Nonabelian Anyons attached to Superconducting Islands in FQH Liquids

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The idea that topologically protected quantum states, such as anyons, may be attached to super/semiconductor heterostructures has received enormous attention, but experimental signatures in 1D systems remain elusive. Here we revisit theoretical underpinnings of anyons in 2D fractional quantum Hall (FQH) systems, whose signatures have been experimentally observed by independent groups. Invoking novel theorems about the Hopfion or $\mathbb{C}P^1$-model understood as flux-quantization in 2-Cohomotopy, we demonstrate a robust prediction for possibly nonabelian anyonic states induced by superconducting islands.

Keywords: fractional quantum Hall effect, nonabelian anyons, superconducting islands, flux quantization, Hopfion model, $\mathbb{C}P^1$ model, Cohomotopy

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I. INTRODUCTION

Anyonic topological order ([1], cf. [2, §II.B]) is the holy grail of quantum materials research (cf. [3, 4]): Degenerate gapped quantum ground states unitarily transforming under adiabatic parameter monodromy. Besides their fundamental theoretical interest, such materials are candidate platforms for intrinsically stabilized quantum computing hardware (cf. [5]), possibly necessary for mitigating the decoding bottleneck faced by quantum error correction at scale (cf. [6]).

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Among proposals for where to find anyonic topological order in materials (as opposed to in quantum simulations), only one stands out to date for having been experimentally observed with confidence ([7], cf. [8]): Fractional quantum Hall systems (FQH, cf. [9, 10]), which are cold electron liquids confined to effectively 2-dimensional surfaces (typically semiconductor junctions) and subject to transverse magnetic fields so strong and yet so fine-tuned that there is an odd integer (generally rational) number $1/\nu$ of flux quanta per electron.

Against the backdrop of such a strongly correlated electron liquid, heuristically understood [11] as a sea of emergent *composite fermion* bound states of electrons and flux quanta, every surplus flux quantum appears as the lack of the ν-th fraction of an electron, thus called a fractional *quasi-hole*. The adiabatic movement of these quasi-holes is theoretically predicted [12] to transform the ground state wave function $|\psi\rangle \in \mathcal{H}$ of the entire system by a complex phase

$$\zeta = e^{\pi \mathrm{i} \nu\, n} \tag{1}$$

which depends solely on the topological *crossing number* $n \in \mathbb{Z}$ of their worldlines, though not on the local nature of the motion.

This is the hallmark property of *topological order*, albeit at this point only in the abelian case (with $\zeta \in \mathrm{U}(1) \subset \mathrm{U}(\mathcal{H})$). Remarkably, this abelian *anyon braiding phase* has been experimentally observed in recent years by independent groups ([7], cf. [8]).

On the other hand, notably for their potential application as *universal quantum gates*, one is interested (cf. [5]) in identifying variant processes in FQH systems which may transform the ground states via *nonabelian* groups of unitaries, $G \subset \mathrm{U}(\mathcal{H})$. A famous proposal argued [13] that this could happen for quasi-holes at $\nu = 5/2$. But industry interest waned when quantum decoherence was found [14] to be overwhelming in these systems.

This has led the community to shift focus to topological states that are argued [15, 16] to appear in 1-dimensional semi/superconductor heterostructures

("Majorana zero modes"). However, despite immense effort and prominent claims, conclusive experimental verification of this proposal remains elusive (cf. [17, 18]).

An alternative that deserves more attention is the possibility of nonabelian anyonic states induced by junctions between 2-dimensional abelian FQH liquids and superconductors, which has been variously argued for in [19–25].

In order to put this proposal, of nonabelian anyons induced by superconducting islands in abelian FQH liquids, on robust theoretical foundations, here we revive a maybe underappreciated global topological formulation of topological states via "Hopfions" [26] (review in [27, §II.C]) which we have recently developed further [28–30] in application to FQH systems (survey in [31]) and also to their crystalline FQAH versions [2, 32].

