Entanglement of Sections:
The pushout of entangled and parameterized quantum information

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Abstract

Recently Freedman & Hastings asked [FH23] for a mathematical theory that would unify quantum entanglement/tensor-structure with parameterized/bundle-structure via their amalgamation (a hypothetical pushout) along bare quantum (information) theory — a question motivated by the role that vector bundles of spaces of quantum states play in the K-theoretic classification of topological phases of matter (there: parameterized over the Brillouin torus).

As a proposed answer to this question, we first make precise a form of the relevant pushout diagram in monoidal category theory. Then we prove that the pushout produces what is known as the *external* tensor product on vector bundles/K-classes, or rather on flat such bundles (flat K-theory), i.e., those equipped with monodromy encoding topological Berry phases. The external tensor product was recently highlighted in discussion of topological phases of matter in [Me20] and through our work in quantum programming theory [MRSS23] but has not otherwise found due attention in quantum theory yet.

The bulk of our result is a further homotopy-theoretic enhancement of the situation to the "derived category" (∞-category) of flat ∞-vector bundles (∞-local systems") equipped with the "derived functor" of the external tensor product. We explain how this serves as categorical semantics for the multiplicative fragment of the homotopically typed quantum programming language `LHoTT` [MRSS23]. This is the generality in which we recently showed [MeSS23] that topological anyonic braid quantum gates are native objects in the `LHoTT`-quantum programming language (in which case the parameterization is over the configuration space of defect anyons in the Brillouin zone).

Contents

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1 Introduction and outline

Entangled quantum processes and Tensor categories. The natural logical framework for pure quantum information theory (e.g. [NS00]) is (this goes back to [Gi87, p. 7][Ye90][Pr92], has been put to practical use since [AC04][AD06], further exposition may be found in [Sl05][Ba06][BS11][CK17][HV19]) the "internal logic" – in fact the "internal type theory" – of the closed monoidal †-category of (finite-dimensional) complex Hilbert spaces, with respect to their usual linear tensor product "$\otimes$":

1. (**Superposition**) The fact that for finite-dimensional Hilbert spaces the classical logical connective of conjunction (interpreted as the cartesian product) merges with the logical disjunction (the coproduct) into a single *biproduct* (the direct sum) effectively encodes the superposition principle of quantum physics.
2. (**No-cloning**) the appearance of another "multiplicative conjunction" interpreted as the *tensor* product reflects, due to its *non-cartesian* nature, the no-cloning/no-deletion constraints on pure quantum processes (interpreted as the absence of diagonal- and projection morphisms).
3. (**Entanglement**) and in combination, these give rise to the all-important phenomenon of entanglement of quantum states (namely the superposition of product states in a tensor product).

In fact, all of these phenomena are governed already by (the internal linear logic of, cf. [Mur14]) the underlying tensor category of complex vector spaces:

Backdrop for *entangled* quantum processes featuring: superposition, no-cloning, and quantum entanglement

non-cartesian monoidal category of complex vector spaces

$$(\mathrm{Mod}_{\mathbb{C}}, \otimes) \quad = \quad \left\{ \begin{array}{ccc} \overset{\text{vector space}}{\mathcal{V}} & \xrightarrow{\quad \phi \quad}_{\text{linear map}} & \mathcal{W} \\ \otimes \ \text{tensor product} & & \otimes \\ \mathcal{V}' & \xrightarrow{\quad \phi' \quad} & \mathcal{W}' \end{array} \right\}$$

The refinement of this situation to Hilbert-spaces serves to provide the:

4. (**Born rule**) The further †-structure on complex *Hilbert* spaces (sending linear operators to their adjoints) reflects the Hermitian inner product in quantum states and hence the probabilistic nature of quantum physics.

One byproduct of our construction here is (see the end of §4) a previously missing natural realization of this Hilbert/dagger-structure inside linear type theory.

So far this concerns coherent quantum processes on pure states, undisturbed by interaction with (such as in quantum measurement) or control by (such as in quantum state preparation) a classical environment. Such classical$\leftrightarrow$quantum interactions are reflected in *parameterized* quantum systems:

Parameterized quantum processes and Bundle categories. More recent developments [Sc14, §6.1][RS18][FKS20][MRSS23] show that the logic of interaction between quantum systems and classical environments is essentially the "internal logic" of categories of Hilbert space-*bundles* over varying classical base spaces (such as plain sets):

1. (**Many worlds**) The fact that different quantum states, or even different Hilbert spaces, may be seen in different classical worlds (in different classical parameter configurations) is reflected in these states being *sections* of *bundles* of state spaces over classical parameter spaces.
2. (**Quantum compulsion**) in which picture the above superposition principle re-appears as the fact that along maps of parameter base spaces with finite pre-images, the left- and right-pushforward of vector bundles coincide.
3. (**Quantum measurement**) In particular, the push-pull of vector bundles along a map with finite pre-image B is a (co)monadic operation whose (co)unit reflects exactly the collapse of quantum states in the Hilbert space $\mathrm{Q}B$ branched according to the classical measurement outcome in B;
4. (**Quantum state preparation**) and, dually, classical quantum state preparation is reflected in the operation where dependent on a classical parameter $b \in B$ we pick in the linear fiber $\mathbb{C}B$ the corresponding basis vector.

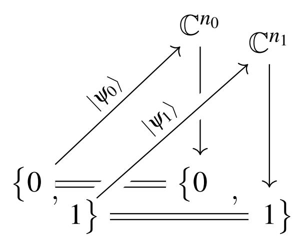
In particular, in this category a classical logical conjunction is restored, the co-cartesian coproduct operation $\sqcup$ on bundles, which interprets the logical connective of having some quantum states in some possible world *or* other quantum states in another possible world:

Backdrop for *parameterized* quantum processes featuring: many worlds, quantum measurement & state collapse, quantum state preparation, etc.

$$\underset{\text{of complex vector bundles}}{\text{co-cartesian monoidal category}} \quad \big(\mathrm{Bun}_{\mathbb{C}}, \sqcup\big) \;=\; \left\{ \begin{tikzcd} \underset{\mathcal{V}}{\text{vector bundle}} \arrow[rr, "\phi", "\text{fiberwise linear map}"'] \arrow[d, "\text{bundle projection}"] && \mathcal{V}' \arrow[d] \\ \underset{\text{base space}}{X} \arrow[rr, "f", "\text{classical map}"'] && X' \end{tikzcd} \right\}$$

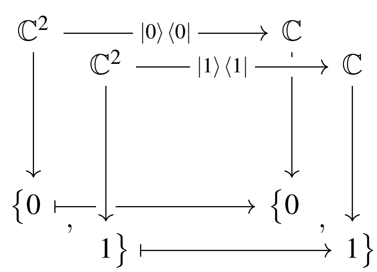
In this context, a quantum state is no longer just an element of a fixed Hilbert space, but is a *section* of a bundle of such spaces (in the language of type theory: a "dependent term of a dependent type") assigning to each parameter value (to each possible classical world) the quantum state seen for that parameter value (in that world).

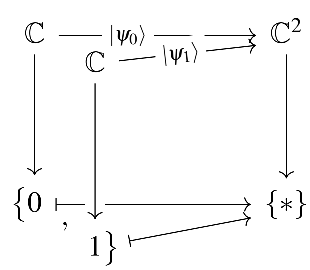
In parameterized quantum theory
a quantum state is a *section*
of a bundle of quantum state spaces
assigning to each parameter value
(to each possible classical world)
the quantum state for that value
(as seen in that particular world).



Crucially, in the category of parameterized bundles of quantum states, *quantum measurement* and the ensuing *quantum state collapse* is naturally reflected by linear projection maps parameterized over the set of possible measurement outcomes, and dually for quantum state preparation:

Process of quantum measurement of qbits $q_0|0\rangle + q_1|1\rangle \in \mathbb{C}^2 \simeq \mathbb{C}[\{0,1\}]$ as a morphism of Hilbert space bundles over spaces of possible classical worlds





State preparation conditioned on classical parameters

In fact ([MRSS23]): the quantum measurement process is the *counit* of the *base change comonad* $\Box_{\{0,1\}} :\equiv p^* p_*$ on the *slice category* of $\mathrm{Bun}_{\mathbb{C}}$ over the 0-bundle $0_{\{0,1\}}$:

$$\begin{tikzcd} 0_{\{0,1\}} \arrow[rr, "p"] && 0_{\{*\}} \end{tikzcd} \qquad \begin{array}{c}\text{map of vector bundles}\\ \text{(here: 0-bundles over sets)}\end{array}$$

$$\begin{array}{c}\textbf{co-monad expressing}\\ \textbf{quantum measurement}\\ \textbf{as a computational effect}\end{array} \quad \begin{tikzcd} \big(\mathrm{Bun}_{\mathbb{C}}\big)_{/0_{\{0,1\}}} \arrow[loop left, "\Box_{\{0,1\}}"] \arrow[rr, shift left=8, "p_!"] \arrow[rr, shift left=4, phantom, "\perp"] \arrow[rr, shift right=4, phantom, "\perp"] \arrow[rr, shift right=8, "p_*"'] && \big(\mathrm{Bun}_{\mathbb{C}}\big)_{/0_{\{*\}}} \arrow[ll, "p^*" description] \end{tikzcd} \quad \begin{array}{c}\text{induced base change adjunction}\\ \text{between slice categories (of}\\ \text{bundles of vector bundles!)}\end{array}$$

It is such base change adjunctions between slices of $\mathrm{Bun}_{\mathbb{C}}$ which interpret the dependent linear type inference rules of quantum programming languages like `Quipper` [RS18] and `LHoTT` [MRSS23].

Unification of entangled and parameterized quantum information. One is naturally led to wonder about *amalgamating* these two fragments of quantum information theory:

1. the non-cartesian monoidal tensor structure on plain vector spaces encoding pure quantum phenomena such as entanglement,
2. the co-cartesian monoidal structure of vector bundles encoding quantum/classical phenomena such as state collapse upon quantum measurement,

by coupling these two theory sectors along their common core of vector spaces of quantum states.

In the language of category theory, such an amalgamation of two objects along a common core would be called a *pushout* (to be abbreviated "po"), which in the present case would mean to ask for the *universal* way of completing the following (for the moment: schematic) diagram to a commuting square, in a suitable sense:

$$\begin{tikzcd}[column sep=small] \begin{array}{l}\textbf{Pure quantum phenomena}\text{:}\\ \text{no-cloning, }\textbf{entanglement}\text{,...}\end{array} & (\mathrm{Mod}_{\mathbb{C}}, \otimes) \arrow[rr, dashed] & & \boxed{(??)} & \begin{array}{l}\textbf{Parameterized quantum phenomena}\\ \textbf{entanglement of sections}\end{array} \\ \begin{array}{l}\textbf{Core quantum phenomena}\text{:}\\ \text{superposition principle}\end{array} & \mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[rr] \arrow[urr, phantom, "\text{(po)}"] & & (\mathrm{Bund}_{\mathbb{C}}, \sqcup) \arrow[u, dashed] & \begin{array}{l}\textbf{Classical} \leftrightarrow \textbf{Quantum phenomena}\text{:}\\ \text{parameterized quantum states: }\textbf{sections}\\ \text{state collapse/preparation in many worlds,}\end{array} \end{tikzcd} \tag{1}$$

Essentially the following natural **Question** was recently raised in [FH23, p. 1]:

(i) How to make this precise?

(ii) What then is the pushout?

(iii) What is its import on quantum information theory?

Here we offer an **Answer.** Informally, our answer says:

> **The amalgamation of**
> **the entanglement tensor product** structure on Hilbert spaces
> **with the parameterized coproduct** structure on Hilbert bundles
> **is the** ***external*** **tensor product** structure on Hilbert bundles.

External tensor product. Here the *external tensor product of vector bundles* (or that induced on their K-theory classes) [At67, §2.6][Bo69, p. 19][Sw75, §13.51][Ka78, §4.9] (cf. also [GHV73, p. 84][Ly01, p. 2][Sh13, p. 7]) forms the cartesian product of parameter base spaces and over each pair of parameter values assigns the tensor product of the corresponding Hilbert spaces:

$$\begin{tikzcd} \begin{array}{c}\textbf{External tensor of}\\ \textbf{vector bundles}\end{array} & \mathcal{V} \boxtimes \mathcal{W} \arrow[d] & \mathcal{V}_x \otimes \mathcal{W}_y \arrow[l, hook', "\textbf{fiber}"'] \arrow[d] & \begin{array}{c}\textbf{Tensor product}\\ \textbf{of vector spaces}\end{array} & \begin{array}{c}\textbf{Entanglement of}\\ \textbf{quantum states}\end{array} \\ \begin{array}{c}\textbf{Classical product}\\ \textbf{of base spaces}\end{array} & X \times Y & \{(x,y)\} \arrow[l, hook', "\textbf{base point}"'] & \begin{array}{c}\textbf{Pair of}\\ \textbf{base points}\end{array} & \begin{array}{c}\textbf{Pairing of}\\ \textbf{classical parameters}\end{array} \end{tikzcd}$$

While, as a mathematical construction, the external tensor product (specifically in topological K-theory) is well-known, its relevance to quantum theory has been considered only recently [Me20][MRSS23] (cf. Rem. 3.37 below).

A key point of the external tensor product is that it *distributes* over the disjoint union of bundles (the co-cartesian product):

$$\mathcal{V}_X \boxtimes \big(\mathcal{W}_Y \sqcup \mathcal{W}'_{Y'}\big) \;\simeq\; \big(\mathcal{V}_X \boxtimes \mathcal{W}_Y\big) \sqcup \big(\mathcal{V}_X \boxtimes \mathcal{W}'_{Y'}\big)\,.$$

In this way $\big(\mathrm{Bun}_{\mathbb{C}}, \sqcup, \boxtimes\big)$ forms what is called a *distributive category* (Def. 2.1):

$$\begin{array}{l}\text{Backdrop for}\\ \textit{parameterized entangled}\\ \text{quantum processes, featuring}\\ \text{entanglement of sections}\end{array} \qquad \begin{array}{c}\textbf{Distributive monoidal category}\\ \textbf{of complex vector bundles}\\ \textbf{with external tensor product}\\ \big(\mathrm{Bund}_{\mathbb{C}}, \sqcup, \boxtimes\big)\end{array} \equiv \left\{ \begin{tikzcd} & \textbf{external tensor}\ \begin{array}{c}\textbf{vector}\\ \textbf{bundle}\\ \mathcal{V}\\ \boxtimes\\ \mathcal{W}\end{array} \arrow[dl] \arrow[rr, "\begin{array}{c}\textbf{fiberwise}\\ \textbf{linear map}\end{array}"] & & \begin{array}{c}\mathcal{V}'\\ \boxtimes\\ \mathcal{W}'\end{array} \arrow[dl] \\ \begin{array}{c}X\\ \times\\ Y\end{array} \arrow[rr, "\textbf{classical map}"'] & & \begin{array}{c}X'\\ \times\\ Y'\end{array}\ \textbf{classical product} & \end{tikzcd} \right\}.$$

Universal property of external tensor product of vector bundles over discrete spaces. Consider the special case of vector bundles over discrete spaces, i.e., over plain sets. (While just a small special case of the general mathematical notion, this already captures all parameterization of quantum processes considered in contemporary quantum information theory). It is useful to understand this category as the unification ("Grothendieck construction") of all categories of vector bundles over fixed sets, which in turn are usefully understood via their fiber-assigning functors:

$$
\begin{array}{ccc}
\mathrm{Mod}_{\mathbb{C}} & \overset{\iota}{\hookrightarrow} & \overbrace{\int_{X \in \mathrm{Set}} \mathrm{Mod}_{\mathbb{C}}^{X}}^{\mathrm{Fam}_{\mathbb{C}}} \\
\mathcal{V} & \mapsto & \begin{bmatrix} \mathcal{V} \\ \downarrow \\ \mathrm{pt} \end{bmatrix}
\end{array}
\quad \text{where} \quad
\begin{array}{ccc}
\mathrm{Set}^{\mathrm{op}} & \longrightarrow & \mathrm{Cat} \\
X & \mapsto & \mathrm{Func}(X, \mathrm{Mod}_{\mathbb{C}}) \equiv \mathrm{Mod}_{\mathbb{C}}^{X} \\
\big\downarrow{\scriptstyle f} & & \big\uparrow{\scriptstyle f^*} \\
Y & \mapsto & \mathrm{Func}(Y, \mathrm{Mod}_{\mathbb{C}}) \equiv \mathrm{Mod}_{\mathbb{C}}^{Y}.
\end{array}
\tag{2}
$$

Notice that every bundle over a discrete space X is the coproduct of its restrictions to the points in the base space:

$$
\mathcal{V}_X \;=\; \begin{array}{c} \mathcal{V}_x \\ \downarrow \\ \{x\} \end{array} \sqcup \begin{array}{c} \mathcal{V}_y \\ \downarrow \\ \{y\} \end{array} \sqcup \begin{array}{c} \mathcal{V}_z \\ \downarrow \\ \{z\} \end{array} \sqcup \cdots .
$$

In fact, vector bundles over sets are the *free coproduct completion* (Ex. A.6) of the plain category of vector spaces. But this means that on such bundles the external tensor product is *completely characterized* by these two properties:

1. Over singletons, it reduces to the ordinary tensor product: $\begin{array}{c} \mathcal{V} \\ \downarrow \\ \mathrm{pt} \end{array} \boxtimes \begin{array}{c} \mathcal{W} \\ \downarrow \\ \mathrm{pt} \end{array} \simeq \begin{array}{c} \mathcal{V} \otimes \mathcal{W} \\ \downarrow \\ \mathrm{pt} \end{array}$

2. It distributes over coproducts: $\begin{array}{c} \mathcal{V} \\ \downarrow \\ \{x\} \end{array} \boxtimes \left(\begin{array}{c} \mathcal{W} \\ \downarrow \\ \{y\} \end{array} \sqcup \begin{array}{c} \mathcal{W}' \\ \downarrow \\ \{y'\} \end{array} \right) = \left(\begin{array}{c} \mathcal{V} \\ \downarrow \\ \{x\} \end{array} \boxtimes \begin{array}{c} \mathcal{W} \\ \downarrow \\ \{y\} \end{array} \right) \sqcup \left(\begin{array}{c} \mathcal{V} \\ \downarrow \\ \{x\} \end{array} \boxtimes \begin{array}{c} \mathcal{W}' \\ \downarrow \\ \{y'\} \end{array} \right).$

This characterization may be reformulated (we make this precise in §2.1) as saying that the following diagram is a *pushout* in a suitable category of plain, monoidal, cocartesian and distributive categories, whose morphisms are strong monoidal functors with respect to the monoidal structure present on their domain category (Thm. 2.14):

$$
\begin{tikzcd}
(\mathrm{Mod}_{\mathbb{C}}, \otimes) \arrow[r, hook, "\iota"] \arrow[dr, phantom, "\text{(po)}"] & \boxed{(\mathrm{Fam}_{\mathbb{C}}, \sqcup, \boxtimes)} \\
\mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[r, hook, "\iota"] & (\mathrm{Fam}_{\mathbb{C}}, \sqcup) \arrow[u]
\end{tikzcd}
\tag{3}
$$

Therefore, this is a first answer to the question (1) for the case of discrete parameter spaces. While this subsumes all of the contemporary quantum information theory, we naturally want to go further.

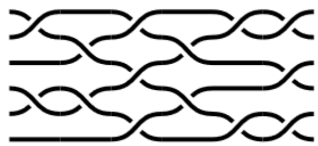
External tensor product of group representations. Some parameters in quantum physics are known not to form discrete sets but to form homotopy 1-types: groupoids [We96]; and the Hilbert bundles over these parameter spaces have *monodromy*. A famous examples are bundles of conformal blocks over configuration spaces of points and equipped with the Knizhnik-Zamolodchikov connection. These are thought to be the bundles of Hilbert spaces for *anyons* (for extensive references to the literature and further discussion related to our perspective here, see [MeSS23]).

$$
\begin{array}{ccccc}
\overset{\substack{\textbf{Hilbert bundle} \\ \textbf{of anyon states} \\ \textbf{("conformal blocks")}}}{\mathcal{H}} & & & & \\
\big\downarrow & & & & \\
\underset{\substack{\textbf{configuration} \\ \textbf{space}}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^2\big)} & \underset{\mathrm{whe}}{\simeq} & \underset{\substack{\textbf{Eilenberg-MacLane} \\ \textbf{space}}}{K\big(\mathrm{PBr}(N), 1\big)} & \underset{\mathrm{whe}}{\simeq} & \underset{\substack{\textbf{classifying} \\ \textbf{space}}}{B\big(\overset{\substack{\textbf{pure braid} \\ \textbf{group}}}{\mathrm{PBr}(N)}\big)}
\end{array}
$$

An elegant way to encode a *flat connection* on such bundles is to consider the *fundamental groupoid* (25) of their base space – which for connected spaces X is equivalently the delooping groupoid $\mathbf{B}\pi_1(X)$ (19) with a single object and the elements of the fundamental group $\pi_1(X)$ as morphisms. Then a flat connection over X is equivalently its monodromy functor $\mathbf{B}\pi_1(X) \to \mathrm{Mod}_{\mathbb{C}}$ ([SW09][Du10]), also known as a "local system" ([De70, §I.1][Sp66, p. 58][Di04, Prop. 2.5.1]):

$$
\begin{array}{ccc}
\mathbf{B}\big(\mathrm{PBr}(N)\big) & \xrightarrow[\text{monodromy}]{\text{Knizhnik-Zamolodchikov}} & \mathrm{Mod}_{\mathbb{C}} \\
\text{braid} \circlearrowright \{1,\cdots,N\} & \mapsto & \text{unitary} \circlearrowright \mathcal{H}
\end{array}
\qquad
\mathcal{H}_1 \xrightarrow{\text{anyon braid quantum gate}} \mathcal{H}_2
\qquad
\begin{array}{c}
\mathcal{H} \\
\downarrow \\
\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\Sigma)
\end{array}
$$

path in config. space of defect points



Generally/equivalently, for G any group, a complex linear G-representation is a functor from $\mathbf{B}G$ to $\mathrm{Mod}_{\mathbb{C}}$:

$$
G\mathrm{Rep}_{\mathbb{C}} \;\simeq\; \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}G} \;=\; \left\{
\begin{array}{ccc}
\mathbf{B}G & \xrightarrow{\mathcal{V}_{(-)}} & \mathrm{Mod}_{\mathbb{C}} \\
g \circlearrowright \mathrm{pt} & \mapsto & \rho_g \circlearrowright \mathcal{V}
\end{array}
\right\}. \tag{4}
$$

There is a classical notion of *external tensor product of group representations* (e.g. [FH91, Exer. 2.36]), which in this groupoid-picture is the *cup-tensor product*

$$
G, G' \in \mathrm{Grp} \qquad \vdash \qquad
\begin{array}{ccc}
G\mathrm{Rep}_{\mathbb{C}} \times G'\mathrm{Rep}_{\mathbb{C}} & \xrightarrow{\boxtimes} & (G \times G')\mathrm{Rep}_{\mathbb{C}} \\
\wr\| & & \wr\| \\
\mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}G} \times \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}G'} & \longrightarrow & \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}(G \times G')}
\end{array}
$$

$$
\mathcal{V} \boxtimes \mathcal{W} \;:\; \mathbf{B}(G \times G') \simeq \mathbf{B}G \times \mathbf{B}G' \xrightarrow{\mathcal{V}_{(-)} \times \mathcal{W}_{(-)}} \mathrm{Mod}_{\mathbb{C}} \times \mathrm{Mod}_{\mathbb{C}} \xrightarrow{\otimes} \mathrm{Mod}_{\mathbb{C}}
$$

This system of group-wise external tensor products is unified on the Grothendieck construction

$$
\mathrm{Rep}_{\mathbb{C}} \;:\equiv\; \int_{G \in \mathrm{Grp}} \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}G}
$$

to a single functor

$$
\mathrm{Rep}_{\mathbb{C}} \times \mathrm{Rep}_{\mathbb{C}} \xrightarrow{\boxtimes} \mathrm{Rep}_{\mathbb{C}}\,.
$$

Universal property of the external tensor product of flat vector bundles. To better understand this situation of group representations over varying groups, we may without restriction focus here on *skeletal groupoids* (26), namely disjoint unions of delooping groupoids, which we may think of as sets each of whose elements is equipped with a group of automorphisms:

$$
\mathrm{Grpd}_{\mathrm{skl}} \;\simeq\; \int_{S \in \mathrm{Set}} \mathrm{Grp}^S \;\simeq\; \big\{ \mathbf{B}G_1 \sqcup \mathbf{B}G_2 \sqcup \mathbf{B}G_3 \sqcup \cdots \big\}\,.
$$

Flat vector bundles on arbitrary base spaces X are equivalently functors on such skeletal groupoids ([SW09][Du10], "local systems" [LY00][Voi02, §I 9.2.1][Di04, §2.5]), namely on the skeletization of their fundamental groupoid, which has one component $\mathbf{B}\pi_1(X_i)$ for each connected component X_i of X:

$$
\underset{\text{over base space } X}{\text{flat complex vector bundle}} \quad \mathrm{Loc}_{\mathbb{C}}(X) \;\simeq\; \mathrm{Mod}_{\mathbb{C}}^{\coprod_{i \in \pi_0(X)} \mathbf{B}\pi_1(X_i)} \;:\equiv\; \mathrm{Func}\big(\mathbf{B}\pi_1(X_1) \sqcup \mathbf{B}\pi_1(X_2) \sqcup \cdots, \mathrm{Mod}_{\mathbb{C}}\big)\,.
$$

Hence replacing, in the previous discussion, bare sets of points with sets of points-with-automorphisms (i.e.: groupoids), we obtain the following category of flat vector bundles over varying base spaces:

$$
\begin{array}{ccc}
G\mathrm{Rep}_{\mathbb{C}} & \xhookrightarrow{\;\;\iota\;\;} & \overbrace{\underset{\mathscr{X}\,\in\,\mathrm{Grpd}}{\int} \mathrm{Mod}_{\mathbb{C}}^{\mathscr{X}}}^{\substack{\text{flat complex vector bundles}\\ \text{over varying base spaces}\\[4pt] \mathrm{Loc}_{\mathbb{C}} \,:\equiv}} \\
(\mathcal{V},\rho) & \longmapsto & \mathcal{V}_{\mathbf{B}G}
\end{array}
\quad \text{where} \quad
\begin{tikzcd}
\mathrm{Grpd}^{\mathrm{op}} \arrow[rr] & & \mathrm{Cat} \\
\mathscr{X} \arrow[d, "f"] & \mapsto & \mathrm{Func}\big(\mathscr{X}, \mathrm{Mod}_{\mathbb{C}}\big) \equiv \mathrm{Mod}_{\mathbb{C}}^{\mathscr{X}} \\
\mathscr{Y} & \mapsto & \mathrm{Func}\big(\mathscr{Y}, \mathrm{Mod}_{\mathbb{C}}\big) \equiv \mathrm{Mod}_{\mathbb{C}}^{\mathscr{Y}} \arrow[u, "{\mathrm{Func}(f,\mathrm{Mod}_{\mathbb{C}})}\ \ f^*"'] .
\end{tikzcd}
\tag{5}
$$

Observe that in this category, every G-representation $\mathcal{V}_{\mathbf{B}G}$ sits in a cartesian square of the following form

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{V}/\!\!/G \arrow[dr] \arrow[dd] & \\
& \mathrm{pt} \arrow[rr] \arrow[dd] & & \mathbf{B}G \arrow[dd] \\
0 \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\
& \mathrm{pt} \arrow[rr] & & \mathbf{B}G
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}_{\mathrm{pt}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_{\mathbf{B}G} \arrow[d] \\
0_{\mathrm{pt}} \arrow[r] & 0_{\mathbf{B}G}
\end{tikzcd}
$$

and that the external tensor product of representations *preserves* the Cartesianness of these squares:

$$
\begin{tikzcd}
\mathcal{V}_{\mathrm{pt}} \boxtimes \mathcal{W}_{\mathbf{B}H} \arrow[r] \arrow[d] & \mathcal{V}_{\mathbf{B}G} \boxtimes \mathcal{W}_{\mathbf{B}H} \arrow[d] \\
0_{\mathrm{pt}} \boxtimes \mathcal{W}_{\mathbf{B}H} \arrow[r] & 0_{\mathbf{B}G} \boxtimes \mathcal{W}_{\mathbf{B}H}
\end{tikzcd}
=
\begin{tikzcd}
\big(\mathcal{V}\otimes\mathcal{W}\big)_{\mathbf{B}(1\times H)} \arrow[r] \arrow[d] & \big(\mathcal{V}\otimes\mathcal{W}\big)_{\mathbf{B}(G\times H)} \arrow[d] \\
0_{\mathbf{B}(1\times H)} \arrow[r] & 0_{\mathbf{B}(G\times H)}
\end{tikzcd}
$$

But this means that on flat vector bundles, the external tensor products of vector bundles and of group representations unify into an external tensor product which is uniquely characterized by these three properties:

1. over singletons, it reduces to the ordinary tensor product: $\begin{array}{c}\mathcal{V}\\ \downarrow \\ \mathrm{pt}\end{array} \boxtimes \begin{array}{c}\mathcal{W}\\ \downarrow \\ \mathrm{pt}\end{array} \simeq \begin{array}{c}\mathcal{V}\otimes\mathcal{W}\\ \downarrow \\ \mathrm{pt}\end{array}$

2. It distributes over

(i) coproducts:
$$
\begin{array}{c}\mathcal{V}/\!\!/G\\ \downarrow \\ \mathbf{B}G\end{array} \boxtimes \left(\begin{array}{c}\mathcal{W}/\!\!/H\\ \downarrow \\ \mathbf{B}H\end{array} \sqcup \begin{array}{c}\mathcal{W}'/\!\!/H'\\ \downarrow \\ \mathbf{B}H'\end{array} \right) = \left(\begin{array}{c}\mathcal{V}/\!\!/G\\ \downarrow \\ \mathbf{B}G\end{array} \boxtimes \begin{array}{c}\mathcal{W}/\!\!/H\\ \downarrow \\ \mathbf{B}H\end{array} \right) \sqcup \left(\begin{array}{c}\mathcal{V}/\!\!/G\\ \downarrow \\ \mathbf{B}G\end{array} \boxtimes \begin{array}{c}\mathcal{W}'/\!\!/H'\\ \downarrow \\ \mathbf{B}H'\end{array} \right).
$$

(ii) homotopy quotients:
$$
\begin{array}{c}\mathcal{V}/\!\!/G\\ \downarrow \\ \mathbf{B}G\end{array} \boxtimes \begin{array}{c}\mathcal{W}/\!\!/H\\ \downarrow \\ \mathbf{B}H\end{array} = \begin{array}{c}\big(\mathcal{V}\otimes\mathcal{W}\big)/\!\!/(G\times H)\\ \downarrow \\ \mathbf{B}(G\times H)\end{array}
$$

which jointly means that it distributes over *homotopy quasi-coproducts* ([HT95], Def. 2.19), to be denoted $\sqcup^{hq}$. In homotopy-theoretic generalization of the previous discussion (§2.1), this property of the external tensor product may then be re-expressed (we make this precise in §2.2) again as a pushout, now understanding cocartesian categories as *homotopy quasi-cocartesian categories* (Thm. 2.31):

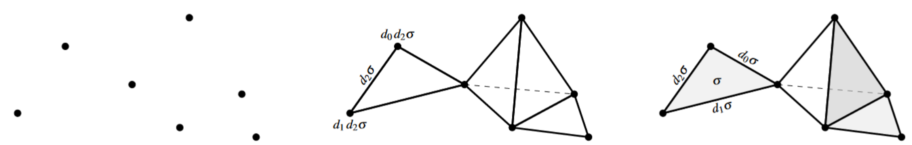
$$
\begin{tikzcd}
\big(\mathrm{Mod}_{\mathbb{C}}, \otimes\big) \arrow[r, hook, "\iota"] \arrow[dr, phantom, "\text{(po)}"] & \boxed{\big(\mathrm{Loc}_{\mathbb{C}}, \sqcup^{hq}, \boxtimes\big)} \\
\mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[r, hook, "\iota"] & \big(\mathrm{Loc}_{\mathbb{C}}, \sqcup^{hq}\big) \arrow[u]
\end{tikzcd}
\tag{6}
$$

This constitutes a satisfactory answer to the question (1). While the structures in (6) capture all of contemporary quantum information theory *and* currently understood topological quantum phenomena, we naturally want to go further:

External tensor product of higher group representations. Fully generally, parameter spaces form *higher* homotopy types (e.g. [Ba95]) and the Hilbert bundles over them are *flat higher bundles* with *higher* monodromy ([To02][PW05][Sc13, §3.5.8], hence “higher local systems”, see §3.1 below), which over connected components are labeled by *higher group representations* (Rem. 3.11). There is a growing sense in the solid state physics community (cf. [MG23]) that such higher group representations (“generalized symmetries”) are key for fundamentally understanding topological phases of quantum matter, cf. [KT17][BBCW19, §D][SS23, §2.3].

Making this general picture precise (§3) requires tools from categorical homotopy theory (e.g. [Ri14][Ric20]). Concretely, the toolbox we shall use here is *simplicial model category* theory [Qu67, §II.2][Hi02, §9][Lu09, §A.3]:

- **simplicial sets** ([Fr12][GJ99, §I]) are systems of sets that capture the idea of sets of $(n+1)$-*morphisms* between n-morphisms, for all $n \in \mathbb{N}$, built by successively attaching to each other: edges, triangles, tetrahedra, and their higher dimensional analogs.



$$
\begin{array}{ccccccc}
\textbf{simplicial set} & \textbf{set of points} & \underset{\text{constant edges}}{\overset{\text{endpoints}}{\leftleftarrows\!\!\!\!\rightarrow}} & \textbf{set of edges} & \underset{\text{thin surfaces}}{\overset{\text{boundary edges}}{\leftleftarrows\!\!\!\!\rightarrow}} & \textbf{set of surfaces} & \leftleftarrows \cdots
\end{array}
$$

$$
\mathscr{X} \;=\; \left(X_0 \;\underset{d_0}{\overset{d_1}{\underset{s_0}{\leftleftarrows\!\!\!\!\rightarrow}}}\; X_1 \;\; \begin{smallmatrix} \xleftarrow{d_2} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \\ \xrightarrow{s_1} \\ \xleftarrow{d_0} \end{smallmatrix} \;\; X_2 \;\; \cdots \right)
$$

We write sSet for the category of simplicial sets, now replacing the base category Set, so:

- sSet**-enriched categories** (e.g. [TV05, §2]) are like categories but with hom-sets enhanced to simplicial sets,
- sSet**-enriched groupoids** ([DK84][EPR21, §3]) are like groupoids but with simplicial hom-sets,
- **simplicial groups** (e.g. [Ma67, §17][GJ99, §V]) are simplicial sets $\mathscr{G}$ equipped with group operations,
- **simplicial delooping groupoids** (e.g. [SS21, §3.1.32]) are one-object sSet-enriched groupoids $\mathbf{B}\mathscr{G}$,
- **simplicial chain complexes** ([RSS01, Cor. 4.6]) model for ∞-vector spaces (Def. 3.2, Thm. 3.3, Rem. 3.1).

- *The homotopy type of any space X is equivalently encoded in a skeletal simplicial groupoid* (Dwyer-Kan theory, see §3.2.)

$$
\underset{\substack{\textbf{fundamental} \\ \textbf{simplicial groupoid}}}{\mathscr{G}(X)} \;\simeq\; \coprod_{i \in \pi_0(X)} \mathbf{B}\big(\underset{\substack{\textbf{simplicial} \\ \textbf{loop group}}}{\mathscr{G}(X_i, x_i)} \big)
$$

- *All (geometric) higher groups* ([Sc13, §3.6.8][SS20, §2.2]) *are equivalently modeled by (presheaves of) simplicial groups* [NS15, Prop. 3.35][SS21, Lem. 3.2.73].

  For example, given a crossed module of groups $H \xrightarrow{\delta} G \xrightarrow{\alpha} \mathrm{Aut}(H)$ presenting a strict 2-group (cf. [ML97, §XII.8][BL04]), then the simplicial nerve $N(G \times H \rightrightarrows G)$ of its action groupoid (cf. [ML97, §XII.2]) is the respective simplicial group.

Therefore, all the previous constructions concerning flat vector bundles (local systems), regarded as functors from fundamental groupoids to vector spaces, have a sensible simplicial generalization to simplicial local systems, given by simplicial functors from fundamental simplicial groupoids to simplicial chain complexes:

- simplicial local systems (§3.1):

$$
\mathscr{G}\mathrm{Rep}_{\mathbb{C}} \;:\equiv\; \mathbf{sCh}_{\mathbb{C}}^{\mathbf{B}\mathscr{G}} \;:\equiv\; \mathbf{sFunc}\big(\mathbf{B}\mathscr{G}, \mathbf{sCh}_{\mathbb{C}}\big)\,, \qquad \mathbf{Loc}_{\mathbb{C}}(X) \;:\equiv\; \prod_{i \in \pi_0(X)} \mathbf{sCh}_{\mathbb{C}}^{\mathbf{B}\mathscr{G}(X_i, x_i)}\,;
$$

- simplicial local systems over varying bases (§3.2):

$$
\mathbf{Rep}_{\mathbb{C}} \;:\equiv\; \int_{\mathscr{G} \,\in\, \mathrm{sGrp}} \mathbf{Mod}_{\mathbb{C}}^{\mathbf{B}\mathscr{G}}\,, \qquad \mathbf{Loc}_{\mathbb{C}} \;:\equiv\; \int_{\mathbf{X} \,\in\, \text{Set-Grpd}_{\mathrm{skl}}} \mathbf{Mod}_{\mathbb{C}}^{\mathbf{X}}\,;
$$

- external tensor of simplicial local systems (§3.3): $\boxtimes : \mathbf{Loc}_{\mathbb{C}} \times \mathbf{Loc}_{\mathbb{C}} \longrightarrow \mathbf{Loc}_{\mathbb{C}}$.

However, one has to check that these enriched 1-categories of simplicial local systems do correctly present the homotopy theory of ∞-category theoretic ∞-local systems:

Theorem: *These simplicial constructions are homotopy-good...* (this is the content of Thms. 3.3, 3.22, 3.40.)
... while retaining the pushout-characterization of the external tensor product.

Entangled quantum phenomena with higher group symmetries

Categorical semantics for multiplicative fragment of `LHoTT` quantum language

$$
\begin{tikzcd}
\big(\mathbf{Rep}_{\mathbb{C}}, \boxtimes\big) \arrow[r, hook] & \boxed{\big(\mathbf{Loc}_{\mathbb{C}}, \sqcup^{hq}, \boxtimes\big)} \\
\mathbf{sCh}_{\mathbb{C}} \arrow[u] \arrow[r, hook, "\iota"] \arrow[ur, phantom, "\text{(po)}"] & \big(\mathbf{Loc}_{\mathbb{C}}, \sqcup^{hq}\big) \arrow[u]
\end{tikzcd}
\tag{7}
$$

Topological (homotopical) core quantum phenomena

Topological (homotopical) classical$\leftrightarrow$quantum phenomena

This constitutes a fairly comprehensive answer to the question (1). One may want to go further and consider non-flat ∞-vector bundles by passing to homotopy theories of (pre)sheaves with coefficients in the categories in (7) ("tangent ∞-stacks"). This is in principle straightforward but spelling it out requires invoking more technology than should be crammed into this article; see the outlook in §4.

Outline.

- In §2 we make precise, with an elementary category-theoretic argument, how the external tensor product is an answer to question (1) in the case of vector bundles over discrete spaces (3) and of flat vector bundles (6).
- In §3, we enhance this answer, via a more advanced model-category theoretic argument, to flat ∞-vector bundles ("∞-local systems").
- In §4 we indicate the relevance of this structure in the practice of quantum information theory and provide further outlook.

Acknowledgement. We thank Dmitri Pavlov for discussion of some technical aspects in §3.

2 The external tensor product as a pushout

Here we make precise the formulation and proof of the claim for vector bundles over discrete spaces (§2.1) and for flat vector bundles over general spaces (§2.2). The discussion here involves fairly elementary category theory. While we find that there is some intrinsic conceptual interest to the theorems presented here, the reader may want to regard this section mainly as a warmup and elementary blueprint for the discussion in §3.

2.1 For vector bundles over discrete spaces

We demonstrate here – in the comparatively simple special case of discrete parameter spaces (the default in quantum information theory) – a precise sense in which there is an amalgamation of the theories of entangled and of parameterized quantum processes, and that it is encoded in an "external tensor product" on bundles of parameterized quantum state spaces (Thm. 2.14 below). We will generalize this theorem to arbitrary base spaces in §3.

