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(New York University, Abu Dhabi & Czech Academy of Science, Prague)

Equivariant super homotopy theory

talk at
Geometry in Modal Homotopy Type Theory
Pittsburgh 2019

based on joint work with

H. Sati, V. Braunack-Mayer, J. Huerta, D. Fiorenza, F. Wellen

Motivation.

Part I.
Super homotopy theory

Part II.
Orbifold cohomology

Motivation

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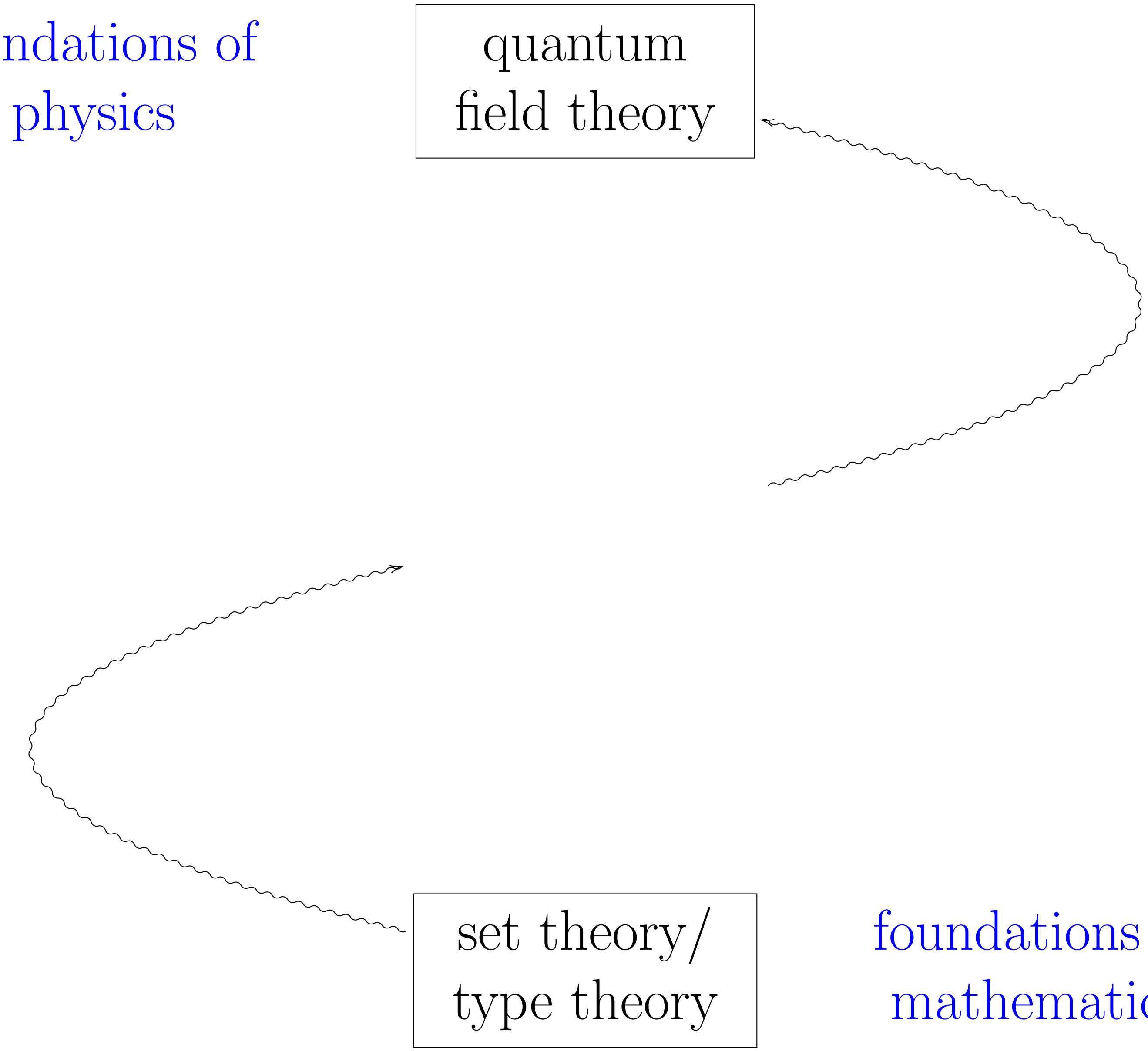
Analytic detours and Synthetic promises



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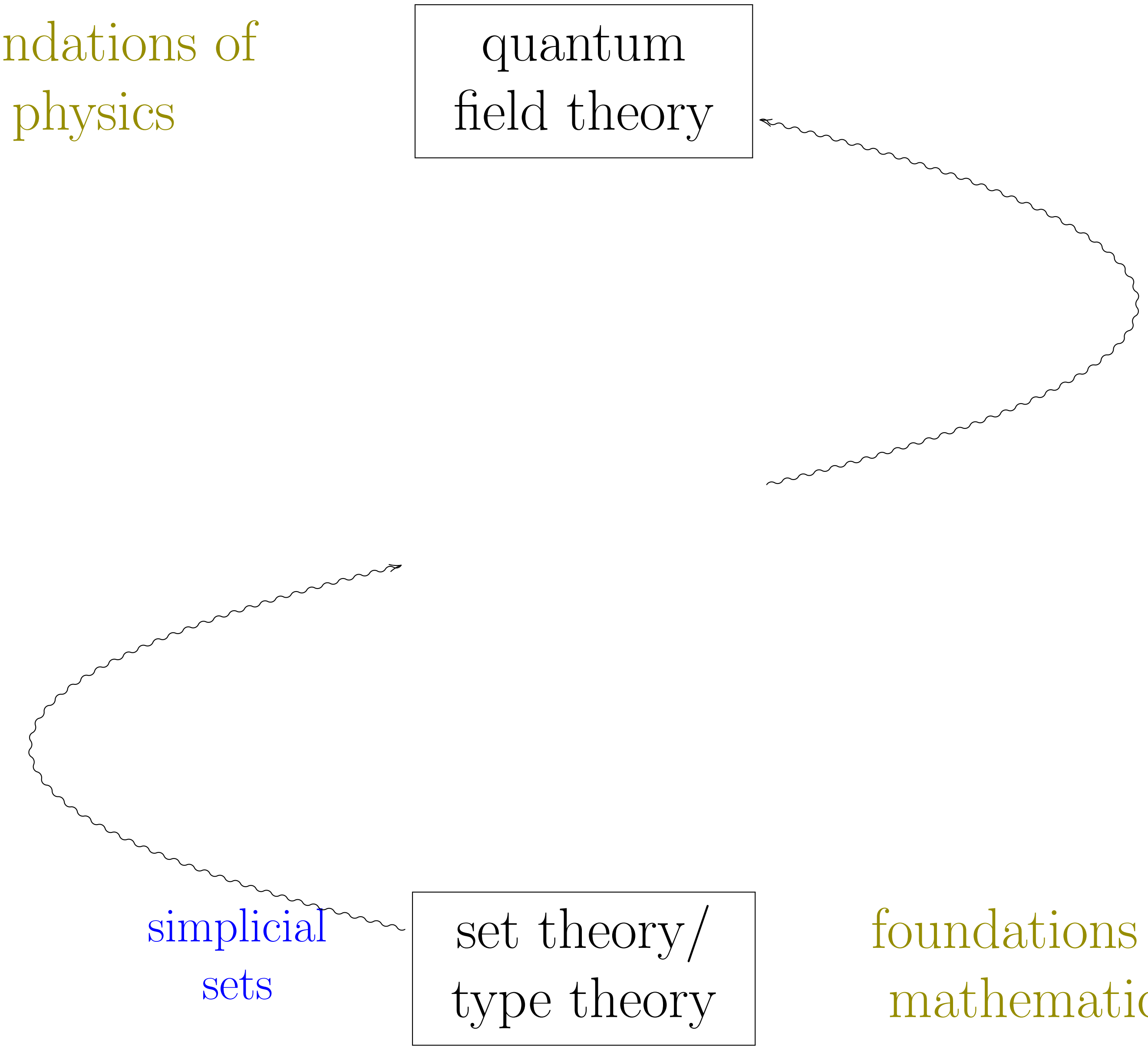

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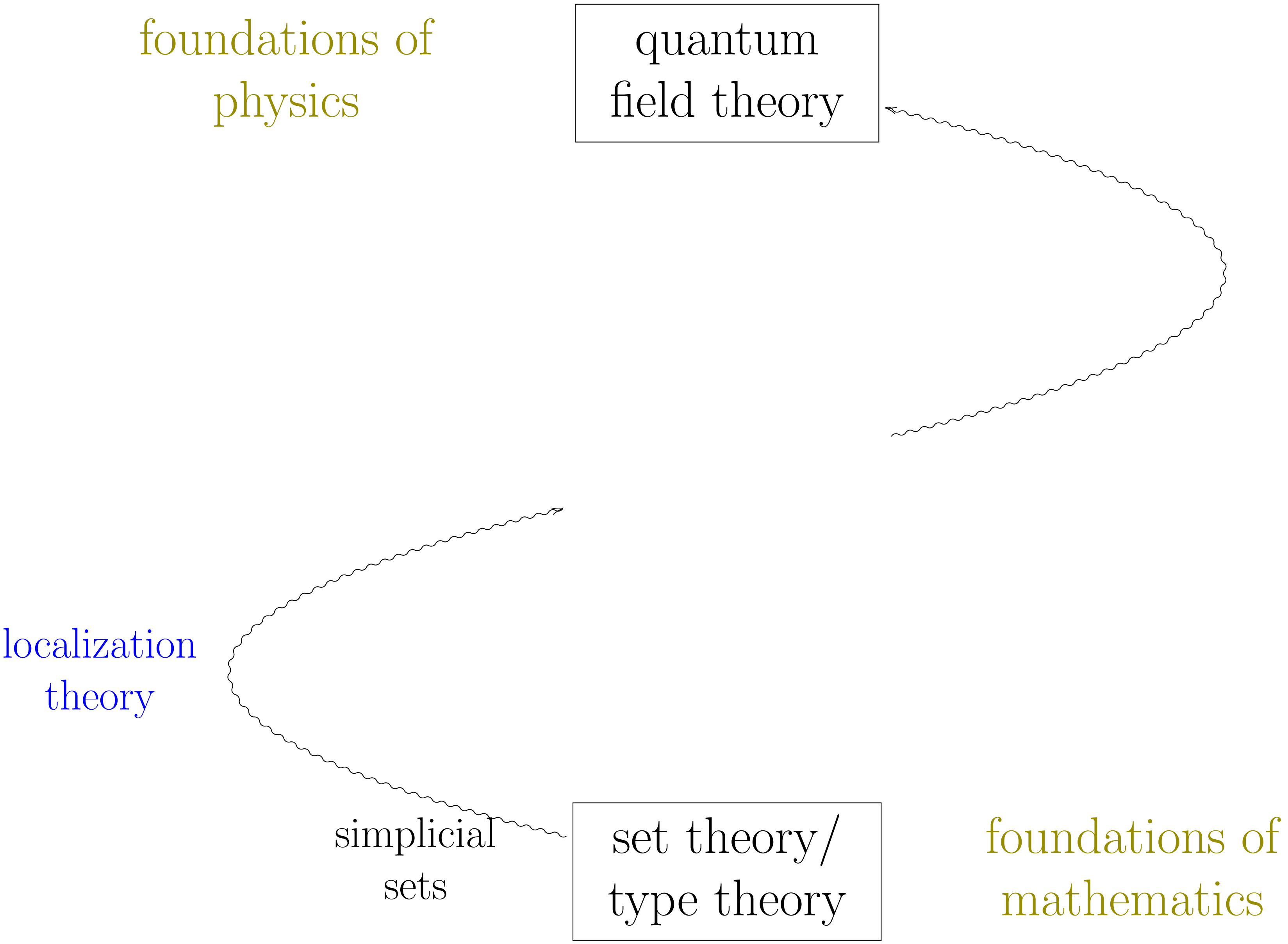

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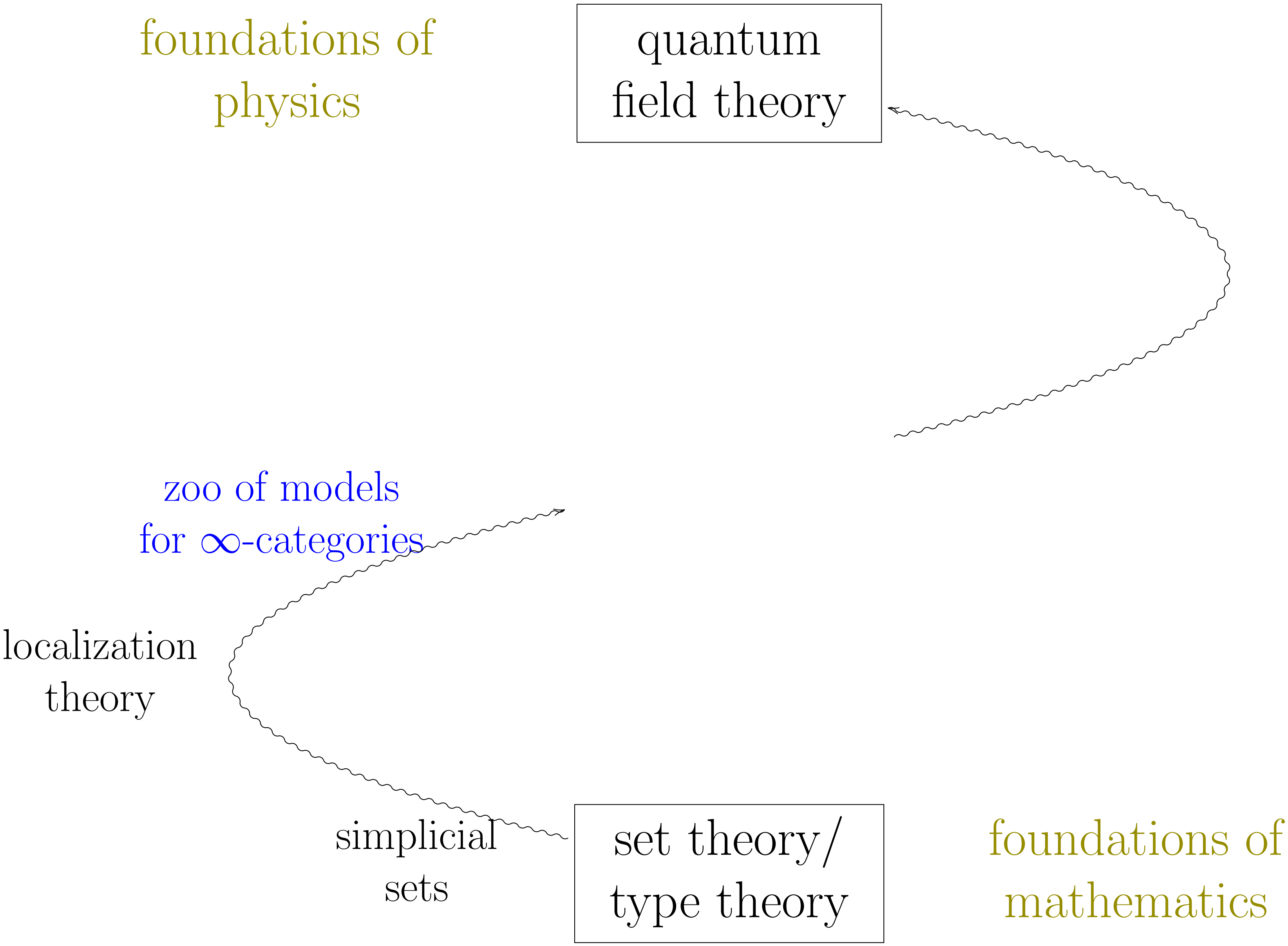

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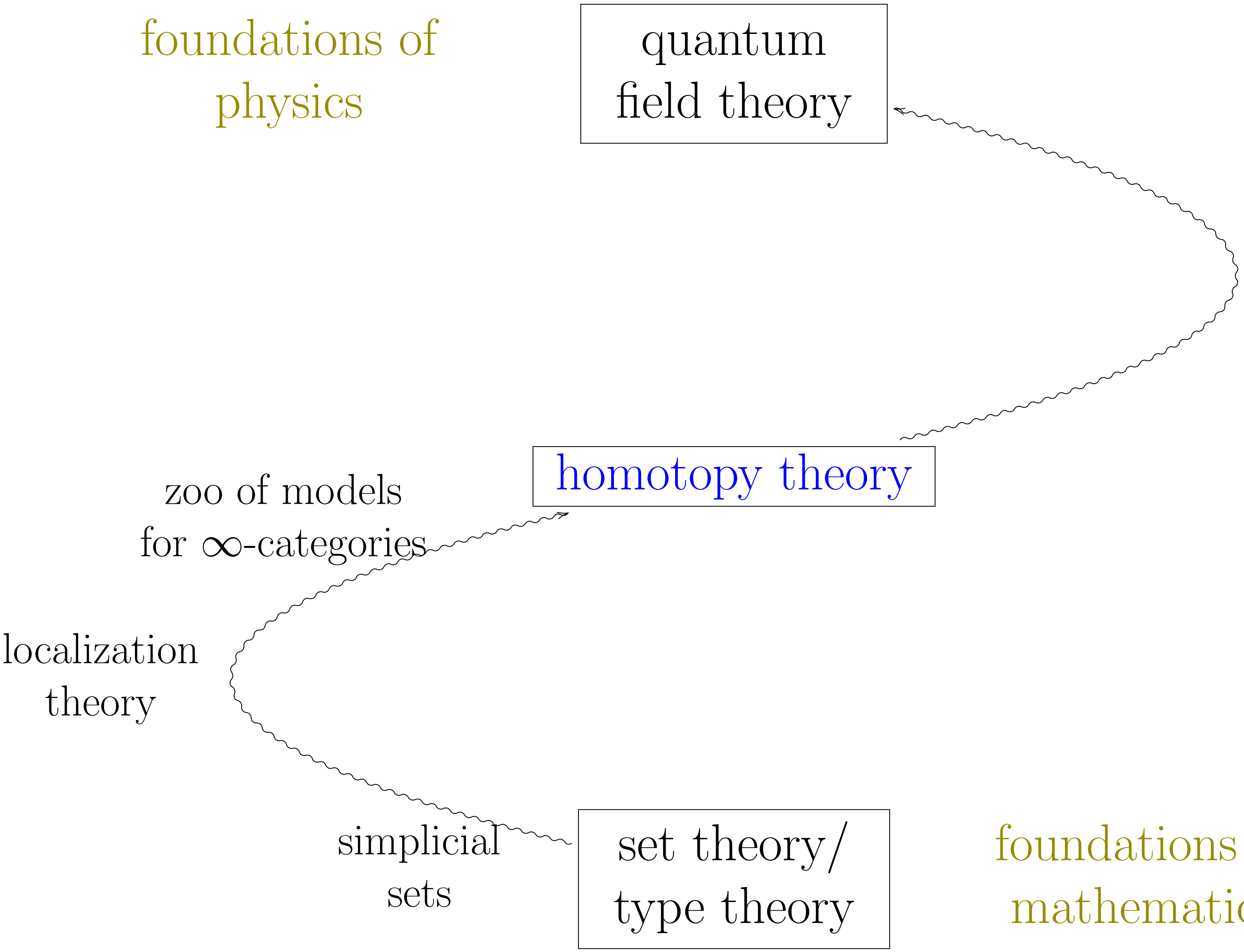

