Flux Quantization on Holographic M5-Branes

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Abstract

In search of a microscopic theory for strongly-coupled quantum phenomena like anyonic topological order – relevant such as for future fault-tolerant quantum computation – the success of AdS/CFT-inspired holography in the qualitative description of quantum materials suggests that fundamental brane dynamics may serve as the missing non-perturbative model. Here it is remarkable that over a decade before modern AdS/CFT duality was formulated, Duff et al. found a candidate microscopic explanation by identifying the CFT fields with fluctuations of probe p-branes stretched out in parallel near the horizon of their own black brane incarnation.

We revisit this form of microscopic holography for the case of M5-branes, by establishing for the first time an explicit super-embedding of M5 probe branes into their own near-horizon geometry exactly at the throat radius. Following our recent discussion of flux quantization on M5-branes, this allows to globally complete the traditional local field content on the M5 by flux quantization laws necessary for capturing fractional (torsion) charged solitons. Choosing flux quantization in co-Homotopy theory ("Hypothesis H") we find from careful analysis of the moduli space that the topological quantum states of solitons stuck at O-planes in "open" holographic M5-branes are those of abelian anyons governed by quantum Chern-Simons theory.

We close with an outlook on applications to quantum materials and quantum computation.

Contents

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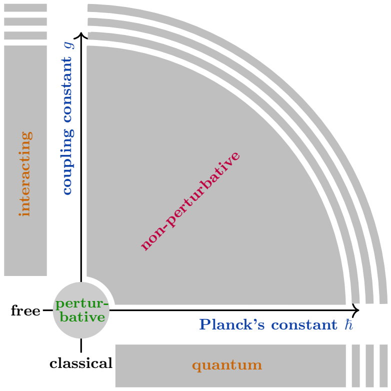
1 Introduction and Overview

The open problem of strongly-coupled quantum physics. The key open problem of contemporary fundamental physics is the general understanding of *strongly-coupled* quantum systems, be it hadronic bound states at room temperature (the problem of confinement, cf. [RS20][Ro21]) or anyonic topologically ordered ground states of quantum materials (cf. [ZCZW19, §III][SS23c], thought to be relevant, if not necessary, for future fault-tolerant quantum computation, cf. [RW18][MySS24]). The traditional toolbox of perturbation- and mean field-theory is largely useless for such systems (cf. [BaSh10]), but general non-perturbative quantum field theory has been missing.

Figure P. To appreciate the scope of the problem of general strongly-coupled quantum physics, it is worth recalling that common perturbative quantum field theory (pQFT), despite its notorious richness, describes only an *infinitesimal* ("formal") neighborhood around the classical free fields in the space of all quantum systems. Away from this familiar but tiny neighborhood the vast range of non-perturbative quantum physics remains to be mapped.

Whatever else string/M-theory has been motivated by at any point in time, its remarkable outcome is the perspective of non-perturbative QFT realized on *branes*, holographically reflected in their ambient gravitational backgrounds.

While exceedingly promising, holographic brane physics has its own open problems. A key one of these – flux quantization – we address here.



Existing approaches to this problem include (besides brute-force computer simulation, i.e., lattice gauge theory) notably the "holographic principle".

The holographic principle. The general success of the *holographic principle* in the guise of AdS/CFT duality — illuminating otherwise elusive strongly-coupled quantum systems by understanding them as "boundary theories" of a higher-dimensional theory of gravity — has been so encompassing that it cannot and need not be reviewed here (see instead e.g. [AGMOO00][Nat15]). The principle works remarkably well also for confined hadrodynamics (cf. [Ah03][BdT09], review in [Er14][DBLM21]) and for aspects of strongly coupled quantum materials [HKSS07], review in [Pi14][ZLSS15][HLS18]. However, a microscopic explanation for this success has been lacking, and with it any understanding of how to apply the principle to more realistic situations, such as beyond the notorious unrealistic large-N limit, which requires (e.g. [IMSY98, Figs. 1-6]) understanding effects of *M-theory branes* (cf. [Du96][Du99a]) in $D = 11$ supergravity (cf. [MiSc06][GSS24a, §3]) on the gravity side of the duality.

Microscopic holography via probe p-branes. Possibly less widely appreciated is the fact that, well before the modern formulation of AdS/CFT duality, a candidate microscopic description had been found by Duff et al., first discussed for the M2-brane [BDPS87][BD88][DFFFTT99] then generalized to include also M5-branes and D-branes [CKvP98][CK$^+$98][PST99][GM00][NP02], review in [Du99b][Du99c] (more recent variations include [DGTZ20][Gu21][Gu24]):

In this *microscopic p-brane holography* – as we shall call it here for lack of an established name – one considers (as indicated in Figure B) probe p-branes (i.e., *light* branes described by sigma-models not back-reacting onto the ambient spacetime, cf. [Si12]) embedded in parallel near the (asymptotically AdS) horizon of their own black-brane incarnation (their *heavy* back-reacted version described by singular solutions of supergravity, cf. [DL94][Du99a, §5]) and finds that their fluctuations about this configuration are described by the conformal field theory (CFT) known from AdS/CFT duality.

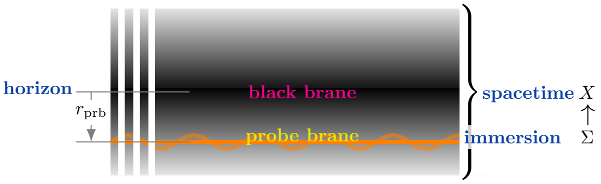
In this picture the otherwise somewhat mysterious holographic duality between (i) quantum systems and (ii) gravity reflects but two perspectives on the expected nature of branes:

(i) as dynamical (fluctuating) physical objects in themselves, and

(ii) as sources of gravitational (and higher gauge-) fields propagating away from the black brane. [1]

Figure B. Schematics of a probe brane worldvolume immersed (embedded) near the horizon of its own black brane incarnation, parallel to it at some coordinate distance r_{prb}. (Precise details on the black M5-brane background are in §2.3 and on the probe M5 in §2.4.)

The curvy line indicates (quantum-)fluctuations about this parallel configuration, thought to incarnate the strongly coupled quantum system holographically encoded in the ambient gravitational field.



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[1] For the case of 0-branes, namely for *particles*, the investigation of these dual perspectives — (1.) as quanta and (2.) as black hole

Global completion and torsion charges by flux quantization. However, as we pointed out in [GSS24b, p. 2], all previous discussions of p-brane sigma-models — and hence in particular of microscopic p-brane holography — have considered only the *local* field content on the brane's worldvolume, that which can be detected and described on a single coordinate chart. This is insufficient (as is well-known already from Dirac charge quantization, cf. [Al85a][Al85a][SS24b, Ex. 3.10]) for capturing global topological charges of the (higher) gauge fields on these branes, such as fractional (torsion) charges relevant notably for modelling anyonic topological order [GSS24b][SS23b][SS23c]. A global completion of the field content requires a choice of *flux quantization law* [SS24b].

We have previously shown [GSS24a] that and how globally completed (flux-quantized) on-shell fields of higher gauge theories, such as on worldvolumes of M5-branes [GSS24b], may be obtained for supergravity and branes defined "*on superspace*" namely on supergeometric enhancements of spacetime and brane worldvolumes (cf. [CDF91] [GSS24c]). This is because:

- **(i)** the process of flux quantization takes care of *and only of* equations of motion that have the form of Bianchi identities ([SS24b, §3] following [FSS23]), but not for instance of Hodge-duality relations, while
- **(ii)** gravitational fields and branes described on super-space miraculously have all their equations of motion indeed given by Bianchi identities: the Hodge duality constraints on ordinary bosonic flux densities become but one super-field component of the Bianchi identities on their super-flux enhancement ([GSS24a, Thm. 3.1][GSS24b, Prop. 3.17], following [CF80][BH80] and [HS97b][So00]).

Anyons on holographic M5-branes. With a flux quantization law finally imposed and thus with the moduli of solitons of the worldvolume (higher gauge) fields actually defined, it becomes possible to rigorously analyze the quantum states of their topological sectors, following [SS23d][GSS24b, §4], and look for anyonic quantum states (cf. [SS23b][SS23c]) signifying topological order in the worldvolume theory of the probe M5-branes. Previous discussions to this extent relied on unproven assumptions about the physics on coinciding M5-branes as well as informal path-integral arguments and required some ad hoc workarounds of ill-defined expressions, see Rem. 4.26 below.

Carrying through this procedure, our main results are (in §2) what seems to be the first actual solution of a holographic M5-brane probe via super-embedding (and we discover that this exists exactly only at the throat radius) and (in §4) a proof of anyonic quantum states of solitons stuck at O-planes in open M5-branes wrapped on the M-theory circle S^1_A, controlled by (abelian) Chern-Simons theory; see the conclusion in §5.

Outline. Our plan is to:

- §2: give a precise and explicit supergeometric form of the super-immersion of probe M5-branes near the horizon of their black brane incarnation;
- §3: use this to obtain the globally completed on-shell field content on these *holographic M5-brane* configurations;
- §4: show that and how this implies anyonic quantum states arising on the holographic M5-worldvolume.

In concluding, we:

- §5: discuss some potential implications for the understanding of topological quantum materials.

2 Holographic M5 super-immersions

Since we need to exhibit the immersions of M5-brane worldvolumes into spacetime as M5 *super-immersions* ([GSS24b, Def. 3.12], essentially the "super-embeddings" of [HS97b][So00]) in order to guarantee that the worldvolume flux quantization (discussed below in §3) is accurate, we give here an explicit constructions of M5 super-immersions into super-AdS$_7\times S^4$, to be called *holographic M5-immersions*, for short:

$$\begin{tikzcd}
\overset{\textbf{super-worldvolume}}{\Sigma^{1,5|2\cdot\mathbf{8}}} \arrow[r, "\phi", "\textbf{M5 super-immersion}"'] & \overset{\textbf{AdS super-spacetime}}{X^{1,10|\mathbf{32}} \equiv \dfrac{\mathrm{OSp}(6,2\,|\,4)}{\mathrm{Spin}(6,1)\times\mathrm{O}(4)}} & \overset{\textbf{its bosonic body}}{\mathrm{AdS}_7\times S^4.} \arrow[l, hook']
\end{tikzcd} \tag{1}$$

Remark 2.1 (**Need for explicit M5 super-immersions**).
(i) The traditional literature [BPSTV95][HS97a][HRS98] [HS97b][So00] (recent review in [BaSo23]) contains arguments that "super-embeddings" (i.e. 1/2BPS super-immersions, [GSS24b, Def. 2.19]) of super p-brane worldvolumes *imply* the equations of motion of the corresponding super p-brane σ-model. However, the converse conclusion — that *no further* contraints than these equations of motion are implied — is far from obvious and has only partially been addressed (e.g. for some aspects of the M2-brane in [BPSTV95, (2.50-52)]). Related to this may be the absence of previously published examples of non-trivial super-embeddings.

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solutions — goes back all the way to [EIH38], and has fascinated authors since, see for instance [AP04][Bu08].

(ii) The analogous issue in the derivation of 11d supergravity (from the superspace torsion constraint) had similarly remained unaddressed in published literature. In this case, we had settled the reverse implication with the substantial help of mechanized computer algebra [GSS24a, Thm. 3.1]. The humongous cancellations that happen to make this work seem nothing less than a miracle, quite reinforcing the idea that 11d supergravity occupies a special point in the space of all field theories.

(iii) A similar miracle may be needed to guarantee that for constructing an M5 super-immersion it is sufficient to solve its equations of motion, plausible as this may otherwise sound, cf. Rem. 2.20 below. In lack of a complete argument to this extent, but to still have the desired implication of the super-flux Bianchi identity ([GSS24b, Prop. 3.17], needed for the flux quantization argument in §3), we have to construct M5 super-immersions explicitly.

This is what we do now for the case of holographic M5 immersions. Apart from its implications to flux quantization in §3, we highlight that this is of interest in its own right as a rare explicit example of a non-trivial 1/2BPS super-immersion ("super-embedding").

Tensor conventions. Our tensor conventions are standard, but since the computations below crucially depend on the corresponding prefactors, here to briefly make them explicit:

- The Einstein summation convention applies throughout: Given a product of terms indexed by some $i \in I$, with the index of one factor in superscript and the other in subscript, then a sum over I is implied: $x_i\, y^i := \sum_{i\in I} x_i\, y^i$.
- Our Minkowski metric is the matrix
$$\big(\eta_{ab}\big)_{a,b=0}^{d} \;=\; \big(\eta^{ab}\big)_{a,b=0}^{d} \;:=\; \big(\mathrm{diag}(-1,+1,+1,\cdots,+1)\big)_{a,b=0}^{d}\,. \tag{2}$$
- Shifting position of frame indices always refers to contraction with the Minkowski metric (2):
$$V^a \;:=\; V_b\,\eta^{ab}\,, \qquad V_a \;=\; V^b\eta_{ab}\,.$$
- Skew-symmetrization of indices is denoted by square brackets ($(-1)^{|\sigma|}$ is sign of the permutation σ):
$$V_{[a_1\cdots a_p]} \;:=\; \tfrac{1}{p!}\sum_{\sigma\in\mathrm{Sym}(n)} (-1)^{|\sigma|} V_{a_{\sigma(1)}\cdots a_{\sigma(p)}}\,.$$
- We normalize the Levi-Civita symbol to
$$\epsilon_{012\cdots} \;:=\; +1 \quad\text{hence}\quad \epsilon^{012\cdots} \;:=\; -1\,. \tag{3}$$
- We normalize the Kronecker symbol to
$$\delta^{a_1\cdots a_p}_{b_1\cdots b_p} \;:=\; \delta^{[a_1}_{[b_1}\cdots\delta^{a_p]}_{b_p]} \;=\; \delta^{a_1}_{[b_1}\cdots\delta^{a_p}_{b_p]} \;=\; \delta^{[a_1}_{b_1}\cdots\delta^{a_p]}_{b_p} \tag{4}$$
so that
$$V_{a_1\cdots a_p}\delta^{a_1\cdots a_p}_{b_1\cdots b_p} \;=\; V_{[b_1\cdots b_p]} \quad\text{and}\quad \epsilon^{c_1\cdots c_p a_1\cdots a_q}\,\epsilon_{c_1\cdots c_p b_1\cdots b_q} \;=\; -\,p!\cdot q!\,\delta^{a_1\cdots a_q}_{b_1\cdots b_q}\,. \tag{5}$$

Spinors in 11d. We briefly recall the following standard facts (proofs and references are given in [GSS24a, §2.2.1]): There exists an $\mathbb{R}$-linear representation $\mathbf{32}$ of $\mathrm{Pin}^+(1,10)$ with generators
$$\Gamma_a \;:\; \mathbf{32}\to\mathbf{32} \tag{6}$$
and equipped with a skew-symmetric bilinear form
$$\big(\overline{(-)}(-)\big) \;:\; \mathbf{32}\otimes\mathbf{32}\longrightarrow\mathbb{R} \tag{7}$$
with the following properties, where as usual we denote skew-symmetrized product of k Clifford generators by
$$\Gamma_{a_1\cdots a_k} \;:=\; \tfrac{1}{k!}\sum_{\sigma\in\mathrm{Sym}(k)} \mathrm{sgn}(\sigma)\,\Gamma_{a_{\sigma(1)}}\cdot\Gamma_{a_{\sigma(2)}}\cdots\Gamma_{a_{\sigma(n)}}\;: \tag{8}$$

- The Clifford generators square to plus the Minkowski metric (2)
$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a \;=\; +2\,\eta_{ab}\,\mathrm{id}_{\mathbf{32}}\,. \tag{9}$$
- The Clifford product is given on the basis elements (8) as
$$\Gamma^{a_j\cdots a_1}\,\Gamma_{b_1\cdots b_k} \;=\; \sum_{l=0}^{\min(j,k)} \pm l!\binom{j}{l}\binom{k}{l}\,\delta^{[a_1\cdots a_l}_{[b_1\cdots b_l}\,\Gamma^{a_j\cdots a_{l+1}]}{}_{b_{l+1}\cdots b_k]}\,. \tag{10}$$
- The Clifford volume form equals the Levi-Civita symbol (3):
$$\Gamma_{a_1\cdots a_{11}} \;=\; \epsilon_{a_1\cdots a_{11}}\mathrm{id}_{\mathbf{32}}\,. \tag{11}$$
- The Clifford generators are skew self-adjoint with respect to the pairing (7)
$$\overline{\Gamma_a} \;=\; -\Gamma_a \qquad\text{in that}\qquad \underset{\phi,\psi\in\mathbf{32}}{\forall}\;\big(\overline{(\Gamma_a\phi)}\,\psi\big) \;=\; -\big(\overline{\phi}\,(\Gamma_a\psi)\big)\,, \tag{12}$$
so that generally
$$\overline{\Gamma_{a_1\cdots a_p}} \;=\; (-1)^{p+p(p-1)/2}\,\Gamma_{a_1\cdots a_p}\,. \tag{13}$$

- The $\mathbb{R}$-vector space of $\mathbb{R}$-linear endomorphisms of $\mathbf{32}$ has a linear basis given by the $\leq$ 5-index Clifford elements
$$\mathrm{End}_{\mathbb{R}}\big(\mathbf{32}\big) \;=\; \big\langle 1,\, \Gamma_{a_1},\, \Gamma_{a_1 a_2},\, \Gamma_{a_1 a_2 a_3},\, \Gamma_{a_1 \cdots a_4},\, \Gamma_{a_1 \cdots a_5} \big\rangle_{a_i = 0,1,\cdots}\,. \tag{14}$$
- The $\mathbb{R}$-vector space space of *symmetric* bilinear forms on $\mathbf{32}$ has a linear basis given by the expectation values with respect to (7) of the 1-, 2-, and 5-index Clifford basis elements:
$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32} \otimes \mathbf{32})_{\mathrm{sym}},\, \mathbb{R} \Big) \;\simeq\; \Big\langle \big((\overline{-}) \Gamma_a (-) \big)\,,\;\; \big((\overline{-}) \Gamma_{a_1 a_2} (-) \big)\,,\;\; \big((\overline{-}) \Gamma_{a_1 \cdots a_5} (-) \big) \Big\rangle_{a_i = 0,1,\cdots\,,} \tag{15}$$
while a basis for the skew-symmetric bilinear forms is given by
$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32} \otimes \mathbf{32})_{\mathrm{skew}},\, \mathbb{R} \Big) \;\simeq\; \Big\langle \big((\overline{-}) (-) \big)\,,\;\; \big((\overline{-}) \Gamma_{a_1 a_2 a_3} (-) \big)\,,\;\; \big((\overline{-}) \Gamma_{a_1 \cdots a_4} (-) \big) \Big\rangle_{a_i = 0,1,\cdots\,.} \tag{16}$$
- Any linear endomorphism $\phi \in \mathrm{End}_{\mathbb{R}}(\mathbf{32})$ is uniquely a linear combination of Clifford elements as:
$$\phi \;=\; \tfrac{1}{32} \sum_{p=0}^{5} \tfrac{(-1)^{p(p-1)/2}}{p!} \mathrm{Tr}\big(\phi \circ \Gamma_{a_1 \cdots a_p} \big) \Gamma^{a_1 \cdots a_p}\,, \qquad a_i \in \{0, \cdots, 5', 6, 7, 8, 9\}\,. \tag{17}$$

Background formulas for 11d Supergravity. Our notation and conventions for super-geometry and for on-shell 11d supergravity on super-space follow [GSS24a, §2.2 & §3], to which we refer for further details and exhaustive referencing.

We denote the local data of a super-Cartan connection on (a surjective submersion $\widetilde{X}$ of) (super-)spacetime X, representing a super-gravitational field configuration, as[2]
$$\begin{array}{lll}
\textbf{Graviton} & \big(E^a\big)_{a=0}^{D-1} & \in\;\; \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathbb{R}^{1,D-1}\big) \\
\textbf{Gravitino} & \big(\Psi^\alpha\big)_{\alpha=1}^{N} & \in\;\; \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathbf{N}_{\mathrm{odd}}\big) \\
\begin{array}{c}\textbf{Spin-}\\ \textbf{connection}\end{array} & \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{D-1} & \in\;\; \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathfrak{so}(1, D-1)\big)
\end{array} \tag{18}$$
and the corresponding Cartan structural equations (cf. [GSS24a, Def. 2.78]) for the supergravity field strengths as
$$\begin{array}{llllll}
\begin{array}{c}\textbf{Super-}\\ \textbf{Torsion}\end{array} & \big(\,T^a & := & \mathrm{d}\,E^a & -\; \Omega^a{}_b\, E^b - \big(\,\overline{\Psi}\,\Gamma^a\,\Psi\,\big)\,\big)_{a=0}^{D-1} \\
\begin{array}{c}\textbf{Gravitino}\\ \textbf{field strength}\end{array} & \big(\,\rho & := & \mathrm{d}\,\Psi & -\; \tfrac{1}{4}\Omega^{ab}\,\Gamma_{ab}\psi\,\big)_{\alpha=1}^{N} \\
\textbf{Curvature} & \big(\,R^{ab} & := & \mathrm{d}\,\Omega^{ab} & -\; \Omega^a{}_c\,\Omega^{cb}\,\big)_{a,b=0}^{D-1}\,.
\end{array} \tag{19}$$
Finally, we denote the corresponding components in the given local super-coframe (E, Ψ) by [GSS24a, (127-8)]:
$$\begin{array}{lll}
T^a & \equiv & 0 \\
\rho & =: & \tfrac{1}{2}\rho_{ab}\, E^a\, E^b \;+\; H_a \Psi\, E^a \\
R^{a_1 a_2} & =: & \tfrac{1}{2} R^{a_1 a_2}{}_{b_1 b_2}\, E^{a_1}\, E^{a_2} \;+\; \big(\overline{J}^{a_1 a_2}{}_b \Psi\big) E^b \;+\; \big(\,\overline{\Psi}\, K^{a_1 a_2}\,\Psi\,\big)\,,
\end{array} \tag{20}$$
where all components not explicitly appearing vanish identically by the superspace torsion constraints [GSS24a, (121), (137)]. In addition, shortly we will assume that also $\rho_{ab} = 0$ (29) whence also $J^{a_1 a_2}{}_b = 0$ (30).

