Supplementary Material for: Classifying Fragile Crystalline Fractional Chern Phases by Equivariant 2-Cohomotopy

We compile some facts needed for the computation of nonabelian equivariant cohomology of 2-tori, such as for the computation of unstable equivariant 2-cohomotopy, which in the companion article [1] is used to classify fragile crystalline Chern phases of quantum materials. While nothing here will come as a surprise to experts in equivariant homotopy theory, the details appear not have to been made explicit before, and the methods appear not to have been considered in the topological phases community before.

CONTENTS

I. EQUIVARIANT CELL STRUCTURE OF 2-TORI

a. Result. We determine (Table I) minimal equivariant cell decompositions for finite group G actions (minimal G-CW complex structures, cf. [2, §I.1-2][3, Ex. 1.3.6] and §II B below) on the 2-torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ according to the 13 symmorphic 2D space groups (symmorphic wallpaper groups, cf. [4, §26][5, §2]).

Point group G	**Space group** $G \ltimes \mathbb{Z}^2$	**minimal cell structure on** $G \circlearrowright (\mathbb{R}^2/\mathbb{Z}^2)$
$\mathbb{Z}_{/1}$	p1	Fig. 1
Dih_1	pm	Fig. 2
Dih_1	cm	Fig. 3
$\mathbb{Z}_{/2}$	p2	Fig. 4
Dih_2	pmm	Fig. 5
Dih_2	cmm	Fig. 6
$\mathbb{Z}_{/3}$	$p3$	Fig. 7
Dih_3	p31m	Fig. 8
Dih_3	p3m1	Fig. 9
$\mathbb{Z}_{/4}$	p4	Fig. 10
Dih_4	p4m	Fig. 11
$\mathbb{Z}_{/6}$	p6	Fig. 12
Dih_6	p6m	Fig. 13

TABLE I. The names of the 13 symmorphic 2D space groups, arranged according to their point groups G and referenced to the cell decomposition of the corresponding G-torus. Here $\mathbb{Z}_{/n} := \mathbb{Z}/n$ denotes the cyclic group of order n and $\mathrm{Dih}_n \simeq \mathbb{Z}_{/n} \rtimes \mathbb{Z}_{/2}$ the corresponding dihedral group of order $2n$ (obtained by adjoining a reflection generator σ subject to $\sigma^2 = \mathrm{e}$ and $\sigma \cdot [n] \cdot \sigma = [-n]$, cf. [4, §4]).

b. Proof. The proof of the following cell decompositions (according to Table I) is by immediate inspection: it may take work to find a cell decomposition but it is easy to recognize one when found. The rule is that cells appear in G-orbits (coset spaces G/H for subgroups $H \subset G$, as shown on the right of the following figures) subject to no further constraint except that their boundaries are equivariantly glued to lower-dimensional cells (as shown on the left of the following figures).

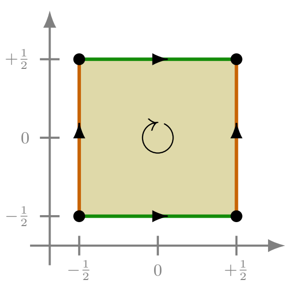
c. Status. The concept of G-equivariant cell complexes is fundamental in equivariant homotopy theory, and equivariant cell decomposition of a domain G-space is pivotal for general computation of its equivariant (generalized) cohomology. Moreover, cell decompositions of G-tori are among the simplest non-trivial examples of the general concept and of key importance for computations of crystalline topological phases (where they appear as Brillouin tori of crystal momenta subject to crystal symmetries). And yet we are not aware that the following cell decompositions (Table I.) have been spelled out before.

d. Terminology. What in equivariant topology are called *fixed points* — orbits of 0-cells of the form $D^0 \times G/G$ — are what in crystallography are known as the *high symmetry points*. Larger orbits of 0-cells $D^0 \times G/H$ are accordingly referred to in crystallography by the subsymmetry H which fixes them (such as "rotation center of order 3" for an orbit $D^0 \times \mathbb{Z}_{/6}/\mathbb{Z}_{/3}$). Crystallographic terminology for higher dimensional G-cells, and hence for G-cell complexes as such, seems to be largely missing.

e. Remarks. While there are crystals with non-symmorphic symmetry in their ordinary "position space", crystal symmetries of Brillouin tori "reciprocal spaces" are all expected to be symmorphic. Therefore the list in Table I is complete for the purpose of classifying 2D crystalline topological phases.

FIG. 1. Minimal cell structure on the torus without symmetry, corresponding to the space group p1.

$$\begin{aligned} \mathbb{Z}_{/1} \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p1}} (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) &\longmapsto [x,y] \end{aligned}$$



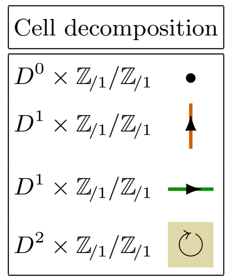


FIG. 2. Minimal equivariant cell structure on the torus with mirror symmetry corresponding to the space group pm.

$$\begin{aligned} \mathbb{Z}_{/2} \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{pm}} (\mathbb{R}^2/\mathbb{Z}^2) \\ \big(\sigma,[x,y]\big) &\longmapsto [-x,y] \end{aligned}$$

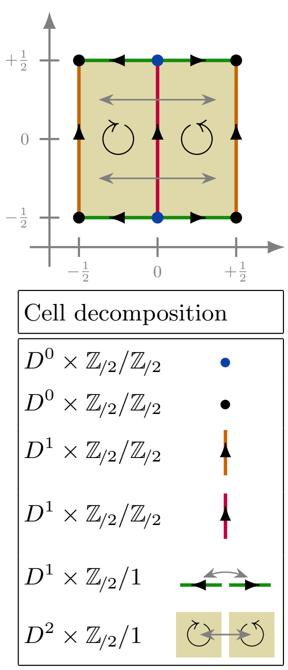


FIG. 3. Minimal equivariant cell structure on the torus with mirror symmetry corresponding to the space group cm.

