Finite-dimensional exceptional tangent spaces. We ask to which extent the bosonic part of **the M-algebra** [DF82, §6][To95, (13)][Se97]

$$
\begin{aligned}
\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \;&\simeq\; \mathbf{11} \oplus \mathbf{55} \oplus \mathbf{462} \qquad\qquad \in \mathrm{Rep}_{\mathbb{R}}(\mathfrak{so}_{1,10}) \\
&\simeq\; \mathbb{R}^{1,10} \oplus \wedge^2(\mathbb{R}^{1,10})^* \oplus \wedge^5(\mathbb{R}^{1,10})^*
\end{aligned}
\tag{1}
$$

may serve (as suggested by [Va07][Ba17][FSS20]) as the bosonic typical fiber of an "exceptional tangent bundle" (cf. [BB20, (4.4)]) for *full* reduction of 11d SuGra on fibers of dimensions $n \in \{0, \cdots, 11\}$.

We first observe – in the spirit of [Hull98, (2.12)] – that, by Hodge dualizing temporal components of the M-brane charges, the M-algebra (1) is:

$$
\simeq \;\; \mathbb{R}^{1,0} \oplus \Big(\mathbb{R}^{10} \oplus \wedge^2(\mathbb{R}^{10})^* \oplus \wedge^5(\mathbb{R}^{10})^* \oplus \wedge^6\mathbb{R}^{10} \oplus \wedge^9\mathbb{R}^{10}\Big)\,. \tag{2}
$$

Moreover, restricting the term in parenthesis along $\mathbb{R}^n \hookrightarrow \mathbb{R}^{10}$ for $n \in \{4,5,6,7\}$, this is just the typical fiber of the exceptional tangent bundles of [Hull07, §4].

Higher n were disregarded before, because the dimensions of the basic reps of $\mathfrak{e}_{n(n)}$ no longer matched.

But we next observe that for $n \geq 8$ the full exceptional tangent space instead carries the basic rep of the **maximal compact subalgebra** $\mathfrak{k}_{n(n)} \subset \mathfrak{e}_{n(n)}$ (the top & left part of the following table is essentially [dWN01, (2)]; the general observation that $\mathfrak{k}_{n(n)}$ has non-trivial finite dimensional irreps is expanded on in [KKLN22]):

n	$\dim\big(\mathbb{R}^n \oplus \wedge^2(\mathbb{R}^n)^* \oplus \wedge^5(\mathbb{R}^n)^* \oplus \wedge^6\mathbb{R}^n \oplus \wedge^9\mathbb{R}^n\big)$		basic rep	of (max cmpt sub-alg of)		excptnl Lie alg
4	$4 + 6$	$\rightsquigarrow$	$\mathbf{10}$			$\mathfrak{sl}_{5(5)}$
5	$5 + 10 + 1$	$\rightsquigarrow$	$\mathbf{16}$			$\mathfrak{so}_{5,5}$
6	$6 + 15 + 6 + 1$	$\rightsquigarrow$	$\mathbf{27} \oplus \mathbf{1}$			$\mathfrak{e}_{6(6)}$
7	$7 + 21 + 21 + 7$	$\rightsquigarrow$	$\mathbf{56}$			$\mathfrak{e}_{7(7)}$
8	$8 + 28 + 56 + 28$	$\rightsquigarrow$	$\mathbf{120}$	$\mathfrak{so}_{16}$	$\subset$	$\mathfrak{e}_{8(8)}$
9	$9 + 36 + 126 + 84 + 1$	$\rightsquigarrow$	$\mathbf{256}$	$\mathfrak{k}_{9(9)}$	$\subset$	$\mathfrak{e}_{9(9)}$
10	$10 + 45 + 252 + 210 + 10$	$\rightsquigarrow$	$\mathbf{527}$	$\mathfrak{k}_{10(10)}$	$\subset$	$\mathfrak{e}_{10(10)}$
1+10	$\dim\big(\mathbb{R}^{1,10} \oplus \wedge^2(\mathbb{R}^{1,10})^* \oplus \wedge^5(\mathbb{R}^{1,10})^*\big)$	$\rightsquigarrow$	$\mathbf{528}$	$\mathfrak{k}_{11(11)}$	$\subset$	$\mathfrak{e}_{11(11)}$

(3)

Here we used:

- $\mathbf{n = 4, 5, 6, 7}$:
  classical, e.g. [dWN01, §2][Hull07, §4]
- $\mathbf{n = 8}$:
  the $\mathbf{248}$ of $\mathfrak{e}_{8(8)}$ branches as $\mathbf{120} \oplus \mathbf{128}$ of the maximal compact $\mathfrak{so}_{16}$ —
  as a representation-theoretic statement this is classical (e.g. [HS14c, p. 4]))
  but as part of a change in pattern from $\mathfrak{e}_{n(n)}$ to $\mathfrak{k}_{n(n)}$ this may not have been appreciated
  ([dWN01, p. 3] instead sees it as a partial change of pattern to the automorphism algebra of the extended susy algebra).

and then the following more novel facts (all due to the exceptional AEI group in Potsdam):