II. METHODS

A. Critique of Effective Lagrangians

With a careful re-analysis of FQH anyon quantum states in mind, we critically review a couple of commonly stated but dubious steps in the literature:

It is common to model FQH liquids at long wavelengths by an effective Chern-Simons type Lagrangian density of the form [1, (2.11)][33, (7.3.10)]:

$$L := \tfrac{k}{2} a \wedge \mathrm{d}a - A \wedge \mathrm{d}a - a \wedge j\,, \tag{2}$$

where A is the electromagnetic gauge potential, $J := \mathrm{d}a$ models the electron current density and j the quasi-particle current density. This *Ansatz* is justified by the fact that the resulting Euler-Lagrange equations of motion

$$\frac{\delta L}{\delta a} J = 0 \qquad \Leftrightarrow \qquad J = \tfrac{1}{k}(F + j) \tag{3}$$

correctly express, at filling fraction $\nu = 1/k$, the Hall conductivity law ($J_x = \frac{1}{k} E_y$) and the fact ($J_0 = \frac{1}{k}(B + j_0)$) that there is one electron per k flux quanta (background and surplus).

Beyond this classical analysis, it is common to assume that, upon quantization, the Chern-Simons term $\frac{k}{2} a \wedge \mathrm{d}a$ in (2) induces quantum U(1)-Chern-Simons theory (cf. [34]) for the "statistical gauge field" a and with it the signatures of abelian topological order (cf. [35, §5.2.3]).

Even setting aside the issue of artificially dropping the remaining summands in (2), we highlight a conceptual tension with this Ansatz: Since F and j are necessarily integral forms (whose periods count background and surplus magnetic flux quanta, respectively, by Dirac flux quantization of the magnetic field, cf. [36, §2.1]), equation (3) implies that $J = \mathrm{d}a$ is fractional. While this accommodates the fractional quasi-particle charge that motivates (2), it generally *invalidates* $\frac{k}{2} a \wedge \mathrm{d}a$ as a quantum-consistent Chern-Simons term. An aspect of just this problem has been noticed in [37, p 35][35, p 159], whose resolution is admitted to be an "open question" in [38, p. 40].

We point out that the problem goes deeper still: Even with the source terms in (2) disregarded and with $\mathrm{d}a$ assumed integral and hence quantum-consistent, a more careful analysis of the effective FQH field theory shows that the statistical gauge field generally has a kinetic Maxwell term besides the Chern-Simons term (cf. [39, (11.41)][40, (402)]):

$$L_{\mathrm{MCS}_3} := \tfrac{1}{T} \mathrm{d}a \wedge \star_3 \mathrm{d}a + \tfrac{k}{2} a \wedge \mathrm{d}a\,. \tag{4}$$

The Maxwell term is commonly alleged to be "irrelevant", and it certainly vanishes in the strong coupling limit $T \longrightarrow \infty$ where the Chern-Simons interaction energy dominates the kinetic energy.

However, for all finite T the *phase space* of the MCS theory (whose canonical coordinate is a) is different from that of CS theory (whose canonical coordinate is only a_1 for $\mathrm{d}a = 0$), whence the $T \longrightarrow \infty$ limit of the MCS quantum theory is *different* from plain Chern-Simons theory: it still contains excitation modes, cf. [41, §IX] and [42] based on [43]. Beyond these references, this subtle strong coupling limit of MCS theory may not have found due attention.