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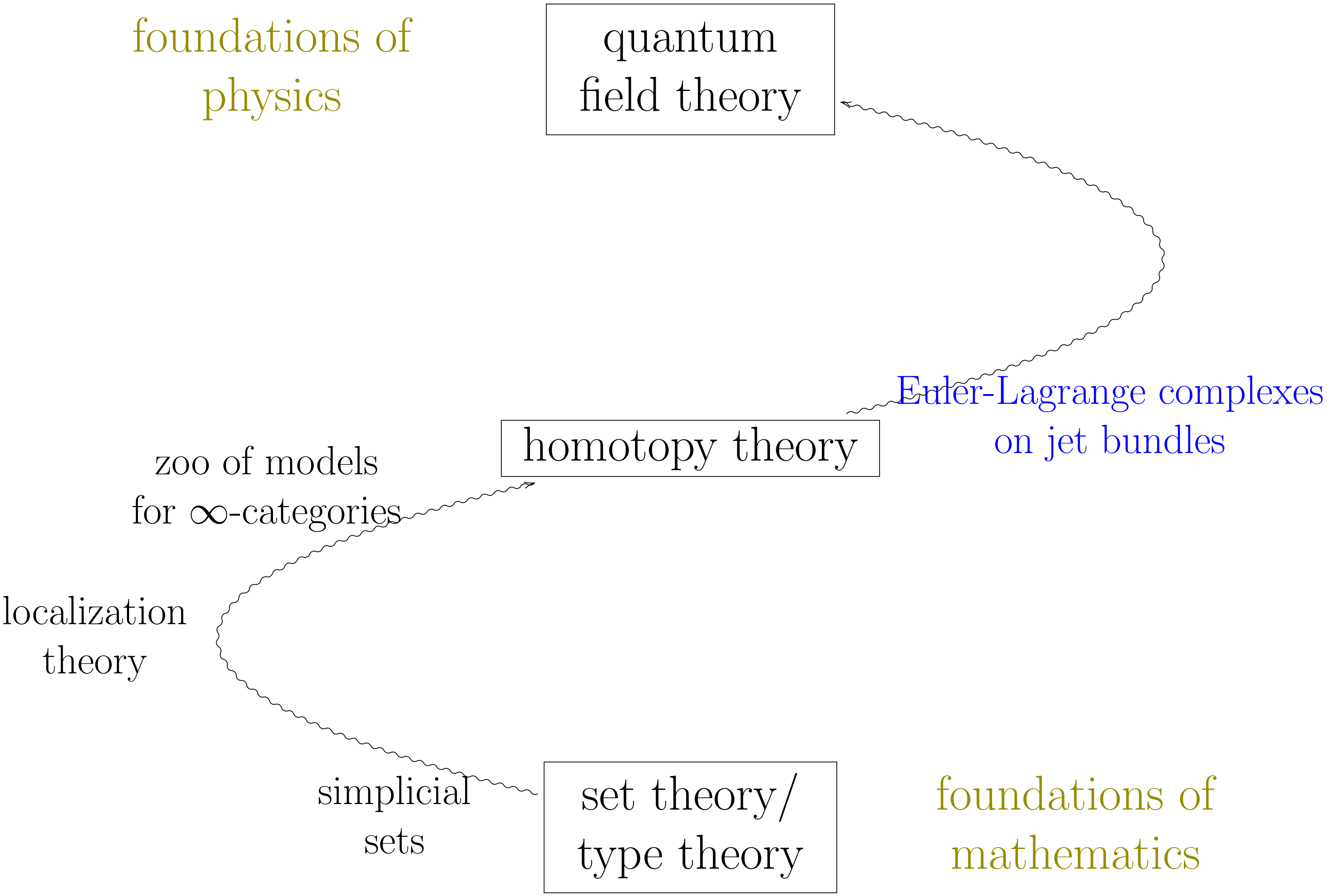

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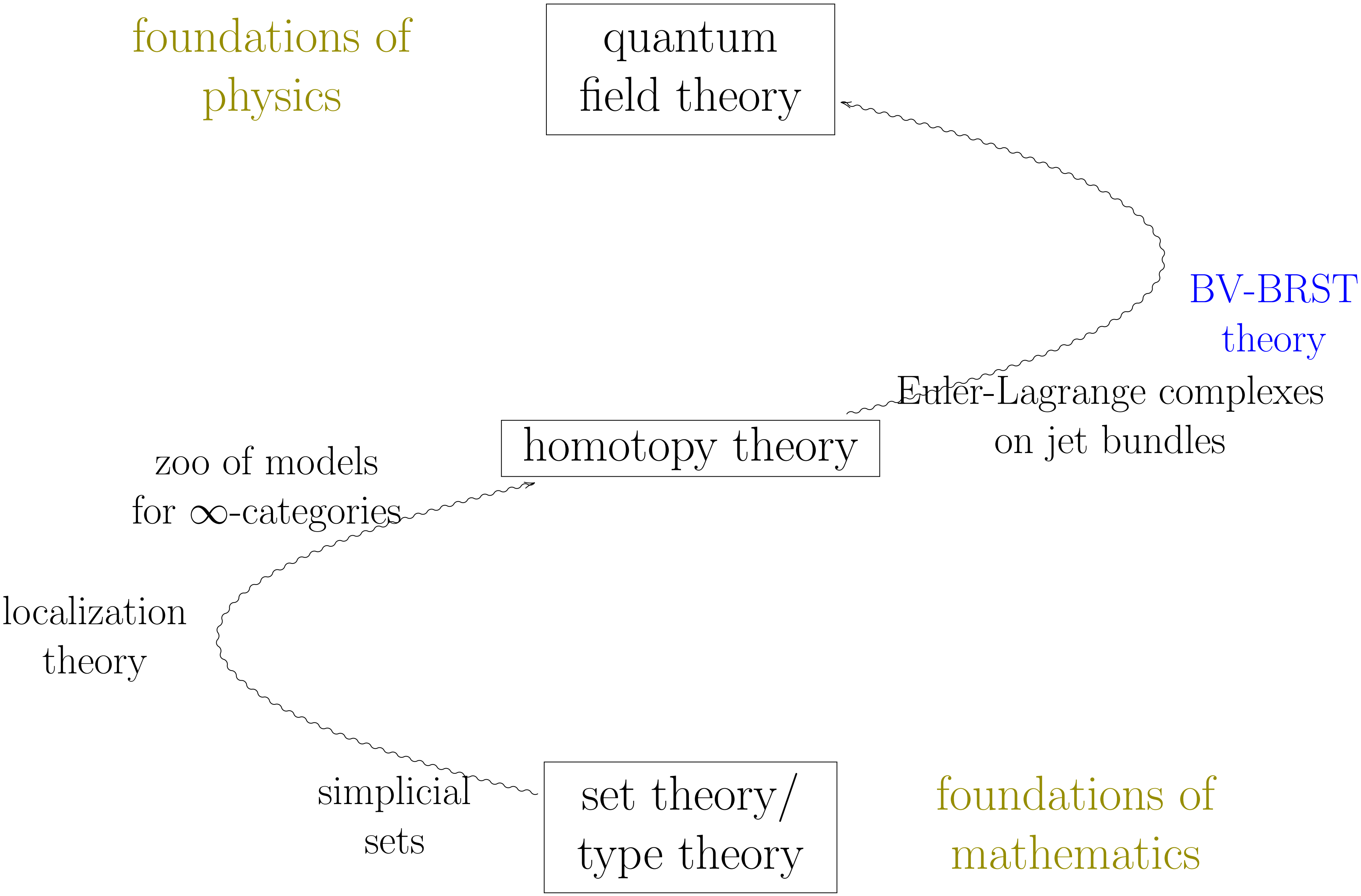

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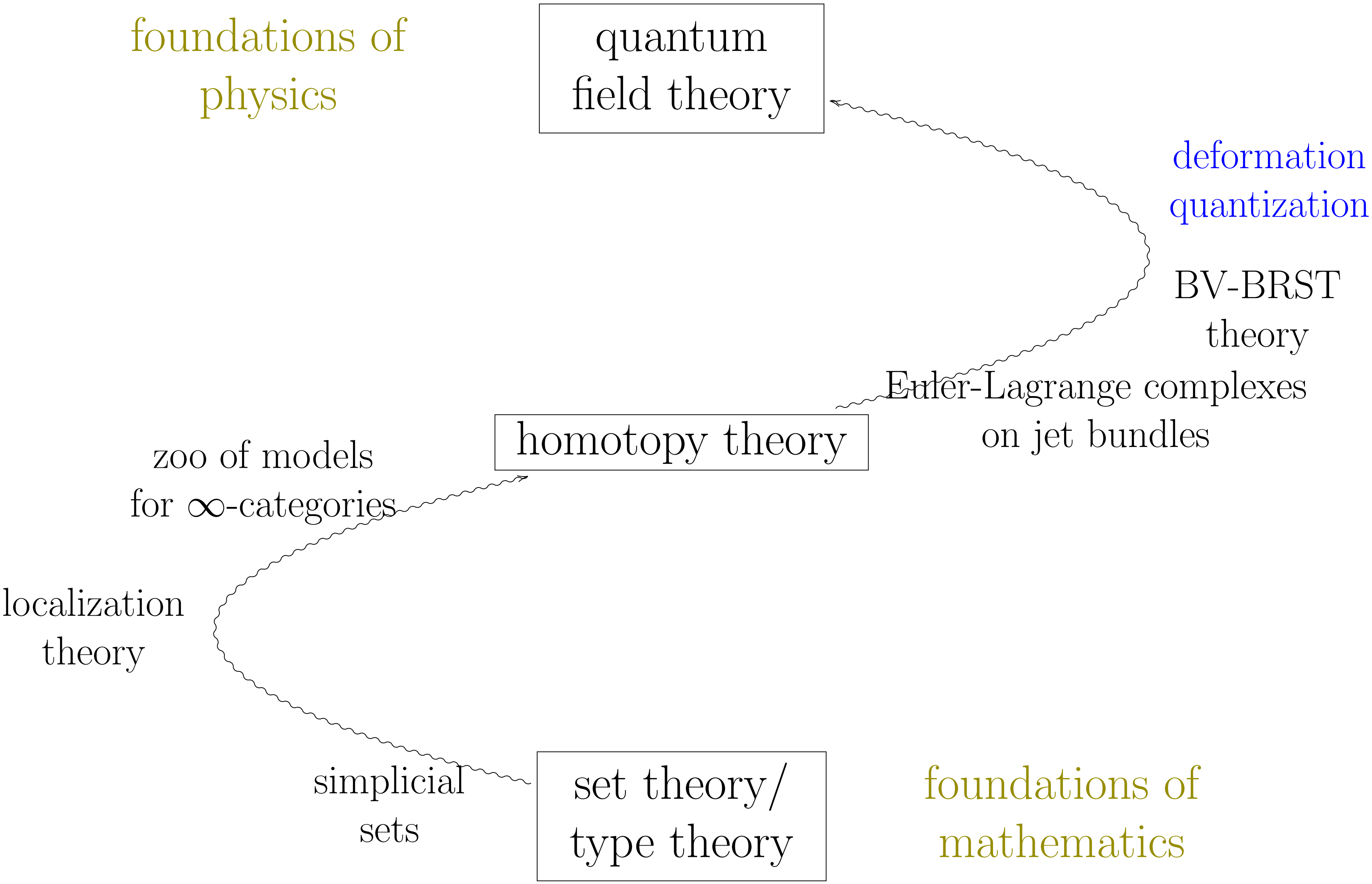

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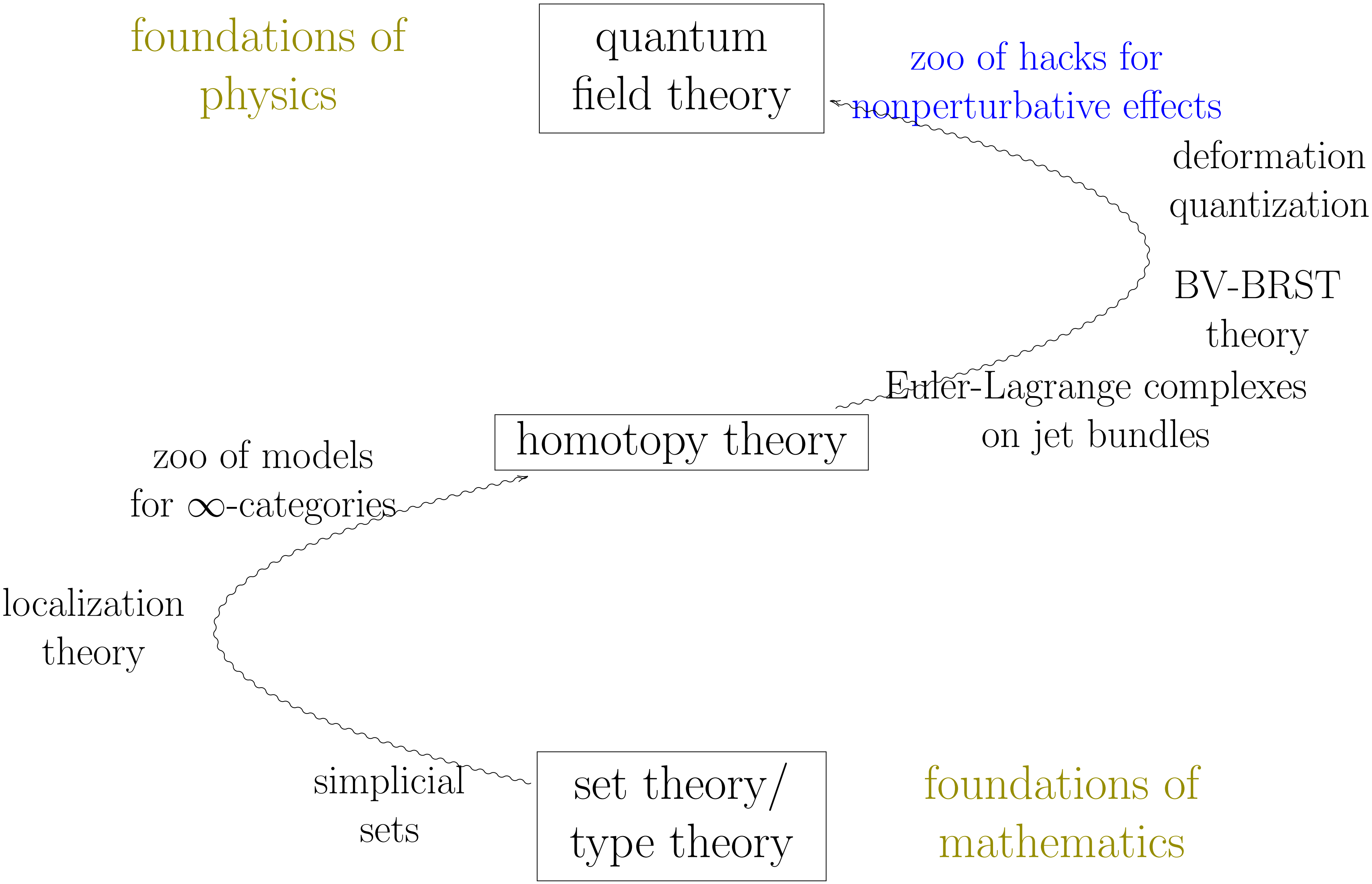

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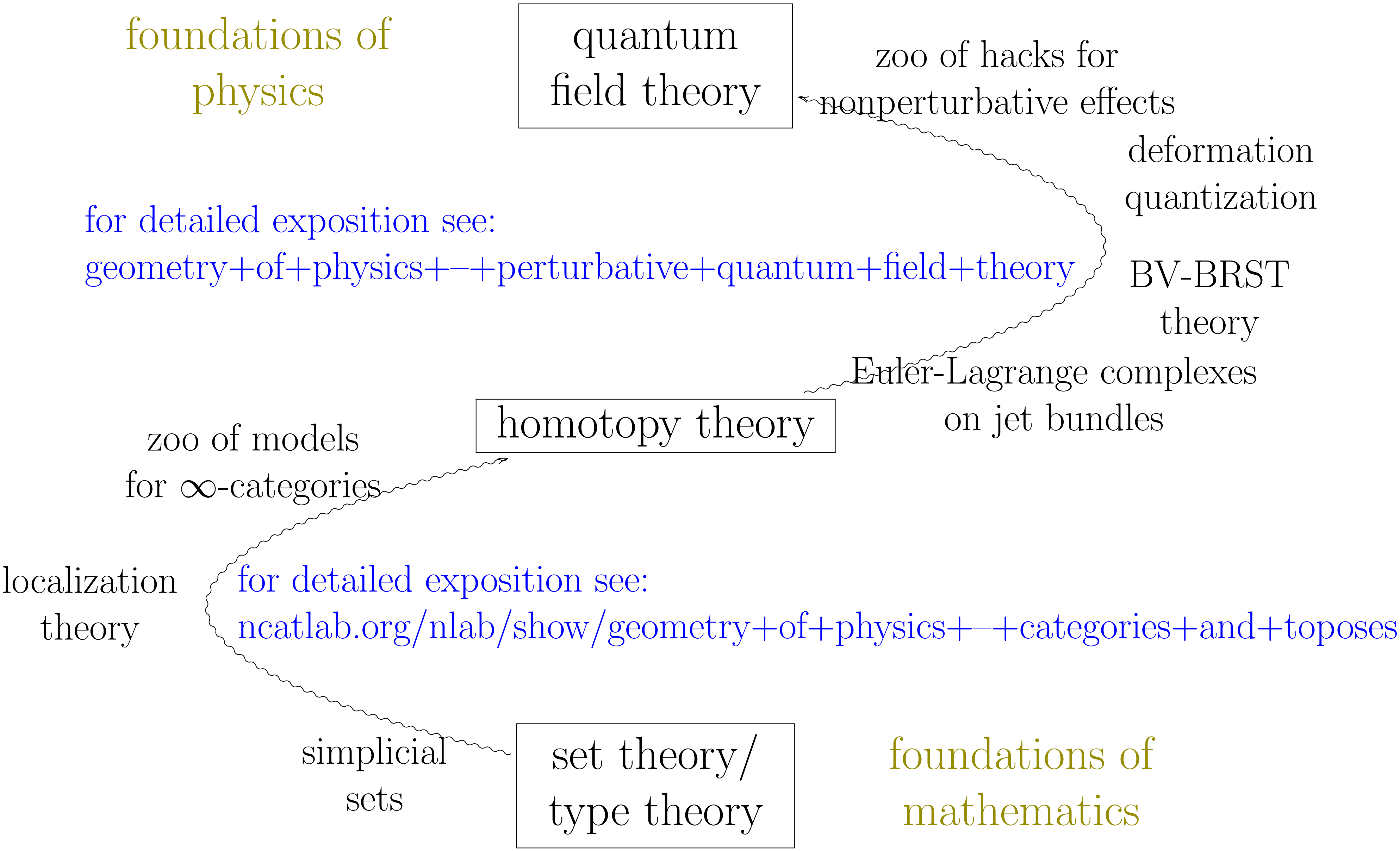

foundations of physics

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zoo of hacks for nonperturbative effects

deformation quantization

for detailed exposition see:
geometry+of+physics+–+perturbative+quantum+field+theory

