Extended higher cup-product Chern-Simons theories

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Abstract

It is well known that the proper action functional of $(4k+3)$-dimensional $U(1)$-Chern-Simons theory including the instanton sectors is given on gauge equivalence classes of fields by the fiber integration of the cup product square of classes in degree-$(2k+2)$ differential cohomology. We first refine this statement from gauge equivalence classes to the full higher smooth moduli stack of fields, to which the higher-order-ghost BRST complex is the infinitesimal approximation. Then we generalize the refined formulation to cup product Chern-Simons theories of nonabelian and higher nonabelian gauge fields, such as the nonabelian String$^{\mathbf{c}}$-2-connections appearing in quantum-corrected 11-dimensional supergravity and M-branes [FSS12a, FSS12b]. We discuss aspects of the off-shell extended geometric pre-quantization (in the sense of extended or multi-tiered QFT) of these theories [∞Quant, Sc12b], where there is a prequantum circle k-bundle (equivalently: $(k-1)$-bundle gerbe) in each codimension k. Examples we find include moduli stacks for differential T-duality structures as well as the anomaly line bundles of higher electric/magnetic charges, such as the 5-brane charges appearing in heterotic supergravity, as line bundles with connection on the smooth higher moduli stacks field configurations.

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1 Introduction and Overview

It has become a familiar fact, known from various examples, that there should be an n-dimensional topological quantum field theory $Z_{\mathbf{c}}$ associated to the following data:

1. a *gauge group G*: a Lie group such as $U(n)$; or more generally a higher smooth group, such as the smooth *circle n-group* $\mathbf{B}^{n-1}U(1)$ or the *String 2-group* or the smooth *Fivebrane 6-group* [SSS09b, FSS10];

2. a universal characteristic class $[c] \in H^{n+1}(BG, \mathbb{Z})$ and/or its image ω in real/de Rham cohomology,

where $Z_{\mathbf{c}}$ is a G-gauge theory defined naturally over all closed oriented n-dimensional smooth manifolds Σ_n, and such that whenever Σ_n happens to be the boundary of some manifold Σ_{n+1} the action fuctional on a field configuration ϕ is given by the integral of the pullback form $\hat{\phi}^*\omega$ (made precise below) over Σ_{n+1}, for some extension $\hat{\phi}$ of ϕ. These are *Chern-Simons type* gauge theories.

Notably for G a connected and simply connected simple Lie group, for $c \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$ any integer – the "level" – and hence for $\omega = \langle -, - \rangle$ the Killing form on the Lie algebra $\mathfrak{g}$, this quantum field theory is the original and standard Chern-Simons theory introduced in [Wi89]. See [Fr95] for a comprehensive review. Familiar as this theory is, there is an interesting aspect of it that has not yet found attention, and which is an example of our constructions here.

To motivate this, it is helpful to look at the 3d Chern-Simons action functional as follows: if we write $H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ for the set of gauge equivalence classes of G-principal connections ∇ on Σ_3, then the (exponentiated) action functional of 3d Chern-Simons theory over Σ_3 is a function of sets

$$\exp(iS(-)) : H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) \to U(1).$$

Of course this function acts by picking a representative of the gauge equivalence class, given by a smooth 1-form $A \in \Omega^1(\Sigma_3, \mathfrak{g})$ and sending that to the element $\exp(2\pi i k \int_{\Sigma_3} \mathrm{CS}(A)) \in U(1)$, where $\mathrm{CS}(A) \in \Omega^3(\Sigma_3)$ is the Chern-Simons 3-form of A [CS74], that gives the whole theory its name. That this is well defined is the fact that for every gauge transformation $g : A \to A^g$, for $g \in C^\infty(\Sigma_3, G)$, both A as well as its gauge transform A^g, are sent to the same element of $U(1)$. A natural formal way to express this is to consider the *groupoid* $\mathbf{H}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ whose objects are gauge fields A and whose morphisms are gauge transformations g as above. Then the fact that the Chern-Simons action is defined on individual gauge field configurations while being invariant under gauge transformations is equivalent the statement that it is a *functor*, hence a morphism of groupoids,

$$\exp(iS(-)) : \mathbf{H}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) \to U(1),$$

where the set underlying $U(1)$ is regarded as a groupoid with only identity morphisms. Hence the fact that $\exp(iS(-))$ has to send every morphism on the left to a morphism on the right is the gauge invariance of the action.

Furthermore, the action functional has the property of being *smooth*. It takes any *smooth family* of gauge fields, over some parameter space U, to a corresponding smooth family of elements of $U(1)$ and such that these assignmens are compatible with precomposition of smooth functions $U_1 \to U_2$ between parameter spaces. The formal langauge that expresses this concept is that of *stacks on the site of smooth manifolds* (see section 2.1 below for a review and pointers to the literature): to say that for every U there is a groupoid, as above, of smooth U-families of gauge fields and smooth U-families of gauge transformations between them, in a consistent way, is to say that there is a *smooth moduli stack*, denoted $[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}]$, of gauge fields on Σ_3. Finally, the fact that the Chern-Simons action functional is not only gauge invariant but also smooth is the fact that it refines to a morphism of smooth stacks

$$\exp(i\mathbf{S}(-)) : [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \to U(1),$$

where now $U(1)$ is regarded as a smooth stack by declaring that a smooth family of elements is a smooth function with values in $U(1)$.

It is useful to think of a smooth stack simply as being a *smooth groupoid*. Lie groups and Lie groupoids are examples (and are called "differentiable stacks" when regarded as special cases of smooth stacks) but there are important smooth groupoids which are not Lie groupoids in that they have not a smooth *manifold* but a more general smooth space of objects and of morphisms. Just as Lie groups have an infinitesimal approximation given by Lie algebras, so smooth stacks/smooth groupoids have an infinitesimal approximation given by *Lie algebroids*. The smooth moduli stack $[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}]$ of gauge field configuration on Σ_3 is best known in the physics literature in the guise of its underlying Lie algebroid: this is the formal dual of the (off-shell) *BRST complex* of the G-gauge theory on Σ_3: in degree 0 this consists of the functions on the space of gauge fields on Σ_3, and in degree 1 it consists of functions on infinitesimal gauge transformations between these: the "ghost fields".

The smooth structure on the action functional is of course crucial in field theory: in particular it allows to define the *differential* $d\exp(i\mathbf{S}(-))$ of the action functional and hence its critical locus, characterized by the Euler-Lagrange equations of motion. This is the *phase space* of the theory, which is a substack

$$[\Sigma_2, \flat \mathbf{B}G] \hookrightarrow [\Sigma_2, \mathbf{B}G_{\mathrm{conn}}]$$

equipped with a presymplectic 2-form. To formalize this, write $\Omega^2_{\mathrm{cl}}(-)$ for the smooth stack of closed 2-forms (without gauge transformations), hence the rule that sends a parameter manifold U to the set $\Omega^2_{\mathrm{cl}}(U)$ of smooth closed 2-forms on U. This may be regarded as the *smooth moduli 0-stack* of closed 2-forms in that for every smooth manifold X the set of morphisms $X \to \Omega^2_{\mathrm{cl}}(-)$ is in natural bijection to the set $\Omega^2_{\mathrm{cl}}(X)$ of closed 2-forms on X. This is an instance of the *Yoneda lemma*. Similarly, a smooth 2-form on the moduli stack of field configurations is a morphism of smooth stacks of the form

$$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \to \Omega^2_{\mathrm{cl}}(-)\,.$$

Explicitly, for Chern-Simons theory this morphism sends for each smooth parameter space U a given smooth U-family of gauge fields $A \in \Omega^1(\Sigma_2 \times U, \mathfrak{g})$ to the 2-form

$$\int_{\Sigma_2} \langle d_U A \wedge d_U A \rangle \in \Omega^2_{\mathrm{cl}}(U)\,.$$

Notice that if we restrict to *genuine* families A which are functions of U but vanish on vectors tangent to U (technically these are elements in the *concretification* of the moduli stack) then this 2-form is the *fiber integral* of the Poincaré 2-form $\langle F_A \wedge F_A \rangle$ along the projection $\Sigma_2 \times U \to U$, where $F_A := dA + \frac{1}{2}[A \wedge A]$ is the curvature 2-form of A. This is the first sign of a general pattern, which we highlight in a moment.

There is more fundamental smooth moduli stack equipped with a closed 2-form: the moduli stack $\mathbf{B}U(1)_{\mathrm{conn}}$ of $U(1)$-gauge fields, hence of smooth circle bundles with connection. This is the rule that sends a smooth parameter manifold U to the groupoid $\mathbf{H}(U, \mathbf{B}U(1)_{\mathrm{conn}})$ of $U(1)$-gauge fields ∇ on U, which we have already seen above. Since the curvature 2-form $F_\nabla \in \Omega^2_{\mathrm{cl}}(U)$ of a $U(1)$-principal connection is gauge invariant, the assignment $\nabla \mapsto F_\nabla$ gives rise to a morphism of smooth stacks of the form

$$F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \to \Omega^2_{\mathrm{cl}}(-)\,.$$

In terms of this morphism the fact that every $U(1)$-gauge field ∇ on some space X has an underlying field strength 2-form ω is expressed by the existence of a commuting diagram of smooth stacks of the form

Conversely, if we regard the bottom morphism ω as given, and regard this closed 2-form as a (pre)symplectic form, then a *choice of lift* ∇ in this diagram is a choice of refinement of the 2-form by a circle bundle with

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connection, hence the choice of a *prequantum circle bundle* in the language of geometric quantization (see for instance section II in [Br93] for a review of geometric quantization).

Applied to the case of Chern-Simons theory this means that a smooth (off-shell) prequantization of the theory is a choice of dashed morphism in a diagram of smooth stacks of the form

$$
\begin{array}{ccc}
 & & \mathbf{B}U(1)_{\mathrm{conn}} \\
 & \nearrow & \downarrow {\scriptstyle F_{(-)}} \\
[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] & \xrightarrow[\int_{\Sigma_2} \langle F_{(-)}, F_{(-)} \rangle]{} & \Omega^2_{\mathrm{cl}}(-)
\end{array} \quad .
$$

Similar statements apply to on-shell geometric (pre)quantization of Chern-Simons theory, which has been so successfully applied in the original article [Wi89]. In summary, this means that in the context of smooth stacks the Chern-Simons action functional and its prequantization are as in the following table:

dimension		moduli stack description
$k = 3$	action functional (0-bundle)	$\exp(i\mathbf{S}(-)) : [\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \to U(1)$
$k = 2$	prequantum circle 1-bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}U(1)_{\mathrm{conn}}$

There is a precise sense, discussed in section 2.3 below, in which a $U(1)$-valued function is a *circle k-bundle with connection* for $k = 0$. If we furthermore regard an ordinary $U(1)$-principal bundle as a *circle 1-bundle* then this table says that in dimension k Chern-Simons theory appears as a *circle $(3 - k)$-bundle with connection* – at least for $k = 3$ and $k = 2$.

Formulated this way, it should remind one of what is called *extended* or *multi-tiered* topological quantum field theory (formalized and classified in [Lu09a]) which is the full formalization of *locality* in the Schrödinger picture of quantum field theory. This says that *after quantization*, an n-dimensional topological field theory should be a rule that to a closed manifold of dimension k assigns an $(n-k)$-categorical analog of a vector space of quantum states. Since ordinary geometric quantization of Chern-Simons theory assigns to a closed Σ_2 the vector space of *polarized sections* (holomorphic sections) of the line bundle associated to the above circle 1-bundle, this suggests that there should be an *extended* or *multi-tiered* refinement of geometric (pre)quantization of Chern-Simons theory, which to a closed oriented manifold of dimension $0 \leq k \leq n$ assigns a *prequantum circle $(n - k)$-bundle* (bundle $(n - k - 1)$-gerbe) on the moduli stack of field configurations over Σ_k, modulated by a morphism $[\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}^{(n-k)}U(1)_{\mathrm{conn}}$ to a moduli $(n - k)$-stack of circle $(n - k)$-bundles with connection (details on this are below in section 2.3).

In particular for $k = 0$ and Σ_0 connected, hence $\Sigma_0 = *$ the point, we have that the moduli stack of fields on Σ_0 is the *universal* moduli stack itself, $[*, \mathbf{B}G_{\mathrm{conn}}] \simeq \mathbf{B}G_{\mathrm{conn}}$, and so a *fully extended prequantization* of 3-dimensional G-Chern-Simons theory would have to involve a *universal characteristic* morphism

$$
\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}
$$

of smooth moduli stacks, hence a smooth circle 3-bundle with connection on the universal moduli stack of G-gauge fields. This indeed naturally exists: an explicit construction is given in [FSS10]. This morphism of smooth higher stacks is a differential refinement of a smooth refinement of the level itself: forgetting the connections and only remembering the underlying (higher) gauge bundles, we still have a morphism of smooth higher stacks

$$
\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3 U(1) \,.
$$

This expression should remind one of the continuous map of topological spaces

$$
c : BG \to B^3 U(1) \simeq K(\mathbb{Z}, 4)
$$

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from the classifying space BG to the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$, which represents the level as a class in integral cohomology $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. Indeed, there is a canonical *derived functor* or ∞-*functor*

$$|-| : \mathbf{H} \to \mathrm{Top}$$

from smooth higher stacks to topological spaces [Sc12], derived left adjoint to the operation of forming *locally constant higher stacks*, and under this map we have

$$|\mathbf{c}| \simeq c \,.$$

In this sense $\mathbf{c}$ is a *smooth refinement* of $[c] \in H^4(BG, \mathbb{Z})$ and then $\mathbf{c}_{\mathrm{conn}}$ is a further *differential refinement* of $\mathbf{c}$.