B. Proper Topological Limit

However, this last problem suggests a natural resolution of the issue, as follows:

In order to avoid taking a problematic limit in a putative space of quantum field theories and instead to take an ordinary limit of observables in a single QFT, we may consider higher-dimensional *5d Maxwell-Chern-Simons* theory, with Lagrangian density (cf. [44, §II])

$$L_{\mathrm{MCS}_5} = \tfrac{1}{T'} \mathrm{d}a' \wedge \star_5 \mathrm{d}a' + \tfrac{k'}{6} a' \wedge \mathrm{d}a'\,, \tag{5}$$

and dimensionally reduce it (cf. [29, §3]) on a 2-dimensional fiber of volume V carrying flux Φ (a "flux compactification"). A standard computation readily shows that the reduced Lagrangian density in 3D is again of the ordinary MCS_3 form (4), but now with *dynamically* rescaled parameters:

$$T = T'/V \quad \text{and} \quad k = k'\Phi\,. \tag{6}$$

This way the strong coupling limit in 3D is now recast equivalently as the dimensional reduction limit $V \longrightarrow 0$ in 5D.

C. Proper Flux Quantization

This reformulation of the problem has the profound consequence that we may now securely access the effective formulation of the topological FQH quantum phase by understanding the topological quantum observables of 5D MCS theory. This has recently been discussed in [29] (review in [45], based on [36]).

The key point is that the Gauss laws of 5D MCS theory are non-linear

$$\mathrm{d}b_2 \;=\; 0 \qquad \mathrm{d}e_2 \;=\; \tfrac{1}{2}b_2 \wedge b_2\,, \tag{7}$$

which implies [46] that magnetic (b_2) and electric (e_2) flux densities can no longer be quantized in ordinary 2-cohomology, since that fails to implement the nonlinear relation in (7). Instead, the basic admissible flux quantization law in this case is (by [36, p 40]) *2-Cohomotopy* (cf. [47, §VII][48, Ex. 2.7]):

This means that where ordinary electromagnetic gauge fields A are classified by maps to the usual classifying space $B\mathrm{U}(1) \simeq \mathbb{C}P^\infty$, the 5D gauge field may be classified by maps to its 2-sphere subspace

$$\underset{\text{5D MCS fluxes}}{\text{classifying space for}}\; \mathbb{C}P^1 \subset \mathbb{C}P^\infty \;\underset{\text{Maxwell fluxes.}}{\text{classifying space for}} \tag{8}$$

(The reason for this is that the *minimal Sullivan model* of the space $\mathbb{C}P^1$ reflects exactly the differential relations (7), cf. review in [36, §3.2] based on [48].)

Concretely, to model solitonic field configurations whose field strength *vanishes at infinity*, we are to consider (cf. [36, §2.2][49, §A.2]):

(i) the spacetime domain X_{dom} as pointed by the *point at infinity*, $\infty \in X_{\mathrm{dom}}$,

(ii) the charge classifying space $\mathcal{A}$ as pointed by the vanishing charge classifier, $0 \in \mathcal{A}$,

so that the basepoint-preserving classifying maps

$$\chi \in \mathrm{Maps}^*(X_{\mathrm{dom}}, \mathcal{A}) \tag{9}$$

literally take the value 0 at ∞:

$$\begin{array}{ccc} X_{\mathrm{dom}} & \xrightarrow{\chi} & \mathcal{A} \\ \infty & \mapsto & 0 \end{array} \tag{10}$$

For instance, an FQH liquid occupies an ϵ-thin slab of material $X^3 \equiv \Sigma^2 \times [-\epsilon, +\epsilon]$, for Σ^2 a surface, whence the spatial domain of the uplift to MCS_5 is pointed homotopy equivalent to

$$\begin{aligned} X_{\mathrm{dom}} &\equiv \big(\mathbb{R}^1 \times X^3\big)_{\cup\{\infty\}} \\ &\simeq \big(\mathbb{R}^1 \times \Sigma^2 \times [-\epsilon, +\epsilon]\big)_{\cup\{\infty\}} \\ &\sim S^1 \wedge \Sigma^2_{\cup\{\infty\}}\,, \end{aligned} \tag{11}$$

where "$\wedge$" denotes the *smash product* of pointed spaces (their product space with all points identified that are at infinity in one or the other factor).