BV-BRST theory

Euler-Lagrange complexes on jet bundles

homotopy theory

zoo of models for ∞-categories

localization theory

for detailed exposition see:
ncatlab.org/nlab/show/geometry+of+physics+–+categories+and+toposes

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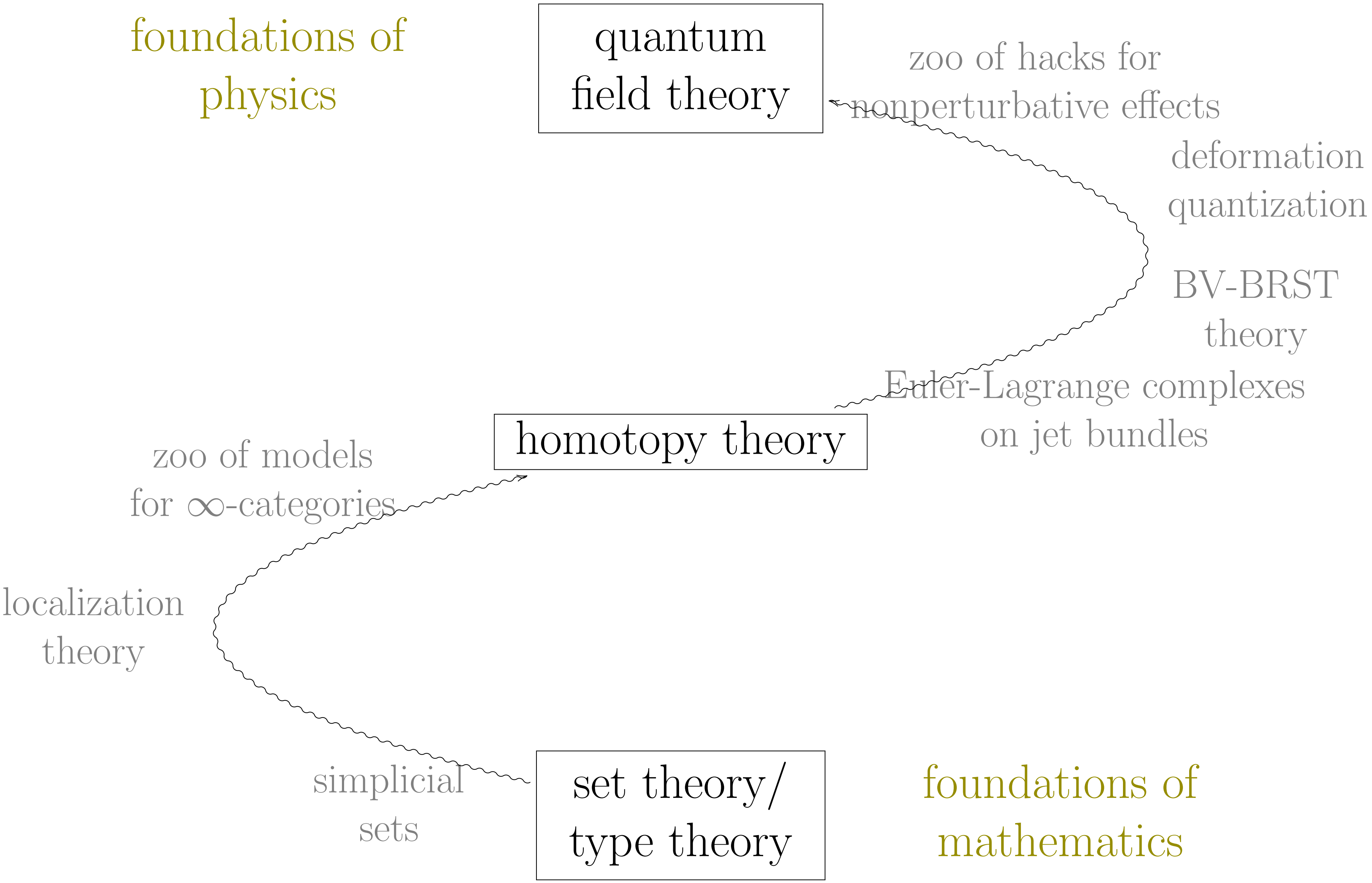

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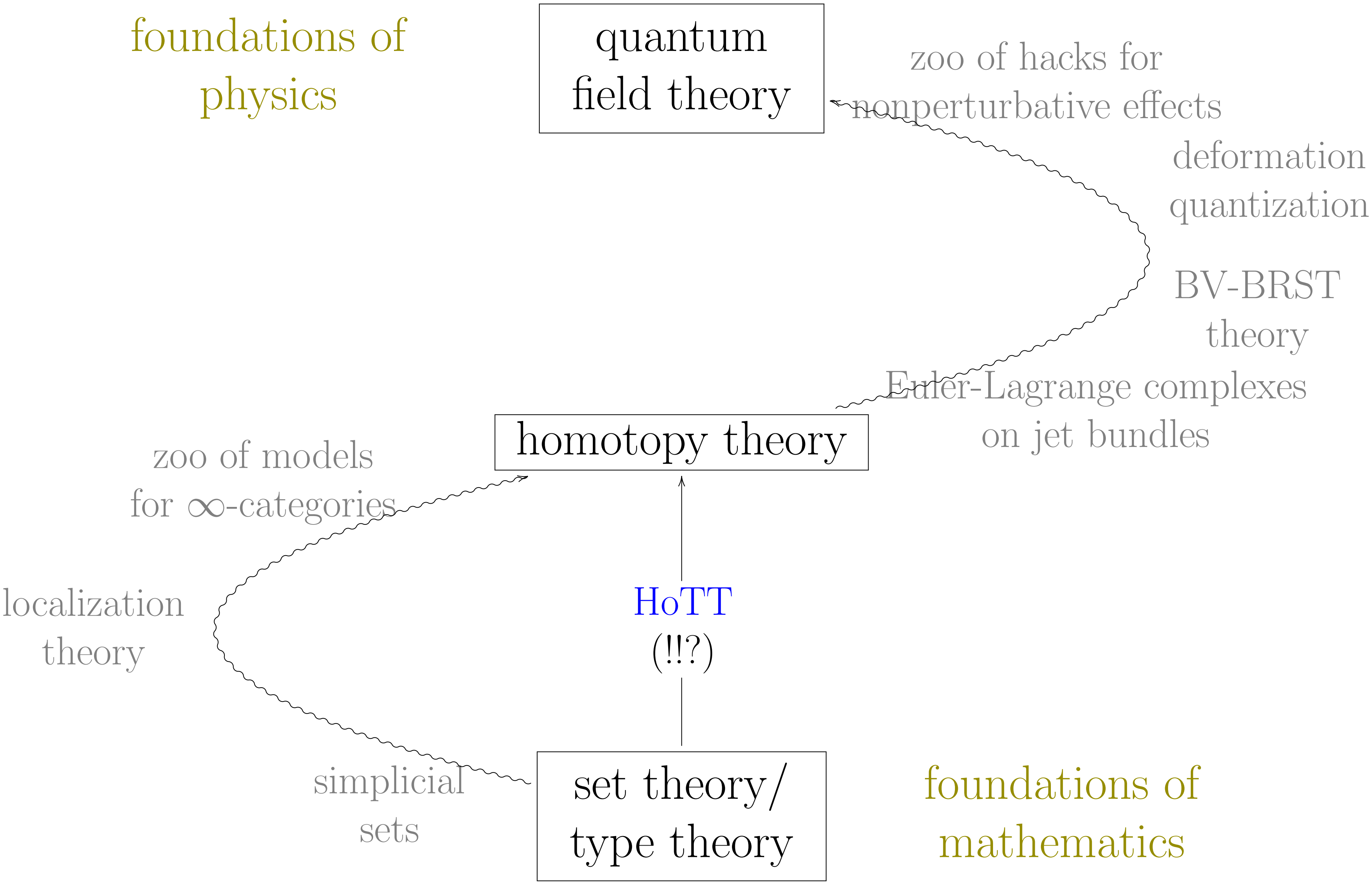

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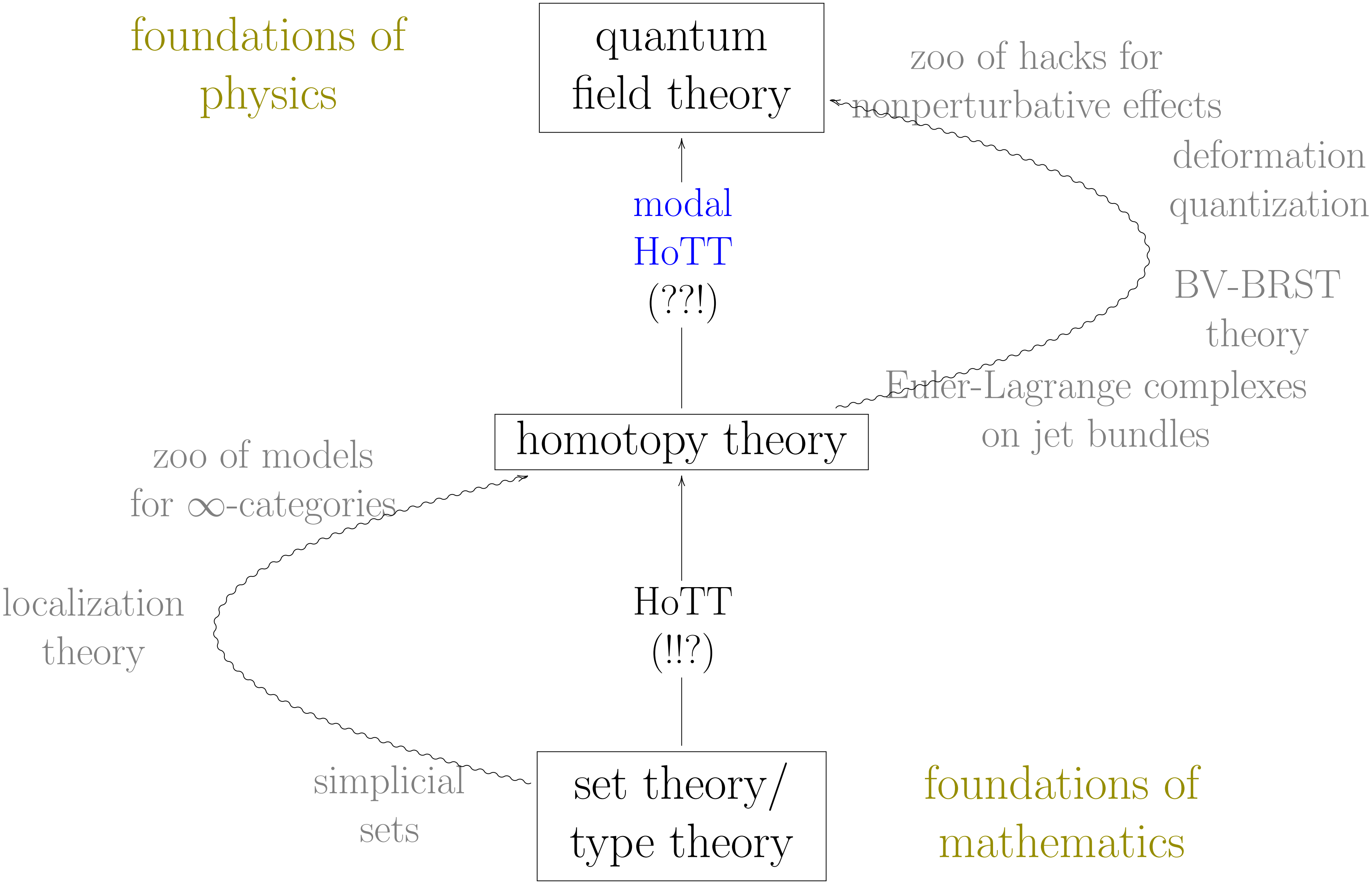

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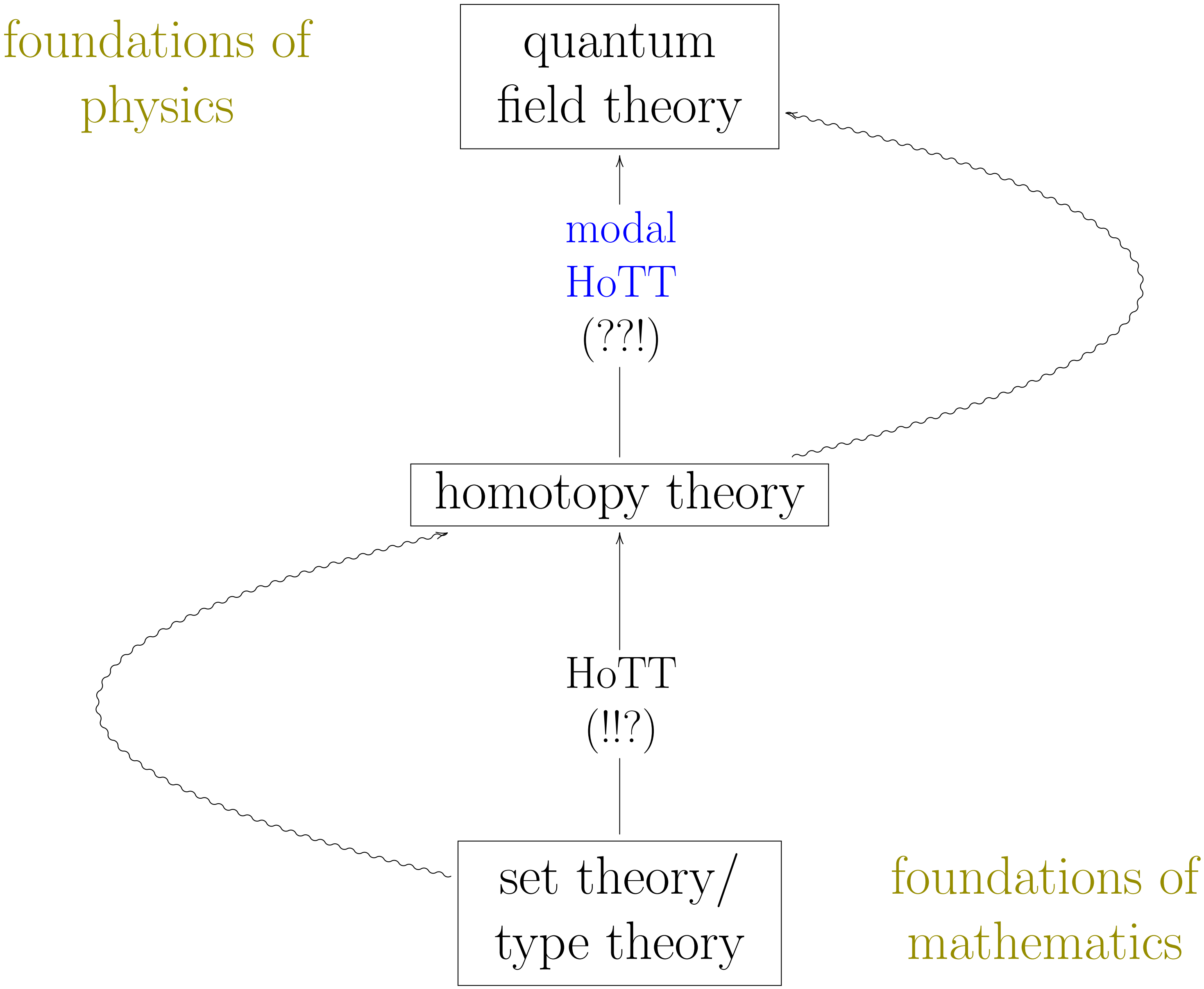

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Part I.