$$\begin{aligned} \mathrm{Dih}_1 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{cm}} (\mathbb{R}^2/\mathbb{Z}^2) \\ \big(\sigma,[x,y]\big) &\longmapsto [y,x] \end{aligned}$$

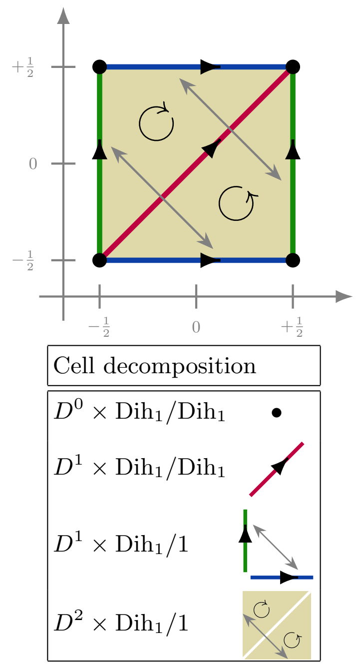


FIG. 4. Minimal equivariant cell structure on torus with 2-fold rotation symmetry corresponding to the space group p2.

$$\begin{array}{ccc} \mathbb{Z}_{/2}\times(\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p2}} & (\mathbb{R}^2/\mathbb{Z}^2)\\ \big([1],[x,y]\big) & \longmapsto & [-x,-y]\end{array}$$

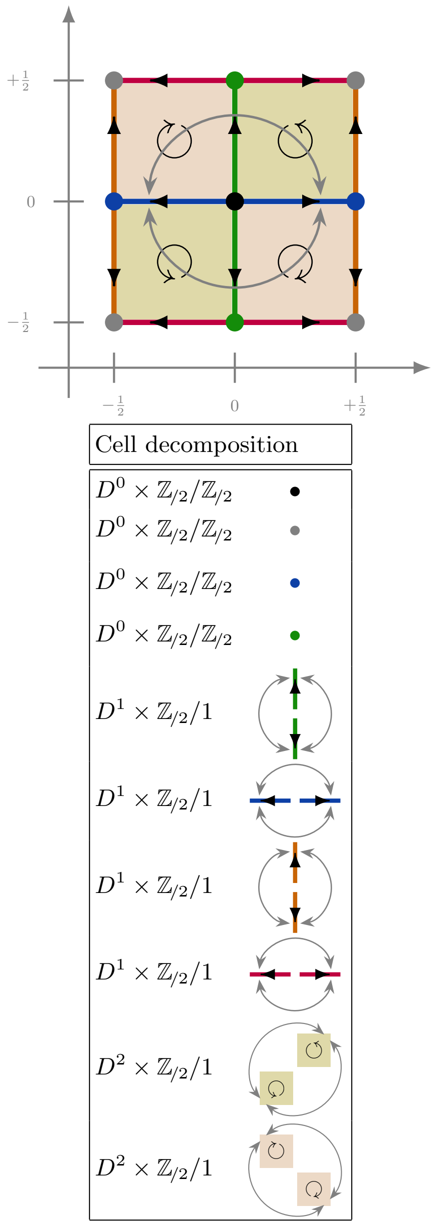


FIG. 5. Minimal equivariant cell structure on torus with 2-fold dihedral symmetry corresponding to the space group pmm.

$$\begin{array}{ccc} \mathrm{Dih}_2\times(\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{pmm}} & (\mathbb{R}^2/\mathbb{Z}^2)\\ \big([1],[x,y]\big) & \longmapsto & [-x,-y]\\ \big(\sigma,[x,y]\big) & \longmapsto & [x,y]\end{array}$$

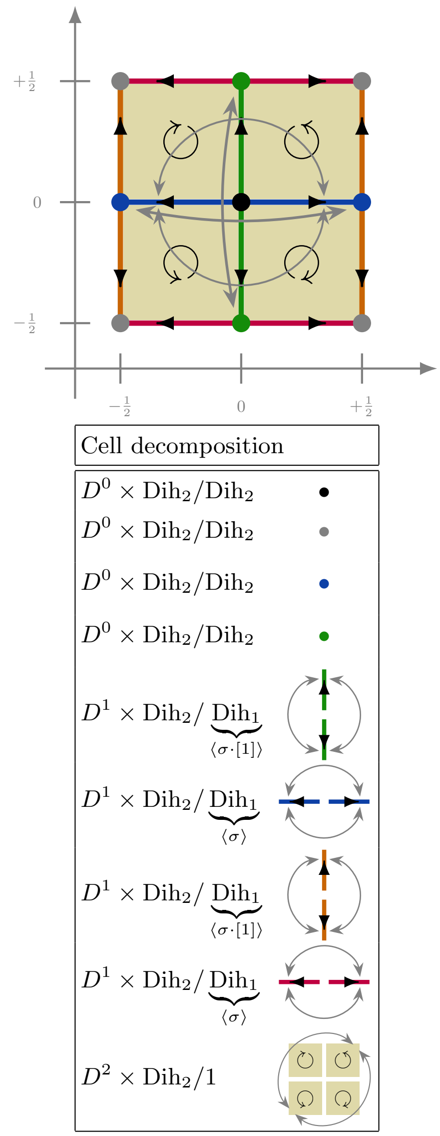


FIG. 6. Minimal equivariant cell structure on torus with 2-fold dihedral symmetry corresponding to the space group cmm.