2.1 Explicit rheonomy

Here we present explicit formulas for extending solutions of 11d supergravity from ordinary spacetime to super-spacetime, in those cases where the (Ψ^0)-component of the gravitino field strength vanishes (29) – which are of course essentially all cases of interest (cf. [FvP12, §12.6]).

This extension process (or the property that it exists) has been called *rheonomy* [CDF91, §III.3.3], alluding to the idea that the ordinary fields "flow" in the odd coordinate directions from the bosonic submanifold over the full supermanifold, to become super-fields. Explicit such formulas have been claimed for the special case of coset-spacetimes (like $\mathrm{AdS}_{p+2} \times S^{D-p+2}$) by [dWPPS98, p. 156][Cl99] (following [KRR98][CK99]), and a derivation in full generality has been given by [Ts04].

We closely follow the latter but find that the specialization (29) to vanishing gravitino field strength (which still subsumes all the former examples) gives a substantial improvement in transparency and usability that may be of interest in its own right. Additionally, we provide full details in order to secure the relative prefactors in the formulas.

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[2] Our use of different letters for the even and odd components of a super co-frame follows e.g. [CDF91]. Other authors write E^α for what we denote Ψ^α, e.g. [BaSo23]. While it is of course part of the magic of supergravity that E^a and E^α/Ψ^α are unified into a single object, we find that for reading and interpreting formulas it is helpful to use different symbols.

The strategy of the construction is to expand the super-fields and their structural equations in a suitable gauge on a suitable super-coordinate chart in order to obtain explicit differential equations for the flow along the odd coordinate directions. Therefore we start by considering:

Coordinate-components of superfields. On a super-chart with coordinates (X, Θ) we have the expansion of the super-gravitational fields (18) first into their coefficients of the coordinate-differentials and then further their super-field expansion as polynomials in the odd coordinates (with index convention as shown on the right),

	Even	**Odd**
Frame	$a \in \{0, \cdots, 10\}$	$\alpha \in \{1, \cdots, 32\}$
Coord.	$r \in \{0, \cdots, 10\}$	$\rho \in \{1, \cdots, 32\}$

$$
\begin{array}{lllll}
E^a & =: & E^a_r \, \mathrm{d}X^r & + & E^a_\rho \, \mathrm{d}\Theta^\rho \\
\Psi^\alpha & =: & \Psi^\alpha_r \, \mathrm{d}X^r & + & \Psi^\alpha_\rho \, \mathrm{d}\Theta^\rho \\
\Omega^{ab} & = & \Omega^{ab}_r \, \mathrm{d}X^r & + & \Omega^{ab}_\rho \, \mathrm{d}\Theta^\rho
\end{array}
\qquad
\begin{array}{lllll}
E^a_{r/\rho} & =: & \sum_{n=0}^{32} \big(E^{(n)}\big)^a_{r/\rho} & =: & \sum_{n=0}^{32} \frac{1}{n!} \, \Theta^{\rho_1} \cdots \Theta^{\rho_n} \, \big(E^{(n)}_{\rho_1 \cdots \rho_n}\big)^a_{r/\rho} \\
\Psi^\alpha_{r/\rho} & =: & \sum_{n=0}^{32} \big(\Psi^{(n)}\big)^\alpha_{r/\rho} & =: & \sum_{n=0}^{32} \frac{1}{n!} \, \Theta^{\rho_1} \cdots \Theta^{\rho_n} \, \big(\Psi^{(n)}_{\rho_1 \cdots \rho_n}\big)^\alpha_{r/\rho} \\
\Omega^{ab}_{r/\rho} & =: & \sum_{n=0}^{32} \big(\Omega^{(n)}\big)^{ab}_{r/\rho} & =: & \sum_{n=0}^{32} \frac{1}{n!} \, \Theta^{\rho_1} \cdots \Theta^{\rho_n} \, \big(\Omega^{(n)}_{\rho_1 \cdots \rho_n}\big)^{ab}_{r/\rho} \,,
\end{array}
\tag{21}
$$

whose coefficients are functions on the underlying bosonic manifold which are skew-symmetric in their indices:

$$
\begin{pmatrix}
E^{(n)}_{\rho_1 \cdots \rho_n} \\
\Psi^{(n)}_{\rho_1 \cdots \rho_n} \\
\Omega^{(n)}_{\rho_1 \cdots \rho_n}
\end{pmatrix}
\;:\; \overset{\rightsquigarrow}{X} \longrightarrow \mathfrak{iso}\big(\mathbb{R}^{1,10|\mathbf{32}}\big),
\qquad
\begin{array}{lll}
E^{(n)}_{\rho_1 \cdots \rho_n} & = & E^{(n)}_{[\rho_1 \cdots \rho_n]} \\
\Psi^{(n)}_{\rho_1 \cdots \rho_n} & = & \Psi^{(n)}_{[\rho_1 \cdots \rho_n]} \\
\Omega^{(n)}_{\rho_1 \cdots \rho_n} & = & \Omega^{(n)}_{[\rho_1 \cdots \rho_n]} \,.
\end{array}
\tag{22}
$$

Notice that this implies:

$$
\big(E^{(n)}_{[\rho' \, \rho_2 \cdots \rho_n}\big)^a_{\rho]} \;=\; \tfrac{1}{n+1} \Big(n \big(E^{(n)}_{\rho' \, [\rho_2 \cdots \rho_n}\big)^a_{\rho]} - \big(E^{(n)}_{\rho \, \rho_2 \cdots \rho_n}\big)^a_{\rho'} \Big) \,.
\tag{23}
$$

Also notice the $\mathbb{N} \times \mathbb{Z}_2$ bi-degrees (cf. [GSS24a, §2.1.1]) of the Ψ-components,

$$
\begin{array}{rcccccc}
 & \Psi^\alpha & = & \Psi^\alpha_r & \mathrm{d}X^r & + \;\; \Psi^\alpha_\rho & \mathrm{d}\Theta^\rho \\
\mathrm{deg:} & (1,1) & & (0,1) & (1,0) & (0,0) & (1,1) \,,
\end{array}
\tag{24}
$$

which implies in particular that the component functions Ψ^α_ρ commute with all other terms.

Wess-Zumino-Tsimpis gauge. On these components, we may impose the following gauge conditions ([Ts04, (39-42)], following [McA84, (A.3-4)][AD87, (17-18)]):

Definition 2.2 (**Wess-Zumino-Tsimpis gauge** [3]**).** The *WZT gauge* is given by the following conditions:

$$
\begin{array}{lll}
\big(E^{(0)}\big)^a_\rho & \equiv & 0 \\
\big(\Psi^{(0)}\big)^\alpha_\rho & \equiv & \delta^\alpha_\rho \\
\big(\Omega^{(0)}\big)^{ab}_\rho & \equiv & 0
\end{array}
\qquad \text{and} \qquad
\underset{n \in \{1, \cdots, 32\}}{\forall}
\left\{
\begin{array}{lll}
\big(E^{(n)}_{[\rho_1 \cdots \rho_n}\big)^a_{\rho]} & \equiv & 0 \\
\big(\Psi^{(n)}_{[\rho_1 \cdots \rho_n}\big)^\alpha_{\rho]} & \equiv & 0 \\
\big(\Omega^{(n)}_{[\rho_1 \cdots \rho_n}\big)^{ab}_{\rho]} & \equiv & 0 \,.
\end{array}
\right.
\tag{25}
$$

Lemma 2.3 (**Direct implications of WZT gauge**)**.** *The WZT gauge conditions (25) imply:*

$$
\begin{array}{lll}
\Theta^\rho E^a_\rho & = & 0 \\
\Theta^\rho \, \Psi^\alpha_\rho & = & \Theta^\rho \, \delta^\alpha_\rho \;=:\; \Theta^\alpha \\
\Theta^\rho \, \Omega_\rho{}^{ab} & = & 0
\end{array}
\qquad \text{and} \qquad
\underset{n \in \{1, \cdots, 32\}}{\forall}
\left\{
\begin{array}{lll}
\Theta^\rho \, \partial_{\rho'} \big(E^{(n)}\big)^a_\rho & = & \big(E^{(n)}\big)^a_{\rho'} \\
\Theta^\rho \, \partial_{\rho'} \big(\Psi^{(n)}\big)^\alpha_\rho & = & \big(\Psi^{(n)}\big)^\alpha_{\rho'} \\
\Theta^\rho \, \partial_{\rho'} \big(\Omega^{(n)}\big)^{ab}_\rho & = & \big(\Omega^{(n)}\big)^{ab}_{\rho'} \,.
\end{array}
\right.
\tag{26}
$$

Proof. The implications on the left of (26) are immediate (cf. [Ts04, (43-44)]). To see the equations on the right of (26) we may proceed as follows:

---

[3] Recall (e.g. [BK95, §3.4.3]) that the *Wess-Zumino gauge* on chiral superfields constrains their dependence on the super-coordinates, hence their auxiliary super-components, but not the physical fields. The suggestion to think of this, in the context of curved super-space/supergravity, as a special case of fermionic Riemann normal coordinates may be due to [AD87], and the higher component generalization (25) is due to [Ts04].

$$
\begin{aligned}
\Theta^\rho \, \partial_{\rho'} \big(E^{(n+1)}\big)^a_\rho
&= \tfrac{1}{n!} \, \Theta^\rho \, \Theta^{\rho_2} \cdots \Theta^{\rho_{n+1}} \big(E^{(n+1)}_{\rho' \, [\rho_2 \cdots \rho_{n+1}}\big)^a_{\rho]} && \text{by (21)}
\\
&= \tfrac{1}{(n+1)!} \, \Theta^\rho \, \Theta^{\rho_2} \cdots \Theta^{\rho_{n+1}} \big(E^{(n+1)}_{\rho \, \rho_2 \cdots \rho_{n+1}}\big)^a_{\rho'} && \text{by (23) \& (25)}
\\
&= \big(E^{(n+1)}_{\rho \, \rho_2 \cdots \rho_{n+1}}\big)^a_{\rho'} && \text{by (21)} \,,
\end{aligned}
\tag{27}
$$

and verbatim so also for E replaced by Ψ or Ω. □

Remark 2.4 (**Fermionic normal coordinates and Rheonomy).** The WZT gauge of Def. 2.2 may be understood as a fermionic form of Riemann normal coordinates [McA84, (A.3-4)][AD87, (17-18)]. In particular the implication $\Theta^\rho \, \Omega_\rho{}^{ab} = 0$ (26) has the further consequence that for translations along the odd coordinate direction ("rheonomy" [CDF91, §III.3.3]) the covariant derivative reduces to the plain coordinate derivative:

$$
\Theta^\rho \, \nabla_\rho \;=\; \Theta^\rho \, \partial_\rho \,.
\tag{28}
$$

Gravitino-flat supergravity solutions on super-space. For our purpose here we focus on solutions to 11d supergravity for which the ordinary component of the gravitino field strength (20) vanishes,

$$
\rho_{ab} \equiv 0
\tag{29}
$$

(which is the case for essentially all supergravity solutions of interest, cf. [FvP12, §12.6]).

With ρ_{ab} also the super-curvature component $J^{a_1 a_2}{}_b$ vanishes (cf. [GSS24a, (161)]) so that on gravitino-flat solutions the super-field strengths (20) have the form

$$
\begin{aligned}
T^a &= 0
\\
\rho &= H_a \Psi \, E^a
\\
R^{a_1 a_2} &= \tfrac{1}{2} R^{a_1 a_2}{}_{b_1 b_2} \, E^{a_1} \, E^{a_2} \;+\; \big(\overline{\Psi} \, K^{a_1 a_2} \, \Psi\big) \,.
\end{aligned}
\tag{30}
$$

Lemma 2.5 (**Θ-independence of field components).** *For gravitino-flat* (29) *super-space solutions of 11d SuGra in WZT gauge* (Def. 2.2) *the following super-field strength components* (30) *are all independent of the odd coordinates* Θ^ρ*:*

$$
\begin{array}{ll}
\textbf{The flux densities} & \partial_\rho\big((G_4)_{a_1 \cdots a_4}\big) \;=\; 0 \,, \qquad \partial_\rho\big((G_7)_{a_1 \cdots a_7}\big) \;=\; 0 \,,
\\[6pt]
\begin{array}{l}\textbf{Odd co-frame component of} \\ \textbf{the gravitino field strength}\end{array} & \partial_\rho\big(H_a\big) \;=\; 0 \,,
\\[6pt]
\begin{array}{l}\textbf{Odd co-frame components} \\ \textbf{of the super-curvature}\end{array} & \partial_\rho\big(K^{a_1 a_2}\big) \;=\; 0 \,.
\end{array}
\tag{31}
$$

Proof. This follows by use of the well-known super-space constraints, which we quote from [GSS24a] (where full derivation and referencing is given). First, the Θ-independence of G_4 follows by

$$
\begin{aligned}
\Theta^\rho \, \partial_\rho\big((G_4)_{a_1 \cdots a_4}\big) &= \Theta^\rho \, \nabla_\rho\big((G_4)_{a_1 \cdots a_4}\big) && \text{by (28)}
\\
&= 12\big(\overline{\Theta} \, \Gamma_{[a_1 a_2} \, \rho_{a_2 a_3]}\big) && \text{by [GSS24a, (136)]}
\\
&= 0 && \text{by (29).}
\end{aligned}
$$

But the remaining components in (31) are linear functions of $(G_4)_{a_1 \cdots a_4}$:

$$
\begin{aligned}
H_a &= \tfrac{1}{6}\tfrac{1}{3!}(G_4)_{a \, b_1 b_2 b_3} \, \Gamma^{b_1 b_2 b_3} \;-\; \tfrac{1}{12}\tfrac{1}{4!}(G_4)^{b_1 \cdots b_4} \, \Gamma_{a \, b_1 \cdots b_4} && \text{[GSS24a, (135)]}
\\
&= \tfrac{1}{6}\tfrac{1}{3!}(G_4)_{a \, b_1 b_2 b_3} \, \Gamma^{b_1 b_2 b_3} \;+\; \tfrac{1}{12}\tfrac{1}{6!}(G_7)_{a \, c_1 \cdots c_6} \, \Gamma^{c_1 \cdots c_6} && \text{[GSS24a, (148)]}
\\[8pt]
K^{a_1 a_2} &= -\tfrac{1}{6}\Big((G_4)^{a_1 a_2 \, b_1 b_2} \Gamma_{b_1 b_2} \;+\; \tfrac{1}{4!}(G_4)_{b_1 \cdots b_4} \Gamma^{a_1 a_2 \, b_1 \cdots b_4}\Big) && \text{[GSS24a, (162)]}
\\
&= -\tfrac{1}{6}\Big((G_4)^{a_1 a_2 \, b_1 b_2} \Gamma_{b_1 b_2} \;+\; \tfrac{1}{5!}(G_7)^{a_1 a_2 \, b_1 \cdots b_5} \Gamma_{b_1 \cdots b_5}\Big)
\end{aligned}
\tag{32}
$$

and hence their Θ-dependence vanishes with that of G_4 and G_7. □

Supergravity field extension to super-space. We now consider solutions to the rheonomy equations for extending on-shell 11d supergravity fields to superspace, cast into recursion relations in the polynomial order of their odd coordinate field dependence as in [Ts04] (similar to [dWPPS98, (3.9)]), but specialized to the case of gravitino-flat spacetimes (29).

Lemma 2.6 (**Rheonomy for the graviton**)**.** *In WZT gauge* (25) *the following recursion relations hold for the bosonic coframe field components* (21)*, recursing in their odd coordinate degree* $n+1 \in \{1, \cdots, 32\}$*:*

$$\begin{aligned} (E^{(n+1)})^a_\rho &= \tfrac{2}{n+2}\big(\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}_\rho\big)\,, \\ (E^{(n+1)})^a_r &= \tfrac{2}{n+1}\big(\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}_r\big) \end{aligned} \tag{33}$$

(cf. [Ts04, (58, 59)].[4]

Proof. The $\mathrm{d}\Theta^\rho$-component of (33) follows as:

$$\begin{aligned} & & \mathrm{d}\,E^a &= \Omega^a{}_b\,E^b + \big(\overline{\Psi}\,\Gamma^a\,\Psi\big) && \text{from (19)} \\ &\Rightarrow & \Theta^\rho\,\partial_{(\rho}\,E^a_{\rho')} &= \Theta^\rho\,\big(\Omega^a{}_b\big)_{(\rho}E^b_{\rho')} + \Theta^\rho\,\Psi^\alpha_{(\rho}\,\Psi^{\alpha'}_{\rho')}\Gamma^a_{\alpha\alpha'} && \text{by (21)} \\ &\Leftrightarrow & \Theta^\rho\partial_{(\rho}\,E^a_{\rho')} &= \Theta^\rho\delta^\alpha_\rho\,\Psi^{\alpha'}_{\rho'}\Gamma^a_{\alpha\alpha'} && \text{by (26) \& (24)} \\ &\Rightarrow & \underbrace{\Theta^\rho\partial_{(\rho}\,\big(E^{(n+1)}\big)^a_{\rho')}}_{\frac{(n+2)}{2}\left(E^{(n+1)}\right)^a_{\rho'}} &= \underbrace{\Theta^\alpha(\Psi^{(n)})^{\alpha'}_{\rho'}\Gamma^a_{\alpha\alpha'}}_{\left(\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}\right)} && \text{by (21) \& (26),} \end{aligned}$$

and the $\mathrm{d}X^r$-component as:

$$\begin{aligned} & & \mathrm{d}\,E^a &= \Omega^a{}_b\,E^b + \big(\,\overline{\Psi}\,\Gamma^a\,\Psi\big) && \text{from (19)} \\ &\Rightarrow & \Theta^\rho\,\partial_\rho E^a_r &= \Theta^\rho\big(\Omega^a{}_b\big)_\rho\,E^b_r - \Theta^\rho\big(\Omega^a{}_b\big)_r\,E^b_\rho + 2\,\Theta^\rho\,\Psi^\alpha_\rho\,\Psi^{\alpha'}_r\Gamma^a_{\alpha\alpha'} && \text{by (21)} \\ &\Leftrightarrow & \Theta^\rho\partial_\rho E^a_r &= 2\,\Theta^\rho\delta^\alpha_\rho\,\Psi^{\alpha'}_r\Gamma^a_{\alpha\alpha'} && \text{by (26)} \\ &\Rightarrow & \underbrace{\Theta^\rho\partial_\rho\big(E^{(n+1)}\big)^a_r}_{(n+1)\left(E^{(n+1)}\right)^a_r} &= 2\,\big(\,\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}_r\big) && \text{by (21) \& (26).} \end{aligned}$$

□

Lemma 2.7 (**Rheonomy for the spin-connection**)**.** *On gravitino-flat* (29) *super-spacetimes in WZT gauge* (25) *we have the following recursion relations for the spin connection* (21)*, recursing in the odd coordinate degree* $n+1 \in \{1, \cdots, 32\}$*:*

$$\begin{aligned} \big(\Omega^{(n+1)}\big)^{a_1a_2}_\rho &= \tfrac{2}{n+2}\big(\overline{\Theta}\,K^{a_1a_2}\,\Psi^{(n)}_\rho\big) \\ \big(\Omega^{(n+1)}\big)^{a_1a_2}_r &= \tfrac{2}{n+1}\big(\overline{\Theta}\,K^{a_1a_2}\,\Psi^{(n)}_r\big) \end{aligned} \tag{34}$$

(cf. [Ts04, (61, 64)][5] noticing our (31)).

Proof. In (34) the $\mathrm{d}\Theta^\rho$-component follows by:

$$\begin{aligned} & & \mathrm{d}\,\Omega^{a_1a_2} &= \Omega^{a_1}{}_b\,\Omega^{ba_2} + R^{a_1a_2} && \text{from (19)} \\ &\Rightarrow & \Theta^{\rho'}\,\partial_{(\rho'}\,(\Omega^{a_1a_2})_{\rho)} &= \Theta^{\rho'}\,\delta^{\alpha'}_{\rho'}\,\Psi^\alpha_\rho\,K^{a_1a_2}_{\alpha'\alpha} && \text{by (30), (21) \& (26)} \\ &\Rightarrow & \underbrace{\Theta^{\rho'}\,\partial_{(\rho'}\,\big(\Omega^{(n+1)}\big)^{a_1a_2}_{\rho)}}_{\frac{(n+2)}{2}\left(\Omega^{(n+1)}\right)^{a_1a_2}_{\rho}} &= \big(\,\overline{\Theta}\,K^{a_1a_2}\,\Psi^{(n)}_\rho\big) && \text{by (21), (26) \& (31),} \end{aligned}$$

---

[4] The factor of “i/2” by which our (33) differs from [Ts04, (58, 59)] is absorbed by our convention for the spacetime signature, the Clifford algebra and the Majorana spinor: Our Γ-matrices are i times the Gamma matrices there (which makes all expressions in Majorana spinors manifestly real, cf. [GSS24a, Rem. 1.7])), and we do not include a factor of 1/2 multiplying the (Ψ^2)-term in the definition of the super-torsion (19).