Equivariant super homotopy theory

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Super homotopy theory

and the Atom of Superspace

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	physics	**mathematics**
	gauge principle	homotopy theory
&	Pauli exclusion	super-geometry
=		**super homotopy theory**

for detailed exposition see:
ncatlab.org/nlab/show/geometry+of+physics+–+supergeometry

The sites of super homotopy theory

1. CartSp – cartesian spaces
2. FormalCartSp – formal cartesian spaces
3. SuperFormalCartSp – super formal cartesian spaces

The sites of super homotopy theory

1. CartSp – cartesian spaces
2. FormalCartSp – formal cartesian spaces
3. SuperFormalCartSp – super formal cartesian spaces

$$\mathrm{CartSp} \;:=\; \left\{ \begin{array}{ll} \text{objects:} & \mathbb{R}^n\text{-s for } n \in \mathbb{N} \\ \\ \text{morphisms:} & \mathbb{R}^{n_1} \underset{\text{smooth}}{\xrightarrow{f}} \mathbb{R}^{n_2} \\ \\ \text{coverings:} & \text{good open covers} \end{array} \right\}$$

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$$\mathrm{CartSp} \;:=\; \left\{ \begin{array}{ll} \text{objects:} & \mathbb{R}^n\text{-s for } n \in \mathbb{N} \\ \\ \text{morphisms:} & \mathbb{R}^{n_1} \xrightarrow[\text{smooth}]{f} \mathbb{R}^{n_2} \\ \\ \text{coverings:} & \text{good open covers} \end{array} \right\}$$

Fact ("Milnor's exercise"):
Sending Cartesian spaces
to their $\mathbb{R}$-algebras of smooth real-valued functions
is fully faithful:

$$C^\infty(-) \;:\; \mathrm{CartSp} \hookrightarrow {\mathrm{CommAlg}_{\mathbb{R}}}^{\mathrm{op}}$$

The sites of super homotopy theory

1. CartSp – cartesian spaces
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$$
\mathrm{FormalCartSp} \;:=\; \left\{ \begin{array}{ll} \text{objects:} & \mathbb{R}^n \times \mathbb{D}\text{-s} \\ & C^\infty\big(\mathbb{R}^n \times \mathbb{D}\big) := C^\infty\big(\mathbb{R}^n\big) \otimes_{\mathbb{R}} \Big(\mathbb{R} \oplus \begin{array}{c}\text{fin dim}\\ \text{nilpotent ideal}\end{array} \Big) \\ \text{morphisms:} & C^\infty\big(\mathbb{R}^n \times \mathbb{D}\big) \xleftarrow[\text{alg.homom.}]{f^*} C^\infty\big(\mathbb{R}^{n'} \times \mathbb{D}'\big) \\ & \\ \text{coverings:} & \text{good open covers } \times \mathrm{id}_{\mathbb{D}} \end{array} \right\}
$$

Fact ("Milnor's exercise"):

Sending Cartesian spaces
to their $\mathbb{R}$-algebras of smooth real-valued functions
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$$
C^\infty(-) \;:\; \mathrm{CartSp} \hookrightarrow \mathrm{CommAlg}_{\mathbb{R}}{}^{\mathrm{op}}
$$

The sites of super homotopy theory

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$$\mathrm{SuperFormalCartSp} \;:=\; \left\{ \begin{array}{ll} \text{objects:} & \mathbb{R}^n \times \mathbb{D}\text{-s} \\ & C^\infty\big(\mathbb{R}^n \times \mathbb{D}\big) := C^\infty\big(\mathbb{R}^n\big) \otimes_{\mathbb{R}} \Big(\mathbb{R} \oplus \underset{\text{nilpotent ideal}}{\text{fin dim}}\Big) \\ \text{morphisms:} & C^\infty\big(\mathbb{R}^n \times \mathbb{D}\big) \underset{\text{alg.homom.}}{\xleftarrow{f^*}} C^\infty\big(\mathbb{R}^{n'} \times \mathbb{D}'\big) \\ & \\ \text{coverings:} & \text{good open covers } \times \mathrm{id}_{\mathbb{D}} \end{array} \right\}$$

Fact (“Milnor’s exercise”):
Sending Cartesian spaces
to their $\mathbb{R}$-algebras of smooth real-valued functions
is fully faithful:

$$C^\infty(-) \;:\; \mathrm{CartSp} \hookrightarrow \mathrm{CommAlg}_{\mathbb{R}}{}^{\mathrm{op}} \hookrightarrow \mathrm{SuperCommAlg}_{\mathbb{R}}{}^{\mathrm{op}}$$

The sites of super homotopy theory

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e.g.

super Cartesian space $\mathbb{R}^{D|N}$ defined by

$$C^\infty\big(\mathbb{R}^{D|N}\big) \;:=\; C^\infty\big(\mathbb{R}^D\big) \,\otimes_{\mathbb{R}}\, \mathbb{R}\big[\{\theta^\alpha\}_{\alpha=1}^N\big]/\big(\theta^\alpha\theta^\beta = -\theta^b\theta^\alpha\big)$$

Fact ("Milnor's exercise"):
Sending Cartesian spaces
to their $\mathbb{R}$-algebras of smooth real-valued functions
is fully faithful:

$$C^\infty(-) \;:\; \mathrm{CartSp} \hookrightarrow \mathrm{CommAlg}_{\mathbb{R}}{}^{\mathrm{op}} \hookrightarrow \mathrm{SuperCommAlg}_{\mathbb{R}}{}^{\mathrm{op}}$$

The sites of super homotopy theory

1. CartSp – cartesian spaces
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Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp} \arrow[r, shift left=8, "\mathrm{even}"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[from=1-2, to=1-1, hook', "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[from=1-3, to=1-2, shift right=3, hook', "\iota_{\mathrm{inf}}"'] \arrow[r, phantom, "\perp"] \arrow[r, shift right=3, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[r, shift left=3, "\Pi"] \arrow[r, phantom, "\perp"] \arrow[from=1-4, to=1-3, shift left=3, hook', "\mathrm{Disc}"'] & *
\end{tikzcd}$$

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp} \arrow[r, shift left=10ex, "\mathrm{even}"] \arrow[r, phantom, shift left=7.5ex, "\perp"] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, shift left=0ex, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[l, hook', shift right=5ex, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, shift left=0ex, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[l, hook', shift right=5ex, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=1ex, "\Pi"] \arrow[r, phantom, "\perp"] & * \arrow[l, hook', shift left=1ex, "\mathrm{Disc}"]
\end{tikzcd}$$

Include point as $\mathbb{R}^0$.

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}
\mathrm{SuperFormalCartSp} \arrow[r, shift left=8, "\mathrm{even}"] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, shift right=2, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[l, hook', shift right=3, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, shift right=2, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[l, hook', shift right=3, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=3, "\Pi"] \arrow[r, phantom, "\perp"] & * \arrow[l, hook, shift left=3, "\mathrm{Disc}"]
\end{tikzcd}$$

Contract $\mathbb{R}^n$ to point.

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}
\mathrm{SuperFormalCartSp} \arrow[r, shift left=7, "\mathrm{even}"] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[l, hook', shift right=3.5, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[l, hook', shift right=3.5, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=2, "\Pi"] \arrow[r, phantom, "\perp"] & * \arrow[l, hook', shift left=2, "\mathrm{Disc}"]
\end{tikzcd}$$

Contract $\mathbb{R}^n \times \mathbb{D}$ to $\mathbb{R}^n$.

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp}
\arrow[r, shift left=16pt, "\mathrm{even}"]
\arrow[r, phantom, shift left=11pt, "\perp"]
\arrow[r, phantom, shift left=3pt, "\perp"]
\arrow[r, shift right=2pt, "\Pi_{\mathrm{sup}}"]
& \mathrm{FormalCartSp}
\arrow[l, hook', shift right=7pt, "\iota_{\mathrm{sup}}"']
\arrow[r, phantom, shift left=3pt, "\perp"]
\arrow[r, shift right=2pt, "\Pi_{\mathrm{inf}}"]
& \mathrm{CartSp}
\arrow[l, hook', shift right=7pt, "\iota_{\mathrm{inf}}"']
\arrow[r, shift left=6pt, "\Pi"]
\arrow[r, phantom, "\perp"]
& *
\arrow[l, hook', shift left=6pt, "\mathrm{Disc}"]
\end{tikzcd}$$

Contract $\mathbb{R}^{D|N}$ to $\mathbb{R}^D$.

The sites of super homotopy theory

1. CartSp – cartesian spaces
2. FormalCartSp – formal cartesian spaces
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Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp}
\arrow[r, shift left=24pt, "\mathrm{even}"]
\arrow[r, phantom, shift left=18pt, "\perp"]
\arrow[r, phantom, shift left=6pt, "\perp"]
\arrow[r, shift left=0pt, "\Pi_{\mathrm{sup}}"]
&
\mathrm{FormalCartSp}
\arrow[l, hook', shift right=12pt, "\iota_{\mathrm{sup}}"']
\arrow[r, phantom, shift left=6pt, "\perp"]
\arrow[r, shift left=0pt, "\Pi_{\mathrm{inf}}"]
&
\mathrm{CartSp}
\arrow[l, hook', shift right=12pt, "\iota_{\mathrm{inf}}"']
\arrow[r, shift left=6pt, "\Pi"]
\arrow[r, phantom, "\perp"]
&
*
\arrow[l, hook', shift left=6pt, "\mathrm{Disc}"]
\end{tikzcd}$$

Include $\mathbb{R}^n$ as $\mathbb{R}^n$.

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp}
\arrow[r, shift left=7ex, "\mathrm{even}"]
\arrow[r, phantom, shift left=5ex, "\perp"]
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, shift left=0ex, "\Pi_{\mathrm{sup}}"]
& \mathrm{FormalCartSp}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{sup}}"']
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, shift left=0ex, "\Pi_{\mathrm{inf}}"]
& \mathrm{CartSp}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{inf}}"']
\arrow[r, shift left=1.5ex, "\Pi"]
\arrow[r, phantom, "\perp"]
& *
\arrow[l, hook', shift left=1.5ex, "\mathrm{Disc}"]
\end{tikzcd}$$

Include $\mathbb{R}^n \times \mathbb{D}$ as $\mathbb{R}^n \times \mathbb{D}$.

The sites of super homotopy theory

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Adjunctions

$$\begin{tikzcd}
\mathrm{SuperFormalCartSp} \arrow[r, shift left=10, "\mathrm{even}"] \arrow[r, phantom, shift left=7, "\perp"] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, shift right=2, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[l, hook', shift right=4, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=1, "\perp"] \arrow[r, shift right=2, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[l, hook', shift right=4, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=2, "\Pi"] \arrow[r, phantom, "\perp"] & * \arrow[l, hook', shift left=2, "\mathrm{Disc}"]
\end{tikzcd}$$

Send $\mathbb{R}^{0|2}$ to $\mathbb{D}^1(1)$.

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty\big(\mathrm{CartSp}\big)$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty\big(\mathrm{FormalCartSp}\big)$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{SuperFormalCartSp}\big)$ – super formal smooth ∞-groupoids

Adjunctions

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp}
\arrow[r, shift left=6ex, "\mathrm{even}" description]
\arrow[r, phantom, shift left=4.5ex, "\perp"]
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{sup}}" description]
&
\mathrm{FormalCartSp}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{sup}}" description]
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{inf}}" description]
&
\mathrm{CartSp}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{inf}}" description]
\arrow[r, "\Pi" description]
\arrow[r, phantom, shift right=1.5ex, "\perp"]
&
*
\arrow[l, hook', shift left=3ex, "\mathrm{Disc}" description]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathbf{H}
\arrow[r, shift left=6ex, "\mathrm{even}" description]
\arrow[r, phantom, shift left=4.5ex, "\perp"]
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{sup}}" description]
&
\mathbf{H}_{\rightsquigarrow}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{sup}}" description]
\arrow[r, phantom, shift left=1.5ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{inf}}" description]
&
\mathbf{H}_{\Re}
\arrow[l, hook', shift right=3ex, "\iota_{\mathrm{inf}}" description]
\arrow[r, "\Pi" description]
\arrow[r, phantom, shift right=1.5ex, "\perp"]
&
\mathbf{H}_{\flat} = \infty\mathrm{Grpd}
\arrow[l, hook', shift left=3ex, "\mathrm{Disc}" description]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\mathfrak{R}} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathrm{SuperFormalCartSp} \arrow[r, shift left=24, "\mathrm{even}"] \arrow[r, shift left=16, phantom, "\perp"] \arrow[r, shift left=4, phantom, "\perp"] \arrow[r, shift left=0, "\Pi_{\mathrm{sup}}"] & \mathrm{FormalCartSp} \arrow[l, hook', shift right=10, "\iota_{\mathrm{sup}}"'] \arrow[r, shift left=4, phantom, "\perp"] \arrow[r, shift left=0, "\Pi_{\mathrm{inf}}"] & \mathrm{CartSp} \arrow[l, hook', shift right=10, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=6, "\Pi"] \arrow[r, phantom, "\perp"] & * \arrow[l, hook', shift left=6, "\mathrm{Disc}"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=24, "\mathrm{even}"] \arrow[r, shift left=16, phantom, "\perp"] \arrow[r, shift left=4, phantom, "\perp"] \arrow[r, shift left=0, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=10, "\iota_{\mathrm{sup}}"'] \arrow[r, shift left=4, phantom, "\perp"] \arrow[r, shift left=0, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\mathfrak{R}} \arrow[l, hook', shift right=10, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=6, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_\flat = \infty\mathrm{Grpd} \arrow[l, hook', shift left=6, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=12, "\mathrm{even}"] \arrow[r, phantom, shift left=9, "\perp"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=6, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=6, "\iota_{\mathrm{inf}}"'] \arrow[r, "\Pi"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook, shift left=6, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\mathfrak{R}} := \mathrm{Sh}_\infty\big(\mathrm{CartSp}\big)$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty\big(\mathrm{FormalCartSp}\big)$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{SuperFormalCartSp}\big)$ – super formal smooth ∞-groupoids

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=12, "\mathrm{even}" description] \arrow[r, phantom, shift left=8, "\perp"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, shift left=-1, "\Pi_{\mathrm{sup}}" description] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=-5, "\iota_{\mathrm{sup}}" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, shift left=-1, "\Pi_{\mathrm{inf}}" description] & \mathbf{H}_{\mathfrak{R}} \arrow[l, hook', shift right=-5, "\iota_{\mathrm{inf}}" description] \arrow[r, shift left=3, "\Pi" description] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=3, "\mathrm{Disc}" description]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty\big(\mathrm{CartSp}\big)$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty\big(\mathrm{FormalCartSp}\big)$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{SuperFormalCartSp}\big)$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=16, "\mathrm{even}"] \arrow[r, phantom, shift left=12, "\perp"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, shift left=1, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=8, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, shift left=1, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=8, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=2, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=2, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_{\infty}(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_{\infty}(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_{\infty}(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=10, "\mathrm{even}"] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, shift right=0, shift left=3, "\Pi_{\mathrm{sup}}"'] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift left=3, "\iota_{\mathrm{sup}}"'] \arrow[r, shift left=3, "\Pi_{\mathrm{inf}}"'] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\Re} \arrow[l, hook', shift left=3, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=3, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=3, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

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Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H}
\arrow[r, shift left=24pt, "\mathrm{even}" description]
\arrow[r, phantom, shift left=18pt, "\perp"]
\arrow[r, leftarrow, hook', shift left=12pt, "\iota_{\mathrm{sup}}" description]
\arrow[r, phantom, shift left=6pt, "\perp"]
\arrow[r, "\Pi_{\mathrm{sup}}" description]
&
\mathbf{H}_{\rightsquigarrow}
\arrow[r, leftarrow, hook', shift left=12pt, "\iota_{\mathrm{inf}}" description]
\arrow[r, phantom, shift left=6pt, "\perp"]
\arrow[r, "\Pi_{\mathrm{inf}}" description]
&
\mathbf{H}_{\Re}
\arrow[r, "\Pi" description]
\arrow[r, phantom, shift right=3pt, "\perp"]
\arrow[r, leftarrow, hook', shift right=6pt, "\mathrm{Disc}" description]
&
\mathbf{H}_\flat = \infty\mathrm{Grpd}
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_{\infty}(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_{\infty}(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_{\infty}(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=30, "\mathrm{even}"] \arrow[r, phantom, shift left=24, "\perp"] \arrow[r, phantom, shift left=12, "\perp"] \arrow[r, shift left=6, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=18, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=12, "\perp"] \arrow[r, shift left=6, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=18, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=6, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=6, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=12, "\mathrm{even}"] \arrow[r, phantom, shift left=9, "\perp"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=6, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=6, "\iota_{\mathrm{inf}}"'] \arrow[r, "\Pi"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=6, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=14, "\mathrm{even}"] \arrow[r, phantom, shift left=10, "\perp"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, shift left=1, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=7, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, shift left=1, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=7, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=1, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=1, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

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Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=8, "\mathrm{even}"] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=4, "\iota_{\mathrm{sup}}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=4, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=2, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=2, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_{\infty}\big(\mathrm{CartSp}\big)$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_{\infty}\big(\mathrm{FormalCartSp}\big)$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_{\infty}\big(\mathrm{SuperFormalCartSp}\big)$ – super formal smooth ∞-groupoids

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Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=8, "\mathrm{even}"] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, hookleftarrow, shift left=4, "\iota_{\mathrm{sup}}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[r, hookleftarrow, shift left=4, "\iota_{\mathrm{inf}}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "\Pi_{\mathrm{inf}}"] & \mathbf{H}_{\Re} \arrow[r, shift left=2, "\Pi"] \arrow[r, phantom, "\perp"] \arrow[r, hookleftarrow, shift right=2, "\mathrm{Disc}"'] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd}
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_{\infty}\big(\mathrm{CartSp}\big)$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_{\infty}\big(\mathrm{FormalCartSp}\big)$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_{\infty}\big(\mathrm{SuperFormalCartSp}\big)$ – super formal smooth ∞-groupoids

Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H}
\arrow[r, shift left=5ex, "\mathrm{even}" description]
\arrow[r, phantom, shift left=3.75ex, "\perp"]
\arrow[r, phantom, shift left=1.25ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{sup}}" description]
& \mathbf{H}_{\rightsquigarrow}
\arrow[l, hook', shift right=2.5ex, "\iota_{\mathrm{sup}}" description]
\arrow[r, phantom, shift left=1.25ex, "\perp"]
\arrow[r, "\Pi_{\mathrm{inf}}" description]
& \mathbf{H}_{\Re}
\arrow[l, hook', shift right=2.5ex, "\iota_{\mathrm{inf}}" description]
\arrow[r, shift left=2.5ex, "\Pi" description]
\arrow[r, phantom, "\perp"]
& \mathbf{H}_{\flat} = \infty\mathrm{Grpd}
\arrow[l, hook, shift left=2.5ex, "\mathrm{Disc}" description]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=7ex, "\mathrm{even}" description] \arrow[r, phantom, shift left=5.25ex, "\perp"] \arrow[r, phantom, shift left=1.75ex, "\perp"] \arrow[r, shift left=0ex, "\Pi_{\mathrm{sup}}" description] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=3.5ex, "\iota_{\mathrm{sup}}" description] \arrow[r, phantom, shift left=1.75ex, "\perp"] \arrow[r, shift left=0ex, "\Pi_{\mathrm{inf}}" description] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=3.5ex, "\iota_{\mathrm{inf}}" description] \arrow[r, shift left=1.75ex, "\Pi" description] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=1.75ex, "\mathrm{Disc}" description]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

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Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[rr, shift left=8, "\mathrm{even}"] \arrow[rr, phantom, shift left=6, "\perp"] \arrow[rr, shift left=2, "\Pi_{\mathrm{sup}}"] & & \mathbf{H}_{\rightsquigarrow} \arrow[ll, hook', shift right=4, "\iota_{\mathrm{sup}}"'] \arrow[ll, phantom, shift right=2, "\perp"] \arrow[rr, shift left=2, "\Pi_{\mathrm{inf}}"] \arrow[rr, phantom, shift left=4, "\perp"] & & \mathbf{H}_{\Re} \arrow[ll, hook', shift right=6, "\iota_{\mathrm{inf}}"'] \arrow[rr, shift left=2, "\Pi"] \arrow[rr, phantom, "\perp"] & & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[ll, hook', shift left=2, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=16, "\mathrm{even}"] \arrow[r, phantom, shift left=12, "\perp"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, "\Pi_{\mathrm{sup}}"] & \mathbf{H}_{\rightsquigarrow} \arrow[l, hook', shift right=8, "\iota_{\mathrm{sup}}"'] \arrow[r, "\Pi_{\mathrm{inf}}"] \arrow[r, phantom, shift left=4, "\perp"] & \mathbf{H}_{\Re} \arrow[l, hook', shift right=8, "\iota_{\mathrm{inf}}"'] \arrow[r, shift left=4, "\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{H}_{\flat} = \infty\mathrm{Grpd} \arrow[l, hook', shift left=4, "\mathrm{Disc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} & \mathbf{H}_{\rightsquigarrow} & \mathbf{H}_{\Re} & \mathbf{H}_{\flat} = \infty\mathrm{Grpd}
\arrow[from=1-1, to=1-2, yshift=25ex, "\mathrm{even}"]
\arrow[from=1-1, to=1-2, phantom, yshift=20ex, "\perp"]
\arrow[from=1-2, to=1-1, hook, yshift=15ex, "\iota_{\mathrm{sup}}"]
\arrow[from=1-1, to=1-2, phantom, yshift=10ex, "\perp"]
\arrow[from=1-1, to=1-2, yshift=5ex, "\Pi_{\mathrm{sup}}"]
\arrow[from=1-1, to=1-2, phantom, "\perp"]
\arrow[from=1-2, to=1-1, hook, yshift=-5ex, "\mathrm{Disc}_{\mathrm{sup}}"]
\arrow[from=1-1, to=1-2, phantom, yshift=-10ex, "\perp"]
\arrow[from=1-1, to=1-2, yshift=-15ex, "\Gamma_{\mathrm{sup}}"]
\arrow[from=1-3, to=1-2, hook, yshift=15ex, "\iota_{\mathrm{inf}}"]
\arrow[from=1-2, to=1-3, phantom, yshift=10ex, "\perp"]
\arrow[from=1-2, to=1-3, yshift=5ex, "\Pi_{\mathrm{inf}}"]
\arrow[from=1-2, to=1-3, phantom, "\perp"]
\arrow[from=1-3, to=1-2, hook, yshift=-5ex, "\mathrm{Disc}_{\mathrm{inf}}"]
\arrow[from=1-2, to=1-3, phantom, yshift=-10ex, "\perp"]
\arrow[from=1-2, to=1-3, yshift=-15ex, "\Gamma_{\mathrm{inf}}"]
\arrow[from=1-3, to=1-4, yshift=5ex, "\Pi"]
\arrow[from=1-3, to=1-4, phantom, "\perp"]
\arrow[from=1-4, to=1-3, hook, yshift=-5ex, "\mathrm{Disc}"]
\arrow[from=1-3, to=1-4, phantom, yshift=-10ex, "\perp"]
\arrow[from=1-3, to=1-4, yshift=-15ex, "\Gamma"]
\arrow[from=1-1, to=1-4, phantom, yshift=-20ex, "\perp"]
\arrow[from=1-4, to=1-1, hook, yshift=-25ex, "\mathrm{coDisc}"]
\end{tikzcd}$$