$$
\begin{array}{rcl}
\mathrm{Dih}_2 \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{cmm}} & (\mathbb{R}^2/\mathbb{Z}^2) \\
\big([1],[x,y]\big) & \longmapsto & [-x,-y] \\
\big(\sigma,[x,y]\big) & \longmapsto & [y,x]
\end{array}
$$

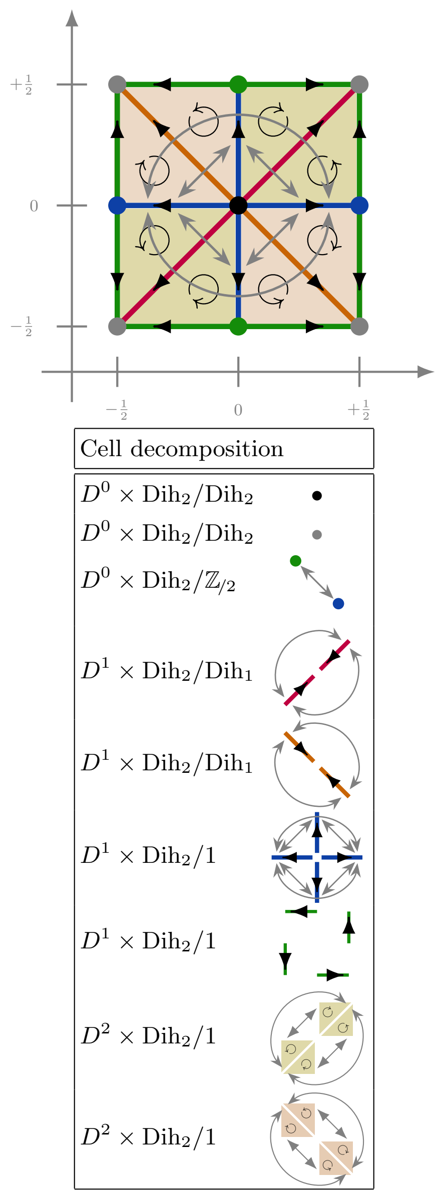


FIG. 7. Minimal equivariant cell structure on torus with 3-fold rotation symmetry corresponding to the space group p3.

$$
\begin{array}{rcl}
\mathbb{Z}_{/3} \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p3}} & (\mathbb{R}^2/\mathbb{Z}^2) \\
\big([1],[x,y]\big) & \longmapsto & [y,-y-x]
\end{array}
$$

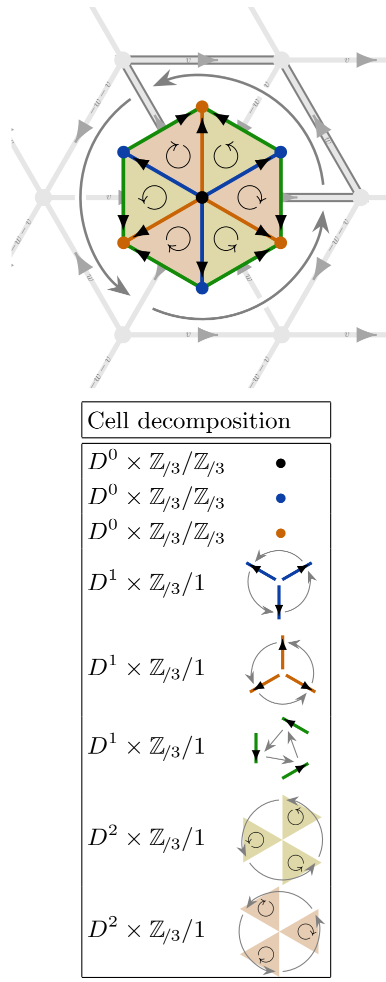
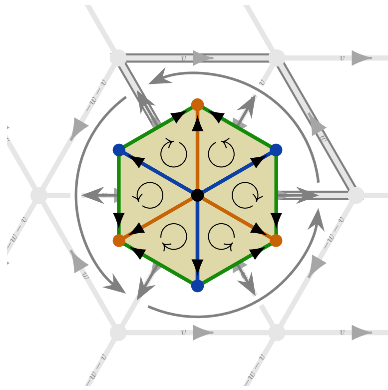


FIG. 8. Minimal equivariant cell structure on torus with 3-fold dihedral symmetry corresponding to the space group p31m.

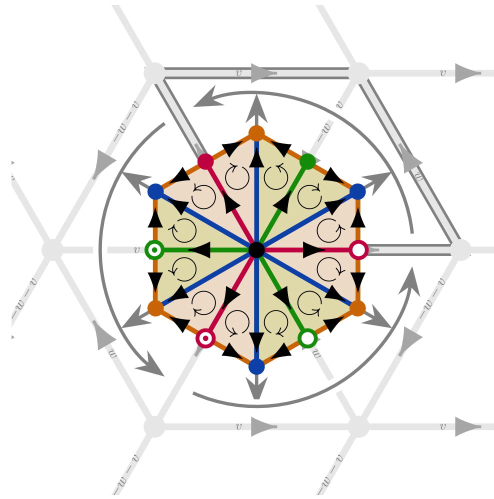
$$\begin{array}{ccc} \mathrm{Dih}_3 \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p31m}} & (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) & \longmapsto & [y,-y-x] \\ \big(\sigma,[x,y]\big) & \longmapsto & [-y,-x] \end{array}$$



Cell decomposition
$D^0 \times \mathrm{Dih}_3/\mathrm{Dih}_3$
$D^0 \times \mathrm{Dih}_3/\mathrm{Dih}_3$
$D^0 \times \mathrm{Dih}_3/\mathrm{Dih}_3$
$D^1 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^1 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\cdot[1]\rangle}$
$D^1 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^2 \times \mathrm{Dih}_3/1$

FIG. 9. Minimal equivariant cell structure on torus with 3-fold dihedral symmetry corresponding to the space group p3m1.

