][BM21]), which is the amalgamation of the the stabilization of all the slices of $\mathbf{H}$:

$$T\mathbf{H} \simeq \int_{\mathscr{B}\in\mathbf{H}} T_{\mathscr{B}}\mathbf{H}\,, \qquad \text{with} \quad T_{\mathscr{B}}\mathbf{H} = \mathrm{Spectra}\big(\mathbf{H}_{/\mathscr{B}}\big) \underset{\Omega^\infty_{\mathscr{B}}}{\overset{\Sigma^\infty_{\mathscr{B}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{H}^{\mathscr{B}/}_{/\mathscr{B}} \underset{\mathrm{undrlg}}{\overset{(-)_+}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{H}_{/\mathscr{B}}\,. \tag{4.148}$$

Notice that for $\mathscr{E} \in T_*\mathbf{H}$ a (sheaf of) ring spectra, this subsumes local coefficients over the Picard groupoid of $\mathscr{E}$, and hence the usual notion of twisting and grading of multiplicative cohomology Whitehead-generalized theories (review and further pointers in [FSS20-TCD, Ex. 2.010]). This means that twisted Whitehead-generalized cohomology theories together with their sheaf-hypercohomology-type generalizations (as in [Bro73]) all naturally are examples [Sc13, §4.1.2.1] of the truly generalized notion of cohomology (4.145).

In fact, if $\mathbf{H}$ is *cohesive* (Def. 3.3.1) then $T\mathbf{H}$ (4.148) is also cohesive [Sc13, Prop. 4.1.9] and knows all about *differential* generalized cohomology theory ([Sc13, §4.1.2.2][BNV16][ADH21]), in that it provides any coefficient object $\mathscr{E} \in T_*\mathbf{H}$ with compatible notions of curvature forms, Chern characters, and flat coefficients.

Last but not least, we have seen (Prop. 3.2.75) that equivariance is just another instance of twisting by slicing (4.147):

$$\begin{array}{ccc} \mathscr{G}\,\mathrm{Act}(\mathbf{H}) & \simeq & \mathbf{H}_{/\mathbf{B}\mathscr{G}} \\ \mathscr{G} \circlearrowright \mathscr{X} & \longmapsto & \mathscr{X} /\!\!/ \mathscr{G}\,. \end{array} \tag{4.149}$$

However – and this is the major subtlety to be dealt with in the conceptual foundations of equivariant generalized cohomology – while the intrinsic generalized cohomology of $\mathbf{H}_{/\mathbf{B}\mathscr{G}}$ itself is of some interest (for instance it subsumes all notions of group cohomology) the traditional notion of *proper*- (as opposed to Borel-) equivariant cohomology is concerned not with the (slice) hom-∞-groupoids of (4.149), but with the *shape* of its *slice mapping stacks* (cf. Ex. 3.2.79):

$$\underbrace{\int \mathrm{Map}\big(\mathscr{X}/\!\!/\mathscr{G}, \mathscr{A}/\!\!/\mathscr{G}\big)_{\mathbf{B}\mathscr{G}}}_{\text{shape of equivariant mapping stack}} \xrightarrow{(3.128)} \underbrace{\mathrm{Map}\Big(\int\mathscr{X}/\!\!/\int\mathscr{G}\,,\, \int\mathscr{A}/\!\!/\int\mathscr{G}\Big)_{\mathbf{B}\int\mathscr{G}}}_{\text{equivariant mapping space of shapes}} \simeq \mathrm{Dsc}\,\mathbf{H}\Big(\int\mathscr{X}/\!\!/\int\mathscr{G}\,,\, \int\mathscr{A}/\!\!/\int\mathscr{G}\Big)_{\mathbf{B}\int\mathscr{G}}. \tag{4.150}$$