However, more is true. Not only is there an extension of the prequantization of 3d G-Chern-Simons theory to the point, but this also induces the extended prequantization in every other dimension by *tracing*: for $0 \leq k \leq n$ and Σ_k a closed and oriented smooth manifold, there is a canonical morphism of smooth higher stacks of the form

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}] \to \mathbf{B}^{n-k} U(1)_{\mathrm{conn}} \,,$$

which refines the fiber integration of differential forms, that we have seen above, from curvature $(n+1)$-forms to their entire prequantum circle n-bundles (we discuss this below in section 2.4). Since, furthermore, the formation of mapping stacks $[\Sigma_k, -]$ is functorial, this means that from a morphism $\mathbf{c}_{\mathrm{conn}}$ as above we get for every Σ_k a composite morphism as such:

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) : \ [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_k, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k} U(1)_{\mathrm{conn}} \,.$$

For 3d G-Chern-Simons theory and $k = n = 3$ this composite *is* the action functional of the theory (down on the set $H(\Sigma_3, \mathbf{B}G_{\mathrm{conn}})$ this is effectively the perspective on ordinary Chern-Simons theory amplified in [CJMSW05]). Therefore, for general k we may speak of this as the *extended action functional*, with values not in $U(1)$ but in $\mathbf{B}^{n-k} U(1)_{\mathrm{conn}}$.

This way we find that the above table, containing the Chern-Simons action functional together with its prequantum circle 1-bundle, extends to the following table that reaches all the way from dimension 3 down to dimension 0.

dim.		prequantum $(3-k)$-bundle	
$k = 0$	differential fractional first Pontrjagin	$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$	[FSS10]
$k = 1$	WZW background B-field	$[S^1, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[S^1, \mathbf{c}_{\mathrm{conn}}]} [S^1, \mathbf{B}^3 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2 U(1)_{\mathrm{conn}}$	[∞WZW]
$k = 2$	off-shell CS prequantum bundle	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_2, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_2, \mathbf{B}^3 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} \mathbf{B}U(1)_{\mathrm{conn}}$	[Sc12]
$k = 3$	3d CS action functional	$[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_3, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_3, \mathbf{B}^3 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$	[FSS10]

For each entry of this table one may compute the *total space* object of the corresponding prequantum k-bundle. This is now in general itself a higher moduli stack. In full codimension $k = 0$ one finds [∞Quant] that this is the moduli 2-stack of $\mathrm{String}(G)$-2-connections described in [SSS09b, FSS12b]. This we discuss in section 3.1.2 below.

It is clear now that this is just the first example of a general class of theories which we may call *higher extended prequantum Chern-Simons theories* or just ∞-*Chern-Simons theories*, for short [∞CS]. These are defined by a choice of

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1. a smooth higher group G;

2. a smooth universal characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$;

3. a differential refinement $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$.

An example of a 7-dimensional such theory on String-2-form gauge fields is discussed in [FSS12a], given by a differential refinement of the second fractional Pontrjagin class to a morphism of smooth moduli 7-stacks

$$\tfrac{1}{6}(\mathbf{p}_2)_{\mathrm{conn}} : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}} \,.$$

We expect that these ∞-Chern-Simons theories are part of a general procedure of *extended geometric quantization* (*multi-tiered* geometric quantization) which proceeds in two steps, as indicated in the following table.

classical system	geometric prequantization	quantization
char. class c of deg. $(n+1)$ with de Rham image ω: invariant polynomial/ n-plectic form	prequantum circle n-bundle on moduli ∞-stack of fields $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$	extended quantum field theory $Z_{\mathbf{c}} : \Sigma_k \mapsto \left\{ \begin{array}{l} \text{polarized sections of} \\ \text{prequantum } (n-k)\text{-bundle} \\ \exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) \end{array} \right\}$

Here we are concerned with the first step, the discussion of n-dimensional Chern-Simons gauge theories (higher gauge theories) in their incarnation as prequantum circle n-bundles on their universal moduli ∞-stack of fields. A dedicated discussion of higher geometric prequantization, including the discussion of higher Heisenberg groups, higher quantomorphism groups, higher symplectomorphisms and higher Hamiltonian vector fields, and their action on higher prequantum spaces of states by higher Heisenberg operators, is in [∞Quant], see also [Sc12b]. As shown there, plenty of interesting physical information turns out to be captured by extended prequantum n-bundles. For instance, if one regards the B-field in type II superstring backgrounds as a prequantum 2-bundle, then its extended prequantization knows all about twisted Chan-Paton bundles, the Freed-Witten anomaly cancellation condition for type II superstrings on D-branes and the associated anomaly line bundle on the string configuration space.

Generally, all higher Chern-Simons theories that arise from extended action functionals this way enjoy a collection of very good formal properties. Effectively, they may be understood as constituting examples of a fairly extensive generalization of the *refined* Chern-Weil homomorphism with coefficients in *secondary characteristic cocycles*. Moreover, we have shown previously that the class of theories arising this way is large and contains not only several familiar theories, some of which are not traditionally recognized to be of this good form, but also contains various new QFTs that turn out to be of interest within known contexts, e.g. [FSS12a, FSS12b]. Here we further enlarge the pool of such examples.

Notably, here we are concerned with examples arising from *cup product* characteristic classes, hence of ∞-Chern-Simons theories which are decomposable or non-primitive secondary characteristic cocycles, obtained by cup-ing more elementary characteristic cocycles. The most familiar example of these is again ordinary 3-dimensional Chern-Simons theory, but now for the non-simply connected gauge group $U(1)$. In this case a gauge field configuration in $\mathbf{H}(\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}})$ is not necessarily given by a globally defined 1-form $A \in \Omega^1(\Sigma_3)$, instead it may have a non-vanishing "instanton number", the Chern-class of the underlying circle bundle. Only if that happens to vanish is the value of the action functional again given by the simple expression $\exp(2\pi i k \int_{\Sigma_3} A \wedge d_{\mathrm{dR}} A)$ as before. But in view of the above we are naturally led to ask: which circle 3-bundle (bundle 2-gerbe) with connection over Σ_3, depending naturally on the $U(1)$-gauge field, has $A \wedge d_{\mathrm{dR}} A$ as its connection 3-form in this special case, so that the correct action functional in generality is again the *volume holonomy* of this 3-bundle (see section 3.1.1 below)? The answer is that it is the *differential cup square* of the gauge field with itself. As a fully extended action functional this is a natural morphism of higher moduli stacks of the form

$$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \,.$$

We explain this below in section 2.3. This morphism of higher stacks is characterized by the fact that under forgetting the differential refinement and then taking geometric realization as before, it is exhibited as a differential refinement of the ordinary cup square on Eilenberg-MacLane spaces

$$(-)^{\cup^2} : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4)$$

and hence on ordinary integral cohomology. By the above general procedure, we obtain a well-defined action functional for $3d$ $U(1)$-Chern-Simons theory by the expression

$$\exp(2\pi i \int_{\Sigma_3} [\Sigma_3, (-)^{\cup^2_{\mathrm{conn}}}]) : [\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}] \to U(1)$$

and this is indeed the action functional of the familiar $3d$ $U(1)$-Chern-Simons theory, also on non-trivial instanton sectors, see section 3.2.2 below.

In terms of this general construction, there is nothing particular to the low degrees here, and we have generally a differential cup square / extended action functional for a $(4k + 3)$-dimensional Chern-Simons theory

$$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \to \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$$

for all $k \in \mathbb{N}$, which induces an ordinary action functional

$$\exp(2\pi i \int_{\Sigma_3} [\Sigma_{4k+3}, (-)^{\cup^2_{\mathrm{conn}}}]) : [\Sigma_{4k+3}, \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}] \to U(1)$$

on the moduli $(2k + 1)$-stack of $U(1)$-$(2k + 1)$-form gauge fields, given by the fiber integration on differential cocycles over the differential cup product of the fields. This is discussed in section 3.2.3 below.

Forgetting the smooth structure on $[\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}]$ and passing to gauge equivalence classes of fields yields the cohomology group $H^{2k+2}_{\mathrm{conn}}(\Sigma_{4k+3})$. This is what is known as *ordinary differential cohomology* and is equivalent to the group of *Cheeger-Simons differential characters*, a review with further pointers is in [HS05]. That gauge equivalence classes of higher degree $U(1)$-gauge fields are to be regarded as differential characters and that the $(4k + 3)$-dimensional $U(1)$-Chern-Simons action functional on these is given by the fiber integration of the cup product is discussed in detail in [FP89], also mentioned notably in [Wi96, Wi98b] and expanded on in [Fr00]. Effectively this observation led to the general development of differential cohomology in [HS05]. Or rather, the main theorem there concerns a shifted version of the functional of $(4k+3)$-dimensional $U(1)$-Chern-Simons theory which allows to further divide it by 2. We have discussed the refinement of this to smooth moduli stacks of fields in [FSS12b]. These developments were largely motivated from the relation of $(4k+3)$-dimensional $U(1)$-Chern-Simons theories as the holographic duals to theories of self-dual forms in dimension $(4k + 2)$ (see [BM06] for survey and references): a choice of conformal structure on a Σ_{4k+2} naturally induces a polarization of the prequantum 1-bundle of the $(4k + 3)$-dimensional theory, and for every choice the resulting space of quantum states is naturally identified with the corresponding conformal blocks (correlators) of the $(4k + 2)$-dimensional theory.

Therefore we have that regarding the differential cup square on smooth higher moduli stacks as an extended action functional yields the following table of familiar notions under extended geometric prequantization.

dim.		prequantum $(4k + 3 - d)$-bundle	
$d = 0$	differential cup square	$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \to \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$	
$\vdots$	$\vdots$	$\vdots$	
$d = 4k + 2$	"pre-conformal blocks" of self-dual $2k$-form field	$[\Sigma_{4k+2}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{[\Sigma_{4k+2}, (-)^{\cup^2_{\mathrm{conn}}}]} [\Sigma_{4k+2}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_{4k+2}}(-))} \mathbf{B}U(1)_{\mathrm{conn}}$	
$d = 4k + 3$	CS action functional	$[\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{[\Sigma_{4k+3}, (-)^{\cup^2_{\mathrm{conn}}}]} [\Sigma_{4k+3}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_{4k+3}}(-))} U(1)$	

This fully extended prequantization of $(4k+3)$-dimensional $U(1)$-Chern-Simons theory allows for instance to ask for and compute the total space of the prequantum circle $(4k+3)$-bundle. This is now itself a higher smooth moduli stack. For $k = 0$, hence in $3d$-Chern-Simons theory it turns out to be the moduli 2-stack of *differential T-duality structures*. This we discuss in section 3.2.2 below.

More generally, as the name suggests, the *differential cup square* is a specialization of a general *differential cup product*. As a morphism of bare homotopy types this is the familiar cup product of Eilenberg-MacLane spaces

$$(-) \cup (-) : K(\mathbb{Z}, p+1) \times K(\mathbb{Z}, q+1) \to K(\mathbb{Z}, p+q+2)$$

for all $p, q \in \mathbb{N}$. Its smooth and then its further differential refinement is a morphism of smooth higher stacks of the form

$$(-) \cup_{\mathrm{conn}} (-) : \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^1 U(1)_{\mathrm{conn}} \to \mathbf{B}^{p+q+1} U(1)_{\mathrm{conn}} \,,$$

which, as before, we describe below in section 2.3.

By the above discussion this now defines a higher extended gauge theory in dimension $p + q + 1$ of *two different* species of higher $U(1)$-gauge fields. One example of this is the *higher electric-magnetic coupling anomaly* in higher (Euclidean) $U(1)$-Yang-Mills theory, as explained in section 2 of [Fr00]. In this example one considers on an oriented smooth manifold X (here assumed to be closed, for simplicity) an *electric current* $(p+1)$-form $J_{\mathrm{el}} \in \Omega_{\mathrm{cl}}^{p+1}(X)$ and a *magnetic current* $(q+1)$-form $J_{\mathrm{mag}} \in \Omega_{\mathrm{cl}}^{q+1}(X)$, such that $p + q = \dim(X)$ is the dimension of X. A *prequantization* of these current forms in our sense of higher geometric quantization [∞Quant] is a lift to differential cocycles

and here this amounts to electric and magnetic *charge quantization*, respectively: the electric charge is the universal integral cohomology class of the circle p-bundle underlying the electric charge cocycle: its *higher Dixmier-Douady class* $[\widehat{J}_{\mathrm{el}}] \in H_{\mathrm{cpt}}^{p+1}(X, \mathbb{Z})$ (see section 3.1.1 below); and similarly for the magnetic charge. Accordingly, the higher mapping stack $[X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}]$ is the smooth higher moduli stack of charge-quantized electric and magnetic currents on X. Recall that this assigns to a smooth test manifold U the higher groupoid whose objects are U-families of pairs of charge-quantized electric and magnetic currents, namely such currents on $X \times U$. As [Fr00] explains in terms of such families of fields, the $U(1)$-principal bundle with connection that in the present formulation is the one modulated by the morphism

$$\nabla_{\mathrm{an}} := \exp(2\pi i \int_X [X, (-) \cup_{\mathrm{conn}} (-)]) : [X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \to \mathbf{B} U(1)_{\mathrm{conn}}$$

is the *anomaly line bundle* of $(p-1)$-form electromagnetism on X, in the presence of electric and magnetic currents subject to charge quantization. In the language of ∞-Chern-Simons theory as above, this is equivalently the off-shell prequantum 1-bundle of the higher cup product Chern-Simons theories on pairs of $U(1)$-gauge p-form and q-form fields.