$$\begin{array}{ccc} \mathrm{Dih}_3 \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p3m1}} & (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) & \longmapsto & [y,-y-x] \\ \big(\sigma,[x,y]\big) & \longmapsto & [x,-y-x] \end{array}$$



Cell decomposition
$D^0 \times \mathrm{Dih}_3/\mathrm{Dih}_3$
$D^0 \times \mathrm{Dih}_3/\mathbb{Z}_{/3}$
$D^0 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^0 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^1 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^1 \times \mathrm{Dih}_3/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^1 \times \mathrm{Dih}_3/1$
$D^1 \times \mathrm{Dih}_3/1$
$D^2 \times \mathrm{Dih}_3/1$
$D^2 \times \mathrm{Dih}_3/1$

FIG. 10. Minimal equivariant cell structure on torus with 4-fold rotation symmetry corresponding to the space group p4.

$$\begin{array}{ccc} \mathbb{Z}_{/4} \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p4}} & (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) & \longmapsto & [y,-x] \end{array}$$

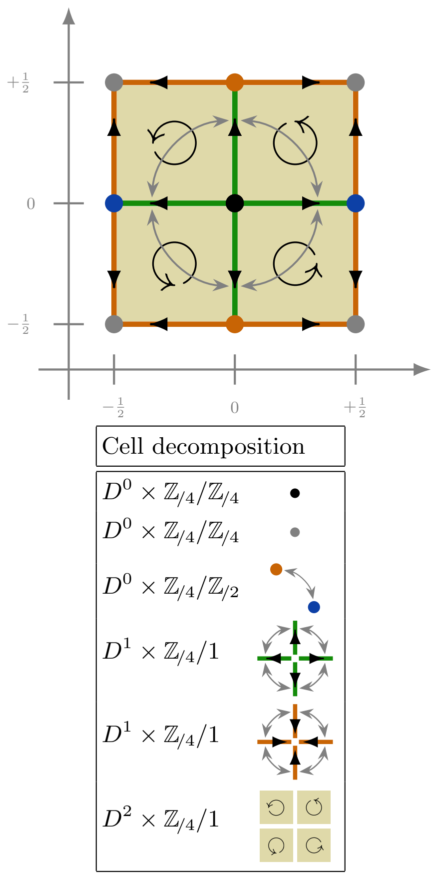


FIG. 11. Minimal equivariant cell structure on torus with 4-fold dihedral symmetry corresponding to the space group p4m.

$$\begin{array}{ccc} \mathrm{Dih}_4 \times (\mathbb{R}^2/\mathbb{Z}^2) & \xrightarrow{\text{p4m}} & (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) & \longmapsto & [y,-x] \\ \big(\sigma,[x,y]\big) & \longmapsto & [x,-y] \end{array}$$

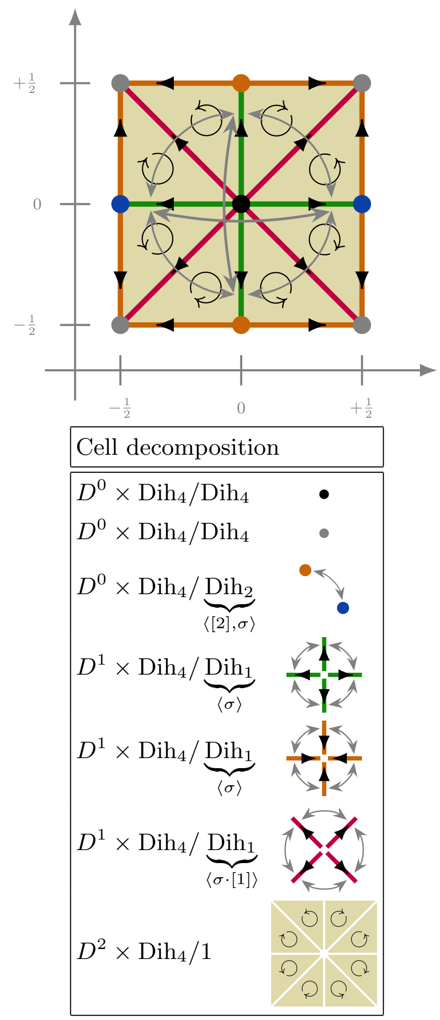
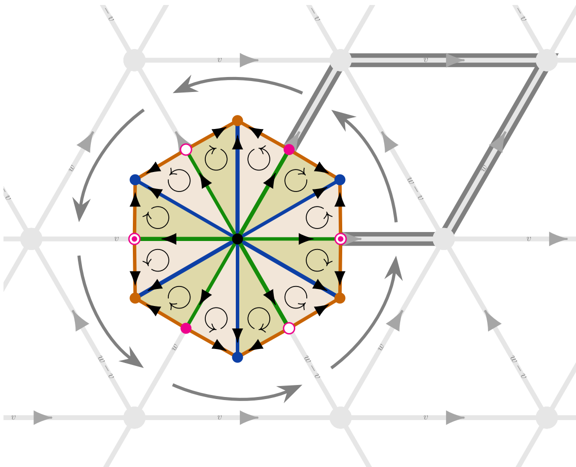


FIG. 12. Minimal equivariant cell structure on torus with 6-fold rotation symmetry corresponding to the space group p6.

$$\begin{aligned} \mathbb{Z}_{/6} \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p6}} (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) &\longmapsto [y, y-x] \end{aligned}$$



Cell decomposition
$D^0 \times \mathbb{Z}_{/6}/\mathbb{Z}_{/6}$
$D^0 \times \mathbb{Z}_{/6}/\mathbb{Z}_{/3}$
$D^0 \times \mathbb{Z}_{/6}/\mathbb{Z}_{/2}$
$D^1 \times \mathbb{Z}_{/6}/1$
$D^1 \times \mathbb{Z}_{/6}/1$
$D^1 \times \mathbb{Z}_{/6}/1$
$D^2 \times \mathbb{Z}_{/6}/1$
$D^2 \times \mathbb{Z}_{/6}/1$