Hence with this spacetime domain and with $\mathcal{A} \equiv \mathbb{C}P^1$ (8), the topological quantum observables, being the compactly supported functions $\mathbb{C}[-]$ on the set of solitonic field sectors, form the group algebra

$$\begin{aligned} \mathrm{Obs} &:= \mathbb{C}\Big[\pi_0\,\mathrm{Maps}^*\big(X_{\mathrm{dom}}, \mathbb{C}P^1\big)\Big] \\ &\simeq \mathbb{C}\Big[\pi_1\,\mathrm{Maps}^*\Big(\Sigma^2_{\cup\{\infty\}}\,\mathbb{C}P^1\Big)\Big] \end{aligned} \tag{12}$$

and hence topological quantum states fall into the corresponding irreps (cf. [49, §3][45, §2]).

D. Relation to Hopfion/$\mathbb{C}P^1$-Model

In the default case of an FQH liquid on the plane/disk, where $\Sigma^2 \simeq \mathbb{R}^2$ in (11) with

$$\mathbb{R}^2_{\cup\{\infty\}} \simeq S^2 \tag{13}$$

(by *stereographic projection*), the topological soliton sectors (12) are labeled by integer multiples of the *Hopf fibration*,

$$\pi_1\,\mathrm{Maps}^*\Big(\mathbb{R}^2_{\cup\{\infty\}}, \mathbb{C}P^1\Big) \simeq \pi^3\big(S^2\big) \simeq \mathbb{Z}\,, \tag{14}$$

as such known from the *Hopfion model* for abelian anyons [26][27, §II.C].

Indeed, closer analysis via the classical *Pontrjagin theorem* (cf. [50, §3.2]) shows [28] that the left hand side of (14) classifies framed oriented links L (of anyon worldlines) by their *total crossing number* (or *writhe*) $\#L$, which is exactly [29, Thm. 1] the exponent of the renormalized expectation value $\langle\nu| - |\nu\rangle$ of Wilson loop observables in U(1)-Chern-Simons theory at level $k = 1/\nu$:

$$\begin{tikzcd} \mathrm{Map}^*\big(X^3_{\mathrm{dom}}, \mathbb{C}P^1\big) \arrow[r, two heads, "\pi_0"] \arrow[rr, bend left, "\langle\nu|-|\nu\rangle"] & \pi^3\big(S^2\big) \arrow[r] & \mathbb{C} \\[-15pt] L & \mapsto \; \#L & \mapsto \; e^{\pi \mathrm{i} \nu\, \#L}\,, \end{tikzcd} \tag{15}$$

reflecting exactly the anyonic braiding phases (1).

E. Covariantized Observables

But the topological observables of MCS_5 flux-quantization in 2-Cohomotopy go beyond the Hopfion model.

First, we may consider other FQH geometries. Notably for the torus, $\Sigma^2 \equiv T^2$, the topological observables (12) turn out to be [30, Prop. 3.19][51, Thm. 3.3]:

$$\begin{aligned} \mathrm{Obs} &= \mathbb{C}\Big[\pi_1\mathrm{Map}\big(T^2, S^2\big)\Big] \\ &\simeq \mathbb{C}[W_a, W_b, \zeta]\Big/\Big(\begin{smallmatrix} W_a W_b = \zeta^2 W_b W_a, \\ \zeta\ \text{central} \end{smallmatrix}\Big) \end{aligned} \tag{16}$$

which is exactly the algebra of quantum observables expected for abelian anyons on the torus ([52, (4.9)], cf. [35, (5.28)]). That this follows as the fundamental group algebra of maps from the torus to $\mathbb{C}P^1$ is a remarkable fact with deep relevance [2, 32] also for the putative crystalline ("anomalous") form of FQH anyons.