Definition 2.1 (Categories of monoidal categories)**.** Consider the following very large categories (cf. Def. A.1):

(i) $\boxed{\mathrm{Cat}}$ of categories
with morphisms all functors,

(ii) $\boxed{\mathrm{MonCat}}$ of *monoidal categories* (e.g. [Ke82, §1.1][ML97, §VII.1][EGN15, §2])
with morphisms functors that admit the structure of (strong) monoidal functors (e.g. [ML97, §XI.2][EGN15, §2.4]),

(iii) $\boxed{\mathrm{CoCartCat}}$ of *co-cartesian categories* i.e., monoidal categories whose monoidal operation is the coproduct $\sqcup$
with morphisms functors that admit coproduct-preserving structure,

(iv) $\boxed{\mathrm{DistMonCat}}$ of *distributive monoidal categories* (e.g. [BJT97, p. 1][La03]), i.e., of monoidal categories $(\mathscr{C}, \otimes)$ with (set-indexed) coproducts $\coprod$ whose tensor product distributes over the coproduct in each variable, in that for any index set I and indexed set $(A_i)_{i\in I}$ of objects, and any other object B, the canonical comparison maps are isomorphisms

$$\coprod_{i\in I}\big(A_i \otimes B\big) \xrightarrow[\sim]{(q_i\otimes \mathrm{id}_B)_{i\in I}} \Big(\coprod_{i\in I} A_i\Big)\otimes B\,, \qquad \coprod_{i\in I}\big(B\otimes A_i\big) \xrightarrow[\sim]{(\mathrm{id}_B\otimes q_i)_{i\in I}} B\otimes\Big(\coprod_{i\in I} A_i\Big) \tag{8}$$

and with morphisms in DistMonCat being functors that admit (strong) monoidal structure for both products.

We are interested for now in the following quadruple of examples:

Example 2.2 (Category of complex vector spaces)**.** We write $\mathrm{Mod}_{\mathbb{C}} \in \mathrm{Cat}$ for the usual category whose objects are complex vector spaces and whose morphisms are complex-linear maps between these.

Example 2.3 (Tensor category of complex vector spaces)**.** We write $\big(\mathrm{Mod}_{\mathbb{C}}, \otimes_{\mathbb{C}}\big) \in \mathrm{MonCat}$ for the category of complex vector spaces from Ex. 2.2, but now regarded as a monoidal category by equipping it with the usual linear tensor product $\otimes_{\mathbb{C}}$ of complex vector spaces (whose tensor unit is $\mathbb{C}$ regarded as a vector space over itself).

The following Ex. 2.4 serves to prepare concepts and notation for the main Ex. 2.5 and Ex. 2.8 further below.

Example 2.4 (Set as distributive monoidal category)**.** We write $\big(\mathrm{Set}, \sqcup, \times\big) \in \mathrm{DistMonCat}$ for the category of sets regarded as a distributive cartesian monoidal category.

An abstract way to see that the cartesian product distributes over the coproduct is to notice that the product functors $Y\times(-),\ (-)\times Y : \mathrm{Set} \longrightarrow \mathrm{Set}$ have a right adjunction (forming function sets $(-)^Y$), which implies that they preserve all colimits and hence, in particular, the corproducts involved in distributivity.

Also notice that every set is isomorphic to the coproduct indexed by its elements, of the singleton set

$$X \in \mathrm{Set} \qquad \vdash \qquad X \simeq \coprod_{x\in X} * \,. \tag{9}$$

Example 2.5 (Category of complex vector bundles over discrete spaces)**.** We write

$$\mathrm{Fam}_{\mathbb{C}} \;:=\; \int_{X\in\mathrm{Set}} \mathrm{Mod}_{\mathbb{C}}^{X}$$

for the category of complex vector *bundles* over varying sets (i.e., over varying discrete topological spaces), hence for the Grothendieck construction (Def. A.4) on the pseudofunctor (Def. A.3)

$$\begin{array}{rcl}
\mathrm{Mod}_{\mathbb{K}}^{(\text{-})} \;:\; \mathrm{Set}^{\mathrm{op}} & \longrightarrow & \mathrm{Cat} \\
X & \longmapsto & \mathrm{Func}(X, \mathrm{Mod}_{\mathbb{C}}) \\
\Big\downarrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle f^* := (-)\circ f} \\
Y & \longmapsto & \mathrm{Func}(Y, \mathrm{Mod}_{\mathbb{C}})
\end{array}$$

and regarded as a co-cartesian monoidal category (in fact, this is the *free coproduct completion* of $\mathrm{Mod}_{\mathbb{C}}$; cf. Ex. A.6). Explicitly this means the following, where on the right we show the corresponding construction of topological vector bundles.

- its objects are pairs $\mathcal{V}_X$ consisting of a base $X \in \mathrm{Set}$ and a functor $\mathcal{V}_{(-)}$ from X regarded as a discrete groupoid to the category $\mathrm{Mod}_{\mathbb{C}}$ of complex vector spaces, hence equivalently a vector bundle (necessarily and uniquely flat) over X:

$$\begin{array}{rcl}
\mathcal{V}_{(-)} \;:\; X & \longrightarrow & \mathrm{Mod}_{\mathbb{C}} \\
x & \mapsto & \mathcal{V}_x
\end{array}
\qquad \longleftrightarrow \qquad
\begin{tikzcd}
\mathcal{V}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_X \arrow[d] \\
\{x\} \arrow[r, hook] & X
\end{tikzcd}$$

- morphisms $\phi_f : \mathcal{V}_X \longrightarrow \mathcal{W}'_Y$ are pairs consisting of a map $f : X \longrightarrow Y$ of base spaces and a natural transformation from $\mathcal{V}_X$ to $f^*\mathcal{W}_Y$, hence equivalently morphisms of vector bundles covering maps of base spaces:

$$x \;\mapsto\; \mathcal{V}_x \xrightarrow{\;\phi_x\;} \mathcal{W}_{f(x)}
\qquad \longleftrightarrow \qquad
\begin{tikzcd}
\mathcal{V}_X \arrow[r, "\phi_f"] \arrow[d] & \mathcal{W}_Y \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

- the co-cartesian pairing $\mathcal{V}_X \sqcup \mathcal{V}'_{X'}$ of a pair of objects is

$$\begin{array}{rcl}
X \sqcup X' & \xrightarrow{\mathcal{V}_X \sqcup \mathcal{V}'_{X'}} & \mathrm{Mod}_{\mathbb{C}} \\
x & \mapsto & \mathcal{V}_x \\
x' & \mapsto & \mathcal{V}'_{x'}
\end{array}
\qquad \longleftrightarrow \qquad
\begin{array}{ccc}
\mathcal{V}_X & \sqcup & \mathcal{V}'_{X'} \\
\Big\downarrow & & \Big\downarrow \\
X & \sqcup & X'
\end{array} \tag{10}$$

The evident full inclusion of the category of plain vector spaces (Ex. 2.2) into bundles of vector spaces, given by regarding the former as the bundles over the singleton set $\{*\} \in \mathrm{Set}$, we denote as follows:

$$\begin{array}{rcl}
\mathrm{Mod}_{\mathbb{C}} & \overset{\iota}{\hookrightarrow} & \mathrm{Fam}_{\mathbb{C}} \\
\mathcal{V} & \mapsto & \mathcal{V}_{\{*\}}
\end{array} \tag{11}$$

Remark 2.6 (Abstract characterization of the construction)**.** The Grothendieck construction $\mathrm{Fam}_{\mathbb{C}} := \int_{S\in\mathrm{Set}} \mathrm{Mod}_{\mathbb{K}}^{S}$ in Ex. 2.5 may be understood as a *free coproduct completion* (Ex. A.6), here applied to $\mathrm{Mod}_{\mathbb{C}}$ but the construction exists for any category. Applied to any symmetric closed monoidal categories and then regarded as categorical semantics for dependent linear-typed quantum programming languages, this has been considered in [RS18, §3.2][FKS20, Def. 2.10]. From §2.2 on we are discussing homotopy-theoretic generalizations.

Remark 2.7 (Coproducts of bundles over singletons)**.** Since every $X \in \mathrm{Set}$ is the disjoint union of the singleton sets $\{x\}$ on its elements $x \in X$ (Ex. 2.4), it follows that every object in $\int_{S \in \mathrm{Set}} \mathrm{Mod}_{\mathbb{C}}^{S}$ (Ex. 2.5) is the coproduct (10) of its restrictions $\mathcal{V}_{\{x\}}$ to these singletons:

$$\mathcal{V}_X \in \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Set}} \qquad \vdash \qquad \mathcal{V}_X \;\simeq\; \coprod_{x \in X} \mathcal{V}_{\{x\}} \,. \tag{12}$$

Example 2.8 (Distributive monoidal category of vector bundles)**.** We write $\big(\mathbb{C}\mathrm{ModMod}_{\mathrm{Set}}, \sqcup, \boxtimes\big) \in \mathrm{DistMonCat}$ for the co-cartesian monoidal category of vector bundles over sets, from Ex. 2.5, but now in addition equipped with a further monoidal structure given by the following *external tensor product* of vector bundles, defined as the result of pulling back to the cartesian product of bases spaces and there forming the usual fiberwise tensor product of bundles:

$$
\begin{array}{l}
\mathcal{V}_X \boxtimes \mathcal{V}'_{X'} \;:\; X \times X' \longrightarrow \mathrm{Mod}_{\mathbb{C}} \\
\qquad\qquad (x, x') \;\longmapsto\; \mathcal{V}_x \otimes \mathcal{V}'_{x'}
\end{array}
\qquad \longleftrightarrow \qquad
\begin{tikzcd}[column sep=small]
 & & \big((\mathrm{pr}_X)^* \mathcal{V}_X\big) \otimes \big((\mathrm{pr}_{X'})^* \mathcal{V}'_{X'}\big) \arrow[dd] & & \\
\mathcal{V}_X \arrow[ddr] & (\mathrm{pr}_X)^* \mathcal{V}_X \arrow[l] \arrow[dr] & & (\mathrm{pr}_{X'})^* \mathcal{V}'_{X'} \arrow[r] \arrow[dl] & \mathcal{V}'_{X'} \arrow[ddl] \\
 & & X \times X' \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_{X'}"] & & \\
 & X & & X' &
\end{tikzcd}
\tag{13}$$

This external tensor product indeed distributes over the co-cartesian product (10) in each variable, in the sense required in (8), in a fiberwise covering of how the cartesian product of base sets distributes over the disjoint union of sets

$$
\begin{array}{ccccc}
 & \mathcal{V}^i_{x_i} \otimes \mathcal{V}'_{x'} & = = = = = = & \mathcal{V}^i_{x_i} \otimes \mathcal{V}'_{x'} \\
 & \shortparallel & & \shortparallel \\
(x_i, x') \;\mapsto & \Big(\big(\mathcal{V}^1_{X_1} \sqcup \mathcal{V}^2_{X_2}\big) \boxtimes \mathcal{V}'_{X'}\Big)_{(x_i, x')} & \longrightarrow & \Big(\big(\mathcal{V}^1_{X_1} \boxtimes \mathcal{V}'_{X'}\big) \sqcup \big(\mathcal{V}^2_{X_2} \boxtimes \mathcal{V}'_{X'}\big)\Big)_{(x_i, x')} \\
 & \big(X_1 \sqcup X_2\big) \times X' & \xrightarrow{\;\;\sim\;\;} & \big(X_1 \times X'\big) \sqcup \big(X_2 \times X'\big) \\
 & (x_i, x') & \longmapsto & (x_i, x')
\end{array}
$$

Indeed, a converse statement holds:

Proposition 2.9 (Characterization of external tensor product)**.** *Up to isomorphism, the external tensor product* (13) *is the unique functor*

$$(-) \boxtimes (-) \;:\; \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Set}} \times \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Set}} \longrightarrow \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Set}}$$

such that:

(i) *It distributes over coproducts* (10) *in each variable, in the sense of* (8)*.*

(ii) *Restricted to plain vector spaces via* (11)*, it coincides with the ordinary tensor product (Ex. 2.3).*

Proof. Let for the moment $\boxtimes$ denote any monoidal product satisfying the above two assumptions. Then it is fixed, up to isomorphism, by the following formula:

$$
\begin{array}{rcll}
\mathcal{V}_X \boxtimes \mathcal{V}'_{X'} & \simeq & \Big(\coprod\limits_{x \in X} \mathcal{V}_{\{x\}}\Big) \boxtimes \Big(\coprod\limits_{x' \in X'} \mathcal{V}'_{\{x'\}}\Big) & \text{by (12)} \\
 & \simeq & \coprod\limits_{x \in X} \bigg(\mathcal{V}_{\{x\}} \boxtimes \Big(\coprod\limits_{x' \in X'} \mathcal{V}'_{\{x'\}}\Big)\bigg) & \text{by assumption (i) in first variable} \\
 & \simeq & \coprod\limits_{x \in X} \coprod\limits_{x' \in X'} \Big(\mathcal{V}_{\{x\}} \boxtimes \mathcal{V}'_{\{x'\}}\Big) & \text{by assumption (i) in second variable} \\
 & \simeq & \coprod\limits_{x \in X} \coprod\limits_{x' \in X'} \Big(\iota\big(\mathcal{V}_x \otimes \mathcal{V}_{x'}\big)\Big) & \text{by assumption (ii)} \\
 & \simeq & \coprod\limits_{(x, x') \in X \times X'} \Big(\iota\big(\mathcal{V}_x \otimes \mathcal{V}_{x'}\big)\Big) & \text{just to make the base space manifest,}
\end{array}
\tag{14}$$

which is manifestly isomorphic to the operation of the external tensor product according to (13). □

We proceed to show that the content of Prop. 2.9 is equivalently exhibited by a pushout of the form requested in (1).

Remark 2.10 (Incrementally forgetting distributive monoidal structure)**.** The forgetful functors between the categories-of-categories from Def. 2.1, i.e., those which act as the identity on the underlying categories $\mathscr{C}$ but forget the presence of either or any monoidal structure, evidently arrange into a commuting square as follows:

$$
\begin{tikzcd}
(\mathscr{C},\otimes) \arrow[ddd, mapsto] & & & (\mathscr{C},\sqcup,\otimes) \arrow[lll, mapsto] \arrow[ddd, mapsto] \\
& \mathrm{MonCat} \arrow[d] & \mathrm{DistMonCat} \arrow[l] \arrow[d] & \\
& \mathrm{Cat} & \mathrm{CoCartCat} \arrow[l] & \\
\mathscr{C} & & & (\mathscr{C},\sqcup) \arrow[lll, mapsto]
\end{tikzcd}
\tag{15}
$$

Definition 2.11. Write $\boxed{\mathrm{AnyMonCat}}$ for the Grothendieck construction (Def. A.4) on the square (15), hence for the very large category-of-categories whose:

- objects are categories $\mathscr{C}$ equipped *either* with no monoidal structure *or* with any monoidal structure $\otimes$ *or* with co-cartesian monoidal structure $\sqcup$ *or* with co-cartesian monoidal structure and any further monoidal structure $\otimes$ distributing over it;
- morphisms are functors whose codomain category carries at least the kind of monoidal structures that the domain carries and which are strong monoidal with respect to the kind of monoidal structures that the domain carries.

Example 2.12 (The candidate commuting diagram for a pushout)**.** We have the following commuting diagram in AnyMonCat (Def. 2.11):

$$
\begin{tikzcd}
\text{Ex. 2.3} & (\mathrm{Mod}_{\mathbb{C}},\otimes) \arrow[r, "\iota"] & (\mathrm{Fam}_{\mathbb{C}},\sqcup,\boxtimes) & \text{Ex. 2.8} \\
\text{Ex. 2.2} & \mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[r, "\iota"] & (\mathrm{Fam}_{\mathbb{C}},\sqcup) \arrow[u] & \text{Ex. 2.5}
\end{tikzcd}
\tag{16}
$$

where

- the underlying functor of the vertical morphisms is the respective identity functor,
- the underlying functor of both horizontal morphisms is (11),
- the top horizontal morphism is strong monoidal essentially by construction
  (or alternatively as a special case of Prop. 2.9),
- the right identity functor is tautologically strong monoidal.

Therefore, the underlying diagram of functors clearly commutes and there is no non-trivial composition of strong-monoidal structure involved, hence the diagram commutes in AnyMonCat.

Remark 2.13 (On morphisms in AnyMonCat)**.**
(i) By Definition 2.11, even if the underlying functors are identities, when regarded as morphisms in AnyMonCat they must point in a direction such that no monoidal structure is "forgotten" along the way.
(ii) For instance, for none of the morphisms shown in the diagram (16) does there exist a reverse morphism in AnyMonCat.
(iii) More importantly: If the left and bottom part of the diagram (16) is given, then the only possibility to complete it to a square in AnyMonCat is by having a distributive monoidal category in the top right corner, because only such a structure can receive morphisms in AnyMonCat from both a monoidal category (top left) and a co-cartesian category (bottom right).

We may now state and prove the conclusion of this discussion:

Theorem 2.14 (Pushout characterization of the external tensor product of vector bundles over sets)**.**
The diagram in Ex. 2.12 is a pushout.

Proof. We check the defining universal property of the pushout. To that end, consider any extension of the square to a cocone diagram as shown by solid arrows in the following (where the tip of the cocone is necessarily a distributive monoidal category, as shown, by Rem. 2.13):

$$\begin{tikzcd} & & \big(\mathscr{C}, \sqcup, \otimes_{\mathscr{C}}\big)\,. \\ \big(\mathrm{Mod}_{\mathbb{C}}, \otimes\big) \arrow[r, "\iota"] \arrow[urr, bend left] & \big(\mathrm{Fam}_{\mathbb{C}}, \sqcup, \boxtimes\big) \arrow[ur, dashed] & \\ \mathrm{Mod}_{\mathbb{C}} \arrow[r, "\iota"] \arrow[u] & \big(\mathrm{Fam}_{\mathbb{C}}, \sqcup\big) \arrow[u] \arrow[uur, bend right, "F"'] & \end{tikzcd} \tag{17}$$

We need to demonstrate that there exists a unique dashed morphism making the full diagram commute. First observe that, since the underlying functor of the vertical morphisms are identity functors, the dashed morphism, if it exists at all, is uniquely fixed to also be given by F, and so the underlying functor of the top morphisms must necessarily be $F \circ \iota$. Hence we really have a diagram as follows

$$\begin{tikzcd} & & \big(\mathscr{C}, \sqcup, \otimes_{\mathscr{C}}\big)\,, \\ \big(\mathrm{Mod}_{\mathbb{C}}, \otimes\big) \arrow[r, "\iota"] \arrow[urr, bend left, "F\circ\iota"] & \big(\mathrm{Fam}_{\mathbb{C}}, \sqcup, \boxtimes\big) \arrow[ur, dashed, "F"] & \\ \mathrm{Mod}_{\mathbb{C}} \arrow[r, "\iota"] \arrow[u] & \big(\mathrm{Fam}_{\mathbb{C}}, \sqcup\big) \arrow[u] \arrow[uur, bend right, "F"'] & \end{tikzcd} \tag{18}$$

whose underlying diagram of functors commutes. Therefore we are reduced to showing that the dashed morphism is well-defined as a morphism in AnyMonCat, which means to show that it intertwines the external tensor product $\boxtimes$ on $\mathrm{Fam}_{\mathbb{C}}$ with the given tensor product $\otimes_{\mathscr{C}}$ on $\mathbb{C}$. This is verified by the following sequence of natural isomorphisms:

$$\begin{aligned} F\big(\mathcal{V}_X \boxtimes \mathcal{V}'_{X'}\big) &\simeq F\Big(\coprod_{(x,x')\in X\times X'} \iota\big(\mathcal{V}_x \otimes \mathcal{V}_{x'}\big)\Big) && \text{by (14)} \\ &\simeq \coprod_{(x,x')\in X\times X''} F\Big(\iota\big(\mathcal{V}_x \otimes \mathcal{V}_{x'}\big)\Big) && \text{since } F \text{ preserves coproducts, by assumption (17)} \\ &\simeq \coprod_{(x,x')\in X\times X''} F\big(\iota(\mathcal{V}_x)\big) \otimes_{\mathscr{C}} F\big(\iota(\mathcal{V}'_{x'})\big) && \text{via strong monoidal structure on } F\circ\iota\text{, by (18)} \\ &\simeq \Big(\coprod_{x\in X} F\big(\iota(\mathcal{V}_x)\big)\Big) \otimes_{\mathscr{C}} \Big(\coprod_{x'\in X'} F\big(\iota(\mathcal{V}'_{x'})\big)\Big) && \text{since } \otimes_{\mathscr{C}} \text{ is distributive, by assumption (17)} \\ &\simeq F\Big(\coprod_{x\in X} \big(\iota(\mathcal{V}_x)\big)\Big) \otimes_{\mathscr{C}} F\Big(\coprod_{x'\in X'} \big(\iota(\mathcal{V}'_{x'})\big)\Big) && \text{since } F \text{ preserves coproducts, by assumption (17)} \\ &\simeq F(\mathcal{V}_X) \otimes_{\mathscr{C}} F(\mathcal{V}'_{X'}) && \text{by (14).} \end{aligned}$$

□

Remark 2.15 (Doubly closed monoidal structure on vector bundles and its "bunched" classical/quantum logic)**.**
(i) The category $\mathrm{Fam}_{\mathbb{C}}$ (Ex. 2.5) also carries a cartesian product — the "external cartesian product" (Ex. A.10).
(ii) Since both this and the external tensor product are closed (see Prop. 3.36 below), jointly they make for a "doubly closed monoidal category" [OP99, §3][O'H03, §2.2] which one may want to think of as providing categorical semantics for both classical propositional logic as well as for the multiplicative fragment of linear logic. Since the antecedents in such a mixed classical/quantum logic are no longer plain lists of classical products of classical contexts, but more generally nested ("bunched") trees obtained by alternatively using the (external) tensor product, this idea of combined classical/linear logic has originally been advertised as a logic of "bunched implications" [OP99][Py02] and came to be known as *bunched logic*, for short (e.g. [ZBHYY21], where also the quantum aspect of bunched logic is considered).

(iii) However, there have all along been subtle technical difficulties with promoting the broad idea of bunched logic to a satisfactory formal language; these problems have been highlighted in [Py08, p. 5] and further in [Ri22, pp. 19 and Rem. 1.4.1]. The claim of [Ri22] is that these problems are finally fixed by enhancing linear type theory to what we call Linear Homotopy Type Theory, `LHoTT` [MRSS23]. On the side of the categorical semantics this requires promoting the doubly closed monoidal category of vector bundles over sets to a suitably doubly monoidal model category presenting ∞-local systems over general homotopy types. This is the task we turn to in §2.2 and more generally in §3; for further discussion see §4.

2.2 For flat vector bundles over general spaces

We generalize the previous discussion from vector bundles over discrete spaces to flat vector bundles over arbitrary base spaces. The previous discussion in §2.1 was essentially a variation of the theme that $\mathrm{Fam}_{\mathbb{C}}$ (Ex. 2.5) is the free coproduct completion of $\mathrm{Mod}_{\mathbb{C}}$ (Ex. A.6). In looking for a first homotopy-theoretic generalization of this notion, one may observe that coproducts are, of course, just the colimits over diagrams of the shape of a discrete category in $\mathrm{Set} \hookrightarrow \mathrm{Cat}$.

Therefore, we are naturally led to ask more generally for completion of categories (notably of $\mathrm{Mod}_{\mathbb{C}}$) under colimits over diagrams of the shape of skeletal groupoids $\mathrm{Grpd}_{\mathrm{skl}} \hookrightarrow \mathrm{Cat}$: This should combine formation of coproducts (indexed by the set of connected components of a given skeletal groupoid) with the formation of *quotients by group actions* indexed by the automorphisms group of any connected component). More precisely, we should ask here for *homotopy quotients* over group actions. This is what we make precise in Def. 2.19 below.

Groupoids. In all of the following we write Grpd (Def. A.1) for the 1-category of small strict groupoids, regarded as a monoidal category under the cartesian product.

Example 2.16 (Basic examples of groupoids and notation (e.g [SS21, §1.2])). For $(G, \mu, \mathrm{e}) \in \mathrm{Grp}$, we write

- $\mathbf{B}G :\equiv (G \rightrightarrows \mathrm{pt}) \in \mathrm{Grpd}$ for the *delooping groupoid* of G, with composition given by reverse group multiplication

$$
\begin{tikzcd}
 & \mathrm{pt} \arrow[dr, "g_{23}"] & \\
\mathrm{pt} \arrow[ur, "g_{12}"] \arrow[rr, "\mu(g_{23}, g_{12})"'] & & \mathrm{pt}
\end{tikzcd}
\tag{19}
$$

  so that (left) G-actions are equivalently functors out of the delooping:

$$
\frac{G \xrightarrow[\text{homom.}]{} \mathrm{Hom}_{\mathscr{C}}(\mathcal{V}, \mathcal{V})}{\begin{array}{ccc} \mathbf{B}G & \xrightarrow{\text{funct.}} & \mathscr{C} \\ \mathrm{pt} & \mapsto & \mathcal{V} \end{array}}
\tag{20}
$$

- $\mathbf{E}G :\equiv (G \times G \rightrightarrows G) \in \mathrm{Grpd}$ for the action groupoid of G acting on itself by left multiplication, so that we have a forgetful functor

$$
\begin{array}{ccc}
\mathbf{E}G & \xrightarrow{q} & \mathbf{B}G \\
g & \mapsto & \mathrm{pt} \\
\big\downarrow{\scriptstyle g_{12}} & & \big\downarrow{\scriptstyle g_{12}} \\
\mu(g_{12}, g) & \mapsto & \mathrm{pt}
\end{array}
\tag{21}
$$

  and a remaining G-action by right inverse multiplication

$$
\begin{array}{ccc}
\mathbf{B}G & \longrightarrow & \mathrm{Grpd} \\
\mathrm{pt} & \mapsto & \mathbf{E}G \\
\big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle \mu(-, g^{-1})} \\
\mathrm{pt} & \mapsto & \mathbf{E}G
\end{array}
\tag{22}
$$

  whose colimiting cocone is $q : \mathbf{E}G \to \mathbf{B}G$ (21).

- $\mathrm{CoDisc}(S) := (S \times S \rightrightarrows) \in \mathrm{Grpd}$ for the *pair groupoid* on some $S \in \mathrm{Set}$, i.e., the groupoid whose objects are the elements of S and which has a unique morphism between any pair of objects. For example

$$\mathrm{CoDisc}\big(\{1,2,3,4\}\big) \;\equiv\; \left\{ \begin{tikzcd} 2 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & 3 \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ 1 \arrow[r, leftrightarrow, bend right] & 4 \end{tikzcd} \right\}. \tag{23}$$

These codiscrete groupoids serve as *contractible resolutions of the point*, since their terminal functor is an equivalence (either in the sense of categorical equivalence or in the sense of homotopy equivalence):

$$\mathrm{CoDisc}(S) \xrightarrow[\text{equivalence}]{} * \tag{24}$$

Notice that $\mathbf{E}G$ (21) is isomorphic to a codiscrete groupoid: $\mathbf{E}G \simeq \mathrm{CoDisc}(G)$.

Last, but not least, for a topological space X we have the *fundamental groupoid* (e.g. [Hi71, §6])

$$\Pi_1(X) \;\equiv\; \left\{ \begin{tikzcd} & x_2 \arrow[dr, rightsquigarrow, "\gamma_{23}"] & \\ x_1 \arrow[ur, rightsquigarrow, "\gamma_{12}"] \arrow[rr, rightsquigarrow, "\mathrm{conc}(\gamma_{23},\gamma_{12})"'] & & x_3 \end{tikzcd} \right\} \tag{25}$$

whose objects are the elements $x \in X$, whose morphisms are homotopy classes γ of continuous paths in X (fixing their endpoints) and whose composition operation is given by concatenation of paths.

Definition 2.17 (Skeletal groupoids, cf. e.g. [ML97, p. 91][Ric20, §2.6])**.** A groupoid is called *skeletal* if it is a disjoint union of delooping groupoids (19):

$$\mathscr{X} \text{ is skeletal} \qquad \Leftrightarrow \qquad \mathscr{X} \underset{\text{iso}}{\simeq} \coprod_{x \in \mathrm{Obj}(\mathscr{X})} \mathbf{B}\big(\mathscr{X}(x,x)\big)\,. \tag{26}$$

It is a standard fact (assuming the Axiom of Choice in the underlying set theory, as usual) that every groupoid is adjoint equivalent (in the sense of equivalence of categories) to a skeletal one (Def. 2.17); we recall this statement in the greater generality of simplicial groupoids below in Lem. 3.15 below. But for the purposes here it is useful to rephrase this as follows, using a 1-category theoretic "model" for the notion of equivalence, in view of (24):

Lemma 2.18 (Connected is delooping times codiscrete)**.** *Every connected groupoid is isomorphic to the product of a delooping groupoid* (19) *with a codiscrete groupoid* (23)*:*

$$\left.\begin{array}{l} \mathscr{X} \in \mathrm{Grpd} \\ \pi_0(\mathscr{X}) \simeq * \\ x_0 \in \mathrm{Obj}(X) \end{array}\right\} \qquad \vdash \qquad \mathscr{X} \underset{\text{iso}}{\simeq} \mathrm{CoDisc}\big(\mathrm{Obj}(\mathscr{X})\big) \times \mathbf{B}\big(\mathscr{X}(x_0,x_0)\big).$$

Proof. Choose for each object $x \in \mathrm{Obj}(\mathscr{X})$ a morphism $\gamma : x_0 \to x$ (which exists by the assumption that $\mathscr{X}$ is connected). This gives the following isomorphism:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\;\sim\;} & \mathrm{CoDisc}\big(\mathrm{Obj}(\mathscr{X})\big) \times \mathbf{B}\big(\mathscr{X}(x_0,x_0)\big)\,. \\ x & \longmapsto & (x, \mathrm{pt}) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle ((x,x'),\, \gamma_{x'}^{-1} \circ f \circ \gamma_x)} \\ x' & \longmapsto & (x', \mathrm{pt}) \end{array}$$

□

Free homotopy quasi-coproduct completion.

In mild variation of [HT95, §1.3][1], we say:

Definition 2.19 (Homotopy quasi-coproducts)**.** A *category with homotopy quasi-coproducts* is a category $\mathscr{C}$

(i) equipped with a tensoring over Grpd

$$\mathrm{Grpd} \times \mathscr{C} \xrightarrow{(-)\cdot(-)} \mathscr{C} \tag{27}$$

(ii) which has all colimits over diagrams of shapes of skeletal groupoids of the following form:

$$\begin{tikzcd} \mathscr{X} \arrow[rr] \arrow[d, equal] & & \mathscr{C} \\ \coprod\limits_{i \in I} \mathbf{B}G_i \arrow[rr, "\left(\mathbf{E}G_i, \mathcal{V}_{(-)}\right)_{i \in I}"] & & \mathrm{Grpd} \times \mathscr{C} \arrow[u, "(-)\cdot(-)"'] \end{tikzcd} \tag{28}$$

This means, equivalently, that $\mathscr{C}$ has

(a) all set-indexed coproducts,

(b) all "Borel constructions", namely all quotients of diagonal actions of any group G on objects of the form $(\mathbf{E}G) \cdot \mathcal{V}$, where $\mathcal{V}$ is equipped with any G-action (20) and where $\mathbf{E}G$ carries the canonical action (22).

We write $\mathrm{CoCartCat}^{hq}$ for the (very large) category (Def. A.1) whose objects are categories with homotopy quasi-coproducts and whose morphisms are functors between them preserving this structure.

Definition 2.20 (Free homotopy quasi-coproduct completion)**.** Given a category $\mathscr{C}$ we say that its *free homotopy quasi-coproduct completion* is a full inclusion of $\mathscr{C}$ in a category with homotopy quasi-coproducts (Def. 2.19) such that any functor out of the latter which preserves the Grpd-tensoring (27) and the homotopy quasi-coproducts (28) is already fixed by its restriction to $\mathscr{C}$.

We will show that the following construction realizes this free homotopy quasi-coproduct completion, at least for cocomplete categories:

Definition 2.21 (Category of local systems with coefficients in any category)**.** For $\mathscr{C}$ a category, we write

$$\mathrm{Loc}_{\mathscr{C}} \;:\equiv\; \int_{\mathscr{X} \in \mathrm{Grpd}} \mathscr{C}^{\mathscr{X}} \tag{29}$$

for the Grothendieck construction (Def. A.4) on the pseudo-functor of functor categories from groupoids into $\mathscr{C}$, as in (5). We regard this as equipped with the full inclusion of objects of $\mathscr{C}$ regarded as constant functors on the terminal groupoid

$$\begin{array}{ccc} \mathscr{C} & \overset{\iota}{\hookrightarrow} & \mathrm{Loc}_{\mathscr{C}} \\ \mathcal{V} & \mapsto & \mathcal{V}_{\mathrm{pt}} \end{array} \tag{30}$$

and with the Grpd-tensoring given by

$$\begin{array}{ccc} \mathrm{Grpd} \times \mathrm{Loc}_{\mathscr{C}} & \xrightarrow{(-)\cdot(-)} & \mathrm{Loc}_{\mathscr{C}} \\ \left(\mathscr{X}\,,\, \mathcal{W}_{\mathscr{Y}}\right) & \longmapsto & \left((\mathrm{pr}_{\mathscr{Y}})^* \mathcal{V}\right)_{\mathscr{X} \times \mathscr{Y}} \end{array} \tag{31}$$

[1] Our notion of homotopy quasi-coproducts (Def. 2.19) is a special case of the notion of *quasi-coproducts* of [HT95, §1.3] in that we require the respective actions not just to be free, but to be free *qua* the Borel construction. But our definition is also slightly stronger in that we in addition require categories with homotopy quasi-coproducts be tensored over Grpd. This may be understood as imposing the further requirement that homotopy quasi-coproducts over constant diagrams are well-behaved.

Example 2.22 (Group representations as local systems)**.** For each $G \in \mathrm{Grp}$ there is a full inclusion of the category of G-actions on objects of $\mathscr{C}$ into the category of local systems (29), given by the correspondence (20):

$$\begin{array}{ccccc} G\mathrm{Act}(\mathscr{C}) & \xrightarrow{\sim} & \mathscr{C}^{\mathbf{B}G} & \hookrightarrow & \mathrm{Loc}_{\mathscr{C}} . \\ G \circlearrowright \mathcal{V} & & \longmapsto & & \mathcal{V}_{\mathbf{B}G} \end{array} \tag{32}$$

However (if there is a zero object $0 \in \mathscr{C}$ in $\mathrm{Loc}_{\mathscr{C}}$, or at least a terminal object) after regarding it inside $\mathrm{Loc}_{\mathscr{C}}$, then any such group representation may be "decomposed" into:

(i) the underlying group G,
(ii) the underlying object $\mathcal{V} \in \mathscr{C}$ and
(iii) the action itself, in that it fits into a pullback square of this form:

$$\begin{tikzcd} \mathcal{V}_{\mathrm{pt}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_{\mathbf{B}G} \arrow[d] \\ 0_{\mathrm{pt}} \arrow[r] & 0_{\mathbf{B}G} \end{tikzcd} \quad \in \quad \mathrm{Loc}_{\mathscr{C}} .$$

At least for $\mathscr{C}$ such that $\mathrm{Loc}_{\mathscr{C}}$ embeds continuously into an ∞-topos, such squares exhibit $\mathcal{V}_{\mathbf{B}G}$ as the *homotopy quotient* of $\mathcal{V}$ by its G-action, and the map $\mathcal{V}_{\mathbf{B}G} \to 0_{\mathbf{B}G}$ as the $\mathcal{V}$-fiber bundle associated to the universal G-principal bundle $0_{\mathbf{E}G} \to 0_{\mathbf{B}G}$; see [SS20, §2.2][SS21, Prop. 0.2.1].

We are mainly interested in the specialization of Def. 2.21 to the case that $\mathscr{C}$ is a cocomplete monoidal category such as $\mathrm{Mod}_{\mathbb{C}}$. When $\mathscr{C}$ is cocomplete, then it is canonically tensored over Set and all base change operations f^* (5) have left adjoints $f_!$ (left Kan extension, Ex. A.18, Ex. A.18)

$$\mathscr{C}\ \text{cocomplete} \quad \vdash \quad \begin{array}{ccc} \mathrm{Set} \times \mathscr{C} & \xrightarrow{(-)\cdot(-)} & \mathscr{C} \\ (S, \mathcal{V}) & \longmapsto & \coprod_{s \in S} \mathcal{V} \end{array} \quad \text{and} \quad \begin{tikzcd} \mathrm{Grpd} \arrow[rr, "\mathscr{C}^{(-)}"] & & \mathrm{Cat}_{\mathrm{adj}} \\ \mathscr{X} \arrow[d, "f"] & \mapsto & \mathscr{C}^{\mathscr{X}} \arrow[d, shift right, "f_!"'] \arrow[d, phantom, "\dashv"] \\ \mathscr{Y} & \mapsto & \mathscr{C}^{\mathscr{Y}} \arrow[u, shift right, "f^*"'] \end{tikzcd} \tag{33}$$

Lemma 2.23 (Pushforward along quotient projection on $\mathbf{E}G$)**.** *If $\mathscr{C}$ is cocomplete, then the push-forward operation* (33) *along $q : \mathbf{E}G \to \mathbf{B}G$* (21) *is given by*

$$\begin{array}{ccc} \mathscr{C}^{\mathbf{E}G} & \xrightarrow{q_!} & \mathscr{C}^{\mathbf{B}G} \\ \mathcal{V}_{\mathbf{E}G} & \longmapsto & (G \cdot \mathcal{V}_{\mathrm{e}})_{\mathbf{B}G} \simeq (G \cdot 1)_{\mathbf{B}G} \boxtimes \mathcal{V}_{\{\mathrm{e}\}} . \end{array}$$

(On the right, using the tensoring (33), the G-action (20) is via the left multiplication action of G on itself, which on the far right we are transparently re-expressing, when $\mathscr{C}$ is monoidal, through the external tensor product (35).)