Regarded as an anomaly bundle, one calls its curvature the *local anomaly* and its *holonomy* the "global anomaly". In our contex the holonomy of ∇_{an} is (discussed again in section 3.1.1 below) the morphism

$$\mathrm{hol}(\nabla_{\mathrm{an}}) = \exp(2\pi i \int_{S^1} [S^1, \nabla_{\mathrm{an}}]) : [S^1, [X, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}]] \to U(1)$$

from the loop space of the moduli stack of fields to $U(1)$. By the characteristic universal propery of higher mapping stacks, together with the "Fubini-theorem"-property of fiber integration, this is equivalently the morphism

$$\exp(2\pi i \int_{X \times S^1} [X \times S^1, (-) \cup_{\mathrm{conn}} (-)]) : [X \times S^1, \mathbf{B}^p U(1)_{\mathrm{comm}} \times \mathbf{B}^q U(1)_{\mathrm{conn}}] \to U(1) \,.$$

But from the point of view of ∞-Chern-Simons theory this is the *action functional* of the higher cup product Chern-Simons field theory induced by $\cup_{\text{conn}}$. The situation is now summarized in the following table.

dim.		prequantum $(\dim(X) + 1 - k)$-bundle
$k = 0$	differential cup product	$(-)^{\cup_{\text{conn}}^2} : \mathbf{B}^p U(1)_{\text{conn}} \mathbf{B}^q U(1)_{\text{conn}} \to \mathbf{B}^{d+2} U(1)_{\text{conn}}$
$\vdots$	$\vdots$	$\vdots$
$k = \dim(X)$	higher E/M-charge anomaly line bundle	$\exp(2\pi i \int_X [X, (-) \cup_{\text{conn}} (-)]) : [X, \mathbf{B}^p U(1)_{\text{conn}} \times \mathbf{B}^q U(1)_{\text{conn}}] \longrightarrow \mathbf{B} U(1)_{\text{conn}}$
$k = \dim(X) + 1$	global anomaly	$\exp(2\pi i \int_{X \times S^1} [X \times S^1, (-) \cup_{\text{conn}} (-)]) : [X \times S^1, \mathbf{B}^p U(1)_{\text{conn}} \times \mathbf{B}^q U(1)_{\text{conn}}] \to U(1)$

These higher electric-magnetic anomaly Chern-Simons theories are of particular interest when the higher electric/magnetic currents are themselves induced by other gauge fields. Namely if we have any two ∞-Chern-Simons theories given by extended action functionals $\mathbf{c}_{\text{conn}}^1 : \mathbf{B}G_{\text{conn}}^1 \to \mathbf{B}^p U(1)_{\text{conn}}$ and $\mathbf{c}_{\text{conn}}^2 : \mathbf{B}G_{\text{conn}}^2 \to \mathbf{B}^q U(1)_{\text{conn}}$, respectively, then composition of these with the differential cup product yields an extended action functional of the form

$$\mathbf{c}_{\text{conn}}^1 \cup_{\text{conn}} \mathbf{c}_{\text{conn}}^2 : \mathbf{B}(G^1 \times G^2)_{\text{conn}} \xrightarrow{(\mathbf{c}_{\text{conn}}^1, \mathbf{c}_{\text{conn}}^2)} \mathbf{B}^p U(1)_{\text{conn}} \times \mathbf{B}^1 U(1)_{\text{conn}} \xrightarrow{(-) \cup_{\text{conn}} (-)} \mathbf{B}^{p+q+1} U(1)_{\text{conn}} \,,$$

which describes extended topological field theories in dimension $p + q + 1$ on two species of (possibly non-abelian, possibly higher) gauge fields, or equivalently describes the higher electric/magnetic anomaly for higher electric fields induced by $\mathbf{c}^1$ and higher magnetic fields induced by $\mathbf{c}^2$.

For instance for heterotic string backgrounds $\mathbf{c}_{\text{conn}}^2$ is the differential refinement of the first fractional Pontrjagin class $\frac{1}{2} p_1 \in H^4(B\text{Spin}, \mathbb{Z})$ [SSS09b, FSS10] of the form

$$\mathbf{c}_{\text{conn}}^2 = \widehat{J}_{\text{mag}}^{\text{NS5}} = \tfrac{1}{2}(\mathbf{p}_1)_{\text{conn}} : \mathbf{B}\text{Spin}_{\text{conn}} \to \mathbf{B}^3 U(1)_{\text{conn}} \,,$$

formalizing the *magnetic NS5-brane charge* needed to cancel the fermionic anomaly of the heterotic string by way of the Green-Schwarz mechanism. It is curious to observe, going back to the very first example of this introduction, that this $\widehat{J}_{\text{mag}}^{\text{NS5}}$ is at the same time the extended action functional for 3d Spin-Chern-Simons theory.

Still more generally, we may differentially cup in this way more than two factors. Examples for such *higher order cup product theories* appear in 11-dimensional supergravity. We discuss this in section 3.3. Notably plain classical 11d supergravity contains an 11-dimensional cubic Chern-Simons term whose extended action functional in our sense is

$$(-)^{\cup_{\text{conn}}^3} : \mathbf{B}^3 U(1)_{\text{conn}} \to \mathbf{B}^{11} U(1)_{\text{conn}} \,.$$

Here for X the 11-dimensional spacetime, a field in $[X, \mathbf{B}^3 U(1)]$ is a first approximation to a model for the *supergravity C-field*. If the differential cocycle happens to be given by a globally defined 3-form C, then the induced action functional $\exp(2\pi i \int_X [X, (-)^{\cup_{\text{conn}}^3}])$ sends this to element in $U(1)$ given by the familiar expression

$$\exp(2\pi i \int_X [X, (-)^{\cup_{\text{conn}}^3}]) : C \mapsto \exp(2\pi i \int_X C \wedge d_{\text{dR}} C \wedge d_{\text{dR}} C) \,.$$

More precisely this model receives quantum corrections from an 11-dimensional Green-Schwarz mechanism. In [FSS12a, FSS12b] we have discussed in detail relevant corrections to the above extended cubic cup-product action functional on the moduli stack of flux-quantized C-field configurations.

This paper is meant to be of interest to both mathematicians and theoretical/mathematical physicists. It provides some basic constructions and variations on theories that are familiar to the former, and illustrates this with reduction to explicit examples familiar to the latter. Our aim is to show and illustrate by further classes of interesting examples how Chern-Weil theory interpreted in higher geometry, hence ∞-*Chern-Simons theory*, usefully unifies a wealth of structures that are of interest both in themselves as well as in

the role they play in quantum field theory and string theory. A more general and encompassing discussion should appear in [∞Quant, ∞CS, ∞WZW].

2 General theory

In this section we describe the general formal definition and construction of higher extended cup-product Chern-Simons theories defined on their full higher moduli stacks of fields. This is the conceptual basis for the discussion of the examples below in section 3.

2.1 Smooth higher stacks

We briefly indicate the context of *smooth higher stacks* (equivalently: *smooth ∞-groupoids* or *smooth homotopy types*) in which we place our discussions of differential cohomology and extended action functionals. We initiated this approach in [SSS09b] (with an unpublished precursor set of notes [SSSS08], presented at [Sc09]), and so the reader can find more detailed surveys with emphasis on different aspects in the series of papers [SS10, FSS10, FRS11, FSS12a, FSS12a, NSS12a, NSS12b]. A comprehesive account is in [Sc12]; an introductory lecture series with emphasis on applications to string theory is in [Sc12a]. The basic idea has then also been propagated at the end of [Ho11][1], together with the statement this is the context in which the seminal article [HS05] was eventually meant to be considered. The following should serve to fix our notation and terminology for the present purpose and to give the reader unfamiliar with the details a quick idea of the conceptual background.

Higher geometry is determined by a choice of *geometric test spaces* or *affine spaces* forming a category $\mathcal{C}$ equipped with a notion of cover: a *site*. The relevant example for the present application to *higher differential geometry* is the category

$$\mathcal{C} := \mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd}$$

of *Cartesian spaces*, i.e., of finite-dimensional smooth manifolds diffeomorphic to $\mathbb{R}^n$ for some n, with smooth maps as morphisms. These form manifestly a local model for smooth manifolds. Namely, every local construction in differential geometry, i.e., every construction which can be expressed in terms of local charts, is actually a construction taking place over CartSp. More precisely, smooth manifolds are presheaves of sets over the site of Cartesian space, and precisely those that are *locally affine* with respect to CartSp. Once this point of view has been adopted, one has an immediate generalization from sets to simplicial sets, and this provides a powerful and natural language to generalize from smooth manifolds to smooth orbifolds, to smooth stacks. Every such object X determines a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{sSet}$ to the category of simplicial sets – a *simplicial presheaf* – which is to be thought of as sending a test space U in $\mathcal{C}$ to the simplicial set whose vertices are the smooth maps $U \to X$, whose edges are the smooth (orbifold-like-)transformations between two such smooth maps, and so on. Since these transformations can be composed and inverted, the simplicial set obtained this way is an *Kan complex*, a combinatorial model for an *∞-groupoid*.

We write $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$ for the category whose objects are such simplicial presheaves, and whose morphisms are natural transformations between them, and – since Kan complexes are precisely the fibrant objects in the standard model category structure on simplicial sets – we write $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}]$ for the subcategory of presheaves taking values in Kan complexes. We say a morphism $f : X \to Y$ of Kan-complex valued presheaves is a *local homotopy equivalence* if it is *stalkwise* a homotopy equivalence of Kan complexes, hence if for every manifold U and every point $x \in U$ there is a neighbourhood $x \in U_x \subset U$ such that $f(U_x) : X(U_x) \to Y(U_x)$ is a homotopy equivalence of Kan complexes. We then write

$$\mathbf{H} := \mathrm{Sh}_{\infty}(\mathcal{C}) := L_W[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}]$$

for the Kan complex-enriched category which is universal with the property that local homotopy equivalences in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}]$ become actual homotopy equivalences. For X and A any two simplicial presheaves, we

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[1]We are grateful to Alexander Kahle for pointing out this talk to us at *String-Math 2012*.

write $\mathbf{H}(X, A)$ for the resulting ∞-groupoid of morphisms between them. This construction is called the *simplicial localization* of the category of simplicial presheaves at the local homotopy equivalences $W \subset$ $\mathrm{Mor}([\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}])$. More abstractly, the resulting $\mathbf{H}$ is called the ∞-*topos* of ∞-*stacks* over $\mathcal{C}$. This is the context for higher geometry modeled on $\mathcal{C}$. Given a simplicial presheaf $X \in [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$, its image in $\mathbf{H}$ is also called its ∞-*stackification*.

Every stack on smooth manifolds is naturally an object in $\mathbf{H}$, and in particular so are *differentiable stacks* and hence Lie groupoids. A basic example is obtained from a Lie group G. This defines a presheaf of Kan complexes which sends a test manifold U to the 1-groupoid with a single object and the (discrete) group of smooth functions $C^\infty(U, G)$ as morphisms from that object to itself. We write $\mathbf{B}G \in \mathbf{H}$ for the corresponding smooth stack. This is the *moduli stack for G-principal bundles*; namely, a morphism $X \to \mathbf{B}G$ in $\mathbf{H}$ modulates a smooth G-principal bundle $P \to X$. We write

$$\mathbf{H}(X, \mathbf{B}G) \simeq G\mathrm{Bund}(X)$$

for the *cocycle groupoid* whose objects are morphisms $X \to \mathbf{B}G$ and whose morphisms are homotopies between such maps. This is equivalently the groupoid of G-principal bundles and smooth gauge transformations between these.

Generally, an ∞-stack G with group structure (up to higher homotopy: a groupal A_∞-structure) determines and is determined by a moduli ∞-stack $\mathbf{B}G$ which *modulates G-principal ∞-bundles* in this way [NSS12a]. For such a G and for $X \in \mathbf{H}$, we call $\mathbf{H}(X, \mathbf{B}G)$ the *cocycle ∞-groupoid* of G-cocycles on X. Its set of connected components is

$$H^1(X, G) := \pi_0 \mathbf{H}(X, \mathbf{B}G) \,,$$

the *degree-1 nonabelian cohomology* of X with coefficients in G. If $\mathbf{B}G$ itself again has a group structure, we may form $\mathbf{B}^2 G$, and so on. Generally, if an object A is *n-times deloopable* this way we write

$$H^n(X, A) := \pi_0 \mathbf{H}(X, \mathbf{B}^n A)$$

for the degree-n cohomology of X with coefficients in A.

In order to compute $\mathbf{H}(X, A)$ in concrete situations, we invoke various tools in homotopy theory, notably the fact that the stalkwise weak homotopy equivalences of simplicial presheaves are the weak equivalences in the projective local model structure on simplicial presheaves. Details of this the reader may find in [Sc12, NSS12b]. Here we just remark that if X happens to be a smooth manifold, and A an abelian Lie group, then $\mathbf{H}(X, \mathbf{B}^n A)$ is equivalent to the simplicial function complex of maps of simplicial presheaves $C(\{U_i\}) \to \mathrm{DK}(A[n])$, where $C(\{U_i\})$ is the Cech nerve simplicial presheaf of any *differentially good cover* $\{U_i \to X\}$. This says that $\mathbf{H}(X, A)$ may be computed by *nonabelian hyper-Cech cohomology*.

A useful tool for producing ∞-stacks A with *abelian* ∞-group structure such that the delooping $\mathbf{B}^n A$ exists for all $n \in \mathbb{N}$ is the *Dold-Kan correspondence*, which we here briefly recall. First, at an algebraic level, we have the classical Dold-Kan correspondence

$$\mathrm{Ch}_\bullet^+ \xrightarrow[\simeq]{\Gamma} \mathrm{sAb},$$

which establishes an equivalence of categories between chain complexes concentrated in non-negative degrees and simplicial abelian groups. Then there is the forgetful functor

$$F : \mathrm{sAb} \to \mathrm{sSet}_{\mathrm{fib}} \hookrightarrow \mathrm{sSet}$$

which forgets the group structure on a simplicial abelian group and just remembers the underlying simplicial set, which in turn is guaranteed to be a Kan complex. This is such that the elements in degree k of a chain complex label the extension of k-cells in the corresponding simplicial set; and the chain homology group in degree k identifies with the simplicial homotopy group in the same degree

$$H_k(V) \simeq \pi_k(F(\Gamma(V))) \,.$$

All this prolongs directly to presheaves of chain complexes and presheaves of abelian groups, and we will use the same symbols and write

$$\mathrm{DK} : \ [\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet] \xrightarrow{\ \Gamma\ } [\mathcal{C}^{\mathrm{op}}, \mathrm{sAb}] \xrightarrow{\ F\ } [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}] \ .$$

for the composite and refer to it as the *Dold-Kan map*. A crucial property of the Dold-Kan map is the following.