[5] As in footnote 4, the difference of [Ts04, (61, 64)] from (34) by a factor of i/2 is due to our spinor convention.

and the $\mathrm{d}X^r$-component by:

$$\begin{array}{rrcll}
 & \mathrm{d}\,\Omega^{a_1 a_2} & = & \Omega^{a_1}{}_b\,\Omega^{b a_2} + R^{a_1 a_2} & \text{from (19)} \\[6pt]
\Rightarrow & \Theta^\rho\,\partial_\rho(\Omega^{a_1 a_2})_r & = & 2\,\Theta^\rho\,\Psi^\alpha_\rho\,\Psi^{\alpha'}_r\,K^{a_1 a_2}_{\alpha\alpha'} & \text{by (30), (21), \& (26)} \\[6pt]
\Rightarrow & \underbrace{\Theta^\rho\,\partial_\rho\big(\Omega^{(n+1)}\big)^{a_1 a_2}_r}_{(n+1)\left(\Omega^{(n+1)}\right)^{a_1 a_2}_r} & = & 2\left(\overline{\Theta}\,K^{a_1 a_2}\,\Psi^{(n)}\right) & \text{by (21), (26) \& (31).}
\end{array}$$

☐

Lemma 2.8 (**Rheonomy for the gravitino).** *On gravitino-flat (29) super-spacetimes in WZT gauge (25) the following recursion relations hold for the odd coordinate dependence of the gravitino field (21):*

$$\begin{aligned}
\big(\Psi^{(n+1)}\big)^\alpha_\rho &= +\tfrac{1}{n+2}\tfrac{1}{4}\big(\Gamma_{ab}\Theta\big)^\alpha\big(\Omega^{(n)}\big)^{ab}_\rho + \tfrac{1}{n+2}\,(H_a\Theta)^\alpha\big(E^{(n)}\big)^a_\rho \\
\big(\Psi^{(n+1)}\big)^\alpha_r &= -\tfrac{1}{n+1}\tfrac{1}{4}(\Gamma_{ab}\Theta)^\alpha\big(\Omega^{(n)}\big)^{ab}_r + \tfrac{1}{n+1}(H_a\Theta)^\alpha\big(E^{(n)}\big)^a_r\,.
\end{aligned} \tag{35}$$

Proof. In (35) the $\mathrm{d}\Theta^\rho$-component follows by:

$$\begin{array}{rrcll}
 & \mathrm{d}\,\Psi^\alpha & = & \tfrac{1}{4}\Omega^{ab}\,(\Gamma_{ab}\Psi)^\alpha + \rho^\alpha & \text{from (19)} \\[6pt]
\Rightarrow & \Theta^{\rho'}\,\partial_{(\rho'}\,\Psi^\alpha_{\rho)} & = & \tfrac{1}{4}\Theta^{\rho'}(\Omega^{ab})_{(\rho'}(\Gamma_{ab}\Psi_{\rho)})^\alpha + \Theta^{\rho'}(H_a\Psi_{(\rho'})^\alpha\,E^a_{\rho)} & \text{by (30), (21), \& (26)} \\[6pt]
\Rightarrow & \underbrace{\Theta^{\rho'}\,\partial_{(\rho'}\big(\Psi^{(n+1)}\big)^\alpha_{\rho)}}_{\frac{n+2}{2}\left(\Psi^{(n+1)}\right)^\alpha_\rho} & = & \tfrac{1}{2}\tfrac{1}{4}\big(\Gamma_{ab}\Theta\big)^\alpha\big(\Omega^{(n)}\big)^{ab}_\rho + \tfrac{1}{2}\,(H_a\Theta)^\alpha\big(E^{(n)}\big)^a_\rho & \text{by (21), (26) \& (31),}
\end{array}$$

and the $\mathrm{d}X^a$-component by:

$$\begin{array}{rrcll}
 & \mathrm{d}\,\Psi^\alpha & = & \tfrac{1}{4}\Omega^{ab}(\Gamma_{ab}\Psi)^\alpha + \rho^\alpha & \text{from (19)} \\[6pt]
\Rightarrow & \Theta^\rho\,\partial_\rho\Psi^\alpha_r & = & -\Theta^\rho\,\tfrac{1}{4}\Omega^{ab}_r(\Gamma_{ab}\Psi_\rho) + \Theta^\rho(H_a\Psi_\rho)^\alpha\,E^a_r & \text{by (30), (21), \& (26)} \\[6pt]
\Rightarrow & \underbrace{\Theta^\rho\partial_\rho\big(\Psi^{(n+1)}\big)^\alpha_r}_{(n+1)\left(\Psi^{(n+1)}\right)^\alpha_r} & = & -\tfrac{1}{4}(\Gamma_{ab}\Theta)\big(\Omega^{(n)}\big)^{ab}_r + (H_a\Theta)^\alpha\,\big(E^{(n)}\big)^a_r & \text{by (21), (26) \& (31).}
\end{array}$$

Notice here how the sign in the second line appears since only the coefficient of $\mathrm{d}X^r\,\mathrm{d}\Theta^\rho$ contributes in the first term, which picks up a sign $\mathrm{d}X^r\,\mathrm{d}\Theta^\rho = -\,\mathrm{d}\Theta^\rho\,\mathrm{d}X^r$ in comparison to the left hand side. ☐

By inserting these recursion relations into each other we may decouple them (resulting in a formulation similar to [dWPPS98, (3.9)]):

Lemma 2.9 (**Decoupled rheonomy recursion relations).** *On gravitino-flat (29) super-spacetimes in WZT gauge (25) the following decoupled recursion relations hold for the odd coordinate dependence of the super-fields:*

$$\begin{aligned}
\big(\Psi^{(n+2)}\big)^\alpha_\rho &= +\tfrac{1}{n+4}\tfrac{2}{n+3}\tfrac{1}{4}\big(\Gamma_{a_1 a_2}\Theta\big)^\alpha\Big(\overline{\Theta}\,K^{a_1 a_2}\,\Psi^{(n)}_\rho\Big) + \tfrac{1}{n+4}\tfrac{2}{n+3}(H_a\Theta)^\alpha\Big(\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}_\rho\Big) \\
\big(\Psi^{(n+2)}\big)^\alpha_r &= -\tfrac{1}{n+2}\tfrac{1}{n+1}\tfrac{1}{4}(\Gamma_{a_1 a_2}\Theta)^\alpha\Big(\overline{\Theta}\,K^{a_1 a_2}\,\Psi^{(n)}_r\Big) + \tfrac{1}{n+2}\tfrac{1}{n+1}(H_a\Theta)^\alpha\Big(\overline{\Theta}\,\Gamma^a\,\Psi^{(n)}_r\Big)
\end{aligned} \quad \begin{array}{l}\text{by inserting}\\ \text{(34) \& (33)}\\ \text{into (35).}\end{array} \tag{36}$$

2.2 Spinors on M5-branes

We briefly recall and record some properties of spinors in 6d among spinors in 11d, following [GSS24b, §3.2], which we will need below. In particular, we establish a Fierz identity (in Lem. 2.10 below), which is crucial in the proof of the M5-immersion in §2.4 below. In contrast to existing literature, we do not use a matrix representation of the 6d Clifford algebra but instead use projection operators (37) to algebraically carve it out of the 11d Clifford algebra. We find that this helps considerably with providing proofs in the following sections.

Spinors in 6d form 11d. Following [GSS24b, §3.2] we conveniently identify the chiral $\mathrm{Spin}(1,5)$-representations $2 \cdot \mathbf{8}_{\pm} \in \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,5)\big)$ with the linear subspaces of the $\mathrm{Spin}(1,10)$-representation $\mathbf{32}$ (6) which are the images of the projection operators ([GSS24b, (92)])

$$\left.\begin{array}{rcl} P &:=& \tfrac{1}{2}\big(1 + \Gamma_{5'6789}\big) \\ \overline{P} &:=& \tfrac{1}{2}\big(1 - \Gamma_{5'6789}\big) \end{array}\right\} \;:\; \mathbf{32} \longrightarrow \mathbf{32}\,, \tag{37}$$

respectively, satisfying the following evident but consequential relations (cf. [GSS24b, (89)]):

$$\begin{array}{rcl} P\,P &=& P \\ \overline{P}\,\overline{P} &=& \overline{P} \\ \overline{P}\,P &=& 0 \\ P\,\overline{P} &=& 0 \end{array} \qquad \begin{array}{rcl} \Gamma^a\,P &=& \overline{P}\,\Gamma^a \\ \Gamma^a\,\overline{P} &=& P\,\Gamma^a \\ \Gamma^{5'}\,P &=& P\,\Gamma^{5'} \\ \Gamma^i\,P &=& P\,\Gamma^i \end{array} \qquad \begin{array}{l} a \in \{0,1,2,3,4,5\} \\ \\ \\ i \in \{6,7,8,9\} \end{array} \qquad \begin{array}{rcl} \Gamma_{5'6789}\,P &=& +P \\ \Gamma_{5'6789}\,\overline{P} &=& -\overline{P} \\ \\ \Gamma_{6789}P &=& \Gamma_{5'}P\,, \end{array} \tag{38}$$

where we suggestively denote the 11d Clifford generators as follows:

$$\begin{tikzcd}[row sep=small]
\overbrace{\Gamma_0 \;\; \Gamma_1 \;\; \Gamma_2 \;\; \Gamma_3 \;\; \Gamma_4 \;\; \Gamma_5}^{\text{tangential}} \arrow[d, bend right, "{\substack{\overline{P}(-)P \\ +P(-)\overline{P}}}"'] & \overbrace{\Gamma_{5'}}^{\text{radial}} & \overbrace{\Gamma_6 \;\; \Gamma_7 \;\; \Gamma_8 \;\; \Gamma_9}^{\text{transversal}} & \in \mathrm{Pin}^+(1,10) \;\subset\; \mathrm{End}_{\mathbb{R}}(\mathbf{32}) \\
\gamma_0 \;\; \gamma_1 \;\; \gamma_2 \;\; \gamma_3 \;\; \gamma_4 \;\; \gamma_5 & & & \in \mathrm{Pin}^+(1,5) \;\subset\; \mathrm{End}_{\mathbb{R}}\big(2\cdot\mathbf{8}_+ \oplus 2\cdot\mathbf{8}_-\big)\,,
\end{tikzcd} \tag{39}$$

in that under the corresponding inclusion

$$\mathrm{Spin}(1,5) \hookrightarrow \mathrm{Spin}(1,10)$$

there are isomorphisms [GSS24b, (86-90)]

$$\begin{array}{rcl} 2\cdot\mathbf{8} \;:=\; 2\cdot\mathbf{8}_+ &\simeq& P(\mathbf{32}) \\ 2\cdot\mathbf{8}_- &\simeq& \overline{P}(\mathbf{32})\,. \end{array} \tag{40}$$

Combined with the vector representation of $\mathrm{Spin}(1,10)$ and $\mathrm{Spin}(1,5)$ on $\mathbb{R}^{1,10}$ and $\mathbb{R}^{1,5}$, respectively, we may regard P (37) as a projector of super-vector spaces

$$\begin{tikzcd}
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, two heads] \arrow[rr, bend left, "P"] & \mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}} \arrow[r, hook] & \mathbb{R}^{1,10\,|\,\mathbf{32}}
\end{tikzcd} \qquad \begin{array}{rcl} P &:=& \tfrac{1}{2}\big(1 + \Gamma_{5'6789}\big) \\ \overline{P} &:=& \tfrac{1}{2}\big(1 - \Gamma_{5'6789}\big)\,, \end{array} \tag{41}$$

which is convenient for unifying the conditions on tangential and transversal super-coframe components in a 1/2BPS super-immersion (Def. 2.14 below).

Lemma 2.10 (A Fierz identity in 6d). *Elements $\theta \in (2\cdot\mathbf{8})_{\mathrm{odd}}$ satisfy*

$$\gamma_a \theta\,\overline{\theta}\gamma^a \;=\; 0\,. \tag{42}$$

Proof. Recall from (40) that we may and do regard $\theta = P\theta \in 2\cdot\mathbf{8} \subset \mathbf{32}$ as an 11d spinor but constrained to be in the image of the projector $P := \tfrac{1}{2}\big(1 + \Gamma_{5'6789}\big)$, see (37). With this, we may use the formula for Clifford expansion (17) of general endomorphisms $\phi \in \mathrm{End}_{\mathbb{R}}(\mathbf{32})$ in the case where

$$\begin{array}{rcl} \phi \;\equiv\; \theta\,\overline{\theta} : \mathbf{32} &\longrightarrow& \mathbf{32} \\ \Phi &\longmapsto& \theta\big(\overline{\theta}\,\Phi\big)\,, \end{array}$$

with the spinor pairing (7) on the right.

But since θ (as opposed to $\mathrm{d}\theta$, cf. [GSS24a, Rem. 2.62]) is a skew-commuting variable, it is only the skew-symmetric Clifford basis elements among $\Gamma_{a_1 \cdots a_p}$ $(p \leq 5)$ which are non-vanishing when evaluated in $(\overline{\theta} - \theta)$, and these are precisely those with 0, 3 or 4 indices (16). Hence (17) specializes to:

$$\theta\,\overline{\theta} \;=\; -\tfrac{1}{32}\Big(\big(\overline{\theta}\,\theta\big) \;-\; \tfrac{1}{3!}\big(\overline{\theta}\,\Gamma_{a_1 a_2 a_3}\,\theta\big)\,\Gamma^{a_1 a_2 a_3}\Big) \;+\; \tfrac{1}{4!}\big(\overline{\theta}\,\Gamma_{a_1 \cdots a_4}\,\theta\big)\,\Gamma^{a_1 \cdots a_4}\Big)\,, \qquad a_i \in \{0,\cdots,5', 6, 7, 8, 9\}\,.$$

Moreover, since the only Clifford elements which remain non-vanishing when sandwiched in $\overline{P} - P$ are those carrying an odd number of tangential (6d) indices, by (38), this reduces further to

$$\begin{array}{rcl} \theta\,\overline{\theta} &=& \tfrac{1}{32}\Big(\tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{a_1 a_2 a_3}\,\theta\big)\,\gamma^{a_1 a_2 a_3} \;-\; \tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{a_1 a_2 a_3}\Gamma_i\,\theta\big)\,\gamma^{a_1 a_2 a_3}\Gamma^i \\ && \quad +\,\tfrac{1}{2}\big(\overline{\theta}\,\gamma_a\Gamma_{i_1 i_2}\,\theta\big)\,\gamma^a\Gamma^{i_1 i_2} \;-\; \tfrac{1}{3!}\big(\overline{\theta}\,\gamma_a\Gamma_{i_1 i_2 i_3}\,\theta\big)\,\gamma^a\Gamma^{i_1 i_2 i_3}\Big) \end{array} \qquad \begin{array}{l} a_i \in \{0,\cdots,5\} \\ i_i \in \{5', 6, 7, 8, 9\}\,. \end{array} \tag{43}$$

But finally, by Hodge duality in the transverse directions

$$\Gamma_{i_1 i_2 i_3}P \;\underset{(38)}{=}\; \Gamma_{i_1 i_2 i_3}\Gamma_{5'6789}P \;=\; \pm\tfrac{1}{2}\epsilon_{i_1 i_2 i_3\, i_4 i_5}\Gamma^{i_4 i_5}P \qquad i_i \in \{5, 6, 7, 8, 9\}\,, \tag{44}$$

we have for the last summand in (43):

$$\begin{aligned} \tfrac{1}{3!}\big(\overline{\theta}\gamma_a\Gamma_{i_1 i_2 i_3}\theta\big)\gamma^a\Gamma^{i_1 i_2 i_3} &= \tfrac{1}{3!}\tfrac{1}{2\cdot 2}\epsilon_{i_1 i_2 i_3\, i_4 i_5}\epsilon^{i_1 i_2 i_3\, j_4 j_5}\big(\overline{\theta}\gamma_a\Gamma^{i_4 i_5}\theta\big)\gamma^a\Gamma_{j_4 j_5} && \text{by (44)} \\ &= \tfrac{1}{3!}\tfrac{3!\cdot 2!}{2\cdot 2}\delta^{j_4 j_5}_{i_4 i_4}\big(\overline{\theta}\gamma_a\Gamma^{i_4 i_5}\theta\big)\gamma^a\Gamma_{j_4 j_5} && \text{by (5)} \\ &= \tfrac{1}{2}\big(\overline{\theta}\gamma_a\Gamma^{i_4 i_5}\theta\big)\gamma^a\Gamma_{i_4 i_5} && \text{by (5),} \end{aligned}$$

whereby that the last two summands in (43) cancel each other, and we are left with:

$$\theta\,\overline{\theta} \;=\; \tfrac{1}{32}\Big(\tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{a_1 a_2 a_3}\,\theta\big)\,\gamma^{a_1 a_2 a_3} \;-\; \tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{a_1 a_2 a_3}\Gamma_i\,\theta\big)\,\gamma^{a_1 a_2 a_3}\Gamma^i\Big)\,, \qquad \begin{array}{l} a_i \in \{0,\cdots,5\} \\ i_i \in \{5', 6, 7, 8, 9\}\,. \end{array} \tag{45}$$

Now observing (by decomposing the sum and making a simple case analysis) that

$$\gamma_b\,\gamma_{a_1 a_2 a_3}\gamma^b \;=\; 0\,, \qquad a_i, b \in \{0,\cdots,5\}\,, \tag{46}$$

the claim (42) follows:

$$\begin{aligned} \gamma_a\theta\,\overline{\theta}\gamma^a &= \tfrac{1}{32}\Big(\tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{b_1 b_2 b_3}\,\theta\big)\,\underbrace{\gamma_a\gamma^{b_1 b_2 b_3}\gamma^a}_{=0} + \tfrac{1}{3!}\big(\overline{\theta}\,\gamma_{b_1 b_2 b_3}\Gamma_i\,\theta\big)\,\underbrace{\gamma_a\gamma^{b_1 b_2 b_3}\gamma^a}_{=0}\Gamma^i\Big) && \text{by (45)} \\ &= 0 && \text{by (46).} \end{aligned}$$

□

2.3 Super AdS_7-spacetime

With the result of §2.1 in hand we may give explicit formulas for super $\mathrm{AdS}_7 \times S^4$-spacetime by first recalling the ordinary bosonic AdS-geometry and then rheonomically extending to super-spacetime.

Near-horizon geometry of black M5-branes. The bosonic near-horizon geometry of N black M5-brane is (cf. [CKvP98, (6.6)][AFHS00, §2.1.2], following [GT93][DGT94]) represented on a chart of the form

$$\mathbb{R}^{1,10} \setminus \mathbb{R}^{1,5} \;\underset{\text{diff}}{\simeq}\; \mathbb{R}^{1,5} \times \big(\mathbb{R}^5 \setminus \{0\}\big) \;\underset{\text{diff}}{\simeq}\; \mathbb{R}^{1,5} \times \mathbb{R}_{>0} \times S^4 \tag{47}$$

with its canonical coordinate functions

$$\begin{array}{lcll} X^a & : & \mathbb{R}^{1,5} \longrightarrow \mathbb{R} & \quad \text{for } a \in \{0, 1, \cdots, 5\} \\ r & : & \mathbb{R}_{>0} \hookrightarrow \mathbb{R} & \end{array} \tag{48}$$

by the AdS_7-metric in "Poincaré coordinates" (cf. [Bl22, §39.3.7]) plus the metric on the round S^4:

$$\mathrm{d}s^2_{N\mathrm{M5}} \;=\; \tfrac{r^2}{N^{2/3}}\mathrm{d}s^2_{\mathbb{R}^{1,5}} + \tfrac{N^{2/3}}{r^2}\mathrm{d}r^2 \;+\; \tfrac{N^{2/3}}{4}\,\mathrm{d}s^2_{S^4} \tag{49}$$

(where $R_{N\mathrm{M5}}/2 := N^{1/3}/2$ is the radius of the 4-sphere in Planck units $2\pi^{1/3}\,\ell_P$, cf. (62) below). So the singular brane locus[6] $\simeq \mathbb{R}^{1,5}$ is (or would be) at $r = 0$. The C-field flux density G_4 supporting this is a multiple of the volume form on S^4 pulled back to the chart along the projection map:

$$G_4 \;=\; c\,\mathrm{dvol}_{S^4_{N\mathrm{M5}}} \in \Omega^4_{\mathrm{dR}}\big(S^4\big) \longhookrightarrow \Omega^4_{\mathrm{dR}}\big(\mathbb{R}^{1,5} \times \mathbb{R}_{>0} \times S^4\big)\,, \tag{50}$$

for some prefactor c which is determined, up to its sign, by the Einstein equations, see (63) below, and determined including its sign by the existence of ½BPS M5-immersions, see (82) below.

For the near-horizon geometry (48) one says that:

- $r \to 0$ is the *horizon*, cf. footnote 6;
- $r \to \infty$ is the *conformal boundary* (e.g. [Bl22, p. 904]),
  at which $\lim\limits_{r\to\infty}\big(\tfrac{1}{r^2}\mathrm{d}s^2_{N\mathrm{M5}}\big) \;=\; \mathrm{d}s^2_{\mathbb{R}^{1,5}}$ is the Minkowski metric on $\mathbb{R}^{1,5}$ (and zero on the remaining $\mathbb{R}_{>0} \times S^4$).

This makes it natural to identify the $\mathbb{R}^{1,5}$-factor at finite r with the worldvolume of a probe M5-brane, to be called a *holographic M5-brane* (cf. [Gu21][Gu24]):

Chart around a holographic M5-brane embedding. We pick a point $s_{\mathrm{prb}} \in S^4 \subset \mathbb{R}^5 \setminus \{0\}$ to designate the direction in which we wish to consider a probe M5-brane worldvolume immersed into this background, at some

---

[6] The locus $r = 0$ is not actually a curvature singularity of the near horizon geometry – as essentially first highlighted by [GHT95] and manifest below in (58) – just a coordinate singularity of the Poincaré chart (49) — but it *is* a singularity of the C-field flux $c\,\mathrm{dvol}_{S^4_{N\mathrm{M5}}}$ (50) *per* unit metric 4-volume $r^4\,\mathrm{dvol}_{S^4_{N\mathrm{M5}}}$, witnessing $r = 0$ as the necessarily singular source of this flux.

coordinate distance r_{prb} from the M5 singularity (cf. [CKvP98, (5.22)][GM00, §8] and Figure B):

$$\begin{tikzcd}[row sep=small] \text{probe M5} \\[-1em] \text{worldvolume} & \mathbb{R}^{1,5} \arrow[rr, "\phi", "\text{embedding}"'] & & \mathbb{R}^{1,5} \times \mathbb{R}_{>0} \times S^4 \\ & x \arrow[rr, mapsto] & & \big(x, r_{\mathrm{prb}}, s_{\mathrm{prb}}\big). \end{tikzcd} \tag{51}$$

Around this point, we may pick a coordinate chart for S^4

$$\begin{tikzcd} \{0\} \arrow[r, "\sim"] \arrow[d, hook] & \{s_{\mathrm{prb}}\} \arrow[d, hook] \\ \mathbb{D}^4 \arrow[r, hook, "\iota"] & S^4 \end{tikzcd}$$

on which we find globally defined co-frame forms $(E^i)_{i=1}^4$ which are orthonormal for the round metric $\mathrm{d}s^2_{S^4}$ on S^4 and torsion-free with respect to the corresponding Levi-Civita connection:

$$\big(E^i \in \Omega^1_{\mathrm{dR}}(\mathbb{D}^4)\big)_{i=1}^4\,, \qquad \text{such that} \quad \mathrm{d}E^i = \big(\iota^*\Omega^{ij}_{S^4}\big)E_j \quad \text{and} \quad \iota^*\mathrm{d}s^2_{S^4} \;=\; \sum_{i=6}^{9} E^i \otimes E^i\,, \tag{52}$$

and such that

$$\iota^*\mathrm{dvol}_{S^4_{NM5}} \;=\; \tfrac{1}{4!}\,\epsilon_{i_1\cdots i_4}\,E^{i_1}\cdots E^{i_4}\,. \tag{53}$$

Using this, we obtain a contractible coordinate chart of the near horizon geometry (47):

$$\mathbb{R}^{1,5} \times \mathbb{R}_{>0} \times \mathbb{D}^4 \xhookrightarrow{\mathrm{id}\times\iota} \mathbb{R}^{1,5} \times \mathbb{R}_{>0} \times S^4\,. \tag{54}$$

Since this is a neighborhood of the worldvolume submanifold (51), for the purpose of establishing its super-embedding it is sufficient to consider this chart.