Proof. We may check the $(q_! \dashv q^*)$ hom-isomorphism: First, speaking equivalently in terms of group actions via Ex. 2.22, since $G \cdot \mathcal{V}_{\mathrm{e}}$ carries the free group action on the underlying object, the G-equivariant morphisms $G \cdot \mathcal{V}_{\mathrm{e}} \to \mathcal{W}$ are in natural bijection with the underlying such morphisms $f : \mathcal{V}_{\mathrm{e}} \to \mathcal{W}$ in $\mathscr{C}$. These, in turn, are in natural bijection with morphisms from $\mathcal{V}_{\mathbf{E}G}$ to $q^*\mathcal{W}$, namely with natural transformations given as follows:

$$\begin{tikzcd} \mathbf{E}G \arrow[rr] & & \mathscr{C} & \\ \mathrm{e} \arrow[d] & \longmapsto & \mathcal{V}_{\mathrm{e}} \arrow[r, "f"] \arrow[d, "\mathcal{V}_{(\mathrm{e},g)}"'] & \mathcal{W} \arrow[d, "\mathcal{W}_{\mu(g', g^{-1})}"] \\ g & \longmapsto & \mathcal{V}_g \arrow[r, "\exists !"] & \mathcal{W} . \end{tikzcd}$$

Here the bottom morphisms clearly exist uniquely for all $g \in G$, thus establishing the claimed bijection. ☐

External tensor product on local systems. The following Prop. 2.26 is fairly immediate, below in §3.3 we spell this out in more detail and in greater generality. In generalization of (8), the following Def. 2.24 is a lightweight version of the notion of *monoidal enriched categories* (cf. Def. A.26) which we use in this section here in order not to overburden the elementary discussion:

Definition 2.24. We say that a Grpd-*monoidal category* is a monoidal category $(\mathscr{C}, \otimes, 1)$ equipped with

(i) a Grpd-tensoring (27) $\;\mathrm{Grpd} \times \mathscr{V} \xrightarrow{(-)\cdot(-)} \mathscr{C}$

(ii) natural isomorphism

$$\left.\begin{array}{l} \mathscr{X} \in \mathrm{Grpd}, \\ \mathcal{V}, \mathcal{W} \in \mathscr{C} \end{array}\right\} \qquad \vdash \qquad \big(\mathscr{X} \cdot \mathcal{V}\big) \otimes \mathcal{W} \;\simeq\; \mathscr{X} \cdot \big(\mathcal{V} \otimes \mathcal{W}\big). \tag{34}$$

Definition 2.25 (Homotopy quasi-distributive categories)**.** We say that a Grpd-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ (Def. 2.24) which has homotopy quasi-coproducts (Def. 2.19) is *homotopy quasi-distributive* if the tensor product is compatible in each variable with the homotopy quasi-coproducts (28) in that the canonical comparison maps are isomorphisms:

$$\begin{aligned} \underrightarrow{\lim}\Big(\mathscr{X} \to \mathscr{C} \xrightarrow{(-)\otimes_{\mathscr{C}} \mathcal{W}} \mathscr{C}\Big) &\xrightarrow[\sim]{} \Big(\underrightarrow{\lim}\big(\mathscr{X} \to \mathscr{C}\big)\Big) \otimes_{\mathscr{C}} \mathcal{W}\,, \\ \underrightarrow{\lim}\Big(\mathscr{X} \to \mathscr{C} \xrightarrow{\mathcal{W} \otimes_{\mathscr{C}} (-)} \mathscr{C}\Big) &\xrightarrow[\sim]{} \mathcal{W} \otimes_{\mathscr{C}} \Big(\underrightarrow{\lim}\big(\mathscr{X} \to \mathscr{C}\big)\Big). \end{aligned}$$

Proposition 2.26 (External tensor product on local systems)**.** *Given a cocomplete closed monoidal category $(\mathscr{C}, 1, \otimes)$, the category of $\mathscr{C}$-valued local systems (Def. 2.21) becomes a homotopy quasi-distributive category (Def. 2.25) under the external tensor product*

$$\begin{array}{ccc} \mathrm{Loc}_{\mathscr{C}} \times \mathrm{Loc}_{\mathscr{C}} & \xrightarrow{\qquad\qquad \boxtimes \qquad\qquad} & \mathrm{Loc}_{\mathscr{C}} \\ \big(\mathcal{V}_{\mathscr{X}}, \mathcal{W}_{\mathscr{Y}}\big) & \longmapsto & \Big(\big((\mathrm{pr}_{\mathscr{X}})^* \mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathscr{Y}})^* \mathcal{W}\big)\Big)_{\mathscr{X} \times \mathscr{Y}} \end{array} \tag{35}$$

with respect to the canonical Grpd*-tensoring* (31)*, hence in particular such that the inclusion* (30) *is strong monoidal*

$$\iota\big(\mathcal{V} \otimes \mathcal{W}\big) \;\simeq\; \iota(\mathcal{V}) \boxtimes \iota(\mathcal{W})\,. \tag{36}$$

Proof. Observing that the Grpd-tensoring (31) equals the restriction of the external tensor product to unit systems

$$\left.\begin{array}{l} \mathscr{X} \in \mathrm{Grpd} \\ \mathcal{W}_{\mathscr{X}} \in \mathrm{Loc}_{\mathscr{C}} \end{array}\right\} \qquad \vdash \qquad \mathscr{X} \cdot \mathcal{W}_{\mathscr{Y}} \;=\; 1_{\mathscr{X}} \boxtimes \mathcal{W}_{\mathscr{Y}}\,, \tag{37}$$

the Grpd-monoidality structure (34) is given identically by (37). Moreover, under the assumption that the tensor product $\otimes$ on $\mathscr{C}$ is closed, hence a left adjoint in each variable and as such colimit-preserving, it follows that also the external tensor product preserves all colimits in each variable (we spell this out in greater generality as Prop. 3.34 below), hence in particular it preserves quasi-coproducts. □

Local systems as the free homotopy quasi-coproduct completion of vector spaces.

The following Lemmas 2.27, 2.28 should be thought of as saying that in $\mathrm{Loc}_{\mathscr{C}}$ every object is *homotopy equivalent* to a coproduct of *homotopy quotients*, where the tensoring with the contractible groupoids $\mathrm{CoDisc}(S)$, $\mathbf{E}G$ (24) is used to model these homotopy-theoretic notions in 1-category theoretic terms.

Lemma 2.27 (Every local system is coproduct of Grpd tensoring of group representation)**.** *Given an object $\mathcal{V}_{\mathscr{X}} \in \mathrm{Loc}_{\mathscr{C}}$* (29)*, it is isomorphic to a coproduct of tensorings* (31) *with coskeletal groupoids (Ex. 2.16) of group representations* (32)*:*

$$\mathcal{V}_{\mathscr{X}} \in \mathrm{Loc}_{\mathscr{C}} \qquad \vdash \qquad \mathcal{V}_{\mathbf{B}G} \underset{\mathrm{iso}}{\simeq} \coprod_{i \in \pi_0(\mathscr{X})} \Big(\mathrm{CoDisc}\big(\mathrm{Obj}(\mathscr{X}_i)\big) \cdot \big(\iota_{x_i}^* \mathcal{V}\big)_{\mathbf{B}\mathscr{X}(x_i, x_i)}\Big)\,.$$

Proof. On underlying groupoids, this is the statement of Lem 2.18. Therefore, it is sufficient to see that any local system $\mathcal{V}_{(-)}$ on $\mathrm{CoDisc}(X) \cdot \mathbf{B}G$ is isomorphic to one pulled back from $\mathbf{B}G$, via the following natural isomorphism

$$
\begin{tikzcd}
\mathrm{CoDisc}(X) \times \mathbf{B}G \arrow[rr] & & \mathscr{C} & \\
(x, \mathrm{pt}) \arrow[d, "{((x,x'),g)}"'] & \longmapsto & \mathcal{V}_x \arrow[r, "\mathcal{V}_{((x,x_i),\mathrm{e})}"'] \arrow[d, "\mathcal{V}_{((x,x'),g)}"'] & \mathcal{V}_{x_i} \arrow[d, "\mathscr{V}_{(\mathrm{id}_{x_i},g)}"] \\
(x', g) & \longmapsto & \mathcal{V}_x \arrow[r, "\mathcal{V}_{((x',x_i),\mathrm{e})}"] & \mathcal{V}_{x_i}\,,
\end{tikzcd}
$$

for any choice of basepoint $x_i \in X$. ☐

Lemma 2.28 (Group representation as homotopy quasi-coproduct)**.** *Given a cocomplete category $\mathscr{C}$, and $G \in \mathrm{Grp}$, any group action $\rho : \mathbf{B}G \to \mathscr{C}$ (20) is, when regarded as an object of $\mathrm{Loc}_{\mathscr{C}}$ (Rem. 2.22), a homotopy quasi-coproduct (Def. 2.19) in that it is the colimit in $\mathrm{Loc}_{\mathscr{C}}$ over the following diagram:*

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr] & & \mathrm{Loc}_{\mathscr{C}} \\
\mathrm{pt} \arrow[d, "g"] & \longmapsto & (\mathbf{E}G) \cdot \mathcal{V}_{\mathrm{pt}} \arrow[d, "\mu(-, g^{-1}) \cdot \rho(g)"] \\
\mathrm{pt} & \longmapsto & (\mathbf{E}G) \cdot \mathcal{V}_{\mathrm{pt}}
\end{tikzcd}
$$

Proof. By the assumption that $\mathscr{C}$ is cocomplete, the colimit exists and is given (Prop. A.9) on underlying groupoids by the quotient coprojection $q : \mathbf{E}G \to \mathbf{B}G$ (21) and on $\mathscr{C}$-components by the colimit over the following diagram:

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr] & & \mathscr{C} & & \\
\mathrm{pt} \arrow[d, "g"'] & \longmapsto & q_!\big(\mathbf{E}G \cdot \mathcal{V}_{\mathrm{pt}}\big) \arrow[d, "q_!\left(\mu(-,g^{-1}) \cdot \rho(g)_{\mathrm{pt}}\right)"] & \simeq & (G \cdot 1)_{\mathbf{B}G} \boxtimes \mathcal{V}_{\mathrm{pt}} \arrow[d, "\left(\mu(-,g^{-1}) \cdot 1\right)_{\mathbf{B}G} \boxtimes \mathcal{V}_{\rho(g)}"] \\
\mathrm{pt} & \longmapsto & q_!\big(\mathbf{E}G \cdot \mathcal{V}_{\mathrm{pt}}\big) & \simeq & (G \cdot 1)_{\mathbf{B}G} \boxtimes \mathcal{V}_{\mathrm{pt}}\,,
\end{tikzcd}
$$

where on the right we used Lem. 2.23. Since this colimit is taken in a functor category, $\mathscr{C}^{\mathbf{B}G}$, it is computed on underlying objects in $\mathscr{C}$, where it is following cocone

$$
\begin{tikzcd}
G \cdot \mathcal{V} \arrow[rr, "\mu(-,g^{-1}) \cdot \rho(g)"] \arrow[dr, "\rho"'] & & G \cdot \mathcal{V} \arrow[dl, "\rho"] \\
& \mathcal{V} &
\end{tikzcd}
$$

and the induced action of morphisms in $\mathbf{B}G$ is thus given by the universal dashed morphism in

$$
\begin{tikzcd}
G \cdot \mathcal{V} \arrow[r, "\mu(g,-)"] \arrow[d, "\rho"] & G \cdot \mathcal{V} \arrow[d, "\rho"] \\
\mathcal{V} \arrow[r, dashed, "\rho(g)"] & \mathcal{V}\,.
\end{tikzcd}
$$

Under the equivalence (32), this is the object $\mathcal{V}_{\mathbf{B}G} \in \mathrm{Loc}_{\mathscr{C}}$, as claimed. ☐

In conclusion:

Theorem 2.29 (Qusi-product completion)**.** *Given a cocomplete category $\mathscr{C}$ then* $\mathrm{Loc}_{\mathscr{C}}$ *(Def. 2.21) is its free homotopy quasi-coproduct completion (Def. 2.20).*

Proof. First, due to the assumption that $\mathscr{C}$ is cocomplete, so is $\mathrm{Loc}_{\mathscr{C}}$ (by Prop. A.9) and hence, in addition to the Grpd-tensoring (31), it in particular has all homotopy quasi-coproducts (28).

Now every object of $\mathrm{Loc}_{\mathscr{C}}$ is isomorphic to a coproduct of Grpd-tensorings of group representations (Lem. 2.27) and every group representation is a homotopy quasi-coproduct of a constant local system (Lem. 2.28). Therefore any functor out of $\mathrm{Loc}_{\mathscr{C}}$ which preserves the Grpd-tensoring and homotopy quasi-coproducts is already fixed by its restrict to constant local systems (30). ☐

Pushout-characterization of the external tensor product on local systems. In generalization of Def. 2.11:

Definition 2.30. Write $\boxed{\mathrm{AnyMonCat}^{hq}}$ for the Grothendieck construction on the following diagram of forgetful functors

$$
\begin{tikzcd}
(\mathscr{C},\otimes) \arrow[d, mapsto] & & & (\mathscr{C},\sqcup^{hq},\otimes) \arrow[lll, mapsto] \arrow[d, mapsto] \\
& \mathrm{MonCat} \arrow[d] & \mathrm{DistMonCat}^{hq} \arrow[l] \arrow[d] & \\
& \mathrm{Cat} & \mathrm{CoCartCat}^{hq} \arrow[l] & \\
\mathscr{C} & & & (\mathscr{C},\sqcup^{hq}) \arrow[lll, mapsto]
\end{tikzcd}
\tag{38}
$$

where now $\mathrm{CoCartCat}^{hq}$ is from Def. 2.19 and $\mathrm{DistMonCat}^{hq}$ from Def. 2.25.

Now we are ready to state and prove the first homotopy-theoretic generalization of the pushout theorem 2.14:

Theorem 2.31 (Pushout-characterization of the external tensor product on local systems)**.** *The following is a pushout diagram in* $\mathrm{AnyMonCat}^{hq}$ *(Def. 2.30), where the structure on the right is from Prop. 2.26:*

$$
\begin{tikzcd}
\big(\mathrm{Mod}_{\mathbb{C}},\otimes\big) \arrow[r, "\iota"] \arrow[dr, phantom, "\text{(po)}"] & \big(\mathrm{Loc}_{\mathbb{C}},\sqcup^{hq},\boxtimes\big) \\
\mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[r, "\iota"] & \big(\mathrm{Loc}_{\mathbb{C}},\sqcup^{hq}\big) \arrow[u]
\end{tikzcd}
\tag{39}
$$

Proof. As before in the proof of Thm. 2.14, we demonstrate the unique existence of a dashed arrow, now in $\mathrm{AnyMonCat}^{hq}$, given a solid diagram as shown here:

$$
\begin{tikzcd}
& & \big(\mathscr{C},\sqcup^{hq},\otimes_{\mathscr{C}}\big) \\
\big(\mathrm{Mod}_{\mathbb{C}},\otimes\big) \arrow[r, "\iota"] \arrow[urr, bend left, "F\circ\iota"] & \big(\mathrm{Loc}_{\mathbb{C}},\sqcup^{hq},\boxtimes\big) \arrow[ur, dashed] & \\
\mathrm{Mod}_{\mathbb{C}} \arrow[u] \arrow[r, "\iota"] & \big(\mathrm{Loc}_{\mathbb{C}},\sqcup^{hq}\big) \arrow[u] \arrow[uur, bend right, "F"'] &
\end{tikzcd}
\tag{40}
$$

The proof proceeds along the same lines as before in Thm. 2.14, now using the stronger homotopy quasi-distributivity property to factor out the richer homotopical quasi-coproduct structure of the objects. Namely, we need to see that the given functor F already intertwines the external tensor product on $\mathrm{Loc}_{\mathscr{C}}$ with the tensor product on $\mathscr{C}$, and this is obtained by the following sequence of natural isomorphisms:

$$
\begin{aligned}
& F\big(\mathcal{V}_{\mathscr{X}} \boxtimes \mathcal{W}_{\mathscr{Y}}\big) \\
\simeq\ & F\Big(\Big(\textstyle\coprod_i \mathrm{CoDisc}\big(X_i\big)\cdot\big(\iota_{x_i}^{*}\mathcal{V}\big)_{\mathbf{B}G_i}\Big)\boxtimes\Big(\textstyle\coprod_j \mathrm{CoDisc}\big(Y_j\big)\cdot\big(\iota_{y_j}^{*}\mathcal{W}\big)_{\mathbf{B}G_j}\Big)\Big) && \text{by Lem. 2.27} \\
\simeq\ & F\Big(\Big(\textstyle\coprod_i \mathrm{CoDisc}\big(X_i\big)\cdot\varinjlim_{\mathbf{B}G_i}\big((\mathbf{E}G_i)\cdot\mathcal{V}_{x_i}\big)\Big)\boxtimes\Big(\textstyle\coprod_j \mathrm{CoDisc}\big(Y_j\big)\cdot\varinjlim_{\mathbf{B}G_i}\big((\mathbf{E}G_j)\cdot\mathcal{W}_{y_j}\big)\Big)\Big) && \text{by Lem. 2.28} \\
\simeq\ & F\Big(\Big(\textstyle\coprod_{i,j} \mathrm{CoDisc}\big(X_i\times Y_j\big)\cdot\varinjlim_{\mathbf{B}(G_i\times G_j)}\Big(\mathbf{E}(G_i\times G_j)\cdot\big(\iota(\mathcal{V}_{x_i})\boxtimes\iota(\mathcal{V}_{y_j})\big)\Big)\Big)\Big) && \text{by Prop. 2.26} \\
\simeq\ & F\Big(\Big(\textstyle\coprod_{i,j} \mathrm{CoDisc}\big(X_i\times Y_j\big)\cdot\varinjlim_{\mathbf{B}(G_i\times G_j)}\Big(\mathbf{E}(G_i\times G_j)\cdot\iota\big(\mathcal{V}_{x_i}\otimes\mathcal{W}_{x_i}\big)\Big)\Big)\Big) && \text{by (36)} \\
\simeq\ & \textstyle\coprod_{i,j} \mathrm{CoDisc}\big(X_i\times Y_j\big)\cdot\varinjlim_{\mathbf{B}(G_i\times G_j)}\Big(\mathbf{E}(G_i\times G_j)\cdot F\Big(\iota\big(\mathcal{V}_{x_i}\otimes\mathcal{W}_{x_i}\big)\Big)\Big) && \text{as } F \text{ preserves hq-coproducts (40)} \\
\simeq\ & \textstyle\coprod_{i,j} \mathrm{CoDisc}\big(X_i\times Y_j\big)\cdot\varinjlim_{\mathbf{B}(G_i\times G_j)}\Big(\mathbf{E}(G_i\times G_j)\cdot F\Big(\iota\big(\mathcal{V}_{x_i}\big)\Big)\otimes_{\mathscr{C}}F\Big(\iota\big(\mathcal{W}_{x_i}\big)\Big)\Big) && \text{as } F\circ\iota \text{ preserves tensor products (40)} \\
\simeq\ & \Big(\textstyle\coprod_i \mathrm{CoDisc}(X_i)\cdot\varinjlim_{\mathbf{B}G_i}\Big(\mathbf{E}G_i\cdot F\big(\iota(\mathcal{V}_{x_i})\big)\Big)\Big)\otimes_{\mathscr{C}}\Big(\textstyle\coprod_j \mathrm{CoDisc}(Y_j)\cdot\varinjlim_{\mathbf{B}G_j}\Big(\mathbf{E}G_j\cdot F\big(\iota(\mathcal{W}_{x_j})\big)\Big)\Big) && \text{as } \otimes_{\mathscr{C}} \text{ preserves hq-coproducts (40)} \\
\simeq\ & F\Big(\textstyle\coprod_i \mathrm{CoDisc}(X_i)\cdot\varinjlim_{\mathbf{B}G_i}\big(\mathbf{E}G_i\cdot\iota(\mathcal{V}_{x_i})\big)\Big)\otimes_{\mathscr{C}}F\Big(\textstyle\coprod_j \mathrm{CoDisc}(Y_j)\cdot\varinjlim_{\mathbf{B}G_j}\big(\mathbf{E}G_j\cdot\iota(\mathcal{W}_{x_j})\big)\Big) && \text{as } F \text{ preserves hq-coproducts (40)} \\
\simeq\ & F\big(\mathcal{V}_{\mathscr{X}}\big)\otimes_{\mathscr{C}}F\big(\mathcal{W}_{\mathscr{Y}}\big) && \text{by Prop. 2.26.}
\end{aligned}
$$

□

While Thm. 2.31 does provide an answer to the question (1) for flat vector bundles over general parameter base spaces, of course flat vector bundles are only sensitive to the homotopy 1-type of the underlying space. This has the pleasant effect that the theory of ordinary local systems can be described entirely in ordinary category/groupoid theory, as we have done in this subsection, but now we want to generalize further to "∞-local systems" which are sensitive to the full homotopy type of the underlying parameter spaces.

Doing so requires the larger toolbox of "homotopical category theory" and for the result to be tractable in practice we need strong tools from simplicial model category theory. Using these, we can give a discussion of *simplicial local systems* which, conceptually, closely parallels the discussion of ordinary local systems we just gave, while being much richer. This we lay out in §3.

After the dust has settled, we arrive in (125) at a pushout diagram just as in (39) but with groupoids (homotopy 1-types) generalizing throughout to simplicial sets (general homotopy types), and thus producing the external tensor product in the generality of ∞-local systems.

3 The external tensor product as a derived functor

We construct the "derived functor" of the above external tensor product (Thm. 3.40 below), thus generalizing from vector bundles over discrete spaces to (flat) vector bundles over general spaces – and in fact to a "derived category" of flat ∞-vector bundles (Thm. 3.22). The key technical ingredient is a good model for ∞-vector spaces by simplicial chain complexes (Thm. 3.3). In fact, we give a presentation of these derived constructions by (model-)categories for which the pushout-characterization of the external tensor product (Thm. 2.14) is retained (due to Prop. 3.35 below):

$$
\begin{tikzcd}
\underset{\text{Rem. 3.10}}{\overset{\text{Simplicial group representations}}{\text{with external tensor product}}}
\Big(\int_{\mathscr{G} \in \mathrm{sGrp}} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} , \boxtimes \Big)
\arrow[r, hook, "\iota"]
& \overset{\text{Thm. 3.22}}{\Big(\overbrace{\int_{\mathbf{X} \in \mathrm{sSet\text{-}Grpd}_{\mathrm{skl}}} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}}^{\mathbf{Loc}_{\mathbb{K}} :\equiv} , \sqcup, \boxtimes \Big)}
\underset{\text{(Def. 3.29)}}{\overset{\text{Simplicial local systems}}{\text{with external tensor product}}}
\\
\underset{\text{(Def. 3.2)}}{\text{simplicial chain complexes}} \;
\mathbf{sCh}_{\mathbb{K}}
\arrow[u]
\arrow[r, hook, "\iota"]
\arrow[ur, phantom, "\text{(po)}"]
& \Big(\int_{\mathbf{X} \in \mathrm{sSet\text{-}Grpd}_{\mathrm{skl}}} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} , \sqcup \Big)
\arrow[u]
\; \underset{\text{(Def. 3.5, (84))}}{\text{simplicial local systems}}
\end{tikzcd}
$$

Since the homotopy theories of $\mathbb{K}$-chain complexes is equivalent to that of $H\mathbb{K}$-module spectra [Sh07] we may think of the integral model category $\mathbf{Loc}_{\mathbb{K}}$ as a global model for parameterized $H\mathbb{K}$-module spectra. As such, it may be compared with the global model categories for parameterized plain spectra ($\mathbb{S}$-module spectra) given in [HSS20] and [Mal23, Rem. 5.4.3]. See also the discussion in §4.

3.1 Flat ∞-vector bundles (∞-local systems)

The idea of a *flat ∞-vector bundles* (∞-local system) over any space X is that

(i) to any point $x \in X$ is assigned a chain complex V_x (cf. Def. 3.2 below);

(ii) to any path $p : \; x \rightsquigarrow y$ is assigned a chain map $\phi_p : V_x \to V_y$;

(iii) to any path-of-paths
$$
\begin{tikzcd}
& y \arrow[dr, squiggly, "p'"] & \\
x \arrow[ur, squiggly, "p"] \arrow[rr, squiggly, "p''"'] & \arrow[u, phantom, "\Downarrow\sigma"] & z
\end{tikzcd}
$$
is assigned a chain homotopy
$$
\begin{tikzcd}
& V_y \arrow[dr, "\phi_{p'}"] & \\
V_x \arrow[ur, "\phi_p"] \arrow[rr, "\phi_{p''}"'] & \arrow[u, phantom, "\Downarrow\sigma"] & V_z
\end{tikzcd}
$$

(In the special case that all chain complexes here are concentrated in degree 0, then these chain homotopies are necessarily identities and the structure reduces to that of an ordinary flat vector bundle.)

(∞) "and so on"...

But the explicit continuation of this pattern is not as straightforward as the previous steps, since there is not *directly* a notion of higher chain homotopy.

On the other hand, in the conceptual perspective of ∞-category theory it is evident that, abstractly, flat ∞-vector bundles should equivalently be ∞-functors from the fundamental ∞-groupoid of their base space to the ∞-category of chain complexes:

$$
\overset{\text{∞-local systems}}{\mathrm{LocSys}_\infty(\underset{\text{base space}}{X})} \;:=\; \overset{\text{∞-functors}}{\mathrm{Func}_\infty}\Big(\underset{\substack{\text{fundamental}\\ \text{∞-groupoid}}}{\textstyle\int X} \;,\; \underset{\substack{\text{∞-category of}\\ \text{chain complexes}}}{N\mathrm{Ch}_{\mathbb{K}}[\mathrm{W}_{\mathrm{qi}}^{-1}]} \Big) \;\simeq\; N\Big(\overset{\text{simplicial functors}}{\mathrm{sFunc}}\big(\underset{\substack{\text{fundamental}\\ \text{simplicial groupoid}}}{\mathscr{G}(\textstyle\int X)} \;,\; \underset{\substack{\text{simplicial model category}\\ \text{of chain complexes}}}{\mathrm{sCh}_{\mathbb{K}}} \big)^{\circ} \Big) . \tag{41}
$$

A quick but unwieldy way to define such ∞-functors is as maps from the singular simplicial complex $\int X$ of X into the suitably-defined "homotopy-coherent simplicial nerve" of $\mathrm{Ch}_{\mathbb{K}}$ (which is essentially tantamount to defining a simplicial notion of "higher chain homotopy"). Taking this to be the "dg-nerve" [Lu17, §1.3.1.6] of the dg-self-enrichement of $\mathrm{Ch}_{\mathbb{K}}$, this is the approach taken in the existing literature on ∞-local systems [BS14, §2][RZ20, §5].

However, here we need tighter control over these ∞-categories of ∞-local systems, in that we need a good model category presentation by simplicial functors — which had been missing but which we establish now.

Remark 3.1 (Equivalence of models of ∞-local systems)**.** That our new construction of ∞-local systems (in Def. 3.5 below) is equivalent, as an ∞-category, to the existing constructions ([BS14, §2][RZ20, §5]) can be seen via the comparison maps between the dg-nerve and the simplicial nerve established in [Lu17, Prop. 1.3.4.5][GP18, Prop. 5.17].[2]

Simplicial model categories. In the following, we make free use of notions and facts of model category theory [Qu67] (textbook accounts include [Ho99][Hi02]) in particular as concerns (combinatorial) simplicial model categories (see [Lu09, §A] and especially the comprehensive introductory account in [Ri14]).

∞-Vector spaces. We consider a particularly good model category presentation of the ∞-category of ∞-vector spaces modeled by chain complexes of vector spaces. While the model category $\mathrm{Ch}_{\mathbb{K}}$ of unbounded chain complexes is fairly familiar (recalled below), in its plain form as an ordinary category it is not a useful ingredient in the construction of flat ∞-vector bundles (∞-local systems), since the ordinary functors from 1-groupoids into it only model flat vector bundles over homotopy 1-types. This may be one reason why existing literature on ∞-local systems has made no use of model category theoretic tools. However, there is a well-known general approach to such situations which does stay within (and hence retains the power of) model category theory: This is to find an enhancement to a *combinatorial simplicial model* category whose underlying ordinary model category is Quillen equivalent to the original one: In this case, the category of simplicial functors with this codomain still presents the desired ∞-functor ∞-category but now does inherit itself a supporting model category structure. The fact that $\mathrm{Ch}_{\mathbb{K}}$ does admit such a simplicial enhancement is essentially well-known, though some of the details, such as its compatibility with the monoidal structure, are not explicit in the literature; therefore we spell it out:

Definition 3.2 (Category of simplicial chain complexes[3])**.**
For $\mathbb{K}$ a field (not necessarily of characteristic zero), we write:

(i) $\boxed{(\mathrm{Mod}_{\mathbb{K}}, \otimes)}$ for the category of $\mathbb{K}$-vector spaces with $\mathbb{K}$-linear maps between them (which below we frequently think of as the special case of R-modules for $R = \mathbb{K}$ a field, whence the notation), and equipped with the usual closed monoidal category structure given by the ordinary tensor product whose tensor unit is $\mathbb{K}$. We denote the linear mapping vector space between a pair of vector spaces by angular brackets

$$(-)\otimes(-) \,:\, \mathrm{Mod}_{\mathbb{K}} \times \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}\,, \qquad [-,-] \,:\, \mathrm{Mod}_{\mathbb{K}}^{\mathrm{op}} \times \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}} \tag{42}$$

and will successively overload this notation as this category is incrementally generalized in the following. The linear mapping space is of course the *internal hom* for this closed monoidal category, in that we have natural isomorphims of hom-sets of this form:

$$\mathrm{Mod}_{\mathbb{K}}\big(T \otimes V, W\big) \;\simeq\; \mathrm{Mod}_{\mathbb{K}}\big(T, [V, W]\big)\,. \tag{43}$$

The category $\mathrm{Mod}_{\mathbb{K}}$ is complete and cocomplete, in particular it is canonically *tensored* and *powered* over Set

$$\begin{array}{ccc} \mathrm{Set} \times \mathrm{Mod}_{\mathbb{K}} & \xrightarrow{\;(-)\cdot(-)\;} & \mathrm{Mod}_{\mathbb{K}}\,, \\ (S,V) & \longmapsto & \coprod_{s \in S} V \end{array} \qquad \begin{array}{ccc} \mathrm{Set}^{\mathrm{op}} \times \mathrm{Mod}_{\mathbb{K}} & \xrightarrow{\;(-)^{(-)}\;} & \mathrm{Mod}_{\mathbb{K}} \\ (S,V) & \longmapsto & \prod_{s \in S} V \end{array} \tag{44}$$

such that there are natural isomorphisms of hom-sets of the following form

$$S \in \mathrm{Set}, V, W \in \mathrm{Mod}_{\mathbb{K}} \quad \vdash \quad \mathrm{Mod}_{\mathbb{K}}\big(S \cdot V, W\big) \;\simeq\; \mathrm{Set}\big(S, \mathrm{Mod}_{\mathbb{K}}(V, W)\big) \;\simeq\; \mathrm{Mod}_{\mathbb{K}}\big(V, W^S\big)\,. \tag{45}$$

This may also be understood as the natural isomorphism (43) partially restricted along the free $\mathbb{K}$-module functor

$$\begin{array}{rcc} \mathbb{K}[-] : \mathrm{Set} & \longrightarrow & \mathrm{Mod}_{\mathbb{K}} \\ S & \longmapsto & \coprod_{s \in S} \mathbb{K} \end{array}$$

[2] We thank Dmitri Pavlov for pointing out this result.

[3] Beware that some authors say "simplicial chain complex" for the chain complexes that compute singular homology groups. Here we properly mean "simplicial objects in the category of chain complexes" — which is of course not unrelated but different and/or more general.

in that

$$
\begin{tikzcd}
\mathrm{Set} \times \mathbf{Mod}_{\mathbb{K}} \arrow[rr, "{\mathbb{K}[-] \times \mathrm{id}}"] \arrow[dr, "(-)\cdot(-)"'] & \arrow[d, phantom, "\Swarrow\sim"] & \mathbf{Mod}_{\mathbb{K}} \times \mathbf{Mod}_{\mathbb{K}} \arrow[dl, "(-)\otimes(-)"] \\
& \mathbf{Mod}_{\mathbb{K}} &
\end{tikzcd}
\qquad
\mathbb{K}[S] \otimes V \;\simeq\; S \cdot V\,. \tag{46}
$$

In particular, for $G \in \mathrm{Grp} \to \mathrm{Set}$ a set equipped with a group structure $\mu : G \times G \to G$, $\mathrm{e} : * \to G$, whose group algebra is $\mathbb{K}(G) \in \mathrm{Alg}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$, a $\mathbb{K}[G]$-module structure on a vector space may equivalently be thought of as a "tensoring action" via a morphism $\rho : G \cdot V \longrightarrow V$ making the following diagrams commute:

$$
\begin{tikzcd}
G \cdot (G \cdot V) \simeq (G \times G) \cdot V \arrow[r, "\mu \cdot \mathrm{id}"] \arrow[d, "\mathrm{id} \cdot \rho"'] & G \cdot V \arrow[d, "\rho"] \\
G \cdot V \arrow[r, "\rho"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
* \cdot \mathcal{V} \arrow[rr, "\mathrm{e} \cdot \mathrm{id}"] \arrow[dr, "\sim"'] & & G \cdot \mathcal{V} \arrow[dl, "\rho"] \\
& \mathcal{V} &
\end{tikzcd}
$$

(ii) $\boxed{(\mathrm{Ch}_{\mathbb{K}}, \otimes)}$ for the category of (unbounded) chain complexes of $\mathbb{K}$-vector spaces with chain maps between these as morphisms (cf. [We94, §1])

$$
\begin{array}{ccc}
V & :\equiv & \Big[\cdots \xrightarrow{\partial_1^V} V_1 \xrightarrow{\partial_0^V} V_0 \xrightarrow{\partial_{-1}^V} V_{-1} \xrightarrow{\partial_{-2}^V} \cdots \Big] \\
\big\downarrow{\scriptstyle \phi} & & \cdots \quad \big\downarrow{\scriptstyle \phi_1} \quad \big\downarrow{\scriptstyle \phi_0} \quad \big\downarrow{\scriptstyle \phi_{-1}} \quad \cdots \\
W & :\equiv & \Big[\cdots \xrightarrow{\partial_1^W} W_1 \xrightarrow{\partial_0^W} W_0 \xrightarrow{\partial_{-1}^W} W_{-1} \xrightarrow{\partial_{-2}^W} \cdots \Big]
\end{array}
\tag{47}
$$

and equipped with monoidal category structure given by the usual tensor product of chain complexes (cf. [We94, §2.7]):

$$
V \otimes W \;:\equiv\; \Big[\cdots \longrightarrow \bigoplus_{n \in \mathbb{Z}} \big(V_{n+1} \otimes W_{-n}\big) \xrightarrow[\;]{\substack{\partial^V \otimes \mathrm{id} - \\ (-1)^n \mathrm{id} \otimes \partial^W}} \bigoplus_{n \in \mathbb{Z}} \big(V_n \otimes W_{-n}\big) \xrightarrow[\;]{\substack{\partial^V \otimes \mathrm{id} - \\ (-1)^n \mathrm{id} \otimes \partial^W}} \bigoplus_{n \in \mathbb{Z}} \big(V_{n-1} \otimes W_{-n}\big) \longrightarrow \cdots \Big] \tag{48}
$$

whose tensor unit is

$$
\mathbb{1} \;:\equiv\; \big[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \underset{\deg = 0}{\mathbb{K}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \big], \tag{49}
$$

and which is closed with internal-hom ("mapping complex") given by:

$$
[V, W] \;:\equiv\; \Big[\cdots \longrightarrow \bigtimes_{n \in \mathbb{X}} \big[V_n, W_{n+1}\big] \xrightarrow[\;]{\substack{\partial^W \circ (-) - \\ (-1)^n (-) \circ \partial^V}} \bigtimes_{n \in \mathbb{X}} \big[V_n, W_n\big] \xrightarrow[\;]{\substack{\partial^W \circ (-) - \\ (-1)^n (-) \circ \partial^V}} \bigtimes_{n \in \mathbb{X}} \big[V_n, W_{n-1}\big] \longrightarrow \cdots \Big], \tag{50}
$$

where the tensor symbol and the angular brackets on the right denote the corresponding operations (42) on component vector spaces.

This category is (co)complete with (co)limits formed degreewise in $\mathrm{Mod}_{\mathbb{K}}$; in particular it is (co)tensored over Set, degreewise as in (44)

$$
\begin{array}{ccc}
\mathrm{Set} \times \mathrm{Ch}_{\mathbb{K}} & \xrightarrow{(-)\cdot(-)} & \mathrm{Ch}_{\mathbb{K}} \\
(S, V) & \longmapsto & \coprod_{s \in S} V
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Set}^{\mathrm{op}} \times \mathrm{Ch}_{\mathbb{K}} & \xrightarrow{(-)^{(-)}} & \mathrm{Ch}_{\mathbb{K}} \\
(S, V) & \longmapsto & \prod_{s \in S} V
\end{array}
\tag{51}
$$

(iii) $\boxed{(\mathrm{sCh}_{\mathbb{K}}, \otimes) := (\mathrm{Ch}_{\mathbb{K}}, \otimes)^{\Delta^{\mathrm{op}}}}$ for the category of *simplicial objects* (cf. [Ma67, §1][KT06, §A.1]) in the previous category of unbounded chain complexes:

$$\mathcal{V} :\equiv \left[\begin{array}{cccccccccc}
\vdots & & & & \vdots & & \vdots & & \vdots & \\
\mathcal{V}_2 & :\equiv & \big[\cdots & \xrightarrow{\partial_1^{\mathcal{V}_2}} & \mathcal{V}_{2,1} & \xrightarrow{\partial_0^{\mathcal{V}_2}} & \mathcal{V}_{2,0} & \xrightarrow{\partial_{-1}^{\mathcal{V}_2}} & \mathcal{V}_{2,-1} & \xrightarrow{\partial_{-2}^{\mathcal{V}_2}} \cdots\big] \\
{\scriptstyle d_1^0}\big\downarrow\;\big\uparrow{\scriptstyle s_1^0}\;\big\downarrow{\scriptstyle d_1^1}\;\big\uparrow{\scriptstyle s_1^1}\;\big\downarrow{\scriptstyle d_1^2} & & & & {\scriptstyle d_{1,1}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{1,1}^0}\;\big\downarrow{\scriptstyle d_{1,1}^1}\;\big\uparrow{\scriptstyle s_{1,1}^1}\;\big\downarrow{\scriptstyle d_{1,1}^2} & & {\scriptstyle d_{1,0}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{1,0}^0}\;\big\downarrow{\scriptstyle d_{1,0}^1}\;\big\uparrow{\scriptstyle s_{1,0}^1}\;\big\downarrow{\scriptstyle d_{1,0}^2} & & {\scriptstyle d_{1,-1}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{1,-1}^0}\;\big\downarrow{\scriptstyle d_{1,-1}^1}\;\big\uparrow{\scriptstyle s_{1,-1}^1}\;\big\downarrow{\scriptstyle d_{1,-1}^2} & \\
\mathcal{V}_1 & :\equiv & \big[\cdots & \xrightarrow{\partial_1^{\mathcal{V}_1}} & \mathcal{V}_{1,1} & \xrightarrow{\partial_0^{\mathcal{V}_1}} & \mathcal{V}_{1,0} & \xrightarrow{\partial_{-1}^{\mathcal{V}_1}} & \mathcal{V}_{1,-1} & \xrightarrow{\partial_{-2}^{\mathcal{V}_1}} \cdots\big] \\
{\scriptstyle d_0^0}\big\downarrow\;\big\uparrow{\scriptstyle s_0^0}\;\big\downarrow{\scriptstyle d_0^1} & & & & {\scriptstyle d_{0,1}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{0,1}^0}\;\big\downarrow{\scriptstyle d_{0,1}^1} & & {\scriptstyle d_{0,0}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{0,0}^0}\;\big\downarrow{\scriptstyle d_{0,0}^1} & & {\scriptstyle d_{0,-1}^0}\big\downarrow\;\big\uparrow{\scriptstyle s_{0,-1}^0}\;\big\downarrow{\scriptstyle d_{0,-1}^1} & \\
\mathcal{V}_0 & :\equiv & \big[\cdots & \xrightarrow{\partial_1^{\mathcal{V}_0}} & \mathcal{V}_{0,1} & \xrightarrow{\partial_0^{\mathcal{V}_0}} & \mathcal{V}_{0,0} & \xrightarrow{\partial_{-1}^{\mathcal{V}_0}} & \mathcal{V}_{0,-1} & \xrightarrow{\partial_{-2}^{\mathcal{V}_0}} \cdots\big]
\end{array}\right]$$

with simplicial chain maps between these as morphisms, i.e., arrays of linear maps respecting all these structure maps:

$$\begin{array}{c}\mathcal{V}\\ \big\downarrow{\scriptstyle\phi}\\ \mathcal{W}\end{array} \quad :\equiv \quad \left(\begin{array}{c}\mathcal{V}_i\\ \big\downarrow{\scriptstyle\phi_i}\\ \mathcal{W}_i\end{array} \in \mathrm{Mor}\big(\mathrm{Ch}_{\mathbb{K}}\big)\right)_{i\in\mathbb{N}} \quad\equiv\quad \left(\begin{array}{c}\mathcal{V}_{i,j}\\ \big\downarrow{\scriptstyle\phi_{i,j}}\\ \mathcal{W}_{i,j}\end{array} \in \mathrm{Mor}\big(\mathrm{Mod}_{\mathbb{K}}\big)\right)_{\substack{i\in\mathbb{N}\\ j\in\mathbb{Z}}}$$

and equipped with the simplicial-degree-wise tensor product of chain complexes from above:

$$\mathcal{V}\otimes\mathcal{W} \quad :\equiv \quad \left[\begin{array}{c}
\vdots\\
\mathcal{V}_1\otimes\mathcal{W}_1\\
{\scriptstyle d_0^0\otimes d_0^0}\big\downarrow\;\big\uparrow{\scriptstyle s_0^0\otimes s_0^0}\;\big\downarrow{\scriptstyle d_0^1\otimes d_0^1}\\
\mathcal{V}_0\otimes\mathcal{W}_0
\end{array}\right],$$

which is[4] monoidal closed with internal hom (“simplicial mapping complex”) given by the end-formula

$$\mathcal{V},\mathcal{W} \in \mathrm{sCh}_{\mathbb{K}} \quad\vdash\quad [\mathcal{V},\,\mathcal{W}] := \left([k] \longmapsto \int_{[s]\in\Delta} \Big[\Delta[k]_s\cdot\mathcal{V}_s,\,\mathcal{W}_s\Big]\right) \in \mathrm{sCh}_{\mathbb{K}}, \tag{52}$$

(where in the integrand on the right the angular brackets refer to the mapping complex (50) and the dot to the tensoring (51) of component chain complexes).

(iv) $\boxed{\mathbf{sCh}_{\mathbb{K}}}$ for the sSet-enriched category (“simplicial category”) whose objects are those of $\mathrm{sCh}_{\mathbb{K}}$ and whose simplicial hom-sets are[5]

$$\mathcal{V},\mathcal{W} \in \mathrm{sCh}_{\mathbb{K}} \quad\vdash\quad \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V},\,\mathcal{W}\big) := \left([k] \mapsto \int_{[s]\in\Delta} \mathrm{Ch}_{\mathbb{K}}\big(\Delta[k]_s\cdot\mathcal{V}_s,\,\mathcal{W}_s\big)\right) \in \mathrm{sSet}, \tag{53}$$

[4]This follows by using Lem. A.25 for: $\mathbf{V} :\equiv \mathbf{C} :\equiv \mathrm{Ch}_{\mathbb{K}}$ and $\mathbf{X} :\equiv \Delta^{\mathrm{op}}$ regarded as a Set $\xhookrightarrow{\mathbb{K}[-]}$ $\mathrm{Ch}_{\mathbb{K}}$-enriched category).