Proposition 2.1.1. *Let A, B and C be presheaves of chain complexes concentrated in non-negative degrees, and let $\cup : A \otimes B \to C$ be a morphism of presheaves of chain complexes. Then the Dold-Kan map induces a natural morphism of simplicial presheaves $\cup_{\mathrm{DK}} : \mathrm{DK}(A) \times \mathrm{DK}(B) \to \mathrm{DK}(C)$*

Proof. Both the categories $\mathrm{Ch}_\bullet^+$ and sAb are monoidal categories under the respective standard tensor products (on $\mathrm{Ch}_\bullet^+$ this is given by direct sums of tensor products of abelian groups with fixed total degree and on sAb by the degreewise tensor product of abelian groups), and the functor Γ is lax monoidal with respect to these structures, i.e., for any $V, W \in \mathrm{Ch}_\bullet^+$ we have natural weak equivalences

$$\nabla_{V,W} : \Gamma(V) \otimes \Gamma(W) \to \Gamma(V \otimes W) \,.$$

The forgetful functor F is the right adjoint to the functor forming degreewise the free abelian group on a set, therefore it preserves products and hence there are natural isomorphisms

$$F(V \times W) \xrightarrow{\cong} F(V) \times F(W) \,,$$

for all $V, W \in \mathrm{sAb}$. Finally, by the definition of tensor product, there are universal natural quotient maps $V, W \in \mathrm{sAb}$

$$p_{V,W} : V \times W \to V \otimes W \,.$$

The morphism $\cup_{\mathrm{DK}}$ is then defined as the composition indicated in the following diagram:

$$
\begin{array}{ccc}
\mathrm{DK}(A) \times \mathrm{DK}(B) & \xrightarrow{\qquad\qquad\qquad \cup_{\mathrm{DK}} \qquad\qquad\qquad} & \mathrm{DK}(C) \\
\| & & \| \\
F(\Gamma(A)) \times F(\Gamma(B)) & & \\
\downarrow{\simeq} & & \\
F(\Gamma(A) \times \Gamma(B)) \xrightarrow{F(p)} F(\Gamma(A) \otimes \Gamma(B)) \xrightarrow{F(\nabla)} F(\Gamma(A \otimes B)) \xrightarrow{F(\Gamma(\cup))} & & F(\Gamma(C)) \ .
\end{array}
$$

$\square$

Given the presentation $\mathbf{H} \simeq L_W [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$, for every presheaf of chain complexes A on $\mathcal{C}$ we obtain a corresponding ∞-stack, the ∞-stackification of the image of A under the Dold-Kan map, which we will denote by the same symbol: $\mathrm{DK}(A) \in \mathbf{H}$.

Definition 2.1.1. For $A \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}]$ a sheaf of abelian groups, we write $A[n] \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+]$ for the corresponding presheaf of chain complexes concentrated on A in degree n, and

$$\mathbf{B}^n A \simeq \mathrm{DK}(A[n]) \in \mathbf{H}$$

for the corresponding ∞-stack.

In this case the corresponding cohomology

$$H^n(X, A) = \pi_0 \mathbf{H}(X, \mathbf{B}^n A)$$

is the traditional *sheaf cohomology* of X with coefficients in A. More generally, if $A \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+]$ is a sheaf of chain complexes not necessarily concentrated in one degree, then

$$H^0(X, A) := \pi_0 \mathbf{H}(X, A)$$

is what traditionally is called the *sheaf hypercohomology* of X with coefficients in A. The central coefficient object in which we are interested here is the sheaf of chain complexes called the *Deligne complex*, to which we now turn.

2.2 Deligne cohomology and the cup product

Ordinary degree-2 integral cohomology $H^2(X, \mathbb{Z})$ on a smooth manifold X classifies smooth circle-bundles on X. Ordinary *differential cohomology* $H^2_{\mathrm{conn}}(X, \mathbb{Z})$ classifies smooth circle bundles with connection. In more detail, there is a groupoid $\mathbf{H}^1_{\mathrm{conn}}(X, U(1))$ whose objects are circle bundles with connection on X, and whose morphisms are smooth gauge transformations on X, such that $H^2_{\mathrm{conn}}(X, \mathbb{Z}) = \pi_0 \mathbf{H}^1_{\mathrm{conn}}(X, U(1))$. Generalized to arbitrary degree, one obtains n-groupoids $\mathbf{H}^n_{\mathrm{conn}}(X, U(1))$ whose objects are interpreted as circle n-bundles/bundle $(n-1)$-gerbes with connection, whose morphisms as smooth gauge transformations, whose 2-morphisms as gauge-of-gauge transformations, and so on.

A famous model for these n-groupoids using chain complexes is due to Deligne and Beilinson, known as the *Deligne complex*, which we briefly review, together with its cup product operation. In the context of differential geometry the use of Deligne cohomology was amplified notably by Brylinski [Br93].

Definition 2.2.1. Write $\mathbb{Z}[n+1]_D^\infty \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+]$ for the sheaf of chain complexes given by

$$\mathbb{Z}[n+1]_D^\infty := \left[\mathbb{Z} \lhook\joinrel\longrightarrow C^\infty(-, \mathbb{R}) \xrightarrow{d_{\mathrm{dR}}} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-) \right],$$

with the constant sheaf of integers $\mathbb{Z}$ in degree $(n+1)$, including into the sheaf of smooth real functions in degree n and with all further differential being the de Rham differential on sheaves of differential forms. This is the *Deligne complex* in degree $(n+1)$. The sheaf hypercohomology with coefficients in $\mathbb{Z}[n+1]_D^\infty$ is accordingly *Deligne cohomology*.

Remark 2.2.1. In the literature the Deligne complex is traditionally regarded as a sheaf on a fixed space X, instead of on the category of all Cartesian spaces. We see below that this difference translates into that between the moduli of circle n-bundles on a given space and the *universal* moduli n-stack of circle n-bundles.

The *Beilinson-Deligne cup product* is an explicit presentation of the cup product in ordinary differential cohomology for the case that the latter is modeled by the Čech-Deligne cohomology.

Definition 2.2.2. The Beilinson-Deligne cup product is the morphism of sheaves of chain complexes

$$\cup_{\mathrm{BD}} : \mathbb{Z}[p+1]_D^\infty \otimes \mathbb{Z}[q+1]_D^\infty \longrightarrow \mathbb{Z}[(p+1)+(q+1)]_D^\infty,$$

given on homogeneous elements α, β as follows:

$$\alpha \cup_{\mathrm{BD}} \beta := \begin{cases} \alpha \wedge \beta = \alpha\beta & \text{if } \deg(\alpha) = p+1 . \\ \alpha \wedge d_{\mathrm{dR}}\beta & \text{if } \deg(\alpha) \leq p \text{ and } \deg(\beta) = 0 . \\ 0 & \text{otherwise} . \end{cases}$$

Remark 2.2.2. When restricted to the diagonal in the case that $p = q$, this means that the cup product sends a p-form α to the $(2p+1)$-form $\alpha \wedge d_{\mathrm{dR}}\alpha$. This is of course the local Lagrangian for cup product Chern-Simons theory of p-forms. We discuss this case in detail in section 3.2.3.

The Beilinson-Deligne cup product is associative and commutative up to homotopy, so it induces an associative and commutative cup product on ordinary differential cohomology. A survey of this can be found in [Br93] (around Prop. 1.5.8 there).

2.3 Moduli n-stacks of circle n-connections

Under the simplicial localization $\mathbf{H} \simeq L_W[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}]$ together with the Dold-Kan correspondence DK : $[\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}_\bullet^+] \to [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{fib}}]$ from section 2.1, the Deligne chain complexes of section 2.2 present smooth ∞-stacks. We briefly discuss these and their induced cup product.

We have already mentioned that $\mathbf{B}^n U(1) \simeq \mathrm{DK}(C^\infty(-, U(1))[n])$ is the moduli n-stack for circle n-bundles.

Definition 2.3.1. Write $\mathbf{B}^n U(1)_{\mathrm{conn}} := \mathrm{DK}(\mathbb{Z}[n+1]_D^\infty)$ for the smooth n-stack presented by the Deligne complex, definition 2.2.1, under definition 2.1.1.

This n-stack sits in a diagram

$$
\begin{array}{ccc}
\mathbf{B}^n U(1)_{\mathrm{conn}} & \xrightarrow{\ F_{(-)}\ } & \Omega_{\mathrm{cl}}^{n+1}(-) \\
\chi \downarrow & \overset{\simeq}{\diagup\!\diagup} & \downarrow \\
\mathbf{B}^n U(1) & \xrightarrow[\ \mathrm{curv}\ ]{} & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)
\end{array}
$$

of n-stacks in $\mathbf{H}$, where

1. $\mathbf{B}^n U(1) \simeq \mathrm{DK}(C^\infty(-, U(1))[n])$ is the moduli n-stack of circle n-bundles (without connection); and χ is induced under DK from the evident morphism of chain complexes;

2. $\Omega_{\mathrm{cl}}^{n+1}(-)$ is the ordinary sheaf of closed $(n+1)$-forms;

3. $\flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \simeq \mathrm{DK}([\Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \longrightarrow \Omega_{\mathrm{cl}}^{n+1}(-)])$ is induced from the truncated de Rham complex, and curv – the universal curvature characteristic – is induced from a zig-zag of chain complexes such that it equips a circle n-bundle with a *pseudo-connection* and then produces the corresponding pseudo-curvature.

In fact this diagram is a homotopy pullback and characterizes $\mathbf{B}^n U(1)_{\mathrm{conn}}$ as the moduli n-stack for *curvature-twisted flat n-bundles*. We call it the *moduli n-stack of circle n-bundles with connection*.

Proposition 2.3.1. *Given a morphism* $\mathbf{c} : X \to \mathbf{B}^n U(1)$, *hence a class in* $H^n(X, U(1))$ *(for X a smooth manifold or itself a smooth ∞-stack such as $\mathbf{B}G$), the total space object $P \to X$ (total higher stack) of the circle n-bundle modulated by this morphism is the homotopy fiber of this morphism, the object universally fitting into a square*

$$
\begin{array}{ccc}
P & \longrightarrow & * \\
\downarrow & \diagup\!\diagup^{\simeq} & \downarrow \\
X & \xrightarrow[\ \mathbf{c}\ ]{} & \mathbf{B}^n U(1)
\end{array} \quad .
$$

This is a special case of the first main theorem in [NSS12a].

Proposition 2.3.2. *For a morphism* $\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ *modulating a circle n-bundle with* connection, *the total space of the underlying circle n-bundle is equivalently the homotopy pullback*

$$
\begin{array}{ccc}
P & \longrightarrow & \Omega^{1 \le \bullet \le n}(-) \\
\downarrow & \diagup\!\diagup^{\simeq} & \downarrow \\
X & \xrightarrow[\ \nabla\ ]{} & \mathbf{B}^n U(1)_{\mathrm{conn}}
\end{array} \quad .
$$

where the right vertical morphism includes the image under DK *of the chain complex* $[\Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-)]$, *hence includes the modli* $(n-1)$*-stack of circle* n*-connections whose underlying circle* n*-bundle is trivial.*

Proof. By Prop. 2.3.1 the total space object is the homotopy pullback

$$
\begin{array}{ccc}
P & \longrightarrow & * \\
\downarrow & \overset{\simeq}{\diagup\diagup} & \downarrow \\
X & \underset{\chi(\nabla)}{\longrightarrow} & \mathbf{B}^n U(1)
\end{array} \ ,
$$

where the bottom morphism is the composite

$$
\chi(\nabla) : \ X \xrightarrow{\ \nabla\ } \mathbf{B}^n U(1)_{\mathrm{conn}} \xrightarrow{\ \chi\ } \mathbf{B}^n U(1)
$$

that projects out the moduli of the circle n-bundle underlying the given n-connection. By the pasting law for homotopy pullbacks the pullback along such a composite map may be computed by iteratively pulling back along the two components, hence by forming the following pasting composite of homotopy pullback squares:

$$
\begin{array}{ccccc}
P & \longrightarrow & \Omega^{n \geq \bullet \geq 1}(-) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & \mathbf{B}^n U(1)_{\mathrm{conn}} & \xrightarrow{\ \chi\ } & \mathbf{B}^n U(1)
\end{array} \ .
$$

Since (see [Sc12] for details of such arguments)

1. the map DK is right Quillen for the global projective model structure on simplicial presheaves,

2. homotopy pullbacks in the local model structure may be computed in the global model structure (since ∞-stackification is left exact),

3. the pre-image under DK of the morphism χ is manifesty a fibration,

we may compute the homotopy pullback on the right as the ordinary pullback of presheaves of chain complexes, under DK$(-)$. Moreover, since these are computed as objectwise and degreewise pullbacks of abelian groups, this manifestly yields the fiber $\Omega^{1 \leq \bullet \leq n}$ as indicated. Hence by the pasting law we obtain the homotopy pullback on the left as claimed. $\qquad\square$

Remark 2.3.1. In the context of higher geometry such total space objects P may have a deeper meaning than in ordinary geometry: if X is a higher moduli stack, say of G-∞-connections $X = \mathbf{B}G_{\mathrm{conn}}$ for some smooth higher group G, then P in the above is itself also a higher moduli stack: namely that of those G-gauge fields equipped with a trivialization of their underlying class χ. Noteworthy examples of this phenomenon are discussed below in sections 3.1.2, 3.1.3 and 3.2.1.

Definition 2.3.2. For $p, q \in \mathbb{N}$ the morphism of simplicial presheaves

$$
\cup_{\mathrm{conn}} : \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^q U(1)_{\mathrm{conn}} \to \mathbf{B}^{p+q+1} U(1)_{\mathrm{conn}}
$$

is the morphism associated to the Beilinson-Deligne cup product $\cup_{\mathrm{BD}} : \mathbb{Z}[p{+}1]_D^\infty \otimes \mathbb{Z}[q{+}1]_D^\infty \longrightarrow \mathbb{Z}[p{+}q{+}2]_D^\infty$ by Proposition 2.1.1.

Since the Beilinson-Deligne cup product is associative up to homotopy, this induces a well defined morphism

$$\mathbf{B}^{n_1}U(1)_{\mathrm{conn}} \times \mathbf{B}^{n_2}U(1)_{\mathrm{conn}} \times \cdots \times \mathbf{B}^{n_{k+1}}U(1)_{\mathrm{conn}} \to \mathbf{B}^{n_1+\cdots+n_{k+1}+k}U(1)_{\mathrm{conn}}.$$

In particular, if $n_1 = \cdots = n_{k+1} = 3$, we find

$$\left(\mathbf{B}^3 U(1)_{\mathrm{conn}}\right)^{k+1} \to \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}.$$

Furthermore, we see from the explicit expression of the Beilinson-Deligne cup product that, on a local chart U, if the 3-form datum of a connection on a $U(1)$-3-bundle is the 3-form C, then the $4k+3$-form local datum for the corresponding connection on the associated $U(1)$-$(4k+3)$-bundle is

$$C \wedge \underbrace{dC \wedge \cdots \wedge dC}_{k\ \mathrm{times}}. \tag{2.3.1}$$

We will illustrate the above constructions with various (classes of) examples arising from string theory and M-theory.