Cartan geometry around the holographic M5. On the chart (54), we evidently have the following coframe forms

$$\begin{array}{llllll} \textbf{AdS} & E^a & := & \tfrac{r}{N^{1/3}}\,\mathrm{d}X^a & \text{tangential} & a \in \{0,1,2,3,4,5\} \\ & E^{5'} & := & \tfrac{N^{1/3}}{r}\,\mathrm{d}r & \text{radial} & a \in \{5'\} \\ \textbf{S} & E^a & = & \tfrac{N^{1/3}}{2}\delta^a_i E^i & \text{transversal} & a \in \{6,7,8,9\} \text{ via (52)}\,, \end{array} \tag{55}$$

which are orthonormal for the metric (49) in that $\mathrm{d}s^2_{\mathrm{NM5}} \;=\; \eta_{ab}\,E^a \otimes E^b$, and make the C-field flux density (50) appear as

$$G_4 \;=\; \tfrac{c}{4!}\,\epsilon_{i_1\cdots i_4}E^{i_1}\cdots E^{i_4}\,, \tag{56}$$

for some constant c, determined in a moment in (63) below.

For the following formulas, we may focus on the AdS-factor in (55). Hence we let the indices a_i, b_i run only through $\{0,1,2,3,4,5\}$, to be called the *tangential* index values – namely tangential to the worldvolume (51) – with the further *radial* index $5'$ carried along separately.

The torsion-free **spin connection** on the AdS-factor of (55), characterized by

$$\mathrm{d}E^a \;=\; \Omega^a{}_b\,E^b + \Omega^a{}_{5'}\,E^{5'}\,, \qquad \mathrm{d}E^{5'} \;=\; \Omega^{5'}{}_a\,E^a\,,$$

is readily seen to have as only non-vanishing components:

$$\Omega^{a5'} \;=\; -\Omega^{5'a} \;=\; -\tfrac{r}{N^{2/3}}\,\mathrm{d}X^a \quad \text{tangential } a. \tag{57}$$

Therefore its **curvature** 2-form has non-vanishing components

$$\begin{aligned} R^{a5'} &= \mathrm{d}\Omega^{a5'} \;=\; -\tfrac{1}{N^{2/3}}\,\mathrm{d}r\,\mathrm{d}X^a \;=\; -\tfrac{1}{N^{2/3}}\,E^{5'}\,E^a \\ &= +\tfrac{1}{N^{2/3}}E^a\,E^{5'} \\ R^{a_1a_2} &= -\Omega^{a_1}{}_{5'}\,\Omega^{5'a_2} \;=\; +\tfrac{r^2}{N^{4/3}}\mathrm{d}X^{a_1}\,\mathrm{d}X^{a_2} \\ &= +\tfrac{1}{N^{2/3}}\,E^{a_1}\,E^{a_1}\,. \end{aligned} \tag{58}$$

Hence the **Riemann tensor** has non-vanishing components (cf. our normalization of δ in (4))

$$\begin{aligned} R^{a5'}{}_{b5'} &= +\tfrac{1}{N^{2/3}}\,\delta^a_b \\ R^{a_1a_2}{}_{b_1b_2} &= +\tfrac{2}{N^{2/3}}\,\delta^{a_1a_2}{}_{b_1b_2}\,, \end{aligned} \tag{59}$$

and the **Ricci tensor** is proportional to the metric tensor, as befits an Einstein manifold:

$$\begin{array}{rcl} \mathrm{Ric}_{a_1 a_2} & = & R_{a_1}{}^{b}{}_{b a_2} + R_{a_1}{}^{5'}{}_{5' a_2} \\ & = & -\tfrac{(6-1)}{N^{2/3}}\,\eta_{a_1 a_2} - \tfrac{1}{N^{2/3}}\,\eta_{a_1 a_2} \\ & = & -\tfrac{6}{N^{2/3}}\,\eta_{a_1 a_2} \\ \mathrm{Ric}_{5'5'} & = & R^{5' a}{}_{a 5'} \\ & = & -\tfrac{6}{N^{2/3}}\,, \end{array} \tag{60}$$

similar to the Ricci tensor of the 4-sphere factor (e.g. [Lee18, Cor. 11.20]):

$$\mathrm{Ric}_{i_1 i_2} \;=\; +\tfrac{3}{N^{2/3}/4}\,\delta_{i_1 i_2}\,. \tag{61}$$

Therefore the **Einstein equation** with source the C-field flux density (56) has non-vanishing components (cf. [GSS24a, (174-5)])

$$\begin{array}{lrcl} & \mathrm{Ric}_{a_1 a_2} & = & -\tfrac{1}{12}\tfrac{1}{12}(G_4)_{i_1\cdots i_4}(G_4)^{i_1\cdots i_4}\,\eta_{a_1 a_2} \\ \Leftrightarrow & -\tfrac{6}{N^{2/3}}\,\eta_{a_1 a_2} & = & -\tfrac{1}{6}c^2\,\eta_{a_1 a_2} \\ \\ & \mathrm{Ric}_{5'5'} & = & -\tfrac{1}{12}\tfrac{1}{12}(G_4)_{i_1\cdots i_4}(G_4)^{i_1\cdots i_4}\,\eta_{5'5'} \\ \Leftrightarrow & -\tfrac{6}{N^{2/3}} & = & -\tfrac{1}{6}c^2 \\ \\ & \mathrm{Ric}_{i_1 i_2} & = & \tfrac{1}{12}(G_4)_{i_1\, j_1 j_2 j_3}(G_4)_{i_2}{}^{j_1 j_2 j_3} \;-\; \tfrac{1}{12}\tfrac{1}{12}(G_4)_{j_1\cdots j_4}(G_4)^{j_1\cdots j_4}\,\delta_{i_1 i_2} \\ \Leftrightarrow & +\tfrac{3}{N^{2/3}/4}\,\delta_{i_1 i_2} & = & \tfrac{1}{2}c^2\,\delta_{i_1 i_2} \;-\; \tfrac{1}{6}c^2\,\delta_{i_1 i_2} \\ & & = & +\tfrac{1}{3}c^2\,\delta_{i_1 i_2} \end{array} \tag{62}$$

hence is solved (NB: the last line is the reason that the radius of S^4 has to be *half* that of AdS_7 in (49)) by

$$c \;=\; \pm\frac{6}{N^{1/3}}\,, \qquad \underset{(50)}{\text{hence}} \qquad G_4 \;=\; \pm\tfrac{6}{N^{1/3}}\mathrm{dvol}_{S^4_{\mathrm{NM5}}}\,. \tag{63}$$

At this point both of the signs in (63) are equally admissible, but we see below in Rem. 2.20 that the + sign is singled out by the existence of holographic M5-brane embedding.

Super-Cartan geometry near M5 horizons. In now passing to the super-spacetime enhancement of $\mathrm{AdS}_7 \times S^4$, we use the notation and conventions for 6d spinors among 11d spinors form [GSS24b, §3.2], recalled in §2.2.

In particular, we denote the Minkowski frame of Clifford generators adapted to the 1+5+1+4 dimensional split of the tangent space to $\mathrm{AdS}_7 \times S^4$ in Poincaré coordinates (47) by [GSS24b, (85)]

$$\begin{array}{rl} \begin{array}{r} \overline{P}(-)P \\ +P(-)\overline{P} \end{array} \Big(& \begin{array}{l} \overbrace{\Gamma_0\;\;\Gamma_1\;\;\Gamma_2\;\;\Gamma_3\;\;\Gamma_4\;\;\Gamma_5}^{\mathbb{R}^{1,5}}\;\;\overbrace{\Gamma_{5'}}^{\mathbb{R}_{>0}}\;\;\overbrace{\Gamma_6\;\;\Gamma_7\;\;\Gamma_8\;\;\Gamma_9}^{\mathbb{D}^4} \;\;\in \mathrm{Pin}^+(1,0) \subset \mathrm{End}_{\mathbb{R}}(\mathbf{32}) \\ \gamma_0\;\;\gamma_1\;\;\gamma_2\;\;\gamma_3\;\;\gamma_4\;\;\gamma_5 \qquad\qquad\qquad\qquad\qquad\;\; \in \mathrm{Pin}^+(1,5) \subset \mathrm{End}_{\mathbb{R}}\big(2\cdot\mathbf{8}_+ \oplus 2\cdot\mathbf{8}_-\big)\,, \end{array} \end{array} \tag{64}$$

where [GSS24b, (86-90)]

$$\begin{array}{rcll} P\Psi & := & \tfrac{1}{2}\big(1+\Gamma_{5'6789}\big)\Psi & \qquad P(\mathbf{32}) \;\simeq\; 2\cdot\mathbf{8}_+ \;\in\; \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,5)\big) \\ \overline{P}\Psi & := & \tfrac{1}{2}\big(1-\Gamma_{5'6789}\big)\Psi\,, & \qquad \overline{P}(\mathbf{32}) \;\simeq\; 2\cdot\mathbf{8}_- \;\in\; \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,5)\big)\,. \end{array} \tag{65}$$

Super-Cartan geometry around holographic M5s. We now obtain the super-extension of the above Cartan geometry (55). Inserting the bosonic AdS Cartan geometry (55) (57) into the initial conditions for WZT gauge (25) means that

$$\begin{array}{rclcl} \big(E^{(0)}\big)^a & = & \tfrac{r}{N^{1/3}}\,\mathrm{d}X^a & \Leftrightarrow & \Big(\big(E^{(0)}\big)^a_r \;=\; \tfrac{r}{N^{1/3}}, \quad \big(E^{(0)}\big)^a_\rho \;=\; 0\Big) \\ \big(E^{(0)}\big)^{5'} & = & \tfrac{N^{2/3}}{r}\,\mathrm{d}X^{5'} & \Leftrightarrow & \Big(\big(E^{(0)}\big)^{5'}_r \;=\; \tfrac{N^{2/3}}{r}, \quad \big(E^{(0)}\big)^{5'}_\rho \;=\; 0\Big) \\ \big(\Psi^{(0)}\big)^\alpha & = & \mathrm{d}\Theta^\alpha & \Leftrightarrow & \Big(\big(\Psi^{(0)}\big)^\alpha_r \;=\; 0, \quad \big(\Psi^{(0)}\big)^\alpha_\rho \;=\; \delta^\alpha_\rho\Big) \\ \big(\Omega^{(0)}\big)^{5'a} & = & \tfrac{r}{N^{2/3}}\,\mathrm{d}X^a & \Leftrightarrow & \Big(\big(\Omega^{(0)}\big)^{5'a}_r \;=\; \tfrac{r}{N^{2/3}}, \quad \big(\Omega^{(0)}\big)^{5'a}_\rho \;=\; 0\Big)\,. \end{array} \tag{66}$$

Moreover, inserting the flux density (56) into the super-field strength components (32) yields

$$
\begin{array}{rcl}
H_a &=& -\tfrac{c}{12}\Gamma_a\,\Gamma_{6789}\\
H_{5'} &=& -\tfrac{c}{12}\,\Gamma_{5'6789}\\
H_i &=& \tfrac{c}{6}\tfrac{1}{3!}\,\epsilon_{i\,i_1 i_2 i_3}\Gamma^{i_1 i_2 i_3}\\[6pt]
K^{a_1 a_2} &=& -\tfrac{c}{6}\Gamma^{a_1 a_2}\Gamma_{6789}\\
K^{5'a} &=& +\tfrac{c}{6}\,\Gamma^a\Gamma_{5'6789}\\
K^{i_1 i_2} &=& -\tfrac{c}{6}\epsilon^{i_1 i_2\, i_3 i_4}\Gamma_{i_3 i_4}\\
K^{ia} &=& 0\\
K^{5'i} &=& 0\,.
\end{array}
\qquad \text{for} \qquad
\begin{array}{l}
a_i \in \{0,1,2,3,4,5\}\\
i_i \in \{6,7,8,9\}
\end{array}
\tag{67}
$$

From this we now obtain the super-field extension of the supergravity fields on $\mathrm{AdS}_7 \times S^4$.

Example 2.11 (**Spacetime super-fields to first Θ-order).** Based on the 0th-order expressions (66), we obtain to first order in Θ (similar to [dWPPS98, (3.11)]):

$$
\begin{array}{rcrcrcll}
E^a &=& \tfrac{r}{N^{1/3}}\,\mathrm{d}X^a &+& \big(\overline{\Theta}\,\Gamma^a\,\mathrm{d}\Theta\big) &+& \mathcal{O}\big(\Theta^2\big) & \text{by (33)}\\
E^{5'} &=& \tfrac{N^{1/3}}{r}\,\mathrm{d}X^{5'} &+& \big(\overline{\Theta}\,\Gamma^{5'}\,\mathrm{d}\Theta\big) &+& \mathcal{O}\big(\Theta^2\big) & \text{by (33)}\\
\Omega^{5'a} &=& \tfrac{r}{N^{2/3}}\,\mathrm{d}X^a &+& \tfrac{c}{6}\,\big(\overline{\Theta}\,\Gamma^a\,\Gamma_{5'6789}\,\mathrm{d}\Theta\big) &+& \mathcal{O}\big(\Theta^2\big) & \text{by (34), (67) \& (25)}\\
\Psi^\alpha &=& \mathrm{d}\Theta^\alpha &+& \big(-\tfrac{1}{2}\tfrac{r}{N^{2/3}}(\Gamma_{5'a}\Theta)^\alpha - \tfrac{c}{12}(\Gamma_a\Gamma_{6789}\Theta)^\alpha\big)\,\mathrm{d}X^a &&&\\
&&&+& -\tfrac{c}{12}(\Gamma_{5'6789}\Theta)^\alpha\,\mathrm{d}X^{5'} &&&\\
&&&+& \tfrac{c}{6}\tfrac{1}{3!}\epsilon_{i\,i_1 i_2 i_3}(\Gamma^{i_1 i_2 i_3}\Theta)^\alpha\,\mathrm{d}X^i &+& \mathcal{O}\big(\Theta^2\big) & \text{by (35) \& (67),}
\end{array}
\tag{68}
$$

where $a \in \{0,\cdots,5\}$.

Some components can readily be deduced to all orders in Θ. For instance:

Lemma 2.12 (**Mixed spin connection vanishes in all Θ-orders).** *$\Omega^{ia} = 0$ to all orders in Θ:*

$$
\Omega^{ia} \;=\; 0 \quad \text{for} \quad \begin{cases} a \in \{0,1,2,3,4,5\}\\ i \in \{6,7,8,9\}. \end{cases}
\tag{69}
$$

Proof. It clearly vanishes in $\mathcal{O}\big(\Theta^0\big)$ (by the Riemannian product nature of $\mathrm{AdS}_7 \times S^4$) and is in each positive order $\mathcal{O}\big(\Theta^{n+1}\big)$ proportional to K^{ia}, by (34), which however vanishes by (67). □

2.4 Holographic M5 immersion

With the background super-spacetime in hand, we inspect the BPS super-immersions of holographic M5-branes.

Our main result here is Thm. 2.21, which says that the evident super-immersion of an M5-brane worldvolume into the Minkowski-part of the Poincaré chart of the near-horizon super-geometry of N black M5-branes is 1/2BPS (hence is a "super-embedding") *iff* its radial distance from the horizon equals the black M5's throat diameter: $r_{\mathrm{prb}} = R_{N\mathrm{M5}}$, see Rem. 2.20 for discussion.

1/2BPS super-immersions. Recall (e.g. [Va04, p. 27], cf. [GSS24b, Rem. 2.10, Def. 2.18]) that:

Definition 2.13 (**Super-immersions).** A map of supermanifolds (e.g. [GSS24a, Ex. 2.13])

$$
\begin{array}{c}\textbf{super-}\\ \textbf{worldvolume}\end{array}\;\;
\Sigma^{1,p\,|\,\mathbf{n}} \xrightarrow[\textbf{immersion}]{\phi} X^{1,d\,|\,\mathbf{N}}
\;\;\begin{array}{c}\textbf{super-}\\ \textbf{spacetime}\end{array}
\tag{70}
$$

is a *super-immersion* if it induces injections on all super-tangent spaces

$$
\begin{tikzcd}
& \mathbb{R}^{1,p|\mathbf{n}} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{R}^{1,d|\mathbf{N}} \arrow[d, "\wr"] \\
\underset{\sigma \in \overset{\rightsquigarrow}{\Sigma}}{\forall} & T_\sigma \Sigma^{1,p\,|\,\mathbf{n}} \arrow[r, "\mathrm{d}\phi_\sigma"] & T_{\phi(\sigma)} X^{1,d\,|\,\mathbf{N}}\,.
\end{tikzcd}
$$

We say, building on [GSS24b, §2.2], that:

Definition 2.14 (**$1/2$BPS super-immersions).** A super-immersion ϕ (70) is *$1/2$BPS* if for a linear projection operator P from the target super-space onto the "*tangential*" worldvolume super-dimensions (with $\overline{P} := 1 - P$ the "*transversal*" projection), projecting onto the fixed locus of a $\mathrm{Pin}^+(1,d)$-element (a *p-brane involution* [HSS19, Def. 4.4])

$$\begin{tikzcd}
\mathbb{R}^{1,d\,|\,\mathbf{N}} \arrow[r, two heads] \arrow[rr, bend left, "P"] & \mathbb{R}^{1,p\,|\,\mathbf{n}} \arrow[r, hook] & \mathbb{R}^{1,d\,|\,\mathbf{N}} \,,
\end{tikzcd} \tag{71}$$

there exists an orthonormal local co-frame field (E, Ψ) (18) on X which is *super-Darboux* with respect to ϕ in that:

(i) the tangential coframe pulls back to a local coframe field on Σ:

$$(e, \psi) \;:=\; \phi^*\big(PE,\, P\Psi\big) \quad \text{is a coframe field} \tag{72}$$

(ii) the transversal bosonic coframe field pulls back to zero

$$\phi^* \overline{P} E \;=\; 0 \tag{73}$$

(iii) the transversal fermionic coframe field pulls back to

$$\phi^* \overline{P} \Psi \;=\; \mathrm{Sh} \cdot \psi \tag{74}$$

for some *fermionic shear* field Sh on Σ, i.e. pointwise valued in $\mathrm{Spin}(d-p)$-equivariant linear maps

$$\underset{\sigma \,\in\, \overset{\rightsquigarrow}{\vec{\Sigma}}}{\forall} \quad \mathrm{Sh}_\sigma \;:\; \mathbf{n} \simeq P\mathbf{N} \longrightarrow \overline{P}\mathbf{N}\,. \tag{75}$$

Example 2.15 (**M5 super-immersions).** If the projection operator (71) is that from (41) then we have the case of *M5-brane super-immersions* ([GSS24b, §3], going back to [HS97b]).

Remarkably (cf. Rem. 2.23 below), the shear map (75) turns out to encode the flux density any higher gauge field on the worldvolume Σ. If this vanishes (as it does in the example holographic M5-branes presented in a moment) the definition simplifies to:

Definition 2.16 (**Fluxless $1/2$BPS super-immersion).** A $1/2$BPS super-immersion (Def. 2.14) is *fluxless* if its super-Darboux coframes (E, Ψ) are characterized more simply by

$$\begin{array}{rccl}
\textbf{Tangential condition:} & (e, \psi) & := & \phi^*\big(PE,\, P\Psi\big) \quad \text{is a coframe field} \\
\textbf{Transversal condition:} & 0 & = & \phi^*\big(\overline{P}E,\, \overline{P}\Psi\big)\,.
\end{array} \tag{76}$$

This is manifestly super-analogous to classical Darboux coframe theory (recalled in [GSS24b, §2.1]) and this is what we establish for holographic M5-branes in Thm. 2.21 below.

Remark 2.17 (**Relation to the literature).**

(i) The conditions (72) and (73) on a $1/2$BPS super-immersion are (for more details see [GSS24b, Rem. 2.23]) a slight strengthening of the "super-embedding" condition used by [So00], following [BPSTV95][HS97a][HS97b][HRS98].

(ii) In particular, (e, ψ) being a super-coframe field (72) entails that $\phi^* PE =: e$ has no component along ψ, which is the "basic super-embedding condition" of [HS97a, (6)][HRS98, (2)], earlier known as the "geometrodynamical condition" [BPSTV95, (2.23)].

(iii) The difference is that more generally one may allow the pullback of the transversal gravitino to have also a bosonic component τ (cf. [GSS24b, Rem. 3.13]), generalizing (74) to

$$\phi^* \overline{P} \Psi \;=\; \mathrm{Sh} \cdot \psi \;+\; \tau_a e^a\,. \tag{77}$$

However, it seems suggestive that (only) with the component τ required to vanish:

(1.) the expected form of the worldsheet Bianchi identity follows [GSS24b, Rem. 3.19],

(2.) the definition has a pleasantly slick reformulation [GSS24b, pp. 17].

In any case, for the example of holographic M5-branes obtained in Thm. 2.21 below, this component does not appear (and no other explicit examples seem to have been discussed in the literature before).