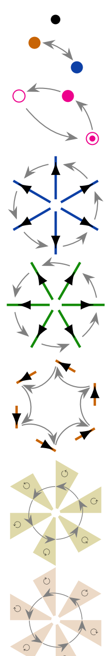
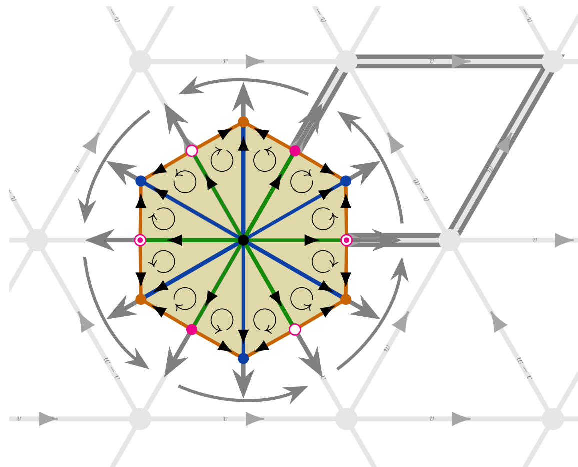
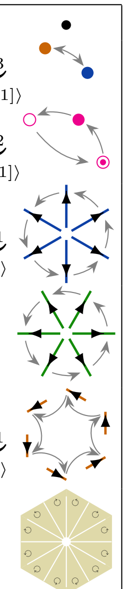


FIG. 13. Minimal equivariant cell structure on torus with 6-fold dihedral symmetry corresponding to the space group p6m.

$$\begin{aligned} \mathrm{Dih}_6 \times (\mathbb{R}^2/\mathbb{Z}^2) &\xrightarrow{\text{p6m}} (\mathbb{R}^2/\mathbb{Z}^2) \\ \big([1],[x,y]\big) &\longmapsto [y, y-x] \\ \big(\sigma,[x,y]\big) &\longmapsto [x, x-y] \end{aligned}$$



Cell decomposition
$D^0 \times \mathrm{Dih}_6/\mathrm{Dih}_6$
$D^0 \times \mathrm{Dih}_6/\underbrace{\mathrm{Dih}_3}_{\langle[2],\sigma\cdot[1]\rangle}$
$D^0 \times \mathrm{Dih}_6/\underbrace{\mathrm{Dih}_2}_{\langle[3],\sigma\cdot[1]\rangle}$
$D^1 \times \mathrm{Dih}_6/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\cdot[1]\rangle}$
$D^1 \times \mathbb{Z}_{/6}/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\rangle}$
$D^1 \times \mathrm{Dih}_6/\underbrace{\mathrm{Dih}_1}_{\langle\sigma\cdot[1]\rangle}$
$D^2 \times \mathrm{Dih}_6/1$



II. MATHEMATICAL BACKGROUND

For reference, we briefly recall some basics from homotopy theory and algebraic topology that are used in the main text. Introductory references include [6] and [7]. For introduction in our context see [8, §A.3][9, §1] and specifically for the case of equivariant homotopy theory see [3].

A. Basic homotopy theory

By a *space* $X \in \mathrm{Top}$ we mean a *compactly generated* topological space (cf. [3, Ntn.1.0.16]), and we speak of *maps* $f : X \longrightarrow Y$ for the continuous maps between these. The set of path-connected components of a space is denoted $\pi_0(X) \in \mathrm{Set}$.

First recall some basic notions of category theory (for pointers in our context see [10][8, §A.2]). Given a pair of maps into the same space, the *pullback* (pb) of one along the other is their universal completion to a commuting square:

$$
\begin{tikzcd}
A' \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\forall Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\forall"] \arrow[ddr, bend right, "\forall"'] & & \\
& A'' \arrow[r] \arrow[d] & A \arrow[d] \\
& B' \arrow[r] & B\,.
\end{tikzcd}
\tag{1}
$$

For instance, the pullback of a map along a point inclusion $* \simeq \{x\} \hookrightarrow B$ is its *fiber* A_x over that point:

$$
\begin{tikzcd}
A_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d] \\
\{x\} \arrow[r] & B\,.
\end{tikzcd}
\tag{2}
$$

Dually, for a pair of maps out of the same space, their *pushout* (po) is again the universal completion to a commuting square

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(po)}"] & B \arrow[d, dashed] \\
A' \arrow[r, dashed] & B'
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left, "\forall"] & \\
A' \arrow[r] \arrow[drr, bend right, "\forall"'] & B' \arrow[dr, dashed, "\exists!"] & \\
& & \forall P\,.
\end{tikzcd}
\tag{3}
$$

For instance, the pushout along the inclusion of an $n-1$-sphere S^{n-1} as the boundary of a closed n-ball D^n is an *n-cell attachment*:

$$
\begin{tikzcd}
S^{n-1} \arrow[r, "f", "\text{attaching map}"'] \arrow[d, hook] \arrow[dr, phantom, "\text{(po)}"] & X \arrow[d] \\
D^n \arrow[r] & D^n \cup_f X\,.
\end{tikzcd}
\tag{4}
$$

Constructing a space by iterative cell attachments starting with $S^{-1} \equiv \varnothing$ means to give it a *cell complex structure* or a *cell decomposition*.

Now, a *homotopy* $\eta : f \Rightarrow g$ between a pair of maps is a map

$$
\eta : X \times [0,1] \longrightarrow Y \qquad \text{s.t.} \begin{cases} \eta(-,1) \;=\; f(-)\,, \\ \eta(-,0) \;=\; g(-)\,. \end{cases} \tag{5}
$$

For (pointed) spaces $X, A \in \mathrm{Top}^{(*)}$ we write $\mathrm{Map}(X,A)$ for the *mapping space* of all maps $X \longrightarrow Y$, equipped with the *compact-open topology*, and we write

$$
\mathrm{Map}^*(X,A) \subset \mathrm{Map}(X,A) \tag{6}
$$

for the subspace of maps that preserve the basepoints.