2.4 Fiber integration and extended higher Chern-Simons actions

We discuss fiber integration in ordinary differential cohomology refined to smooth higher stacks and how this turns every differential characteristic maps into a tower of extended higher Chern-Simons action functionals in all codimensions.

One of the basic properties of ∞-toposes such as our $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{CartSp})$ is that they are *cartesian closed*. This means that:

Fact 2.4.1. *For every two objects $X, A \in \mathbf{H}$ – hence for every two smooth higher stacks – there is another object denoted $[X, A] \in \mathbf{H}$ that behaves like the "space of smooth maps from X to A." in that for every further $Y \in \mathbf{H}$ there is a natural equivalence of cocycle ∞-groupoids of the form*

$$\mathbf{H}(X \times Y, A) \simeq \mathbf{H}(Y, [X, A]),$$

saying that cocycles with coefficients in $[X, A]$ on Y are naturally equivalent to A-cocycles on the product $X \times Y$.

Remark 2.4.1. The object $[X, A]$ is in category theory known as the *internal hom* object, but in applications to physics and stacks it is often better known as the "families version" of A-cocycles on Y, in that for each smooth parameter space $U \in \mathrm{SmthMfd}$, the elements of $[X, A](U)$ are "U-parameterized families of A-cocycles on X", namely A-cocycles on $X \times U$. This follows from the above characterizing formula and the Yoneda lemma:

$$[X, A](U) \xrightarrow[\mathrm{Yoneda}]{\simeq} \mathbf{H}(U, [X, A]) \xrightarrow{\simeq} \mathbf{H}(X \times U, A).$$

Notably for G a smooth ∞-group and $A = \mathbf{B}G_{\mathrm{conn}}$ a moduli ∞-stack of smooth G-principal ∞-bundles with connection the object

$$[\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \in \mathbf{H}$$

is the smooth higher moduli stack of G-connection *on* Σ_k. It assigns to a test manifold U the ∞-groupoid of U-parameterized families of G-∞-connections, namely of G-∞-connections on $X \times U$. This is the smooth higher stack incarnation of the configuration space of higher G-gauge theory on Σ_k.

Example 2.4.1. In the discussion of anomaly polynomials in heterotic string theory over a 10-dimensional spacetime X one encounters degree-12 differential forms $I_4 \wedge I_8$, where I_i is a degree i polynomial in characteristic forms. Clearly these cannot live on X, as every 12-form on X, given by an element in the hom-∞-groupoid

$$\mathbf{H}(X, \Omega^{12}(-)) \xrightarrow[\mathrm{Yoneda}]{\simeq} \Omega^{12}(X)$$

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is trivial. Instead, these differential forms are elements in the internal hom $[X, \Omega^{12}(-)]$, which means that for every choice of smooth parameter space U there is a smooth 12-form on $X \times U$, such that this system of forms transforms naturally in U.

Below in section 3.2.4 we discuss how such anomaly forms appear from morphisms of higher moduli stacks

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{11}U(1)_{\mathrm{conn}}$$

for $\mathbf{B}G_{\mathrm{conn}}$ the higher moduli stack of supergravity field configurations by sending the families of moduli of field configurations on spacetime X to their anomaly form:

$$[X, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[X, \mathbf{c}_{\mathrm{conn}}]} [X, \mathbf{B}^{11}U(1)_{\mathrm{conn}}] \xrightarrow{[X, \mathrm{curv}]} [X, \Omega^{12}(-)] \ .$$

We now discuss how such families of n-cocycles on some X can be integrated over X to yield $(n - \dim(X))$-cocycles.

Proposition 2.4.1. *Let Σ_k be a closed (= compact and without boundary) oriented smooth manifold of dimension k. Then for every $n \geq k$ there is a natural morphism of smooth higher stacks*

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}] \to \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

from the moduli n-stack of circle n-bundles with connection on Σ_k to the moduli $(n - k)$-stack of smooth circle $(n - k)$-bundles with connection such that

1. *for $k = n$ this yields a $U(1)$-valued gauge invariant smooth function*

$$\exp(2\pi i \int_{\Sigma_k} (-)) : [\Sigma_n, \mathbf{B}^nU(1)_{\mathrm{conn}}] \to U(1) \ ,$$

   which is the n-volume holonomy of a circle n-connection over the "n-dimensional Wilson volume" Σ_n;

2. *for $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 \leq n$ we have*

$$\exp(2\pi i \int_{\Sigma_{k_1}} (-)) \circ \exp(2\pi i \int_{\Sigma_{k_2}} (-)) \simeq \exp(2\pi i \int_{\Sigma_{k_1} \times \Sigma_{k_2}} (-)) \ .$$

Proof. Since $\mathbf{B}^nU(1)_{\mathrm{conn}}$ is fibrant in the projective local model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ (since every circle n-bundle with connection on a Cartesian space is trivializable) the mapping stack $[\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}]$ is presented for any choice of good open cover $\{U_i \to \Sigma_k\}$ by the simplicial presheaf

$$U \mapsto [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\mathcal{U}) \times U, \mathbf{B}^nU(1)_{\mathrm{conn}}) \ ,$$

where $\check{C}(\mathcal{U})$ is the Čech nerve of the open cover $\{U_i \to \Sigma_k\}$. Therefore a morphism as claimed is given by natural fiber integration of Deligne hypercohomology along product bundles $\Sigma_k \times U \to U$ for closed Σ_k. This has been constructed for instance in [GT1]. $\qquad \square$

Definition 2.4.1. Let $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^nU(1)_{\mathrm{conn}}$ be a differential characteristic map. Then for Σ_k a closed smooth manifold of dimension $k \leq n$, we call

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) : \ [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_k, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

the *off-shell prequantum* $(n-k)$-*bundle of extended* $\mathbf{c}_{\text{conn}}$-∞-*Chern-Simons theory*. For $n = k$ we have a *circle 0-bundle*

$$\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}]): [\Sigma_n, \mathbf{B}G_{\text{conn}}] \xrightarrow{[\Sigma_n, \mathbf{c}_{\text{conn}}]} [\Sigma_n, \mathbf{B}^n U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_n}(-))} U(1) \ ,$$

which we call the *action functional* of the theory.

This construction subsumes several fundamental aspects of Chern-Simons theory:

1. gauge invariance and smoothness of the (extended) action functionals, remark 2.4.2;

2. inclusion of instanton sectors (nontrivial gauge ∞-bundles), remark 2.4.3;

3. level quantization, remark 2.4.4;

4. definition on non-bounding manifolds and relation to (higher) topological Yang-Mills on bounding manifolds, remark 2.4.5.

We discuss these in more detail in the following remarks, as indicated.

Remark 2.4.2 (Gauge invariance and smoothness). Since $U(1) \in \mathbf{H}$ is an ordinary manifold (after forgetting the group structure), a 0-stack with no non-trivial morphisms (no gauge transformation), the action functional $\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}])$ takes every morphism in the moduli stack of field configurations to the identity. But these morphisms are the *gauge transformations*, and so this says that $\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}])$ is *gauge invariant*, as befits a gauge theory action functional. To make this more explicit, notice that

$$\mathbf{H}(\Sigma_n, \mathbf{B}G_{\text{conn}}) \simeq [\Sigma_n, \mathbf{B}G_{\text{conn}}](*)$$

is the evaluation of the moduli stack on the point, hence the ∞-groupoid of smooth families of field configurations which are trivially parameterized. Moreover

$$H^1_{\text{conn}}(\Sigma_n, G) := \pi_0 \mathbf{H}(\Sigma_n, \mathbf{B}G_{\text{conn}})$$

is the set of gauge equivalent such field configurations. Then the statement that the action functional is both gauge invariant and smooth is the statement that it can be extended from $H^1_{\text{conn}}(\Sigma_n, G)$ (supposing that it were given there as a function $\exp(iS(-))$ by other means) via $\mathbf{H}(\Sigma_n, \mathbf{B}G_{\text{conn}})$ to $[\Sigma_n, \mathbf{B}G_{\text{conn}}]$

Remark 2.4.3 (Definition on instanton sectors). Ordinary 3-dimensional Chern-Simons theory is often discussed for the special case only when the gauge group G is connected and simply connected. This yields a drastic simplification compared to the general case; since for every Lie group the second homotopy group $\pi_2(G)$ is trivial, and since the homotopy groups of the classifying space BG are those of G shifted up in degree by one, this implies that BG is 3-connected and hence that every continuous map $\Sigma_3 \to BG$ out of a 3-manifold is homotopic to the trivial map. This implies that every G-principal bundle over Σ_3 is trivializable. As a result, the moduli stack of G-gauge fields on Σ_3, which a priori is $[\Sigma_3, \mathbf{B}G_{\text{conn}}]$, becomes

in this case equivalent to just the moduli stack of trivial G-bundles with (non-trivial) connection on Σ_3, which is identified with the groupoid of just $\mathfrak{g}$-valued 1-forms on Σ_3, and gauge transformations between these, which is indeed the familiar configurations space for 3-dimensional G-Chern-Simons theory.

One should compare this to the case of 4-dimensional G-gauge theory on a 4-dimensional manifold Σ_4, such as G-Yang-Mills theory. By the same argument as before, in this case G-principal bundles may be nontrivial, but are classified enirely by the second Chern class (or first Pontrjagin class) $[c_2] \in H^4(\Sigma_4, \pi(G))$. In Yang-Mills theory with $G = SU(n)$, this class is known as the *instanton number* of the gauge field.

The simplest case where non-trivial classes occur already in dimension 3 is the non-simply connected gauge group $G = U(1)$, discussed in section 3.2.2 below. Here the moduli stack of fields $[\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}]$ contains configurations which are not given by globally defined 1-forms, but by connections on non-trivial circle bundles. By analogy with the case of $SU(n)$-Yang-Mills theory, we will loosely refer to such field configurations as instanton field congurations, too. In this case it is the first Chern class $[c_1] \in H^2(X, \mathbb{Z})$ that measures the non-triviality of the bundle. If the first Chern-class of a $U(1)$-gauge field configurations happens to vanish, then the gauge field is again given by just a 1-form $A \in \Omega^1(\Sigma_3)$, the familiar gauge potential of electromagnetism. The value of the 3d Chern-Simons action functional on such a non-instanton configuration is simply the familiar expression

$$\exp(iS(A)) = \exp(2\pi i \int_{\Sigma_3} A \wedge d_{\mathrm{dR}}A),$$

where on the right we have the ordinary integration of the 3-form $A \wedge dA$ over Σ_3.

In the general case, however, when the configuration in $[\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}]$ has non-trivial first Chern class, the expression for the value of the action functional on this configuration is more complicated. If we pick a good open cover $\{U_i \to \Sigma_3\}$, then we can arrange that locally on each patch U_i the gauge field is given by a 1-form A_i and the contribution of the action functional over U_i by $\exp(2\pi i \int_{\Sigma_3} A_i \wedge dA_i)$ as above. But in such a decomposition there are further terms to be included to get the correct action functional. This is what the construction in Prop. 2.4.1 achieves.

Remark 2.4.4 (Level quantization). Traditionally, Chern-Simons theory in 3-dimensions with gauge group a connected and simply connected group G comes in a family parameterized by a *level* $k \in \mathbb{Z}$. This level is secretly the cohomology class of the differential characteristic map

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

(constructed in [FSS10]) in

$$H^3_{\mathrm{smooth}}(BG, U(1)) \simeq H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}.$$

So the traditional level is a cohomological shadow of the differential characteristic map that we interpret as the off-shell prequantum n-bundle in full codimension n (down on the point). Notice that for a general smooth ∞-group G the cohomology group $H^{n+1}(BG, \mathbb{Z})$ need not be equivalent to $\mathbb{Z}$ and so in general the level need not be an integer. For for every smooth ∞-group G, and given a morphism of moduli stacks $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, also every integral multiple $k\mathbf{c}_{\mathrm{conn}}$ gives an n-dimensional Chern-Simons theory, "at k-fold level". The converse is in general hard to establish: whether a given $\mathbf{c}_{\mathrm{conn}}$ can be divided by an integer. For instance for 3-dimensional Chern-Simons theory division by 2 may be possible for Spin-structure. For 7-dimensional Chern-Simons theory division by 6 may be possible in the presence of String-structure [FSS12a].

Remark 2.4.5. Ordinary 3-dimensional Chern-Simons theory is often defined on bounding 3-manifolds Σ_3 by

$$\exp(iS(\nabla)) = \exp(2\pi i k \int_{\Sigma_4} \langle F_{\widehat{\nabla}} \wedge F_{\widehat{\nabla}} \rangle),$$

where Σ_4 is any 4-manifold with $\Sigma_3 = \partial\Sigma_4$ and where $\widehat{\nabla}$ is any extension of the gauge field configuration from Σ_3 to Σ_4. Similar expressions exist for higher dimensional Chern-Simons theories. If one takes these expressions to be the actual definition of Chern-Simons action functional, then one needs extra discussion for which manifolds (with desired structure) are bounding, hence which vanish in the respective cobordism ring, and, more seriously, one needs to include those which are not bounding from the discussion. For example, in type IIB string theory one encounters the cobordism group $\Omega^{\mathrm{Spin}}_{11}(K(\mathbb{Z},6))$ [Wi96], which is proven to vanish in [KS05], meaning that all the desired manifolds happen to be bounding.

We emphasize that our formula in Prop. 2.4.1 applies generally, whether or not a manifold is bounding. Moreover, it is guaranteed that *if* Σ_n happens to be bounding after all, then the action functional is equivalently given by integrating a higher curvature invariant over a bounding $(n+1)$-dimensional manifold. At the level of differential cohomology classes $H^n_{\mathrm{conn}}(-, U(1))$ this is the well-known property (a review and further pointers are given in [HS05]) which is an explicit axiom in the equivalent formulation by Cheeger-Simons differential characters: a Cheeger-Simons differential character of degree $(n+1)$ is by definition a group homomorphism from closed n-manifolds to $U(1)$ such that whenever the n-manifold happens to be bounding, the value in $U(1)$ is given by the exponentiated integral of a smooth $(n+1)$-form over any bounding manifold.