[5]This follows using Lem. A.25 for $\mathbf{V} :\equiv \mathrm{Set}$, $\mathbf{C} :\equiv \mathrm{Ch}_{\mathbb{K}}$ and $\mathbf{X} :\equiv \Delta^{\mathrm{op}}$.

and with (co)tensoring given by

$$\begin{array}{ccc} \mathrm{sSet} \times \mathrm{sCh}_{\mathbb{K}} & \xrightarrow{(-)\cdot(-)} & \mathrm{sCh}_{\mathbb{K}} \\ (\mathscr{S}, \mathcal{V}) & \mapsto & \big([k] \mapsto \mathscr{S}_k \cdot \mathcal{V}_k\big), \end{array} \qquad \begin{array}{ccc} \mathrm{sSet}^{\mathrm{op}} \times \mathrm{sCh}_{\mathbb{K}} & \xrightarrow{(-)^{(-)}} & \mathrm{sCh}_{\mathbb{K}} \\ (\mathscr{S}, \mathcal{V}) & \mapsto & \Big([k] \mapsto \int_{s \in \Delta} \big(\mathcal{V}_s\big)^{(\mathscr{S} \times \Delta[k])_s}\Big), \end{array} \tag{54}$$

$$\mathscr{S} \in \mathrm{sSet};\, \mathcal{V}, \mathcal{W} \in \mathrm{sCh}_{\mathbb{K}} \quad \vdash \quad \mathbf{sCh}_{\mathbb{K}}\big(\mathscr{S} \cdot \mathcal{V}, \mathcal{W}\big) \simeq \mathbf{sSet}\big(\mathscr{S}, \mathbf{sCh}(\mathcal{V}, \mathcal{W})\big) \simeq \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V}, \mathcal{W}^{\mathscr{S}}\big) \tag{55}$$

and regarded as an sSet-*enriched monoidal category* (Def. A.26) by enhancement of the previous tensor functor to an sSet-enriched functor with the following components:

$$\begin{array}{ccc} \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V}, \mathcal{W}\big) \times \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V}', \mathcal{W}'\big) & \xrightarrow{\otimes_{(\mathcal{V},\mathcal{V}'),(\mathcal{W},\mathcal{W}')}} & \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V} \otimes \mathcal{V}', \mathcal{W} \otimes \mathcal{W}'\big) \\ \Big(\big(\Delta[k] \cdot \mathcal{V} \xrightarrow{\phi} \mathcal{W}\big), \big(\Delta[k] \cdot \mathcal{V}' \xrightarrow{\phi'} \mathcal{W}'\big)\Big) & \mapsto & \Big(\Delta[k] \cdot \big(\mathcal{V} \otimes \mathcal{V}'\big) \xrightarrow{\mathrm{diag}\cdot(\cdots)} \big(\Delta[k] \cdot \mathcal{V}\big) \otimes \big(\Delta[k] \cdot \mathcal{V}'\big) \xrightarrow{\phi \otimes \phi'} \mathcal{W} \otimes \mathcal{W}'\Big) \end{array} \tag{56}$$

and dually for the simplicial enhancement of the internal hom-functor:

$$\begin{array}{ccc} \mathbf{sCh}_{\mathbb{K}}^{\mathrm{op}}\big(\mathcal{V}, \mathcal{W}\big) \times \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V}', \mathcal{W}'\big) & \xrightarrow{[-,\,-]_{(\mathcal{V},\mathcal{V}'),(\mathcal{W},\mathcal{W}')}} & \mathbf{sCh}_{\mathbb{K}}\big([\mathcal{V}, \mathcal{V}'], [\mathcal{W}, \mathcal{W}']\big) \\ \Big(\big(\Delta[k] \cdot \mathcal{W} \xrightarrow{\phi} \mathcal{V}\big), \big(\mathcal{V}' \xrightarrow{\phi'} (\mathcal{W}')^{\Delta[k]}\big)\Big) & \mapsto & \Big([\mathcal{V}, \mathcal{W}] \xrightarrow{[\phi, \phi']} \big[\Delta[k] \cdot \mathcal{V}', (\mathcal{W}')^{\Delta[k]}\big] \xrightarrow{(\cdots)^{\mathrm{diag}}} [\mathcal{V}, \mathcal{W}]\Big). \end{array} \tag{57}$$

(v) $\boxed{\mathrm{Ch}_{\mathbb{K}} \underset{\mathrm{ev}_0}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{sCh}_{\mathbb{K}}}$ for the pair of strong monoidal adjoint functors, where

- the left adjoint, const, sends a chain complex V to the simplicial chain complex all whose entries are V and all whose simplicial maps are id_V,
- the right adjoint, ev_0, sends a simplicial chain complex $\mathcal{V}$ to its 0-component $\mathcal{V}_0$ (this being the limit over the simplicial diagram),

(vi) $\boxed{\mathrm{sCh}_{\mathbb{K}} \xrightarrow{\mathrm{tot}} \mathrm{Ch}_{\mathbb{K}}}$ for the *total chain complex* functor

$$\mathrm{tot}(\mathcal{V}) \;:\equiv\; \Big[\cdots \longrightarrow \bigoplus_{s+t=1} \mathcal{V}_{s,t} \xrightarrow{\partial + \sum_s (-1)^s d^s} \bigoplus_{s+t=0} \mathcal{V}_{s,t} \xrightarrow{\partial + \sum_s (-1)^s d^s} \bigoplus_{s+t=-1} \mathcal{V}_{s,t} \longrightarrow \cdots\Big]. \tag{58}$$

Theorem 3.3 (Model category of simplicial chain complexes)**.** *The categories of chain complexes of vector spaces from Def. 3.2 carry the following model category structures:*

(i) $\boxed{(\mathrm{Ch}_{\mathbb{K}}, \otimes)}$ *carries a monoidal model category structure with the following properties:*

- *the model structure has*
  - *weak equivalences the quasi-isomorphisms (the isomorphisms on chain homomology),*
  - *fibrations the degreewise surjections (in particular all objects are fibrant),*
  - *cofibrations the degreewise injections (in particular all objects are cofibrant),*
- *the model structure is:*
  - *proper,*
  - *combinatorial*
    with sets of generating (acyclic) cofibrations $\mathrm{I}_{\mathbb{K}} := \{i_n \,|\, n \in \mathbb{Z}\}$ ($\mathrm{J}_{\mathbb{K}} := \{j_n \,|\, n \in \mathbb{Z}\}$) *given by:*

$$\begin{array}{ccl} S^{n-1} & :\equiv & [\cdots \to 0 \to 0 \to 0 \to \mathbb{K} \to 0 \to \cdots] \\ \big\downarrow{\scriptstyle i_n} & & \qquad\;\;\, \downarrow \quad\; \downarrow \quad\; \downarrow \quad\; \downarrow{\scriptstyle \mathrm{id}} \;\; \downarrow \\ D^n & :\equiv & [\cdots \to 0 \to 0 \to \underset{\deg = n}{\mathbb{K}} \xrightarrow{\mathrm{id}} \mathbb{K} \to 0 \to \cdots] \end{array} \qquad \begin{array}{ccl} 0 & :\equiv & [\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots] \\ \big\downarrow{\scriptstyle j_n} & & \qquad\;\;\, \downarrow \quad\; \downarrow \quad\; \downarrow \quad\; \downarrow \quad\; \downarrow \\ D^n & :\equiv & [\cdots \to 0 \to 0 \to \underset{\deg = n}{\mathbb{K}} \xrightarrow{\mathrm{id}} \mathbb{K} \to 0 \to \cdots] \end{array} \tag{59}$$

(ii) $\boxed{(\mathbf{sCh}_{\mathbb{K}}, \otimes)}$ *carries a simplicial monoidal model category structure with the following properties:*

- *the model structure has*
  - *weak equivalences the total-quasi-isomorphisms, i.e. the quasi-isos on total chain complexes* (58)*,*
  - *all objects cofibrant;*
- *the adjunction* $\mathrm{const} : \mathrm{Ch}_{\mathbb{K}} \rightleftarrows \mathrm{sCh}_{\mathbb{K}} : \mathrm{ev}_0$ *is a monoidal Quillen equivalence;*
- *the model structure is:*
  - *left proper,*
  - *combinatorial,*
    with sets of generating (acyclic) cofibrations to be denoted

$$\mathrm{sI}_{\mathbb{K}}\,, \quad \mathrm{sJ}_{\mathbb{K}}\,. \tag{60}$$

Proof. We discuss the claims successively:

Combinatorial model structure. Generally, for R a commutative ring, the projective model structure on unbounded chain complexes of R-modules (i.e., with weak equivalences the quasi-isomorphisms and fibrations the degreewise surjection) exists as a proper and cofibrantly generated model category by arguments given in [HPS, after Thm. 9.3.1][Ho99, p. 41] [ScSh00, p. 7][Fa06, Thm. 3.2]. For the special case that all submodules of free R-modules are themselves free (such as for $R = \mathbb{Z}$ but also in our case where $R = \mathbb{K}$ is a field) an alternative proof is spelled out in [Str20].

That the underlying category is locally presentable, hence that this cofibrantly generated model structure is in fact combinatorial, follows from classical facts: A category of R-modules is a Grothendieck abelian category (e.g. [Jo77, Thm. 8.11]), the category of chain complexes in a Grothendieck abelian category is itself Grothendieck abelian (e.g. [Ho99a, p. 3]) and every Grothendieck abelian category is locally presentable, by [Be00, Prop. 3.10] (cf., e.g., [Kr15, Cor. 5.2]).

Characterization of the cofibrations. Still in the generality of any commutative ground ring R, [Ho99, Prop. 2.3.9] shows that the cofibrations in this projective model structure are the degreewise *split*-injections *with cofibrant cokernel*, and [Ho99, Lem 2.3.6] shows that at least all bounded-below cochain complexes of *projective* modules are cofibrant.

Now to specialize this characterization to our case where R is a field $\mathbb{K}$, so that the above R-modules become $\mathbb{K}$-vector spaces: Here, by the basis theorem, every module is free and in particular projective, and every short exact sequence splits. So in this case it follows, first, that at least every bounded-below chain complex is cofibrant and that the cofibrations are the degreewise injections with cofibrant cokernel. So to see that in this case the cofibrations in fact coincide with all degreewise injections it is now sufficient to see that actually every chain complex over a field is projectively cofibrant: Observe that every chain complex V is the colimit over its cotower of k-connective covers $V \simeq \mathrm{colim}\big(\mathrm{cn}_0 V \hookrightarrow \mathrm{cn}_{-1} V \hookrightarrow \mathrm{cn}_{-2} V \hookrightarrow \cdots\big)$, which, by the remarks just made, is a transfinite composition of cofibrations, so that the cocone $\mathrm{cn}_0 V \hookrightarrow V$ is itself a cofibration. Since we already know that $\mathrm{cn}_0 V$ is cofibrant (being bounded below) it follows that also V is.

Monoidal model structure. That the tensor product of chain complexes makes the above model category into a monoidal model category is discussed in [Ho01, Cor. 3.7][Fa06, Thm. 6.1] and in [Str20] (there in the generality where submodules of free R-modules are themselves free). Explicitly, notice that for checking the pushout-product axiom (158) in a closed tensor product on a cofibrantly generated model category, it is sufficient to check it on generating (acyclic) cofibrations, which in our case (59) is fairly immediate.

Much of the model category structure listed so far, that is not specific to R being a field, is also summarized in [MR19, §1].

Simplicial model structure. Again in the generality of any commutative ground ring, [RSS01, p. 10] show that a simplicial enhancement of the projective model structure on unbounded chain complexes is given, by the Reedy model structure on sCh_R left Bousfield-localized at the total-quasi-isomorphisms, making the adjoint pair $\mathrm{const} \dashv \mathrm{ev}_0$ a Quillen equivalence.

For us, it remains to see that over a ground field $R = \mathbb{K}$ all objects in this simplicial model structure are cofibrant. But since left Bousfield localization does not change the class of cofibrations, and since we already saw above that all objects in $\mathrm{Ch}_{\mathbb{K}}$ are cofibrant, it is sufficient to see that: [6]

Every simplicial diagram in a model category with underlying abelian category and all whose objects are cofibrant is itself Reedy-cofibrant. This follows by appeal to the Dold-Kan correspondence, which exhibits any such simplicial object degreewise as a direct sum of objects of degenerate and of non-degenerate simplices. Inspection shows that these summands of degenerate simplices are isomorphically the "latching objects" appearing in the definition of the Reedy model structure, which implies that a simplicial diagram in the given case is Reedy cofibrant as soon as the degreewise sub-objects of non-degenerate simplices are cofibrant.

Monoidal simplicial model structure.[7] First, the plain Reedy model structure on simplicial objects in a symmetric monoidal model category is itself monoidal model under the degree-wise tensor product, by [Ba10, Thm. 3.51]. To check that this monoidal model structure is preserved by left Bousfield localization at the total-quasi-isomorphisms we check the sufficient criterion given in [Ba10, Thm. 3.51]. Indeed, observing that:

- every object $\mathcal{V} \in \mathrm{sCh}_{\mathbb{K}}$ is a homotopy colimit of simplicially constant objects $\mathrm{const}(V)$ (since these are Reedy cofibrant, by the above, so that $\mathrm{hocolim}_{[k]\in\Delta}\mathrm{const}(\mathcal{V}_k)$ is computed by the coend $\int^{[k]\in\Delta}\Delta[k]\cdot\mathrm{const}(\mathcal{V})_k$, which is $\mathcal{V}$),
- for a Reedy fibrant object $\mathcal{W}$ to be local in $\mathrm{sCh}_{\mathbb{K}}$ with respect to total-quasi-isomorphisms means [RSS01] to be *homotopically constant* in that all the simplicial maps $d_i : \mathcal{W}_{i+i} \longrightarrow \mathcal{W}_i$ and $s_i : \mathcal{W}_i \longrightarrow \mathcal{W}_{i+1}$ are quasi-isomorphisms,

this criterion says it is sufficient to check that for $\mathcal{W}$ homotopically constant, also the internal hom $\big[\mathrm{const}(V),\, \mathcal{W}\big]$ (52) is homotopically constant. Now for constant domain, the internal hom reduces (essentially by an incarnation of the Yoneda Lemma) to

$$
\begin{aligned}
\big[\mathrm{const}(V),\, \mathcal{W}\big] \;:\; [k] \;\longmapsto\; \int_{[s]\in\Delta}\Big[\big(\Delta[k]\cdot\mathrm{const}(V)\big)_s,\, \mathcal{W}_s\Big] \;&\simeq\; \int_{[s]\in\Delta}\big[\Delta[k]_s\cdot V,\, \mathcal{W}_s\big] \\
&\simeq\; \int_{[s]\in\Delta}\Big[V,\, (\mathcal{W}_s)^{\Delta[k]_s}\Big] \\
&\simeq\; \Big[V,\, \int_{[s]\in\Delta}(\mathcal{W}_s)^{\Delta[k]_s}\Big] \\
&\simeq\; \big[V,\, \mathcal{W}_k\big]\,.
\end{aligned}
$$

Therefore, it is sufficient now to observe that $[V,-] : \mathrm{Ch}_{\mathbb{K}} \longrightarrow \mathrm{Ch}_{\mathbb{K}}$ preserves all quasi-isomorphism. But this is the case because, by the above discussion, (1.) all objects in $\mathrm{Ch}_{\mathbb{K}}$, such as V here, are cofibrant so that $[V,-]$ is a right Quillen functor by the pullback-power axioms satisfied in the monoidal model category, and (2.) all objects, such as the $\mathcal{W}_k$ here, are also fibrant, so that weak equivalences between them are preserved by right Quillen functors, according to Ken Brown's lemma A.19.

It just remains to observe that the Quillen equialence $\mathrm{const} \dashv \mathrm{ev}_0$ is a monoidal Quillen adjunction (according to [Ho99, Def. 4.2.16]), which is immediate since const is already a strong monoidal functor and since the tensor unit is cofibrant (like all objects).

Left proper combinatorial simplicial model structure. Finally, that this model structure on $\mathrm{sCh}_{\mathbb{K}}$ is left proper and combinatorial follows by general results from the above fact that $\mathrm{Ch}_{\mathbb{K}}$ is so:

1. Any Reedy model category with coefficients in a locally presentable model category (with small domains of generating cofibrations) is itself locally presentable, by [Hi02, Thm. 15.6.27].
2. Any functor category out of a small category into a locally presentable category is itself locally presentable [AR94, Cor. 1.54].
3. Any Reedy model category with coefficients in a left (or right) proper model category is itself left (or right) proper [Hi02, Thm. 15.3.4 (2)].
4. Any left Bousfield localization of a left proper combinatorial model category is itself left proper combinatorial [Ba10, Thm. 4.7]. □

[6] This argument was pointed out to us by Charles Rezk, and we thank Dmitri Pavlov for further discussion. The details may be found spelled out at: `ncatlab.org/nlab/show/Reedy+model+structure#WithValuesInAnAbelianCategory`.

[7] We thank Dmitri Pavlov for pointing out the results of [Ba10] used here.

The upshot of Theorem 3.3 is that the simplicial monoidal model category $\mathbf{sCh}_{\mathbb{K}}$ is a good model category theoretic enhancement of the coefficient ∞-category needed to define flat ∞-vector bundles. Beyond giving a good handle on ∞-local systems over fixed base space, we use this below to construct the global theory of flat ∞-vector bundles over *varying* parameter spaces (Thm. 3.22 below), on which the $\boxtimes$-tensor product will exist as a decently homotopical functor (Thm. 3.40 below).

Remark 3.4 (Fibrant replacement)**.** Since every object of $\mathrm{sCh}_{\mathbb{K}}$ is cofibrant, the notion of higher chain homotopy encoded by this model category is all given by Reedy-*fibrant* replacement of chain complexes $V \in \mathrm{Ch}_{\mathbb{K}} \xhookrightarrow{\mathrm{const}} \mathrm{sCh}_{\mathbb{K}}$.

Linear ∞-representations of ∞-categories. Specifically, we can now invoke the following general constructions:

Definition 3.5 (Category of simplicial local systems)**.** Given $\mathbf{X} \in \mathrm{sSet\text{-}Cat}$ a small simplicial category, we write $\boxed{\big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} :\equiv \mathbf{sFunc}(\mathbf{X}, \mathbf{sCh}_{\mathbb{K}}), \otimes_{\mathbf{X}}\big)}$ for the closed monoidal simplicial category whose [8]

- objects are sSet-enriched functors from $\mathbf{X}$ to $\mathbf{sCh}_{\mathbb{K}}$ (Def. 3.2), to be denoted
$$\begin{array}{rccc} \mathcal{V}_{\mathbf{X}} : & \mathbf{X} & \longrightarrow & \mathbf{sCh}_{\mathbb{K}} \\ & x & \longmapsto & \mathcal{V}_x \end{array}$$
- hom-complexes are
$$\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\big(\mathcal{V}_{\mathbf{X}}, \mathcal{W}_{\mathbf{X}}\big) \;:\equiv\; \int_{x \in \mathbf{X}} \mathbf{sCh}\big(\mathcal{V}_x, \mathcal{W}_x\big) \;\in\; \mathrm{sSet},$$
  and[9]
  - equipped with the cup-tensor product induced from the tensor product carried by $\mathbf{sCh}_{\mathbb{K}}$
$$\begin{array}{ccc} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \times \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} & \xrightarrow{\quad\otimes_{\mathbf{X}}\quad} & \mathrm{sCh}_{\mathbb{K}}^{\mathbf{X}} \\ \big(\mathcal{V}_{\mathbf{X}}, \mathcal{V}'_{\mathbf{X}}\big) & \longmapsto & \mathbf{X} \xrightarrow{\mathrm{diag}} \mathbf{X} \times \mathbf{X} \xrightarrow{\mathcal{V}_{\mathbf{X}} \times \mathcal{V}'_{\mathbf{X}}} \mathbf{sCh}_{\mathbb{K}} \times \mathbf{sCh}_{\mathbb{K}} \xrightarrow{\otimes} \mathbf{sCh}_{\mathbb{K}} \end{array} \tag{61}$$
  - and similarly with the following cup-tensoring
$$\begin{array}{ccc} \mathbf{sSet}^{\mathbf{X}} \times \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} & \xrightarrow{\quad(-)\cdot_{\mathbf{X}}(-)\quad} & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \\ \big(S_{\mathbf{X}}, \mathcal{V}_{\mathbf{X}}\big) & \longmapsto & \mathbf{X} \xrightarrow{\mathrm{diag}} \mathbf{X} \times \mathbf{X} \xrightarrow{S_{\mathbf{X}} \times \mathcal{V}'_{\mathbf{X}}} \mathbf{sSet} \times \mathbf{sCh}_{\mathbb{K}} \xrightarrow{(-)\cdot(-)} \mathbf{sCh}_{\mathbb{K}} \end{array} \tag{62}$$
  - whose corresponding internal hom is given by
$$\begin{array}{ccc} \big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\big)^{\mathrm{op}} \times \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} & \xrightarrow{\quad[-,-]_{\mathbf{X}}\quad} & \mathrm{sCh}_{\mathbb{K}}^{\mathbf{X}} \\ \big(\mathcal{V}_{\mathbf{X}}, \mathcal{W}_{\mathbf{X}}\big) & \longmapsto & \Big(x \mapsto \int_{x' \in \mathbf{X}} \big[\mathbf{X}(x', x) \cdot \mathcal{V}_{x'}, \mathcal{W}_{x'}\big]\Big) \end{array} \tag{63}$$
$$\mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \qquad \vdash \qquad \begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[r, shift left, "\mathcal{V}\otimes_{\mathbf{X}}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[l, shift left, "{[\mathcal{V},-]_{\mathbf{X}}}"] \end{tikzcd}. \tag{64}$$

Proposition 3.6 (Model category of simplicial local systems over a fixed base space)**.**
For $\mathbf{X} \in \mathrm{sSet\text{-}Cat}$ *a small simplicial category, the monoidal simplicial functor category (Def. 3.5)* $\boxed{\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}}$ *carries a model category structure with the following properties:*

(i) *the model structure has*
- *weak equivalences the* $\mathbf{X}$*-objectwise weak equivalences in* $\mathrm{sCh}_{\mathbb{K}}$*,*
- *fibrations the* $\mathbf{X}$*-objectwise fibrations in* $\mathrm{sCh}_{\mathbb{K}}$ *(both according to Prop. 3.3);*

[8] This uses Lem. A.25 for $\mathbf{V} :\equiv \mathrm{sSet}$, $\mathbf{C} :\equiv \mathrm{sCh}_{\mathbb{K}}$ and the given $\mathbf{X}$.

[9] Now using Lem. A.25 for $\mathbf{V} :\equiv \mathbf{C} :\equiv \mathrm{sCh}_{\mathbb{K}}$ and $\mathbf{X}$ regarded as a $\mathrm{sSet} \xhookrightarrow{\mathbb{K}[-]} \mathrm{sCh}_{\mathbb{K}}$-enriched category.

(ii) *the model structure is*

– *combinatorial*

with sets of generating (acyclic) cofibrations those of (60) *tensored to representables:*

$$\mathrm{sI}_{\mathbb{K}}^{\mathbf{X}} := \big\{ \mathbf{X}(x,-)\cdot i \,\big|\, x \in \mathbf{X},\, i \in \mathrm{sI}_{\mathbb{K}} \big\}, \qquad \mathrm{sJ}_{\mathbb{K}}^{\mathbf{X}} := \big\{ \mathbf{X}(x,-)\cdot j \,\big|\, x \in \mathbf{X},\, j \in \mathrm{sJ}_{\mathbb{K}} \big\}. \tag{65}$$

Proof. Since $\mathbf{sCh}_{\mathbb{K}}$ is simplicial combinatorial, by Prop. 3.3, this is the existence statement of the projective model structure on enriched functors, see for instance [Lu09, Prop. A.3.3.2]. □

Remark 3.7 (Base change between model structures of simplicial local systems)**.** For $\mathbf{f} : \mathbf{X} \longrightarrow \mathbf{X}'$ a morphism of simplicial groupoids, the induced pair of adjoint functors (150)

$$\begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[r, shift left=2, "\mathbf{f}_!"] \arrow[r, phantom, "\perp"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}'} \arrow[l, shift left=2, "\mathbf{f}^*"] \end{tikzcd}$$

is a Quillen adjunction with respect to the model structure from Prop. 3.6 (since the right adjoint $\mathbf{f}^*$, which acts by precomposition, clearly preserves the objectwise defined weak equivalences and fibrations).

Remark 3.8 (Linear ∞-category-representations)**.** In the following, we focus on the specialization of Prop. 3.6 to domains $\mathscr{X}$ which model ∞-groupoids. But it is clear that Prop. 3.6 is relevant more generally. For example, already in one of simplest examples of a small (simplicial) category $\mathscr{X}$ which is not a (simplicial) groupoid, namely the category $\mathrm{FinSet}_{\mathrm{inj}}$ of finite sets and *injective* maps between them, Prop. 3.6 provides the homotopy-coherent enhancement of the notion of *FI-modules*. The latter is known to have some deep relation to braid group representations and hence to topological quantum computation.

∞-Group representations. With this in hand and by the fact that every ∞-group is presented by a simplicial group, we immediately obtain a model category of "$\mathbb{K}$-linear ∞-representation of ∞-groups", identified with ∞-local systems over a delooping ∞-groupoid:

Definition 3.9 (Simplicial delooping groupoids)**.** For a simplicial group $\mathscr{G} \in \mathrm{Grp}(\mathrm{sSet})$ we write

$$\mathbf{B}\mathscr{G} \in \mathrm{sSet\text{-}Grpd} \hookrightarrow \mathrm{sSet\text{-}Cat}$$

for the simplicial groupoid which has

- a single object,
- single hom-object identified with the simplicial group, whose composition operation and identity element is given by the group operation and the neutral element on $\mathscr{G}$.

Remark 3.10 (Simplicial functors between $\mathbf{B}\mathscr{G}$s are simplicial group homomorphisms)**.**
The sSet-enriched functors between simplicial delooping groupoids (Def. 3.9) correspond naturally to homomorphisms between the corresponding simplicial groups

$$\mathrm{sSet\text{-}Grpd}\big(\mathbf{B}\mathscr{G}, \mathbf{B}\mathscr{G}'\big) \;\simeq\; \mathrm{sGrp}(\mathscr{G}, \mathscr{G}').$$

Remark 3.11 (Simplicial local systems on $\mathbf{B}\mathscr{G}$ are simplicial group representations)**.**
By the (co)tensoring adjunctions, a simplicial functor $\mathcal{V}_{\mathbf{B}\mathscr{G}} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}$ on the simplicial delooping groupoid $\mathbf{B}\mathscr{G}$ (Def. 3.9) is equivalently a simplicial $\mathscr{G}$-action $\rho_{\mathcal{V}}$ on some object $\mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}$:

$$\begin{array}{ccc} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} & \xrightarrow{\sim} & \mathscr{G}\mathrm{Act}\big(\mathbf{sCh}_{\mathbb{K}}\big) \\ \mathcal{V}_{\mathbf{B}\mathscr{G}} & \mapsto & \big(\mathcal{V}, \rho_{\mathcal{V}}\big) \end{array} \qquad \begin{array}{c} \mathscr{G}\cdot\mathcal{V} \xrightarrow{\quad \rho_{\mathcal{V}} \quad} \mathcal{V} \\ \hline \mathcal{V}_{\mathbf{B}\mathscr{G}} : \mathscr{G} \xrightarrow{\qquad} \mathbf{sCh}_{\mathbb{K}}\big(\mathcal{V}, \mathcal{V}\big) \\ \hline \mathcal{V} \xrightarrow{\quad \tilde{\rho}_{\mathcal{V}} \quad} \mathcal{V}^{\mathscr{G}} \end{array} \tag{66}$$

Proposition 3.12 (Closed monoidal structure of simplicial local systems over simplicial delooping groupoids)**.** *Over a simplicial delooping groupoid, the monoidal structure from Def. 3.5 has*
(i) *tensor product given by*

$$
\begin{array}{rcl}
\mathcal{V}_{\mathbf{B}\mathscr{G}} \otimes \mathcal{W}_{\mathbf{B}\mathscr{G}} \;:\; \mathscr{G} & \longrightarrow & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}\big(\mathcal{V}\otimes\mathcal{W}\,,\,\mathcal{V}\otimes\mathcal{W}\big) \\
\big(\Delta[k] \xrightarrow{g} \mathscr{G}\big) & \longmapsto & \Big(\Delta[k]\cdot\mathcal{V}\otimes\mathcal{W} \xrightarrow{\mathrm{diag}\cdot\mathcal{V}\otimes\mathcal{W}} \big(\Delta[k]\cdot\mathcal{V}\big)\otimes\big(\Delta[k]\cdot\mathcal{W}\big) \xrightarrow{\tilde{\rho}_{\mathcal{V}}\otimes\tilde{\rho}_{\mathcal{W}}} (\mathcal{G}\cdot\mathcal{V})\otimes(\mathcal{G}\cdot\mathcal{W}) \xrightarrow{g\,\otimes g} \mathcal{V}\otimes\mathcal{W}\Big)
\end{array}
$$

(ii) *and internal hom given by*

$$
\begin{array}{rcl}
\big[\mathcal{V}_{\mathbf{B}\mathscr{G}},\,\mathcal{W}_{\mathbf{B}\mathscr{G}}\big] \;:\; \mathscr{G} & \longrightarrow & \mathbf{sCh}_{\mathbb{K}}\big([\mathcal{V},\,\mathcal{W}],[\mathcal{V},\,\mathcal{W}]\big) \\
\big(\Delta[k] \xrightarrow{g} \mathscr{G}\big) & \longmapsto & \Big([\mathcal{V},\,\mathcal{W}] \xrightarrow{[\rho_{\mathcal{V}},\,\tilde{\rho}_{\mathcal{W}}]} \big[\mathscr{G}\cdot\mathcal{V},\,\mathcal{V}^{\mathscr{G}}\big] \xrightarrow{[g^{-1}\cdot\mathcal{V},\,\mathcal{V}^{g}]} \big[\Delta[k]\cdot\mathcal{V},\,\mathcal{W}^{\Delta[k]}\big] \xrightarrow{[\mathcal{V},\,\mathcal{W}]^{\mathrm{diag}}} [\mathcal{V},\,\mathcal{W}]^{\Delta[k]}\Big).
\end{array}
$$

Proof. The first statement follows readily by unwinding the definitions. This makes the adjunction property of the second formula essentially manifest. ☐

Skeletal simplicial groupoids. Intermediate between general sSet-enriched groupoids in Prop. 3.6 and simplicial delooping groupoids in Rem. 3.11 are skeletal simplicial groupoids.

Definition 3.13 (Skeletal simplicial groupoids)**.** **(i)** A simplicial groupoid $\mathbf{X} \in$ sSet-Grpd is *skeletal* if its only non-empty hom-complexes are those from a given object to itself. In other words, if and only if it is isomorphic to a disjoint union of simplicial delooping groupoids (Def. 3.9):

$$
\mathbf{X} \in \text{sSet-Grpd} \qquad \vdash \qquad \mathbf{X} \text{ is skeletal} \quad \Leftrightarrow \quad \mathbf{X} \underset{\text{iso}}{\simeq} \coprod_{[x]\in\pi_0(\mathbf{X})} \mathbf{B}\big(\mathbf{X}(x,x)\big)\,. \tag{67}
$$

(ii) We denote the full subcategory of skeletal simplicial groupoids by

$$
\text{sSet-Grpd}_{\text{skl}} \hookrightarrow \text{sSet-Grpd}\,. \tag{68}
$$

Given any $\mathbf{X} \in$ sSet-Grpd as say that a *skeleton of* $\mathbf{X}$ is a full inclusion (hence a Dwyer-Kan equivalence) out of a skeletal groupoid (67)

$$
\mathbf{X}_{\text{skl}} \underset{\in \mathrm{W}_{\mathrm{DK}}}{\hookrightarrow} \mathbf{X}\,. \tag{69}
$$

Lemma 3.14 (Skeletal implies fibrant)**.** *Every skeletal simplicial groupoid (Def. 3.13) is fibrant.*

Proof. Unwinding the definitions, this is tantamount to saying that underlying any simplicial group is a Kan-fibrant simplicial set. This is the case by Moore's theorem [Mo54, Thm. 3, p. 18-04][Qu67, §II 3.8]. ☐

The following Lemmas 3.15, 3.17, which are standard arguments, use the Axiom of Choice in the underlying category of sets, which we assume throughout, as usual.

Lemma 3.15 (Skeletization of simplicial groupoids)**.** *Every* $\mathbf{X} \in$ sSet-Grpd *admits an adjoint equivalence and deformation retraction onto a skeleton* (69)*.*

Proof. If $\mathbf{X}$ is empty then the statement is trivial. So assume $\mathbf{X}$ is inhabited, whence it is the disjoint union of its inhabited connected components

$$
\mathbf{X} \underset{\text{iso}}{\simeq} \coprod_{i\in\pi_0(\mathbf{X})} \mathbf{X}_i\,.
$$

Choosing a base-point in each component

$$
i \in \pi_0(\mathbf{X}) \quad \vdash \quad x_i \in \mathrm{Obj}(\mathbf{X}_i)
$$

induces a sSet-enriched full inclusion

$$
\iota \;:\; \coprod_{i\in\pi_0(\mathbf{X})} \mathbf{B}\big(\mathbf{X}(x_i,x_i)\big) \hookrightarrow \mathbf{X}
$$

and choosing a 1-morphism from each object to this basepoint:

$$x \in \mathrm{Obj}(\mathbf{X}) \qquad \vdash \qquad \gamma_x : * \to \mathbf{X}(x_{[x]}, x)$$

(where by$[x] \in \pi_0(\mathbf{X})$ we denote the connected component of the given object x) induces a reverse enriched functor

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "\mathbf{p}"] & & \coprod\limits_{i \in \pi_0(\mathbf{X})} \mathbf{B}\big(\mathbf{X}(x_i, x_i)\big) \\ \mathbf{X}(x,y) \arrow[rr, "\mathbf{p}_{x,y}"] \arrow[d, "\wr"] & & \mathbf{X}\big(x_{[x]}, x_{[y]}\big) \\ * \times \mathbf{X}(x,y) \times * \arrow[rr, "(\gamma_y)^{-1} \times \mathrm{id} \times \gamma_x"] & & \mathbf{X}\big(y, x_{[y]}\big) \times \mathbf{X}(x,y) \times \mathbf{X}\big(x_{[x]}, x\big) \arrow[u, "\circ"] \end{tikzcd}$$

such that the γ_x serve as components of an sSet-enriched natural transformation $\gamma : \iota \circ \mathbf{p} \to \mathrm{id}_{\mathbf{X}}$.

Similarly, the inverse components define a converse transformation, but if we choose, as we may, $\gamma_{[x]} = \mathrm{id}_{[x]}$, then there is already an equality $\mathbf{p} \circ \iota = \mathrm{id}$. This means that we have a *deformation retraction* of $\mathbf{X}$ into its skeleton

$$\begin{tikzcd} \mathbf{X}_{\mathrm{sk}} \arrow[r, "\iota"] \arrow[rr, bend left, "\mathrm{id}"] & \mathbf{X} \arrow[r, "\mathbf{p}"] & \mathbf{X}_{\mathrm{skl}}, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X} \arrow[r, "\mathbf{p}"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbf{X}_{\mathrm{skl}} \arrow[r, "\iota"] \arrow[d, phantom, "\Downarrow\gamma"] & \mathbf{X} \\ & {} & \end{tikzcd} \tag{70}$$

manifestly satisfying

$$\mathbf{p}(\gamma_x) = \mathrm{id}_x \quad \text{and} \quad \gamma_{\mathbf{p}(x)} = \mathrm{id}_{\mathbf{p}(x)}$$

and thus exhibiting ι as the left adjoint in an enriched adjoint equivalence. □

Lemma 3.16 (Bifibrant resolution by skeletal simplicial groupoids)**.** *Every* $\mathbf{X} \in$ sSet-Grpd *(Prop. 3.20) admits a bifibrant replacement by a skeletal simplicial groupoid (Def. 3.13).*

Proof. By the existence of the model structure, all objects of sSet-Grpd admit some cofibrant resolution, and by Lem. 3.15 this in turn admits a deformation retraction along a weak equivalence onto a skeletal object. The latter is still cofibrant since cofibrations are closed under retractions and it is fibrant by Lem. 3.14. □

Lemma 3.17 (Skeletalization of simplicial local systems)**.**
Given $\mathbf{X} \in$ sSet-Grpd*, with skeleton* $\mathbf{X}_{\mathrm{skl}} \in$ sSet-Grpd$_{\mathrm{skl}}$*,* $\mathbf{X}_{\mathrm{skl}} \underset{\in \mathrm{W_{DK}}}{\overset{\iota}{\hookrightarrow}} \mathbf{X}$ (69)*:*
(i) *We have an adjoint equivalence of categories of local systems*

$$\begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_{\mathrm{skl}}} \arrow[r, shift left=2, "\iota_!"] \arrow[r, phantom, "\perp_{\simeq}"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}, \arrow[l, shift left=2, "\iota^*"] \end{tikzcd} \tag{71}$$

which is a Quillen equivalence with respect to the projective model structures (from Prop. 3.6).
(ii) *Moreover, given a morphism* $\mathbf{f} : \mathbf{X}' \to \mathbf{X}$ *then these equivalences may be chosen such as to make a commuting square of adjunctions commute*

$$\begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}'_{\mathrm{skl}}} \arrow[r, shift left=2, "\mathbf{f}_!"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "(\iota')_!"] \arrow[d, phantom, "\dashv\wr"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_{\mathrm{skl}}} \arrow[l, shift left=2, "\mathbf{f}^*"] \arrow[d, shift left=2, "\iota_!"] \arrow[d, phantom, "\dashv\wr"] \\ \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}'} \arrow[r, shift left=2, "\mathbf{f}_!"] \arrow[r, phantom, "\perp"] \arrow[u, shift left=2, "(\iota')^*"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[l, shift left=2, "\mathbf{f}^*"] \arrow[u, shift left=2, "\iota^*"] \end{tikzcd} \tag{72}$$

Proof. The adjoint equivalence form the proof of Lem. 3.15 induces the claimed adjoint equivalence on local systems with $\iota_! = \mathbf{p}^*$

$$\big(\mathbf{p}^* \iota^* \mathcal{V}\big)_x \;\simeq\; \mathcal{V}_{\iota \circ \mathbf{p}(x)} \xrightarrow{\;\mathcal{V}_\gamma\;} \mathcal{V}_x\,.$$

Moreover, with $\iota_! \dashv \iota^*$ being an adjoint equivalence so is $\iota^* \dashv \iota_!$ and since the projective model structure on $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_{\mathrm{skl}}}$ is evidently right-transferred along $i_! = p^*$ from that of $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}$ it follows (Prop. A.20) that $(\iota^* \dashv p^*)$ is a Quillen equivalence, whence also $(p^* \dashv \iota^*) \simeq (\iota_! \dashv \iota^*)$ is a Quillen equivalence:

$$\begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[r, shift right, "\iota^*"'] \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_{\mathrm{sk}}} \arrow[l, shift right, "p^*"'] \arrow[r, shift right, "p^*"'] \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[l, shift right, "\iota^*"'] \end{tikzcd}\,. \tag{73}$$

While this construction is far from natural, due to the choices of x_i involved, these choices can be made consistently with respect to a single map $\mathbf{f} : \mathbf{X}' \to \mathbf{X}$, by choosing $x'_{i'} \in \mathbf{f}^{-1}\big(\{x_i\}\big)$ for $i' \in [\mathbf{f}]^{-1}\big(\{i\}\big)$, this producing commuting diagrams of this form:

$$\begin{tikzcd} \mathbf{B}\big(\mathbf{X}'(x'_{i'}, x'_{i'})\big) \arrow[r, "\mathbf{B}\left(\mathbf{f}_{x'_{i'}, x'_{i'}}\right)"] \arrow[d, hook, "\iota'"] & \mathbf{B}\big(\mathbf{X}(x_i, x_i)\big) \arrow[d, hook, "\iota"] \\ \mathbf{X}' \arrow[r, "\mathbf{f}"] & \mathbf{X}\,. \end{tikzcd}$$

This is enough to obtain the diagram (72) commuting up to enriched natural isomorphism. But furthermore we may choose $\gamma'_{x'} \in \mathbf{f}^{-1}\big(\{\gamma_x\}\big)$ for $x' \in f^{-1}\big(\{x\}\big)$, which makes the diagram commute strictly. $\square$

Simplicial local systems over skeletal simplicial groupoids.

