Higher geometric prequantum theory II
L_∞-Algebras of local observables

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Based on d.f., Chris Rogers and Urs Schreiber L_∞*-algebras of local observables from higher prequantum bundles*, `arXiv:1304.6292`

$$\begin{array}{ccc} & & C^\infty(X;\mathbb{R}),\ \{,\} \\ (X,\omega) & & \\ \text{symplectic manifold} & \rightsquigarrow & \downarrow \\ & & \mathfrak{X}_{\mathrm{Ham}}(X) \end{array}$$

(X, ω)
pre-n-plectic manifold $\rightsquigarrow$

$$\begin{array}{c} L_\infty(X;\omega) \\ \downarrow \\ \mathfrak{X}_{\mathrm{Ham}}(X) \end{array}$$

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A pre-n-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1 closed form on X

The vector space of Hamiltonian pairs on a symplectic manifold (X,ω) is

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The vector field $[v, w]$ is Hamiltonian with Hamiltonian $n-1$-form $\iota_{v\wedge w}\omega$.

$$\iota\omega : \mathfrak{X}(X) \to \Omega^n(X)$$

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Is $\iota\omega$ a morphism of Lie algebras?

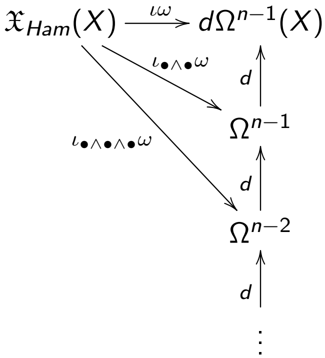
$$\iota_{[v,w]}\omega \stackrel{?}{=} [\iota_v\omega, \iota_w\omega]$$

$$\iota_{[v,w]}\omega \stackrel{?}{=} \underbrace{[\iota_v\omega, \iota_w\omega]}_{0}$$

$$\iota_{[v,w]}\omega = \underbrace{[\iota_v\omega, \iota_w\omega]}_{0} + d\iota_{v\wedge w}\omega$$

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota\omega} d\Omega^{n-1}(X)$$

$$
\begin{array}{ccc}
\mathfrak{X}_{Ham}(X) & \xrightarrow{\iota\omega} & d\Omega^{n-1}(X) \\
& \searrow^{\iota_\bullet \wedge \bullet \omega} & \uparrow d \\
& & \Omega^{n-1}
\end{array}
$$


$\mathfrak{X}_{Ham}(X)$
$\iota\omega$
$d\Omega^{n-1}(X)$
$\iota_{\bullet\wedge\bullet}\omega$
d
Ω^{n-1}
$\iota_{\bullet\wedge\bullet\wedge\bullet}\omega$
d
Ω^{n-2}
d
$\vdots$

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So we can ask ourselves: is $\iota_\bullet\omega$ an L_∞-morphism?

An L_∞-morphism between a Lie algebra $\mathfrak{g}$ and a chain complex $\mathfrak{h}$ is a sequence of multilinear maps $\varphi_k : \wedge^k\mathfrak{g} \to \mathfrak{h}[1-k]$ such that

$$d_{\mathfrak{h}}\varphi_k(v_1\wedge\cdots\wedge v_k) = \sum_{i<j} \pm\varphi_{k-1}([v_i, v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k).$$

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega \stackrel{?}{=} \sum_{i<j} \pm\iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

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For $\omega \in \Omega^{n+1}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}(X)$ we have

$$\begin{aligned} d\iota_{v_1\wedge\cdots\wedge v_k}\omega = &\sum_{i<j} \pm\iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k}\omega \\ &+ \sum_{i} \pm\iota_{v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge v_k}\mathcal{L}_{v_i}\omega \\ &\pm \iota_{v_1\wedge\cdots\wedge v_k} d\omega \end{aligned}$$

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For $\omega \in \Omega^{n+1}_{cl}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}(X)$ we have

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For $\omega \in \Omega^{n+1}_{cl}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}_{Ham}(X)$ we have

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Now that we have an L_∞-morphism...

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota_\bullet \omega} \left(\Omega^0(X) \cdots d\Omega^{n-1}\right)$$

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$$\begin{array}{ccc} L_\infty(X,\omega) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{X}_{Ham}(X) & \xrightarrow{\iota_\bullet \omega} & \left(\Omega^0(X) \cdots d\Omega^{n-1}(X)\right) \end{array}$$

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Luckily in the above situation one can use the following recognition principle

Let $\mathfrak{g}$ be an L_∞-algebra, A be a dgla and $f_\infty : \mathfrak{g} \to A$ an L_∞ morphism. Let $p_A : B \to A$ be a fibrant replacement of the zero morphism $0 \to A$ in the category of dglas. If

$$\begin{array}{ccc} (\mathfrak{g} \times_A B, Q) & \xrightarrow{\pi_{B,\infty}} & B \\ {\scriptstyle \pi_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathfrak{g} & \xrightarrow{f_\infty} & A \end{array}$$

is a commutative diagram of L_∞-algebras whose linear part is a pullback diagram of chain complexes, then $(\mathfrak{g} \times_A B, Q)$ is a model for the homotopy fiber of f_∞.