The holographic M5 super-immersion. We may now define and analyze the super-geometric enhancement of the immersion of M5-worldvolumes parallel and near to the horizon of their own black brane incarnation (cf. again Figure B):

Definition 2.18 (**Holographic super-immersion**)**.** We extend the holographic immersion (51) to a super-immersion (Def. 2.13) in the evident way:

$$\begin{tikzcd}
\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}} \arrow[r, "\phi"] & \mathbb{R}^{1,5} & \times\ \mathbb{R}_{>0} & \times\ \mathbb{D}^4 & \times\ \mathbb{R}^{0|2\cdot\mathbf{8}_+} & \times\ \mathbb{R}^{0|2\cdot\mathbf{8}_-} & \subset\ \mathbb{R}^{1,10\,|\,\mathbf{32}} \\
x^a & X^a \arrow[l, mapsto, "\phi^*"'] & & & & & \\
r_{\mathrm{prb}} & & r \arrow[ll, mapsto, "\phi^*"'] & & & & \\
s^i_{\mathrm{prb}} & & & X^i \arrow[lll, mapsto, "\phi^*"'] & & & \\
\theta^\alpha & & & & \big(P\Theta\big)^\alpha \arrow[llll, mapsto, "\phi^*"'] & & \\
0 & & & & & \big(\overline{P}\Theta\big)^\alpha\,, \arrow[lllll, mapsto, "\phi^*"'] &
\end{tikzcd} \tag{78}$$

where $P = \tfrac{1}{2}\big(1 + \Gamma_{5'6789}\big)$, cf. (37), which defines the super-coordinates on the worldvolume to be the projected pullbacks of those of target space, and where

$$r_{\mathrm{prb}},\, s^i_{\mathrm{prb}} \;\in\; \mathbb{R} \hookrightarrow C^\infty(\mathbb{R}^{1,5}) \hookrightarrow C^\infty(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}}) \qquad \text{for} \qquad i \in \{6,7,8,9\}$$

are the chosen constants parametrizing the transverse position of the immersion (cf. Figure B).

Lemma 2.19 (**Worldvolume super-fields to first θ-order**)**.**
(i) *Under the holographic super-immersion* (78)*, the first-order super-fields* (68) *pull back to*

$$\begin{array}{rclclcl}
e^a & := & \phi^* E^a & = & \tfrac{r_{\mathrm{prb}}}{N^{1/3}}\,\mathrm{d}x^a & + & \big(\overline{\theta}\,\gamma^a\,\mathrm{d}\theta\big) & + & \mathcal{O}\big(\theta^2\big) \\
 & & \phi^* E^{5'} & = & & & & & \mathcal{O}\big(\theta^2\big) \\
\psi^\alpha & := & \phi^*(P\Psi)^\alpha & = & & & \mathrm{d}\theta^\alpha & + & \mathcal{O}\big(\theta^2\big) \\
 & & \phi^*\big(\overline{P}\Psi\big)^\alpha & = & \big(\tfrac{1}{2}\tfrac{r_{\mathrm{prb}}}{N^{2/3}} - \tfrac{c}{12}\big)(\Gamma_{a5'}\theta)^\alpha\,\mathrm{d}x^a & & & + & \mathcal{O}\big(\theta^2\big) \\
e^{b_1}\mathrm{I\!I}^{5'}_{b_1 b_2} + \psi^\beta \mathrm{I\!I}^{5'}_{\beta\, b_2} & := & \phi^*\Omega^{5'}{}_{b_2} & = & \tfrac{r_{\mathrm{prb}}}{N^{2/3}}\delta_{b_1 b_2}\,\mathrm{d}x^{b_1} & + & \tfrac{1}{6}\big(\overline{\theta}\,\gamma^a\,\mathrm{d}\theta\big) & + & \mathcal{O}\big(\theta^2\big)\,.
\end{array} \tag{79}$$

(ii) *The 2nd fundamental super-form* $\mathrm{I\!I}^{5'}$ (cf. [GSS24b, (67)]) *has the following components:*

$$\begin{array}{rclcl}
\mathrm{I\!I}^{5'}_{b_1 b_2} & = & \tfrac{1}{N^{1/3}}\delta_{b_1 b_2} & + & \mathcal{O}\big(\theta^2\big) \\
\mathrm{I\!I}^{5'}_{\beta\, b_2} & = & \big(\tfrac{1}{N^{1/3}} - \tfrac{c}{6}\big)\big(\overline{\theta}\,\gamma_{b_1}\big)_\beta & + & \mathcal{O}\big(\theta^2\big)\,.
\end{array} \tag{80}$$

Proof. The first line in (79) is evident. For the second line just note that $\big(\overline{\theta}\,\Gamma^{5'}\,\mathrm{d}\theta\big) = 0$ by (38). For the third line notice similarly that

$$\phi^*(P\Psi)^\alpha \;=\; \mathrm{d}\theta^\alpha + \Big(-\tfrac{1}{2}\tfrac{r_{\mathrm{prb}}}{N^{2/3}}\underbrace{(P\,\Gamma_{5'a}\theta)^\alpha}_{=0} - \tfrac{c}{12}\underbrace{(P\,\Gamma_a\Gamma_{6789}\theta)^\alpha}_{=0}\Big)\,\mathrm{d}x^a \;\; + \mathcal{O}(\theta^2)\,,$$

where the terms over the braces vanish by (38):

$$P\Gamma_{5'a}\theta \;=\; \Gamma_{5'a}\overline{P}\theta \;=\; =\; \Gamma_{5'a}\overline{P}P\theta \;=\; 0 \qquad \text{and} \qquad P\Gamma_a\Gamma_{6789}\theta \;=\; \Gamma_a\Gamma_{6789}\overline{P}\theta \;=\; \Gamma_a\Gamma_{6789}\overline{P}P\theta \;=\; 0\,.$$

The fourth line works analogously but complementarily:

$$\phi^*\big(\overline{P}\Psi\big)^\alpha \;\;=\;\; \Big(-\tfrac{1}{2}\tfrac{r_{\mathrm{prb}}}{N^{2/3}}\underbrace{\big(\overline{P}\,\Gamma_{5'a}\theta\big)^\alpha}_{-\Gamma_{a5'}\theta} - \tfrac{c}{12}\underbrace{\big(\overline{P}\,\Gamma_a\Gamma_{6789}\theta\big)^\alpha}_{\Gamma_{a5'}\theta}\Big)\,\mathrm{d}x^a \;\; + \mathcal{O}(\theta^2)\,, \tag{81}$$

where under the braces we again used (38):

$$\overline{P}\,\Gamma_{5'a}\theta \;=\; \Gamma_{5'a}P\theta \;=\; \Gamma_{5'a}\theta \;=\; -\Gamma_{a5'}\theta \qquad \text{and} \qquad \overline{P}\,\Gamma_a\Gamma_{6789}\theta \;=\; \Gamma_a\Gamma_{6789}P\theta \;=\; \Gamma_{a5'}\theta\,.$$

Finally, the fifth line follows again similarly, now using that $\Gamma_{5'6789}P = P$, again by (38). From this, the last statement (80) is checked by expanding out:

$$\begin{array}{rcll}
e^{b_1}\,\mathrm{I\!I}^{5'}_{b_1 b_2} + \psi^\beta\,\mathrm{I\!I}^{5'}_{\beta b_2} & = & \Big(\tfrac{r_{\mathrm{prb}}}{N^{1/3}}\mathrm{d}x^{b_1} + \big(\overline{\theta}\,\gamma^{b_1}\,\mathrm{d}\theta\big)\Big)\tfrac{1}{N^{1/3}}\delta_{b_1 b_2} + \mathrm{d}\theta^\beta\Big(\tfrac{1}{N^{1/3}} - \tfrac{c}{6}\Big)\big(\overline{\theta}\,\gamma_{b_2}\big)_\beta \;\; + \mathcal{O}\big(\theta^2\big) & \text{by (79) \& (80)} \\
 & = & \tfrac{r_{\mathrm{prb}}}{N^{1/3}}\delta_{b_1 b_2}\mathrm{d}x^{b_2} + \tfrac{c}{6}\big(\overline{\theta}\,\gamma_{b_2}\,\mathrm{d}\theta\big) \;\; + \mathcal{O}\big(\theta^2\big) & \\
 & = & \phi^*\,\Omega^{5'}{}_{b_2} \;\; + \mathcal{O}\big(\theta^2\big) & \text{by (79)}\,.
\end{array}$$

□

Remark 2.20 (Critical distance of holographic M5-brane probe from black M5 horizon). Since $c = \pm 6/N^{1/3}$ (63), Lem. 2.19 implies that the holographic super-immersion (78) is (fluxless) ½BPS (Def. 2.16) to first order in θ iff

(i) the background C-field flux density (63) is positive [7] and

(ii) the M5-brane probe sits at the throat radius $r_{\mathrm{prb}} = N^{1/3}$:

in that, by (79) (81):

$$\phi^*\big(\overline{P}\Psi\big) \;=\; \mathcal{O}\big(\theta^2\big) \qquad \Leftrightarrow \qquad \left\{\begin{array}{l} c > 0, \text{ i.e. } G_4 \;=\; +\tfrac{6}{N^{1/3}}\mathrm{dvol}_{S^4_{N\mathrm{M5}}} \\ r_{\mathrm{prb}} \;=\; R_{N\mathrm{M5}} \;:=\; N^{1/3}\,. \end{array}\right. \tag{82}$$

Away from this critical radius, the super-immersion picks up exactly a contribution of the parameter called $\tau = \tau_a e^a$ in (77), whose presence would at least complicate the induction argument in Thm. 2.21 below, see footnote 9 there. On the other hand, Thm. 2.21 shows that the characterization (82) of the critical radius holds in fact to all orders of θ.

Such critical radii have previously been discussed for the analogous situation of M2-brane probes near the horizon of black M2-branes, where they have been argued to depend on which exact Killing vector field is used in defining a static probe brane embedding (our holographic embedding, rectilinear with respect to the Poincaré chart (49), being just one example of a static embedding), see [CK⁺98, (2.23) & §A] following [BDPS87, below (15)].

Next, from the first-order formulas (79), we now proceed by induction to the full computation of the worldvolume fields. For this, let now

$$(E, \Psi) \;\in\; \Omega^1_{\mathrm{dR}}\big(\mathbb{R}^{1,5}\times\mathbb{R}_{>0}\times\mathbb{D}^4\times\mathbb{R}^{0|\mathbf{32}};\, \mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \tag{83}$$

denote the super coframe fields (55) on the Poincaré neighborhood (54) of $\mathrm{AdS}_7\times S^4$ uniquely extended to superspace via WZT gauge (Def. 2.2), to all orders in Θ.

Now we are ready for the main statement of this section:

Theorem 2.21 (Existence of fluxless ½BPS holographic M5-brane probes). *The holographic super-immersion* (78) *of an M5-brane probe near the horizon of N coincident black M5-branes is (fluxless) ½BPS* (Def. 2.16) *if* [8] *the radial position of the M5-probe from the horizon equals the throat radius*

$$r_{\mathrm{prb}} \;=\; R_{N\mathrm{M5}} \;\equiv\; N^{1/3} \qquad \Rightarrow \qquad \phi \textit{ is } ½BPS\,. \tag{84}$$

Proof. By Lem. 2.19 with Rem. 2.20 the statement holds to first order in the odd worldvolume coordinates. Hence it is sufficient to check that all higher contributions actually vanish.

First, the vanishing of the higher orders of the transversal gravitino,

$$\phi^*\overline{P}\Psi \;=\; 0\,, \qquad \text{equivalently} \qquad \phi^*\Psi \;=\; P\phi^*\Psi \tag{85}$$

(using throughout that $\phi^*\circ P \;=\; P\circ\phi^*$ and similarly for $\overline{P}$) follows via the decoupled recursion relations from Lem. 2.9 by induction on the θ-order:

- For the even component by

$$\begin{array}{lll}
\phi^*\big(\overline{P}\,\Psi_r^{(n+2)}\big)^\alpha\cdot(n+2)(n+1) & & \\
= -\tfrac{1}{4}\big(\overline{P}\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,\phi^*\Psi_r^{(n)}\big) + \big(\overline{P}H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,\phi^*\Psi_r^{(n)}\big) & a_i\in\{0,\cdots,5,5',6,\cdots,9\} & \text{by (36) \& (78)} \\
= -\tfrac{1}{4}\big(\overline{P}\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,P\phi^*\Psi_r^{(n)}\big) + \big(\overline{P}H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_r^{(n)}\big) & a_i\in\{0,\cdots,5,5',6,\cdots,9\} & \text{by induction assumption} \\
= -\tfrac{1}{2}\big(\overline{P}\,\Gamma_{5'a}\theta\big)^\alpha\big(\overline{\theta}\,K^{5'a}\,P\phi^*\Psi_r^{(n)}\big) + \big(\overline{P}\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_r^{(n)}\big) & a\in\{0,\cdots,5\} & \text{by (38)} \\
= -\tfrac{1}{2}\tfrac{c}{6}\big(\overline{P}\,\underbrace{\Gamma_{5'a}}_{-\Gamma_{a5'}}\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_r^{(n)}\big) - \tfrac{c}{12}\big(\overline{P}\,\Gamma_{a5'}\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_r^{(n)}\big) & a\in\{0,\cdots,5\} & \text{by (67) \& (38)} \\
= 0\,. & &
\end{array}$$

---

[7] Since the difference of signs in (63) signifies the difference between black branes and black *anti*-branes that source the C-field flux, and if in the spirit of microscopic p-brane holography (p. 2) we think of the black brane and its holographic probe as two aspects of the same physical system, then the requirement (Rem. 2.20) of the positive sign for the existence of the holographic probe characterizes this as an actual brane instead of an anti-brane.

[8] The proof of Thm. 2.21 shows also the converse implication, but *only for the chosen super-coframe* (83). In order to have a general logical equivalence in (84) (instead of just an implication) one would have to show that for $r_{\mathrm{prb}} \neq R_{N\mathrm{M5}}$ there is *no other* choice of super-coframe – e.g. not using the WZT gauge (25) – with respect to which such ϕ is ½BPS. While this seems likely, we do not attempt to prove it here. See also footnote 9.

- For the odd component by use of the Fierz identity from Lem. 2.10:

$$
\begin{aligned}
&\phi^*\big(\overline{P}\,\Psi_\rho^{(n+2)}\big)^\alpha \cdot (n+4)(n+3)\tfrac{1}{2} \\
&= \tfrac{1}{4}\big(\overline{P}\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,\phi^*\Psi_\rho^{(n)}\big) + \big(\overline{P}\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,\phi^*\Psi_\rho^{(n)}\big) && a_i \in \{0,\cdots,5,5',6,\cdots,9\} && \text{by (36) \& (78)} \\
&= \tfrac{1}{4}\big(\overline{P}\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,P\phi^*\Psi_\rho^{(n)}\big) + \big(\overline{P}\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_\rho^{(n)}\big) && a_i \in \{0,\cdots,5,5',6,\cdots,9\} && \text{by induction assumption} \\
&= \tfrac{1}{2}\big(\overline{P}\,\Gamma_{5'a}\theta\big)^\alpha\big(\overline{\theta}\,K^{5'a}\,P\phi^*\Psi_\rho^{(n)}\big) + \big(\overline{P}\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_\rho^{(n)}\big) && a \in \{0,\cdots,5\} && \text{by (38)} \\
&= \tfrac{1}{2}\tfrac{c}{6}\big(\overline{P}\,\Gamma_{5'a}\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_\rho^{(n)}\big) - \tfrac{c}{12}\big(\overline{P}\,\Gamma_{a5'}\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,P\phi^*\Psi_\rho^{(n)}\big) && a \in \{0,\cdots,5\} && \text{by (67) \& (38)} \\
&= \tfrac{c}{6}\big(\overline{P}\,\Gamma_{5'}\,\underbrace{\gamma_a\theta\big)^\alpha\big(\overline{\theta}\,\gamma^a}_{=0}\,P\phi^*\Psi_\rho^{(n)}\big) \\
&= 0 && && \text{by (42)}\ .
\end{aligned}
$$

From this it then follows that:

- The pullback of the radial & transversal vielbein vanishes to all orders:

$$\phi^*\overline{P}E \;=\; 0 \tag{86}$$

because we now have for $E^{5'}$ that

$$
\begin{aligned}
&\phi^*\big(E^{(n+1)}\big)_r^{5'} \\
&= \tfrac{2}{n+1}\big(\overline{\theta}\,\Gamma^{5'}\phi^*\Psi_r^{(n)}\big) && \text{by (33) \& (78)} \\
&= \tfrac{2}{n+1}\big(\overline{\theta}\,\Gamma^{5'}P\phi^*\Psi_r^{(n)}\big) && \text{by (85)} \\
&= 0 && \text{by (38),}
\end{aligned}
\qquad
\begin{aligned}
&\phi^*\big(E^{(n+1)}\big)_\rho^{5'} \\
&= \tfrac{2}{n+2}\big(\overline{\theta}\,\Gamma^{5'}\phi^*\Psi_\rho^{(n)}\big) && \text{by (33) \& (78)} \\
&= \tfrac{2}{n+2}\big(\overline{\theta}\,\Gamma^{5'}P\phi^*\Psi_\rho^{(n)}\big) && \text{by (85)} \\
&= 0 && \text{by (38),}
\end{aligned}
$$

and verbatim so for E^i.

- The fermionic component of the tangential coframe field equals

$$\psi \;=\; \mathrm{d}\theta \tag{87}$$

to all orders in θ, because it does so to first order by (79) and all higher orders vanish (now $a_i \in \{0,\cdots,9\}$):

$$
\begin{aligned}
&\big(\psi^{(n+2)}\big)_r^\alpha \;:=\; \phi^*\big(P\,\Psi^{(n+2)}\big)_r^\alpha \\
&= -\tfrac{1}{n+2}\tfrac{1}{n+1}\tfrac{1}{4}\big(P\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,\phi^*\Psi_r^{(n)}\big) + \tfrac{1}{n+2}\tfrac{1}{n+1}\big(P\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,\phi^*\Psi_r^{(n)}\big) && \text{by (36) \& (78)} \\
&= -\tfrac{1}{n+2}\tfrac{1}{n+1}\tfrac{1}{4}\big(P\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}P\phi^*\Psi_r^{(n)}\big) + \tfrac{1}{n+2}\tfrac{1}{n+1}\big(P\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^aP\phi^*\Psi_r^{(n)}\big) && \text{by (85)}\ ^{9} \\
&= 0 && \text{by (67) \& (38)}
\end{aligned}
\tag{88}
$$

and

$$
\begin{aligned}
&\big(\psi^{(n+2)}\big)_\rho^\alpha \;:=\; \phi^*\big(P\,\Psi^{(n+2)}\big)_\rho^\alpha \\
&= -\tfrac{1}{n+4}\tfrac{1}{n+3}\tfrac{1}{4}\big(P\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}\,\phi^*\Psi_\rho^{(n)}\big) + \tfrac{1}{n+4}\tfrac{1}{n+3}\big(P\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^a\,\phi^*\Psi_\rho^{(n)}\big) && \text{by (36) \& (78)} \\
&= -\tfrac{1}{n+4}\tfrac{1}{n+3}\tfrac{1}{4}\big(P\,\Gamma_{a_1a_2}\theta\big)^\alpha\big(\overline{\theta}\,K^{a_1a_2}P\phi^*\Psi_\rho^{(n)}\big) + \tfrac{1}{n+4}\tfrac{1}{n+3}\big(P\,H_a\theta\big)^\alpha\big(\overline{\theta}\,\Gamma^aP\phi^*\Psi_\rho^{(n)}\big) && \text{by (85)} \\
&= 0 && \text{by (67) \& (38)}\,.
\end{aligned}
$$

Notice that in the last step, in both cases, we observe from (67) that $K^{a_1a_1}$ and H_a have for all index values the same parity (with respect to the projectors P, $\overline{P}$) as $\Gamma_{a_1a_2}$ and Γ^a, respectively, so that the two terms $P\,\Gamma_{a_1a_2}P$ and $\overline{P}K^{a_1a_2}P$ can never both be non-vanishing, and similarly for PH_aP and $\overline{P}\,\Gamma^aP$.

- The bosonic component of the tangential coframe field equals

$$e^a \;=\; \mathrm{d}x^a + \big(\overline{\theta}\,\gamma^a\,\mathrm{d}\theta\big) \tag{89}$$

to all orders in θ, because it does so to first order by (79) and by assumption (84), and since all higher orders

---

[9] The second step in (88) fails if one were to go away from the critical radius, $r_{\mathrm{prb}} \neq N^{1/3}$ (82), where $\psi_r^{(0)} \neq 0$ (79), in which case the analogue of (88) instead says that there are potentially contributions to ψ_r in every even order of θ. It would remain to be checked if (e,ψ) is still a coframe field in this case, hence (Rem. 2.17) if the "basic super-embedding condition" would still hold away from the critical radius. (This is tacitly claimed around [GM00, (8.2)], but any higher θ-corrections to ψ seem to be ignored there.)

vanish, as follows:

$$\begin{aligned} \big(e^{(n+1)}\big)^a_r &:= \phi^\ast\big(E^{(n+1)}\big)^a_r \\ &= \tfrac{2}{n+1}\big(\overline{\theta}\,\gamma_a\,\phi^\ast\Psi^{(n)}_r\big) && \text{by (33) \& (78)} \\ &= 0 && \text{by (85) \& (79)}\,, \end{aligned} \qquad \begin{aligned} \big(e^{(n+2)}\big)^a_\rho &:= \phi^\ast\big(E^{(n+2)}\big)^a_\rho \\ &= \tfrac{2}{n+3}\big(\overline{\theta}\,\gamma_a\,\phi^\ast\Psi^{(n+1)}_\rho\big) && \text{by (33) \& (78)} \\ &= 0 && \text{by (85) \& (79)}\,. \end{aligned}$$

To conclude:

- the statements (86) and (85) establish the transversal condition in (76) that was to be shown, namely that $\phi^\ast\big(\overline{P}E,\,\overline{P}\Psi\big) = 0$.
- The statements (87) and (89) establish the tangential condition in (76) that was to be shown, namely that (e,ψ) is a coframe field, manifestly so by expanding the coordinate differentials in their (e,ψ)-components as

$$\begin{aligned} \mathrm{d}x^a &= \tfrac{N^{1/3}}{r_{\mathrm{prb}}}\,e^a \;-\; \big(\overline{\theta}\,\gamma^a\,\psi\big) \\ \mathrm{d}\theta^\alpha &= \psi^\alpha . \end{aligned}$$

This completes the check that ϕ (78) is a (fluxless) 1/2BPS super-immersion (Def. 2.16), hence that the holographic probe M5-brane really exists – at the critical radius $r_{\mathrm{prb}} = R_{N\mathrm{M5}} \equiv N^{1/3}$ (Rem. 2.20). ☐

Remark 2.22 (Bianchi identity and vanishing H_3-flux density).
(i) For §3, the key point of establishing the 1/2BPS property of the holographic M5-brane immersion, via Thm. 2.21, is that this establishes a solution to the equations of motion of the H_3-flux density on the worldvolume ([GSS24b, Prop. 3.17]), namely the appropriate self-duality, the Bianchi identity, and rheonomy.