Remark 3.18 (Simplicial local systems on skeletal groupoids)**.** Over a skeletal simplicial groupoid (Def. 3.13), the model category of simplicial local systems (Prop. 3.6) is the product model structure on the product of categories of simplicial local systems on the connected components:

$$\mathbf{X} \simeq \coprod_{s \in S} \mathbf{B}\mathscr{G}_s \qquad \vdash \qquad \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \simeq \prod_{s \in S} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}_s} \;\in\; \mathrm{ModCat}.$$

This is immediate from the fact that $\mathbf{Func}\big(\coprod_s \mathbf{D}_s, \mathbf{C}\big) \simeq \prod_s \mathbf{Func}(\mathbf{D}_s, \mathbf{C})$ and since the weak equivalences and fibrations in the projective model structure on functors are defined objectwise.

Proposition 3.19 (Monoidal model structure on simplicial local systems)**.** *For* $\mathbf{X} \in \mathrm{sSet\text{-}Grpd}$*, the simplicial model structure* $\boxed{\big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}, \otimes\big)}$ *from Prop. 3.6 is monoidal model (with respect to the monoidal structure from Prop. 3.5).*

Proof. By the Quillen equivalences (73) it is sufficient to show this for skeletal $\mathbf{X}$ (Def. 3.13). Moreover, by Rem. 3.18 and since the tensor product is defined objectwise, the simplicial local systems over skeletal $\mathbf{X}$ form a product model category equipped factorwise with the closed monoidal structure from Prop. 3.12:

$$\big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}, \otimes_{\mathbf{X}}\big) \;\simeq\; \prod_{i \in I} \big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}_i}, \otimes_{\mathbf{B}\mathscr{G}_i}\big)\,.$$

Therefore, it is in fact sufficient to check that a model category of simplicial local systems over a simplicial delooping groupoid $\big(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}_i}, \otimes_{\mathbf{B}\mathscr{G}_i}\big)$ is monoidal as a model category.

This follows by [BM06, p. 6]. For the record, we spell out the argument. The point is that over a delooping groupoid $\mathbf{B}\mathscr{G}$ the generating (acyclic) cofibrations (60) are given by tensoring a generating (acyclic) cofibration of $\mathbf{sCh}_{\mathbb{K}}$ with $\mathscr{G}$ equipped with its own multiplication action:

$$\mathrm{sI}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} = \big\{\mathscr{G} \cdot i \,\big|\, i \in \mathrm{sI}_{\mathbb{K}}\big\}, \quad \mathrm{sJ}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} = \big\{\mathscr{G} \cdot j \,\big|\, j \in \mathrm{sJ}_{\mathbb{K}}\big\}\,. \tag{74}$$

This happens to coincide with the free construction (forming simplicial "regular representations") which is left adjoint to the functor undrl that forgets the $\mathscr{G}$-action (66):

$$\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} \simeq \mathscr{G}\mathrm{Act}\big(\mathbf{sCh}_{\mathbb{K}}\big) \underset{\text{undrl}}{\overset{\mathscr{G}\cdot\mathcal{V} \leftarrow\!\shortmid \mathcal{V}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sCh}_{\mathbb{K}} \tag{75}$$

Another conclusion from (74) is that the underlying functor is also *left Quillen*

$$\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}} \simeq \mathscr{G}\mathrm{Act}\big(\mathbf{sCh}_{\mathbb{K}}\big) \underset{(*\to\mathbf{B}\mathscr{G})_*}{\overset{\text{undrl}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot_{\mathrm{Qu}}}}}} \mathbf{sCh}_{\mathbb{K}} \tag{76}$$

since $\mathscr{G}\cdot i$ (resp. $\mathscr{G}\cdot j$) are still (acyclic) cofibrations in $\mathbf{sCh}_{\mathbb{K}}$ due to its sSet-enriched model structure (Prop. 3.3).

With these preliminaries in hand, we check the pushout-product axiom (158): Consider a pair of generating cofibrations $\mathcal{V} \to \mathcal{V}'$ and $\mathcal{W} \to \mathcal{W}'$ in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}$; we need to show that their pushout-product morphism on the far left of the following diagrams is a cofibration, which equivalently means that for any acyclic fibration $\mathcal{R} \to \mathcal{R}'$ and commuting diagrams as on the left, there exists a lift as shown by the dashed arrow on the left, and by a standard argument (e.g. [Lu09, Rem. A.3.1.6]) this exists if and only if a lift in the corresponding diagram on the right exists:

$$\begin{array}{ccc} (\mathcal{V}\otimes\mathcal{W}') \underset{}{\overset{\mathcal{V}\otimes\mathcal{W}}{\coprod}} (\mathcal{V}'\otimes\mathcal{W}) & \longrightarrow & \mathcal{R} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{V}'\otimes\mathcal{W}' & \longrightarrow & \mathcal{R}' \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} \mathcal{V} & \longrightarrow & [\mathcal{W}',\mathcal{R}] \\ \downarrow & \nearrow & \downarrow \\ \mathcal{V}' & \longrightarrow & [\mathcal{W}',\mathcal{R}'] \underset{[\mathcal{W},\mathcal{R}']}{\prod} [\mathcal{W},\mathcal{R}] \end{array}$$

But by the previous observation, the left morphism in the diagram on the right is in the image of the left adjoint functor (75), which finally means that the dashed lift on the right exists in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}$ as soon as such exists for the underlying diagram in $\mathbf{sCh}_{\mathbb{K}}$. But by definition of the projective model structure in Prop. 3.6 the underlying map of $\mathcal{R} \to \mathcal{R}'$ is still a fibration, and using that $\mathbf{sCh}_{\mathbb{K}}$ is sSet-enriched model (Prop. 3.3) it follows also that underlying the generating cofibrations (74) are cofibrations in $\mathbf{sCh}_{\mathbb{K}}$, and then that lifting in the diagram on the right above is that of a cofibration against an acyclic fibration in $\mathbf{sCh}_{\mathbb{K}}$ and hence exists. Verbatim the argument with the evident substitutions shows that the same kind of lifts exist if $\mathcal{W} \to \mathcal{W}'$ is actually an acyclic cofibration and $\mathcal{R} \to \mathcal{R}'$ is any fibration. In summary this establishes the pushout-product axiom in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}$.

It remains to check the unit axiom, namely that for $Q\mathbb{K} \xrightarrow{p} \mathbb{K}$ any cofibrant replacement of the trivial $\mathscr{G}$-representation, and for $\mathcal{V}$ any cofibrant $\mathscr{G}$ representation, the tensor product morphism $(Q\mathbb{K})\otimes\mathcal{V} \xrightarrow{p\otimes\mathcal{V}} \mathbb{K}\otimes\mathcal{V}$ is still a weak equivalence. Now since weak equivalences are those on underlying maps, and since all objects in $\mathbf{sCh}_{\mathbb{K}}$ are cofibrant (Prop. 3.3), it is sufficient, via Ken Brown's lemma A.19, to see that $\mathcal{V}\otimes(-)$ is left Quillen on $\mathbf{sCh}_{\mathbb{K}}$. By the pushout-product axiom satisfied in $\mathbf{sCh}_{\mathbb{K}}$ this follows as soon as the underlying object of the cofibrant representation $\mathcal{V}$ is still cofibrant, which is the case by (76). □

3.2 Parameterization over varying base spaces

We now glue all the model categories of simplicial local systems over fixed base spaces to an integral model structure on simplicial local systems over varying base spaces. First, to recall:

Proposition 3.20 (Dwyer-Kan model structures)**.**
(i) *We have the following classical model categories:*

(a) *The category of simplicial sets*
$\boxed{\text{sSet}}$ *carries the Kan-Quillen model structure whose*

- *weak equivalences are the simplicial weak homotopy equivalences,*
- *fibrations are the Kan fibrations,*
- *cofibrations are the monomorphisms*

(b) *The category of (small)* sSet*-enriched categories (often known as "simplicial categories")*
$\boxed{\text{sSet-Cat}}$ *carries a model structure whose*

– *weak equivalences are the Dwyer-Kan equivalences, namely the* sSet*-functors which on isomorphism classes of objects in the homotopy category are surjective and on all hom-complexes are simplicial weak homotopy equivalences of underlying simplicial sets,*
– *fibrations are the* sSet*-functors which are isofibrations on homotopy categories and on all hom-complexes are Kan fibrations of underlying simplicial sets.*

(c) *The category of small* sSet*-enriched groupoids (often known as "simplicial groupoids")* $\boxed{\mathrm{sSet\text{-}Grpd}}$ *carries a model structure whose*
– *weak equivalences are the Dwyer-Kan equivalences as above,*
– *fibrations are the maps that admit lifting of 1-morphisms and are Kan fibrations on underlying simplicial sets of all automorphism groups,*
– *cofibrations are in particular injective on objects and degreewise on all hom-complexs.*

(d) *The category of simplicial groups* $\boxed{\mathrm{sGrp} := \mathrm{Grp}(\mathrm{sSet})}$ *carries a model structure whose*
– *weak equivalences are the simplicial weak homotopy equivalences of underlying simplicial sets,*
– *fibrations are the Kan fibrations of underlying simplicial sets,*
– *cofibrations are the retracts of "almost free" (cf. [GJ99, p. 270]) simplicial group inclusions, in particular all cofibrations are monomorphisms (simplicial subgroup inclusions).*

(ii) *We have the following functors relating these:*

- *The canonical full inclusions are compatible with this model structure*

$$\begin{tikzcd} \mathrm{sGrp} \arrow[r, hook] & \mathrm{sSet\text{-}Grpd} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, hook, shift right=1.5ex, "\iota"'] & \mathrm{sSet\text{-}Cat} \arrow[l, shift right=1.5ex, "\mathrm{Loc}"'] \\ \mathscr{G} \arrow[r, mapsto] & \mathbf{B}\mathscr{G} & \end{tikzcd} \tag{77}$$

in that
– $\mathrm{Loc} \dashv \iota$ *is a Quillen adjunction*
(here Loc *is degreewise the free groupoid construction on or equivalently the full localization of a category);*
– $\mathbf{B}(-)$ *(Def. 3.9) preserves weak equivalences and fibrations (but has no left adjoint).*

- *There is a Quillen equivalence*

$$\begin{tikzcd} \mathrm{sSet\text{-}Grpd} \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] & \mathrm{sSet} \arrow[l, shift right=1.5ex, "\mathbf{G}"'] \arrow[l, leftarrow, shift left=1.5ex, "\overline{\mathrm{W}}"'] \end{tikzcd} \tag{78}$$

and a Quillen adjunction

$$\begin{tikzcd} \mathrm{sSet\text{-}Grpd} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=1.5ex, "\widetilde{\mathbf{G}} :\equiv \mathrm{Loc}\circ\mathfrak{C}"] & \mathrm{sSet} \arrow[l, leftarrow, shift left=1.5ex, "\widehat{\mathrm{W}} :\equiv N\circ\iota"] \end{tikzcd} \;:\equiv\; \begin{tikzcd} \mathrm{sSet\text{-}Grpd} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=1.5ex, "\mathrm{Loc}"] & \mathrm{sSet\text{-}Cat} \arrow[l, leftarrow, shift left=1.5ex, "\iota"] \arrow[r, phantom, "\perp"] \arrow[r, leftarrow, shift left=1.5ex, "\mathfrak{C}"] & \mathrm{sSet} \arrow[l, leftarrow, shift left=1.5ex, "N"] \end{tikzcd}, \tag{79}$$

such that
– *there exists a natural transformation*

$$\mathscr{X} \in \mathrm{sSet} \qquad \vdash \qquad \mathrm{Loc}\circ\mathfrak{C}(\mathscr{X}) \xrightarrow[\in \mathrm{W}_{\mathrm{DK}}]{} \mathbf{G}(\mathscr{X}) \tag{80}$$

which is a Dwyer-Kan equivalence,
– *the natural transformation*

$$\mathscr{X}, \mathscr{Y} \in \mathrm{sSet} \qquad \vdash \qquad \mathfrak{C}(\mathscr{X}\times\mathscr{Y}) \xrightarrow[\in \mathrm{W}_{\mathrm{DK}}]{(\mathfrak{C}(\mathrm{pr}_{\mathscr{X}}), \mathfrak{C}(\mathrm{pr}_{\mathscr{Y}}))} \mathfrak{C}(\mathscr{X})\times\mathfrak{C}(\mathscr{Y}) \tag{81}$$

is a Dwyer-Kan equivalence.

Proof. **(i)** The Kan-Quillen model structure on sSet is, of course, due to [Qu67, §II.3], see also for instance [GJ99, §I.11]. The model structure on sGrp is due to [Qu67, §II 3.7], see [GJ99, §V].

The model structure on sSet-Grpd is due to [DK84, §2.5]. Their [DK84, §2.4 with §2.3 (i)] asserts that the cofibrations are in particular retracts of degreewise injections of sets (of objects and of morphisms). But since injections of sets are closed under retracts this means that all cofibrations are in particular degreewise injections.

The model structure on sSet-Cat is due to [Be07a], see also [Lu09, Thm. A.3.2.4].

(ii) From this, it is immediate that the functors in (77) preserve the structure as stated; the Quillen adjunction on the right of (77) is also made explicit in [MRZ23, Prop. 2.8].

The Quillen equivalence $\mathscr{G} \dashv \overline{\mathrm{W}}$ (78) is due to [DK84, Thm. 3.3] reviewed in [GJ99, Thm. 7.8].

The adjunction $\mathfrak{C} \dashv N$ on the right of (79) is actually a Quillen equivalence with respect to the Joyal model structure on simplicial sets [Be07b, Thm. 7.8][Lu09, Thm. 2.2.5.1]:

$$\mathrm{sSet\text{-}Cat} \underset{N}{\overset{\mathfrak{C}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot_{\mathrm{Qu}}}}}} \mathrm{sSet}_{\mathrm{Joy}}\,. \tag{82}$$

But since the Joyal model structure has the same cofibrations as the Kan-Quillen model structure this implies with (77) that $\widetilde{\mathbf{G}} \equiv \mathrm{Loc} \circ \mathfrak{C}$ preserves cofibrations. To see that it also preserves weak equivalences, and hence is a left Quillen functor as claimed on the left of (79), notice that (80) – which is due to [MRZ23, Thm. 1.1] – implies commuting squares

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow[\in \mathrm{W}]{f} & \mathscr{X}' \\ \\ \widetilde{\mathbf{G}}(\mathscr{X}) & \xrightarrow{\widetilde{\mathbf{G}}(f)} & \widetilde{\mathbf{G}}(\mathscr{X}) \\ \downarrow & & \downarrow \\ \mathbf{G}(\mathscr{X}) & \xrightarrow{\mathbf{G}(f)} & \mathbf{G}(\mathscr{Y})\,, \end{array}$$

where the vertical maps are Dwyer-Kan equivalences. But also the bottom map is a Dwyer-Kan equivalence by Ken Brown's Lemma A.19, since $\mathbf{G}$ is a left Quillen functor (78) on a model category all whose objects are cofibrant, whence also $\widetilde{\mathbf{G}}(f)$ is a weak equivalence, by the 2-out-of-3 property satisfied by weak equivalences.

Finally, the property (81) is due to [Lu09, Cor. 2.2.5.6], see also [DS11, Prop. 6.2]. □

Remark 3.21 (Dwyer-Kan simplicial fundamental groupoids)**.** The classical Dwyer-Kan functor $\mathbf{G} : \mathrm{sSet} \longrightarrow \mathrm{sSet\text{-}Grpd}$ (78) may be thought of as forming simplicial *fundamental groupoids* of spaces and hence so may be its Dwyer-Kan-equivalent version $\widetilde{\mathbf{G}}$ (79).

The integral model structure on simplicial local systems. Our interest is now in the pseudofunctor assigning model categories (Def. A.21) of simplicial local systems (Prop. 3.6) to simplicial fundamental groupoids $\widetilde{\mathbf{G}}(-)$ (Rem. 3.21) of simplicial sets, which exists as a bivariant pseudofunctor by Ex. A.18 and with values in ModCat (Def. A.21) by Rem. 3.7:

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, "\mathbf{G}"] & \mathrm{sSet\text{-}Grpd} \arrow[r, "\mathbf{sCh}_{\mathbb{K}}^{(-)}"] & \mathrm{ModCat} \\ \mathscr{X} \arrow[d, "f"] \arrow[r, phantom, "\longmapsto"] & \mathbf{X} \arrow[d, "\mathbf{f}"] \arrow[r, phantom, "\longmapsto"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[d, shift right=2, "\mathbf{f}_!"'] \arrow[d, phantom, "\dashv"] \\ \mathscr{X}' \arrow[r, phantom, "\longmapsto"] & \mathbf{X}' \arrow[r, phantom, "\longmapsto"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}'} \arrow[u, shift right=2, "\mathbf{f}^*"'] \end{tikzcd} \tag{83}$$

Theorem 3.22 (Integral model structure on simplicial local systems over varying bases)**.**
The integral model structures (Def. A.22) on the Grothendieck constructions (Def. A.4) on the pseudofunctors (83) *exist and are Quillen equivalent, to be denoted as follows:*

$$\boxed{\mathbf{Loc}_{\mathbb{K}}^{\mathrm{sSet}} := \int_{\mathscr{X} \in \mathrm{sSet}} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{G}(\mathscr{X})} \underset{\widehat{\mathrm{W}}}{\overset{\widehat{\mathbf{G}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq_{\mathrm{Qu}}}}}} \int_{\mathbf{X} \in \mathrm{sSet\text{-}Grpd}} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} =: \mathbf{Loc}_{\mathbb{K}}} \tag{84}$$

Proof. First regarding the existence of the model structures, given maps $f : X \to X'$ in sSet (resp. $\mathbf{f} : \mathbf{X} \to \mathbf{Y} \in \mathbf{X}'$ in sSet-Grpd) we need to show (by Prop. A.23) the following three properties:

1. **If f (resp. $\mathbf{f}$) is a weak equivalence then $\mathbf{G}(f)_! \dashv \mathbf{G}(f)^*$ (resp. $\mathbf{f}_! \dashv \mathbf{f}^*$) is a Quillen equivalence.**
   Since $\mathbf{G}$ preserves weak equivalences (by Ken Brown's lemma A.19, being a left Quillen functor (78) on a category with all objects cofibrant) it is sufficient to see that $\mathbf{sCh}_{\mathbb{K}}^{(-)}$ has this property.
   This is a special case of the following general statement, which may be of interest in its own right.

   Lemma 3.23. *Let* $\mathbf{C}$ *be a combinatorial simplicial model category and* $\mathbf{f} : \mathbf{X}' \longrightarrow \mathbf{X}$ *a Dwyer-Kan equivalence (Prop. 3.20) of small* KanCplx*-enriched categories. Then* $\mathbf{f}_! : \mathbf{C}^{\mathbf{X}'} \rightleftarrows \mathbf{C}^{\mathbf{X}} : \mathbf{f}^*$ *is a Quillen equivalence between the projective model structures on the enriched functor categories.*

   Proof of Lemma. We demonstrate this claim by appeal to the ∞-categories (quasi-categories) [Jo08][Lu09] presented by the model structure, via the homotopy coherent nerve functor N : sSet-Cat $\longrightarrow$ sSet (82) applied to full simplicial subcategories $(\text{-})^\circ$ of bifibrant objects in simplicial model categories: By [Lu09, Prop. 4.2.44] there are natural transformations as on the top of the following diagram, which restrict on bifibrant objects to weak equivalences of quasi-categories, as shown at the bottom:

$$
\begin{tikzcd}
N\big(\mathbf{C}^{\mathbf{X}}\big) \arrow[r] & N(\mathbf{C})^{N(\mathbf{X})} \\
N\big(\big(\mathbf{C}^{\mathbf{X}}\big)^\circ\big) \arrow[u, hook] \arrow[r, "\in \mathrm{W}_{\mathrm{Joy}}"'] & N(\mathbf{C}^\circ)^{N(\mathbf{X})} \arrow[u, hook]
\end{tikzcd}
\tag{85}
$$

   Consider then the following diagram of (large) simplicial sets:

$$
\begin{tikzcd}
N\big(\big(\mathbf{C}^{\mathbf{X}}\big)^\circ\big) \arrow[rrrr, "\in \mathrm{W}_{\mathrm{Joy}}"'] \arrow[dr, hook] \arrow[dddd, "N(\mathbb{R}(\mathbf{f}^*))"'] & & & & N(\mathbf{C}^\circ)^{N(\mathbf{X})} \arrow[dl, hook'] \arrow[dddd, "N(\mathbf{f})^*"] \\
& N\big(\mathbf{C}^{\mathbf{X}}\big) \arrow[rr] \arrow[d, "N(\mathbf{f}^*)"] & & N(\mathbf{C})^{N(\mathbf{X})} \arrow[d, "N(\mathbf{f})^*"'] & \\
& N\big(\mathbf{C}^{\mathbf{X}'}\big) \arrow[rr] \arrow[d, "N(Q)"'] \arrow[dr, "\mathrm{id}"] & & N(\mathbf{C})^{N(\mathbf{X}')} & \\
& N\big(\mathbf{C}^{\mathbf{X}'}\big) \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\Nearrow"] & N\big(\mathbf{C}^{\mathbf{X}'}\big) \arrow[ur] & & \\
N\Big(\big(\mathbf{C}^{\mathbf{X}'}\big)^\circ\Big) \arrow[ur, hook] \arrow[rrrr, "\in \mathrm{W}_{\mathrm{Joy}}"'] & & & & N(\mathbf{C}^\circ)^{N(\mathbf{X}')} \arrow[uul, hook']
\end{tikzcd}
\tag{86}
$$

   where Q denotes a *functorial* cofibrant replacement functor (which exists by [Du01, Prop. 2.3][Ba10, Prop. 2.5] since $\mathbf{C}^{\mathbf{X}'}$ is combinatorial) and the double arrow denotes (the image under the right adjoint functor N of) the corresponding natural transformation whose components are the resolution equivalences $Q(\text{-}) \xrightarrow[\in \mathrm{W}]{} (\text{-})$.
   In this diagram (86): the top and bottom squares are instances of (85), the middle square commutes by naturality, the right square commutes evidently and the left square commutes by the usual construction of derived functors of right Quillen functors. With this, we have a natural transformation of ∞-functors filling the full diagram and we observe that this is a natural equivalence: This follows by [Jo08, §5, Thm C (p. 125)] from the fact that its objectwise components are (resolution-)equivalences, by construction of $N(Q)$.
   In conclusion, this shows that the right derived functor $\mathbb{R}\mathbf{f}^*$ represents the precomposition ∞-functor $N(\mathbf{f})^*$ up to natural equivalence; in particular, both coincide on homotopy categories up to natural isomorphism.
   But for $\mathbf{f}$ a DK-equivalence between KanCplx-enriched categories the ∞-functor $N(\mathbf{f})$ is an equivalence of ∞-categories by [Lu09, Thm. 2.2.5.1] (with Ken Brown's lemma A.19) and therefore $N(\mathbf{f})^*$ is an equivalence by [Lu09, Prop. 1.2.7.3 (3)], hence in particular is an equivalence of homotopy categories, whence so is $\mathbb{R}\mathbf{f}^*$, which finally means that $\mathbf{f}_! \dashv \mathbf{f}^*$ is a Quillen equivalence. ☐

2. **If f (resp. $\mathbf{f}$) is an acyclic fibration then $\mathbf{G}(f)^*$ (resp. $\mathbf{f}^*$) preserves weak equivalences.**
   This is immediate and in fact holds for all maps $\mathbf{f}$ since $\mathbf{f}^*$ acts by precomposition and weak equivalences are given objectwise.
3. **If f (resp. $\mathbf{f}$) is an acyclic cofibration then $\mathbf{G}(f)_!$ (resp. $\mathbf{f}_!$) preserves weak equivalences.**
   Since $\mathbf{G}$, being a left Quillen functor (79), preserves acyclic cofibrations, it is sufficient to show the claim for any acyclic cofibration $\mathbf{f}$.
   Moreover, by the 2-out-of-3 property applied to the diagram (72) in Lem. 3.17 it is sufficient to show the claim for the (co)restriction $\mathbf{f}_{\mathrm{skl}}$ of $\mathbf{f}$ to compatibly chosen skeletons of the domain and codomain:

$$\begin{tikzcd} \mathbf{X}_{\mathrm{skl}} \arrow[r, "\mathbf{f}_{\mathrm{skl}}"] \arrow[d, hook, "\iota"] & \mathbf{X}'_{\mathrm{skl}} \arrow[d, hook, "\iota'"] \\ \mathbf{X} \arrow[r, "\mathbf{f}"] & \mathbf{X}' \end{tikzcd}$$

   Since the cofibrations in sSet-Grpd are, in particular, injections in all degrees (by Prop. 3.20), we have that also $\widetilde{\mathbf{G}}(f)$ is an injection both on objects as well as on hom-complexes. By Rem. 3.18 this means that it is in fact sufficient to check the claim for $\widetilde{\mathbf{G}}(f)$ replaced by the delooping of a simplicial subgroup inclusion $\mathscr{H} \hookrightarrow \mathscr{G}$

$$\mathbf{f} : \mathbf{B}\mathscr{H} \to \mathbf{B}\mathscr{G}\,.$$

   This is the case we check now, using the identification of simplicial local systems over simplicial delooping groupoids with simplicial group representations (Rem. 3.11).
   On general grounds, in this case $\mathbf{f}_!$ acts by forming left-induced representations, namely by

$$\mathbf{f}_! : \mathscr{V} \mapsto \mathscr{G} \cdot_{\mathscr{H}} \mathscr{V} := \big(\mathscr{G} \cdot \mathscr{V}\big)/\mathscr{H} \;:\; [n] \mapsto \big(\mathscr{G}_n \cdot \mathscr{V}_n\big)/\mathscr{H}_n\,,$$

   where on the right the tensoring $\mathscr{G} \cdot \mathscr{V}$ is equipped with the diagonal $\mathcal{H}$-action which on $\mathscr{G}$ is given by right inverse multiplication; and we have notationally highlighted that quotients of simplicial objects are computed degreewise.
   Now we use that $\mathbf{f} : \mathbf{B}\mathscr{H} \longrightarrow \mathbf{B}\mathscr{G}$ is simplicial-degreewise a subgroup inclusion

$$n : \mathbb{N} \qquad \vdash \qquad \phi_n : \mathscr{H}_n \hookrightarrow \mathscr{G}_n\,.$$

   Since the weak equivalences in the local model structure $\mathrm{sCh}_{\mathbb{K}}$ (from Thm. 3.3) *include* (either by definition of left Bousfield localization or else by [Hi02, Prop. 3.1.5]) the global Reedy equivalences which are the simplicial-degreewise weak equivalences between objects $\mathscr{V}_n \in \mathrm{Ch}_{\mathbb{K}}$, it is sufficient now to observe that for each $n \in \mathbb{N}$ there is an isomorphism (*not* necessarily natural in n, but it does not need to be) of the form

$$\big(\mathscr{G}_n \cdot \mathscr{V}_n\big)/\mathcal{H}_n \simeq \big(\mathscr{G}_n/\mathscr{H}_n\big) \cdot \mathscr{V}_n \quad \in \mathrm{Ch}_{\mathbb{K}}\,. \tag{87}$$

   This concludes the argument, because the tensoring with any set — as on the right of (87) — is a left Quillen functor on $\mathrm{Ch}_{\mathbb{K}}$, and since all objects in $\mathrm{Ch}_{\mathbb{K}}$ are cofibrant (both by Thm. 3.3) this Quillen functor preserves all weak equivalences, by Ken Brown's lemma A.19 (or more concretely: because a direct sum of quasi-isomorphisms is itself a quasi-isomorphism).

   For completeness, we comment further on the isomorphism (87). Under the identifications (46), this is the familiar statement from representation theory that the tensor product of any group representation V with the regular G-representation is isomorphic to the $\dim(V)$-fold direct sum of the regular representation with itself; but for the record we make the isomorphism explicit by an elementary argument:
   Using the Axiom of Choice in our underlying category Set, we may choose a section as follows (and the arbitrariness in this choice makes the construction be non-natural):

$$\begin{tikzcd} & \mathscr{G}_n \arrow[d] \\ \mathscr{G}_n/\mathscr{H}_n \arrow[ur, dashed, "\sigma_n"] \arrow[r, equal] & \mathscr{G}_n/\mathscr{H}_n \end{tikzcd}$$

which determines a function

$$
\begin{array}{rccc}
\sigma[\text{-}]\backslash(\text{-}) : \mathscr{G}_n & \longrightarrow & \mathscr{H}_n \\
g & \longmapsto & \sigma[g]^{-1} g \,.
\end{array}
$$

Using this, the desired isomorphism and its inverse are given by the diagonal morphisms in the following diagram (the top of which shows the coequalizer defining the global quotient, just for context):

$$
\begin{tikzcd}
\mathscr{H}_n \cdot \big(\mathscr{G}_n \cdot \mathcal{V}_n\big)
\arrow[r, shift left, "{(h,(g,v)) \mapsto (g,\, v)}"]
\arrow[r, shift right, "{(h,(g,v)) \mapsto (g \cdot h^{-1}, h \cdot v)}"']
&
\mathscr{G}_n \cdot \mathcal{V}_n
\arrow[rr, "{(g,v) \mapsto [g,v]}", "\mathrm{coeq}"']
\arrow[d, "{(g,v) \mapsto ([g], \sigma[g]\backslash g \cdot v)}"']
&&
\big(\mathscr{G}_n \cdot \mathcal{V}_n\big)/\mathscr{H}_n \,.
\arrow[dll, dashed, bend left, "{[g,v] \mapsto ([g], \sigma[g]\backslash g \cdot v)}"]
\\
&
\big(\mathscr{G}_n/\mathscr{H}_n\big) \cdot \mathcal{V}_n
\arrow[urr, "{([g],v) \mapsto [\sigma[g],v]}"']
\end{tikzcd}
$$

This shows that both integral model structures exist. It remains to see the see the Quillen equivalence in (84):

The underlying pair of adjoint functors $\widehat{\mathbf{G}} \dashv \widehat{\mathrm{W}}$ is given by Ex. A.8. Furthermore, $\widehat{\mathrm{W}}$ is clearly a right Quillen functor because (recall Def. A.22) the underlying functor is right Quillen by Prop. 3.20 while the action on component morphisms by pullback is right Quillen by Rem. 3.7. Finally to see that this Quillen adjunction is a Quillen equivalence it is sufficient for cofibrant $\mathcal{V}_{\mathscr{X}} \in \mathbf{Loc}_{\mathbb{K}}^{\mathrm{sSet}}$ and fibrant $\mathcal{V}'_{\mathbf{X}'} \in \mathbf{Loc}_{\mathbb{K}}$ to check that

$$
\mathcal{V}_{\mathscr{X}} \xrightarrow{\phi_f} \widehat{R}\big(\mathcal{V}'_{\mathbf{X}'}\big) \quad \text{is a weak equivalence iff its adjunct} \quad \widehat{\mathbf{G}}\big(\mathcal{V}_{\mathscr{X}}\big) \xrightarrow{\phi_{\tilde{f}}} \mathcal{V}'_{\mathbf{X}'} \quad \text{is a weak equivalence}
$$

But on underlying morphisms this is the case because $\mathbf{G} \dashv \overline{\mathrm{W}}$ is a Quillen equivalence (78), while on component morphisms both $\widehat{G}(-)$ (141) as well as $\varepsilon \circ (-)$ (141) are the identity operation, whence so is, on components, the passage $\varepsilon \circ \widehat{\mathbf{G}}(-)$ to adjuncts. □

Definition 3.24 (Notation for simplicial local systems)**.** We denote the objects and morphisms in the category $\mathbf{Loc}_{\mathbb{K}}$ (84) of simplicial local systems as follows:

- objects are denoted
$$
\mathcal{V}_{\mathbf{X}} \;\; :\equiv \;\; \Big(\mathbf{X} \in \text{sSet-Grpd}, \; \mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\Big) \tag{88}
$$
which is suggestively read as "$\mathcal{V}$ is a simplicial local system over $\mathbf{X}$";
- morphisms are denoted by their components in the underlying *contravariant* pseudofunctor
$$
\big(\phi_{\mathbf{f}} : \mathcal{V}_{\mathbf{X}} \longrightarrow \mathcal{V}'_{\mathbf{X}'}\big) \quad :\equiv \quad \left(\begin{array}{l} \mathcal{V} \xrightarrow{\;\;\phi\;\;} \mathbf{f}^* \mathcal{V}' \\ \mathbf{X} \xrightarrow{\;\;\mathbf{f}\;\;} \mathbf{X}' \end{array}\right), \tag{89}
$$
as opposed to the adjunct form of the component ϕ, which we indicate by a $\widetilde{\phi}$:
$$
\frac{\mathcal{V} \;-\!\phi\!\rightarrow\; \mathbf{f}^* \mathcal{V}'}{\mathbf{f}_! \mathcal{V} \;-\!\widetilde{\phi}\!\rightarrow\; \mathcal{W}} \tag{90}
$$

Remark 3.25 (Choice of component morphisms)**.** The choice (89) is motivated by the fact that $\mathbf{f}^*$ but not $\mathbf{f}_!$ is a (strong) monoidal functor (Prop. 3.27 below), which means that in the notational convention (89) the external tensor product (Def. 3.29) below is *manifestly* given by the obvious formula. On the other hand, some computation shows (Prop. 3.32 below) that with the other convention, the analogous obvious formula will still hold (even if far from manifestly so) so that in the end the choice in (89) is as arbitrary as one would hope it is.

Example 3.26 (Decomposing group representations)**.** In $\mathbf{Loc}_{\mathbb{K}}$ (84), every simplicial group representation (Ref. 3.11) decomposes along a homotopy cartesian squares of this form

$$\left.\begin{array}{l} \mathscr{G} \in \mathrm{sGrp} \\ \mathcal{V} \in \left(\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}\right)^{\mathrm{fib}} \end{array}\right\} \quad \vdash \quad \begin{tikzcd} \mathrm{pt}^*\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_{\mathbf{B}\mathscr{G}} \arrow[d, "0_{\mathrm{id}}"', "\in\mathrm{Fib}"] \\ 0_{\mathrm{pt}} \arrow[r, "0_{\mathrm{pt}}"'] & 0_{\mathbf{B}\mathscr{G}} \end{tikzcd} \quad \equiv: \quad \begin{tikzcd} \mathcal{V} \arrow[r] & \mathcal{V}/\!\!/\mathscr{G} \arrow[d] \\ & \mathbf{B}\mathscr{G} \end{tikzcd} \tag{91}$$

exhibiting $\mathcal{V}_{\mathbf{B}\mathscr{G}}$ as a homotopy quotient of $\mathcal{V}_{\mathrm{pt}}$ by a $\mathscr{G}$-action.

Proof. First, that the square is Cartesian follows by Prop. A.9:

$$\begin{tikzcd} \mathrm{pt} \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}\mathscr{G} \arrow[d, "\mathrm{id}"] \\ \mathrm{pt} \arrow[r] & \mathbf{B}\mathscr{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{pt}^*\mathcal{V} \arrow[r, "\mathrm{id}"] \arrow[d, "0"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{pt}^*\mathcal{V} \arrow[d, "\mathrm{pt}^*0"] \\ 0 \arrow[r, "0"] & \mathrm{pt}^*0 \end{tikzcd}$$

Finally, the right vertical map is a fibration since the identity on $\mathbf{B}\mathscr{G}$ is a fibration in sSet-Grpd and since $\mathcal{V} \to 0$ is a fibration in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{B}\mathscr{G}}$ iff it is in $\mathbf{sCh}_{\mathbb{K}}$, which is the case by assumption. □

3.3 The external tensor of flat ∞-vector bundles

We discuss here the construction (Def. 3.29) and its homotopical properties (Thm. 3.40) of the external tensor product on $\mathbf{Loc}_{\mathbb{K}}$ (84), covering the Cartesian product on sSet-Grpd (Prop. A.2).

Motivic yoga on simplicial local systems. The abstract form of the following structures and conditions were essentially all first formulated and named ("Frobenius reciprocity", "Beck-Chevalley condition") in discussion of (categorical semantics for) formal logic/type theory [La70][Se83][Pa91, §1][Pa96], even though the same structures govern what came to be known as *Grothendieck's yoga of six operations* and as such must have originated at around the same time but have been systematically recorded only much later (notably [FHM03], in whose terminology we are dealing with the *Wirthmüller-form* of the yoga) especially once Grothendieck's idea of "motives" was felt to be nailed down by "motivic homotopy theory" [CD19, §A.5][Ho17, p. 4]).

On the other hand, the original discussion in logic was entirely in *classical* logic, while Grothendieck's "yoga" that concerns us now always focused on non-cartesian (hence non-classical, i.e.: linear, "quantum") monoidal categories dependent on classical base spaces — whence here we speak of the "motivic yoga" for short. More recently, essentially the same is referred to as "indexed closed monoidal enriched categories with indexed coproducts" [Sh13], which in its category-theoretic sobriety is again more suggestive (for the cognoscenti) of the logical/computational meaning of such structures: they serve as categorical semantics for the *multiplicative fragment* of *dependent linear/quantum homotopy type theory* [Sc14, §3.2][Ri22, §2.4][MRSS23], for more on this perspective see §4 below.