Here the left hand side here reduces to a traditional definitions of proper *equivariant cohomology* (such as of equivariant K-theory, Ex. 4.5.4 below):

$$H^0_G(-;\mathrm{A}) := \pi_0 \int \mathrm{Map}\big(-/\!\!/G, \mathrm{A}/\!\!/G\big)_{\mathbf{B}G} \;:\; \mathrm{Ho}\big(\mathbf{H}_{/\mathbf{B}G}\big)^{\mathrm{op}} \longrightarrow \mathrm{Set}\,, \tag{4.151}$$

while the right-hand side captures only its coarse form of *Borel-equivariant cohomology*. This brings out
(i) that proper equivariant cohomology theory is intrinsically a concept of *cohesive* homotopy theory and
(ii) that in order for its definition (4.151) to qualify as an instance of truly generalized cohomology (4.145) there must be *some* kind of equivariant Oka principle (along the lines indicated in §0.1) which allows to take the shape modality from outside to inside a mapping stack construction.

The orbi-smooth Oka principle which we have proven above (Thm. 4.1.55) concerns special cases when the comparison map (4.150) restricts to an equivalence. However, if we are content with the right-hand side being *any* generalized cohomology theory (4.145) – not necessarily that in a slice of the given $\mathbf{H}$ – then we can ask for a more general equivariant Oka-like principle which identifies the shape of an equivariant mapping stack with the hom-∞-groupoid of the shape of its arguments regarded in some *other* ∞-topos.

We highlight the perspective that this is what the classical Elmendorf-Dwyer-Kan theorem really accomplishes (recalled as Prop. 4.5.1 below) by identifying (under mild conditions) the cohesive definition (4.151) of proper equivariant cohomology with the intrinsic generalized cohomology (4.145) of the ∞-topos of ∞-presheaves over the corresponding orbit category – hence (in the case of discrete equivariance groups, Ex. 4.5.2 below) with the intrinsic cohomology of the G-equivariant ∞-topos $G\mathbf{H}$ (Def. 3.3.62).

But this means, in view of (4.147) that we have identified the correct general definition of twisted & G-equivariant generalized cohomology: This is the intrinsic cohomology (4.145) of slices of $G(\mathbf{H})$:

$$\underbrace{H^\bullet_G(-;\mathscr{A})}_{\text{twisted } G\text{-equivariant generalized cohomology}} := H^0\big((-,\bullet);(\mathscr{A},p_{\mathscr{A}})\big) \;:\; \mathrm{Ho}\big(G\mathbf{H}_{/\mathscr{B}}\big)^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{4.152}$$

(for given moduli of equivariant twistings $\mathscr{B} \in G(\mathbf{H})_{/\mathscr{B}}$).

For this definition to be practically useful and to reproduce existing definitions (notably of twisted equivariant K-theory, Ex. 4.5.4 below) we just need to verify one consistency condition: The equivariant principal bundles of part II should serve, under suitable conditions, as twists in (4.152), in that twisted equivariant cohomology sets in this abstract sets are given by concordance classes of equivariant sections (as in Prop. 4.2.11) of their associated equivariant fiber bundles. That this is indeed the case, under suitable conditions ("admissible twists". Def. 4.3.11) is the statement of the *twisted EKD theorem*, which we prove as Thm. 4.5.3 below, as an easy consequence of a couple of results in singular-cohesive homotopy theory that we have established above.

The twisted Elemendorf theorem. The classical Elmendorf-Dywer-Kan theorem (recalled in slight singular-cohesive re-formulation as Prop. 4.5.1) below, expresses the shape of equivariant mapping stacks between topological spaces equipped with topological group actions as the hom-∞-groupoid between the shapes of their systems of fixed loci (which are ∞-presheaves over the category of orbits of the equivariance group).

Below we use this classical EDK theorem and the singular-cohesive homotopy theory that we have developed above in order to prove a twisted/parameterized/sliced generalization (Thm. 4.5.3) which identifies the shape of spaces of equivariant *sections* of suitable equivariant fiber bundles with the *slice* hom-∞-groupoid of systems of fixed loci slice over the respective proper equivariant classifying shape (Def. 4.3.1).