The toposes of super homotopy theory

1. $\mathbf{H}_{\Re} := \mathrm{Sh}_\infty(\mathrm{CartSp})$ – smooth ∞-groupoids
2. $\mathbf{H}_{\rightsquigarrow} := \mathrm{Sh}_\infty(\mathrm{FormalCartSp})$ – formal smooth ∞-groupoids
3. $\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SuperFormalCartSp})$ – super formal smooth ∞-groupoids

---

Adjunctions by Kan extension

$$\begin{tikzcd}[column sep=huge]
\mathbf{H}
\arrow[r, shift left=40pt, "\mathrm{even}" description]
\arrow[r, phantom, shift left=32pt, "\perp"]
\arrow[r, phantom, shift left=16pt, "\perp"]
\arrow[r, shift left=8pt, "\Pi_{\mathrm{sup}}" description]
\arrow[r, phantom, "\perp"]
\arrow[r, phantom, shift right=16pt, "\perp"]
\arrow[r, shift right=24pt, "\Gamma_{\mathrm{sup}}" description]
\arrow[rrr, phantom, shift right=32pt, "\perp"]
&
\mathbf{H}_{\rightsquigarrow}
\arrow[l, hook', shift right=24pt, "\iota_{\mathrm{sup}}" description]
\arrow[l, hook', shift left=8pt, "\mathrm{Disc}_{\mathrm{sup}}" description]
\arrow[r, shift left=8pt, "\Pi_{\mathrm{inf}}" description]
\arrow[r, phantom, shift left=16pt, "\perp"]
\arrow[r, phantom, "\perp"]
\arrow[r, phantom, shift right=16pt, "\perp"]
\arrow[r, shift right=24pt, "\Gamma_{\mathrm{inf}}" description]
&
\mathbf{H}_{\Re}
\arrow[l, hook', shift right=24pt, "\iota_{\mathrm{inf}}" description]
\arrow[l, hook', shift left=8pt, "\mathrm{Disc}_{\mathrm{inf}}" description]
\arrow[r, shift left=8pt, "\Pi" description]
\arrow[r, phantom, "\perp"]
\arrow[r, phantom, shift right=16pt, "\perp"]
\arrow[r, shift right=24pt, "\Gamma" description]
&
\mathbf{H}_{\flat} = \infty\mathrm{Grpd}
\arrow[l, hook', shift left=8pt, "\mathrm{Disc}" description]
\arrow[lll, hook', shift left=40pt, "\mathrm{coDisc}" description]
\end{tikzcd}$$

The Modalities of Super homotopy theory

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd} \overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \arrow[rrrr, "\Gamma"] & & & & * \end{tikzcd}$$

The Modalities of Super homotopy theory

$\varnothing$

nothing

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \arrow[r, shift right=2, "\Gamma" description] & {*} \arrow[l, hook', shift right=2, "\Delta" description]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$\varnothing \dashv \ast$

pure being

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \arrow[r, "\Gamma" description] & \ast \arrow[l, hook', shift right=3] \arrow[l, hook, shift left=3, "\nabla" description]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccc} \flat & & \\ \vee & & \\ \emptyset & \dashv & \ast \end{array}}$$

discrete

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[r, "\Gamma" description] & \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook', shift right=4, "\Delta" description] \arrow[l, hook, shift left=4] \arrow[r] & \ast \arrow[l, hook', shift right=4] \arrow[l, hook, shift left=4]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{array}{ccc} \flat & \dashv & \color{blue}{\sharp} \\ \vee & & \vee \\ \varnothing & \dashv & \ast \end{array}$$

continuous

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric} \\ \infty\text{-groupoids}}} \arrow[r, shift left=0, "\color{blue}{\Gamma}" description] & \overbrace{\mathbf{H}_\flat}^{\color{blue}{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}}} \arrow[l, hook', shift right=3] \arrow[l, hook, shift left=3, "\color{blue}{\nabla}" description] \arrow[r] & \ast \arrow[l, hook', shift right=3] \arrow[l, hook, shift left=3]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccc} \int & \dashv & \flat & \dashv & \sharp \\ & & \vee & & \vee \\ & & \varnothing & \dashv & \ast \end{array}}$$

shaped

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \arrow[r, shift left=9, "\Pi"] \arrow[r, shift left=3, hookleftarrow, "\Delta" description] \arrow[r, shift right=3] \arrow[r, shift right=9, hookleftarrow] & \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[r, shift left=3, hookleftarrow] \arrow[r, shift right=3] \arrow[r, shift right=9, hookleftarrow] & \ast
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccc} \Im &&&& \\ \vee &&&& \\ \int & \dashv & \flat & \dashv & \sharp \\ && \vee && \vee \\ && \varnothing & \dashv & * \end{array}}$$

infinitesimally shaped

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}}
\arrow[r, shift left=4.5, "\Pi" description]
\arrow[r, shift right=1.5]
& \overbrace{\mathbf{H}_{\Re}}^{\substack{\text{reduced differential-geometric}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1.5, "\Delta" description]
\arrow[r, shift left=4.5]
\arrow[r, shift right=1.5]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=1.5]
\arrow[ll, hook', shift left=4.5, bend left=0]
\arrow[r, shift right=1.5]
& *
\arrow[l, hook', shift right=1.5]
\arrow[l, hook', shift left=4.5]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{array}{ccccc}
\Im & \dashv & \& & & \\
\vee & & \vee & & \\
\int & \dashv & \flat & \dashv & \sharp \\
 & & \vee & & \vee \\
 & & \varnothing & \dashv & \ast
\end{array}$$

infinitesimally discrete

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}}
\arrow[r, shift left=6]
\arrow[r, shift right=2, "\Gamma"]
&
\overbrace{\mathbf{H}_{\Re}}^{\substack{\text{reduced differential-geometric}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=2, "\Delta"']
\arrow[r, shift left=6]
\arrow[r, shift right=2]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
\arrow[l, hook', shift right=2]
\arrow[ll, hook', shift left=6]
\arrow[r, shift right=2]
&
\ast
\arrow[l, hook', shift right=2]
\arrow[l, hook', shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \vee & & \vee & & \\ & & \int & \dashv & \flat & \dashv & \sharp \\ & & & & \vee & & \vee \\ & & & & \varnothing & \dashv & * \end{array}}$$

reduced

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\text{supergeometric}} \arrow[r, shift left=4, "\Pi" description] \arrow[r, shift right=4] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced differential-geometric}} \arrow[l, hook', shift right=8, "\iota" description] \arrow[l, hook'] \arrow[r, shift left=4] \arrow[r, shift right=4] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[ll, hook', shift left=8] \arrow[l, hook'] \arrow[r, shift right=4] & * \arrow[l, hook'] \arrow[l, hook', shift left=8]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$
\boxed{
\begin{array}{ccccccc}
\rightsquigarrow & & & & & & \\
\vee & & & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \varnothing & \dashv & *
\end{array}
}
$$

bosonic

The terminal functor factors into a system of dualities = adjunctions.

$$
\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{supergeometric}}
  \arrow[r, shift left=1, "\Pi"]
  \arrow[rr, bend right=30]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
  \arrow[l, hook', shift right=5, "\iota"']
  \arrow[l, hook, shift left=3]
  \arrow[r, shift left=1]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
  \arrow[l, hook', shift right=5]
  \arrow[l, hook, shift left=3]
  \arrow[r, shift left=3]
  \arrow[r, shift right=3]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[l, hook, shift left=0]
  \arrow[lll, hook, bend left=30]
  \arrow[r, shift right=3]
& *
  \arrow[l, hook, shift left=0]
  \arrow[l, hook, shift left=6]
\end{tikzcd}
$$

The Modalities of Super homotopy theory

$$\begin{array}{ccccccc}
\rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
\vee & & \vee & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & & \vee & & \\
 & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & & \vee \\
 & & & & \varnothing & \dashv & *
\end{array}$$

rheonomic

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\text{supergeometric}} \arrow[r, "\Pi"] \arrow[rr, bend right=25] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook, bend right=30] \arrow[l, hook, shift left=2, "\Delta"] \arrow[r] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook, bend right=30] \arrow[l, hook, shift left=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}} \arrow[l, hook] \arrow[lll, hook, bend left=30] \arrow[r] & * \arrow[l, hook, shift right=3] \arrow[l, hook, shift left=3]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\textcolor{blue}{\rightrightarrows} & \dashv & \rightsquigarrow & \dashv & \mathrm{R\!\!h} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

even

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=5, "\textcolor{blue}{\epsilon}"] \arrow[r, shift left=1] \arrow[rr, shift right=3] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\textcolor{blue}{\text{bosonic}}} \arrow[l, hook', shift left=3, "\textcolor{blue}{\iota}"'] \arrow[l, hook', shift left=1] \arrow[r, shift left=1] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=3] \arrow[l, hook', shift left=1] \arrow[r, shift left=1] \arrow[r, shift right=3] & \overbrace{\mathbf{H}_{\flat}}^{\text{geometrically discrete }\infty\text{-groupoids}} \arrow[l, hook', shift left=1] \arrow[lll, hook', shift left=5] & \ast \arrow[l, hook', shift left=1] \arrow[l, hook', shift left=5] \\
 & & & \arrow[r, shift right=3, from=1-4, to=1-5] &
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc} {\color{blue}\mathrm{id}} & \dashv & {\color{blue}\mathrm{id}} \\ \vee & & \vee \\ \rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} \\ & & \vee & & \vee \\ & & \Re & \dashv & \Im & \dashv & \& \\ & & & & \vee & & \vee \\ & & & & \int & \dashv & \flat & \dashv & \sharp \\ & & & & & & \vee & & \vee \\ & & & & & & \emptyset & \dashv & \ast \end{array}}$$

super-geometric

The terminal functor factors into a system of dualities = adjunctions.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\color{blue}\text{super-geometric}} \arrow[r, shift left=6] \arrow[r, shift left=1] \arrow[rr, shift right=5]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift right=3] \arrow[l, hook', shift right=7] \arrow[r, shift left=1]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=3] \arrow[l, hook', shift right=7] \arrow[r, shift left=1] \arrow[r, shift right=5]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}} \arrow[l, hook', shift right=3] \arrow[lll, hook', shift left=9] \arrow[r, shift right=5]
& \ast \arrow[l, hook', shift right=3] \arrow[l, hook', shift left=9]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{R\!\!\!\!\mathrm{h}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}
$$

$\mathbb{A}^1$-*local*

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}[column sep=huge]
\overset{\text{super-geometric}}{\overbrace{\mathbf{H}}} \arrow[r, bend left=50] \arrow[r, shift left=2, "\Pi_{\mathrm{sup}}"] \arrow[rr, bend right=30] & \overset{\text{bosonic}}{\overbrace{\mathbf{H}_{\rightsquigarrow}}} \arrow[l, hook, bend right=30] \arrow[l, hook, shift left=2, "\Delta_{\mathrm{sup}}"] \arrow[r, shift left=2, "\Pi_{\mathrm{inf}}"] & \overset{\text{reduced}}{\overbrace{\mathbf{H}_{\Re}}} \arrow[l, hook, bend right=30] \arrow[l, hook, shift left=2, "\Delta_{\mathrm{inf}}"] \arrow[r, shift left=2, "\Pi"] \arrow[r, bend right=30] & \overset{\substack{\text{geometrically discrete} \\ \infty\text{-groupoids}}}{\overbrace{\mathbf{H}_{\flat}}} \arrow[l, hook, shift left=2, "\Delta"] \arrow[lll, hook, bend left=30] \arrow[r, shift right=2] & \ast \arrow[l, hook, shift right=2] \arrow[l, hook, bend left=30]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{R}\!\!\mathrm{h} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}^1} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \emptyset & \dashv & *
\end{array}}$$

continuum-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=6] \arrow[r, shift left=2, "\Pi_{\mathrm{sup}}"] \arrow[rr, bend right=20] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift left=4] \arrow[l, hook', shift left=0, "\Delta_{\mathrm{sup}}"] \arrow[r, shift left=2, "\Pi_{\mathrm{inf}}"] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift left=4] \arrow[l, hook', "\Delta_{\mathrm{inf}}"] \arrow[r, shift left=2, "\Pi"] \arrow[r, shift right=4] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook', "\Delta"] \arrow[lll, hook', bend left=20] \arrow[r, shift right=2] & * \arrow[l, hook', shift right=2] \arrow[l, hook', shift left=6]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}^1} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}}$$

infinitum-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}[column sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}} \arrow[r, shift left=6] \arrow[r, shift left=2, "\Pi_{\mathrm{sup}}"] \arrow[rr, shift right=6, bend right] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift left=4] \arrow[l, hook', shift left=0, "\Delta_{\mathrm{sup}}"] \arrow[r, shift left=2, "\Pi_{\mathrm{inf}}"] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=4] \arrow[l, hook', shift left=0, "\Delta_{\mathrm{inf}}"] \arrow[r, shift left=4] \arrow[r, shift right=4] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}} \arrow[l, hook'] \arrow[lll, hook', shift left=8, bend left] \arrow[r, shift right=4] & \ast \arrow[l, hook'] \arrow[l, hook', shift left=8]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\boxed{\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\mathbb{R}^{0|1}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \& & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}^{1}} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & *
\end{array}}$$

superpoint-local

The central modalities are motivic $\mathbb{A}^1$-localizations.

$$\begin{tikzcd}
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\\text{geometric}}} \arrow[r, shift left=6] \arrow[r, "\Pi_{\mathrm{sup}}", shift left=1] \arrow[rr, shift right=5] & \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift right=3] \arrow[l, hook', "\Delta_{\mathrm{sup}}", shift left=1] \arrow[r, shift left=1] & \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift right=3] \arrow[l, hook', shift left=1] \arrow[r, shift left=1] \arrow[r, shift right=5] & \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\\infty\text{-groupoids}}} \arrow[l, hook', shift left=1] \arrow[lll, hook', shift left=8] \arrow[r, shift right=5] & * \arrow[l, hook', shift left=1] \arrow[l, hook', shift left=8]
\end{tikzcd}$$

The Modalities of Super homotopy theory

$$\begin{tikzcd}
\mathrm{id} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \mathrm{id} \arrow[d, phantom, "\vee"] & & & \\
\rightrightarrows \arrow[r, phantom, "\dashv"] & \rightsquigarrow \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \boxed{\mathbb{R}^{0|1}} \arrow[d, phantom, "\vee"] & & \\
& \Re \arrow[r, phantom, "\dashv"] & \boxed{\mathbb{D}} \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \& \arrow[d, phantom, "\vee"] & \\
& & \boxed{\mathbb{R}^1} \arrow[r, phantom, "\dashv"] & \flat \arrow[r, phantom, "\dashv"] \arrow[d, phantom, "\vee"] & \sharp \arrow[d, phantom, "\vee"] \\
& & & \emptyset \arrow[r, phantom, "\dashv"] \arrow[uuul, squiggly, bend right] & \ast
\end{tikzcd}$$

⇒ emergence of Atom of Superspace from `\nothing`

The Modalities of Super homotopy theory

$$\begin{tikzcd}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\mathbb{R}^{0|1}} & & & & \\
& & \vee & & \vee & & & & \\
& & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \& & & \\
& & & & \vee & & \vee & & \\
& & & & \boxed{\mathbb{R}^1} & \dashv & \flat & \dashv & \sharp \\
& & & & & & \vee & & \vee \\
& & & & & & \varnothing \arrow[uuuuuull, squiggly, bend right] & \dashv & \ast
\end{tikzcd}$$

⇒ emergence of Atom of Superspace from \`nothing`

now apply the microscope of homotopy theory
to discover what emerges, in turn, out of the superpoint...