But more in detail, topological quantum systems ought to be "generally covariant", meaning that any pair of configurations differing only by a diffeomorphism of the spacetime domain ought to be regarded as physically equivalent. For the classifying maps (9) this means [30, Def. 2.11] that their physically relevant space is the *homotopy quotient* $(-)/\!\!/(-)$ (Borel construction) of the mapping space by the action of the diffeomorphism group at least of the surface Σ^2 (11), whence the algebra of *covariantized* topological quantum observables is:

$$\begin{aligned}\mathrm{CovObs} &:= \mathbb{C}\Big[\pi_1\Big(\mathrm{Maps}^*\big(\Sigma^2,\, \mathbb{C}P^1\big)/\!\!/\mathrm{Diff}^+(\Sigma^2)\Big)\Big] \\ &\simeq \mathbb{C}\Big[\pi_1\Big(\mathrm{Maps}^*\big(\Sigma^2,\, \mathbb{C}P^1\big)\Big) \rtimes \mathrm{MCG}(\Sigma^2)\Big]\,.\end{aligned} \tag{17}$$

As shown in the second line, this entails [30, Prop. 2.24] passing to the semidirect product of the fundamental group of maps by the surface's *mapping class group* $\mathrm{MCG}(\Sigma)^2 := \pi_0\mathrm{Diff}^+(\Sigma^2)$ (cf. [53]).

Remarkably, over the torus $\Sigma^2 \equiv T^2$, the condition that topological quantum states, being finite-dimensional irreps of (16), are generally covariant in that they extend to representations of (17), implies [30, §3.4] the restriction of the braiding phase ζ (1) to a root of unity and the expected ground state degeneracy reflecting the topological order of the quantum system.

This shows that the covariantization of the topological observables (17) captures crucial physical properties of the FQH liquid.

III. RESULTS

This discussion (§ II) shows that the assumption that surplus FQH flux is quantized in 2-Cohomotopy implies right away the fine detail of the topological order of the corresponding quasi-hole anyons, whose traditional derivation via effective Chern-Simons Lagrangians is less robust (§ II A) and subject to *ad hoc* renormalization choices [29].

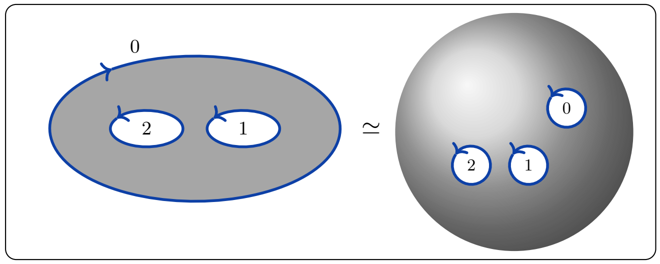
It is thus compelling to use this novel theory to derive predictions about FQH systems not obviously formulatable with Lagrangian effective field theory:

A. Modelling Superconducting Islands

Namely consider now the case that Σ^2 in (11) is a *punctured* surface. The experimentally most relevant case is that of the n-punctured plane/disk

$$\Sigma^2_{0,n} := \mathbb{R}^2 - \{x_1, \cdots, x_n\}\,. \tag{18}$$

FIG. 1. A disk (hosting an FQH fluid) with n islands (punctures) is homeomorphic to a sphere with $n+1$ punctures. From the view of the complement bulk, all punctures are *at infinity* (19), which entails with (10) that magnetic flux does not enter here.



Mathematically, this is a standard choice of surface to plug into the general formula for topological FQH observables found in (17). Let us see what this mathematical model describes physically:

The key observation here is that each puncture in (18) is an "asymptotic boundary" (being the complement of a closed point), which implies that each of the points deleted in (18) is "at infinity" as seen from $\Sigma^2_{0,n}$. This entails that adjoining infinity as a formal new point to the spacetime domain (11) amounts to identifying all these points with the single point-at-infinity, hence is the result of *pinching* the sphere (13) at $n+1$ points (cf. Fig. 1):

$$\big(\Sigma^2_{0,n}\big)_{\cup\{\infty\}} \simeq S^2/\{x_0, \cdots, x_n\}\,. \tag{19}$$

To see the physical meaning of this mathematical result, recall that the flux classifying maps *vanish at infinity* (10). Before we punctured $\mathbb{R}^2$, this encoded just the evident physical condition that the magnetic field is threaded through the FQH liquid under consideration, not extending beyond its boundary. Including these punctures models a scenario in which *magnetic flux is expelled also from the vicinity of each of the punctures*.