With reference to such differential characters Chern-Simons action functions have been formulated for instance in [Wi96, Wi98b]. The sheaf hypercohomology classes of the Deligne complex that we are concerned with here are well known to be equivalent to these differential characters, and Čech-Deligne cohomology has the advantage that with results such as [GT1] invoked in Prop. 2.4.1 above, it yields explict formulas for the action functional on non-bounding manifolds in terms of local differential form data.

3 Examples and applications

Here we list and discuss examples of higher extended cup-product Chern-Simons theories constructed by the general procedure introduced above in section 2. Some of the examples below are known in low codimension, notably from constructions in string theory and M-theory, in that their action functional (codimension 0) and hence their prequantum line bundles (codimension 1) are well known, while others have maybe not been considered before. Already in the known cases our discussion provides the refinement of the action functional

1. to the full higher moduli stacks of fields;

2. to arbitrary codimension.

The titles of the following subsections follow the pattern

$$\text{XYZ Chern-Simons theory} \quad \text{and} \quad \text{ABC theory}$$

where "ABC theory" is an incarnation of the extended Chern-Simons theory XYZ in higher codimension.

Before we proceed to section 3.1, the following list gives an overview of the various types of examples that we consider, and how they conceptually relate to each other as specializations and/or combinations of other classes of examples.

List of classes of examples.

1. **Fully general ∞-Chern-Simons theory.** In full generality, an "∞-Chern-Simons theory" is specified by a smooth gauge ∞-group G and a differential characteristic map of moduli stacks

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}\,.$$

This is such that for Σ_k a k-dimensional smooth manifold, the object $[\Sigma_k, \mathbf{B}G_{\mathrm{conn}}]$ discussed in section 2.4, is the moduli stack of G-gauge fields on Σ_k, and $[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ is the moduli stack of n-form connections. Then if $0 \leq k \leq n$ and Σ_k is closed and oriented, we obtain a morphism

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \mathbf{c}_{\mathrm{conn}}]) : [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \to \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

as in section 2.4, which gives the *off-shell prequantum $(n-k)$-bundle* of an n-dimensional Chern-Simons theory. In particular, for $k = n$ this is the *action functional* of the higher extended Chern-Simons theory specified by $\mathbf{c}_{\mathrm{conn}}$.

2. **Inhomogeneous $U(1)$ cup-product theories.** In this general context, the cup product

$$\cup_{\mathrm{conn}} : \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^q U(1)_{\mathrm{conn}} \to \mathbf{B}^{p+q+1}U(1)_{\mathrm{conn}}$$

from section 2.2 for $p, q \geq 1$, is itself a differential characteristic map, since we may regard it as defining an ∞-Chern-Simons theory with gauge ∞-group the product map(p, q)-group $(\mathbf{B}^{p-1}U(1)) \times (\mathbf{B}^{q-1}U(1))$, hence by reading $\cup_{\mathrm{conn}}$ as

$$\cup_{\mathrm{conn}} : \mathbf{B}\left(\mathbf{B}^{p-1}U(1) \times \mathbf{B}^{q-1}U(1)\right)_{\mathrm{conn}}\right) \to \mathbf{B}^{p+q+1}U(1)_{\mathrm{conn}} .$$

A class of examples of this form that does appear in the physics literature is the electric-magnetic coupling term in higher abelian gauge theory. This is a *section* of the prequantum 1-bundle of this Chern-Simons theory, and the class of that bundle is the electric-magnetic quantum anomaly.

Two variants of this theory are important.

(a) $U(1)$ **cup-square theories.** In the case that $p = q$ we may restrict to the diagonal of the cup pairing, hence taking the two p-form fields to be two copies of one single field. Formally this means that we are considering the differential characteristic map which is the composite

$$(-)^{\cup_{\mathrm{conn}}^2} : \ \mathbf{B}^p U(1)_{\mathrm{conn}} \xrightarrow{\ \Delta\ } \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^p U(1)_{\mathrm{conn}} \xrightarrow{\ \cup_{\mathrm{conn}}\ } \mathbf{B}^{p+q+1}U(1)_{\mathrm{conn}} .$$

For $p = 1$, this yields (the higher codimension-extended version of) traditional 3-dimensional $U(1)$-Chern-Simons theory. For $p = 2k + 1$ it yields the $(4k + 3)$-dimensional $U(1)$-Chern-Simons theory which is the holographic dual of self-dual $2k$-form theory in dimension $4k + 2$.

(b) **Cup product of two nonabelian theories.** Given two possibly nonabelian gauge ∞-groups G_1 and G_2 equipped with two differential characteristic maps $(\mathbf{c}_1)_{\mathrm{conn}}$ and $(\mathbf{c}_1)_{\mathrm{conn}}$, we may form the "cup product of two nonabelian Chern-Simons theories"

$$(\mathbf{c}_1)_{\mathrm{conn}} \cup_{\mathrm{conn}} (\mathbf{c}_2)_{\mathrm{conn}} : \ \mathbf{B}(G_1 \times G_2)_{\mathrm{conn}} \xrightarrow{((\mathbf{c}_1)_{\mathrm{conn}}, (\mathbf{c}_2)_{\mathrm{conn}})} \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^q U(1)_{\mathrm{conn}} \xrightarrow{\cup_{\mathrm{conn}}} \mathbf{B}^{p+q+1}U(1)_{\mathrm{conn}} .$$

This appears for instance in the electric-magnetic anomaly of the heterotic string.

3. **Cup-square of one non-abelian theory.** The two variants above may be combined to yield the cup product of a non-abelian Chern-Simons theory with itself.

4. **Multiple-factor cup-product theories.** Finally, all of this can be considered with three cup factors ("cubic theories") or more cup-factors, instead of just two of them ("quadratic theories"). Examples of cubic Chern-Simons theories appear in 11-dimensional supergravity, for instance.

3.1 Unary examples

Before discussing genuine cup-product higher Chern-Simons theories we consider here some indecomposable theories – *unary* cup-product theories, if one wishes – that serve as building blocks for the cup product theories.

3.1.1 Higher differential Dixmier-Douady class and higher dimensional $U(1)$-holonomy

The *degenerate* or rather *tautological* case of extended ∞-Chern-Simons theories nevertheless deserves special attention, since it appears universally in all other examples: that where the extended action functional is the *identity* morphism

$$(\mathbf{DD}_n)_{\text{conn}} : \; \mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{\;\text{id}\;} \mathbf{B}^n U(1)_{\text{conn}} \; ,$$

for some $n \in \mathbb{N}$. Trivial as this may seem, this is the differential refinement of what is called the (higher) *universal Dixmier-Douady class* the higher universal first Chern class – of circle n-bundles / bundle $(n-1)$-gerbes, which on the topological classifying space $B^n U(1)$ is the weak homotopy equivalence

$$\text{DD}_n : \; B^n U(1) \xrightarrow{\;\simeq\;} K(\mathbb{Z}, n+1) \; .$$

Therefore, we are entitled to consider $(\mathbf{DD}_n)_{\text{conn}}$ as the extended action functional of an n-dimensional ∞-Chern-Simons theory. Over an n-dimensional manifold Σ_n the moduli n-stack of field configurations is that of circle n-bundles with connection on Σ_n. In generalization to how a circle 1-bundle with connection has a *holonomy* over closed 1-dimensional manifolds, we note that a circle n-connection has a *n-volume holonomy* over the n-dimensional manifold Σ_n. This is the ordinary (codimension-0) action functional associated to $(\mathbf{DD}_n)_{\text{conn}}$ regarded as an extended action functional:

$$\text{hol} := \exp(2\pi i \int_{\Sigma_n} [\Sigma_n, (\mathbf{DD}_n)_{\text{conn}}]) : [\Sigma_n, \mathbf{B}^n U(1)_{\text{conn}}] \to U(1) \, .$$

This formulation makes it manifest that, for G any smooth ∞-group and $\mathbf{c}_{\text{conn}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^n U(1)_{\text{conn}}$ any extended ∞-Chern-Simons action functional in codimension n, the induced action functional is indeed the n-volume holonomy of a family of "Chern-Simons circle n-connections", in that we have

$$\exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}]) \simeq \text{hol}_{\mathbf{c}_{\text{conn}}} \, .$$

This is most familiar in the case where the moduli ∞-stack $\mathbf{B}G_{\text{conn}}$ is replaced with an ordinary smooth oriented manifold X (of any dimension and not necessarily compact). In this case $\mathbf{c}_{\text{conn}} : X \to \mathbf{B}^n U(1)_{\text{conn}}$ modulates a circle n-bundle with connection ∇ on this smooth manifold. Now regarding this as an extended Chern-Simons action function in codimension n means to

1. take the moduli stack of fields over a given closed oriented manifold Σ_n to be $[\Sigma_n, X]$, which is simply the space of maps between these manifolds, equipped with its natural ("diffeological") smooth structure (for instance the smooth loop space LX when $n = 1$ and $\Sigma_n = S^1$);

2. take the value of the action functional on a field configuration $\phi : \Sigma_n \to X$ to be the n-volume holonomy of ∇

$$\text{hol}_\nabla(\phi) = \exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{c}_{\text{conn}}]) : \; [\Sigma_n, X] \xrightarrow{[\Sigma_n, \mathbf{c}_{\text{conn}}]} [\Sigma_n, \mathbf{B}^n U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi \int_{\Sigma_n} (-))} U(1) \; .$$

Using the proof of Prop. 2.4.1 to unwind this in terms of local differential form data, this reproduces the familiar formulas for (higher) $U(1)$-holonomy.

3.1.2 Ordinary $3d$ Spin-Chern-Simons theory and String-2-connections

For G any connected and simply connected compact simple Lie group we have $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. In the case that $G = \text{Spin}$ is the spin group (in dimension ≥ 3), the generator (unique up to sign) of this group is called the *first fractional Pontrjagin class*, represented by a map

$$\tfrac{1}{2}p_1 : B\text{Spin} \to B^3 U(1) \simeq K(\mathbb{Z}, 4) \, .$$

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In [Sc12] it is shown that this has a unique (up to equivalence) smooth refinement to a morphism of higher smooth moduli stacks of the form

$$\tfrac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3 U(1)\,.$$

Moreover, in [FSS10] we construct the further differential refinement

$$\tfrac{1}{2}(\mathbf{p}_1)_{\mathrm{conn}} : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

from the moduli stack of Spin-principal bundles with connection to the smooth moduli 3-stack of smooth circle 3-bundles (bundle 2-gerbes) with connection. Regarding this as an extended action functional for an ∞-Chern-Simons theory, it is not hard to see that the corresponding action functional

$$\exp(2\pi i \int_{\Sigma_3}) : [\Sigma_3, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}] \to U(1)$$

is that of ordinary 3d Spin-Chern-Simons theory, as discussed in the Introduction, section 1, (an observation on cohomology classes was first highlighted in [CJMSW05]).

In addition to the comments on ordinary Chern-Simons theory regarded as an extended prequantized theory already made in the Introduction, we here observe the following. The total space 2-stack is of the prequantum circle 3-bundle of this theory, regarded as an 0-1-2-3 extended prequantum Chern-Simons theory, is, by prop. 2.3.2, the homotopy pullback of the form

$$
\begin{array}{ccccc}
\mathbf{B}\mathrm{String}_{\mathrm{conn}'} & \longrightarrow & \Omega^{1\leq\bullet\leq 3} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} & \xrightarrow[\tfrac{1}{2}(\mathbf{p}_1)_{\mathrm{conn}}]{} & \mathbf{B}^3 U(1)_{\mathrm{conn}} & \xrightarrow{\chi} & \mathbf{B}^3 U(1)\,.
\end{array}
$$

Comparison with [FSS12b] shows that this total space is the moduli 2-stack $\mathbf{B}\mathrm{String}_{\mathrm{conn}'}$ of String-principal 2-connections, as indicated. (See the appendix of [FSS12a] for a discussion of how these are nonabelian 2-form connections). If one further restricts along the inclusion $\Omega^3(-) \longrightarrow \Omega^{1\leq\bullet\leq 3}(-)$, then these restrict to the structures discussed in [Wal09]. If, on the other hand, one replaces the twist by 3-forms with the twist of the differential second Chern-class of $E_8 \times E_8$-principal bundles

$$\mathbf{a} : \mathbf{B}E_8 \times E_8 \to \mathbf{B}^3 U(1)$$

then one obtains the moduli 2-stack of String$^{\mathbf{a}}$-connections that control the anomaly-free field content, including the twisted B-field, of the heterotic Green-Schwarz mechanism as discussed in [SSS09b].

Note that there are other effectively unary theories which fall under our formulation; notably, those whose action functional takes the form $\int \Omega \wedge CS$, where CS is a Chern-Simons term, not necessarily three-dimensional, and Ω is an auxiliary form on the underlying manifold, independent of the first term. Since Ω is a fixed form it does not enter into the dynamics and so the whole system is governed by the Chern-Simons term. Examples include

1. Kähler-Chern-Simons theories (see [NS, IKU, LMNS]) where Ω is a Kähler form,

2. holomorphic Chern-Simons theories (see [Wi92, FT]) where Ω is a middle form on a Calabi-Yau manifold, as well as

3. theories that lift M-theory via terms of the form $\int_{M^{27}} \Omega_{16} \wedge CS_{11}$ [Sa09], where CS_{11} is the Chern-Simons term in M-theory (3.3.3), and Ω_{16} is a composite form on the octonionic projective plane.