(ii) In the present case of *vanishing* flux density this may look fairly trivial, but it is still crucial to establish it unambiguously as a solution, because (only) then is flux quantization guaranteed to produce the exact completed field content which may still be non-trivial (namely torsion-charged), as discussed in §3.

(iii) In any case, it is immediate to check the conclusions of [GSS24b, Prop. 3.17] in the present case: In particular, with (56) and (78) we have

$$\phi^\ast G_4 \;=\; 0 \tag{90}$$

so that the general worldvolume Bianchi identity $\mathrm{d}H_3 \;=\; \phi^\ast G_4$ (cf. [GSS24b, (1)]) is un-twisted and becomes

$$\mathrm{d}H_3 \;=\; 0\,,$$

which is clearly satisfied by $H_3 = 0$.

Remark 2.23 (Absence of fluxed 1/2BPS holographic M5-branes). The proof of Thm. 2.21 also readily shows that it is impossible to have non-vanishing worldvolume flux density $H_3 \neq 0$ on a holographic M5-brane (78), while keeping its 1/2BPS- ("super-embedding"-) property (at least with respect to the given coframe field (83), cf. ftn. 8). Namely, by [HS97b, (40)][HSW97, (7)][So00, p. 91] (re-derived in [GSS24b, (126)]) such non-trivial flux corresponds to modifying the super-immersion (78) by a summand $\tilde{\not{H}}_3$

$$\phi^\ast P\Theta \;=\; \theta \,+\, \tilde{\not{H}}_3\theta\,, \qquad \text{for} \quad \tilde{\not{H}}_3 \;\equiv\; \tfrac{1}{3!}(\tilde{H}_3)_{a_1a_2a_3}\gamma^{a_1a_2a_3}\,,$$

which vanishes iff the actual flux density H_3 vanishes (cf. [GSS24b, Rem. 3.18]) – but non-vanishing such $\tilde{H}_3$ immediately fails the Darboux condition (86), by the computation shown right below there. (This is in contrast notably to the case of the rectilinear embedding of the M5-brane into flat Minkowski superspacetime, which allows any constant H_3-flux to be switched on, see [GSS24b, Ex. 3.14]).

This phenomenon naturally leads over to the discussion of flux-quantization on holographic M5-branes in the next section §3. Namely a constraint of vanishing flux density

$$H_3 \;=\; 0$$

trivializes the higher gauge field on holographic M5-branes only locally, on any (contractible) coordinate chart, while the globally completed higher gauge field, controlled by a flux quantization law, may still attain non-trivial configurations carrying non-trivial *torsion charges*.

In other words, while flux quantization completes general gauge field configurations by torsion-charged sectors, this is particularly relevant for configurations with vanishing flux, as found here on holographic M5-branes, in which case the non-trivial higher gauge field content is invisible by traditional local field analysis and is *all* contained in the subtleties of the flux quantization law. This is what we discuss next.

3 Flux quantization on holographic M5-branes

Flux quantization on M5-branes. The point here of having established the holographic M5 super-immersion in §2 is that (by the result of [GSS24b]) it allows to determine the admissible global completions of the worldvolume higher gauge field (the "B-field" with flux density H_3) by a choice of *flux quantization law* (exposition in [SS24b]).

This is relevant in particular for the resulting "torsion charges", i.e. for non-trivial charges encoded in solitonic field configurations which are *not* reflected in the flux density H_3, hence which may exist even in the fluxless case $H_3 = 0$ (as encountered in Thm. 2.21).

Such a situation is familiar in the classical example of vacuum electromagnetism, whose flux-quantization law (going back to Dirac) makes the globally completed electromagnetic field have an underlying charge class in the integral cohomology $H^2(X;\mathbb{Z})$ of spacetime, which may take non-trivial torsion-group values (even) if the electromagnetic flux density vanishes, $F_2 = 0$, such as may happen on cosmological scales if 3-space were a lens space (for which, amusingly there are some mild indications from observational cosmology, cf. [AL12].)

However, away from this familiar special case where flux-quantization is in *ordinary* cohomology, torsion charges in flux-quantized higher gauge fields are rather the rule than the exception, since their flux quantization laws typically need to be given by *generalized* (and non-abelian) cohomology theories (exposition in [SS24b, §3], details in [FSS23]), which generically induce richer charge structure.

In particular, the admissible flux quantization laws for the B-field on M5-branes are (by [GSS24b, p. 6] following [FSS20b, §3.7][FSS21c], see [FSS23, §12]) those whose *rational* ($\sim$ non-torsion) shadow looks like a certain twisted form of the generalized cohomology theory known as *3-Cohomotopy*, denoted $\pi^3(X)$ (in dual analogy with the 3rd homotopy groups, denoted $\pi_3(X)$).

Hypothesis H. Among the infinite set of admissible such laws, one clearly stands out: namely (the suitably twisted form of) 3-Cohomotopy theory itself. The hypothesis that this special choice of flux-quantization is the "correct" one to globally complete the theory of M5-branes has been called "*Hypothesis H*" in [FSS20b][FSS21a][GS21][SS20a] [SS23a], following [Sa13, §2.5]. As discussed in these articles, this hypothesis finds justification in how it *implies* a list of subtle topological (anomaly cancellation-)conditions that are thought need to hold in M-theory and hence for M5-brane physics.

Under this Hypothesis H, the result of [GSS24b] with its specialization to holographic embeddings in §2 implies that the global completion of the field content on holographic M5-branes Σ involves a previously neglected field component which on gauge-equivalence classes is manifested by a class χ in the 3-Cohomotopy of the worldvolume Σ (un-twisted, by (90)):

$$\chi \in \pi^3(\Sigma) \simeq \pi_0 \mathrm{Maps}\big(\Sigma\,, S^3\big)\,. \tag{91}$$

whose image in de Rham cohomology under the *3-cohomotopical character* map ch_π (essentially the pullback of the volume form on S^3, see [FSS20b, §3.7][FSS21a, §3][FSS23, §12]) coincides with the de Rham class of the H_3-flux:

Total flux $[H_3] \;=\; \mathrm{ch}_\pi(\chi)$ **$\mathbb{R}$-rationalization of total charge** i.e. the following diagram commutes

$$\begin{tikzcd} & & \pi^3(\Sigma) \arrow[d, "\mathrm{ch}_\pi"] \\ \ast \arrow[urr, bend left, "\chi"] \arrow[r, "H_3"] & \Omega^3_{\mathrm{dR}}(\Sigma) \arrow[r, two heads, "{[-]}"] & H^3_{\mathrm{dR}}(\Sigma) \end{tikzcd} \tag{92}$$

(The *complete* field content is given by a homotopy-theoretic enhancement of the diagram on the right, which encodes how the flux density H_3 is related to the global charge χ by local gauge potentials B_2, see [GSS24b, §4.1] and see [SS24b, §3.3] for background).

Notice that on holographic M5-branes where the H_3-flux density vanishes (Rem. 2.23) this means that the available charges are the *pure torsion* charges, namely those whose cohomotopical character vanishes.

Charges on holographic M5-branes under Hypothesis H. Here Σ is generally to be understood as including the "point at infinity" — in fact this is mandatory if we want to identify the topology of Σ with that of the *conformal boundary* of AdS_7. Therefore, for plain holographic M5-branes the worldvolume domain Σ on which to compute torsion charges has the homotopy type of the 5-sphere (cf., e.g., [SS23a, Rem. 2.3])

$$\mathbb{R}^{0,1} \times \mathbb{R}^5_{\cup\{\infty\}} \underset{\text{hmtp}}{\simeq} \mathbb{R}^{0,1} \times S^5 .$$

Now on this domain 3-Cohomotopy allows – and hence Hypothesis H predicts – the existence of a non-trivial torsion charge potentially present in all M5-brane physics, namely the one corresponding to the non-trivial 5th homotopy group of the 3-sphere (the *second stable stem*, e.g. [Ki21], cf. [SS23a]):

$$\begin{array}{c}\text{Cohomotopy charges of B-field}\\ \text{vanishing at spatial infinity}\\ \text{on plain holographic M5-branes}\end{array} \quad \pi^3(\Sigma) \;\equiv\; \pi^3\big(\mathbb{R}^1 \times \mathbb{R}^5_{\cup\{\infty\}}\big) \;\simeq\; \pi_0 \mathrm{Maps}(S^5,\, S^3) \;\simeq\; \pi_5(S^3) \;\simeq\; \mathbb{Z}_2\,. \tag{93}$$

Notice that $\mathbb{Z}_2$ being a pure torsion group, its image under the cohomotopical character map is necessarily zero, so that the non-trivial element in (93) indeed has vanishing character, matching the vanishing flux density

$$\big\{0 = [H_3]\big\} \;\simeq\; \mathrm{ch}_\pi\big(\pi^3(S^5)\big)\,.$$

This non-trivial torsion charge potentially appearing on holographic M5-branes is a universal twist whose impact on $\mathrm{AdS}_7/\mathrm{CFT}_6$-duality remains to be determined.

Notice that similar torsion effects have previously been discussed in the case of $\mathrm{AdS}_4/\mathrm{CFT}_3$-duality for M2-branes at A-type orbifold singularities, where already the ordinary cohomology group for the ambient C-field charge is pure torsion

$$H^4\big(\mathbb{R}^{1,2} \times \mathbb{R}_{>0} \times S^7/\mathbb{Z}_k;\, \mathbb{Z}\big) \;\simeq\; H^4\big(S^7/\mathbb{Z}_k;\, \mathbb{Z}\big) \;\simeq\; \mathbb{Z}_k\,,$$

interpreted as the charge carried by "fractional M2-branes" and controlling the level of the Chern-Simons field on the worldvolume [ABJ08].

Instead of further dwelling on this interesting point here, we focus now on a related but more intricate effect of flux quantization on holographic M5-branes which is manifestly relevant for modeling strongly-coupled quantum materials.

4 Anyonic quantum states on holographic M5s

Topological moduli of quantized charges. Beyond the plain charge sectors discussed above, any choice of flux quantization law $\mathcal{A}$ gives rise to the full *moduli stack* of on-shell fields [SS24b, §3.3], of which the charge sectors are only the connected components. This moduli stack is the (higher) Lie-integrated incarnation of the *BRST-complex* of the theory (in generalization of how a Lie group is the integration of its Lie algebra) and as such the correct phase space of the higher gauge theory [SS24a] on which to discuss its (quantum) observables.

Then focusing on the topological quantum observables means [SS23d, pp. 8] to pass (via "topological realization", cf. Rem. 4.21 below) from this moduli stack to the underlying moduli space of topological charges, of which the charge sectors are still the connected components:

$$\begin{tikzcd}
\begin{array}{c}\text{Moduli space}\\ \text{of } \mathcal{A}\text{-charges}\end{array} & \mathbf{H}^1\big(\Sigma,\, \Omega\mathcal{A}\big) \arrow[d, ->>] & \equiv & \mathrm{Maps}^{*/}\big(\Sigma;\, \mathcal{A}\big) & \text{Cocycle space} \\
& & & & \begin{array}{c}\text{quotienting by}\\ \text{coboundaries}\end{array} \\
\begin{array}{c}\text{Set of}\\ \text{charge sectors}\end{array} & H^1\big(\Sigma,\, \Omega\mathcal{A}\big) & \equiv & \pi_0\mathrm{Maps}^{*/}\big(\Sigma;\, \mathcal{A}\big) & \mathcal{A}\text{-cohomology}
\end{tikzcd} \tag{94}$$

In the case of flux quantization in Cohomotopy, where the classifying space $\mathcal{A} \equiv S^n$ is the homotopy type of the n-sphere, these moduli are (pointed) maps from a (worldvolume) manifold (with a point at infinity adjoined) to S^n, whose study was initiated long ago by Pontrjagin [Pon38]:

$$\begin{tikzcd}
\begin{array}{c}\text{Moduli space of}\\ \text{Cohomotopy charges}\end{array} & \boldsymbol{\pi}^n(\Sigma) \arrow[d, ->>] & \equiv & \mathrm{Maps}^{*/}\big(\Sigma;\, S^n\big) & \begin{array}{c}\text{Cohomotopy}\\ \text{cocycle space}\end{array} \\
& & & & \begin{array}{c}\text{quotienting by}\\ \text{coboundaries}\end{array} \\
\begin{array}{c}\text{Set of}\\ \text{Cohomotopy charges}\end{array} & \pi^n(\Sigma) & \equiv & \pi_0\mathrm{Maps}^{*/}\big(\Sigma;\, S^n\big) & n\text{-Cohomotopy}
\end{tikzcd} \tag{95}$$

Cohomotopy of open M5-branes. In order to realize anyonic quantum observables on M5-branes, following [GSS24b, §4.2], we assume that their worldvolume flux H_3 is quantized in 3-Cohomotopy — Hypothesis H (91) — and consider wrapping the M5-branes (both the probes and their black brane incarnation) on a Hořava-Witten orbifold torus $S^1_A \times S^1_H /\!\!/ \mathbb{Z}_2$, by imposing the corresponding cyclic identifications and $\mathbb{Z}_2$-action on the Poincaré chart (47). The resulting worldvolume domain space appropriate for measuring charges of anyonic solitons on the M5 is thus the following orbifold with a point at infinity included [GSS24b, (153)]:

$$\Sigma \;\equiv\; \mathbb{R}^{1,1}_{\sqcup\{\infty\}} \wedge \big(\mathbb{R}^2 \times \mathbb{R}^1_H /\!\!/ \mathbb{Z}_2\big)_{\cup\{\infty\}} \wedge \big(S^1_A\big)_{\sqcup\{\infty\}} \qquad \boxed{\begin{array}{cccc} \mathbb{R}^{1,1} \times & \mathbb{R}^2 \times & S^1_H \times & S^1_A \\ \blacksquare & & & \square \end{array}} \;\; \begin{array}{l}\text{M5-worldvolume}\\ \text{soliton inside}\end{array} \tag{96}$$

(Since after passing to the naive quotient space $S^1_H/\mathbb{Z}_2 \simeq [0,1]$ this looks like M5-brane stretched along an interval, these configurations are known as *open M5-branes* [BGT06, Fig. 3], here further wrapped on the M/IIA circle fiber S^1_A.) In fact, via Hořava-Witten theory this is to be regarded as an *orientifold* which means that its Cohomotopy charge is to be measured in $\mathbb{Z}_2$-*equivariant* 3-Cohomotopy ([SS20a][SS20b, Def. 5.28]) with respect to a reflection action also on one coordinate of the classifying sphere S^3.

The result of [GSS24b, (154)] was that the resulting moduli space of charges is the product space of group-completions $\mathbb{G}\mathrm{Conf}$ of configuration spaces of points, one in dimension 3 (being the solitons that move into the HW-bulk) and one of dimension 2 (being the solitons stuck on the Hořava-Witten O-plane):

$$\begin{array}{c}\textbf{Moduli space of solitonic charges}\\ \textbf{on open holographic M5-branes}\\ \textbf{before wrapping on } S^1_A\end{array} \quad \boldsymbol{\pi}^{2,1}\Big(\big(\mathbb{R}^2\times\mathbb{R}^1_H\big)_{\cup\{\infty\}}\Big) \;\simeq\; \underset{\substack{\textbf{Group-completed config. space}\\ \textbf{of solitons in HW-bulk}}}{\mathbb{G}\mathrm{Conf}(\mathbb{R}^3)} \;\times\; \overset{\substack{\textbf{Group-completed config. space}\\ \textbf{of solitons stuck on HW O-plane}}}{\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)}\,. \tag{97}$$

Namely, the space $\mathrm{Conf}(\mathbb{R}^n)$ of entirely positively charged points in $\mathbb{R}^n$ admits the evident partially defined *topological addition* operation by forming the union of pairs of configurations if they are disjoint [Se73, p. 215], and the group completion $\mathbb{G}\mathrm{Conf}(\mathbb{R}^n)$ is the result of formally adjoining inverses to this operation – hence adjoining anti-branes to the branes represented by the original points.

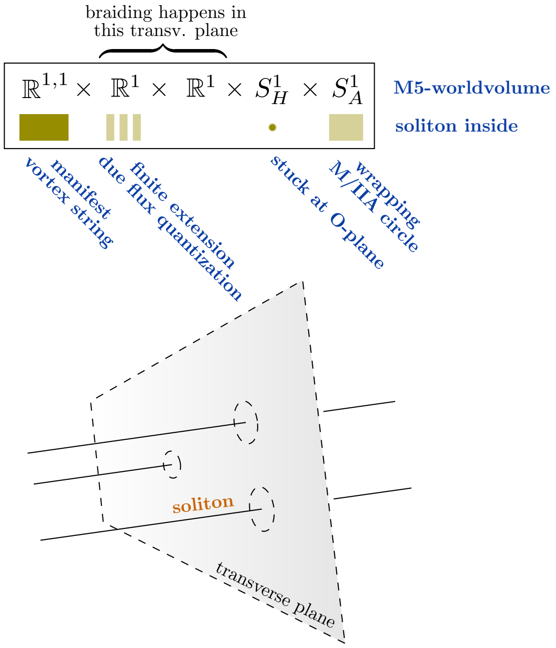

Figure BD. The brane-diagram of the solitons on M5-branes which carry anyonic quantum observables under Hypothesis H. Here, from right to left:

(i) Both the M5 and its worldvolume soliton are wrapped on the M/IIA circle S^1_A in order to admit topological lightcone quantization (cf. §4.2).

(ii) The M5-brane itself is moreover wrapped over the M/HET circle S^1_H, but their worldvolume solitons that we focus on are those (97) that are stuck at an O-plane, i.e. at one of the fixed points in S^1_H (the others escape into the HW bulk and thus will no longer be anyonic).

(iii) Due to a subtle effect of flux quantization in Cohomotopy, these solitons have *finite* extension along one of their would-be transverse directions inside the M5, as explained with Figure Conf.

(iv) Otherwise, after the compactification the solitons look like strings that may move around each other in the transverse plane (not unlike Abrikosov vortex strings in a slab of type II superconducting material, cf. [SS24b, §2.1]).

Remark 4.1 (**Anyons in the sector of vanishing total charge).** Since on holographic M5-branes we are (by Rem. 2.23) necessarily in the zero-charge sector, it follows that the movement of the above solitons is of the peculiar nature where there are non-trivial braiding processes that however all start and end in the vacuum. Remarkably, this is just the situation envisioned in many texts on computational processes based on anyon braiding (e.g. [Kau02, Fig. 17] [FKLW03, Fig. 2][Ro16, Fig. 2][DMNW17, Fig. 2][RW18, Fig. 3][Ro22, Fig. 1]), where an anyon braiding process is essentially assumed to be a *link diagram* (hence a knot diagram if connected), as indicated in Figure C.

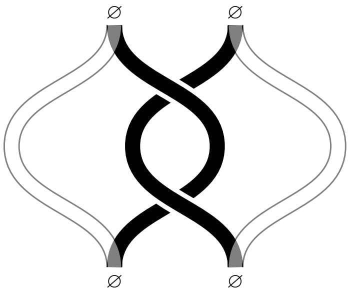

Figure C. A based loop in the configuration space of charged points/strings is an evolution that begins in the vacuum configuration ∅, then proceeds by pair-creation into a configuration where a number of positively charged points/strings and the same number of negatively-charged such objects have appeared; then proceeds by braiding these and finally ends, via pairwise annihilation of all the points/strings, again in the vacuum state ∅. Or rather, along the way any number of further such vacuum diagrams may appear, braid, disappear — not shown in the simple example on the right.

Just such processes are traditionally envisioned as computational processes in texts of topological quantum computation, for the braids regarded as worldlines of anyons.

Notice that this means to assume the sector of *vanishing total charge.*

Therefore we now turn to a careful analysis of the moduli space of solitons on M5-branes, under Hypothesis H.

Configuration spaces of solitonic charges. Naïvely one might expect $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ in (97) – the factor on which we will focus now – to be the configuration space of *signed* points in $\mathbb{R}^2$, where each point carries a charge in $\{\pm 1\}$, with the topology of the configuration space such that oppositely-charge points may undergo pair annihilation/creation. While this is the correct picture on the level of connected components, it turns out *not* to correctly capture the homotopy type of this space, as observed in [McD75, p. 96].

However, something close is true and interesting with respect to our physics interpretation: To get the correct moduli space, the points (hence the worldvolume solitons) need to be regarded as being of finite thickness [CW81] at least in one direction [Ok05], so that the points (which for us are the positions of worldvolume solitons in their transversal space, cf. Figure BD.) are resolved to "strings" carrying charges at their ends.

We now discuss this in more detail (culminating in Thm. 4.18 below).

Beware that the group-completed plain configuration spaces considered here are different from the configuration spaces considered in [GS21]: The spaces there correspond to *intersections* of solitonic branes with codimension=1 branes (which induces an ordering of the points in the configuration, in contrast to the un-ordered configurations considered here).

4.1 Configurations of solitons on M5

We work out key aspects of the moduli space (97) of the configurations of solitons stuck at O-planes in open holographic M5-branes wrapped on S^1_A(Figure BD) subject to flux quantization in 3-Cohomotopy (§3).

Group-completed configuration space of points. For general background on configuration spaces of points see [FH01][Ka24]. The aspects of their group completion which we discuss in a moment, based on [Ok05], seem not to previously have found attention.