But this is just the situation enforced by the *Meissner effect* when (the vicinity of) each puncture represents the laboratory situation of a type-I superconducting island present inside the FQH liquid.

Therefore we are to conclude that using the n-punctured disk (18) in the general formula (17) yields the topological observables, and hence the topological quantum states, of planar FQH liquids hosting n superconducting islands.

B. Deriving the Anyon Attachment

We have thereby reduced the physical analysis to the question of computing the topological observable algebra obtained from plugging (18) and (19) into the general formula (17)

$$\begin{aligned}&\mathrm{CovObs} \simeq \\ &\mathbb{C}\Big[\pi_1\mathrm{Maps}^*\Big(\big(\Sigma^2_{0,n}\big)_{\cup\{\infty\}}, \mathbb{C}P^1\Big) \rtimes \mathrm{MCG}\big(\Sigma^2_{0,n}\big)\Big]\,.\end{aligned} \tag{20}$$

Computing this is now a problem purely in low-dimensional algebraic topology and may be attacked with classical tools of that field. The computation is spelled out in [30, §3.5-6] with the result that:

Proposition III.1. *The topological quantum states on the n-punctured disk according to* (20) *fall into unitary irreps of the subgroup of the* rot*-quotient of the* framed spherical braid group *with* $n+1$ *strands*

$$\mathrm{FBr}_{n+1}(S^2)/\mathrm{rot} := \mathbb{Z}^{n+1} \rtimes \mathrm{Br}_{n+1}(S^2)/\mathrm{rot} \qquad (21)$$

on framed braids whose total framing number is a multiple of $n+1$.

Here the formula (21) unwinds as follows:

(i) One factor of $\mathbb{Z}$ in $\mathbb{Z}^{n+1}$ arises as the Hopfion contribution in (14), accompanied by further factors for each island, signifying that solitonic abelian anyons now have one separate braiding phase (1) for braiding in the vicinity of each island.

(ii) But the striking novel import of (21) is the appearance of the *nonabelian* (for $n > 1$) spherical braid group $\mathrm{Br}_{n+1}(S^2)$ (cf. [54, Lit. 2.20][30, Ex. 2.17]), formally reflecting the motion of possibly nonabelian defect anyons around each other (cf. [5, §II.A]), here associated with the superconducting islands.

(iii) Specifically, the spherical braid group appearing here (cf. Fig. 2) is the quotient of the usual Artin braid group Br_{n+1} (with its set of Artin generators $\{b_i\}_{i=1}^n$) by the braid where one strand goes once around all others (cf. [55, pp 245]):

$$\mathrm{Br}_{n+1}(S^2) \simeq \mathrm{Br}_{n+1}/\big((b_1 \cdots b_n)(b_n \cdots b_1)\big)\,. \qquad (22)$$

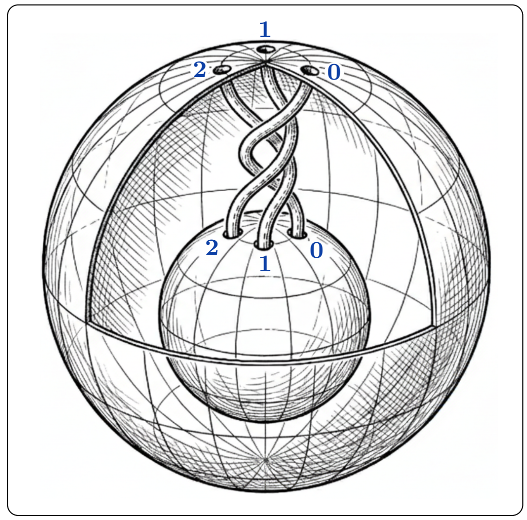
(iv) Furthermore, the element rot (cf. [53, (Fig. 9.6)]) denotes the braid whose strands jointly perform a full rotation. Its square turns out to be equivalent to the trivial braid and hence the expression $(-)/\mathrm{rot}$ in (21) means the quotient by the corresponding $\mathbb{Z}_{/2}$ subgroup.