Proposition 4.5.1 (Elmendorf-Dwyer-Kan theorem [DK84]). *For*

- $\Gamma \in \mathrm{Grp}(\mathrm{kTopSpc})$ *a topological group* (no further conditions whatsoever)*,*
- $\Gamma \circlearrowright \mathrm{P} \in \Gamma\mathrm{CWCplx} \hookrightarrow \Gamma\mathrm{Act}(\mathrm{kTopSpc})$ *a Γ-CW complex with finite stabilizer groups,*
- $\Gamma \circlearrowright \mathrm{A} \in \Gamma\mathrm{Act}(\mathrm{kTopSpc})$ *and topological Γ-space,*

the operation of passing to shapes of fixed loci (Def. 4.3.16) constitutes a natural equivalence of ∞-groupoids

$$\int\Big(\mathrm{Map}\big(\Gamma \circlearrowright \mathrm{P}, \Gamma \circlearrowright \mathrm{A}\big)^{\Gamma}\Big) \xrightarrow{\sim} \mathrm{PSh}_{\infty}\big(\mathrm{Orb}(\Gamma)\big)\Big(\int\mathrm{FxdLoc}(\mathrm{P}), \int\mathrm{FxdLoc}(\mathrm{A})\Big) \tag{4.153}$$

from the shape of the equivariant mapping stack to the hom-∞-groupoid of ∞-presheaves over the category of proper Γ-orbits (Def. 4.3.14, hence here considered for finite stabilizer groups)

Proof. This is a slight re-formulation of the result of [DK84, Thm. 3.1] applied to the computation of derived mapping spaces (see also [CP96], review in [Blu17, Thm. 1.3.8]):

Via Prop. 3.3.24 and Prop. 3.3.40, the left hand side of (4.153) is equivalently the singular simplicial complex of the equivariant mapping space. (Similarly, the shapes appearing in the arguments on the right of (4.153) are equivalently the singular simplicial complexes of the systems of fixed loci.)

By the assumption that P is a Γ-CW complex, it is cofibrant in the simplicial model category structure on Γ-spaces considered in [DK84, §1.2 & Thm. 2.2]. Since every object is fibrant in this model structure, it follows by standard facts of model category theory that the left-hand side of (4.153) is equivalently the derived hom-complex – i.e. the hom-∞-groupoid – in this model structure. Now the Quillen equivalence of [DK84, Thm. 3.1] identifies this hom-∞-groupoid with that (between the singular simplicial complexes of the systems of fixed loci) in the projective model structure on simplicial presheaves (our Ntn. 3.2.26, corresponding to [DK84, §1.3]) which (by Def. 4.3.14 and Prop. 3.2.27, Ntn. 3.2.28) presents the right hand side of (4.153). $\square$

Example 4.5.2 (Elmendorf-Dwyer-Kan theorem for discrete equivariance groups in terms of singular-cohesive homotopy theory)**.** If $\Gamma = G \in \mathrm{Grp}(\mathrm{Set}) \hookrightarrow \mathrm{Grp}(\mathrm{kTopSpc})$ is a discrete group then with the notation and equivalences of §3.3.2, the Elmendorf-Dwyer-Kan theorem (Thm. 4.5.1) reads equivalently as follows (under the assumption that X is a G-CW-complex, Ex. 1.3.6):

$$\int\Big(\mathrm{Map}\big(\mathrm{X}/\!\!/G, \mathrm{A}/\!\!/G\big)_{\mathbf{B}G}\Big) \xrightarrow{\sim} \cup\mathrm{Map}\big(\int\curlyvee(\mathrm{X}/\!\!/G), \int\curlyvee(\mathrm{A}/\!\!/G)\big)_{\underset{\curlyvee}{G}}\,.$$