3.1.3 $7d$ String-**Chern-Simons theory and Fivebrane 6-connections**

The construction of the total space of the fully extended prequantum n-bundle in section 3.1.2 above is just the first step in a whole tower of *higher Spin structure* and (extended) *higher Spin-Chern-Simons theories* that are obtained by a smooth and differential refinement of the *Whitehead tower* of BO. This is the tower of homotopy types on the left vertical axis of the following diagram.
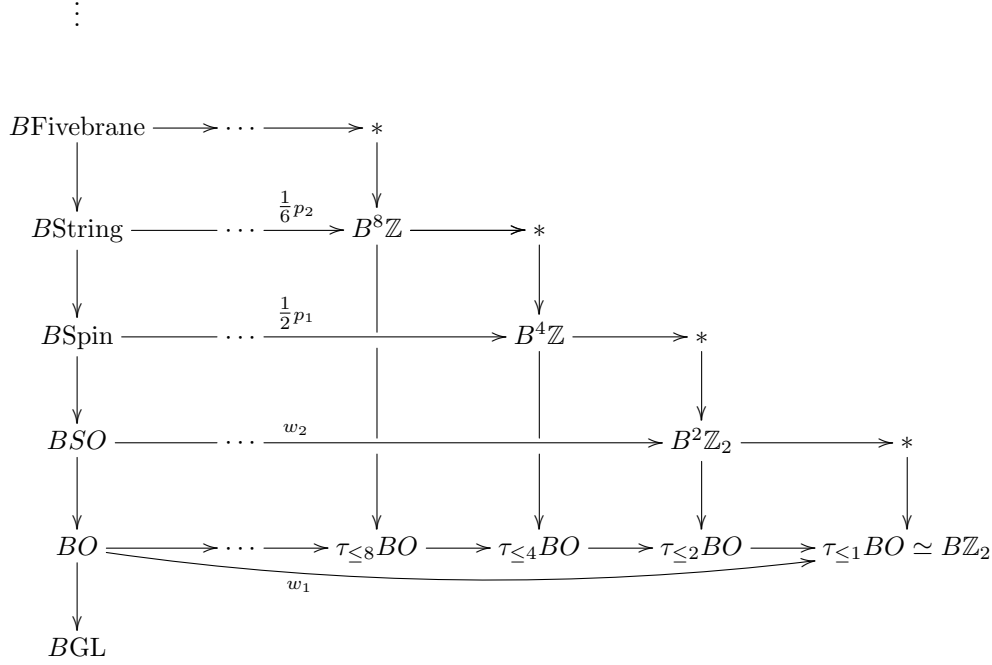
Here the bottom horizontal tower is the Postnikov tower of BO and all rectangles are homotopy pullbacks (see section 4 of [Sc12] for more details).

For X a smooth manifold, there is a canonically given map $X \to BGL$, which classifies the tangent bundle TX. The lifts of this classifying map through the above Whitehead tower correspond to structures on X as indicated in the following diagram:

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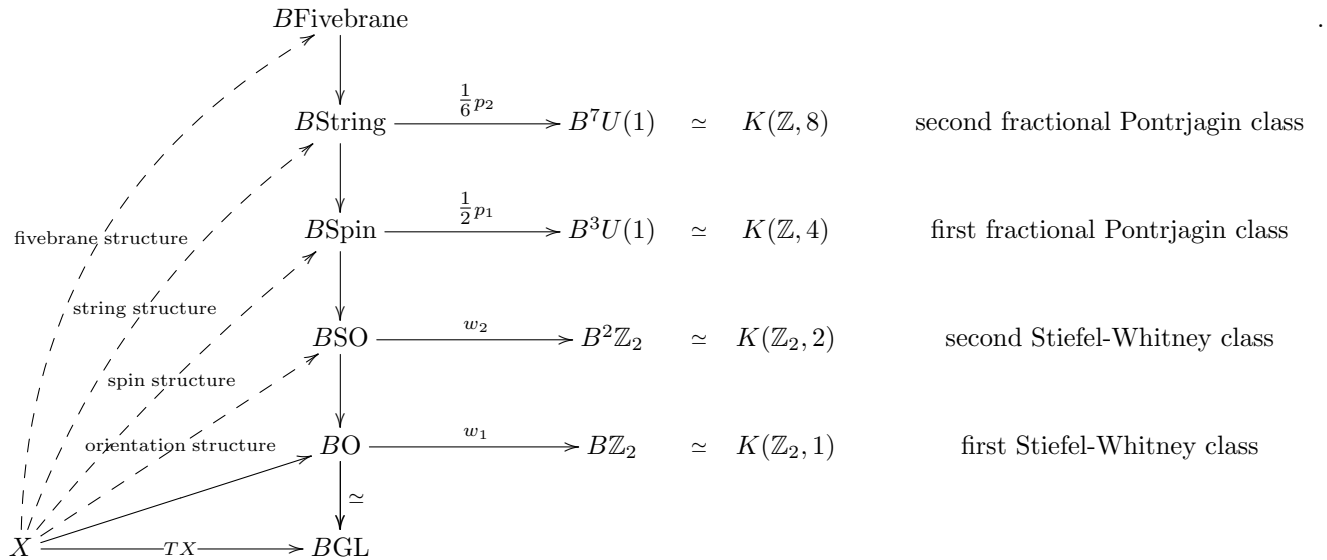
Here the horizontal morphisms denote representatives of universal characteristic classes, such that each sub-diagram of the shape

$$
\begin{array}{ccc}
B\widehat{G} & & \\
\downarrow & & \\
BG & \xrightarrow{\ c\ } & B^n K
\end{array}
$$

is a homotopy fiber sequence. Several variations and twists on the above structures are considered in [Sa11c, Sa11d, Sa12c].

In [FSS10] we gave an explicit construction of the smooth refinement of the second fractional Pontrjagin class to a morphism of smooth moduli stacks

$$\tfrac{1}{6}(\mathbf{p}_2)_{\mathrm{conn}} : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{con}}$$

from that of String 2-connections to that of circle 7-bundles with connection. When regarding this as the fully extended action functional of an ∞-Chern-Simons theory it produces a 7-dimensional theory which in [FSS12a] we argued is part of the holographic dual of the M5-brane theory, see section 3.2.3 below. As before, it is of interest to compute the total space of the prequantum circle 7-bundle on the moduli 2-stack of String-connections. By Prop. 2.3.2 and after comparison with [SSS09b] is the moduli 6-stack of (twisted) Fivebrane-*6-form connections*.

$$
\begin{array}{ccccc}
\mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}'} & \longrightarrow & \Omega^{1\leq\bullet\leq 7} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{B}\mathrm{String}_{\mathrm{conn}} & \xrightarrow[\frac{1}{6}(\mathbf{p}_2)_{\mathrm{conn}}]{} & \mathbf{B}^7 U(1)_{\mathrm{conn}} & \xrightarrow{\ \chi\ } & \mathbf{B}^7 U(1) \ .
\end{array}
$$

Another important example is the Whitehead tower of U(n): the k-connected cover $\mathrm{U}(n)\langle 2k-1\rangle \simeq \mathrm{U}(n)\langle 2k\rangle$ is the natural home for the differential refinement $(\mathbf{c}_{k+1})_{\mathrm{conn}} : \mathbf{B}\mathrm{U}(n)\langle 2k\rangle_{\mathrm{conn}} \to \mathbf{B}^{2k+1}\mathrm{U}(1)_{\mathrm{conn}}$ of the $(k+1)$st Chern class $c_{k+1} \in H^{2k+2}(B\mathrm{U}(n);\mathbb{Z})$. Constructions analogous to those of the orthogonal case follow similarly.

3.1.4 $(2n+1)d$ Chern-Simons (super)gravity and WZW_{2n}-models

The literature contains various proposals of higher-dimensional (super-)Chern-Simons-type theories, all *unary* in our sense here, that are argued to be possible candidates for a theory related to actual (super)gravity [BTZ96, TZ98], see [Za05] for a review. In codimension 1 these theories are known to be related to higher dimensional analogs of the 2d WZW-model in dimension $2n$ [BGH96, GK00]. In the case of M-theory, with $n = 5$, there are candidates that propose to describe the theory based on holography and Chern-Simons theory [Hor, Na, IR].

These unary nonabelian higher dimensional Chern-Simons theories are interesting candidates for extended prequantization as considered here, but whether or in which cases their fully extended prequantizations exist has not been worked out yet.

3.2 Quadratic examples

We now consider examples of extended ∞-Chern-Simons theories that are formed of a differential cup-product of two factors.

3.2.1 3d $U(1)$-theory with two species and differential T-duality

Consider the extended ∞-Chern-Simons action functional given simply by the differential cup product of def. 2.3.2 in the first non-trivial degree:

$$(-)\cup_{\mathrm{conn}}(-): \ \mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1) \ .$$

Its moduli stack of fields $[\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}}]$ consists of pairs of two different $U(1)$-gauge fields on Σ_3. On those field configurations that have trivial underlying integral classes and are hence given by globally defined 1-forms A_1, A_2, the action functional in dimension 3 takes these to

$$\exp(2\pi i \int_{\Sigma_3} [\Sigma_3, (-)\cup_{\mathrm{conn}} (-)]) : (A_1, A_2) \mapsto \exp(2\pi i \int_{\Sigma_3} A_1 \wedge d_{\mathrm{dR}}A_2) = \exp(2\pi i \int_{\Sigma_3} A_2 \wedge d_{\mathrm{dR}}A_1)\,.$$

The "diagonal of this theory", namely the extended action functional obtained by precomposition with the diagonal map $\Delta : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}}$ is the ordinary 3d $U(1)$-Chern-Simons theory of a single gauge field species discussed below in section 3.2.2.

By Prop. 2.3.2 the total space object P of the prequantum circle 3-bundle of the above extended action functional is the homotopy pullback

$$
\begin{array}{ccccc}
P & \longrightarrow & \Omega^{1\leq\bullet\leq 3}(-) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}} & \xrightarrow{\ \cup_{\mathrm{conn}}\ } & \mathbf{B}^3 U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^3 U(1) \ .
\end{array}
$$

By the universal property of the homotopy pullback this means that P is the moduli 2-stack for pairs (∇_1, ∇_2) of circle bundles with connection – hence pairs of *1-torus bundles* with connection – equipped with a smooth trivialization of the cup product

$$\mathbf{c}_1(\nabla_1) \cup \mathbf{c}(\nabla_2) = \chi(\nabla_1) \cup \chi(\nabla_2)$$

of their Chern classes. This is the structure called a *differential T-duality pair* in Def. 2.1 of [KV09], expressing the necessary differential geometric structure for an action of T-duality between two torus fibrations on the *differential K-theory* of the underlying spaces, hence on the charge-quantized RR-fields in type II string theory.

3.2.2 Ordinary 3d $U(1)$-Chern-Simons theory and generalized B_n-geometry

Ordinary 3-dimensional $U(1)$-Chern-Simons theory on a closed oriented manifold Σ_3 contains field configurations which are given by globally defined 1-forms $A \in \Omega^1(\Sigma_3)$ and on which the action functional is given by the familiar expression

$$\exp(iS(A)) = \exp(2\pi i k \int_{\Sigma_3} A \wedge d_{\mathrm{dR}}A)\,.$$

More generally, though, a field configuration of the theory is a connection ∇ on a $U(1)$-principal bundle $P \to \Sigma_3$ and this simple formula is modified, from being the exponential of the ordinary integral of the wedge product of two differential forms, to the fiber integration in differential cohomology, Def. 2.4.1, of the differential cup-product, Def. 2.3.2:

$$\exp(iS(\nabla)) = \exp(2\pi i k \int_{\Sigma_3} \nabla \cup_{\mathrm{conn}} \nabla)\,.$$

This defines the action functional on the set $H^1_{\mathrm{conn}}(\Sigma_3, U(1))$ of equivalence classes of $U(1)$-principal bundles with connection

$$\exp(iS(-)) : H^1_{\mathrm{conn}}(\Sigma_3) \to U(1)\,.$$

That the action functional is gauge invariant means that it extends from a function on gauge equivalence classes to a functor on the groupoid $\mathbf{H}^1_{\mathrm{conn}}(\Sigma_3, U(1))$, whose objects are actual $U(1)$-principal connections, and whose morphsims are smooth gauge transformations between these:

$$\exp(iS(-)) : \mathbf{H}^1_{\mathrm{conn}}(\Sigma_3) \to U(1) \, .$$

Finally, that the action functional depends *smoothly* on the connections means that it extends further to the moduli stack of fields to a morphism of stacks

$$\exp(iS(-)) : [\Sigma_3, \mathbf{B}U(1)_{\mathrm{conn}}] \to U(1) \, .$$

The fully extended prequantum circle 3-bundle of this extended 3d Chern-Simons theory is that of the two-species theory in section 3.2.1, restricted along the diagonal $\Delta : \mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}} \times \mathbf{B}U(1)_{\mathrm{conn}}$. This is the homotopy fiber of the smooth cup square in these degrees.

According to [Hi12] aspects of the differential geometry of the homotopy fiber of a differential refinement of this cup square are captured by the "generalized geometry of B_n-type" that was suggested in section 2.4 of [Ba11]. In view of the relation of the same structure to differential T-duality discussed above in section 3.2.1 one is led to expect that "generalized geometric of B_n-type" captures aspects of the differential cohomology on fiber products of torus bundles that exhibit auto T-duality on differential K-theory. Indeed, such a relation is pointed out in [Bo11][2].

3.2.3 $(4k+3)d$ $U(1)$-Chern-Simons theory and self-dual $(2k+1)$-form field theory

The differential cup square in general degree

$$(-)^{\cup^2_{\mathrm{conn}}} : \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \to \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$$

for any $k \in \mathbb{N}$ reduces in codimension 0 and on cohomology classes to the action functional

$$\exp(2\pi i \int_{\Sigma_{4k+3}} [\Sigma_{4k+3}, (-)^{\cup^2_{\mathrm{conn}}}]) : H^{2k+2}_{\mathrm{conn}}(\Sigma_{4k+3}) \to U(1)$$

on differential cohomology that exhibits $(4k+3)$-dimensional $U(1)$-Chern-Simobs theory, that is considered generally for instance in [FP89, HS05]. For $k = 0$ this is the 3-dimensional system from section 3.2.2. Generally, its spaces of quantum states in codimension 1 produces the conformal blocks of self-dual $(2k+1)$-form gauge theory on Σ_{4k+2} – this is higher Chern-Simons *holography*, as discussed generally in [BM06] and for the case of $k = 1$ famously in [Wi96, Wi98b].

We briefly recall the self-dual theories as k varies.