(v) These two quotients mean that the nonabelian braiding operations appearing here are a little more constrained than usually assumed for nonabelian anyons. For example, for $n = 2$ the monodromy (21) reduces to the *framed symmetric group* (cf. [30, (42)])

$$\mathrm{FBr}(S^2)/\mathrm{rot} \simeq \mathbb{Z}^3 \rtimes \mathrm{Sym}_3 \qquad (23)$$

and hence exhibits the special case of "parastatistical" anyons (cf. [30, Rem. 3.60]). These may deserve more attention as they still implement non-Clifford quantum gates (cf. [30, Prop. 3.61]).

FIG. 2. *Spherical braids* (22) are braids of worldlines of punctures on the 2-sphere (cf. Fig. 1). These are almost the same as ordinary Artin braids of punctures in the plane. (Shown is the element $b_1 b_2 b_1 b_2 b_1$), the only difference being that on the sphere the Artin braid $(b_1 \cdots b_n)(b_n \cdots b_1)$ (where all punctures are fixed except one which is circling the others) is topologically trivial.



While Prop. III.1 does not specify the spherical braid representation in which the ground states will transform (which will depend on microscopic details beyond the presence of superconducting islands), it shows that there may be superselection sectors (irreps) beyond the abelian phase which exhibit nonabelian anyonic braid statistics, in contrast to the abelian situation (14) without superconducting islands.

IV. CONCLUSION

FQH liquids stand out as the only quantum materials currently known with some certainty to exhibit at least abelian anyons. This motivates the search for nonabelian anyons in these systems, potentially serving as much anticipated topological hardware for future scalable quantum computers.

While nonabelian anyons are theoretically expected to appear at FQH filling fraction $\nu = 5/2$, practical interest in this system has waned due to grave stability concerns. As a result, academic and industry focus has shifted to search for topological states in 1-dimensional semi/superconductor junctions. But despite considerable effort, initial hopes for their experimental confirmation have been shattered, also raising concerns about the reliability of the original theoretical prediction.

Here we turned attention to superconducting junctions/islands in 2-dimensional FQH liquids, previously explored by only a few authors, aiming for reliable theory making robust predictions for possible occurrence and nature of anyonic states in this situation.

Observing that these strongly correlated electron systems are non-perturbative and not as reliably captured, at long wavelengths, by 3D Chern-Simons Lagrangian densities as often assumed, our key move was to invoke instead a recently proposed alternative effective description [28–30]: Here we properly deal with the subtle strong coupling limit of the 3D Maxwell-Chern-Simons anyon fluid by lifting to 5D Maxwell-Chern-Simons theory and then globally completing the theory by proper flux quantization in 2-Cohomotopy.

The resulting novel theory turns out to accurately and reliably retrodict fine detail of the familiar abelian FQH anyon states, thus lending weight to the predictions that it makes for the effect of superconducting islands in FQH liquids.

To this end, using computations in low-dimensional algebraic topology which extend the old Hopfion model for abelian anyons, we arrive at the striking result (Prop. III.1) that, besides a proliferation of abelian braiding phases, the system's ground states now fall into irreps of (a mild quotient of) the spherical braid group which exhibits nonabelian anyons attached to the superconducting islands.

While it may be challenging to engineer and sustain type-I superconducting islands in FQH liquids, even more so with controlled mobility, our result suggests that this scenario may be a worthwhile target to further explore in experiment.

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