This is the result of the following sequence of natural equivalences of ∞-groupoids:

$$
\begin{aligned}
& \int\Big(\mathrm{Map}\big(\mathrm{X}/\!\!/G,\, \mathrm{A}/\!\!/G\big)_{\mathbf{B}G}\Big) \\
& \simeq \mathrm{PSh}_\infty\big(\mathrm{Orb}(G)\big)\Big(\int\mathrm{FxdLoc}(\mathrm{P}), \int\mathrm{FxdLoc}(\mathrm{A})\Big) && \text{by (4.153)} \\
& = G\mathrm{Grpd}_\infty\Big(\int\mathrm{FxdLoc}(\mathrm{P}), \int\mathrm{FxdLoc}(\mathrm{A})\Big) && \text{by Def. 3.3.62, Ex. 3.3.63} \\
& = G\mathrm{Grpd}_\infty\Big(G\mathrm{OrbSmth}\big(\int\curlyvee(\mathrm{A}/\!\!/G)\big), G\mathrm{OrbSmth}\big(\int\curlyvee(\mathrm{A}/\!\!/G)\big)\Big) && \text{by (3.215)} \\
& = \mathrm{SnglrGrpd}_{\infty/\underset{\curlyvee}{G}}\Big(\int\curlyvee(\mathrm{A}/\!\!/G), \int\curlyvee(\mathrm{A}/\!\!/G)\Big) && \text{by (3.216)} \\
& = \cup\mathrm{Map}\big(\int\curlyvee(\mathrm{A}/\!\!/G), \int\curlyvee(\mathrm{A}/\!\!/G)\big)_{\underset{\curlyvee}{G}} && \text{by (3.83)}\,.
\end{aligned}
$$

Theorem 4.5.3 (Twisted EDK-theorem)**.** *Let*

– $G \in \mathrm{Grp}(\mathrm{FinSet})$,

– $G \circlearrowright \Gamma \in \mathrm{Grp}\big(G\mathrm{Act}(\mathrm{DTopSpc})\big)$, *such that* $\mathbf{B}(\Gamma \rtimes G)$ *is admissible G-equivariant twisting (Def. 4.3.11),*

– $G \circlearrowright \mathrm{X} \in G\mathrm{CWCplx} \hookrightarrow G\mathrm{Act}(\mathrm{TopSpc})$ *a G-CW-complex (Exp. 1.3.6),*

– $G \circlearrowright \mathrm{P} \in G\mathrm{Equv}\Gamma\mathrm{PrnFibBdl}_{\mathrm{X}}$ *with modulating morphism* $\mathrm{tw}/\!\!/G \in \mathbf{H}(\mathrm{X}/\!\!/G, \mathbf{B}\Gamma/\!\!/G)_{\mathbf{B}G}$,

– $(\Gamma \rtimes G) \circlearrowright \mathrm{A} \in G \circlearrowright \Gamma\mathrm{Act}\big(G\mathrm{Act}(\mathrm{TopSpc})\big)$.

Then, for E *denoting the associated equivariant A-fiber bundle (Def. 4.2.10), we have a natural equivalence of ∞-groupoids*

$$
\begin{aligned}
&\overset{\text{shape of stack of G-equivariant sections of equivariant A-fiber bundle}}{\int\Big(\big(\mathrm{Map}(G\circlearrowright\mathrm{X}, G\circlearrowright\mathrm{E})_{G\circlearrowright\mathrm{X}}\big)^{G}\Big)} \;\overset{\text{Prop. 4.2.11}}{\simeq}\; \underset{\text{shape of slice mapping stack of homotopy quotients from base to typical fiber}}{\int\mathrm{Map}\big((\mathrm{X},\mathrm{tw})/\!\!/G,\, \mathrm{A}/\!\!/(\Gamma\rtimes G)\big)_{\mathbf{B}(\Gamma\rtimes G)}} \\
&\overset{\text{twisted EDK}}{\simeq}\; \overset{\text{slice mapping space of shapes of orbi-singularizations}}{\cup\mathrm{Map}\Big(\int\curlyvee\big((\mathrm{X},\mathrm{tw})/\!\!/G\big), \int\curlyvee\big(\mathrm{A}/\!\!/(\Gamma\rtimes G)\big)\Big)_{\int\curlyvee\mathbf{B}(\Gamma\rtimes G)}} \\
&\simeq\; \underset{\text{slice hom-∞-groupoid of proper G-equivariant homotopy types of equivariant classifying shape}}{G\mathrm{Grpd}_\infty\Big(\int\mathrm{FxdLoc}(\mathrm{X},\mathrm{tw}),\, G\mathrm{OrbSmth}\int\curlyvee\big(\mathrm{A}/\!\!/(\Gamma\rtimes G)\big)\Big)_{B_G\Gamma}}
\end{aligned}
$$