Definition 4.2 (**Plain configuration space of points** [Se73, p. 215])**.** For $n \in \mathbb{N}$, we write $\mathrm{Conf}(\mathbb{R}^n)$ for the topological space of finite subsets of (i.e. configurations of plain points in) $\mathbb{R}^n$. This is a partial topological monoid under the partial operation

$$\mathrm{Conf}(\mathbb{R}^n) \times \mathrm{Conf}(\mathbb{R}^n) \xrightarrow{\sqcup} \mathrm{Conf}(\mathbb{R}^n) \tag{98}$$

which is defined when the pair of configurations is disjoint, in which case it is given by their union. We write

$$\mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;:=\; \Omega\big(B_{\sqcup}\mathrm{Conf}(\mathbb{R}^n)\big) \tag{99}$$

for the topological group completion of this partial monoid, namely the based loop space of the topological realization of its simplicial nerve.

(See [SS23d, §A.2] for the general topology of pointed spaces that we need here.)

Proposition 4.3 (**Group-completed configurations as iterated loops** [Se73, Thm. 1])**.** *The cohomotopy charge map ("scanning map") constitutes a weak homotopy equivalence between the group completion of the configuration space of plain points in $\mathbb{R}^n$ (Def. 4.2) and the n-fold based loop space of the n-sphere:*

$$\mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;\simeq\; \Omega^n S^n. \tag{100}$$

Definition 4.4 (**Configuration space of charged open strings** [Ok05, Def. 3.1-2])**.** For $n \in \mathbb{N}_{\geq 1}$, we write [10] $\mathrm{Conf}^I(\mathbb{R}^n)$ for the quotient by the equivalence relations indicated on the right of Figure Conf of the topological space of disjoint unions of (half-)open/closed line segments in $\mathbb{R}^n$ *parallel to the first coordinate axis*, where in Figure Conf a filled (black) circle indicates that the corresponding point is included in the interval, while an empty (white) circle indicates that it is not.

Proposition 4.5 (**Charged open strings as group-completion of plain points** [Ok05, Thm. 1])**.** *For $n \in \mathbb{N}_{\geq 1}$ there is a weak homotopy equivalence between the configuration space of charged open strings (Def. 4.4) and the group completion of the plain configuration space of plain points (Def. 4.2):*

$$\mathrm{Conf}^I(\mathbb{R}^n) \;\simeq\; \mathbb{G}\mathrm{Conf}(\mathbb{R}^n)\,. \tag{101}$$

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[10] The space we denote $\mathrm{Conf}^I(\mathbb{R}^n)$ in Def. 4.4 would be denoted "$I_n(S^0)_{\mathbb{R}}$" in the notation of [Ok05].

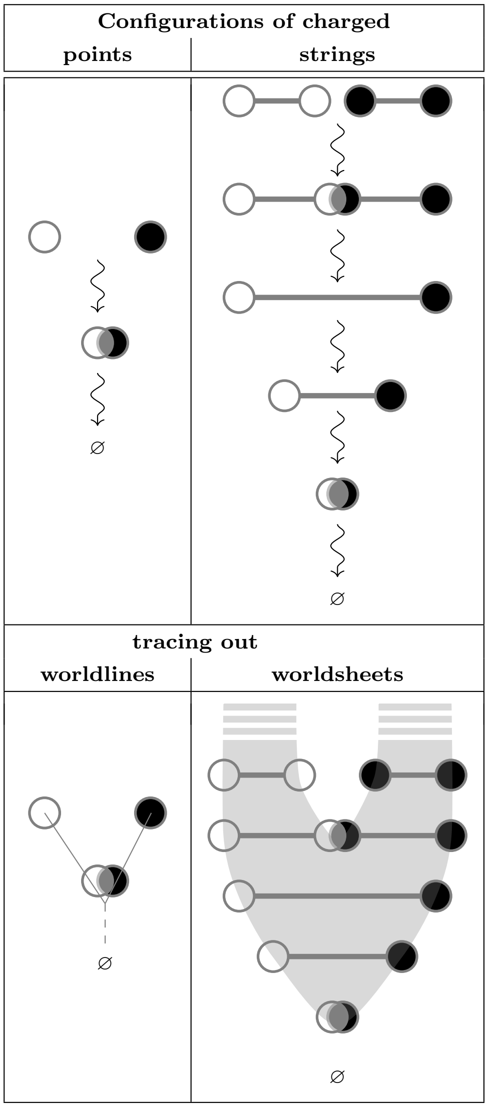


Figure Conf. Indicated in the left column is the equivalence relation ([McD75, p. 94]) controlling the configuration space of charged *points* in some $\mathbb{R}^n$, where configurations involving a positively and a negatively charged point are connected by a continuous path to the corresponding configuration where both of these points are absent (have mutually annihilated). This configuration space is close to but *not* (weak-homotopy) equivalent (by [McD75, p. 6]) to the group-completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^n)$.

Indicated in the right column are the analogous relations (from [Ok05, Def. 3.1-2]) in the configuration space of charged "strings", where charged points are replaced by line segments of finite length, parallel to a fixed coordinate axis, whose endpoints are carrying charges. This configuration space *is* (weak-homotopy) equivalent to the group-completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ (by [Ok05, Thm. 1.1]).
(In both cases, the curvy lines indicate continuous paths in these configuration spaces, here realizing the pair-annihilation processes. Running along these paths in the opposite direction reflects the corresponding pair-creation processes.)

Notice that in both cases the physical processes are *grosso modo* the same — a pair of opposite charges mutually annihilates —, the difference being only that on the right the process is "smoothed out" in the familiar way in which string interactions resolve singularities in particle interactions — only that here we did not postulate this explicitly: it is derived by applying the result of [Ok05] to the consequence (97) of Hypothesis H.

This is curious because it means that what naively looks like (non-supersymmetric) solitonic 2-branes inside the M5-brane worldvolume — indicated by the bars in (96) — is resolved via flux quantization in Cohomotopy to a kind of unstable open 3-branes stretching along finite intervals in one of the naïve three transverse directions, cf. Figure BD.

Remark 4.6 (Charged strings reflecting Cohomotopy moduli). In summary, this identifies the n-Cohomotopy moduli vanishing at infinity on $\mathbb{R}^n$ with the Okuyama configuration space of charged open strings in $\mathbb{R}^n$:

$$\begin{aligned} \underset{\text{charged open strings}}{\text{Configuration space of}} \quad \mathrm{Conf}^I(\mathbb{R}^n) \;&\underset{(101)}{\simeq}\; \mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;\underset{(100)}{\simeq}\; \Omega^n S^n \\ &\simeq\; \mathrm{Maps}^{*/}\big(\mathbb{R}^n_{\cup\{\infty\}},\, S^n\big) \;\equiv\; \boldsymbol{\pi}^n\big(\mathbb{R}^n_{\cup\{\infty\}}\big) \quad \underset{\text{vanishing at infinity}}{\text{Cohomotopy moduli}}\,. \end{aligned} \tag{102}$$

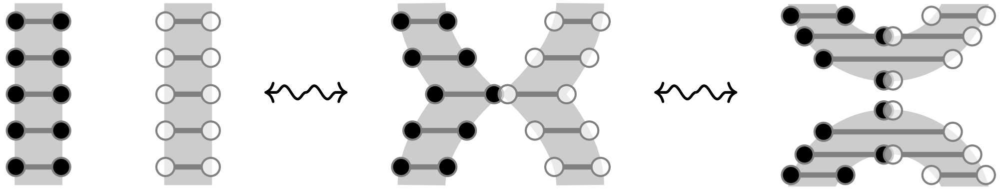
This implies:

Proposition 4.7 (Fundamental group of charged string configurations). *The fundamental group of Okuyama's configuration space of charged open strings in the plane (Def. 4.4) is the group of integers:*

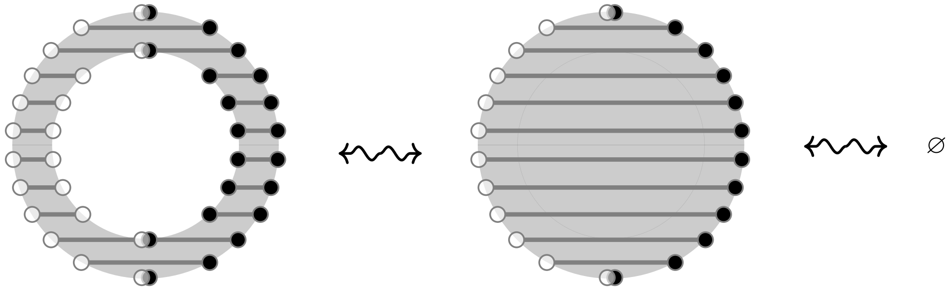
$$\pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big) \;\equiv\; \pi_0\big(\Omega_0\,\mathrm{Conf}^I(\mathbb{R}^2)\big) \underset{(102)}{\simeq} \pi_0\big(\Omega^3 S^2\big) \;\equiv\; \pi_3(S^2) \;\simeq\; \mathbb{Z}\,. \tag{103}$$

The generator on the right of (103) is well-known to be represented by the complex Hopf fibration $h_{\mathbb{C}} : S^3 \to S^2$. Our goal in the following is to understand the corresponding generator on the left, i.e. the unit-charged open string loop whose composites and their reverses are deformation-equivalent to general charged open string loops. A key observation for this identification is the following:

Example 4.8 (Relations between charged open string worldsheets). Continuous deformations of paths of charged open strings, i.e. continuous maps of the form $[0,1]^2 \longrightarrow \mathrm{Conf}^I(\mathbb{R}^n)$, subsume the following two "moves" and their images under exchange of positive and negative charges:

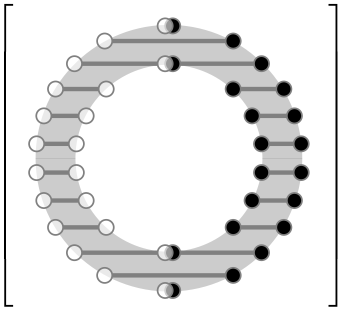


(104)



(105)

Here the second move (105) is a path of based loop and implies that the class of the annulus worldsheet in the fundamental group of the configuration space vanishes:



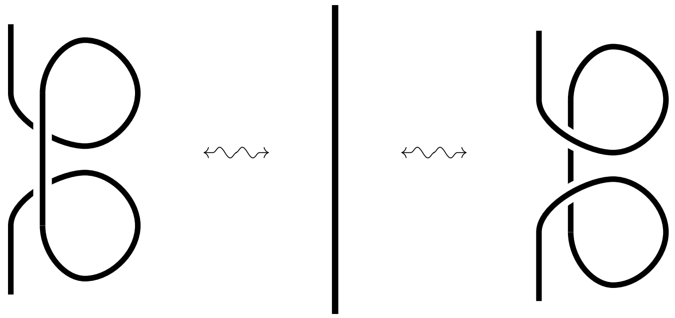
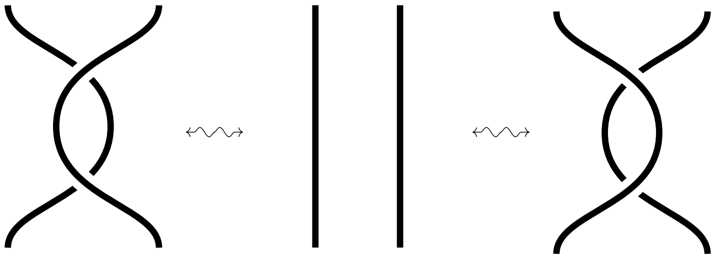
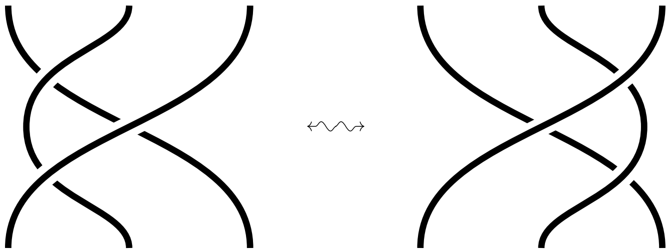
$$= \ast \in \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^n)\big)\,.$$

Hence the annulus is not the generator of $\pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)$ that we are after, and we need to look further:

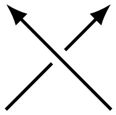
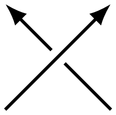
Loops in Okuyama's configuration space as framed oriented links. Our first observation now is that based loops in Okuyama's configuration space of charged open strings (Def. 4.4) may be identified with *framed oriented links* (cf. Figure F). For general discussion of framed links see for instance [Oh1, pp. 15][EHI20].

Definition 4.9 (Framed oriented links).
(i) A *framed oriented link diagram* is an immersion of k oriented circles $\big(S^1\big)^{\sqcup^k}$, for $k \in \mathbb{N}$, into the plane $\mathbb{R}^2$ with isolated crossings at Euclidean distance > 1 from each other, at each of which one of the two crossing segments is labeled as crossing *over*.
(ii) Two framed oriented link diagrams are regarded as equivalent if they may be transformed into each other by a sequence of isotopies (continuous paths in the space of framed link diagrams) and the three *Reidemeister moves* shown in Figure R.
(iii) The *framed oriented links* are the corresponding equivalence classes of framed oriented link diagrams.

Figure R – Reidemeister moves for framed link diagrams (e.g. [Oh1, Thm. 1.8]).

1st	
2nd	
3rd	

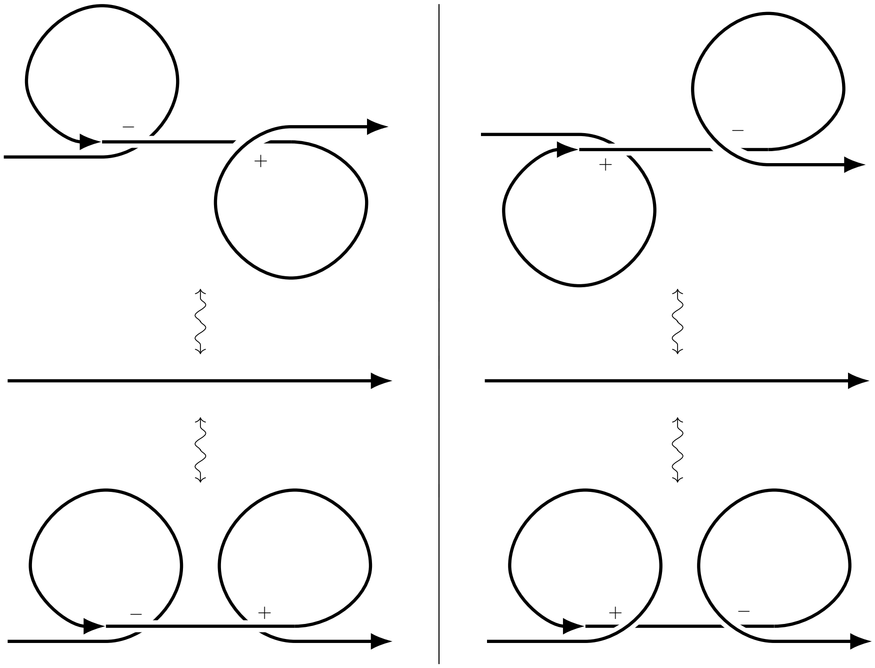
Definition 4.10 (Crossing-, Linking- and Framing numbers).
(i) Up to oriented-preserving diffeomorphism, every crossing in a framed oriented link diagram (Def. 4.9) locally looks like of of the following situations, which we assign the *crossing number* ± 1, respectively, as shown:

$\#\Bigg($  $\Bigg) = +1,$ $\#\Bigg($  $\Bigg) = -1.$ (106)

(ii) The *linking matrix* (e.g. [Oh1, p. 227]) of a framed oriented link L with k connected components is the $k \times k$-matrix whose (i, j)-entry is the sum of the crossing numbers of all crossings between the ith and the jth connected component.
(iii) The link's *total linking* number $\mathrm{lnk}(L) \in \mathbb{Z}$ is half the sum of all off-diagonal entries of the linking matrix.
(iv) The framed link's *framing number* $\mathrm{frm}(L) \in \mathbb{Z}$ is the sum of the diagonal entries.
(v) Hence the sum over all entries of the linking matrix is the framing number plus twice the linking number, this being simply the sum of the crossing numbers of all crossings of L, which we denote as:

$$\#(L) \;:=\; \sum_{\substack{c\,\in \\ \mathrm{crssngs}(L)}} \#(c) \;=\; \mathrm{frm}(L) + 2\,\mathrm{lnk}(L)\,. \tag{107}$$

Example 4.11 (Invariance of framing number). The framing and linking numbers (Def. 4.10) are invariants depending only on the equivalence class of a framed oriented link diagram. The following moves show how successive self-crossings of opposite crossing number cancel out (at the bottom by the 1st Reidemeister move and at the top by the 2nd and 3rd Reidemeister moves):



Definition 4.12 (**Charged string loops as framed links).** Given a framed oriented link diagram (Def. 4.9) we obtain a based loop in Okuyama's configuration space of charged strings in $\mathbb{R}^2$ (Def. 4.4) by thickening the underlying link to a string worldsheet as illustrated in Figure F:

$$\mathrm{FrmdOrntdLnkDgrm} \longrightarrow \Omega_0\, \mathrm{Conf}^I(\mathbb{R}^2)\,. \tag{108}$$

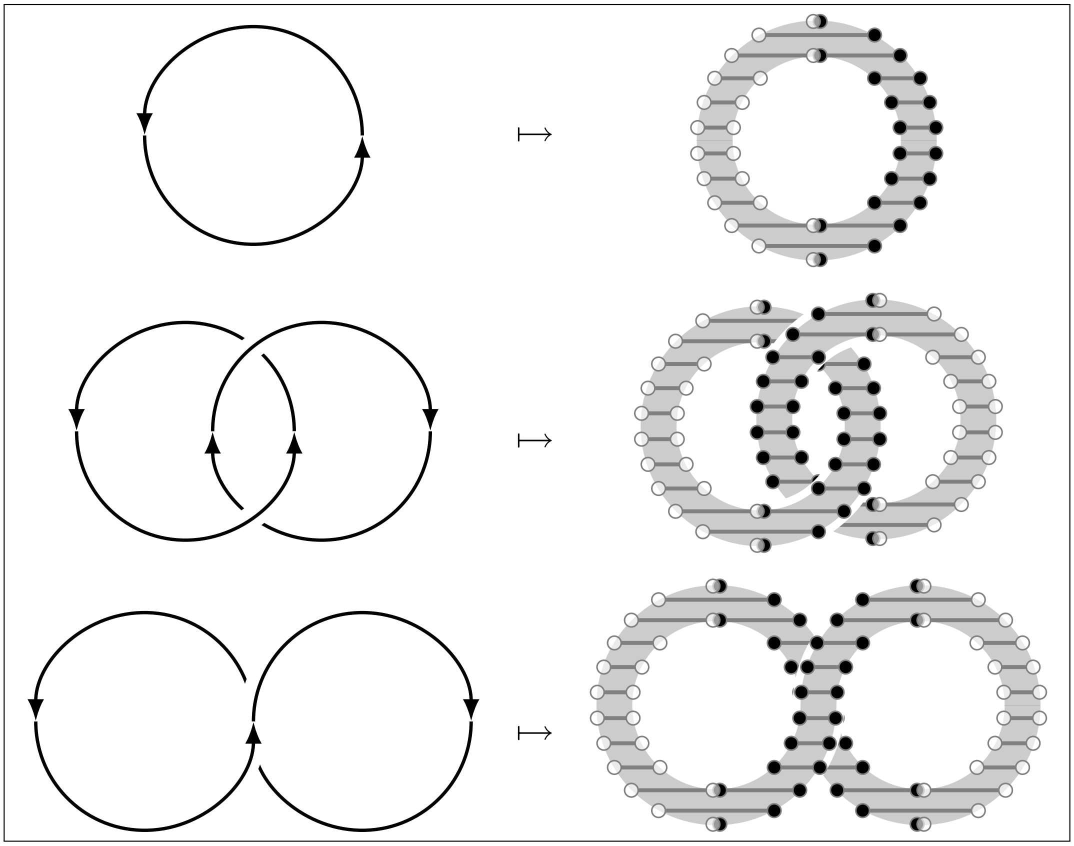
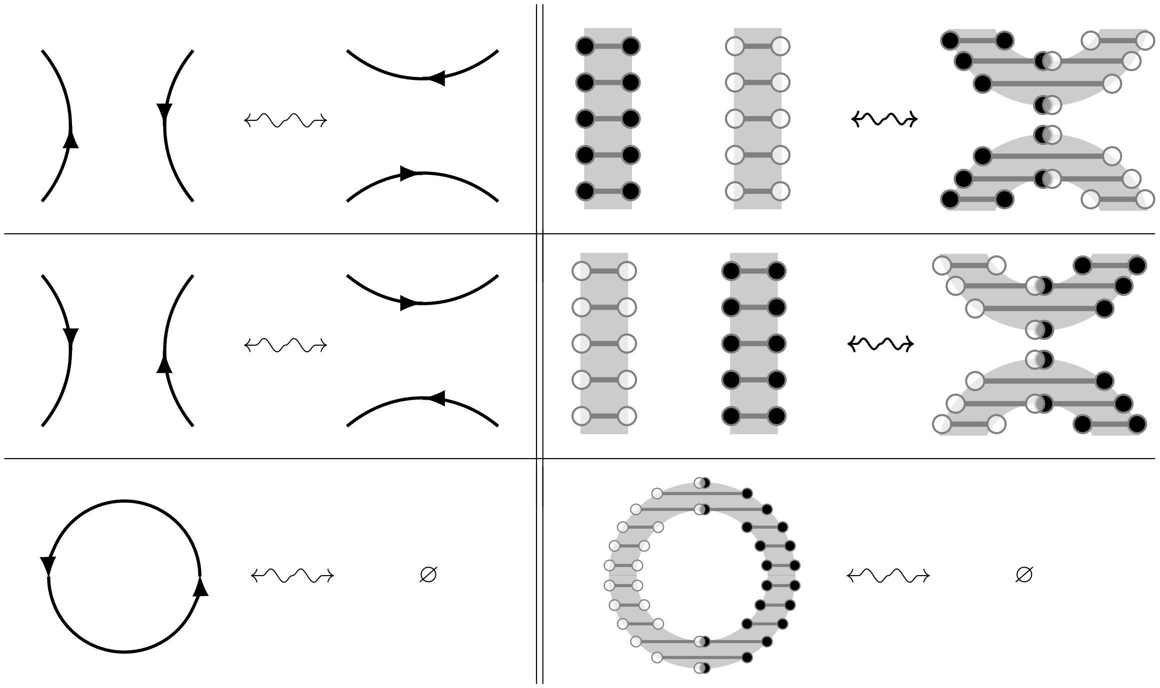


Figure F – Charged string loops as framed oriented links. Notice how it is the stringy nature of the loops of configurations on the right (via Def. 4.4) which reflects the "blackboard framing" of the link diagrams on the left. This framing would be absent for loops of configurations of charged *points* as on the left of Figure Conf.