Homotopies (5) are precisely the continuous paths in these mapping spaces (6), whence their path-connected components are the *homotopy classes* $[-]$ of maps

$$
\big[X \xrightarrow{f} Y \big] \;\in\; \pi_0 \, \mathrm{Map}^{(*)}(X,Y)\,. \tag{7}
$$

For $X = S^1$ the circle, these mapping spaces (6) are the *based* and the *free* loop space, respectively, cf. (20):

$$
\Omega A \subset \mathcal{L}A\,. \tag{8}
$$

Forming mapping spaces is covariantly functorial in the second variable, and forming mapping spaces into a pullback (1) yields again a pullback:

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}
\;\Rightarrow\;
\begin{tikzcd}
\mathrm{Map}^{(*)}(X,A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Map}^{(*)}(X,A) \arrow[d] \\
\mathrm{Map}^{(*)}(X,B') \arrow[r] & \mathrm{Map}^{(*)}(X,B)\,.
\end{tikzcd}
\tag{9}
$$

Dually, forming mapping space is contravariantly functorial in the first variable (reverses the direction of maps), and forming mapping spaces out of a pushout (3) gives a pullback (1):

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(po)}"] & A \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}
\;\Rightarrow\;
\begin{tikzcd}
\mathrm{Map}^{(*)}(A',X) & \mathrm{Map}^{(*)}(A,X) \arrow[l] \\
\mathrm{Map}^{(*)}(B',X) \arrow[u] \arrow[ur, phantom, "\text{(pb)}"] & \mathrm{Map}^{(*)}(B,X)\,. \arrow[l] \arrow[u]
\end{tikzcd}
\tag{10}
$$

The *fundamental group* of a pointed space A is the connected components of its based loop space (8):

$$
\pi_1(A) \;\equiv\; \pi_0(\Omega A)\,, \tag{11}
$$

and iteratively so for the higher *homotopy groups*:

$$
\pi_n(A) \;\equiv\; \pi_0(\Omega^n A) \;\equiv\; \pi_0(\underbrace{\Omega \cdots \Omega A}_{n\text{fold loop space}})\,, \tag{12}
$$

which implies in particular that

$$
\pi_n\big(\Omega^k A\big) \;=\; \pi_{n+k}(A)\,. \tag{13}
$$

For example, the non-torsion homotopy groups of the 2-sphere are

$$
\pi_2(S^2) \simeq \mathbb{Z}\,, \quad \pi_3(S^2) \simeq \mathbb{Z}\,. \tag{14}
$$

A map $f : X \longrightarrow Y$ induces a map of connected components π_0 as well as homomorphisms between all homotopy groups π_n (12) for all basepoints $x \in X$ and induced basepoints $f(x) \in Y$:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
\pi_0(X) & \xrightarrow{\ \pi_0(f)\ } & \pi_0(Y) \\
\pi_n(X,x) & \xrightarrow{\ \pi_n(f)\ } & \pi_n\big(Y, f(x)\big)\,.
\end{array}
\tag{15}
$$

If $\pi_0(f)$ is a bijection and $\pi_n(f,x)$ is an isomorphism for all n and x, then f is called a *weak homotopy equivalence*, to be denoted "$\xrightarrow{\sim}$" and X is said to be (weak homotopy) *equivalent* to Y, denoted

$$
\exists\big(\, f : X \xrightarrow{\sim} Y \,\big) \quad \Leftrightarrow \quad X \sim Y\,.
\tag{16}
$$

A map $f : X \longrightarrow Y$ is a (Serre-)*fibration*, to be denoted "$\twoheadrightarrow$", if all families of paths continuously parameterized by higher dimensional disks D^n may be lifted through f for arbitray lifts of their starting points:

$$
\begin{tikzcd}
D^n \times \{0\} \arrow[r, "\forall"] \arrow[d, hook] & X \arrow[d] \\
D^n \times [0,1] \arrow[r, "\forall"'] \arrow[ur, dashed, "\exists"] & Y\,.
\end{tikzcd}
\tag{17}
$$

For $A \overset{p}{\twoheadrightarrow} B$ a fibration (17) of pointed spaces with fiber F (2), there is induced a long exact sequence of homotopy groups (12) of the form:

$$
\begin{tikzcd}
F \arrow[r, "\mathrm{fib}_p"] & A \arrow[r, two heads, "p"] & B \\
\cdots \arrow[r, dotted] & \pi_3(S) \arrow[r, "\pi_3\, p"] & \pi_3(B) \arrow[dll, "\delta_2"] \\
\pi_2\, F \arrow[r, "\pi_2\, \mathrm{fib}_p"] & \pi_1 A \arrow[r, "\pi_2\, p"] & \pi_2\, B \arrow[dll, "\delta_1"] \\
\pi_1\, F \arrow[r, "\pi_1\, \mathrm{fib}_p"] & \pi_1 A \arrow[r, "\pi_1\, p"] & \pi_1\, B \arrow[dll, "\delta_0"] \\
\pi_0\, F \arrow[r, "\pi_0\, \mathrm{fib}_p"] & \pi_0 A \arrow[r, "\pi_0\, p"] & \pi_0\, B\,.
\end{tikzcd}
\tag{18}
$$

These long exact sequences are *natural* in that a commuting square of pointed maps induces a commuting ladder of homotopy groups:

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_{n+1}\, B' \arrow[r] \arrow[d] & \pi_{n+1}\, B \arrow[d] \\
\pi_n\, F' \arrow[r] \arrow[d] & \pi_n\, F \arrow[d] \\
\pi_n\, A' \arrow[r] \arrow[d] & \pi_n\, A \arrow[d] \\
\pi_n\, B' \arrow[r] & \pi_n\, B\,.
\end{tikzcd}
\tag{19}
$$

For example, the based loop space (8) is the fiber of the point-evaluation map of its free loop space

$$
\underbrace{\mathrm{Map}^*(S^1, A)}_{\Omega A} \xrightarrow{\ \mathrm{fib}_{\mathrm{ev}}\ } \underbrace{\mathrm{Map}(S^1, A)}_{\mathcal{L}A} \underset{\mathrm{cnst}}{\overset{\mathrm{ev}}{\rightleftarrows}} A
\tag{20}
$$

(as well as the fiber of the *based path space fibration*

$$
\underbrace{\mathrm{Map}^*(S^1, A)}_{\Omega A} \xrightarrow{\ \mathrm{fib}_{\mathrm{ev}_1}\ } \mathrm{Map}^*\big(D^1, A\big) \overset{\mathrm{ev}_1}{\twoheadrightarrow} A\,,
\tag{21}
$$

where in the second case $* := 0 \in D^1$ is one endpoint of the interval, while $1 \in D_1$ is the other), and due to the section cnst (20) the induced homotopy long exact sequence (18) splits to yield:

$$
\pi_n \mathcal{L} A \simeq \pi_n \Omega A \times \pi_n A
\tag{22}
$$

(where the base point of the loop spaces is the loop constant on the base points of A).

For instance, for $A = S^2$ this gives, with (14) and (13):

$$
\begin{tikzcd}
\pi_1 \Omega S^2 \arrow[r, "\pi_1 \mathrm{fib}_{\mathrm{ev}}"] \arrow[d, equal] & \pi_1 \mathcal{L} S^2 \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z}\,,
\end{tikzcd}
$$

$$
\begin{tikzcd}
\pi_2 \Omega S^2 \arrow[r, "\pi_2 \mathrm{fib}_{\mathrm{ev}}"] \arrow[d, equal] & \pi_2 \mathcal{L} S^2 \arrow[r, shift right, "\pi_2\, \mathrm{ev}"'] \arrow[d, equal] & \pi_2 S^2 \arrow[l, shift right, "\pi_2\, \mathrm{cnst}"'] \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"] & \mathbb{Z}^2 \arrow[r, shift right, "{[\,0\,1\,]}"'] & \mathbb{Z}\,. \arrow[l, shift right, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"']
\end{tikzcd}
\tag{23}
$$

Note in this context that from long exact sequences of groups there are induced short exact sequences by *truncation*

$$
\begin{array}{l}
A_{-2} \xrightarrow{\ f_{-2}\ } A_{-1} \xrightarrow{\ f_{-1}\ } A_0 \xrightarrow{\ f_1\ } A_1 \\
1 \longrightarrow A_{-1}/A_{-2} \xrightarrow{\ f_{-1}\ } A_0 \xrightarrow{\ f_1\ } \mathrm{im}(f_1) \longrightarrow 1\,.
\end{array}
\tag{24}
$$

More generally, a commuting diagram of spaces is said to be a *homotopy pullback* (hpb) if there exists a factoring via weak homotopy equivalences through an ordinary pullback of a fibration:

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{(hpb)}"] & A \arrow[d, "f"] \\
B' \arrow[r] & B
\end{tikzcd}
\quad \Leftrightarrow \quad \exists \quad
\begin{tikzcd}
A \arrow[r] \arrow[d, dashed, "\wr"] \arrow[ddd, bend right, "f'"'] & A \arrow[d, "\wr"] \arrow[ddd, bend left, "f"] \\
\widehat{A'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{A} \arrow[d, two heads] \\
B' \arrow[r] & B \\
B' \arrow[r] & B
\end{tikzcd}
\tag{25}
$$

For example a general *homotopy fiber* F_b of a map $F \longrightarrow B$ is the homotopy pullback of that map to the given base point b:

$$
\begin{tikzcd}
F_b \arrow[r, "\mathrm{hfib}"] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & F \arrow[d] \\
* \arrow[r, "b"] & B\,,
\end{tikzcd}
\tag{26}
$$

and a based loop space (8) is equivalently the homotopy pullback of the base point inclusion along itself:

$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & * \arrow[d] \\
* \arrow[r] & X\,,
\end{tikzcd}
\tag{27}
$$

while a product is just a homotopy pullback from the point:

$$
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & Y \arrow[d] \\
X \arrow[r] & *\,.
\end{tikzcd}
\tag{28}
$$

Homotopy pullbacks (25) are invariant, up to equivalence, under homotopy (5). For example the space of x_0-based loops in a connected space X is equivalent to the space of paths (maps $D^1 \longrightarrow X$) whose endpoints are pinned to a pair of not necessarily coincident points $x_0, x_1 \in X$:

$$
\begin{tikzcd}
\mathrm{Map}^{*_0,*_1}\big(D^1, X\big) \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & \mathrm{Map}^{*_0}\big(D^1, X\big) \arrow[dd, "\mathrm{ev}_1"] \\
& \Omega_{x_0} X \arrow[ur, bend right] \arrow[d] & \\
* \arrow[r, equal] \arrow[dr, equal] & * \arrow[r, "x_1"] & X\,. \\
& * \arrow[ur, "x_0"'] &
\end{tikzcd}
\tag{29}
$$

Moreover, homotopy pullbacks (25) satisfy the *pasting law* saying that if in a commuting diagram of the form

$$
\begin{tikzcd}
A'' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & A \arrow[d] \\
B'' \arrow[r] & B' \arrow[r] & B\,.
\end{tikzcd}
\tag{30}
$$

the right square is a homotopy pullback, then the left square is so if and only if the total rectangle is.