between the shape of the stack of equivariant sections of the fiber bundle and the slice hom-∞-groupoid between the systems of G-fixed loci, sliced over the equivariant classifying shape $B_G\Gamma$.

Proof. This may be obtained as the composite of the following sequence of natural equivalences:

$$
\begin{aligned}
& \int\mathrm{Map}\big((\mathrm{X},\mathrm{tw})/\!\!/G,\, \mathrm{A}/\!\!/\Gamma\rtimes G\big)_{\mathbf{B}(\Gamma\rtimes G)} \\
& \simeq \int\mathrm{Map}\big(\mathrm{P}/\!\!/\Gamma\rtimes G,\, \mathrm{A}/\!\!/\Gamma\rtimes G\big)_{\mathbf{B}(\Gamma\rtimes G)} && \text{by Ex. 4.1.24} \\
& \simeq \int\Big(\mathrm{Map}\big((\Gamma\rtimes G)\circlearrowright\mathrm{P},\, (\Gamma\rtimes G)\circlearrowright\mathrm{A}\big)^{(\Gamma\rtimes G)}\Big) && \text{by Prop. 3.2.75} \\
& \simeq \mathrm{PSh}_\infty\big(\mathrm{Orb}(\Gamma\rtimes G)\big)\big(\int\mathrm{FxdLoc}(\mathrm{P}), \int\mathrm{FxdLoc}(\mathrm{A})\big) && \text{by Prop. 4.5.1} \\
& \simeq \mathrm{PSh}_\infty\big(\mathrm{Orb}(G)_{/B_G\Gamma}\big)\Big(\int\mathrm{FxdLoc}(\mathrm{X},\mathrm{tw}),\, G\mathrm{OrbSmth}\int\curlyvee\big(\mathrm{A}/\!\!/(\Gamma\rtimes G)\big)\Big) && \text{by Prop. 4.3.15, Ex. 4.3.17} \\
& \simeq \mathrm{PSh}_\infty\big(\mathrm{Orb}(G)\big)\Big(\int\mathrm{FxdLoc}(\mathrm{X},\mathrm{tw}),\, G\mathrm{OrbSmth}\int\curlyvee\big(\mathrm{A}/\!\!/(\Gamma\rtimes G)\big)\Big)_{B_G\Gamma} && \text{by Prop. 3.2.47}\,.
\end{aligned}
$$

Here the key step is the identification of the slice ∞-site over the equivariant classifying shape $B_G\Gamma$ via Prop. 4.3.15, which is the step needing the assumption that $\mathbf{B}(\Gamma\rtimes G)$ is an admissible G-equivariant twisting. $\square$

Example 4.5.4 (Twisted equivariant K-theory)**.** The tradtional definition of twisted G-equivariant complex K-theory is as homotopy classes of sections of $\mathrm{Fred}^{(0)}$-fiber bundles which are associated to stable G-equivariant PU_ω-principal bundles [AS04, Def. 3.2][BEJU14, Def. 4.2]. By Prop. 4.2.11 we may equivalently reformulate this in terms of lifts of the modulating map of the underlying equivariant principal PU_ω-bundle, and with Thm. 4.5.3 (which does apply to PU_ω-twists due to Prop. 4.3.13) we then find that this traditional definition of twisted equivariant K-theory does qualify as a twisted equivariant cohomology theory in the general abstract sense discussed above.