Recall from (150) that for $\mathbf{f} : \mathbf{X} \longrightarrow \mathbf{Y}$ a morphism in sSet-Cat we have an associated adjoint triple of sSet-enriched base change functors of simplicial local systems (Def. 3.5):

$$\begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[r, shift left=3, "\mathbf{f}_!"] \arrow[r, shift right=3, "\mathbf{f}_*"'] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}} \arrow[l, "\mathbf{f}^*"' description] \end{tikzcd} \tag{92}$$

given by precomposition $\mathbf{f}^* \equiv (-) \circ \mathbf{f}$ and its left $\mathbf{f}_!$ and right $\mathbf{f}_*$ Kan extension, respectively (151); and that for each $\mathbf{X}$ we have symmetric closed monoidal category structure (64):

$$\mathbf{X} \in \text{sSet-Grpd}, \;\; \mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \qquad \vdash \qquad \begin{tikzcd} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[r, shift left=2, "\mathcal{V}\otimes_{\mathbf{X}}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \arrow[l, shift left=2, "{[\mathcal{V},-]_{\mathbf{X}}}"] \end{tikzcd}. \tag{93}$$

Proposition 3.27 (Frobenius reciprocity for simplicial local systems)**.** *Pullback of simplicial local systems along maps of simplicial groupoids*

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\ \ \mathbf{f}\ \ } & \mathbf{Y} \\ \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} & \xleftarrow{\ \ \mathbf{f}^*\ \ } & \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}} \end{array}$$

is

(i) *strong monoidal, in that there are natural isomorphisms of this form:*

$$\mathbf{f}^*(\mathbb{1}) \;\simeq\; \mathbb{1} \tag{94}$$

$$\mathcal{V},\, \mathcal{W} \,\in\, \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}} \qquad \vdash \qquad \mathbf{f}^*(\mathcal{V} \otimes_{\mathbf{Y}} \mathcal{W}) \;\simeq\; (\mathbf{f}^*\mathcal{V}) \otimes_{\mathbf{X}} (f^*\mathcal{W}) \tag{95}$$

(ii) *strong closed, in that there are natural isomorphisms of this form:*

$$\mathcal{V},\, \mathcal{W} \,\in\, \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}} \qquad \vdash \qquad \mathbf{f}^*[\mathcal{V},\, \mathcal{W}]_{\mathbf{Y}} \;\simeq\; [\mathbf{f}^*\mathcal{V}, \mathbf{f}^*\mathcal{W}]_{\mathbf{X}} \tag{96}$$

(iii) *and satisfies projection, in that there are natural isomorphisms of this form:*

$$\mathcal{R} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}},\ \mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}} \qquad \vdash \qquad \mathbf{f}_!\big(\mathcal{R} \otimes_{\mathbf{X}} \mathbf{f}^*\mathcal{V}\big) \;\simeq\; (\mathbf{f}_!\mathcal{R}) \otimes_{\mathbf{Y}} \mathcal{V}\,. \tag{97}$$

Proof. By the adjoint equivalences (72) it is sufficient to check this for maps between skeletal simplicial groupoids. That in this case precomposition $\mathbf{f}^*$ is a strong monoidal closed functor is manifest by Prop. 3.12. The projection formula then follows by adjointness (cf. [FHM03]), namely for any $\mathcal{W} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}}$ we have natural isomorphisms

$$\begin{aligned} \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}}\Big(\mathbf{f}_!\big(\mathcal{R} \otimes_{\mathbf{X}} \mathbf{f}^*\mathcal{V}\big),\, \mathcal{W}\Big) \;&\simeq\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\Big(\mathcal{R} \otimes_{\mathbf{X}} \mathbf{f}^*\mathcal{V}, \mathbf{f}^*\mathcal{W}\Big) && \text{by (92)} \\ &\simeq\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\Big(\mathcal{R},\, \big[\mathbf{f}^*\mathcal{V}, \mathbf{f}^*\mathcal{W}\big]_{\mathbf{X}}\Big) && \text{by (93)} \\ &\simeq\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}\Big(\mathcal{R}, \mathbf{f}^*\big[\mathcal{V},\, \mathcal{W}\big]_{\mathbf{Y}}\Big) && \text{by (96)} \\ &\simeq\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}}\Big(\mathbf{f}_!\mathcal{R},\, \big[\mathcal{V},\, \mathcal{W}\big]_{\mathbf{Y}}\Big) && \text{by (92)} \\ &\simeq\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}}\big((\mathbf{f}_!\mathcal{R}) \otimes_{\mathbf{Y}} \mathcal{V},\, \mathcal{W}\big) && \text{by (93),} \end{aligned}$$

and since these are natural in $\mathcal{W}$, the projection formula (97) follows by the Yoneda Lemma. □

We will need the Beck-Chevalley condition for simplicial local systems, but it will be sufficient to consider the following very special case (which is item (b) in [Se83, p. 511], for $B = *$). To that end, we denote the projections out of a Cartesian product of simplicial groupoids as follows:

$$\mathbf{X}, \mathbf{Y} \,\in\, \text{sSet-Grpd} \qquad \vdash \qquad \begin{tikzcd} & \mathbf{X} \times \mathbf{Y} \arrow[dl, "\mathrm{pr}_{\mathbf{X}}"'] \arrow[dr, "\mathrm{pr}_{\mathbf{Y}}"] & \\ \mathbf{X} & & \mathbf{Y} \end{tikzcd}$$

Proposition 3.28 (Beck-Chevalley for simplicial local systems along products)**.** *Given a diagram in* sSet-Grpd *of the form*

$$\begin{tikzcd} \mathbf{X} \times \mathbf{Y} \arrow[r, "f \times \mathrm{id}"] \arrow[d, "\mathrm{pr}_{\mathbf{X}}"] & \mathbf{X}' \times \mathbf{Y} \arrow[d, "\mathrm{pr}_{\mathbf{X}'}"] \\ \mathbf{X} \arrow[r, "f"] & \mathbf{Y} \end{tikzcd}$$

then the following two ways of pull/push of local systems through this diagram (92) *are naturally isomorphic:*

$$(f \times \mathrm{id})_! \circ \mathrm{pr}_{\mathbf{X}}^* \;\simeq\; \mathrm{pr}_{\mathbf{X}'}^* \circ f_! \;\;:\;\; \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}} \longrightarrow \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}' \times \mathbf{Y}}. \tag{98}$$

Proof. By coend calculus, we have for $(x', y) \in \mathbf{X}' \times \mathbf{Y}$ the following sequence of natural isomorphisms:

$$
\begin{aligned}
\big((f \times \mathrm{id})_! \circ \mathrm{pr}_{\mathbf{X}}^* \mathcal{V}\big)_{(x',y)} &\simeq \int^{(x,y_0)\in \mathbf{X}\times\mathbf{Y}} \mathbf{X}\big(f(x), x'\big) \times \mathbf{Y}\big(\mathrm{id}(y_0), y\big) \cdot \big(\mathrm{pr}_{\mathbf{X}}^* \mathcal{V}\big)_{(x,y_0)} \\
&\simeq \int^{(x,y_0)\in \mathbf{X}\times\mathbf{Y}} \mathbf{X}\big(f(x), x'\big) \times \mathbf{Y}(y_0, y) \cdot \mathcal{V}_x \\
&\simeq \int^{y_0\in\mathbf{Y}} \int^{x\in\mathbf{X}} \mathbf{Y}(y_0, y) \times \mathbf{X}\big(f(x), x'\big) \cdot \mathcal{V}_x \\
&\simeq \int^{y_0\in\mathbf{Y}} \mathbf{Y}(y_0, y) \cdot \int^{x\in\mathbf{X}} \mathbf{X}\big(f(x), x'\big) \cdot \mathcal{V}_x \\
&\simeq \int^{y_0\in\mathbf{Y}} \mathbf{Y}(y_0, y) \cdot (f_! \mathcal{V})_{x'} \\
&\simeq (f_! \mathcal{V})_{x'} \\
&\simeq \big(\mathrm{pr}_{\mathbf{X}'}(f_! \mathcal{V})\big)_{(x',y)}\,.
\end{aligned}
$$

□

External tensor of simplicial local systems.

Definition 3.29 (External tensor product of simplicial local systems)**.**
The *external tensor product* on the category of simplicial local systems (84) is the following functor:

$$
\begin{array}{ccc}
\mathbf{Loc}_{\mathbb{K}} \times \mathbf{Loc}_{\mathbb{K}} & \xrightarrow{\qquad\qquad\boxtimes\qquad\qquad} & \mathbf{Loc}_{\mathbb{K}} \\
\big(\mathcal{V}_{\mathbf{X}},\, \mathcal{W}_{\mathbf{Y}}\big) & \longmapsto & \Big(\big(\mathrm{pr}_{\mathbf{X}}^* \mathcal{V}\big) \otimes_{\mathbf{X}\times\mathbf{Y}} \big(\mathrm{pr}_{\mathbf{Y}}^* \mathcal{W}\big)\Big)_{\mathbf{X}\times\mathbf{Y}} \\
{\scriptstyle (\phi_f,\, \gamma_g)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Big(\big(\mathrm{pr}_{\mathbf{X}}^*\phi\big) \otimes_{\mathbf{X}\times\mathbf{Y}} \big(\mathrm{pr}_{\mathbf{Y}}^*\gamma\big)\Big)_{f\times g}} \\
\big(\mathcal{V}'_{\mathbf{X}'},\, \mathcal{W}'_{\mathbf{Y}'}\big) & \longmapsto & \Big(\big(\mathrm{pr}_{\mathbf{X}'}^* \mathcal{V}'\big) \otimes_{\mathbf{X}'\times\mathbf{Y}'} \big(\mathrm{pr}_{\mathbf{Y}'}^* \mathcal{W}'\big)\Big)_{\mathbf{X}'\times\mathbf{Y}'},
\end{array}
\tag{99}
$$

where on the right we are leaving the strong monoidal structure isomorphism notationally implicit. In more detail, the component map of the morphism on the right is this composite:

$$
\big(\mathrm{pr}_{\mathbf{X}}^* \mathcal{V}\big) \otimes_{\mathbf{X}\times\mathbf{Y}} \big(\mathrm{pr}_{\mathbf{Y}}^* \mathcal{W}\big) \xrightarrow{(\mathrm{pr}_{\mathbf{X}}^*\phi)\otimes(\mathrm{pr}_{\mathbf{Y}}^*\gamma)} \Big(\big(\underbrace{\mathrm{pr}_{\mathbf{X}}^* f^*}_{(f\times g)^*\mathrm{pr}_{\mathbf{X}'}^*} \mathcal{V}'\big) \otimes_{\mathbf{X}\times\mathbf{Y}} \big(\underbrace{\mathrm{pr}_{\mathbf{Y}}^* g^*}_{(f\times f')^*\mathrm{pr}_{\mathbf{Y}'}^*} \mathcal{W}'\big)\Big) \simeq (f\times g)^*\Big(\big(\mathrm{pr}_{\mathbf{X}'}^* \mathcal{V}'\big) \otimes_{\mathbf{X}'\times\mathbf{Y}'} \big(\mathrm{pr}_{\mathbf{Y}'}^* \mathcal{W}'\big)\Big),
$$

where the under-braces indicate equalities exhibiting the commutativity of this diagram

$$
\begin{tikzcd}
\mathbf{X} \arrow[r, "f"] & \mathbf{X}' \\
\mathbf{X}\times\mathbf{Y} \arrow[u, "\mathrm{pr}_{\mathbf{X}}"] \arrow[r, "f\times g"] \arrow[d, "\mathrm{pr}_{\mathbf{Y}}"'] & \mathbf{X}'\times\mathbf{Y}' \arrow[u, "\mathrm{pr}_{\mathbf{X}'}"'] \arrow[d, "\mathrm{pr}_{\mathbf{Y}'}"] \\
\mathbf{Y} \arrow[r, "g"] & \mathbf{Y}',
\end{tikzcd}
\tag{100}
$$

while the isomorphism "$\simeq$" is the strong monoidal structure of $(f \times f)^*$ (which is in fact still an actual equality, objectwise).

Remark 3.30 (External tensor with a unit system)**.** Since pullback preserves tensor units (94), the pullback of a simplicial local system to a Cartesian product is isomorphic to its external tensor product (99) with the unit system on the other factor:

$$
\mathcal{V}_{\mathbf{X}} \boxtimes \mathbb{K}_{\mathbf{Y}} \;\simeq\; \big(\mathrm{pr}_{\mathbf{X}}^* \mathcal{V}\big)_{\mathbf{X}\times\mathbf{Y}}\,. \tag{101}
$$

In particular, the external tensor product in general may be expressed in terms of external tensoring with tensor units, as:

$$
\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{W}_{\mathbf{Y}} \;\simeq\; \big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathbb{K}_{\mathbf{Y}}\big)_{\mathbf{X}\times\mathbf{Y}} \otimes_{\mathbf{X}\times\mathbf{Y}} \big(\mathbb{K}_{\mathbf{X}} \boxtimes \mathcal{W}_{\mathbf{Y}}\big)_{\mathbf{X}\times\mathbf{Y}}\,. \tag{102}
$$

Lemma 3.31 (Pull/push of external tensors along product maps). *Given maps of simplicial groupoids* $\mathbf{f} : \mathbf{X} \to \mathbf{X}'$ *and* $\mathbf{g} : \mathbf{Y} \to \mathbf{Y}'$ *there are natural isomorphisms* [10]

$$
\begin{aligned}
(\mathbf{f}\times\mathbf{g})^*\big(\mathcal{V}\boxtimes\mathcal{W}\big) &\simeq (\mathbf{f}^*\mathcal{V})\boxtimes(\mathbf{g}^*\mathcal{W})\,, &(103)\\
(\mathbf{f}\times\mathbf{g})_!\big(\mathcal{V}\boxtimes\mathcal{W}\big) &\simeq (\mathbf{f}_!\mathcal{V})\boxtimes(\mathbf{g}_!\mathcal{W})\,. &(104)
\end{aligned}
$$

Proof. The isomorphism (103) may be obtained as the following composite of natural isomorphisms:

$$
\begin{aligned}
(\mathbf{f}\times\mathbf{g})^*\big(\mathcal{V}\boxtimes\mathcal{W}\big) &\simeq (\mathbf{f}\times\mathbf{g})^*\Big(\big(\mathrm{pr}_{\mathbf{Y}}^*\mathcal{V}\big)\otimes_{\mathbf{Y}\times\mathbf{Y}'}\big(\mathrm{pr}_{\mathbf{Y}'}^*\mathcal{W}\big)\Big) && \text{by (99)}\\
&\simeq \big((\mathbf{f}\times\mathbf{g})^*\mathrm{pr}_{\mathbf{Y}}^*\mathcal{V}\big)\otimes_{\mathbf{X}\times\mathbf{X}'}\big((\mathbf{f}\times\mathbf{g})^*\mathrm{pr}_{\mathbf{Y}'}^*\mathcal{W}\big) && \text{by (95)}\\
&\simeq \big(\mathrm{pr}_{\mathbf{X}}^*\mathbf{f}^*\mathcal{V}\big)\otimes_{\mathbf{X}\times\mathbf{X}'}\big(\mathrm{pr}_{\mathbf{X}'}^*\mathbf{g}^*\mathcal{W}\big) && \text{by (100) \& (134)}\\
&\simeq \big(\mathbf{f}^*\mathcal{V}\big)\boxtimes\big(\mathbf{g}^*\mathcal{W}\big) && \text{by (99)}\;\;.
\end{aligned}
$$

For the second isomorphism (104), first observe the special case

$$
\begin{aligned}
(\mathbf{f}\times\mathrm{id})_!\big(\mathcal{V}_{\mathbf{X}}\boxtimes\mathbb{K}_{\mathbf{Y}}\big) &\simeq \Big((\mathbf{f}\times\mathrm{id})_!\big(\mathrm{pr}_{\mathbf{X}}^*\mathcal{V}\big)_{\mathbf{X}'\times\mathbf{Y}'}\Big) && \text{by (101)}\\
&\simeq \big(\mathrm{pr}_{\mathbf{Y}}^*(\mathbf{f}_!\mathcal{V})\big)_{\mathbf{X}'\times\mathbf{Y}} && \text{by (98)}\\
&\simeq (\mathbf{f}_!\mathcal{V})_{\mathbf{X}'}\boxtimes\mathbb{K}_{\mathbf{Y}} && \text{by (101)}
\end{aligned}
\qquad (105)
$$

from which we obtain the general isomorphism as the following composite:

$$
\begin{aligned}
(\mathbf{f}\times\mathbf{g})_!(\mathcal{V}\boxtimes\mathcal{W}) &\simeq (\mathbf{f}\times\mathbf{g})_!\big((\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}})\otimes(\mathbb{K}_{\mathbf{X}}\boxtimes\mathcal{W})\big) && \text{by (102)}\\
&\simeq (\mathbf{f}\times\mathbf{g})_!\Big(\big((\mathrm{id}\times g)^*(\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\big)\otimes(\mathbb{K}_{\mathbf{X}}\boxtimes\mathcal{W})\Big) && \text{by (103) \& (94)}\\
&\simeq (\mathbf{f}\times\mathrm{id})_!(\mathrm{id}\times\mathbf{g})_!\Big(\big((\mathrm{id}\times\mathbf{g})^*(\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\big)\otimes(\mathbb{K}_{\mathbf{X}}\boxtimes\mathcal{W})\Big) && \text{by (134)}\\
&\simeq (\mathbf{f}\times\mathrm{id})_!\Big((\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\otimes\big((\mathrm{id}\times\mathbf{g})_!(\mathbb{K}_{\mathbf{X}}\boxtimes\mathcal{W})\big)\Big) && \text{by (97)}\\
&\simeq (\mathbf{f}\times\mathrm{id})_!\Big((\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\otimes\big(\mathbb{K}_{\mathbf{X}}\boxtimes(\mathbf{g}_!\mathcal{W})\big)\Big) && \text{by (105)}\\
&\simeq (\mathbf{f}\times\mathrm{id})_!\Big((\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\otimes(\mathbf{f}\times\mathrm{id})^*\big(\mathbb{K}_{\mathbf{X}'}\boxtimes(\mathbf{g}_!\mathcal{W})\big)\Big) && \text{by (103) \& (94)}\\
&\simeq \big((\mathbf{f}\times\mathrm{id})_!(\mathcal{V}\boxtimes\mathbb{K}_{\mathbf{Y}'})\big)\otimes\big(\mathbb{K}_{\mathbf{X}'}\boxtimes(\mathbf{g}_!\mathcal{W})\big) && \text{by (97)}\\
&\simeq \big((\mathbf{f}_!\mathcal{V})\boxtimes\mathbb{K}_{\mathbf{Y}'}\big)\otimes\big(\mathbb{K}_{\mathbf{X}'}\boxtimes(\mathbf{g}_!\mathcal{W})\big) && \text{by (105)}\\
&\simeq (\mathbf{f}_!\mathcal{V})\boxtimes(\mathbf{g}_!\mathcal{W}) && \text{by (102).}
\end{aligned}
$$

□

As a direct consequence:

Proposition 3.32 (Pull-push adjunct of external tensor products). *The push/pull adjunct of an external tensor product of morphisms into pullbacks*

$$
\phi\boxtimes\gamma : \mathcal{V}\boxtimes\mathcal{W} \longrightarrow (\mathbf{f}^*\mathcal{V}')\boxtimes(\mathbf{g}^*\mathcal{W}') \simeq (\mathbf{f}\times\mathbf{g})^*(\mathcal{V}'\boxtimes\mathcal{W}')
$$

is the external tensor product of the separate adjuncts:

$$
\widetilde{\phi\boxtimes\gamma} \simeq \widetilde{\phi}\boxtimes\widetilde{\gamma} : (\mathbf{f}\times\mathbf{g})_!(\mathcal{V}\boxtimes\mathcal{W}) \simeq (\mathbf{f}_!\mathcal{V})\boxtimes(\mathbf{g}_!\mathcal{W}) \longrightarrow \mathcal{V}'\boxtimes\mathcal{W}'\,. \qquad (106)
$$

[10] The isomorphism (103) is a fairly immediate consequence of (94), but (104) is not so immediate, it appears mentioned in generality but without proof in [Sh13, p. 624], while in models for parameterized spectra it appears in [MS06, Rem. 2.5.8, Prop. 13.7.2], [Mal19, Lem. 3.4.1], [Mal23, Lem. 2.5.1].

Proposition 3.33 (External pushout-product)**.** *The pushout-product (Def. A.27) of the external tensor (Def. 3.29) in $\mathbf{Loc}_{\mathbb{K}}$ is given by the following formula:*

$$(\phi_{\mathbf{f}}) \widehat{\boxtimes} (\gamma_{\mathbf{g}}) \;\simeq\; \Big(\widetilde{\big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \widehat{\otimes} \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big)} \Big)_{\mathbf{f} \widehat{\times} \mathbf{g}} .$$

Proof. By the general formula for colimits in Grothendieck constructions (Prop. A.9), the underlying colimit is the Cartesian pushout-product of simplicial groupoids

$$\begin{tikzcd} \mathbf{X} \times \mathbf{Y} \arrow[r, "\mathrm{id} \times \mathbf{g}"] \arrow[d, "\mathbf{f} \times \mathrm{id}"'] \arrow[dr, phantom, "(\mathrm{po})"] & \mathbf{X} \times \mathbf{Y}' \arrow[d, "q_r"] \arrow[ddr, bend left, "\mathbf{f} \times \mathrm{id}"] & \\ \mathbf{X}' \times \mathbf{Y} \arrow[r, "q_l"] \arrow[drr, bend right, "\mathrm{id} \times \mathbf{g}"'] & \mathbf{f} \widehat{\times} \mathbf{g} \arrow[dr, dashed] & \\ & & \mathbf{X}' \times \mathbf{Y}' \end{tikzcd} \tag{107}$$

and the linear colimiting component map over the dashed morphism is obtained by pushing the separate linear components along the coprojections $q_\cdot$ to form a cospan in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{f} \widehat{\times} \mathbf{g}}$, whose universal pushout-product morphism, in turn, is the further pushforward of that cocone along the dashed morphism:

$$\begin{aligned}
& \big(\mathbf{f} \widehat{\times} \mathbf{g}\big)_! \bigg(\Big((q_l)_! \Big(\big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}})^* \mathrm{id}_{\mathcal{W}}\big) \Big) \Big) \wedge \Big((q_r)_! \Big(\big((\mathrm{pr}_{\mathbf{X}})^* \mathrm{id}_{\mathcal{V}}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big) \Big) \Big) \bigg) && \text{by (106)} \\
\simeq\; & \bigg(\Big(\big(\mathbf{f} \widehat{\times} \mathbf{g}\big)_! (q_l)_! \Big(\big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}})^* \mathrm{id}_{\mathcal{W}}\big) \Big) \Big) \wedge \Big(\big(\mathbf{f} \widehat{\times} \mathbf{g}\big)_! (q_r)_! \Big(\big((\mathrm{pr}_{\mathbf{X}})^* \mathrm{id}_{\mathcal{V}}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big) \Big) \Big) \bigg) && \text{by (92)} \\
\simeq\; & \bigg(\Big((\mathrm{id} \otimes \mathbf{g})_! \Big(\big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}})^* \mathrm{id}_{\mathcal{W}}\big) \Big) \Big) \wedge \Big((\mathbf{f} \otimes \mathrm{id})_! \Big(\big((\mathrm{pr}_{\mathbf{X}})^* \mathrm{id}_{\mathcal{V}}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big) \Big) \Big) \bigg) && \text{by (107)} \\
\simeq\; & \bigg(\Big(\Big(\big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}})^* \mathrm{id}_{\mathbf{f}_! \mathcal{W}}\big) \Big) \Big) \wedge \Big(\Big(\big((\mathrm{pr}_{\mathbf{X}})^* \mathrm{id}_{\mathbf{f}_! \mathcal{V}}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big) \Big) \Big) \bigg) && \text{by (104)} \\
\simeq\; & \big((\mathrm{pr}_{\mathbf{X}'})^* \widetilde{\phi}\big) \widehat{\otimes} \big((\mathrm{pr}_{\mathbf{Y}'})^* \widetilde{\gamma}\big) && \text{by def.}
\end{aligned}$$

□

External internal hom of simplicial local systems. We discuss the right adjoint to external tensoring with a simplicial local system. Since right adjoints to tensoring functors are called *internal homs* this would by default be named the *external internal hom*, for better or worse.

Proposition 3.34. *The external tensor product (Def. 3.29) with a simplicial local system preserves colimits:*

$$\left.\begin{array}{l} \mathcal{W}_{\mathbf{Y}} \in \mathbf{Loc}_{\mathbb{K}}, \\ \mathscr{V}_{\mathbf{X}} : I \to \mathbf{Loc}_{\mathbb{K}} \end{array}\right\} \quad \vdash \quad \Big(\varinjlim_{i \in I} \mathcal{V}(i)_{\mathbf{X}_i} \Big) \boxtimes \mathcal{W}_{\mathbf{Y}} \;\simeq\; \varinjlim_{i \in I} \big(\mathcal{V}(i)_{\mathbf{X}_i} \boxtimes \mathcal{W}_{\mathbf{Y}} \big). \tag{108}$$

Proof. First notice that the statement holds for the underlying colimit in sSet-Grpd, since $(-) \times \mathbf{Y}$ is a left adjoint (Prop. A.2):

$$\big(\varinjlim_{j \in I} \mathbf{X}_j \big) \times \mathbf{Y} \;\simeq\; \varinjlim_{j \in I} \big(\mathbf{X}_j \times \mathbf{Y} \big) .$$

Now denoting the coprojections of this underlying colimit by:

$$\mathbf{X}_i \xrightarrow{\;q^{\mathbf{X}_i}\;} \varinjlim_{j \in I} \mathbf{X}_j\,, \qquad \begin{tikzcd} \mathbf{X}_i \times \mathrm{Y} \arrow[r, "q^{\mathbf{X}_i} \times \mathrm{id}"] \arrow[d, "\mathrm{pr}_{\mathbf{X}_i}"] & \big(\varinjlim_{j \in I} \mathbf{X}_j\big) \times \mathrm{Y} \arrow[d, "\mathrm{pr}_{\varinjlim \mathbf{X}}"] \\ \mathbf{X}_i \arrow[r, "q^{\mathbf{X}_i}"] & \big(\varinjlim_{j \in I} \mathbf{X}_j\big) \end{tikzcd} \tag{109}$$

we identify, via Prop. A.9, the full colimit by the following sequence of natural isomorphisms:

$$
\begin{aligned}
\varinjlim\big(\mathcal{V}_{\mathbf{X}}\big) \boxtimes \mathcal{W}_{\mathbf{Y}} \;&\simeq\; \Big(\varinjlim\,(q^{\mathbf{X}})_!\,\mathcal{V}\Big)_{\varinjlim \mathbf{X}} \boxtimes \mathcal{W}_{\mathbf{Y}} && \text{by (A.9)}\\
&\simeq\; \Big(\big((\mathrm{pr}_{\varinjlim \mathbf{X}})^*\varinjlim\,(q^{\mathbf{X}})_!\,\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)_{\varinjlim \mathbf{X}\times\mathbf{Y}} && \text{by defn. (3.29)}\\
&\simeq\; \Big(\big(\varinjlim\,(\mathrm{pr}_{\varinjlim \mathbf{X}})^*\,(q^{\mathbf{X}})_!\,\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)_{\varinjlim \mathbf{X}\times\mathbf{Y}} && \text{since (-)}^* \text{ is left adjoint (92)}\\
&\simeq\; \Big(\big(\varinjlim\,(q^{\mathbf{X}}\times\mathrm{id}_{\mathbf{Y}})_!\,(\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)_{\varinjlim \mathbf{X}\times\mathbf{Y}} && \text{by Beck-Chevalley (98) for (109)}\\
&\simeq\; \bigg(\varinjlim\Big(\big((q^{\mathbf{X}}\times\mathrm{id}_{\mathbf{Y}})_!\,(\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)\bigg)_{\varinjlim \mathbf{X}\times\mathbf{Y}} && \text{since (-)}\otimes\cdots\text{ preserves colimits}\\
&\simeq\; \bigg(\varinjlim\,(q^{\mathbf{X}}\times\mathrm{id}_{\mathbf{Y}})_!\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)\bigg)_{\varinjlim \mathbf{X}\times\mathbf{Y}} && \text{projection formula (97)}\\
&\simeq\; \varinjlim\bigg(\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big)\otimes\big((\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}\big)\Big)_{\mathbf{X}\times\mathbf{Y}}\bigg) && \text{by (A.9)}\\
&\simeq\; \varinjlim\big(\mathcal{V}_{\mathbf{X}}\boxtimes\mathcal{W}_{\mathbf{Y}}\big) && \text{by defn. (3.29)}\,.
\end{aligned}
$$

□

Proposition 3.35 (Distributive coproducts of simplicial local systems)**.** *Any simplicial local system over a skeletal simplicial groupoid is the coproduct in $\mathbf{Loc}_{\mathbb{K}}$ of its restrictions to the connected components:*

$$
\begin{array}{l}
\mathbf{Y} \equiv \coprod_{i\in I}\mathbf{Y}_i \in \text{sSet-Grpd}_{\text{skl}}\,,\\
\mathcal{W} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{Y}}
\end{array}
\quad\vdash\quad
\mathcal{W}_{\mathbf{Y}} \;\simeq\; \coprod_{i\in I}\mathcal{W}_{\mathbf{Y}_i} \;\;\in\; \mathbf{Loc}_{\mathbb{K}}\,. \tag{110}
$$

and the external tensor product (Def. 3.29) with any $\mathcal{V}_{\mathbf{X}} \in \mathbf{Loc}_{\mathbb{K}}$ distributes over (these) coproducts:

$$
\mathcal{V}_{\mathbf{X}} \boxtimes \coprod_{i\in I}\mathcal{W}_{\mathbf{Y}_i} \;\simeq\; \coprod_{i\in I}\big(\mathcal{V}_{\mathbf{X}}\boxtimes\mathcal{W}_{\mathbf{Y}_i}\big). \tag{111}
$$

Proof. This is essentially the general phenomenon of free coproduct completion of connected objects (Prop. A.7) only that base sets are now replaced by skeletal simplicial groupoids. For the record, we spell it out. In the following we discuss binary coproducts just for convenience of notation; The argument immediately generalizes to general set-indexed coproducts.

First, observe – readily by adjointness (92), alternatively by the Kan extension formula (151) — that pushforward of simplicial local systems along a coprojection into a coproduct of simplicial groupoids is extension by zero to the other connected component:

$$
\left.\begin{array}{l}
\mathbf{X}_1, \mathbf{X}_2 \in \text{sSet-Grpd}\\
\mathcal{V} \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_1},\ \mathcal{V}' \in \mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}_2}
\end{array}\right\}
\;\vdash\;
\begin{tikzcd}
\mathbf{X}_1 \arrow[d, hook, "q_1"'] \arrow[drr, "\mathcal{V}"] & & \\
\mathbf{X}_1 \sqcup \mathbf{X}_2 \arrow[rr, "(q_1)_!\mathcal{V}"] & & \mathbf{sCh}_{\mathbb{K}} \\
\mathbf{X}_2 \arrow[u, hook', "q_2"] \arrow[urr, "0"'] & &
\end{tikzcd}
\;\;\text{and}\;\;
\begin{tikzcd}
\mathbf{X}_1 \arrow[d, hook, "q_1"'] \arrow[drr, "0"] & & \\
\mathbf{X}_1 \sqcup \mathbf{X}_2 \arrow[rr, "(q_2)_!\mathcal{V}"] & & \mathbf{sCh}_{\mathbb{K}} \\
\mathbf{X}_2 \arrow[u, hook', "q_2"] \arrow[urr, "\mathcal{V}'"'] & &
\end{tikzcd}
$$

from which it follows that the coproduct of a pair of such push-forwards is given by

$$
\begin{tikzcd}
\mathbf{X}_1 \arrow[d, hook, "q_1"'] \arrow[drr, "\mathcal{V}"] & & \\
\mathbf{X}_1 \sqcup \mathbf{X}_2 \arrow[rr, "(q_1)_!\mathcal{V}\sqcup(q_2)_!\mathcal{V}'"] & & \mathbf{sCh}_{\mathbb{K}}\,. \\
\mathbf{X}_2 \arrow[u, hook', "q_2"] \arrow[urr, "\mathcal{V}'"'] & &
\end{tikzcd}
$$

This implies the first formula (111) by the general formula for colimits in Grothendieck constructions (Prop. A.9), which gives that

$$\big((q_1)_!\mathcal{V} \sqcup (q_2)_!\mathcal{V}'\big)_{\mathbf{X}_1 \sqcup \mathbf{X}_2} \;\simeq\; \mathcal{V}_{\mathbf{X}_1} \coprod \mathcal{V}'_{\mathbf{X}_2} \tag{112}$$

Moreover, in the situation

$$\begin{tikzcd}
 & & \mathbf{X} \times \mathbf{Y}_i \arrow[dl, "q_i"'] \arrow[dr, "\mathrm{pr}_{\mathbf{Y}_i}"] \arrow[dll, bend right, "\mathrm{pr}_{\mathbf{X}}"'] & \\
 & \mathbf{X} \times \big(\mathbf{Y}_1 \sqcup \mathbf{Y}_2\big) \arrow[dl, "\mathrm{pr}_{\mathbf{X}}"] \arrow[dr, "\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2}"'] & & \mathbf{Y}_i \arrow[dl, "q_i"] \\
\mathbf{X} & & \mathbf{Y}_1 \sqcup \mathbf{Y}_2 &
\end{tikzcd}$$

it follows that the corresponding Beck-Chevalley condition is satisfied

$$(\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^* \circ (q_i)_! \;\simeq\; (q_i)_! \circ (\mathrm{pr}_{\mathbf{Y}_i})^* \tag{113}$$

and that

$$\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((q_i)_!(\mathrm{pr}_{\mathbf{Y}_i})^*\mathcal{W}\big) \;\simeq\; (q_i)_!\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}_i})^*\mathcal{W}\big)\Big). \tag{114}$$

With this, we establish the second statement:

$$\begin{aligned}
&\mathcal{V}_{\mathbf{X}} \boxtimes \big(\mathcal{W}_{\mathbf{Y}_1} \coprod \mathcal{W}'_{\mathbf{Y}_2}\big) \\
&\simeq \Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes (\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^*\big((q_1)_!\mathcal{W} \sqcup (q_2)_!\mathcal{W}'\big)\Big)_{\mathbf{X} \times \big(\mathbf{Y}_1 \sqcup \mathbf{Y}_2\big)} && \text{by (112)} \\
&\simeq \bigg(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \Big(\big((\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^*(q_1)_!\mathcal{W}\big) \sqcup \big((\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^*(q_2)_!\mathcal{W}'\big)\Big)\bigg)_{\mathbf{X} \times \mathbf{Y}_1 \sqcup \mathbf{X} \times \mathbf{Y}_1} && \text{(-)}^* \text{ is left adjoint} \\
&\simeq \bigg(\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^*(q_1)_!\mathcal{W}\big)\Big) \sqcup \Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}_1 \sqcup \mathbf{Y}_2})^*(q_2)_!\mathcal{W}'\big)\Big)\bigg)_{\mathbf{X} \times \mathbf{Y}_1 \sqcup \mathbf{X} \times \mathbf{Y}_1} && \cdots \otimes \text{(-) is left adjoint} \\
&\simeq \bigg(\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((q_1)_!(\mathrm{pr}_{\mathbf{Y}_1})^*\mathcal{W}\big)\Big) \sqcup \Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((q_2)_!(\mathrm{pr}_{\mathbf{Y}_2})^*\mathcal{W}'\big)\Big)\bigg)_{\mathbf{X} \times \mathbf{Y}_1 \sqcup \mathbf{X} \times \mathbf{Y}_1} && \text{by (113)} \\
&\simeq \bigg((q_1)_!\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}_1})^*\mathcal{W}\big)\Big) \sqcup (q_2)_!\Big(\big((\mathrm{pr}_{\mathbf{X}})^*\mathcal{V}\big) \otimes \big((\mathrm{pr}_{\mathbf{Y}_2})^*\mathcal{W}'\big)\Big)\bigg)_{\mathbf{X} \times \mathbf{Y}_1 \sqcup \mathbf{X} \times \mathbf{Y}_1} && \text{by (114)} \\
&\simeq \big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{W}_{\mathbf{Y}_1}\big) \coprod \big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{W}'_{\mathbf{Y}_2}\big) && \text{by (112)}\,.
\end{aligned}$$

□

Proposition 3.36 (External internal hom of simplicial local systems)**.**
(i) *Forming the external tensor product with a simplicial local system over a discrete space has a right adjoint functor:*

$$\begin{array}{l} \mathrm{Y} \in \mathrm{Set} \hookrightarrow \mathrm{sSet\text{-}Grpd}_{\mathrm{skl}} \\ \mathcal{R} \in \mathbf{sCh}_{\mathbb{K}}^{\mathrm{Y}} \end{array} \qquad \vdash \qquad \begin{tikzcd} \mathbf{Loc}_{\mathbb{K}} \arrow[r, shift left, "\mathcal{R}_{\mathrm{Y}} \boxtimes (\text{-})"] \arrow[r, phantom, "\perp"] & \mathbf{Loc}_{\mathbb{K}} \arrow[l, shift left, "\mathcal{R}_{\mathrm{Y}} \multimap\!\!\!\square (\text{-})"] \end{tikzcd}. \tag{115}$$

(ii) *This is given by*

$$\mathcal{W}_{\mathbf{Z}} \in \mathbf{Loc}_{\mathbb{K}} \qquad \vdash \qquad \mathcal{R}_{\mathrm{Y}} \multimap\!\!\!\square\, \mathcal{W}_{\mathbf{Z}} \;:\equiv\; \bigg(\prod_{y \in \mathrm{Y}} \mathrm{ev}_y^*\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big]\bigg)_{\mathbf{Z}^{\mathrm{Y}}}, \tag{116}$$

(iii) *which is such that over a map* $f : \mathrm{Y} \longrightarrow \mathbf{Z}$, *hence when restricted along* $\{f\} \hookrightarrow \mathbf{Z}^{\mathrm{Y}}$, *it is given by*

$$\{f\}^*\big(\mathcal{R}_{\mathrm{Y}} \multimap\!\!\!\square\, \mathcal{W}_{\mathbf{Z}}\big) \;\simeq\; (p_{\mathrm{Y}})_*\big[\mathcal{R}, f^*\mathcal{W}\big]\,. \tag{117}$$

Proof. First, consider the special case that Y is a singleton set $\{y\}$. Then we have the following sequence of natural isomorphisms:

$$
\begin{aligned}
\mathbf{Loc}_{\mathbb{K}}\big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{R}_{\{y\}},\, \mathcal{W}_{\mathbf{Z}}\big) &\simeq \big(f \in \text{sSet-Grpd}(\mathbf{X},\mathbf{Z})\big) \times \mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\Big(\mathcal{V} \otimes \big((\mathrm{pr}_{\{y\}})^*\mathcal{R}\big),\, f^*\mathcal{W}\Big) && \text{by (137)} \\
&\simeq \big(f \in \text{sSet-Grpd}(\mathbf{X},\mathbf{Z})\big) \times \mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\Big(\mathcal{V} \otimes \big((p_{\mathbf{X}})^*\mathcal{R}\big),\, f^*\mathcal{W}\Big) && \text{by (119)} \\
&\simeq \big(f \in \text{sSet-Grpd}(\mathbf{X},\mathbf{Z})\big) \times \mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\Big(\mathcal{V},\, \big[(p_{\mathbf{X}})^*\mathcal{R},\, f^*\mathcal{W}\big]\Big) && \text{by (64)} \\
&\simeq \big(f \in \text{sSet-Grpd}(\mathbf{X},\mathbf{Z})\big) \times \mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\Big(\mathcal{V},\, \big[f^*(p_{\mathbf{Z}})^*\mathcal{R},\, f^*\mathcal{W}\big]\Big) && \text{by (119)} \\
&\simeq \big(f \in \text{sSet-Grpd}(\mathbf{X},\mathbf{Z})\big) \times \mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\Big(\mathcal{V},\, f^*\big[(p_{\mathbf{Z}})^*\mathcal{R},\, \mathcal{W}\big]\Big) && \text{by (96)} \\
&\simeq \mathbf{Loc}_{\mathbb{K}}\big(\mathcal{V}_{\mathbf{X}},\, \big[(p_{\mathbf{Z}})^*\mathcal{R},\, \mathcal{W}\big]_{\mathbf{Z}}\big) && \text{by (137),}
\end{aligned}
\tag{118}
$$

where we have made use of notation corresponding to the following commuting diagram:

$$
\begin{tikzcd}
\mathbf{X} \times \{y\} \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_{\{y\}}"'] & \mathbf{X} \arrow[r, "f"] \arrow[d, "p_{\mathbf{X}}"'] & \mathbf{Z} \arrow[d, "p_{\mathbf{Z}}"] \\
\{y\} \arrow[r, "\sim"] & * \arrow[r, equal] & *.
\end{tikzcd}
\tag{119}
$$

From this, we obtain the general statement (116) as follows:

$$
\begin{aligned}
\mathbf{Loc}_{\mathbb{K}}\big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{R}_{\mathrm{Y}},\, \mathcal{W}_{\mathbf{Z}}\big) &\simeq \mathbf{Loc}_{\mathbb{K}}\Big(\mathcal{V}_{\mathbf{X}} \boxtimes \Big(\coprod_{y \in Y} \mathcal{R}_{\{y\}}\Big),\, \mathcal{W}_{\mathbf{Z}}\Big) && \text{by (110)} \\
&\simeq \mathbf{Loc}_{\mathbb{K}}\Big(\coprod_{y \in Y} \big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{R}_{\{y\}}\big),\, \mathcal{W}_{\mathbf{Z}}\Big) && \text{by (111)} \\
&\simeq \prod_{y \in Y} \mathbf{Loc}_{\mathbb{K}}\Big(\mathcal{V}_{\mathbf{X}} \boxtimes \mathcal{R}_{\{y\}},\, \mathcal{W}_{\mathbf{Z}}\Big) && \text{hom-functors preserve limits} \\
&\simeq \prod_{y \in Y} \mathbf{Loc}_{\mathbb{K}}\Big(\mathcal{V}_{\mathbf{X}},\, \big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big]_{\mathbf{Z}}\Big) && \text{by (118)} \\
&\simeq \mathbf{Loc}_{\mathbb{K}}\Big(\mathcal{V}_{\mathbf{X}},\, \prod_{y \in Y}\Big(\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big]_{\mathbf{Z}}\Big)\Big) && \text{hom-functors preserve limits} \\
&\simeq \mathbf{Loc}_{\mathbb{K}}\Big(\mathcal{V}_{\mathbf{X}},\, \Big(\prod_{y \in \mathrm{Y}} \mathrm{ev}_y^*\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big]\Big)_{\mathbf{Z}^{\mathrm{Y}}}\Big) && \text{by Prop. A.9.}
\end{aligned}
$$

Finally, the formula (117) is obtained as follows:

$$
\begin{aligned}
\{f\}^* \prod_{y \in \mathrm{Y}} \mathrm{ev}_y^*\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big] &\simeq \prod_{y \in \mathrm{Y}} \{f\}^* \mathrm{ev}_y^*\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big] && \text{since (-)}^* \text{ is right adjoint (92)} \\
&\simeq \prod_{y \in \mathrm{Y}} \{f(y)\}^*\big[(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \mathcal{W}\big] && \text{by (120)} \\
&\simeq \prod_{y \in \mathrm{Y}} \big[\{f(y)\}^*(p_{\mathbf{Z}})^*\mathcal{R}_{\{y\}},\, \{f(y)\}^*\mathcal{W}\big] && \text{by (96)} \\
&\simeq \prod_{y \in \mathrm{Y}} \big[\mathcal{R}_{\{y\}},\, \mathcal{W}_{\{f(y)\}}\big] && \text{by (120)} \\
&\simeq \int_{y \in \mathrm{Y}} \big[\mathcal{R}_{\{y\}},\, (f^*\mathcal{W})_{\{y\}}\big] \\
&\simeq (p_{\mathrm{Y}})_*\big[\mathcal{R},\, f^*\mathcal{W}\big] && \text{by (151)},
\end{aligned}
$$

where we made use of the notation in the following commuting diagram

$$
\begin{tikzcd}
\{f\} \arrow[rr, hook] \arrow[dr, "f(y)"'] & & \mathbf{Z}^{\mathrm{Y}} \arrow[dl, "\mathrm{ev}_y"] \\
& \mathbf{Z} \arrow[d, "p_{\mathbf{Z}}"'] & \\
& * &
\end{tikzcd}
\tag{120}
$$

□

Example 3.37 (External internal hom of vector bundles over discrete spaces)**.** In the special case when the local systems in question take values in plain vector spaces

$$\mathcal{R}, f^*\mathcal{W} : \mathrm{Y} \longmapsto \mathrm{Mod}_{\mathbb{K}} \hookrightarrow \mathrm{Ch}_{\mathbb{K}} \xhookrightarrow{\mathrm{const}} \mathrm{sCh}_{\mathbb{K}}$$

then the expression in (117) reduces to the vector space of vector bundle morphisms $\mathcal{R} \to f^*\mathcal{W}$ over Y. This way we recover the expression for the $\boxtimes$-adjoint internal hom given in [MRSS23, p. 6] (there denoted "$\multimap$" instead of "$\multimap\!\!\!\square$").