- $\mathbf{n = 9}$:
  the (infinite-dimensional) basic rep of $\mathfrak{e}_{9(9)}$ branches as $\mathbf{256} \oplus$ higher-parabolic-levels under $\mathfrak{k}_{9(9)}$ —
  this is just very recently discussed in [Kö24, pp 38, 41, 42]
- $\mathbf{n = 10}$:
  remarkably, there is an irrep $\mathbf{527}$ of $\mathfrak{k}_{10(10)}$: it appears in the symmetric square of a spinorial $\mathbf{32}$ irrep [dBHP05, §8][KN06] as:
  $\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \simeq \mathbf{1} \oplus \mathbf{527}$ [DKN06, p. 37]
  which exactly matches the interpretation here, where the bosonic dimension of the M-algebra is the same expression $\dim(\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32})$ — the remaining $\mathbf{1}$ is the first summand (the time axis) in (2)

- $\mathbf{n = 1 + 10}$:
  finally, re-including this temporal component and hence going back to the unbroken bosonic M-algebra (1) we need an irrep **528** of $\mathfrak{k}_{11(11)}$
  remarkably, this also exists, by [GKP19, p. 29], and it is isomorphic to the symmetric square
  $\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \simeq \mathbf{528}$ (cf. [BKS19, §D])
  of a fermionic irrep **32** which under $\mathfrak{so}_{1,10} \hookrightarrow \mathfrak{k}_{11(11)}$ becomes the usual Majorana spinor [BKS19, p. 42] that originally entered the M-algebra in (1).

Moreover, the $\mathbf{527} \oplus \mathbf{1}$ of $\mathfrak{k}_{10(10)}$ is the branching of the **528** of $\mathfrak{k}_{11(11)}$ [Kl24].

Therefore, in summary it looks like the **528** of $\mathfrak{k}_{11(11)}$ is the root of the tower of exceptional tangent bundles (3) by the following sequence of branchings:

$$\begin{array}{ccccccc} \mathfrak{k}_{8(8)} & \hookrightarrow & \mathfrak{k}_{9(9)} & \hookrightarrow & \mathfrak{k}_{10(10)} & \hookrightarrow & \mathfrak{k}_{11(11)} \\ \mathbf{120} \oplus \cdots & \longleftarrow & \mathbf{256} \oplus \cdots & \longleftarrow & \mathbf{527} \oplus \cdots & \longleftarrow & \mathbf{528}\,. \end{array}$$

Remark: Local U-duality. It remains to ask what is lost – or gained – by having the maximal compact U-duality algebra $\mathfrak{k}_{n(n)}$ act on the exceptional tangent spaces instead of all of $\mathfrak{e}_{n(n)}$.

Here it is noteworthy that $\mathfrak{k}_{n(n)}$ is the *local symmetry* where $\mathfrak{e}_{n(n)}$ is the *global symmetry* of U-duality.

For low $n = 7$ and $n = 8$ this distinction is seen in the original constructions of U-duality covariant sugra Lagrangians with local $\mathfrak{k}(\mathfrak{e}_{7(7)}) \simeq \mathfrak{su}_8$-symmetry [CJ79][dWN86] and local $\mathfrak{k}(\mathfrak{e}_{8(8)}) \simeq \mathfrak{so}_{16}$ symmetry [Cr81, p 6][Ni87]. For $n = 9$ the fact of the maximal compact $\mathfrak{k}_{9(9)}$ playing the role of the local symmetry is observed in [KNP07, p 4]. The general statement for all n is made explicit in [KN21, p 4] [1]. In fact, the emphasis on exceptional *local* symmetries is already visible in [Ni99, p 150].

Super-space enhancement. With the bosonic part of the M-algebra thus being a plausible candidate for the full exceptional tangent space, we next ask to which extent the full M-algebra, including also its further fermionic generators [DF82, §6][BDIPV04][Se97] [2], serves as a combined exceptional- and super-geometric tangent space for M-theory ("super-exceptional geometry" as envisioned in [FSS20]).

Concretely, the extra fermionic generators of the M-algebra are of the form (e.g. [Va07, (2.35-7)])

$$\big(Z_\alpha\big)_{\alpha \in \{1,\cdots,32\}},\ \big(Z_{a\,\alpha}\big)_{\substack{\alpha \in \{1,\cdots,32\} \\ a \in \{0,\cdots 10\}}},\ \big(Z_{[a_1 \cdots a_4]\,\alpha}\big)_{\substack{\alpha \in \{1,\cdots,32\} \\ a_i \in \{0,\cdots 10\}}},$$

where the first clearly transforms as **32** – but the second cannot transform as the Γ-traceless **320** of $\mathrm{so}_{1,10}$ (since vanishing Γ-trace is incompatible with its algebra operations) hence spans a reducible $352 \simeq \mathbf{320} \oplus \mathbf{32}$-dimensional representation of $\mathfrak{so}_{1,10}$.

Interestingly, this matches the situation for $\mathfrak{k}_{11(11)}$-reps, among which is a **352** that branches to $\mathbf{320} \oplus \mathbf{32}$ of $\mathfrak{k}_{10(10)}$ [BKS19, p 42].

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[2] [Se97] considered in addition further bosonic generators for the M-algebra. [Va07] argued that these are partially gauge-redundant and in other part pathological. In any case, our discussion here suggests that we cannot well add bosonic generators on top of the **528** of $\mathfrak{e}_{11(11)}$.

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