Concretely, With Ex. 4.4.2, the class $[\tau]$ of any stable equivariant bundle with structure group $\mathrm{PU}^{\mathrm{gr}}_{\omega'} \rtimes \mathbb{Z}_{/2}$ is represented by one with structure group $\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_{/2}$ (Thm. 4.1.52 (2)), and since the latter has a canonical conjugation action (1.97) on the space of Fredholm operators (1.96) on the graded Hilbert space $\mathscr{H} \otimes \mathbb{C}^2 = L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}_{+} \oplus L^2(S^3_{\mathrm{rm}})^{\otimes^\infty}_{+}$ (4.90), we may use these equivariant principal bundles to define twisted equivariant K-theory in the general stacky form of [SS20-Orb, Rem. 2.94], namely as the homotopy-classes of sections of the stacky $\mathrm{Fred}^{(0)}$-fiber bundle associated (as in [NSS12a, §4]) to the given equivariant principal bundle τ:

$$
\begin{aligned}
\overset{\substack{\text{twisted equivariant}\\ \text{KR-cohomology}}}{\mathrm{KR}^\tau_G(\mathrm{X})} \;:=\;& \overset{\text{connected components of shape of sliced mapping stack}}{\tau_0 \textstyle\int \mathrm{Map}\Big((\mathrm{X}/\!\!/G, \tau),\, \mathrm{Fred}^{(0)}/\!\!/(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}) \Big)_{\mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2})}}
\\
=\;& \left\{
\begin{tikzcd}
 & & \mathrm{Fred}^{(0)}/\!\!/(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}) \arrow[d, "\substack{\text{stacky universal}\\ \text{Fredholm bundle}}"] \\
\overset{\substack{\text{Real good}\\ \text{orbifold}}}{\mathrm{X}/\!\!/G} \arrow[urr, dashed, "\substack{\text{cocycle in}\\ \text{twisted equivariant Real K-theory}}"] \arrow[rr, "\tau", "\substack{\text{modulation of stable Real-equivariant}\\ \text{projective graded bundle}}"'] \arrow[dr, "\text{Real structure}"'] & & \mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}) \arrow[dl] \\
 & \mathbf{B}\mathbb{Z}_{/2} &
\end{tikzcd}
\right\}_{/\sim_{\mathrm{cncrdnc}}}
\end{aligned}
\tag{4.154}
$$

Notice here that the sliced mapping space in (4.154) is formed in the cohesive ∞-topos $\mathrm{SmthGrpd}_\infty$:
(1) This makes the dashed section be an equivariant section of topological bundles, while the image of this section in plain Grpd_∞ under the shape operation $\int \widetilde{\mathrm{ev}}$ (3.128) sees only the coarser Borel-equivariant K-theory. This cohesive-homotopy theoretic construction of twisted equivariant KR-theory generalizes a similar approach in [Pv14, §3.19] to the twisted and Real case.
(2) In particular, the bundle of Fredholm operators on the right of (4.154) is a stacky universal associated bundle over the moduli stack $\mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2})$ (as for the stacky universal principal bundles in §4.2), instead of a plain universal bundle just over the classifying space $B(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2})$ as considered in [LU14, §15] (whose local triviality had remained open, see [EU16, p. 2]).

By way of outlook, we briefly highlight how the Real-equivariant projective graded bundles from Ex. 4.4.2 neatly encode the structure that has been argued to capture:

– quantum symmetries of gapped topological quantum systems (Rem. 4.5.5 below),

– B-field configurations on orbi-orientifold stacetimes in string theory (Rem. 4.5.6 below).