$k = 0$: the self-dual scalar in 3 dimensions. The action for the scalar field ϕ in two dimensions is $d\phi \wedge *d\phi$. The partition function of this field can be described via 3-dimensional Chern-Simons theory, which takes the form

$$i \int_{Y^3} \mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A) = i \int_{Y^3} \mathrm{tr}(A) \wedge F_A, \tag{3.2.1}$$

where the curvature 2-form F_A is a representative for the first Chern of a complex line bundle.

---

[2]Thanks, once more, to Alexander Kahle, for discussion of this point, at *String-Math 2012*.

$k = 1$: the $6d$ self-dual theory on the M5-brane. The action functional of classical 11-dimensional supergravity contains a *cubic* abelian Chern-Simons, recalled below in section 3.3.2. After compactification on a four-sphere S^4 this becomes an abelian 7-dimensional quadratic Chern-Simons term, an example of the above system for $k = 1$. In [Wi96, Wi98b] it is argued that this topological term alone in the full supergravity action functional determines the conformal blocks of the $(0, 2)$-superconformal field theory on a single M5-brane under $\mathrm{AdS}_7/\mathrm{CFT}_6$-duality. But if the 11-dimensional quantum corrections are taken into account, the 11-dimensional Chern-Simons term is accompanied by further terms which after reduction to 7 dimensions involve a cup product of a nonabelian 3d Chern-Simons theory as in section 3.1.2 with itself, whose action thus locally reads [Wi98b]

$$-i\tfrac{N}{4\pi}\int_{Y^7} \mathrm{CS}_3(A) \cup d\mathrm{CS}_3(A) \;=\; -i\tfrac{N}{4\pi}\int_{Y^7}\left(\langle A, dA\rangle + \tfrac{1}{3}\langle A, [A, A]\rangle\right) \wedge \langle F_A, F_A\rangle. \qquad (3.2.2)$$

as well as an indecomposable 7-dimensional term. In [FSS12a, FSS12b] we argued that if furthermore the *flux quantization* of the supergravity C-field is taken into account, then the quantum-corrected 7d Chern-Simons action that is holographically dual to the M5-brane theory is defined on *String 2-form fields* as in section 3.1.

$k = 2$: Ramond-Ramond fields in type IIB string theory. Type II RR fields are self-dual. The relation between the RR partition function to the Chern-Simons theory in eleven dimensions is explained in [BM06] (see also [Sa10]). The action is of the form $\int_{Y^{11}} F_5 \wedge dF_5$ and the quantization condition of the Ramond-Ramond fields implies that these fields are given essentially by the Chern character: $F_5 = \mathrm{ch}(E)\sqrt{A(X)}$, where E is the Chan-Paton bundle [MW]. The way the Chern character is to be interpreted is by extending by a circle to one dimension higher. Alternatively, one can view F_5 as a "composite connection" for a degree six field strength [Wi96]. Identifying F_5 with the Chern-Simons 5-form CS_5, shows that the Chern-Simons action is indeed of the form $\int_{Y^{11}} \mathrm{CS}_5 \wedge d\mathrm{CS}_5$.

$k = 3$: Fivebrane structures and 15-dimensional theories. One could continue this pattern in the obvious way. For example, one could consider $\mathrm{CS}_7(A) \cup d\mathrm{CS}_7(A)$ with $d\mathrm{CS}_7(A)$ an 8-form representative for the second Pontrjagin class of a String bundle [SSS08, SSS09a]. With the right normalization constant κ, one associates this 15-dimensional action to the Fivebrane structure [SSS08]. A lift of this to sixteen dimensions would take the form $x_8 \cup x_8$, an instance of which is studied in [Sa09] in the lift of M-theory to higher dimensions.

3.2.4 The cup-product of two extended CS theories and the higher charge anomaly

We have already discussed in the introduction section 1 the interpretation of regarding the differential cup product from def. 2.3.2 as an extended action functional

$$(-) \cup_{\mathrm{conn}} (-) : \mathbf{B}^p U(1)_{\mathrm{conn}} \times \mathbf{B}^1 U(1)_{\mathrm{conn}} \to \mathbf{B}^{p+q+1} U(1)_{\mathrm{conn}}.$$

By itself this encodes higher Maxwell charge anomalies in terms of extended Chern-Simons theory. We briefly recall what this looks like in heterotic string theory, which in the counting of the previous section coresponds to the pair of degrees $(k_1, k_2) = (1, 3)$. See the third section of [Sc12a] for more exposition in the present context.

The *local anomaly* term (the curvature of the fully extended action functional on the moduli stack of fields) is here a 12-form $I_4(F, R) \wedge I_8(F, R)$, where $I_4(F, R)$ and $I_8(F, R)$ are the Green-Schwarz anomaly polynomials in degree 4 and 8, respectively, in terms of the curvature R of the tangent bundle and the curvature F of the gauge bundle. These terms are given essentially by a difference of first Pontrjagin classes and a difference of second Pontrjagin classes, respectively. Thus, in eleven dimensions, this is a cup product Chern-Simons theory, which can be written as $\int_Y \mathrm{CS}_3(\nabla) \cup d\mathrm{CS}_7(A)$ or, dually as $\int_Y \mathrm{CS}_7(A) \cup d\mathrm{CS}_3(\nabla)$, where A is the connection of the curvature F on the gauge bundle and ∇ is the Spin connection; see [SSS09a].

Notice that, since both the gauge bundle and the tangent bundle are involved, the Chern-Simons action term is of the mixed type. Such a Chern-Simons theory can be reduced by one dimension, as is the case in the above systems. The reduction to the boundary of Y should be done in the context of manifolds of corners of codimension 2, as explained in [Sa11a]. On the boundary one then has a term of the form $CS_3(\nabla) \cup CS_7(A)$; see [Sa12b] for more details.

3.3 Cubic and higher order examples

We have seen so far examples that are the cup products of two copies of the same or different Chern-Simons theories. One might wonder whether more than two terms can naturally occur. There are at least two remarkable examples of abelian Chern-Simons theories where there are three terms in the action.

3.3.1 $5d$ supergravity

The topological part of pure five-dimensional $N = 2$ supergravity resembles that of M-theory, except that a connection 1-form A_1 replaces the C-field. That term is locally given by

$$\int_{Y^5} A_1 \wedge F_2 \wedge F_2 \, , \tag{3.3.1}$$

where $F_2 = dA_1$ is the curvature of the $U(1)$-connection A_1. Considering the refinement to differential cohomology, we interpret this as the three-term cup-product

$$\int_{Y^5} CS_1 \cup dCS_1 \cup dCS_1 \, , \tag{3.3.2}$$

i.e. a 3-fold Chern-Simons theory. Thus this falls under our formulation and hence admits a refinement to the corresponding moduli stacks of supergravity fields.

3.3.2 $11d$ supergravity

The topological aspects of this supergravity theory allows for a glimpse at the elusive M-theory. An ingredient which allows for this is the Chern-Simons term for the C-field given by

$$\tfrac{1}{6} \int_{Y^{11}} C_3 \wedge G_4 \wedge G_4 \, , \tag{3.3.3}$$

where G_4 is the field strength of the C-field 3-form C_3. Geometrically, this can be seen as the curvature 4-form of a connection on a $U(1)$-2-gerbe. Therefore, refined to differential cohomology, the above action takes the form of a three-term cup-product of the type (2.3.1) for $k = 2$. Note that the C-field is essentially a Chern-Simons 3-form $CS_3(A)$ for a connection 1-form A which admits a refinement to moduli 3-stacks (see [FSS12b]). The total term (3.3.3) thus admits a refinement in the our sense of higher cup-product Chern-Simons theories.

3.3.3 Higher order examples

We now consider the situation when we have four or more terms in the cup product. We will describe a pattern that emerges. Consider a generalization of the heterotic anomaly cancellation discussed above, where the anomaly takes the form of the wedge product of two Chern characters ch_{n_1} and ch_{n_2}, to more terms, that is to

$$S_Z = \int_{Z^{n_1+n_2+\cdots n_k}} \mathrm{ch}_{n_1} \wedge \mathrm{ch}_{n_2} \wedge \cdots \wedge \mathrm{ch}_{n_k} \, . \tag{3.3.4}$$

With the local formula $\mathrm{ch}_{n_i} = dCS_{2n_i+1}$ and passing to differential cohomology, we can write each of the factors in (3.3.4) in terms of CS_{2n_i+1}, for $i = 1, \cdots, k$. This involves using a type of Stokes formula. With

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k such operations, we are considering using a Stokes formula for various faces in codimension k, in the setting advocated in [Sa11a, Sa12a, Sa12b]. That is, we take $Z^{n_1+n_2+\cdots+n_k}$ to admit a codimension-k corner $X^{n_1+n_2+\cdots+n_k-k}$, on which the action takes the form

$$S_X = \int_{X^{n_1+n_2+\cdots+n_k-k}} CS_{2n_1+1}(A_1) \cup CS_{2n_2+1}(A_2) \cup \cdots \cup CS_{2n_k+1}(A_k) \,. \tag{3.3.5}$$

This is a n_k-fold Chern-Simons theory.

We express this again, now for extended action functionals on higher moduli stacks. Let G be a compact and simply connected simple Lie group and let c the characteristic class given by the canonical generator of $H^4(BG; \mathbb{Z})$. Then we have the $(k+1)$-fold cup product of c with itself defining a degree $4k+4$ integral cohomology class $c \cup \cdots \cup c$. In terms of characteristic maps, this corresponds to the composition

$$c \cup \cdots \cup c : \ BG \xrightarrow{(c,c)} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 4) \xrightarrow{\cup} K(\mathbb{Z}, 4k+4) \,. \tag{3.3.6}$$

Since G is simply connected, the characteristic map c is induced by the canonical Lie algebra 3-cocycle on G. Then it has, by [FSS10], a differential refinement to a morphism of stacks

$$\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \,. \tag{3.3.7}$$

By itself, this induces ordinary 3d Chern-Simons theory. Then the $(k+1)$-fold differential cup product of $\mathbf{c}_{\mathrm{conn}}$ with itself induces a $(4k+3)$-dimensional theory. Namely, we have a differential refinement of the ordinary integral cup product $c \cup c \cup \cdots \cup c$ to a morphism of smooth $(4k+3)$-stacks

$$\hat{\mathbf{c}} \,\hat{\cup}\, \hat{\mathbf{c}} \,\hat{\cup} \cdots \,\hat{\cup}\, \hat{\mathbf{c}} : \ \mathbf{B}G_{\mathrm{conn}} \xrightarrow{(\hat{\mathbf{c}},\ldots,\hat{\mathbf{c}})} \mathbf{B}^3 U(1)_{\mathrm{conn}} \times \mathbf{B}^3 U(1)_{\mathrm{conn}} \times \cdots \times \mathbf{B}^3 U(1)_{\mathrm{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}} \,.$$

Thus if Σ_{4k+3} is a closed oriented smooth manifold of dimension $4k+3$, we have a *cup product Chern-Simons theory* induced by $\mathbf{c}$: its Chern-Simons functional is

$$\exp(iS_{c \cup \cdots \cup c}) : \ \mathbf{H}(\Sigma_{4k+3}, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\hat{\mathbf{c}} \,\hat{\cup}\, \cdots \,\hat{\cup}\, \hat{\mathbf{c}}} \mathbf{H}(\Sigma, \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}}) \xrightarrow{\exp(2\pi i \int_{\Sigma_{4k+3}}(-))} U(1) \,. \tag{3.3.8}$$

On those gauge field configurations for which the underlying G-principal bundle on Σ_{4k+3} is topologically trivial, this action has a particularly simple expression: if $\mathfrak{g}$ denotes the Lie algebra of G, then the datum of a $\mathfrak{g}$-connection on G is the datum of a $\mathfrak{g}$-valued 1-form A on Σ, and the Chern-Simons functional $\exp(iS_{c \cup \cdots \cup c})$ is

$$\exp\left(2\pi i \int_{\Sigma} CS_3(A) \wedge \langle F_A, F_A \rangle \wedge \cdots \wedge \langle F_A, F_A \rangle \right), \tag{3.3.9}$$

where $\langle\,,\,\rangle$ is the Killing form of $\mathfrak{g}$, F_A is the curvature 2-form of A and $CS_3(A)$ is the Chern-Simons 3-form of A

$$CS_3(A) := \langle A, dA \rangle + \tfrac{1}{3}\langle A, [A, A] \rangle.$$

One can consider more generally a compact connected Lie group G with $\pi_1(G) \simeq \mathbb{Z}$ and the lift

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}} \tag{3.3.10}$$

of the generator c of $H^2(BG, \mathbb{Z})$ to a morphism of stacks. For instance if $G = U(N)$ then $c = c_1$ is the first Chern class, and the lift $\hat{\mathbf{c}}_1$ is induced by the group homomorphism $\det : U(N) \to U(1)$. The Chern-Simons 1-form $CS_1(A)$ is just the trace of the connection form A in this case, while $dCS_1(A) = F_A$. Hence the first Chern class induces $(2k+1)$-dimensional $(k+1)$-fold product Chern-Simons theories, whose action – in the particular case of a topologically trivial $U(1)$-bundle – reads

$$\exp\left(\kappa i \int_{\Sigma} \mathrm{tr}(A) \wedge F_A \wedge \cdots \wedge F_A \right). \tag{3.3.11}$$

Remark on classification of abelian Spin n-fold Chern-Simons theories. Classification of general Chern-Simons theories is a formidable task. Three-dimensional abelian Spin Chern-Simons theories with structure group $U(1)^N$ have been classified by Belov and Moore [BM05]. This classification of quantum theories involves three invariants, one of which is a quadratic form. It is natural to ask what the corresponding classification for cup product of such theories would be. We do not attempt a complete answer to this question, but merely point out that that such an extension should involve a correspondence with higher forms, that is beyond quadratic forms. The 2-fold theories, as in [BM05], require an extension to a bounding manifold. On the other hand, the n-fold theories will require an extension to a bounding manifold in the sense of manifolds with corners, such that the original manifold is a manifold with corners of codimension-n. The corresponding forms will have degree n, that is cubic for $n = 3$, quartic for $n = 4$, etc.

Acknowledgements

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