Rational
Super homotopy theory

and the fundamental super p-Branes

back to Part I

Higher super Lie theory and Rational homotopy

$$\left.\begin{array}{c}\text{infinitesimal}\\ \text{rational}\end{array}\right\}\text{approximation of super-homotopy by}\left\{\begin{array}{c}\text{higher Lie integration}\\ \text{Sullivan construction}\end{array}\right.$$

Definition.

$$\underbrace{\mathrm{FDAs}}_{\substack{\text{terminology}\\ \text{common in}\\ \text{supergravity}\\ \text{([?])}}} := \underbrace{\mathrm{dgcSuperAlg}_{\mathbb{R},\mathrm{cn}}}_{\substack{\infty\text{-category of}\\ \text{differential}\\ \text{graded-commutative}\\ \text{superalgebras}}} \underset{\simeq}{\xleftarrow{\mathrm{CE}}} \Big(\underbrace{\mathrm{Super}L_\infty\mathrm{Algbd}^{\mathrm{nil}}_{\mathbb{R},\mathrm{cn}}}_{\substack{\infty\text{-category of}\\ \text{nilpotent}\\ \text{super } L_\infty\text{-algebroids}}}\Big)^{\mathrm{op}}$$

Theorem. ([BM18], see Sch18)

Sullivan rational homotopy theory

$$\mathrm{Super}L_\infty\mathrm{Algbd}^{\mathrm{nil}}_{\mathbb{R},\mathrm{cn}} \simeq \overbrace{\mathrm{FDAs}^{\mathrm{op}}}^{\substack{\text{rational super-}\\ \infty\text{-groupoids}}} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} \overbrace{\mathbf{H}}^{\substack{\text{supergeometric}\\ \infty\text{-groupoids}}} \underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}} \overbrace{\mathbf{H}_\flat}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}$$

higher super Lie integration

$$\underbrace{\mathbb{R}^{0|1}}_{\substack{D=0,\ \mathcal{N}=1\\ \text{supersymmetry}\\ \text{super Lie algebra}}} \longmapsto \underbrace{\mathbb{R}^{0|1}}_{\text{superpoint}}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

the Atom of Superspace

$\mathbb{R}^{0|\mathbf{1}}$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$\mathbb{R}^{0|\mathbf{1}}$

$\boxed{\text{Type I}}$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[HS17]

$$\begin{tikzcd}
 & \mathbb{R}^{2,1|\mathbf{2}} \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

universal central extension: 3d super-Minkowski spacetime

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[HS17]

$$\begin{tikzcd}
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{3,1|\mathbf{4}} \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

[HS17]

universal invariant central extension: 4d super-Minkowski spacetime

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & {\color{blue}\mathbb{R}^{5,1|\mathbf{8}}} \arrow[dl] \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \color{blue}{\mathbb{R}^{9,1|\mathbf{16}}} \arrow[dr] & \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & {\color{blue}\mathbb{R}^{10,1|\mathbf{32}}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift right] \arrow[r, shift left] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift right] \arrow[r, shift left] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & \\
\boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}}
\end{tikzcd}$$

[HS17]

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\color{blue}{\mathfrak{string}_{\mathrm{IIB}}} \arrow[dr] & \text{[FSS13]} & \color{blue}{\mathfrak{string}_{\mathrm{I}}} \arrow[d] & & \color{blue}{\mathfrak{string}_{\mathrm{IIA}}} \arrow[dl] \\
 & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
 & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & \\
 & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \color{olive}{\text{emergent spacetime}} & \text{[HS17]} \\
 & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \\
 & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & \\
 & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} &
\end{tikzcd}$$

universal *higher* central invariant extension: stringy extended super-spacetimes

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
& & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
& & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
& & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \begin{array}{c}\text{emergent}\\ \text{spacetime}\end{array} & & & \text{[HS17]} \\
& & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
& & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
& & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m2brane} \arrow[d] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \begin{array}{c}\text{emergent}\\ \text{spacetime}\end{array} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & & \text{[FSS15]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[FSS15]

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & &
\end{tikzcd}$$

brane bouquet

[FSS13]

emergent spacetime

[HS17]

Type IIB

Type I

Type IIA

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[d] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & {} \arrow[d] & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] \arrow[dr, phantom, "\text{(pb)}"] & & & \mathfrak{d6brane} \arrow[dl] & \text{Gauge enhancement} \arrow[dl, dotted] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
 & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[dddr] & \mathfrak{d3brane} \arrow[ddd] & \mathfrak{d1brane} \arrow[dddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[dddr] & \mathfrak{d2brane} \arrow[ddd] & \mathfrak{d4brane} \arrow[dddl] & \\
\mathfrak{d7brane} \arrow[ddr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dddr] & & & \mathfrak{d6brane} \arrow[ddl] & \text{Gauge enhancement} \arrow[ddl, dotted] \\
 & & & & \text{(pb)} & & & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
 & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & \text{T-Duality} \arrow[ul, dotted] \arrow[ur, dotted] & & & &
\end{tikzcd}$$

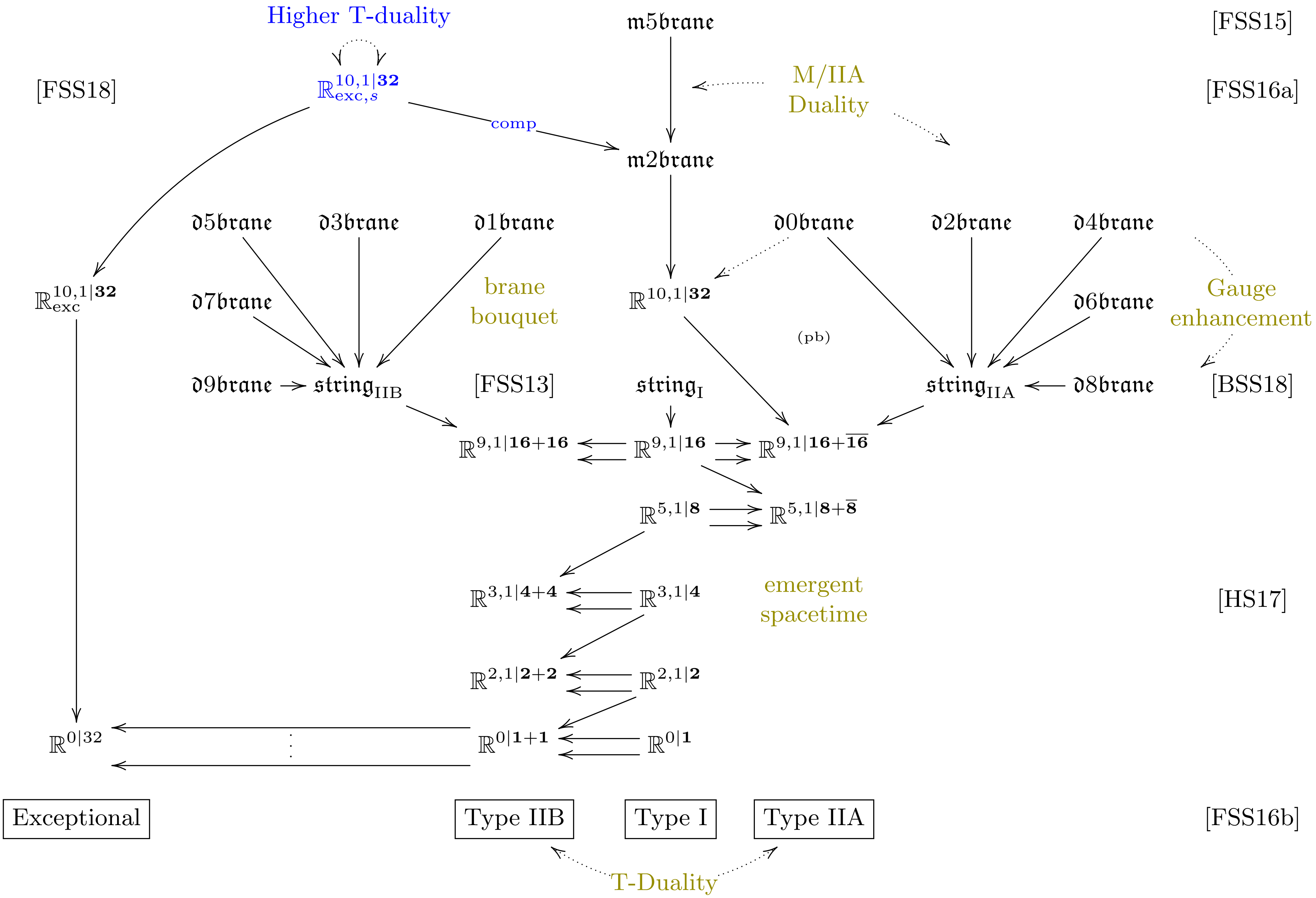
Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
\text{[FSS18]} & & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
 & \mathfrak{d5brane} \arrow[dddr] & \mathfrak{d3brane} \arrow[ddd] & \mathfrak{d1brane} \arrow[dddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[dddr] & \mathfrak{d2brane} \arrow[ddd] & \mathfrak{d4brane} \arrow[dddl] & \\
 & \mathfrak{d7brane} \arrow[ddr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dddr] & & & \mathfrak{d6brane} \arrow[ddl] & \text{Gauge enhancement} \arrow[ddl, dotted] \\
 & & & & & \text{(pb)} & & & \\
 & \mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
\mathbb{R}^{0|32} & & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} \arrow[lll, shift left=2] \arrow[lll, phantom, "\vdots"] \arrow[lll, shift right=2] & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
\boxed{\text{Exceptional}} & & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & & \text{T-Duality} \arrow[ul, dotted] \arrow[ur, dotted] & & & &
\end{tikzcd}$$

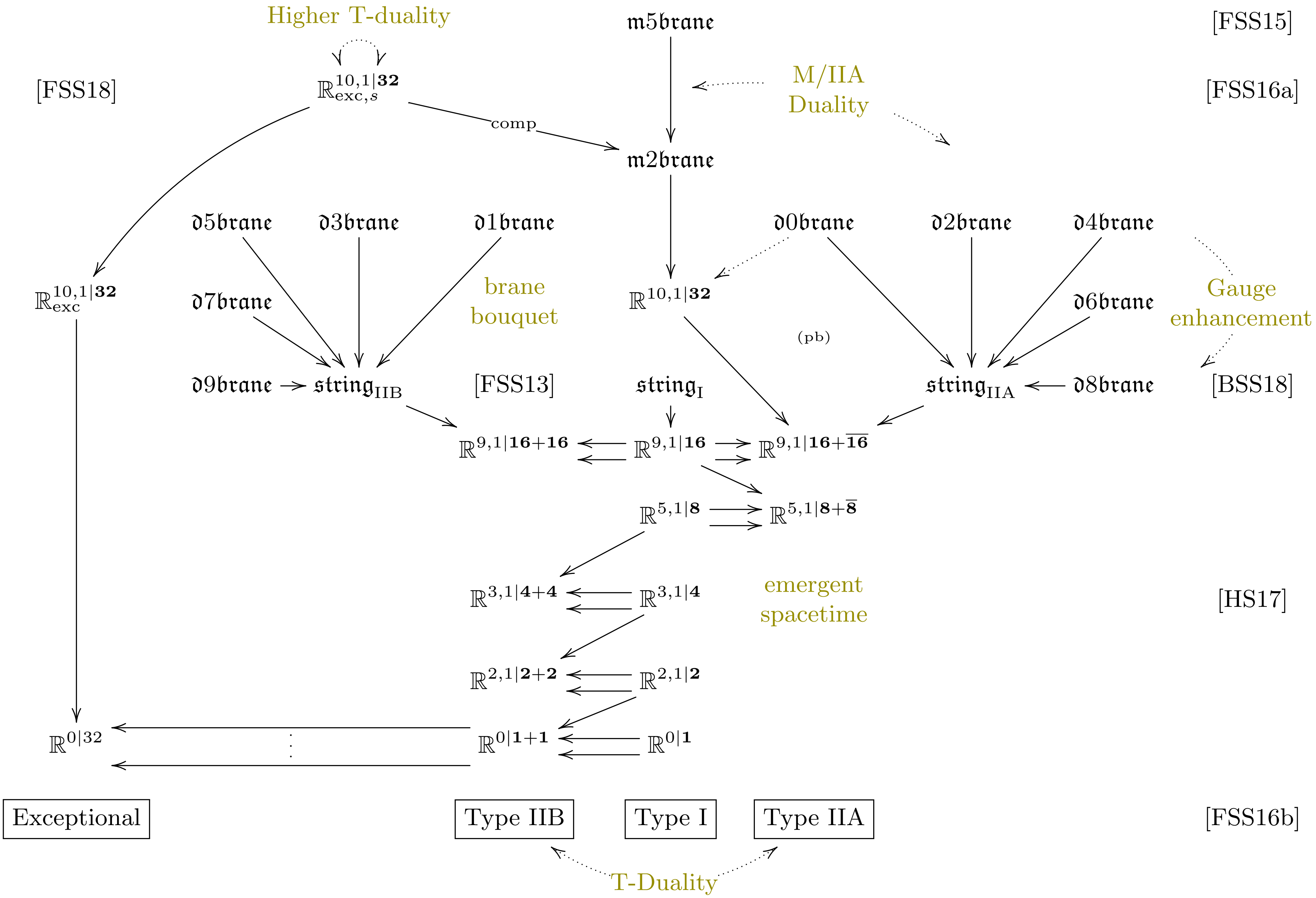
Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & & & & \mathfrak{m5brane} \arrow[dd] & & & & \text{[FSS15]} \\
\text{[FSS18]} & & & & & \text{M/IIA Duality} \arrow[l, dotted] \arrow[dr, dotted] & & & \text{[FSS16a]} \\
 & & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
 & \mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] & \\
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[dddddd] & \mathfrak{d7brane} \arrow[dr] & & \text{brane bouquet} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[ddr] & \text{(pb)} & & \mathfrak{d6brane} \arrow[dl] & \text{Gauge enhancement} \arrow[d, dotted] \\
 & \mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & \text{[FSS13]} & \mathfrak{string}_{\mathrm{I}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \text{[BSS18]} \\
 & & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \text{emergent spacetime} & & & \text{[HS17]} \\
 & & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
\mathbb{R}^{0|32} & & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} \arrow[lll, shift left=2] \arrow[lll, shift right=2] \arrow[lll, phantom, "\vdots"] & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & & \\
\boxed{\text{Exceptional}} & & & \boxed{\text{Type IIB}} & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & & & \text{[FSS16b]} \\
 & & & & \text{T-Duality} \arrow[ul, dotted] \arrow[ur, dotted] & & & &
\end{tikzcd}$$

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet



Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet



Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

$$\begin{tikzcd}
 & \mathfrak{m5brane} \arrow[d] & & \text{[FSS15]} \\
 & \mathfrak{m2brane} \arrow[d] & & \\
\text{[FSS13]} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & & \\
 & \mathbb{R}^{9,1|\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \text{[HS17]} \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] \arrow[uuuuuu, squiggly, bend right=60] & &
\end{tikzcd}$$

emergence of fundamental M-branes from the Atom of Superspace

Universal central invariant super-L_∞ extensions of $\mathbb{R}^{0|1}$: Brane bouquet

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] \\
\mathfrak{m2brane} \arrow[d] \\
\mathbb{R}^{10,1|\mathbf{32}}
\end{tikzcd}$$

zoom in on the fundamental M-brane super-extensions

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] & \\
\mathfrak{m2brane} \arrow[d, swap, "\mathrm{hofib}(\mu_{\mathrm{M2}})"] & \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, swap, "\mu_{\mathrm{M2}}"] & \\
 & B^4\mathbb{R}
\end{tikzcd}$$

$$\mu_{\mathrm{M2}} = dL^{\mathrm{WZW}}_{\mathrm{M2}} = \tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right) \wedge e^{a_1} \wedge e^{a_2}$$

the WZW-curvature of the Green-Schwarz-type sigma-model super-membrane

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}\big(\mu_{\mathrm{M2}}\big)"'] & & S^7_{\mathbb{R}} \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{\mathrm{M2}}"'] & & \\
& B^4\mathbb{R} &
\end{tikzcd}$$

$$\mu_{\mathrm{M5}} = dL^{\mathrm{WZW}}_{\mathrm{M5}} = \tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right)\wedge e^{a_1}\wedge\cdots\wedge e^{a_5} + c_3\wedge\tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1a_2}\psi\right)\wedge e^{a_1}\wedge e^{a_2}$$

the WZW-curvature of the Green-Schwarz-type sigma-model super-fivebrane

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}\big(\mu_{\mathrm{M2}}\big)"'] & & S^7_{\mathbb{R}} \arrow[d, "\mathrm{hofib}(c_2)"] \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{\mathrm{M2}}"'] & & S^4_{\mathbb{R}} \arrow[dl, "c_2"] \\
& B^4\mathbb{R} &
\end{tikzcd}$$

the quaternionic Hopf fibration (in rational homotopy theory)

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M5}})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{hofib}\left(\mu_{\mathrm{M2}}\right)"'] & & S^7_{\mathbb{R}} \arrow[d, "\mathrm{hofib}(c_2)"] \arrow[from=2-3, to=3-1, Rightarrow, shorten <=18pt, shorten >=18pt] \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, dashed, "\mu_{\mathrm{M2/M5}}"] \arrow[dr, "\mu_{\mathrm{M2}}"'] & & S^4_{\mathbb{R}} \arrow[dl, "c_2"] \\
& B^4\mathbb{R} &
\end{tikzcd}$$

the unified M2/M5-cocycle

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & & S^4_{\mathbb{R}}
\end{tikzcd}$$

the unified M2/M5-cocycle is in rational Cohomotopy in degree 4

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & & S^4_{\mathbb{R}} \\
\frac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right) \wedge e^{a_1} \wedge e^{a_2} & & G_4 \arrow[ll, mapsto] \\
\frac{1}{5!}\left(\overline{\psi}\Gamma_{a_1 \cdots a_5}\psi\right) \wedge e^{a_1} \cdots e^{a_5} & & G_7 \arrow[ll, mapsto]
\end{tikzcd}$$

Sullivan model: $\mathcal{O}\left(S^4_{\mathbb{R}}\right) \;\simeq\; \mathbb{R}[G_4, G_7]/\left(\begin{smallmatrix} dG_4 = 0 \\ dG_7 = -\frac{1}{2} G_4 \wedge G_4 \end{smallmatrix}\right)$

= 11d supergravity equations of motion of the C-field ([Sati13, Sect. 2.5])

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r, "\mu_{\mathrm{M2/M5}}"] & S^4_{\mathbb{R}}
\end{tikzcd}$$

the unified M2/M5-cocycle

The fundamental M2/M5-brane cocycle

[FSS15]

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2/M5}}"] & {} \arrow[d, "\text{double dimensional reduction \& gauge enhancement}" description] & S^4_{\mathbb{R}} \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[rr, "\mu_{F1/D_{2p}}"'] & {} & \mathrm{ku}_{/\!/ B^2\mathbb{R}}
\end{tikzcd}$$

D-brane charge in twisted K-theory, rationally
[BSS18]

The rational conclusion.

In $\left\{ \begin{array}{c} \text{infinitesimal} \\ \text{rational} \end{array} \right\}$ approximation

brane charge quantization follows from first principles

and reveals this situation:

brane species	**cohomology theory of charge quantization**
D-branes	twisted K-theory
M-branes	Cohomotopy in degree 4

The rational conclusion.