Proposition 3.38. *The external tensor product preserves the Cartesian squares* (91)*.*

$$\begin{tikzcd}
\mathcal{V}_{\mathrm{pt}} \boxtimes \mathcal{W}_{\mathbf{Y}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{V}_{\mathbf{B}\mathscr{G}} \boxtimes \mathcal{W}_{\mathbf{Y}} \arrow[d] \\
0_{\mathrm{pt}} \boxtimes \mathcal{W}_{\mathbf{Y}} \arrow[r] & 0_{\mathbf{B}\mathscr{G}} \boxtimes \mathcal{W}_{\mathbf{Y}}
\end{tikzcd}$$

Homotopical properties of the external tensor product. We establish in Thm. 3.40 homotopical properties of the external tensor product $\boxtimes$ of simplicial local systems (from Def. 3.29). The first key point is that $\boxtimes$ preserves all weak equivalences and in this sense already coincides with its derived functor.

Beyond that, one would wish that $\boxtimes$ were a Quillen bifunctor (Def. A.30) so that also its right adjoint internal hom were homotopically well behaved. However, on $\mathbf{Loc}_{\mathbb{K}}$ this fails in general simply because the underlying category sSet-Grpd is, while cartesian monoidal as a category, not cartesian monoidal as a *model category*, i.e., the Cartesian product here in general fails the Quillen bifunctor property. However, the second item of the following theorem shows that this is the only problem in that $\boxtimes$ generally satisfies the Quillen bifunctor property on linear components and hence satisfies it genuinely whenever the Cartesian product on sSet-Grpd does so, which is the case at least when one argument is restricted to simplicial local systems over discrete spaces. [11]

While still less strong than what one might hope for, the results of Thm. 3.40 compare favorably with the state of the art in the literature on parameterized spectra, see Rem. 3.40 below. For further commentary and outlook see §4 below.

Lemma 3.39 (Homotopical properties of Set-tensoring of simplicial groupoids)**.** *The tensoring of* sSet-Grpd *over sets*

$$(-)\cdot(-) \;:\; \mathrm{Set} \times \mathrm{sSet\text{-}Grpd} \hookrightarrow \mathrm{sSet\text{-}Grpd} \times \mathrm{sSet\text{-}Grpd} \xrightarrow{(-)\times(-)} \mathrm{sSet\text{-}Grpd}$$

is a restricted Quillen bifunctor in that (recalling from Prop. 3.20 that the cofibrations in sSet-Grpd *are in particular injections on sets of objects and weak equivalences are in particular bijections on connected components):*

$$\mathrm{X} \xhookrightarrow{f} \mathrm{X}' \in \mathrm{Inj}(\mathrm{Set}),\ \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Y}' \in \mathrm{Cof}\big(\mathrm{sSet\text{-}Grpd}\big) \qquad \vdash \qquad f \,\widehat{\times}\, \mathbf{g} \in \mathrm{Cof}\big(\mathrm{sSet\text{-}Grpd}\big)$$

and the pushout-product on the right is in addition a weak equivalence if $\mathbf{g}$ *is in addition a weak equivalent or if f is an isomorphism.*

In particular, the Cartesian product with a fixed set is a left Quillen functor:

$$S \in \mathrm{Set} \hookrightarrow \mathrm{sSet\text{-}Grpd} \qquad \vdash \qquad \begin{tikzcd} \mathrm{sSet\text{-}Grpd} \arrow[r, shift left, "S\times(-)\,\simeq\,\coprod\limits_{s\in S}(-)"] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{sSet\text{-}Grpd} \arrow[l, shift left, "(-)^S\,\simeq\,\prod\limits_{s\in S}(-)"] \end{tikzcd} . \tag{121}$$

[11] For the Quillen equivalent model category $\mathbf{Loc}_{\mathbb{K}}^{\mathrm{sSet}}$ (84) the situation is somewhat complementary: Here the underlying Cartesian product is a Quillen bifunctor but now there is little control over the action on linear components, since the transfer functor $\mathbf{G} : \mathrm{sSet} \to \mathrm{sSet\text{-}Grpd}$ does not preserve products.

Proof. It is immediate that (121) is a Quillen adjunction, since the class of (acyclic) cofibrations is closed under coproducts in the arrow category. Moreover, the pushout-product diagram in question is of the form

$$
\begin{tikzcd}
\mathrm{X}\times\mathbf{Y} \arrow[r, "\mathrm{id}_{\mathrm{X}}\times\mathbf{g}"] \arrow[d, hook] & \mathrm{X}\times\mathbf{Y}' \arrow[d, hook] \arrow[ddr, bend left, "f\times\mathrm{id}_{Y'}"] & \\
\mathrm{X}\times\mathbf{Y}\coprod(\mathrm{X}'\setminus\mathrm{X})\times\mathbf{Y} \arrow[r] \arrow[drr, bend right, "\mathrm{id}\times\mathbf{g}"'] & \mathrm{X}\times\mathbf{Y}'\coprod(\mathrm{X}'\setminus\mathrm{X})\times\mathbf{Y} \arrow[dr, dashed] & \\
& & \mathrm{X}\times\mathbf{Y}'\coprod(\mathrm{X}'\setminus\mathrm{X})\times\mathbf{Y}'
\end{tikzcd}
$$

showing that the dashed pushout-product morphism is a coproduct (in the arrow category) of $\mathrm{id}_{X\times\mathbf{Y}'}$ (which is trivially an acyclic cofibration) with copies of $\mathbf{g}$, hence is itself a cofibration, by the previous comment, and an acyclic cofibration if $\mathbf{g}$ is. ☐

Theorem 3.40 (Homotopical properties of external tensor product on simplicial local systems)**.** *The external tensor product (Def. 3.29) on the integral model category of simplicial local systems (Prop. 3.22)*

$$\mathbf{Loc}_{\mathbb{K}}\times\mathbf{Loc}_{\mathbb{K}}\xrightarrow{\boxtimes}\mathbf{Loc}_{\mathbb{K}}$$

has the following properties:

(i) *It is a homotopical functor, in that it sends weak equivalences in* $\mathbf{Loc}_{\mathbb{K}}\times\mathbf{Loc}_{\mathbb{K}}$ *to weak equivalences in* $\mathbf{Loc}_{\mathbb{K}}$*:*

$$\mathrm{W}\boxtimes\mathrm{W}\subset\mathrm{W}, \tag{122}$$

hence it passes immediately to its derived functor.

(ii) *It is "linear-componentwise a Quillen bifunctor" in that it satisfies the pushout-product axiom* (158) *on linear component maps:*

$$
\begin{array}{lcl}
\phi_{\mathbf{f}}\in\mathrm{Cof}_{\mathbf{f}},\ \gamma_{\mathbf{g}}\in\mathrm{Cof}_{\mathbf{g}} & \vdash & (\phi_{\mathbf{f}})\widehat{\boxtimes}(\gamma_{\mathbf{g}})\in\mathrm{Cof}_{\mathbf{f}\widehat{\times}\mathbf{g}} \\
\phi_{\mathbf{f}}\in\mathrm{Cof}_{\mathbf{f}},\ \gamma_{\mathbf{g}}\in(\mathrm{Cof}\cap\mathrm{W})_{\mathbf{g}} & \vdash & (\phi_{\mathbf{f}})\widehat{\boxtimes}(\gamma_{\mathbf{g}})\in(\mathrm{Cof}\cap\mathrm{W})_{\mathbf{f}\widehat{\times}\mathbf{g}}\,.
\end{array}
\tag{123}
$$

In particular, for fixed base objects this means that:

$$\mathbf{X},\mathbf{Y}\in\mathrm{sSet\text{-}Grpd}_{\mathrm{skl}}\quad\vdash\quad\mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}\times\mathbf{sCh}^{\mathbf{Y}}_{\mathbb{K}}\xrightarrow{\boxtimes}\mathbf{sCh}^{\mathbf{X}\times\mathbf{Y}}_{\mathbb{K}}\ \textit{ is a Quillen bifunctor.} \tag{124}$$

(iii) *It is a left Quillen functor when restricted in one argument to local systems over a discrete space, hence with right Quillen adjoint* (115)*:*

$$
\begin{array}{l}
\mathrm{X}\in\mathrm{Set}\hookrightarrow\mathrm{sSet\text{-}Grpd}, \\
\mathcal{V}_{\mathrm{X}}\in\mathbf{sCh}^{X}_{\mathbb{K}}
\end{array}
\quad\vdash\quad
\begin{tikzcd}
\mathbf{Loc}_{\mathbb{K}} \arrow[r, shift left=1.5, "\mathcal{V}_{\mathrm{X}}\boxtimes(\text{-})"] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathbf{Loc}_{\mathbb{K}} \arrow[l, shift left=1.5, "\mathcal{V}_{\mathrm{X}}\multimap\!\!\!\Box(\text{-})"]
\end{tikzcd}\,.
$$

Proof. **(i)** By symmetry of the external tensor product and definition of product categories, it is sufficient to check that external tensor product $\mathcal{W}_{\mathbf{X}}\boxtimes(-)$ with a fixed object preserves weak equivalences in the other argument. Now, due to the objectwise definition of the weak equivalences and fibrations in the projective model structure on each $\mathbf{sCh}^{\mathbf{Y}}_{\mathbb{K}}$, the precomposition functors $\mathbf{f}^*$ preserve fibrant replacements. Therefore a morphism $\phi_{\mathbf{f}}:\mathcal{V}_{\mathbf{X}}\longrightarrow\mathcal{V}'_{\mathbf{X}'}$ is an integral weak equivalence (Def. A.22) iff $\mathbf{f}:\mathbf{X}\longrightarrow\mathbf{X}'$ is a weak equivalence in sSet-Grpd and $\phi:\mathcal{V}\longrightarrow\mathbf{f}^*\mathcal{V}$ is a weak equivalence in $\mathbf{sCh}^{\mathbf{X}}_{\mathbb{K}}$. Given such an integral weak equivalence, we need to see that also:

(a) $\mathbf{f}\times\mathrm{id}_{\mathbf{Y}}:\mathbf{X}\times\mathbf{Y}\to\mathbf{X}'\times\mathbf{Y}$ is a weak equivalence in sSet-Grpd, hence a Dwyer-Kan equivalence. This is hom-wise the condition that Cartesian product with a simplicial set preserves weak equivalences of simplicial sets, which is the case (for instance by Ken Brown's Lemma A.19 using that sSet is cartesian monoidal model and all objects are cofibrant).
(Beware that even so $\mathbf{X}\times(-)$ is homotopical, it is far from being left Quillen on sSet-Grpd.)

(b) $\big((\mathrm{pr}_{\mathbf{X}})^*\phi\big) \otimes \mathrm{id}_{(\mathrm{pr}_{\mathbf{Y}})^*\mathcal{W}}$ is a weak equivalence in $\mathbf{sCh}_{\mathbb{K}}^{\mathbf{X}}$.
First, $(\mathrm{pr}_{\mathbf{X}})^*\phi$ itself is a weak equivalence, since these are defined objectwise and hence preserved by the precomposition functors $(-)^*$. Similarly, the tensor product over $\mathbf{X} \times \mathbf{Y}$ is a weak equivalence if and only if for all $(x,y) \in \mathbf{X} \times \mathbf{Y}$ its component $\phi_x \otimes \mathrm{id}_{\mathcal{W}_y}$ is a weak equivalence, hence if tensoring $\mathcal{W} \otimes (-)$ preserves weak equivalences in $\mathbf{sCh}_{\mathbb{K}}$. This follows by Ken Brown's Lemma A.19 since $\mathcal{W} \otimes (-)$ is a left Quillen functor on and all objects are cofibrant in $\mathbf{sCh}_{\mathbb{K}}$, by Prop. 3.3.

(ii) By Prop. 3.33, we are immediately reduced to proving the special case (124). For this, it is sufficient to check the pushout-product axiom on generating (acyclic) cofibrations. But for these (60), the relevant diagram (where we are abbreviating representable simplicial copresheaves by $\underline{x} := \mathbf{X}(x,-) : \mathbf{X} \longrightarrow \mathbf{sSet}$)

$$\begin{tikzcd}
\big(\underline{x}\cdot\mathcal{V}\big) \boxtimes \big(\underline{y}\cdot\mathcal{W}\big) \arrow[r, "\mathrm{Id}\boxtimes(\mathrm{id}\cdot g)"] \arrow[d, "(\mathrm{id}_{\underline{x}}\cdot f)\boxtimes\mathrm{Id}"'] \arrow[dr, phantom, "\text{(po)}"] & \big(\underline{x}\cdot\mathcal{V}\big) \boxtimes \big(\underline{y}\cdot\mathcal{W}'\big) \arrow[d] \arrow[ddr, bend left, "(\mathrm{id}_{\underline{x}}\cdot g)\boxtimes\mathrm{Id}"] & \\
\big(\underline{x}\cdot\mathcal{V}'\big) \boxtimes \big(\underline{y}\cdot\mathcal{W}\big) \arrow[r] \arrow[drr, bend right, "\mathrm{Id}\boxtimes(\mathrm{id}_{\underline{y}}\cdot g)"'] & \big(\mathrm{id}_{\underline{x}}\cdot f\big)\widehat{\boxtimes}\big(\mathrm{id}_{\underline{y}}\cdot g\big) \arrow[dr, dashed] & \\
& & \big(\underline{x}\cdot\mathcal{V}'\big) \boxtimes \big(\underline{y}\cdot\mathcal{W}'\big)
\end{tikzcd}$$

reduces to the tensoring with $\underline{(x,y)}$ of the analogous diagram for the tensor product in $\mathbf{sCh}_{\mathbb{K}}$:

$$\begin{tikzcd}
\underline{(x,y)}\cdot\mathcal{V}\otimes\mathcal{W} \arrow[r, "\mathrm{id}\cdot\mathrm{id}\otimes g"] \arrow[d, "\mathrm{id}\cdot f\otimes\mathrm{id}"'] \arrow[dr, phantom, "\text{(po)}"] & \underline{(x,y)}\cdot\mathcal{V}\otimes\mathcal{W}' \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\cdot f\otimes\mathrm{id}"] & \\
\underline{(x,y)}\cdot\mathcal{V}'\otimes\mathcal{W} \arrow[r] \arrow[drr, bend right, "\mathrm{id}\cdot\mathrm{id}\otimes g"'] & \underline{(x,y)}\cdot(f\,\widehat{\otimes}\,g) \arrow[dr, dashed] & \\
& & \underline{(x,y)}\cdot\mathcal{V}'\otimes\mathcal{W}'
\end{tikzcd}
\quad = \quad
\underline{(x,y)}\cdot\left(
\begin{tikzcd}
\mathcal{V}\otimes\mathcal{W} \arrow[r, "\mathrm{id}\otimes g"] \arrow[d, "f\otimes\mathrm{id}"'] \arrow[dr, phantom, "\text{(po)}"] & \mathcal{V}\otimes\mathcal{W}' \arrow[d] \arrow[ddr, bend left, "f\otimes\mathrm{id}"] & \\
\mathcal{V}'\otimes\mathcal{W} \arrow[r] \arrow[drr, bend right, "\mathrm{id}\otimes g"'] & f\,\widehat{\otimes}\,g \arrow[dr, dashed] & \\
& & \mathcal{V}'\otimes\mathcal{W}'
\end{tikzcd}
\right).$$

Since $(\mathbf{sCh}_{\mathbb{K}}, \otimes)$ is sSet-enriched monoidal model (by Prop. 3.3), the tensoring $\underline{(x,y)} \cdot (-)$ on the right is a left Quillen functor and hence the claim follows.

(iii) The previous item (ii) establishes that the only obstacle to $\boxtimes$ being a left Quillen bifunctor is the failure of the underlying Cartesian product being left Quillen bifunctorial in sSet-Grpd. But restricted in one argument to discrete groupoids (sets) it is by Lem. 3.39, and so claim (iii) follows. $\square$

Remark 3.41 (Comparison to model categories of parametrized spectra)**.** Thm. 3.40 may be compared to analogous discussion for parameterized plain spectra (i.e. for $\mathbb{S}$-module spectra instead of the $H\mathbb{K}$-module spectra equivalently discussed here): An analogue of (122) is [Mal23, Rem. 6.1.4] (which, being a little weaker than the statement here, is referred to there as the "perhaps most convenient category of parameterized spectra"); the analog of (124) appears as [Mal19, Lem. 6.4.3][Mal23, Lem. 5.4.5], following [MS06]. See also the discussion in §4.

4 Conclusion and outlook

The derived external tensor product as a pushout. The underlying sSet-enriched category $\mathbf{Loc}_{\mathbb{K}}$ of simplicial local systems (Thm. 3.22) equipped with the simplicial external tensor product (Def. 3.29) is conceptually a straightforward generalization of the hq-distributive monoidal category of ordinary local systems from Prop. 2.26, under passage from groupoids (homotopy 1-types) to general simplicial sets (general homotopy types): The simplicial version of the external tensor product still distributes over coproducts (Prop. 3.35) and is compatible with the homotopy-quotient nature of simplicial group representations (Prop 3.38).

Accordingly, there is an evident enhancement of the pushout diagram (39) to sSet-enriched (hq-distributive) monoidal categories

$$
\begin{tikzcd}
\big(\mathbf{sCh}_{\mathbb{K}}, \otimes\big) \arrow[r, "\iota"] & \big(\mathbf{sCh}_{\mathbb{K}}, \sqcup^{hq}, \boxtimes\big) \\
\mathbf{sCh}_{\mathbb{K}} \arrow[u] \arrow[r, "\iota"] \arrow[ur, phantom, "\text{(po)}"] & \big(\mathbf{Loc}_{\mathbb{K}}, \sqcup^{hq}\big) \arrow[u]
\end{tikzcd}
\tag{125}
$$

and that this is again a pushout follows verbatim with the same proof as of Thm. 2.31 subject only to the evident substitutions. The main Theorem 3.40 serves to show that the external tensor product $\boxtimes$ appearing via this pushout does have appropriate homotopy-theoretic properties. (More ambitiously one would want to say and show that the above pushout diagram is a *homotopy pushout* in a suitable model category of monoidal simplicial model categories, but this is well beyond the scope of the present article.)

As such, the pushout (125) provides a quite comprehensive answer to the question (1). In fact, it goes a long way towards providing "categorical semantics" for the recently presented homotopically-typed quantum programming language `LHoTT`:

Categorical semantics for Linear Homotopy Type Theory. Informally speaking, higher categorical semantics for the homotopically-typed quantum programming language `LHoTT` [MRSS23] ought to be given [Sc14][RFL21] by ∞-categories of parameterized R-module spectra (the "R-linearized tangent ∞-toposes") regarded as equipped with the bireflective sub-∞-category of plain homotopy types and as doubly closed monoidal (Rem. 2.15) with respect to (external-)cartesian and external R-tensor product. But available notions of formalization of this idea (for pointers see [MeSS23, around (107)]) must proceed through suitable model 1-categories which on the one hand present such ∞-categories under simplicial localization while on the other hand compatibly providing ordinary 1-categorical semantics (the only kind of categorical semantics currently understood) for the type theory.

The model category $\mathbf{Loc}_{\mathbb{K}}$ constructed in Thm. 3.22 and equipped with the homotopical external tensor product of Thm. 3.40 provides such semantics for the "multiplicative fragment" of `LHoTT` (essentially the content of [Ri22, §2.4] as envisioned in [Sc14, §3.2]) and (only) rudimentarily for the remaining classical cartesian fragment (cf. Prop. 3.36). The reason for (both of) these is the use of the model of classical base types by sSet-enriched groupoids instead of simplicial sets:

$$
\begin{array}{l}\text{classical}\\ \text{homotopy types}\\ \text{(Prop. 3.20)}\end{array}
\quad
\begin{tikzcd}
\mathrm{sSet\text{-}Grpd} \arrow[r, hook, "\perp_{\mathrm{Qu}}"'] & \mathbf{Loc}_{\mathbb{K}} \arrow[l, shift right=3ex] \arrow[l, shift left=3ex, "\perp_{\mathrm{Qu}}"'] \arrow[loop right, "\natural"]
\end{tikzcd}
\quad
\begin{array}{l}\text{classical modality on}\\ \text{parameterized quantum}\\ \text{homotopy types}\end{array}
\tag{126}
$$

This choice makes the theory of parameterized quantum homotopy types modeled as simplicial local systems *over* classical homotopy types flow naturally via simplicial model category theory, but the model category sSet-Grpd (in contrast to sSet) is not itself cartesian monoidal model: only its Set-tensoring remains a Quillen bifunctor. (In the Quillen equivalent model $\mathbf{Loc}_{\mathbb{K}}^{\mathrm{sSet}}$ (84) the situation is complementary: Here the pushout-product axiom holds on the underlying base types, but now it fails for the linear components since the fundamental simplicial groupoid functor $\mathbf{G}(-)$ does not preserve products.)

On the other hand, discrete parameter base spaces are all that traditional quantum information theory has ever used so far, so that the semantics for the homotopically-multiplicative & rudimentarily-classical fragment of

LHoTT provided by $\mathbf{Loc}_{\mathbb{K}}$ is still a considerable homotopy-theoretic generalization of previously existing models of dependent linear types (cf. [RS18][FKS20] and Rem 2.6). In particular, $\mathbf{Loc}_{\mathbb{K}}$ should be more than sufficient for modeling the quantum programming with topological quantum gates that is laid out in [MeSS23]; cf. Ex. 3.37.

Nevertheless, ultimately one will want stronger model categorical semantics of LHoTT. An alternative route toward this goal might be the setup of [Mal23] following [MS06] (cf. Rem. 3.41), another might be the external-monoidal model category of parameterized spectra presented in [HSS20]: The latter does seem to satisfy the *monoid axiom* (though this would need to be scrutinized) which should imply that for a monoid object R over a contractible base type in this category, the corresponding model structure on R-modules should exist, would present parameterized R-module spectra, and might plausibly provide semantics for a larger or even complete fragment of LHoTT. On the other hand, even if this works, ordinary vector bundles would be recognizable in this model only through the "stable monoidal Dold-Kan correspondence" of [Sh07] needed to identify $R := H\mathbb{K}$-module spectra with $\mathbb{K}$-chain complexes, which possibly would make it unwieldy to use in the practice of quantum information theory.

The construction we presented here is designed to avoid these complications with module spectra by building the intended semantics in actual (chain complex of) vector spaces right into the model. We hope to further discuss, develop and refine this model elsewhere.

Hilbert space structure via local systems. One interesting application of local systems (§2.2) as higher homotopy models for quantum data types, which we highlight by way of outlook, is that they allow implementing Hermitian inner product structure, hence finite-dimensional Hilbert space structure and hence "dagger-structure", in the linear type theory. This is of course absolutely crucial for a full model of quantum information theory, but is missing from existing non-homotopic models of dependent linear types. We briefly explain how this works:

Over the ground field of real numbers, $\mathbb{K} := \mathbb{R}$, consider the complex numbers as a real vector space equipped with the linear involution given by complex conjugation,

$$\begin{array}{ccc} \mathbf{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{Mod}_{\mathbb{R}} \\ \mathrm{pt} & \mapsto & \mathbb{C} \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \overline{(-)}} \\ \mathrm{pt} & \mapsto & \mathbb{C}\,, \end{array}$$

hence as a linear $\mathbb{Z}_2$-representation $\mathbb{Z}_2 \circlearrowright \mathbb{C} \in \mathrm{Rep}_{\mathbb{R}}(\mathbb{Z}_2)$ and hence (4) (32) as a local system, which we want to regard in the slice category of local systems over $0_{\mathbf{B}\mathbb{Z}_2}$ (Ex. 3.26):

$$\mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \;\in\; (\mathrm{Loc}_{\mathbb{R}})_{/0_{\mathbf{B}\mathbb{Z}_2}}\,.$$

Since the multiplication of complex numbers is equivariant under complex conjugation ($\overline{z \cdot w} = \overline{z} \cdot \overline{w}$) and its unit is invariant ($\overline{1} = 1$), it makes this local system into a *monoid object* (algebra object) in the slice category understood with its fiberwise tensor product:

$$\begin{array}{ccc} \mathbb{R}_{\mathbf{B}\mathbb{Z}_2} & \xrightarrow{\text{unit}} & \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \\ \mathbb{R} & \hookrightarrow & \mathbb{C} \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \overline{(-)}} \\ \mathbb{R} & \hookrightarrow & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \otimes_{\mathbf{B}\mathbb{Z}_2} \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} & \longrightarrow & \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \big\downarrow{\scriptstyle \overline{(-)} \otimes_{\mathbb{R}} \overline{(-)}} & & \big\downarrow{\scriptstyle \overline{(-)}} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array} \tag{127}$$

and thus we may ask for *module objects* over this algebra object. To that end, consider a finite-dimensional complex vector space $\mathcal{H}$ equipped with a Hermitian inner product $\langle - | - \rangle$ and its induced complex *anti*-linear isomorphism to its dual vector space

$$\begin{array}{ccc} \mathcal{H} & \xleftrightarrow{(-)^\dagger} & \mathcal{H}^* \\ |\psi\rangle :\equiv \psi & \mapsto & \langle \psi | - \rangle \equiv: \langle \psi | \end{array} \tag{128}$$

The fundamental reason that the literature on categorical quantum theory invokes †-category structure on the category of Hilbert spaces is that these isomorphisms (128) are *not* $\mathbb{C}$-linear, hence not actually morphisms in $\mathrm{Mod}_{\mathbb{C}}$, hence need to be formulated by some extra structure.

Our observation is that, in the homotopy-theoretic context, this extra structure need not be imposed "by hand" but is automatically provided. Namely, (128) is of course $\mathbb{R}$-linear and hence induces an involution on the underlying real vector space of the direct sum of $\mathcal{H}$ with its Hermitian dual:

$$\begin{array}{ccc} \mathbf{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{Mod}_{\mathbb{R}} \\ \mathrm{pt} & \mapsto & \mathcal{H} \oplus \mathcal{H}^* \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle (-)^\dagger} \\ \mathrm{pt} & \mapsto & \mathcal{H} \oplus \mathcal{H}^* \end{array}$$

and as such realizes Hermitian vector spaces as local systems in the slice over $0_{\mathbf{B}\mathbb{Z}_2}$:

$$(\mathcal{H} \oplus \mathcal{H}^*)_{\mathbf{B}\mathbb{Z}_2} \;\;\in\;\; (\mathrm{Loc}_{\mathbb{R}})_{/0_{\mathbf{B}\mathbb{Z}_2}}$$

and in fact as module objects over $\mathbb{C}_{\mathbf{B}\mathbb{Z}_2}$ (127):

$$\begin{array}{ccc} \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \otimes_{\mathbf{B}\mathbb{Z}_2} \big(\mathcal{H} \oplus \mathcal{H}^*\big)_{\mathbf{B}\mathbb{Z}_2} & \longrightarrow & \big(\mathcal{H} \oplus \mathcal{H}^*\big)_{\mathbf{B}\mathbb{Z}_2} \\ \big(c, |\psi\rangle\big) & \mapsto & c|\psi\rangle \\ \updownarrow & & \updownarrow \\ \big(\bar{c}, \langle\psi|\big) & \mapsto & \bar{c}\langle\psi| \end{array}$$

But now observe that the $\mathbb{C}_{\mathbf{B}\mathbb{Z}_2}$-linear morphisms between these slice local systems of $\mathbb{R}$-modules are in natural bijection with $\mathbb{C}$-linear maps $\mathcal{H} \xrightarrow{A} \mathcal{K}$ between the underlying complex vector spaces, due to their $\mathbb{Z}_2$-equivariance:

$$\begin{array}{ccc} \mathcal{H} \oplus \mathcal{H}^* & \longrightarrow & \mathcal{K} \oplus \mathcal{K}^* \\ |\psi\rangle & \mapsto & A|\psi\rangle \\ \updownarrow & & \updownarrow \\ \langle\psi| & \mapsto & \langle\psi|A^\dagger\,. \end{array} \tag{129}$$

This means that we have obtained a full embedding of the category of finite-dimensional Hermitian complex vector spaces into a slice of local systems of real vector spaces

$$\begin{array}{ccc} \mathrm{HermMod}^{\mathrm{fd}}_{\mathbb{C}} & \hookrightarrow & \mathrm{Mod}_{(\mathbb{C}_{\mathbf{B}\mathbb{Z}_2})}\big((\mathrm{Loc}_{\mathbb{R}})_{/0_{\mathbf{B}\mathbb{Z}_2}}\big) \\ \big(\mathcal{H}, \langle\text{-}|\text{-}\rangle\big) & \mapsto & \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \;:\equiv\; \big(\mathcal{H} \oplus \mathcal{H}^*\big)_{\mathbf{B}\mathbb{Z}_2} \end{array}$$

where the canonical †-structure on the left is now embodied in the $\mathbb{Z}_2$-equivariance on the right.

Concretely, after regarding them as local systems this way, the finite-dimensional Hermitian vector spaces become self-dual objects, with evaluation map

$$\begin{array}{ccc} \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \otimes_{\mathbf{B}\mathbb{Z}_2} \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} & \xrightarrow{\mathrm{ev}} & \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} \\ |\psi\rangle\langle\phi| & \mapsto & \langle\phi|\psi\rangle \\ \updownarrow & & \updownarrow \\ |\phi\rangle\langle\psi| & \mapsto & \langle\psi|\phi\rangle \end{array} \tag{130}$$

This allows for characterizing *unitary* maps (and hence unitary quantum gates) in the category of local systems (something that ordinarily requires extra dagger-structure) as those maps (129) which preserve this evaluation map (130):

$$\mathcal{H} \xrightarrow{A} \mathcal{K} \;\; \text{is unitary} \qquad \Leftrightarrow \qquad \begin{tikzcd} \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \otimes_{\mathbf{B}\mathbb{Z}_2} \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \arrow[rr, "\underline{A} \otimes_{\mathbf{B}\mathbb{Z}_2} \underline{A}"] \arrow[dr, "\mathrm{ev}"'] & & \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \otimes_{\mathbf{B}\mathbb{Z}_2} \underline{\mathcal{H}}_{\mathbf{B}\mathbb{Z}_2} \arrow[dl, "\mathrm{ev}"] \\ & \mathbb{C}_{\mathbf{B}\mathbb{Z}_2} & \end{tikzcd} .$$

Proceeding in this manner one finds that the homotopical categorical semantics of parameterized quantum data types constructed here also serves to encode quantum circuits of density matrices and mixed quantum states, an important generalization that previous models of dependent linear/quantum data types had been missing.

Last but not least, as the reader familiar with topological K-theory may have noticed, the structures used in this argument are fundamentally those of twisted "Real" K-theory on spaces equipped with involutions. We believe that this is not a coincidence but a further aspect of the profound relation between topological K-theory and quantum physics and hope to expand on this elsewhere.

A Appendix: Some definitions and facts

For reference, we record some basic facts from the literature and highlight some immediate examples that we use in the main text.

Categories, groupoids and simplicial enrichment. We use basic concepts from category theory (e.g. [ML97]) and enriched category theory (e.g. [Ke82]).

Definition A.1 (Categories and groupoids)**.** With respect to any fixed Grothendieck universe $\mathfrak{U}$ of sets [Schu72, §3.2] of which we assume at least two $\mathfrak{U} < \mathfrak{U}'$, cf. eg. [Le18, p. 4][Sh08, p. 18]:
(i) We write

$$\begin{tikzcd} \mathrm{Grpd} \arrow[r, hook] & \mathrm{Cat} \arrow[l, bend right, "\mathrm{Loc}"'] \arrow[l, bend left, "\mathrm{core}"] \end{tikzcd} \tag{131}$$

for the full inclusion of the 1-category of $\mathfrak{U}$-small groupoids into the 1-category of $\mathfrak{U}$-small categories (e.g. [Schu72, §3]), with left adjoint Loc being the *localization*-construction that universally inverts all morphisms [GZ67, §1.5.4].
(The $\mathfrak{U}'$-small categories are called $\mathfrak{U}$-*large*, whence Cat in this case is "very large" [Sh08, p. 18].
(ii) More generally, we write

$$\begin{tikzcd} \mathrm{sSet\text{-}Grpd} \arrow[r, hook] & \mathrm{sSet\text{-}Cat} \arrow[l, bend right, "\mathrm{Loc}"'] \arrow[l, bend left, "\mathrm{core}"] \end{tikzcd} \tag{132}$$

for the categories of $\mathscr{V}$-enriched categories [Ke82] over [DK80] the category $\mathscr{V} = \mathrm{sSet}$ of simplicial sets [GZ67, §II] (review includes [Ri14]) and for the enriched groupoids [EPR21, §3] among these, traditionally regarded as "simplicial groupoids" with discrete simplicial sets of objects [DK80, §5.5][DK84][GJ99, §V.7][Ja15, §9.3]. Here the three functors in (132) are degreewise those of (131), cf. [MRZ23, Def. 2.7].

Proposition A.2 (Cartesian closure of sSet-enriched groupoids)**.** *Both* sSet-Cat *and* sSet-Grpd *are cartesian closed [ML97, §IV.6], with cartesian product given by forming enriched product categories [Ke82, §1.4] and internal hom given by enriched functor categories [Ke82, §2.2].*

Proof. For sSet-Cat this is the statement of [Ke82, §2.3]. One readily checks that both constructions restrict to simplicial groupoids. ☐

Pseudofunctors and the Grothendieck construction. Given a "coherent system of categories and functors" – namely a pseudo-functorial diagram of categories, Def. A.3 below – the *Grothendieck construction* (Def. A.4 below) is the natural way of merging this data into a single category whose morphisms subsume those of the individual categories but also transfers from one category to the other along one of the given functors.

Definition A.3 (Pseudofunctor [Gr60, §A.1], cf. [Vi05, Def. 3.10])**.**
(i) For $\mathscr{B}$ a category, a *covariant pseudofunctor* to Cat

$$\begin{array}{cccc} \mathbf{C}_{(-)} : & \mathscr{B} & \longrightarrow & \mathrm{Cat} \\ & X_1 & \longmapsto & \mathbf{C}_{X_1} \\ & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_!} \\ & X_2 & \longmapsto & \mathbf{C}_{X_2} \end{array} \tag{133}$$

is an assignment that sends

- each object $B \in \mathrm{Obj}(\mathscr{B})$ to a category $\mathbf{C}_B$,
- each morphism $f : X_0 \to X_1$ to a functor $f_! : \mathbf{C}_{X_0} \to \mathbf{C}_{X_1}$,

- each pair of composable morphisms $X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2$ to a natural isomorphism $(f_{12})_! \circ (f_{01})_! \Rightarrow (f_{12} \circ f_{01})_!$

$$
\begin{tikzcd}
 & X_1 \arrow[dr, "f_{12}"] & \\
X_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] \arrow[rr, phantom, shift left=3, "/\!/"] & & X_2
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
 & \mathbf{C}_{X_1} \arrow[dr, "(f_{12})_!"] & \\
\mathbf{C}_{X_0} \arrow[ur, "(f_{01})_!"] \arrow[rr, "(f_{02})_!"'] \arrow[rr, phantom, shift left=3, "\mu_{f_{01},f_{12}} \Downarrow \wr"] & & \mathbf{C}_{X_2}
\end{tikzcd}
\tag{134}
$$

- and, finally, each identity morphism $\mathrm{id}_X : X \to X$ to a natural isomorphism $(\mathrm{id}_X)_! \Rightarrow \mathrm{id}_{\mathbf{C}_X}$

$$
X \xrightarrow{\ \mathrm{id}_X\ } X \quad \longmapsto \quad
\begin{tikzcd}
\mathbf{C}_X \arrow[r, bend left, "(\mathrm{id}_B)_!"] \arrow[r, "\mathrm{id}_{\mathbf{C}_X}"'] \arrow[r, phantom, shift left=2, "\Downarrow\wr"] & \mathbf{C}_X
\end{tikzcd}
$$

such that these natural isomorphisms satisfy evident associativity and unitality coherences.

(ii) Similarly, a contravariant pseudofunctor is such a pseudofunctor on the opposite category $\mathscr{B}^{\mathrm{op}}$.

$$
\begin{array}{rccc}
\mathbf{C}_{(-)} : & \mathscr{B}^{\mathrm{op}} & \longrightarrow & \mathrm{Cat} \\
 & X_1 & \longmapsto & \mathbf{C}_{X_1} \\
 & \big\downarrow f & & f^* \big\uparrow \\
 & X_2 & \longmapsto & \mathbf{C}_{X_2}
\end{array}
\tag{135}
$$

Definition A.4 (Grothendieck construction [Gr71, §VI.8], cf. [Vi05, §3.1.3])**.**

(i) The *Grothendieck construction* on a covariant pseudofunctor $\mathbf{C}_{(-)} : \mathscr{B} \longrightarrow \mathrm{Cat}$ (133) is the category $\int_{X \in \mathscr{B}} \mathbf{C}_X$ whose

- objects $\mathcal{V}_X$ are pairs $(X, \mathcal{V})$ with $X \in \mathrm{Obj}(\mathscr{B})$ and $\mathcal{V} \in \mathrm{Obj}(\mathbf{C}_X)$,
- morphisms $\phi_f : \mathcal{V}_X \longrightarrow \mathcal{W}_Y$ are pairs (f, ϕ) with $f : X \longrightarrow Y$ in $\mathscr{B}$ and $\boxed{\phi : f_! \mathcal{V} \longrightarrow \mathcal{W} \text{ in } \mathbf{C}_Y}$,

hence the hom-sets of the covariant Grothendieck construction are these dependent products:

$$
\Big(\int_{\mathscr{B}} \mathbf{C}\Big)\Big(\mathcal{V}_X, \mathcal{W}_Y\Big) \;:\equiv\; \big(f \in \mathscr{B}(X,Y)\big) \times \mathbf{C}_Y\big(f_! X, Y\big). \tag{136}
$$

(ii) Dually, the *Grothendieck construction* on a contra-variant pseudofunctor $\mathbf{C}_{(-)} : \mathscr{B}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ (135) is the category $\int_{X \in \mathscr{B}} \mathbf{C}_X$ whose

- objects $\mathcal{V}_X$ are pairs $(X, \mathcal{V})$ with $X \in \mathrm{Obj}(\mathscr{B})$ and $\mathcal{V} \in \mathrm{Obj}(\mathbf{C}_X)$,
- morphisms $\phi_f : \mathcal{V}_X \longrightarrow \mathcal{W}_Y$ are pairs (f, ϕ) with $f : X \longrightarrow Y$ in $\mathscr{B}$ and $\boxed{\phi : \mathcal{V} \longrightarrow f^* \mathcal{W} \text{ in } \mathbf{C}_X}$,

hence the hom-sets of the contravariant Grothendieck construction are these dependent products:

$$
\Big(\int_{\mathscr{B}} \mathbf{C}\Big)\Big(\mathcal{V}_X, \mathcal{W}_Y\Big) \;:\equiv\; \big(f \in \mathscr{B}(X,Y)\big) \times \mathbf{C}_Y\big(X, f^* Y\big). \tag{137}
$$

(iii) Finally, composition of morphisms $\mathcal{V}_X \xrightarrow{\phi_f} \mathcal{W}_Y \xrightarrow{\psi_g} \mathcal{R}_Z$ in the Grothendieck construction is defined by using the pseudo-functoriality of $\mathbf{C}_{(-)}$ to coherently push (or pull) morphisms into the codomain or domain category:

$$
\psi_g \circ \phi_f \;:=\; \big(\psi \circ g_!(\phi) \circ \mu_{f,g}(\mathcal{V})\big)_{g \circ f} \qquad \text{or} \qquad \psi_g \circ \phi_f \;:=\; \big(\mu_{f,g}(\mathcal{R}) \circ g^*(\psi) \circ \phi\big)_{g \circ f}.
$$

Here one is using the coherence isomorphisms (134) to adjust for the identification of composite functors:

$$
(g \circ f)_!(\mathcal{V}) \underset{\sim}{\xrightarrow{\mu_{f,g}}} g_!\big(f_!(\mathcal{V})\big) \xrightarrow{g_!(\phi)} g_!\big(\mathcal{W}\big) \xrightarrow{\ \psi\ } \mathcal{R} \qquad \text{or} \qquad \mathcal{V} \xrightarrow{\ \phi\ } f^*\big(\mathcal{W}\big) \xrightarrow{f^*(\psi)} f^*\big(g^*(\mathcal{R})\big) \xrightarrow{\mu_{f,g}} (g \circ f)^* \mathcal{R}\,.
$$

Remark A.5 (Grothendieck fibration)**.** A key aspect of the Grothendieck construction is that it is a *fibered category* over the original diagram shape, and as such an equivalent incarnation of the pseudo-functor that induced it. While important, here we do not need this aspect and will regard the Grothendieck construction as a plain category, this being the domain category of the corresponding Gorthendieck fibration.