While both of these applications are meanwhile classical and have been widely discussed from a multitude of angles, it seems that the nicely unifying perspective of equivariant principal bundles and for the full projective graded conjugation-equivariant structure group $\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}$ (1.95) has not been fully brought out before:

Remark 4.5.5 (Real-equivariant projective graded bundles as quantum symmetries)**.** Consider the data encoded in the geometric G-fixed locus (4.119) of the equivariant moduli stack (4.115) of the Real-equivariant projective graded unitary bundles in (4.143). Postcomposition with the homomorphism $\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2} \xrightarrow{(c,\mathrm{pr}_2)} \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$ (1.95) projects this onto the groupoid of crossed homomorphisms (Prop. 1.2.18) from the equivariance group to $\mathbb{Z}^2_{/2}$:

$$
G\mathrm{FxdLoc}\big(G \curvearrowright \mathbf{B}\mathrm{PU}^{\mathrm{gr}}_\omega\big) \underset{\text{Lem. 4.3.3}}{\simeq} \mathrm{Map}\big(\mathbf{B}G,\, \mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2})\big)_{\mathbf{B}\mathbb{Z}_{/2}} \xrightarrow{\;(c,ct)_*\;} \mathrm{Map}\big(\mathbf{B}G,\, \mathbf{B}(\mathbb{Z}_{/2} \times \mathbb{Z}_{/2})\big)_{\mathbf{B}\mathbb{Z}_{/2}}\,.
$$

This natural structure reflects exactly the data of "extended QM symmetry classes" according to [FM13, Def. 3.7] (review in [Thi18, pp. 88][SS22-Ord, §2.2]): For $g \in G$ the elements $(c(g), ct(g))$ in target group $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$ (denoted "$\mathscr{C}$" in [FM13, (6.1)]) are interpreted as the *c*harge-conjugation and its product with *t*ime-reversal symmetry, respectively, and their pre-image on the left is a projective representation (Ex. 3.3.30) of the quantum symmetry, including the datum of a $ct =: \phi$*-twisted extension* ([FM13, Def. 1.7]) G^τ of G:

$$
\begin{tikzcd}
\mathrm{U}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\substack{\text{(delooping of)} \\ ct =: \phi\text{-twisted extension of} \\ \text{quantum symmetry group}}}{\mathbf{B}G^\tau} \arrow[rr, "\substack{\text{linear realization of} \\ \text{quantum symmetry group}}"] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}(\mathrm{U}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}) \arrow[d] & \\
\ast \arrow[r] & \mathbf{B}G \arrow[rr, "\tau", "\substack{\text{moduli of real-equivariant} \\ \text{projective graded bundle}}"'] \arrow[dr, "\substack{\text{(anti-)unitarity of} \\ \text{charge+time reversal}}\ \mathbf{B}ct"'] & & \mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_{/2}) \arrow[dl, "\mathbf{B}\mathrm{pr}_2"] \arrow[dr, "\mathbf{B}c\ \substack{\text{sign of} \\ \text{charge conjugation}}"] & \\
& & \mathbf{B}\mathbb{Z}_{/2} & & \mathbf{B}\mathbb{Z}_{/2}
\end{tikzcd}
$$

Remark 4.5.6 (Equivariant $\mathbb{Z}_{/2} \circlearrowright \mathrm{PU}^{\mathrm{gr}}_\omega$-principal bundles as B-fields on orbi-orientifolds)**.**
(i) A famous hypothesis asserts that the massless RR/NS-fields of type II string theory are classified by some form of geometrically twisted K-theory (see [BSS18, §1] for review and pointers and [GS19a] for recent constructions), specifically that geometrically twisted equivariant KR-theory classifies the massless RR/NS-fields on orbi-orientifolds [DFM11b][DFM11a][DMDR14][DMDR15][HSMV16], hence, in particular, that the geometric twists classified by our Ex. 4.4.2, reflect gauge-equivalence classes of the NS B-field in string theory on orbi-orientifolds.
(ii) In fact, this traditional conjecture has been argued to require corrections ([dBD$^+$00, §4.5.2, 4.6.5]). Elsewhere we have shown that such corrections are plausibly captured by replacing twisted equivariant K-theory by twisted equivariant Cohomotopy ([SS19-Tad][BSS19] [SS20-Orb, Rem. 5.30]). The general nature of Thm. 4.3.24 provides the proper theoretical backdrop for analyzing such variant definitions and their comparison. We hope to further discuss this elsewhere.

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