In $\left\{ \begin{array}{c} \text{infinitesimal} \\ \text{rational} \end{array} \right\}$ approximation

brane charge quantization follows from first principles

and reveals this situation:

brane species	**cohomology theory of charge quantization**
D-branes	twisted K-theory
M-branes	Cohomotopy in degree 4

Lift beyond $\left\{ \begin{array}{c} \text{infinitesimal} \\ \text{rational} \end{array} \right\}$ approximation is not unique

but one lift of rational Cohomotopy is *minimal* (in number of cells):

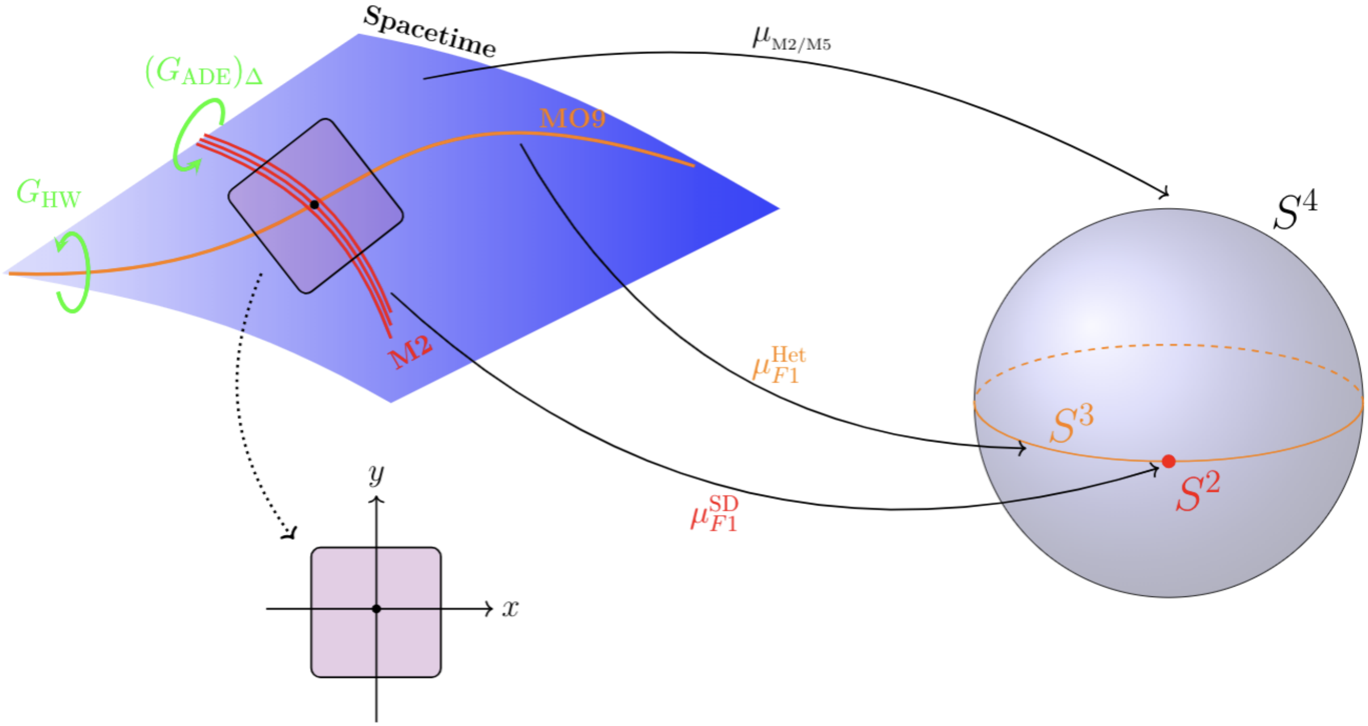
actual Cohomotopy represented by the actual 4-sphere

$$\begin{tikzcd}
 & S^4 \arrow[d, "\text{rationalization}"] \\
X \arrow[ur, "\substack{\text{cocycle in} \\ \text{actual Cohomotopy}}"] \arrow[r, "\substack{\text{cocycle in} \\ \text{rational cohomotopy}}"'] & S^4_{\mathbb{R}}
\end{tikzcd}$$

Towards microscopic M-theory

1. Construct
differential equivariant Cohomotopy $\widehat{S^4_{\gamma}}$
of 11d super-orbifold spacetimes $\mathcal{X}$

2. lifting super-tangent-space-wise
the fundamental M2/M5-brane cocycle.



3. Compare the resulting observables on
M-brane charge quantized supergravity field moduli
with expected limiting corners of M-theory

Part II.

Orbifold cohomology

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Global equivariant Super homotopy theory

and the C-field at singularities

back to Part I

orbifolded

$\rightsquigarrow$

Global equivariant
Super homotopy theory

and the C-field at singularities

back to Part I

The idea of global equivariant homotopy theory

What is an orbifold, really?

---

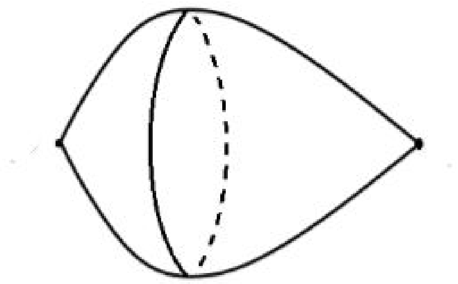
detailed exposition in
ncatlab.org/nlab/show/orbifold+cohomology

The idea of global equivariant homotopy theory

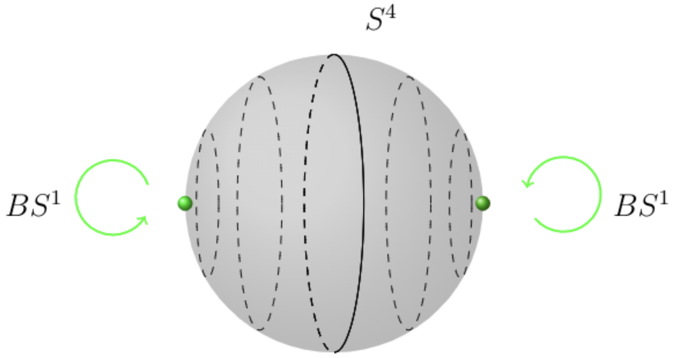
What is an orbifold, really?

Two opposite dual views on orbifold singularities $X \circlearrowright G$:

non-smooth singular points



smooth but stacky quotients



orbifold singularity

$$\begin{tikzcd}
 & \mathbb{B}G \arrow[dl, "\int_{\mathrm{orb}}"'] \arrow[dr, "\flat_{\mathrm{orb}}"] & \\
\ast = \ast/G & & \mathbf{B}G = \ast /\!\!/ G
\end{tikzcd}$$

opposite extreme
aspects of orbifold singularitys

non-smooth
singular point
$\ast = \ast/G$

smooth
stacky quotient
$\mathbf{B}G = \ast /\!\!/ G$

The site of global equivariant homotopy theory

$$\mathrm{Singularities} \;:=\; \left\{ \begin{array}{ll} \text{objects:} & \text{groupoids } \mathbb{B}G := \Big\{ \overset{g}{\overset{\curvearrowright}{\ast}} \;\Big|\; g \in G \Big\} \\ & \text{for finite groups } G \\[1em] \text{morphisms:} & \text{groupoids of functors} \\ & \mathbb{B}G \longrightarrow \mathbb{B}G' \\[1em] \text{coverings:} & \text{identity functors} \end{array} \right\}$$

Elsewhere known as the "global orbit category"

but better thought of as the ∞-category
of models for orbifold singularities.

The toposes of global equivariant homotopy theory

call the base topos

$$\mathbf{H}_{\cup} \;:=\; \mathrm{Sh}_\infty\big(\mathrm{SuperFormalCartSp}\big)$$

set

$$\mathbf{H} \;:=\; \mathrm{Sh}_\infty\big(\mathrm{Singularities},\, \mathbf{H}_{\cup}\big)$$

Adjunctions

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=9, "\Pi_{\mathrm{orb}}" description] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, shift right=3, "\Gamma_{\mathrm{orb}}" description] \arrow[r, phantom, shift right=6, "\perp"] & \mathbf{H}_{\cup} \arrow[l, hook', shift right=3, "\mathrm{Disc}_{\mathrm{orb}}" description] \arrow[l, hook', shift left=9, "\mathrm{coDisc}_{\mathrm{orb}}" description]
\end{tikzcd}$$

Cohesion of global equivariant homotopy theory
highlighted by C. Rezk, *Global homotopy theory and cohesion* (2014)

The modalities of global equivariant homotopy theory

$$\boxed{\begin{array}{ccccc} \rotatebox[origin=c]{70}{$<$} & \dashv & \cup & \dashv & \rotatebox[origin=c]{70}{$\prec$} \\ \\ \text{singular} & & \text{smooth} & & \substack{\text{orbi-}\\ \text{singular}} \end{array}}$$

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}}
 \arrow[r, shift left=7.5] \arrow[r, shift left=1.5] \arrow[rr, shift right=4.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
 \arrow[l, hook', shift right=4.5] \arrow[l, hook', shift left=1.5] \arrow[r, shift left=1.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
 \arrow[l, hook', shift right=4.5] \arrow[l, hook', shift left=1.5] \arrow[r, shift left=1.5] \arrow[r, shift right=4.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
 \arrow[l, hook', shift left=1.5] \arrow[lll, hook', shift left=7.5] \arrow[r, shift right=4.5]
 \arrow[d, shift right=4.5] \arrow[d, shift left=1.5]
&
\ast
 \arrow[l, hook', shift left=1.5] \arrow[l, hook', shift left=7.5]
 \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\substack{\text{super-}\\ \text{geometric}}}
 \arrow[r, shift left=7.5] \arrow[r, shift left=1.5] \arrow[rr, shift right=4.5]
 \arrow[u, hook', shift left=1.5] \arrow[u, hook', shift right=4.5]
&
\overbrace{\mathbf{H}_{\substack{\cup\\ \rightsquigarrow}}}^{\text{bosonic}}
 \arrow[l, hook', shift right=4.5] \arrow[l, hook', shift left=1.5] \arrow[r, shift left=1.5]
 \arrow[u, hook', shift left=1.5] \arrow[u, hook', shift right=4.5]
&
\overbrace{\mathbf{H}_{\substack{\cup\\ \Re}}}^{\text{reduced}}
 \arrow[l, hook', shift right=4.5] \arrow[l, hook', shift left=1.5] \arrow[r, shift left=1.5] \arrow[r, shift right=4.5]
 \arrow[u, hook', shift left=1.5] \arrow[u, hook', shift right=4.5]
&
\overbrace{\mathbf{H}_{\substack{\cup\\ \flat}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
 \arrow[l, hook', shift left=1.5] \arrow[lll, hook', shift left=7.5] \arrow[r, shift right=4.5]
 \arrow[u, hook', shift left=1.5] \arrow[u, hook', shift right=4.5]
&
\ast
 \arrow[l, hook', shift left=1.5] \arrow[l, hook', shift left=7.5]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$$\begin{array}{ccccc} \rotatebox[origin=c]{70}{$<$} & \dashv & \rotatebox[origin=c]{70}{$\subset$} & \dashv & \rotatebox[origin=c]{70}{$\prec$} \\ \text{singular} & & \text{smooth} & & \begin{array}{c}\text{orbi-}\\ \text{singular}\end{array} \end{array}$$

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\substack{\text{super-}\\ \text{geometric}}}
  \arrow[r, shift left=4.5]
  \arrow[r, shift left=1.5]
  \arrow[rr, bend right=20]
  \arrow[d, shift right=4.5, "\Pi"']
  \arrow[d, shift left=1.5]
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
  \arrow[l, hook, shift left=1.5]
  \arrow[l, hook, shift left=4.5]
  \arrow[r, shift left=1.5]
  \arrow[d, shift right=4.5]
  \arrow[d, shift left=1.5]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
  \arrow[l, hook, shift left=1.5]
  \arrow[l, hook, shift left=4.5]
  \arrow[r, shift left=1.5]
  \arrow[r, shift right=1.5]
  \arrow[d, shift right=4.5]
  \arrow[d, shift left=1.5]
& \overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[l, hook, shift left=1.5]
  \arrow[lll, hook, bend left=20]
  \arrow[r, shift right=1.5]
  \arrow[d, shift right=4.5]
  \arrow[d, shift left=1.5]
& \ast
  \arrow[l, hook, shift right=1.5]
  \arrow[l, hook, shift left=4.5]
  \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}}^{\substack{\text{super-}\\ \text{geometric}}}
  \arrow[u, hook, shift right=1.5, "\Delta"]
  \arrow[u, hook, shift left=4.5]
  \arrow[r, shift left=4.5]
  \arrow[r, shift left=1.5]
  \arrow[rr, bend right=20]
& \overbrace{\mathbf{H}_{\substack{\rotatebox[origin=c]{70}{$\subset$}\\ \rightsquigarrow}}}^{\text{bosonic}}
  \arrow[u, hook, shift right=1.5]
  \arrow[u, hook, shift left=4.5]
  \arrow[l, hook, shift left=1.5]
  \arrow[l, hook, shift left=4.5]
  \arrow[r, shift left=1.5]
& \overbrace{\mathbf{H}_{\substack{\rotatebox[origin=c]{70}{$\subset$}\\ \Re}}}^{\text{reduced}}
  \arrow[u, hook, shift right=1.5]
  \arrow[u, hook, shift left=4.5]
  \arrow[l, hook, shift left=1.5]
  \arrow[l, hook, shift left=4.5]
  \arrow[r, shift left=1.5]
  \arrow[r, shift right=1.5]
& \overbrace{\mathbf{H}_{\substack{\rotatebox[origin=c]{70}{$\subset$}\\ \flat}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[u, hook, shift right=1.5]
  \arrow[u, hook, shift left=4.5]
  \arrow[l, hook, shift left=1.5]
  \arrow[lll, hook, bend left=20]
  \arrow[r, shift right=1.5]
& \ast
  \arrow[l, hook, shift right=1.5]
  \arrow[l, hook, shift left=4.5]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$$\vee \quad \dashv \quad \cup \quad \dashv \quad \curlyvee$$

singular

smooth

orbi-singular

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\text{super-geometric}} \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[rr, shift right=6] \arrow[d, shift right=6] \arrow[d, shift left=2, "\Gamma"']
& \overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}} \arrow[l, hook', shift left=4] \arrow[l, hook', shift right=0] \arrow[r, shift left=2] \arrow[d, shift right=6] \arrow[d, shift left=2]
& \overbrace{\mathbf{H}_{\Re}}^{\text{reduced}} \arrow[l, hook', shift left=4] \arrow[l, hook'] \arrow[r, shift left=2] \arrow[r, shift right=4] \arrow[d, shift right=6] \arrow[d, shift left=2]
& \overbrace{\mathbf{H}_{\flat}}^{\text{geometrically discrete }\infty\text{-groupoids}} \arrow[l, hook'] \arrow[lll, hook', shift right=8] \arrow[r, shift right=4] \arrow[d, shift right=6] \arrow[d, shift left=2]
& * \arrow[l, hook'] \arrow[l, hook', shift right=8] \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\text{super-geometric}} \arrow[u, hook, shift right=2, "\Delta"] \arrow[u, hook, shift left=6] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[rr, shift right=6]
& \overbrace{\mathbf{H}_{\underset{\rightsquigarrow}{\cup}}}^{\text{bosonic}} \arrow[u, hook, shift right=2] \arrow[u, hook, shift left=6] \arrow[l, hook', shift left=4] \arrow[l, hook'] \arrow[r, shift left=2]
& \overbrace{\mathbf{H}_{\underset{\Re}{\cup}}}^{\text{reduced}} \arrow[u, hook, shift right=2] \arrow[u, hook, shift left=6] \arrow[l, hook', shift left=4] \arrow[l, hook'] \arrow[r, shift left=2] \arrow[r, shift right=4]
& \overbrace{\mathbf{H}_{\underset{\flat}{\cup}}}^{\text{geometrically discrete }\infty\text{-groupoids}} \arrow[u, hook, shift right=2] \arrow[u, hook, shift left=6] \arrow[l, hook'] \arrow[lll, hook', shift right=8] \arrow[r, shift right=4]
& * \arrow[l, hook'] \arrow[l, hook', shift right=8]
\end{tikzcd}$$

smooth = non-singular

The modalities of global equivariant homotopy theory

$\rotatebox[origin=c]{70}{$<$}$	$\dashv$	$\rotatebox[origin=c]{70}{$\subset$}$	$\dashv$	$\rotatebox[origin=c]{70}{$\prec$}$
singular		smooth		orbi-singular

possibly singular/orbifolded

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overbrace{\mathbf{H}}^{\text{super-geometric}}
  \arrow[r, shift left=6]
  \arrow[r, shift right=2]
  \arrow[rr, bend right=25]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2, "\Gamma"']
&
\overbrace{\mathbf{H}_{\rightsquigarrow}}^{\text{bosonic}}
  \arrow[l, hook', shift right=2]
  \arrow[l, hook', shift left=2]
  \arrow[r, shift right=2]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\Re}}^{\text{reduced}}
  \arrow[l, hook', shift right=6]
  \arrow[l, hook', shift left=2]
  \arrow[r, shift left=2]
  \arrow[r, shift right=6]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\overbrace{\mathbf{H}_{\flat}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[l, hook', shift left=2]
  \arrow[lll, hook', bend left=25]
  \arrow[r, shift right=2]
  \arrow[d, shift right=6]
  \arrow[d, shift left=2]
&
\ast
  \arrow[l, hook', shift left=2]
  \arrow[l, hook', shift right=6]
  \arrow[d, equal]
\\
\overbrace{\mathbf{H}_{\cup}}^{\text{super-geometric}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6, "\nabla"']
  \arrow[r, shift left=6]
  \arrow[r, shift right=2]
  \arrow[rr, bend right=25]
&
\overbrace{\mathbf{H}_{\cup_{\rightsquigarrow}}}^{\text{bosonic}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift right=2]
  \arrow[l, hook', shift left=2]
  \arrow[r, shift right=2]
&
\overbrace{\mathbf{H}_{\cup_{\Re}}}^{\text{reduced}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift right=6]
  \arrow[l, hook', shift left=2]
  \arrow[r, shift left=2]
  \arrow[r, shift right=6]
&
\overbrace{\mathbf{H}_{\cup_{\flat}}}^{\substack{\text{geometrically discrete}\\ \infty\text{-groupoids}}}
  \arrow[u, hook, shift right=2]
  \arrow[u, hook, shift left=6]
  \arrow[l, hook', shift left=2]
  \arrow[lll, hook', bend left=25]
  \arrow[r, shift right=2]
&
\ast
  \arrow[l, hook', shift left=2]
  \arrow[l, hook', shift right=6]
\end{tikzcd}$$

smooth = non-singular

Super-Orbifolds ([Wellen17, Sch13])

Let $\underbrace{V}_{\substack{\text{tangent}\\\text{space}\\\text{model}}}\,,\ \underbrace{G}_{\substack{\text{generic}\\\text{singularity}\\\text{type}}} \in \mathrm{Grp}(\mathbf{H})$ be group objects.

Definition. A **G-orbi V-fold** is

- an object $\mathcal{X} \in \mathbf{H}_{/\mathbf{B}G_{\curlyvee}}$

which is

1. 0-truncated: $\tau_0(\mathcal{X}) \simeq \mathcal{X}$
2. orbi-singular: $\curlyvee(\mathcal{X}) \simeq \mathcal{X}$
3. a V-fold: there exists a V-*atlas* $$\begin{tikzcd} & U \arrow[dl, "p_V"'] \arrow[dr, "p_X"] & \\ V & & \mathcal{X}_{\cup} \end{tikzcd}$$
   (a) p_X is a covering: $(\tau_{-1})_{/X}(p_X) \simeq *$
   (b) p_X is a local diffeomorphism: $\Im_{/X}(p_X) \simeq p_X$
   (c) p_V is a local diffeomorphism: $\Im_{/V}(p_V) \simeq p_V$

The global equivariant 4-sphere

In the following $G := \flat\mathrm{Pin}(5)$
the unoriented spin group in 5d, regarded as geometrically discrete.