Example A.6 (Categories of indexed sets of objects [Bé85, §3], Free coproduct completion [HT95, §2])**.**
For $\mathscr{C}$ any category, there is the contravariant pseudofunctor (Def. A.3) on Set which to a set S assigns the S-fold product category of $\mathscr{C}$ with itself:

$$\mathscr{C} \in \mathrm{Cat} \qquad \vdash \qquad \begin{array}{ccc} \mathrm{Set}^{\mathrm{op}} & \longrightarrow & \mathrm{Cat} \\ S & \longmapsto & \mathrm{Func}(S,\mathscr{C}) \;\equiv\; \mathscr{C}^S \;\simeq\; \prod\limits_{s\in S}\mathscr{C} \\ \big\downarrow{\scriptstyle f} & & \big\uparrow{\scriptstyle f^*} \\ T & \longmapsto & \mathrm{Func}(T,\mathscr{C}) \;\equiv\; \mathscr{C}^T \;\simeq\; \prod\limits_{t\in T}\mathscr{C}\,. \end{array} \tag{138}$$

equivalently, the functor category into $\mathscr{C}$ out of the discrete category on S:

$$\begin{array}{ccc} \mathrm{Func}(S,\mathscr{C}) & \xrightarrow{\;\;\sim\;\;} & \prod\limits_{s\in S}\mathscr{C} \\ (s \mapsto \mathcal{V}_s) & \longmapsto & (\mathcal{V}_s)_{s\in S}\,, \end{array}$$

and whose base change functors are given by precomposition with, hence re-indexing by, the given map of sets:

$$f : S \longrightarrow T \qquad \vdash \qquad \begin{array}{ccc} \mathrm{Func}(S,\mathscr{C}) & \xleftarrow{\;\;f^*\;\;} & \mathrm{Func}(T,\mathscr{C}) \\ \big(\mathcal{V}_{f(s)}\big)_{s\in S} & \longleftarrow\!\shortmid & (\mathcal{V}_t)_{t\in T}\,. \end{array}$$

Accordingly, the Grothendieck construction (Def. A.4) on this pseudofunctor,
$\boxed{\int_{S\in\mathrm{Set}} \mathscr{C}^S}$ has the following description:

- objects $\mathcal{V}_S$ are dependent pairs consisting of a set $S \in \mathrm{Set}$ and an S-tuple $\big(\mathcal{V}_s\big)_{s\in S}$ of objects in $\mathscr{C}$,
- morphisms $\phi_f : \mathcal{V}_S \longrightarrow \mathcal{W}_T$ are S-tuples $\big(\phi_s : \mathcal{V}_s \to \mathcal{W}_{f(s)}\big)_{s\in S}$ of morphisms in $\mathscr{C}$.

Independently of whether or how $\mathscr{C}$ has co-products, this category has set-indexed coproducts $\coprod_i \mathcal{V}(i)_{S_i}$ with underlying set $\coprod_i S_i$ and components $\big(\coprod_i \mathcal{V}(i)_{S_i}\big)_{s_j} = \mathcal{V}(j)_{s_j}$ for $s_j \in S_j$.

But if the category $\mathscr{C}$ is *extensive*, in that it already has coproducts itself and the coproduct-functors between (products of) slice categories are equivalences

$$S \in \mathrm{Set} \qquad \vdash \qquad \begin{array}{ccc} \prod\limits_{s\in S}\mathscr{C}_{/X_s} & \xrightarrow{\;\;\sim\;\;} & \mathscr{C}_{/\coprod_s X_s} \\ \left(\begin{array}{c} E_s \\ \downarrow \\ X_s \end{array}\right)_{s\in S} & \longmapsto & \left(\begin{array}{c} \coprod_s E_s \\ \downarrow \\ \coprod_s X_s \end{array}\right) \end{array}$$

then the construction yields the category of bundles in $\mathscr{C}$ over sets, the latter understood via the unique coproduct-preserving inclusion $\iota_{\mathrm{Set}} : \mathrm{Set} \hookrightarrow \mathscr{C}$, hence the comma category $(\mathrm{id}_{\mathscr{C}}, \iota_{\mathrm{Set}})$:

$$\mathscr{C} \text{ extensive} \qquad \vdash \qquad \int_{S\in\mathrm{Set}} \prod_{s\in S}\mathscr{C} \;\simeq\; (\mathrm{id}_{\mathscr{C}}, \iota_{\mathrm{Set}})$$

whose morphisms $\phi_f : X_S \longrightarrow Y_T$ are commuting diagrams in $\mathscr{C}$ of this form:

$$\begin{array}{ccc} \coprod\limits_{s\in S} X_s & \xrightarrow{\;\;\phi\;\;} & \coprod\limits_{t\in T} Y_t \\ \downarrow & & \downarrow \\ S & \xrightarrow{\;\;f\;\;} & T\,. \end{array}$$

Conversely, if $\mathscr{C}$ is not extensive, then we may understand $\int_{S\in\mathrm{Set}}\mathscr{C}^S$ as the stand-in for the would-be category of "$\mathscr{C}$-fiber bundles" over sets.

Proposition A.7 ([CV98, Lem. 4.2])**.** *A category $\mathscr{C}$ with all set-indexed coproducts each of whose objects is a coproduct of connected objects is the free coproduct completion (Ex. A.6) of its full subcategory of connected objects (i.e., of those objects $X \in \mathscr{C}$ for which $\mathscr{C}(X,-) : \mathscr{C} \to \mathscr{C}$ preserves coproducts).*

Proof. Since, by assumption, every object is already presented by an indexed set of connected objects, it remains to see that also the morphisms $\big(\coprod_s X_s\big) \longrightarrow \big(\coprod_t Y_t\big)$ are in bijection to indexed sets of morphisms of connected objects. This follows by

$$
\begin{aligned}
\mathscr{C}\big(\textstyle\coprod_s X_s, \coprod_t Y_t\big) & \simeq \textstyle\prod_s \mathscr{C}\big(X_s, \coprod_t Y_t\big) \\
& \simeq \prod_{s \in S} \coprod_{t_s \in T} \mathscr{C}\big(X_s, Y_{t_s}\big) \\
& \simeq \coprod_{f : S \to T} \prod_{s \in S} \mathscr{C}\big(X_s, Y_{f(s)}\big),
\end{aligned}
$$

where the first bijection is by general properties of Hom-functors and the second is by the assumption that all X_s are connected. $\square$

Example A.8 (Induced adjunctions between Grothendieck constructions)**.** Given a contravariant pseudofunctor and a left adjoint functor into its domain

$$
\begin{tikzcd}
\mathscr{C} \arrow[r, shift left=1.2ex, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left=1.2ex, "R"]
\end{tikzcd}
\qquad\qquad
\mathscr{B}^{\mathrm{op}} \xrightarrow{\;\mathbf{C}_{(-)}\;} \mathrm{Cat}
$$

there is an induced adjunction between the Grothendieck constructions on $\mathbf{C}_{(-)}$ and on $\mathbf{C}_{L(-)}$, covering the given adjunction:

$$
\begin{tikzcd}
\Big(\int\limits_{c \in \mathscr{C}} \mathbf{C}_{L(c)}\Big) \arrow[r, shift left=1.2ex, "\hat{L}"] \arrow[r, phantom, "\perp"] \arrow[d] & \Big(\int\limits_{b \in \mathscr{B}} \mathbf{C}_b\Big) \arrow[l, shift left=1.2ex, "\hat{R}"] \arrow[d] \\
\mathscr{C} \arrow[r, shift left=1.2ex, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left=1.2ex, "R"]
\end{tikzcd}
\tag{139}
$$

where on components in $\mathbf{C}_{(-)}$ the functor $\widehat{L}$ is the identity while $\widehat{R}$ is pullback along the underlying adjunction counit $\varepsilon^{L \dashv R} : L \circ R \to \mathrm{id}$:

$$
\begin{array}{c} \mathcal{V}_c \\ \Big\downarrow {\scriptstyle \phi_f} \\ \mathcal{V}'_{c'} \end{array}
\;\overset{\widehat{L}}{\longmapsto}\;
\begin{array}{c} \mathcal{V}_{L(c)} \\ \Big\downarrow {\scriptstyle \phi_{L(f)}} \\ \mathcal{V}'_{L(c')} \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \mathcal{V}_b \\ \Big\downarrow {\scriptstyle \phi_f} \\ \mathcal{V}'_{b'} \end{array}
\;\overset{\widehat{R}}{\longmapsto}\;
\begin{array}{c} \mathcal{V}_{R(b)} \\ \Big\downarrow {\scriptstyle (\varepsilon_b^{L \dashv R})^* \phi_{L(f)}} \\ \mathcal{V}'_{R(b')} \end{array}
\tag{140}
$$

The counit of this adjunction is given by the identity component map covering the underlying counit:

$$
\varepsilon^{\widehat{L} \dashv \widehat{R}}_{\mathcal{V}_{\mathbf{x}}} \;:\; \widehat{L}\widehat{R}\big(\mathcal{V}_b\big) \;=\; \big(\varepsilon^{L \dashv R}_{LR(B)} v\big).
\tag{141}
$$

Proposition A.9 (Colimits in a Grothendieck construction [TBG91, §3.2, Thm. 2][HP15, Prop. 2.4.4])**.**
(i) *The Grothendieck construction* $\int_{X \in \mathscr{B}} \mathbf{C}_X$ *(Def. A.4) on a covariant pseudofunctor* $\mathbf{C}_{(-)} : \mathscr{B} \to \mathrm{Cat}$ (A.3) *is cocomplete as soon as the base category* $\mathscr{B}$ *as well as all the fiber categories* $\mathbf{C}_X$*,* $X \in \mathscr{B}$ *are cocomplete. In this case the colimit of a small diagram*

$$
\begin{aligned}
I & \longrightarrow \textstyle\int_{X \in \mathscr{B}} \mathbf{C}_X \\
i & \;\mapsto\;\; \mathcal{V}(i)_{X_i}
\end{aligned}
$$

is given by

$$
\varinjlim_{i \in I} \big(\mathcal{V}(i)_{X_i}\big) \;\;\simeq\;\; \Big(\varinjlim_i \big(q(i)_! \mathcal{V}(i)\big)\Big)_{\varinjlim_i X_i} \;\;\in\; \int_{X \in \mathscr{B}} \mathbf{C}_X \,,
\tag{142}
$$

where

$$
i \in I \qquad \vdash \qquad q(i) \;:\; X_i \longrightarrow \varinjlim_i X_i \;\;\in\; \mathscr{B}
$$

denote the coprojections into the underlying colimit in $\mathscr{B}$*.*
(ii) *The analogous dual statement holds for limits.*

Example A.10 (External cartesian product)**.** Given a contravariant pseudofunctor $\mathbf{C}_{(-)} : \mathscr{B}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ such that both $\mathscr{B}$ as well as all the $\mathbf{C}_{(-)}$ have Cartesian products, then its Gorthendieck construction has cartesian products given by

$$\mathcal{V}_X \times \mathcal{W}_Y \;\simeq\; \Big(\big((\mathrm{pr}_X)^* \mathcal{V}\big) \times \big((\mathrm{pr}_Y)^* \mathcal{W}\big) \Big)_{X \times Y}. \tag{143}$$

More explicitly, the components of the external Cartesian product are

$$\begin{aligned} \big(\mathcal{V}_X \times \mathcal{W}_Y\big)_{(x,y)} &\simeq \{(x,y)\}^* \Big(\big((\mathrm{pr}_X)^* \mathcal{V}\big) \times \big((\mathrm{pr}_Y)^* \mathcal{W}\big) \Big) \\ &\simeq \Big(\big(\{(x,y)\}^* (\mathrm{pr}_X)^* \mathcal{V}\big) \times \big(\{(x,y)\}^* (\mathrm{pr}_Y)^* \mathcal{W}\big) \Big) \\ &\simeq \big(\{x\}^* \mathcal{V}\big) \times \big(\{y\}^* \mathcal{W}\big) \\ &\simeq \mathcal{V}_x \times \mathcal{W}_y \end{aligned}$$

$$\begin{tikzcd} \{x\} \arrow[d] & \{(x,y)\} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, hook] & \{y\} \arrow[d, hook] \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"'] \arrow[r, "\mathrm{pr}_Y"] & Y \end{tikzcd}$$

This gives the following elementary fact, which is crucial in the main text:

Proposition A.11 (Free coproduct completion)**.** *If a category $\mathscr{C}$ has Cartesian products, then its free coproduct completion (Ex. A.17) also has Cartesian products and those distribute* (8) *over the coproducts.*

The 2-category of categories with adjoint functors between them. We extract the gist of the discussion in [ML97, p. 97-103].

Definition A.12 (Conjugate transformation of adjoints [ML97, p. 98])**.** Given a pair of pairs of adjoint functors between the same categories

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "L_i"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left, "R_i"] \end{tikzcd} \qquad i \in \{1,2\},$$

then a *conjugate transformation* between them

$$(\lambda, \rho) \;:\; (L_1 \dashv R_1) \Longrightarrow (L_2 \dashv R_2)$$

is a pair of natural transformations of the form

$$\lambda : L_1 \Rightarrow L_2, \quad \rho : R_2 \Rightarrow R_1$$

such that they make the following square of natural transformations of hom-sets commute, where the horizontal maps refer to the given hom-isomorphisms:

$$\begin{tikzcd} \mathscr{C}\big(L_2(-), -\big) \arrow[r, "\sim"] \arrow[d, "\mathscr{C}\left(\lambda_{(-)}, \mathrm{id}_{(-)}\right)"'] & \mathscr{D}\big(-, R_2(-)\big) \arrow[d, "\mathscr{D}\left(\mathrm{id}_{(-)}, \rho_{(-)}\right)"] \\ \mathscr{C}\big(L_1(-), -\big) \arrow[r, "\sim"] & \mathscr{D}\big(-, R_1(-)\big) \end{tikzcd} \tag{144}$$

Such conjugate transformations compose via composition of their components (λ, ρ), yielding a category of adjoint functors with conjugate transformations between them, which we denote as follows:

$$\mathscr{C}, \mathscr{D} \in \mathrm{Cat} \qquad \vdash \qquad \mathrm{Cat}_{\mathrm{adj}}(\mathscr{C}, \mathscr{D}) \;\in\; \mathrm{Cat}. \tag{145}$$

Proposition A.13 (Uniqueness of conjugate transformations [ML97, p. 98])**.** *Given $L_i \dashv R_i : \mathscr{C} \rightleftarrows \mathscr{D}$ and λ in Def. A.12, there is a* unique *ρ that completes this data to a conjugate transformation. In other words, the forgetful functor from* (145) *to the functor category is a fully faithful sub-category inclusion:*

$$\mathscr{C}, \mathscr{D} \in \mathrm{Cat} \qquad \vdash \qquad \begin{array}{ccc} \mathrm{Cat}_{\mathrm{adj}}(\mathscr{C}, \mathscr{D}) & \hookrightarrow & \mathrm{Cat}(\mathscr{C}, \mathscr{D}) \\ (L_1 \dashv R_1) & \longmapsto & L_1 \\ \big\downarrow{\scriptstyle (\lambda, \rho)} & & \big\downarrow{\scriptstyle \lambda} \\ (L_2 \dashv R_2) & \longmapsto & L_2\,. \end{array} \tag{146}$$

Proposition A.14 (Horizontal composition of conjugate transformations [ML97, p. 102])**.** *The horizontal composition $(-)\cdot(-)$ of the underlying natural transformations of a pair of conjugate transformations Def. A.12 is itself a conjugate transformation, so that the composition functor on functor categories restricts along the inclusions* (146)*:*

$$\mathscr{C},\mathscr{D},\mathscr{E} \in \mathrm{Cat} \qquad \vdash \qquad \begin{array}{ccc} \mathrm{Cat}_{\mathrm{adj}}(\mathscr{D},\mathscr{E}) \times \mathrm{Cat}_{\mathrm{adj}}(\mathscr{C},\mathscr{D}) & \longrightarrow & \mathrm{Cat}_{\mathrm{adj}}(\mathscr{C},\mathscr{E}) \\ \big((\lambda,\rho),(\lambda',\rho')\big) & \longmapsto & \big(\lambda'\cdot\lambda,\rho\cdot\rho'\big). \end{array}$$

Via Prop. A.14, we have:

Definition A.15 (2-category of categories, adjoint functors and conjugate transformations [ML97, p. 102])**.** Write

$$\mathrm{Cat}_{\mathrm{adj}} \longrightarrow \mathrm{Cat} \tag{147}$$

for the (very large) locally full sub-2-category of Cat whose

- objects are categories,
- hom-categories are those (145) of adjoint functors with conjugate transformations between them.

Proposition A.16 (Bivariant pseudofunctors, cf. [Ja98, Lem. 9.1.2][HP15, Prop. 2.2.1][CM20, pp. 10])**.** *Given a covariant pseudofunctor $\mathbf{C}_{(-)}$ (Def. A.3) such that each component functor $f_! : \mathbf{C}_X \longrightarrow \mathbf{C}_Y$ has a right adjoint*

$$\mathbf{C}_{(-)} : \qquad \begin{array}{ccc} \mathscr{B} & \longrightarrow & \mathrm{Cat} \\ X_1 & \longmapsto & \mathbf{C}_{X_1} \\ \big\downarrow{\scriptstyle f} & & {\scriptstyle f_!}\big\downarrow \dashv \big\uparrow{\scriptstyle f^*} \\ X_2 & \longmapsto & \mathbf{C}_{X_2} \end{array} \tag{148}$$

then:

(i) *it factors essentially uniquely through* $\mathrm{Cat}_{\mathrm{adj}}$ (152)*,*
(ii) *hence it induces a contravariant pseudofunctor with component functors f^*,*
(iii) *such that the Grothendieck construction (Def. A.4) on the covariant pseudofunctor is equivalent to that on the corresponding contravariant pseudofunctor via the functor that is the identity on objects and on morphisms is the hom-isomorphism of the given adjoint pairs:*

$$\begin{array}{c} f_!\mathcal{V} \xrightarrow{\ \widetilde{\phi}\ } \mathcal{W} \\ X \xrightarrow{\ f\ } Y \end{array} \qquad \leftrightarrow \qquad \begin{array}{c} \mathcal{V} \xrightarrow{\ \phi\ } f^*\mathcal{W} \\ X \xrightarrow{\ f\ } Y\,. \end{array}$$

Therefore, both construction are still unambiguously denoted by $\int_{X\in\mathscr{B}} \mathbf{C}_X$.

Proof. The first statement is a direct consequence of Prop. A.13, the second then follows by Prop. A.14 and finally the third by the property (144) in Def. A.12. □

In refinement of Ex. A.6, we have:

Example A.17 (Categories of indexed sets of objects with coproducts)**.** If a category $\mathscr{C}$ already has all coproducts, then the pseudofunctor (138) of its product categories has left adjoint component functors given by forming coproducts over fibers of base maps

$$f : S \longrightarrow T \qquad \vdash \qquad \begin{array}{ccc} (\mathcal{V}_s)_{\in S} & \longmapsto & \Big(\coprod_{s\in f^{-1}(\{t\})} \mathcal{V}_s\Big)_{t\in T} \\ \mathrm{Func}\big(S,\mathscr{C}\big) & \underset{f^*}{\overset{f_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp}}}} & \mathrm{Func}\big(T,\mathscr{C}\big) \\ (\mathcal{V}_{f(s)})_{s\in S} & \longleftarrow\!\shortmid & (\mathcal{V}_t)_{t\in T} \end{array} \tag{149}$$

Consequently, here Prop. A.16 says that we have in fact a bivariant pseudofunctor:

$$
\begin{tikzcd}
\mathrm{Set} \arrow[rr] & & \mathrm{Cat}_{\mathrm{adj}} \\
S \arrow[d, "f"] & \longmapsto & \mathrm{Func}(S, \mathscr{C}) \arrow[d, shift right, "f_!"'] \arrow[d, phantom, "\dashv"] & \simeq \prod_{s \in S} \mathscr{C} \\
T & \longmapsto & \mathrm{Func}(T, \mathscr{C}) \arrow[u, shift right, "f^*"'] & \simeq \prod_{t \in T} \mathscr{C} .
\end{tikzcd}
$$

More generally:

Example A.18 (Systems of enriched functor categories)**.** Let $\mathbf{C}$ be an sSet-enriched bicomplete and sSet-(co)tensored category. Then for every sSet-enriched functor $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ between small sSet-enriched categories the precomposition functors $\mathbf{f}^*$ between the sSet-enriched functor categories into $\mathbf{C}$

$$
\mathbf{C}^{(-)} :\equiv \mathbf{sFunc}(\text{-}, \mathbf{C}),
$$

has a left adjoint $\mathbf{f}_!$ and a right adjoint $\mathbf{f}_*$:

$$
\begin{tikzcd}
\mathbf{C}^{\mathbf{X}} \arrow[r, shift left=2, "\mathbf{f}_!"] \arrow[r, shift right=2, "\mathbf{f}_*"'] & \mathbf{C}^{\mathbf{Y}} \arrow[l, "\mathbf{f}^*" description]
\end{tikzcd}, \tag{150}
$$

given by enriched left and right Kan extension, respectively, expressible by the following (co)end formulas [Ke82, (4.24), (4.25)]:

$$
\mathcal{V} \in \mathbf{C}^{\mathbf{X}} \qquad \vdash \qquad \left\{ \begin{array}{l} (f_! \mathcal{V}) \; : \; y \mapsto \int^{x \in \mathbf{X}} \mathbf{Y}\big(\mathbf{f}(x), y\big) \cdot \big(\mathcal{V}_x\big) \\ (\mathbf{f}_* \mathcal{V}) \; : \; y \mapsto \int_{x \in \mathbf{X}} \big(\mathcal{V}_x\big)^{\mathbf{Y}(y, \, \mathbf{f}(x))} \end{array} \right. \tag{151}
$$

In particular, this gives a bivariant pseudo-functor on small sSet-enriched categories:

$$
\begin{tikzcd}
\mathbf{C}^{(-)} \; : \; \mathrm{sSet\text{-}Cat}_{\mathrm{sm}} \arrow[r] & \mathrm{Cat}_{\mathrm{adj}} \\
\mathbf{X} \arrow[d, "\mathbf{f}"] & \mapsto \quad \mathbf{C}^{\mathbf{X}} \arrow[d, shift right, "\mathbf{f}_!"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Y} & \mapsto \quad \mathbf{C}^{\mathbf{Y}} \arrow[u, shift right, "\mathbf{f}^*"']
\end{tikzcd}
$$

Model category theory.

Lemma A.19 (Ken Brown's Lemma)**.** *Let $F : \mathscr{D} \longrightarrow \mathscr{C}$ be a functor between (underlying categories of) model categories. Then:*

(i) *If F sends acyclic fibrations to weak equivalences then it sends all weak equivalences between fibrant objects to weak equivalences.*

(ii) *If F sends acyclic cofibrations to weak equivalences then it sends all weak equivalences between cofibrant objects to weak equivalences.*

As a simple but important special case of right transfer:

Proposition A.20 (Model structure transfer along adjoint equivalence)**.** *Given an adjoint equivalence of categories $\mathscr{D} \underset{\underset{R}{\longrightarrow}}{\overset{\overset{L}{\longleftarrow}}{\bot}} \mathscr{C}$ and a model structure on $\mathscr{C}$, then $\mathscr{D}$ becomes a model category and the adjunction becomes a Quillen equivalence by setting* $\mathrm{W}(\mathscr{D}) \equiv R^{-1}\big(\mathrm{W}(\mathscr{C})\big)$, $\mathrm{Fib}(\mathscr{D}) \equiv R^{-1}\big(\mathrm{Fib}(\mathscr{C})\big)$, $\mathrm{Cof}(\mathscr{D}) \equiv R^{-1}\big(\mathrm{Fib}(\mathscr{C})\big)$.

Model category structures on Grothendieck constructions. We recall the main point of [HP15][CM20], which goes back to [Ro94][St12].

Definition A.21 (2-category of model categories [Ho99, p. 24], cf. [HP15, Def. 2.5.3])**.** Write

$$\mathrm{ModCat} \longrightarrow \mathrm{Cat}_{\mathrm{adj}} \longrightarrow \mathrm{Cat} \tag{152}$$

for the (very large) 2-category whose
- objects are model categories,
- 1-morphisms are Quillen adjunctions regarded in the direction of the left adjoint,
- 2-morphisms are conjugate transformations (Def. A.12) between the underlying adjoint functors,

equipped with its forgetful 2-functor to $\mathrm{Cat}_{\mathrm{adj}}$ (Def. A.15).

Definition A.22 (Integral model structure [HP15, Def. 3.0.4])**.** Given a model category $\mathscr{B}$ and a pseudofunctor (Def. A.3) on $\mathscr{B}$ with values in model categories (Def. A.21)

$$\mathbf{C}_{(-)} : \mathscr{B} \longrightarrow \mathrm{ModCat} \longrightarrow \mathrm{Cat}_{\mathrm{adj}}$$

then we call a morphism

$$\phi_f : \mathcal{V}_X \longrightarrow \mathcal{V}'_{X'} \qquad \in \int_{X \in \mathscr{B}} \mathbf{C}_X$$

in its Grothendieck construction (Def. A.4):

(i) an *integral weak equivalence* if

(a) $f : X \longrightarrow X'$ is a weak equivalence in $\mathscr{B}$,

(b) $f_!(\mathcal{V}^{\mathrm{cof}}) \xrightarrow{f_!(p)} f_!(\mathcal{V}) \xrightarrow{\tilde{\phi}} \mathcal{V}'$ is a weak equivalence in $\mathbf{C}_{X'}$, for $p : \mathcal{V}^{\mathrm{cof}} \to \mathcal{V}$ a cofibrant replacement in $\mathbf{C}_X$,

($\overline{\text{b}}$) which, when $f_! \dashv f^*$ is a Quillen equivalence, is equivalent to:
$\mathcal{V} \xrightarrow{\phi} f^*(\mathcal{V}') \xrightarrow{f^*(q)} f^*(\mathcal{V}'_{\mathrm{fib}})$ is a weak equivalence in $\mathbf{C}_X$, for $q : \mathcal{V}' \to \mathcal{V}'_{\mathrm{fib}}$ a fibrant replacement in $\mathbf{C}_{X'}$;

(ii) an *integral fibration* if

(a) $f : X \longrightarrow X$" is a fibration in $\mathscr{B}$;

(b) $\phi : \mathcal{V} \longrightarrow f^*(\mathcal{V}')$ is a fibration in $\mathbf{C}_X$,

(iii) an *integral cofibration* if

(a) $f : X \longrightarrow X'$ is a cofibration in $\mathscr{B}$,

(b) $\widetilde{\phi} : f_!(\mathcal{V}) \longrightarrow \mathcal{V}$ is a cofibration in $\mathbf{C}_{X'}$.

Proposition A.23 (Existence of integral model structures [HP15, Thm. 3.0.12])**.** *The classes of morphisms in Def. A.22 constitute a model category structure if, given $f : X \to X'$ in $\mathscr{B}$, the following conditions are satisfied:*

(i) *if f is a weak equivalence then $f_! \dashv f^* : \mathbf{C}_X \rightleftarrows \mathbf{C}_{X'}$ is a Quillen equivalence,*

(ii) *if f is an acyclic fibration then $f^* : \mathbf{C}_{X'} \longrightarrow \mathbf{C}_X$ preserves weak equivalences,*

(iii) *if f is an acyclic cofibration then $f_! : \mathbf{C}_X \longrightarrow \mathbf{C}_{X'}$ preserves weak equivalences.*

Example A.24 (Integral model structure over trivial model structure)**.** In the case that the base category $\mathscr{B}$ in Def. A.22 is the "trivial" model structure on a bicomplete category – whose weak equivalences are just the isomorphisms and all whose morphisms are fibrations and cofibrations – then the conditions in Prop. A.23 are satisfied, and hence the integral model structure on any pseudofunctor $\mathbf{C}_{(-)} : \mathscr{B} \longrightarrow \mathrm{ModCat}$ exists.

Lemma A.25 (Enhanced enrichment of functor categories)**.** *Consider*
- $\mathbf{V}$ *a bicomplete symmetric closed monoidal category, regarded as canonically enriched over itself via its internal hom* $[-,-]$*;*
- $(\mathbf{C}, \otimes)$ *a complete* $\mathbf{V}$*-enriched category that is also* $\mathbf{V}$*-(co)tensored;*
- $\mathbf{X}$ *a small* $\mathbf{V}$*-enriched category.*

(i) *Then the enriched functor category* $\mathbf{V}^{\mathbf{X}} := \mathrm{Func}(\mathbf{X},\mathbf{V})$ *carries symmetric closed monoidal category structure with respect to the* $\mathbf{X}$*-objectwise tensor product*

$$\mathscr{S}_{\mathbf{X}}, \mathscr{T}_{\mathbf{X}} \in \mathbf{V}^{\mathbf{X}} \qquad \vdash \qquad \mathscr{S}_{\mathbf{X}} \otimes \mathscr{T}_{\mathbf{X}} \;:=\; \big(x \mapsto \mathscr{S}_x \otimes \mathscr{T}_x\big) \;\;\in\; \mathbf{V}^{\mathbf{T}},$$

whose corresponding internal hom is given by

$$\mathscr{S}_{\mathbf{X}}, \mathscr{T}_{\mathbf{X}} \in \mathbf{V}^{\mathbf{X}} \qquad \vdash \qquad \big[\mathscr{S}_{\mathbf{X}}\,,\,\mathscr{T}_{\mathbf{X}}\big] \;:=\; \Big(x \mapsto \int_{x \in \mathbf{X}} \mathbf{V}\big(\mathbf{X}(x,x') \otimes \mathscr{S}_{x'}\,,\; \mathscr{T}_{x'}\big)\Big) \;\;\in\; \mathbf{V}^{\mathbf{T}}.$$

(ii) *Furthermore,* $\mathbf{C}^{\mathbf{X}} := \mathrm{Func}(\mathbf{X},\mathbf{C})$ *becomes* $\mathbf{V}^{\mathbf{X}}$*-enriched, -tensored and -cotensored via the following end-formulas, respectively:*

$$\begin{aligned}
\mathcal{V}_{\mathbf{X}},\, \mathcal{W}_{\mathbf{X}} \in \mathbf{C}^{\mathbf{X}} \quad &\vdash \quad \mathbf{C}^{\mathbf{X}}\big(\mathcal{V}_{\mathbf{X}},\, \mathcal{W}_{\mathbf{X}}\big) \;:=\; \Big(x \mapsto \textstyle\int_{x' \in \mathbf{X}} \mathbf{C}\big(\mathbf{X}(x,x') \cdot \mathcal{V}_{x'},\, \mathcal{W}_{x'}\big)\Big) \;\;\in\; \mathbf{V}^{\mathbf{X}} \\
\mathscr{S}_{\mathbf{X}} \in \mathbf{V}^{\mathbf{X}},\, \mathcal{W}_{\mathbf{X}} \in \mathbf{C}^{\mathbf{X}} \quad &\vdash \quad \mathscr{S}_{\mathbf{X}} \cdot \mathcal{V}_{\mathbf{X}} \;:=\; \big(x \mapsto \mathscr{S}_x \cdot \mathcal{V}_x\big) \;\;\in\; \mathbf{C}^{\mathbf{X}} \\
\mathscr{S}_{\mathbf{X}} \in \mathbf{V}^{\mathbf{X}},\, \mathcal{W}_{\mathbf{X}} \in \mathbf{C}^{\mathbf{X}} \quad &\vdash \quad \big(\mathcal{V}_{\mathbf{X}}\big)^{\mathscr{S}_{\mathbf{X}}} \;:=\; \Big(x \mapsto \textstyle\int_{x' \in \mathbf{X}} \big(\mathcal{W}_{x'}\big)^{\mathbf{X}(x,x') \cdot \mathscr{S}_{x'}}\Big) \;\;\in\; \mathbf{C}^{\mathbf{X}}.
\end{aligned} \tag{153}$$

Proof. This follows by standard manipulations and may be folklore but hard to cite from the literature; we have spelled out the details at: `ncatlab.org/nlab/show/enriched+functor+category#EnhancedEnrichment`. ☐

sSet**-Enriched categories (aka: simplicial categories).** Most of what we say here applies to enriched categories over more general symmetric monoidal enriching categories than just sSet, but we focus on this case for brevity of notation, since this is what we use in the main text.

Definition A.26 (sSet-enriched monoidal category [BM12, Def. 1][MP19, Def. 2.1][Lu17, §1.6])**.**
A sSet-*enriched monoidal category* $(\mathbf{C},\otimes)$ is an sSet-enriched category equipped with a tensor product given by an sSet-enriched functor

$$X, X', Y, Y' \in \mathrm{Obj}(\mathbf{C}) \qquad \vdash \qquad \mathbf{C}(X,Y) \times \mathbf{C}(X',Y') \xrightarrow{\;\otimes_{X \otimes X', Y \otimes Y'}\;} \mathbf{C}\big(X \otimes X',\, Y \otimes Y'\big)$$

satisfying its coherence laws by sSet-enriched natural isomorphisms.

Monoidal and enriched model categories.

Definition A.27 (Pushout-product)**.** Given a bifunctor $\otimes : \mathscr{D}_1 \times \mathscr{D}_2 \to \mathscr{C}$ into a category with pushouts, then the corresponding *pushout-product* of a pair of morphisms $f : X \to X'$ in $\mathscr{D}_1$ and $g : Y \to Y'$ in $\mathscr{D}_2$ is the universal dashed map in $\mathscr{C}$ given by the following diagram:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, "f \otimes \mathrm{id}"'] \arrow[dr, phantom, "(\mathrm{po})"] & X \otimes Y' \arrow[d, "q_r"] \arrow[ddr, bend left, "f \otimes \mathrm{id}"] & \\
X' \otimes Y \arrow[r, "q_l"] \arrow[drr, bend right, "\mathrm{id} \otimes g"'] & f \,\widehat{\otimes}\, g \arrow[dr, dashed] & \\
& & X' \otimes Y' \,.
\end{tikzcd} \tag{154}$$

Example A.28 (Cartesian pushout-products of sets)**.** In the category Set with respect to the Cartesian product $\mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$, the pushout of $\mathrm{id} \times g$ along $f \times \mathrm{id}$ is the quotient set

$$f \,\widehat{\times}\, g \;\simeq\; \big\{(x,y'),\,(x',y)\big\}\Big/\Big(\big(f(x),y\big) \sim \big(x,g(y)\big)\Big)$$

(where all variables range over the sets denoted by the corresponding capital letters) whose equivalence classes we denote by $[x,y']$ and $[x',y]$, on which the pushout-product map (154) is given by

$$
\begin{array}{ccc}
f \,\widehat{\times}\, g & \dashrightarrow & X' \times Y' \\
{\scriptstyle [x,y']} & \longmapsto & {\scriptstyle \big[f(x),\, y'\big]} \\
{\scriptstyle [x',y]} & \longmapsto & {\scriptstyle \big[x',\, g(y)\big]}
\end{array}
$$

whose fibers are as follows:

$$
\big(f \,\widehat{\times}\, g\big)_{(x',y')} \;\simeq\; \left\{
\begin{array}{c|l}
* & (x',y') \in \mathrm{im}(f) \times \mathrm{im}(g) \\
f^{-1}(\{x'\}) \;\simeq\; \big\{[x,y'] \,\big|\, x \in f^{-1}(\{x'\})\big\} & y' \in Y' \setminus \mathrm{im}(g) \\
g^{-1}(\{y'\}) \;\simeq\; \big\{[x',y] \,\big|\, y \in g^{-1}(\{y'\})\big\} & x' \in X' \setminus \mathrm{im}(f)\,.
\end{array}
\right. \tag{155}
$$

Example A.29 (Pushout-product and bifunctors)**.** With respect to any bifunctor $(-) \otimes (-)$, forming the pushout-product (Def. A.27) with an identity morphism yields the identity morphism on the codomain:

$$
f \,\widehat{\otimes}\, \mathrm{id}_Y \;\simeq\; \mathrm{id}_{X' \otimes Y} \qquad \text{and} \qquad \mathrm{id} \,\widehat{\otimes}\, g \;\simeq\; \mathrm{id}_{X \otimes Y'}\,. \tag{156}
$$

Moreover, if $\mathscr{C}$ has initial objects $\varnothing$ and the bifunctor preserves the initial object in each argument separately, then the pushout-product with an initial morphism is given by $\otimes$:

$$
\big(\varnothing \to X'\big) \,\widehat{\otimes}\, g \;\simeq\; \mathrm{id}_{X'} \otimes g \qquad \text{and} \qquad f \,\widehat{\otimes}\, \big(\varnothing \to Y'\big) \;\simeq\; f \otimes \mathrm{id}_{Y'}\,. \tag{157}
$$

Definition A.30 (Left Quillen bifunctor)**.** Given model categories $\mathscr{D}_1, \mathscr{D}_2$ and $\mathscr{C}$ a functor of the form

$$
\otimes : \;\; \mathscr{D}_1 \times \mathscr{D}_2 \longrightarrow \mathscr{C}
$$

is called a (left) *Quillen bifunctor* if

(i) (*two-variable cocontinuity*): $\otimes$ preserves colimits in each argument separately;

(ii) (*pushout-product axiom*): given a pair of cofibrations, their $\otimes$-pushout product is also a cofibration

$$
\begin{array}{l}
f \;\in\; \mathrm{Cof}(\mathscr{D}_1) \cap \mathrm{W}(\mathscr{D}_1) \\
g \;\in\; \mathrm{Cof}(\mathscr{D}_2) \cap \mathrm{W}(\mathscr{D}_2)
\end{array}
\qquad \vdash \qquad f \,\widehat{\otimes}\, g \in \mathrm{Cof}(\mathscr{C}) \cap \mathrm{W}(\mathscr{C}) \tag{158}
$$

and if, *moreover*, either is a weak equivalence, then so is the pushout-product.

Definition A.31 (Rigidification of quasi-categories)**.** We write, as usual

$$
\begin{tikzcd}
\mathrm{sSet\text{-}Cat} \arrow[r, shift right, "N"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, shift right, "\mathfrak{C}"']
\end{tikzcd} \tag{159}
$$

for the *homotopy coherent nerve* N of simplicial categories and its left adjoint $\mathfrak{C}$ [Lu09, §1.1.5, §2.2], which on quasi-categories $\mathrm{QCat} \hookrightarrow \mathrm{sSet}$ may be understood as *rigidification* [DS11].

Proposition A.32 (Rigidification preserves products up to DK-equivalence [Lu09, Cor. 2.2.5.6][DS11, Prop. 6.2])**.** *For* $S, S' \in \mathrm{sSet}$ *there is a natural isomorphism from the rigidification* (159) *of their Cartesian product to the product* sSet-*categories of their rigidifications*

$$
\mathfrak{C}(S \times S') \longrightarrow \mathfrak{C}(S) \times \mathfrak{C}(S')
$$

which is a Dwyer-Kan equivalence (Prop. 3.20).

Proposition A.33 (Comparing rigidification to Dwyer-Kan fundamental groupoids [MRZ23, Thm. 1.1])**.** *For* $S \in \mathrm{sSet}$ *there is a natural transformation from the localization* (131) *of the rigidification* (159) *to the Dwyer-Kan fundamental simplicial groupoid*

$$
\mathrm{Loc} \circ \mathfrak{C}(S) \longrightarrow \mathscr{G}(S)
$$

which is a Dwyer-Kan equivalence (Prop. 3.20).

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