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The underlying chain complex is

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \mathrm{Ham}^{n-1}(X)$$

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with $Ham^{n-1}(X)$ in degree zero; the bilinear bracket is

$$[x_1,x_2] = \begin{cases} [v_1,v_2] + \iota_{v_1\wedge v_2}\omega & \text{if } \deg x_1, x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

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where $x_i = v_i \oplus \eta_i$ if $\deg x_i = 0$; and, for $k > 2$:

$$[x_1, \ldots, x_k]_k = \begin{cases} \pm\iota_{v_1 \wedge \cdots \wedge v_k}\omega & \text{if } \deg x_1, \ldots, x_k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{array}{ccc}
L_\infty(X,\omega) & \longrightarrow & 0 \\
\big\downarrow \lrcorner & & \big\downarrow \\
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If X is n-connected and $x \in X$ then

$ev_x : \left(\Omega^0(X)\cdots d\Omega^{n-1}(X)\right) \to \mathbb{R}[n]$ is a quasi-isomorphism

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If X is $(n-1)$-connected and $x \in X$ then

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\mathfrak{X}_{Ham}(X) & \xrightarrow{\; ev_x\, \iota_\bullet \omega \;} & \mathbb{R}[n]
\end{array}
$$

By the pasting law for homotopy pullbacks, if X is $(n-1)$-connected, $L_\infty(X,\omega)$ is presented as an abelian extension of $\mathfrak{X}_{Ham}(X)$ by a Kostant-Souriau-type cocycle.

Example. (X, ω) connected symplectic manifold.

$$L_\infty(X, \omega) = Ham^0(X)$$

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One recovers that $C^\infty(X; \mathbb{R}), \{,\}$ is an abelian extension of $\mathfrak{X}_{Ham}(X)$ by the Kostant-Souriau cocycle.

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$$
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Heis^{\rho}(\mathfrak{g}) & \longrightarrow & L_\infty(X,\omega) & \longrightarrow & 0 \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\
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Example. (V, ω) symplectic vector space; $\mathfrak{g} = V$ acting by translations; $x = 0$

$$\begin{array}{ccc} Heis(V,\omega) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ V & \xrightarrow{\omega} & \mathbb{R}[1] \end{array}$$

is the classical Heisenberg algebra of (V, ω).

Example. (G, ω_G) compact simple simply connected Lie group; ω_G standard closed 3-form; $\mathfrak{g}$ the Lie algebra of G; $x = e$

$$\begin{array}{ccc} \mathfrak{string}(\mathfrak{g}) & \longrightarrow & 0 \\ \big\downarrow \;\lrcorner & & \big\downarrow \\ \mathfrak{g} & \xrightarrow{\langle [-,-],- \rangle} & \mathbb{R}[2] \end{array}$$

is the classical string Lie 2-algebra of $\mathfrak{g}$.

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This exists if and only if ω represents an integral cohomology class.

A geometric prequantization of a pre-n-plectic manifold (X, ω) is the choice of a $U(1)$-n-bundle with connection whose curvature $n+1$-form is ω

$$\begin{array}{ccc} & & \mathbf{B}^n U(1)_{conn} \\ & \overset{\nabla}{\nearrow} & \downarrow F \\ X & \xrightarrow{\ \omega\ } & \Omega^{n+1}_{cl} \end{array}$$

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and its infinitesimal version, the Lie n-algebra

$$Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$$

Theorem. If (X, ω) admits a prequantization, then the L_∞-algebra $L_\infty(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$

Theorem. If (X, ω) admits a prequantization, then the L_∞-algebra $L_\infty(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$, i.e., there is a quasi-isomorphism of L_∞-algebras

$$L_\infty(X, \omega) \cong Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$$

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^n U(1)$ instead.

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For $n = 1$ one obtains the Lie algebra of sections of the Atiyah algebroid of a principal $U(1)$-bundle.

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This leads to the Courant and to the Atiyah Lie n-algebras

For $n = 1$ one obtains the Lie algebra of sections of the Atiyah algebroid of a principal $U(1)$-bundle.

For $n = 2$ one obtains the Lie 2-algebra of sections of the Courant Lie 2-algebroid of a $U(1)$-gerbe with connective structure (Collier).