This unifies
ADE-singularities
with
O-plane singularities

$$\begin{tikzcd}
\overbrace{\overset{\text{ADE-singularity}}{\overset{\text{[MFFGME09, MFF10]}}{\overbrace{(G_{\mathrm{ADE}} \times_Z G'_{\mathrm{ADE}})}}} \times \overset{\text{O-plane}}{\overbrace{\mathbb{Z}_2}}}^{\text{[HSS18]}} & & \\
(G_{\mathrm{ADE}} \times_Z G'_{\mathrm{ADE}}) \arrow[d, hook] \quad \times & \mathbb{Z}_2 \arrow[d, equal] & \\
\mathrm{Spin}(4) \quad \times & O(1) \arrow[r, hook] & \mathrm{Pin}(5)
\end{tikzcd}$$

Write $\mathbf{S}^4 \in \mathrm{SmoothManifolds}$ for the smooth 4-sphere.

$$\hookrightarrow \mathbf{H}$$

with $S^4 := \int \mathbf{S}^4 \in \infty\mathrm{Groupoids}$ its shape.

Then

$$\mathbf{S}^4_{\rotatebox[origin=c]{70}{$\prec$}} := \rotatebox[origin=c]{70}{$\prec$}\Big(\mathbf{S}^4 /\!\!/ \flat\mathrm{Pin}(5)\Big)$$ is a $\flat\mathrm{Pin}(5)$-orbi $\mathbb{R}^4$-fold

$$\in \mathbf{H}_{/\mathbf{B}\flat\mathrm{Pin}(5)}$$

$$S^4_{\rotatebox[origin=c]{70}{$\prec$}} := \int \rotatebox[origin=c]{70}{$\prec$}\Big(\mathbf{S}^4 /\!\!/ \mathrm{Pin}(5)^\flat\Big)$$ is its shape orbi-space

Equivariant Cohomotpy of Super-orbifolds

Let

$$\mathbb{R}^{10,1|\mathbf{32}} \in \mathrm{Grp}(\mathbf{H}) \qquad D = 11, \mathcal{N} = 1 \text{ translational supersymmetry}$$

$$\mathcal{X} \in \mathbf{H}_{/\curlyvee\flat\mathbf{B}\mathrm{Pin}(5)} \qquad \text{a } \flat\mathrm{Pin}(5)\text{-orbi } \mathbb{R}^{10,1|\mathbf{32}}\text{-fold.}$$

Definition.

The cocycle space of *equivariant Cohomotopy* of $\mathcal{X}$ is

$$\mathbf{H}_{/\curlyvee\flat\mathbf{B}\mathrm{Pin}(5)}\big(\mathcal{X}\,,\, S^4_{\curlyvee}\big) \;=\; \left\{ \begin{tikzcd} \mathcal{X} \arrow[rr, dashed, "\text{cocycle in}\\\text{equivariant Cohomotopy}"] \arrow[dr, ""{name=L}] & & S^4_{\curlyvee} \arrow[dl, ""'{name=R}] \\ & \curlyvee\flat\mathbf{B}\mathrm{Pin}(5) & \arrow[Rightarrow, from=1-3, to=2-2, phantom, "\Leftarrow"] \end{tikzcd} \right\}$$

and so the cohomology set is

$$H\left(\mathcal{X}, S^4_{\curlyvee}\right) \;:=\; \pi_0\Big(\mathbf{H}_{/\curlyvee\flat\mathbf{B}\mathrm{Pin}(5)}\Big)\left(\mathcal{X}, S^4_{\curlyvee}\right)$$

Super Cartan geometry

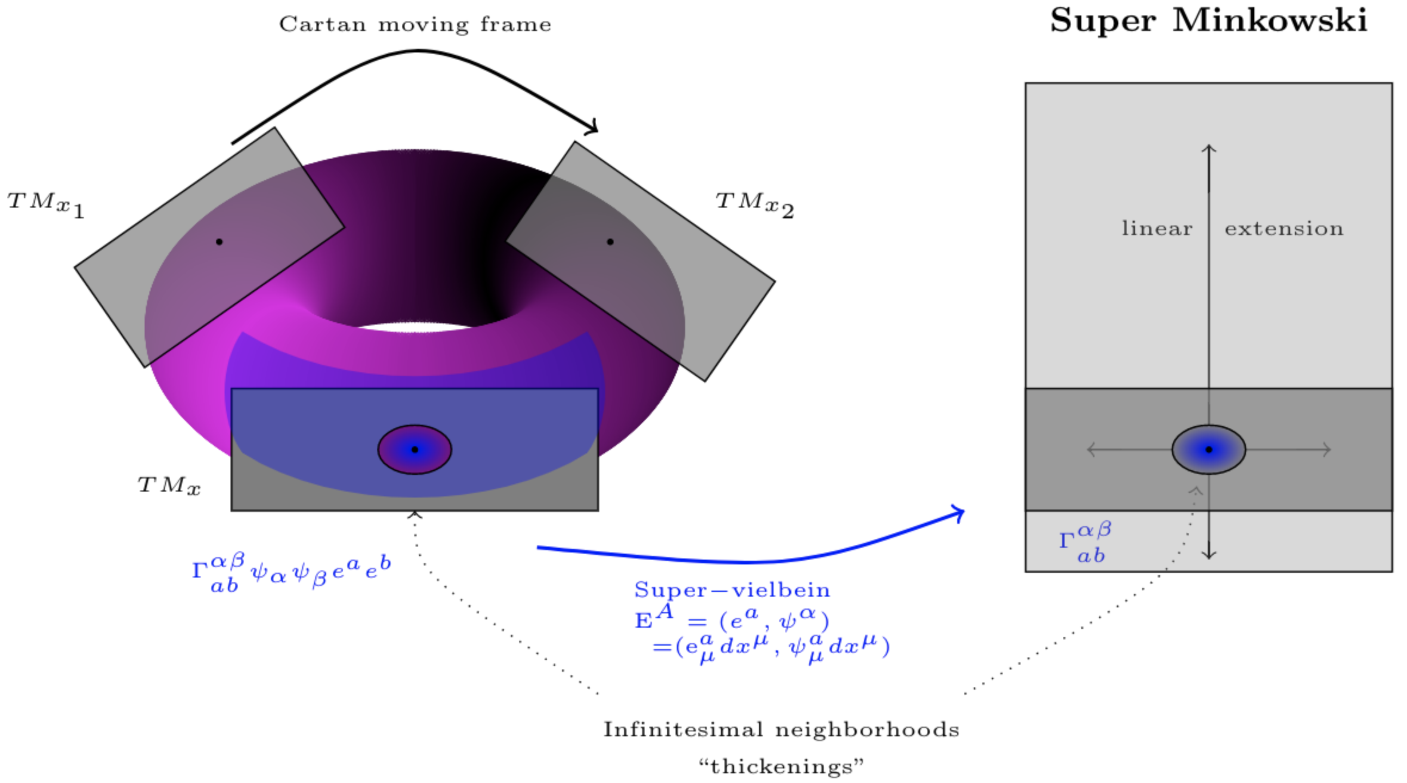
and 11d orbifold supergravity

back to Part I

Cartan geometry formalizes Einstein principle of equivalence

Spacetime is locally equivalent to Minkowski spacetime,
namely in the infinitesimal neighbourhood of every point

We now generalize this
from manifolds to super-orbifolds...



G-**Structures on orbi** V-**folds** ([Wellen17, Sch13])

Def.: infinitesimal disk around origin:

$$\mathbb{D}^V \;:=\; V \underset{\Im(V)}{\times} \{e\} \hookrightarrow V$$

Prop.: every orbi V-fold $\mathcal{X}$ carries
its canonical V-*frame bundle*

$$\mathcal{X}_{\cup} \xrightarrow{\text{frame}} \mathbf{BAut}\left(\mathbb{D}^V\right)$$

Def.: for $G \xrightarrow{\text{homom.}} \mathbf{Aut}\left(\mathbb{D}^V\right)$
a G-structure is a lift
(E is the *vielbein*)

$$\begin{tikzcd}
\mathcal{X}_{\cup} \arrow[rr, dashed] \arrow[dr, "\text{frame}"'] & & \mathbf{B}G \arrow[dl] \\
& \mathbf{BAut}\left(\mathbb{D}^V\right) \arrow[from=1-2, Rightarrow, shorten <=6pt, shorten >=10pt, "E"] &
\end{tikzcd}$$

Prop.: V itself carries
canonical G-structure
given by left translation

$$\begin{tikzcd}
V \arrow[rr, dashed] \arrow[dr, "\text{frame}"'] & & \mathbf{B}G \arrow[dl] \\
& \mathbf{BAut}\left(\mathbb{D}^V\right) \arrow[from=1-2, Rightarrow, shorten <=6pt, shorten >=10pt, "E_{\mathrm{li}}"] &
\end{tikzcd}$$

Def.: a G-structure is *torsion-free and flat*
if it coincides with this canonical one
on each infinitesimal disk

$$E_{|\mathbb{D}^V_x} \;\simeq\; \left(E_{\mathrm{li}}\right)_{|\mathbb{D}^V_e}$$

11d Supergravity from Super homotopy theory

Consider now $V = \mathbb{R}^{10,1|\mathbf{32}}$ and $\mathcal{X}$ an orbi $\mathbb{R}^{10,1|\mathbf{32}}$-fold.

Claim:

$G := \mathbf{Aut}^{\rightsquigarrow}_{\mathrm{Grp}}\big(\mathbb{R}^{10,1|\mathbf{32}}\big)$	$\simeq$	$\mathrm{Spin}(10,1)$
G-structure on $\mathcal{X}$	$\simeq$ $\simeq$	super-vielbein on $\mathcal{X}$ metric/field of gravity
G-structure is torsion-free:	$\Leftrightarrow$ $\underset{\substack{\text{[CaLe93]}\\ \text{[How97]}}}{\Leftrightarrow}$	super-torsion on $\mathcal{X}$ vanishes $\mathcal{X}$ is solution to 11d supergravity with vanishing bosonic flux
G-structure is flat:	$\Leftrightarrow$	$\mathcal{X}$ is a "flat" super-orbifold solution to 11d supergravity

11d Supergravity from Super homotopy theory

Consider now $V = \mathbb{R}^{10,1|\mathbf{32}}$ and $\mathcal{X}$ an orbi $\mathbb{R}^{10,1|\mathbf{32}}$-fold.

Claim:

$G := \mathbf{Aut}^{\rightsquigarrow}_{\mathrm{Grp}}\big(\mathbb{R}^{10,1|\mathbf{32}}\big) \quad \simeq \quad \mathrm{Spin}(10,1)$

G-structure on $\mathcal{X}$ $\quad \simeq \quad$ super-vielbein on $\mathcal{X}$
$\quad \simeq \quad$ metric/field of gravity

G-structure is torsion-free: $\quad \Leftrightarrow \quad$ super-torsion on $\mathcal{X}$ vanishes
$\quad \underset{\substack{\text{[CaLe93]} \\ \text{[How97]}}}{\Leftrightarrow} \quad$ $\mathcal{X}$ is solution to 11d supergravity with vanishing bosonic flux

---

G-structure is flat: $\quad \Leftrightarrow \quad$ $\mathcal{X}$ is a "flat" super-orbifold solution to 11d supergravity

$$\Rightarrow \quad \text{all} \left\{ \begin{array}{c} \text{curvature} \\ \&\ G_4\text{-flux} \end{array} \right\} \text{hence all} \left\{ \begin{array}{c} \text{higher curvature corrections} \\ \&\ \text{flux quantization} \end{array} \right\}$$

crammed into orbifold singularities
and thus taken care of by the *equivariance*
of charge quantization in differential equivariant Cohomotopy

Flat & fluxless except at curvature- & flux- singularities

Plausibility check:

Black M2/M5-brane solutions to SuGra interpolate ([AFFHS98]) between:

Planck-scale curved throat $\xleftrightarrow{\text{near/far horizon geometry}}$ orbifold-singularity in flat & fluxless space

$$\begin{tikzcd}
 & \text{black M2} \arrow[dl, "\ell_P \gg 1"'] \arrow[dr, "\ell_P \ll 1"] & \\
\mathrm{AdS}_4 \times \underbrace{S^7/G_{\mathrm{A}}}_{\text{spherical space form}} & & \mathbb{R}^{2,1} \times \underbrace{\mathbb{R}^8 /\!\!/ G_{\mathrm{A}}}_{\text{orbifold du Val singularity}}
\end{tikzcd}$$

inconsistent:

Planck-scale throat ($\ell_P \gg 1$)
spurious in SuGra ($\ell_P \ll 1$)
(evaded only by
macroscopic $N \gg 1$)

consistent:

all Planck-scale geometry
crammed into orbi-singularity
(necessary for
microscopic $N = 1$)

Hence, indeed, a consistent & complete picture:

1. is flat & fluxless away from singularities,
2. has hidden degrees of freedom inside the singularities.

Flat & fluxless except at curvature- & flux- singularities

Plausibility check:

Black M2/M5-brane solutions to SuGra interpolate ([AFFHS98]) between:

Planck-scale curved throat $\xleftrightarrow{\text{near/far horizon geometry}}$ orbifold-singularity in flat & fluxless space

$$\begin{tikzcd} & \text{black M5} \arrow[dl, "\ell_P \gg 1"'] \arrow[dr, "\ell_P \ll 1"] & \\ \mathrm{AdS}_7 \times \underbrace{S^4 /\!\!/ G}_{\text{[MFF10]}} & & \mathbb{R}^{6,1} \times \underbrace{\mathbb{R}^4 /\!\!/ G}_{\text{orbifold du Val singularity}} \end{tikzcd}$$

inconsistent:
Planck-scale throat ($\ell_P \gg 1$)
spurious in SuGra ($\ell_P \ll 1$)
(evaded only by
macroscopic $N \gg 1$)

consistent:
all Planck-scale geometry
crammed into orbi-singularity
(necessary for
microscopic $N = 1$)

Hence, indeed, a consistent & complete picture:

1. is flat & fluxless away from singularities,
2. has hidden degrees of freedom inside the singularities.

In conclusion, in super homotopy theory emerges:

$$\mathrm{CovariantPhaseSpace} :=$$

$$\coprod_{\substack{[\mathcal{X}] \in \\ \flat\mathrm{Pin}(3) \times \mathrm{Pin}(5)\mathrm{Orbi-} \\ \mathbb{R}^{10,1|\mathbf{32}}\mathrm{Folds}_{/\sim}}} \left(\begin{tikzcd}
 & & \text{gauge field} & \int\Big(\mathbf{S}^7 /\!\!/ \flat\big(\mathrm{Spin}(3) \times \mathrm{Spin}(5)\big)\Big) \arrow[r] \arrow[d] & \int\Big(\mathbf{S}^4 /\!\!/ \flat\mathrm{Spin}(5)\Big) \arrow[d] \\
\mathcal{X} \arrow[urrr, dashed, "{(G_4, G_7)}"] \arrow[rrr, dashed, "{\big(N_{Q_{\mathrm{M5}}} Q_{\mathrm{M2}},\, N_X Q_{\mathrm{M5}}\big)}" description] \arrow[drr, dashed, "{\big(TQ_{\mathrm{M2}},\, N_X Q_{\mathrm{M2}}\big)}" description] \arrow[ddr, dashed, "g"] \arrow[dddr, bend right, "\mathrm{Fr}_{\mathcal{X}}"'] & & & \mathbf{B}\flat\big(\mathrm{Spin}(3) \times \mathrm{Spin}(5)\big) \arrow[r] \arrow[d] & \mathbf{B}\flat\mathrm{Spin}(5) \\
 & & \mathbf{B}\flat\big(\mathrm{Spin}(2,1) \times \mathrm{Spin}(8)\big) \arrow[r] \arrow[d] \arrow[ur, phantom, "\text{gravity}"] & \mathbf{B}\flat\mathrm{Spin}(8) & \\
 & \mathbf{B}\flat\mathbf{Aut}_{\mathrm{Grp}}\big(\mathbb{D}^{10,1|\mathbf{32}}\big) \arrow[r, equal] \arrow[d] & \mathbf{B}\flat\mathrm{Spin}(10,1) & & \\
 & \mathbf{B}\mathbf{Aut}\big(\mathbb{D}^{10,1|\mathbf{32}}\big) & & &
\end{tikzcd} \right)_{/\!\!/ \mathbf{Aut}(\mathcal{X})}$$

$$\mathrm{MotiveOfObservables} \; := \; \Sigma^{\infty}_{\flat\mathrm{Pin}(3) \times \flat\mathrm{Pin}(5)}\big(\mathrm{CovariantPhaseSpace}\big)$$

In conclusion, in super homotopy theory emerges:

This is the core ingredients of what is known as

M-theory on Spin(7)*-manifolds with space-filling M2-branes*

$$\boxed{\begin{array}{ccccccc} & & \mathrm{Pin}(2,1) & & \mathrm{Pin}(3) & & \mathrm{Pin}(5) \\ & & \curvearrowright & & \curvearrowright & & \curvearrowright \\ \mathbb{R}^{10,1} & \simeq & \mathbb{R}^{2,1} & \oplus & \mathbb{R}^{3} & \oplus & \mathbb{R}^{5} \\ \mathrm{M5} & & \times & & \times & & - \\ \mathrm{M2} & & \times & & - & & - \end{array}}$$

T-dual to

F-theory on elliptically fibered 8-manifolds with space-filling D3-branes
(whose near-horizon geometry is $\mathrm{AdS}_5 \times S^5$)

This happens to be
the phenomenologically relevant sector of type II string theory.

Theorem (Fiorenza-Sati-S.):

The C-field charge quantized in twisted Cohomotopy this way, implies cancellation of M-theory anomalies:

1) M5-brane anomaly counterterm:

$$\begin{array}{l} G_4 \in Z^4\big(X, \mathbb{Q}\big) \\ G_7 \in C^7\big(X, \mathbb{Q}\big) \end{array} \quad \text{such that} \quad \begin{array}{l} dG_4 = 0 \\ dG_7 = -\tfrac{1}{2} G_4 \wedge G_4 + \underbrace{\tfrac{1}{8} p_2\big(N_X Q_{\mathrm{M5}}\big)}_{\text{[Witten 96b, (5.7)]}} \end{array}$$

2) Half-integrally shifted C-field flux quantization

$$\underbrace{[G_4] + \Big[\tfrac{1}{4} p_1\big(N_X Q_{\mathrm{M5}}\big)\Big] \in H^4\big(X, \mathbb{Z}\big)}_{\text{[Witten 96a, 2.1 and 2.2]}}$$

3) M-theoretic tadpole cancellation:

$$\underbrace{dG_7 = \tfrac{1}{2}\chi(TX) = \tfrac{1}{4}\big(p_2(TX) - \tfrac{1}{4} p_1(TX)^2\big)}_{\text{[Sethi-Vafa-Witten 96] [Witten 96a, 3]}}$$

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