For example, the left square in the following diagram is a homotopy pullback:

$$
\begin{tikzcd}
* \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & * \arrow[d] \\
X \arrow[r, "\mathrm{cnst}"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathcal{L}X \arrow[r, "\mathrm{ev}"] & X\,,
\end{tikzcd}
\tag{31}
$$

because the right square is so by (20) and the total rectangle is so because the (homotopy) pullback of an identity is (equivalent to) an identity.

B. Equivariant homotopy theory

All of the above generalizes to spaces equipped with an *action* $G \circlearrowright X$ by a finite group G (regarded as a discrete topological group), given by maps

$$
\begin{array}{ccc}
G \times X & \longrightarrow & X \\
(g, x) & \longmapsto & g \cdot x
\end{array}
\tag{32}
$$

such that

$$
\forall_{\substack{g_i \in G \\ x \in X}} \begin{cases} \mathrm{e} \cdot x = x \\ (g_2 \cdot g_1) \cdot x = g_2 \cdot (g_1 \cdot x)\,. \end{cases}
$$

A basic example of G-spaces are the (discrete) coset spaces G/H for subgroups $H \subset G$ (with the action given by left multiplication in G). These are the canonical *orbits* of points under G, in that for a point $x \in X$ with *stabilizer* or *isotropy* group

$$
\mathrm{Stab}(x) := \big\{ g \in G \,\big|\, g \cdot x = x \big\} \subset G
$$

the *G-orbit* of all images of x under the action of G is G-equivariantly identified with the $G/\mathrm{Stab}(X)$.

Pullbacks (1) and pushouts (3) along G-equivariant maps are constructed as for the underlying plain maps and inherit unique G-space structure (cf. [3, Lem 1.1.10]).

For example, a *G-cell attachment* to a G-space $G \circlearrowright X$ is like a plain cell attachment (4) but now by *G-orbits* of cells:

$$
\begin{tikzcd}
S^{n-1} \times G/H \arrow[r, "f", "\text{equiv. attaching map}"'] \arrow[d, hook] \arrow[dr, phantom, "\text{(po)}"] & X \arrow[d] \\
D^n \times G/H \arrow[r] & \big(D^n \times G/H\big) \cup_f X\,.
\end{tikzcd}
\tag{33}
$$

Constructing a space by iterative G-cell attachments starting with $S^{-1} \times G/H \equiv \varnothing$ means to give it a *G-cell complex structure* or a *G-cell decomposition* (cf. [3, Ex. 1.3.6]).

For example, the G/G-orbits of cells in a G-space are its *G-fixed points* which in crystallography, with G a crystallographic point group, are known as the *high symmetry points.*

In equivariantly mapping out of an equivariant cell complex, note that, for $X \in \mathrm{Top}^{(*)}$ a (pointed) space with trivial G-action and $G \circlearrowright Y$ any (pointed) G-space:

- G-Equivariant maps out of a free action are equivalently ordinary maps on the components of the neutral element:

$$
\mathrm{Map}\big(X \times G/1\,,\, Y\big)^G \simeq \mathrm{Map}\big(X, Y\big)\,. \tag{34}
$$

- G-Equivariant maps out of a trivial action are equivalently maps to the G-fixed locus $(-)^G$ (36) inside the domain space:

$$
\mathrm{Map}\big(X \times G/G\,,\, Y\big)^G \simeq \mathrm{Map}\big(X, Y^G\big)\,. \tag{35}
$$

Here, for a G-space $G \circlearrowright X$ and subgroup $H \subset G$, the *H-fixed locus* is the subspace

$$
X^H := \big\{ x \in X \,\big|\, \forall_{g \in G}\; g \cdot x = x \big\}\,. \tag{36}
$$

For example, given (pointed) G-spaces $G \circlearrowright X$ and $G \circlearrowright Y$, the (pointed) mapping space (6) of their underlying spaces naturally carries the G-*conjugation action*

$$
\begin{array}{ccc}
G \times \mathrm{Map}^{(*)}(X,Y) & \longrightarrow & \mathrm{Map}^{(*)}(X,Y) \\
(g,f) & \longmapsto & g \circ f \circ g^{-1}
\end{array}
$$

and the subspace of G-equivariant maps is the corresponding G-fixed locus (36)

$$
\mathrm{Map}(X,Y)^G \subset \mathrm{Map}(X,Y)\,. \tag{37}
$$

Passage to G-fixed loci (36) preserves pullbacks (1):

$$
\begin{tikzcd}
G \circlearrowright A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & G \circlearrowright A \arrow[d] \\
G \circlearrowright B' \arrow[r] & G \circlearrowright B
\end{tikzcd}
\;\Rightarrow\;
\begin{tikzcd}
A'^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & A^G \arrow[d] \\
B'^G \arrow[r] & B^G\,.
\end{tikzcd}
\tag{38}
$$

In particular, with (37) and (10) this implies that forming equivariant maps out of an equivariant pushout yields a pullback:

$$
\begin{array}{c}
\begin{tikzcd}
G \circlearrowright A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{po})"] & G \circlearrowright A \arrow[d] \\
G \circlearrowright B' \arrow[r] & G \circlearrowright B
\end{tikzcd}
\\
\Downarrow
\\
\begin{tikzcd}
\mathrm{Map}^{(*)}(A',X)^G & \mathrm{Map}^{(*)}(A,X)^G \arrow[l] \\
\mathrm{Map}^{(*)}(B',X)^G \arrow[u] & \mathrm{Map}^{(*)}(B,X)^G\,. \arrow[l] \arrow[u] \arrow[ul, phantom, "\scriptstyle(\mathrm{pb})"]
\end{tikzcd}
\end{array}
\tag{39}
$$

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