Example 4.13 (Link cobordism). The two moves of charged open string worldsheets from Ex. 4.8 correspond on framed oriented link diagrams to the following moves shown on the left:



(109)

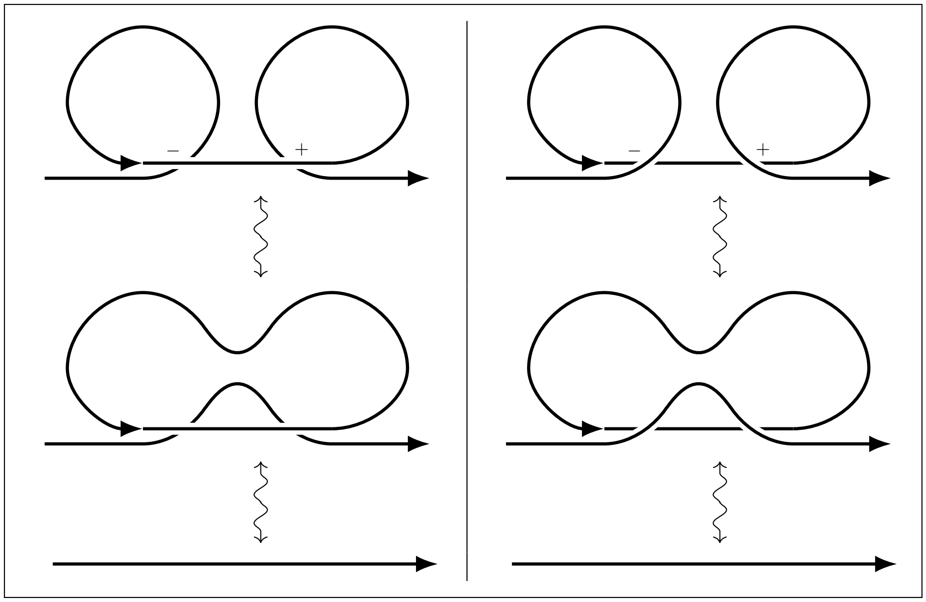
These are known, respectively, as the *birth/death* move and the *fusion moves* ([Kh00, §6.3][Ja04, Fig. 15], cf. [Lo24, Fig. 12]) or *oriented saddle point* moves (e.g. [Kau15, Fig. 16]) generating (on top of usual link diagram equivalence) the relation of *link cobordism*[11].

Proposition 4.14 (Surjection on equivalence classes). *The map* (108) *descends to a surjection on equivalence classes, there sending framed oriented links (instead of their representing diagrams) to elements in the fundamental group of the stringy configuration space:*

$$\mathrm{FrmdOrntdLnk} \longrightarrow\!\!\!\!\rightarrow \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)\,. \tag{110}$$

Proof. It is clear that diagram isotopies and the 2nd and 3rd Reidemeister moves on the left are reflected in continuous paths on the right. What remains to be shown is that also the 1st Reidemeister move on the left is reflected in continuous paths on the right.

It is thus sufficient to show that these extra moves (109) of framed oriented link diagrams *imply* the 1st Reidemeister move. That this is the case is shown on the right. □



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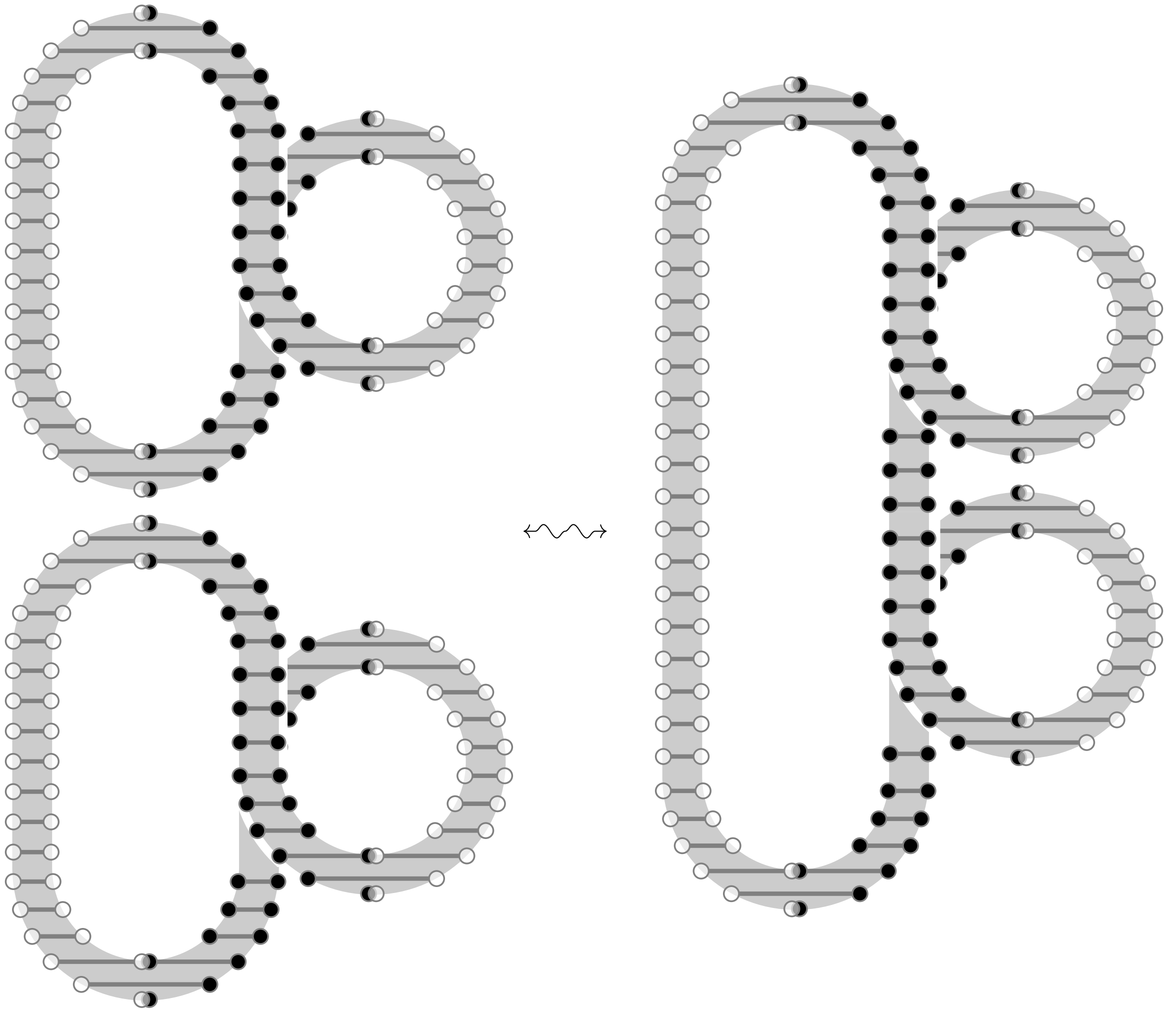
[11] Beware that early authors (e.g. [Ho68][CS80]) say "link cobordism" for what is now called "link concordance", namely for cylindrical cobordisms only. In this case, the corresponding equivalence classes of links are non-trivial.

The modern use of "link cobordism" for actual cobordisms considered here seems to originate with [Kh00, §6.3], cf. [Lo24, Fig 12]. With this notion, all (framed) links are equivalent to (framed) unknots (Lem. 4.16 below) and hence the broader interest in general link cobordism is instead in characterizing the cobordisms themselves, notably through their associated homomorphism between Khovanov homologies [Ja04].

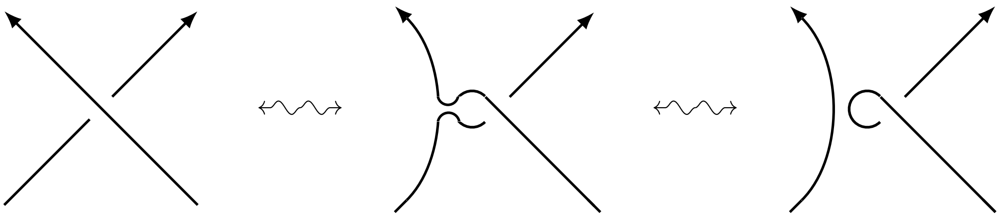
Example 4.15 (**Group of stringy images of framed unknots).** The images of the framed unknots under (110) constitute an integer subgroup $\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)$ (cf. Prop. 4.7) whose group operation corresponds to the addition of framing number (Def. 4.10). For instance, the following is the move corresponding to the equation $1 + 1 = 2$ in this subgroup:



In fact, this subgroup inclusion is surjective (117), hence exhausts the full fundamental group, by the following further analysis:

Lemma 4.16 (**Framed links are cobordant to framed unknots).** *Every framed oriented link is related by the stringy moves* (109) *to a framed oriented unknot.*

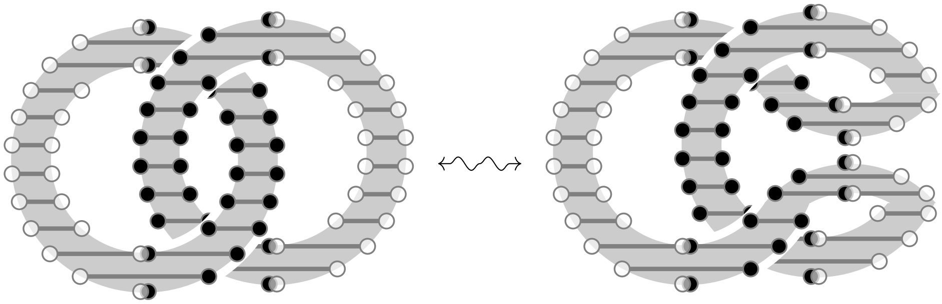
Proof. Using the saddle move, every crossing of two straight segments may be turned into an avoided crossing of a straight edge with a twisted edge, e.g.:



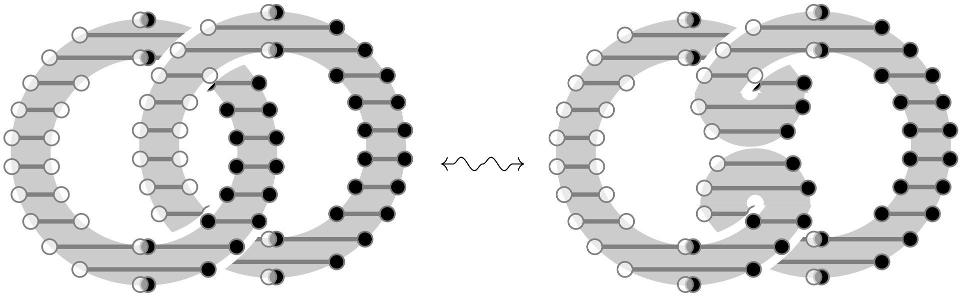
(111)

Applying such a move to all crossings of a given link diagram yields a framed unlink. Then forming the connected sum of its connected components (as in Ex. 4.15) yields a framed unknot. ☐

Example 4.17 (**Framed links turned into framed unknots).** The Hopf link becomes the unknot with framing ±2 by applying the saddle move either on the right or in the middle, depending on the given orientations:

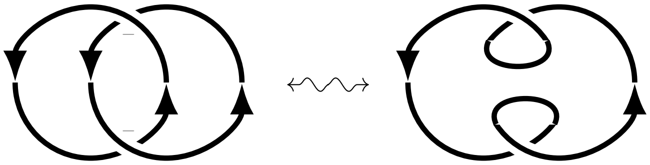


(112)



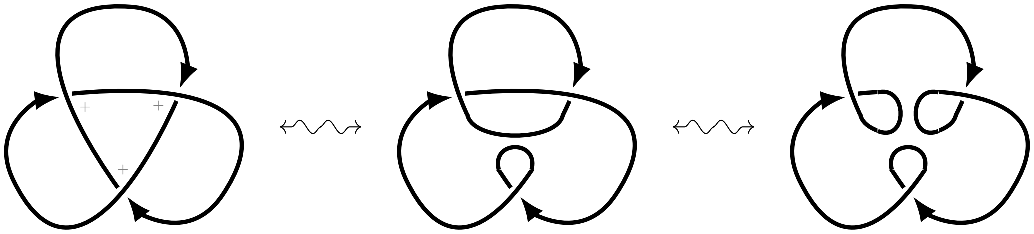
(113)

If we understand the stringy moves applied already to the corresponding framed link diagrams, then we may draw the above example more succinctly as



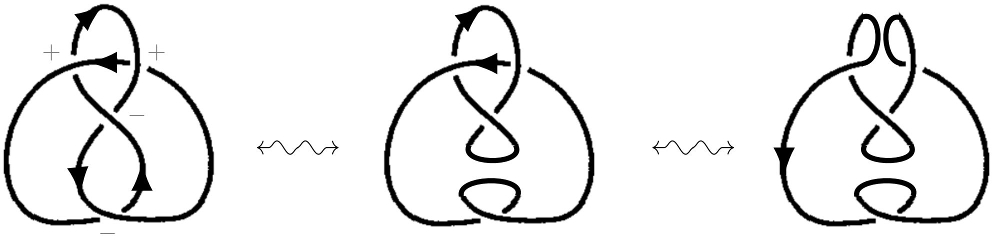
(114)

Further examples in this notation are the following: The trefoil knot becomes



(115)

and the figure-eight knot becomes



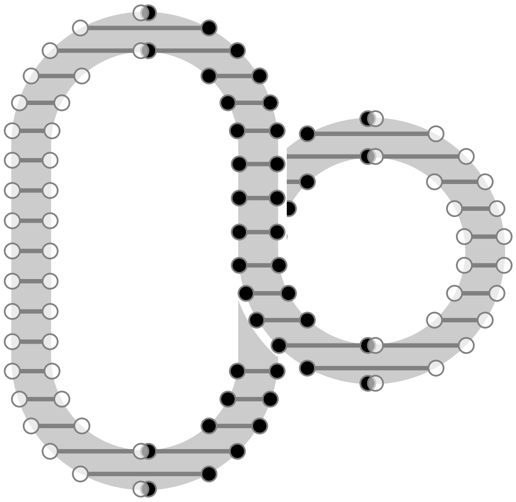
(116)

Using all this, finally we have:

Theorem 4.18 (**Charged open string loops classified by crossing number).** *The map* (110) *from framed oriented links to the fundamental group of Okuyama's configuration space of charged open strings in the plane is, under the latter's identification with the integers* (103)*, given by sending a link* L *to its total crossing number* $\#(L)$ (107)*:*

$$\begin{array}{ccc} \mathrm{FrmdOrntdLnk} & \longrightarrow & \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big) \simeq \mathbb{Z}\,. \\ L & \longmapsto & \#(L) \end{array}$$

Proof. By Lem. 4.16, the image of L is equivalently a framed unknot via the saddle moves (109). Since all framed unknots are multiples of the unit-framed unknot, by Ex. 4.15, this exhibits the unit framed unknot as the generator



$$\left[\quad \right] \;=\; 1 \quad \in \; \mathbb{Z} \;\simeq\; \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)\,. \tag{117}$$

(which hence corresponds to the Hopf fibration under the identification of Prop. 4.7).

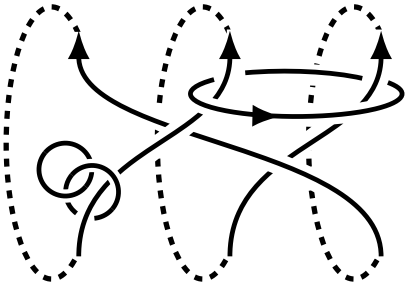
Moreover, since the saddle move (111) used in Lem. 4.16 manifestly preserves total crossing numbers # (107), the total crossing number of the resulting unknot (being its framing number) is that of L (cf. Ex. 4.17), and hence it represents the $\#(L)$-fold multiple of the generator (117). ☐

Remark 4.19 (**Comparison to Pontrjagin theorem**).
(i) Under the equivalences of Prop. 4.7, Thm. 4.18 is similar to the statement of the Pontrjagin theorem (review in [SS23a, §3.2][SS20a, §2.1]) specialized to codimension=2 submanifolds in $\mathbb{R}^3$, which says that Cohomotopy cocycles $\mathbb{R}^3_{\cup\{\infty\}} \to S^2$ essentially correspond to closed 1-dimensional submanifolds in $\mathbb{R}^3$ (hence: links) equipped with *normal framing* and that coboundaries (homotopies) between such cocycles correspond to cobordism between such normally framed links.
(ii) By carefully translating between the different notions of framings – the framing in the sense of framed links as above in Def. 4.9 is not the same as a normal framing, but closely related (and both are of course different from tangential framing of the links) – this statement matches the above, and Thm. 4.18 may be viewed as a re-proof of Pontrjagin's theorem in these dimensions (cf. [Br93, p. 126]) from Okuyama's theorem [Ok05].
(iii) Besides the transparent diagrammatic analysis shown above, for our purposes this re-proof makes manifest the relation both to solitonic 3-branes insides M5-branes (as per Figure B) and to anyon/anti-anyon braids of vanishing total charge (as per Figure C).

Remark 4.20 (**Relation to braiding in non-vanishing charge sectors**).
(i) Since the group completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ is, by construction, a topological group, it follows abstractly that all its connected components are, in particular, weakly homotopy equivalent, hence so are those of the weakly equivalent stringy configuration space $\mathrm{Conf}^I(\mathbb{R}^2)$, by Prop. 4.5, and hence so are the loop spaces based on any of these connected components:

$$\underset{n,n' \in \mathbb{Z}}{\forall} \qquad \Omega_n\, \mathrm{Conf}^I(\mathbb{R}^2) \;\;\simeq\;\; \Omega_{n'}\, \mathrm{Conf}^I(\mathbb{R}^2)\,.$$

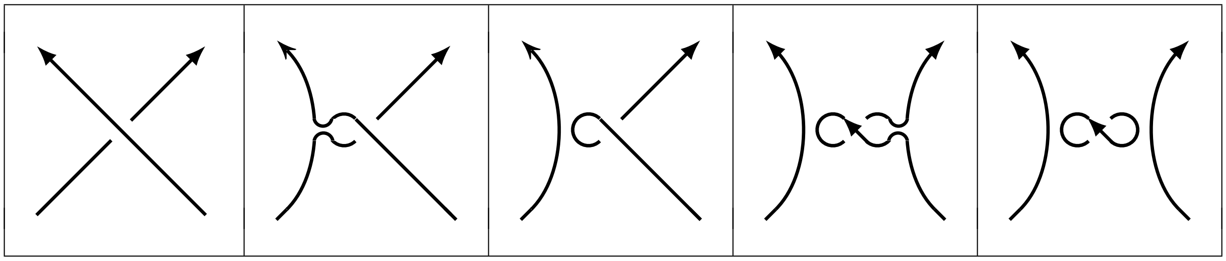
(ii) More concretely, we may now exhibit this equivalence in terms of the interpretation of loops in $\mathrm{Conf}^I(\mathbb{R}^2)$ as framed links that we have established. Or rather, this interpretation applies to the loops in the 0-charge sector, while loops in the charge=n sector may be understood more generally as *braids* on n strands interlinked with any number of links.



$$\in \quad \Omega_3\, \mathrm{Conf}^I(\mathbb{R}^2).$$

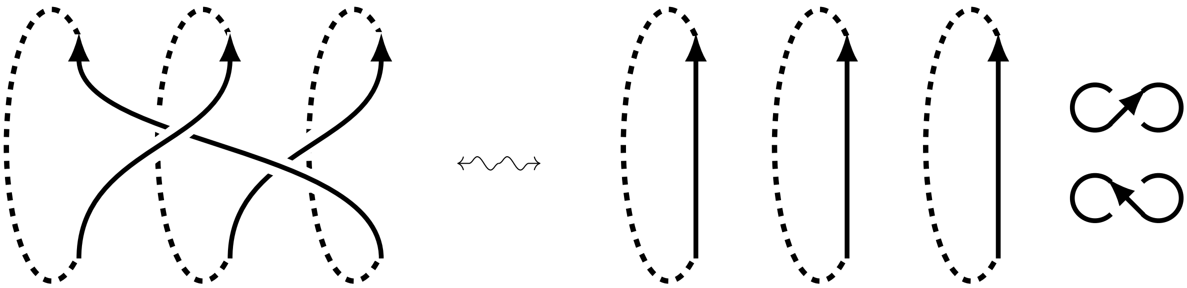
Figure L. An example of a loop in $\mathrm{Conf}^I(\mathbb{R}^2)$ based in the component of total charge $n = 3$.

(iii) To see the homotopy equivalence of the form $\Omega_n \,\mathrm{Conf}^I(\mathbb{R}^2) \simeq \Omega_0 \,\mathrm{Conf}^I(\mathbb{R}^2)$ (and hence also all the others) in terms of such "framed link-braids" (Figure L) being equivalently framed links with un-braids, and hence equivalently just framed links, observe that the saddle move from Lem. 4.16 in the following symmetrized form



(118)

allows to un-braid any braid-link at the cost of picking up a corresponding collection of further framed link components, e.g.:



(119)

4.2 Quantum observables on solitons on M5

We now put all the pieces together and show that the topological soliton sector on holographic M5-branes under consideration is controlled by (abelian) Chern-Simons theory.

Topological Quantum Observables. First, we briefly recall the general notion of (discrete light cone) quantum observables on topological charge sectors of flux-quantized higher gauge fields according to [SS23d] (previously applied to Hanany-Witten brane configurations in [SS22][CSS23][Col23]). Given a flux quantization law $\mathcal{A}$ for higher gauge fields on a spacetime domain X equipped with the structure of an S^1-fibration $X \to Y$ (as in M/IIA duality), the corresponding algebra of light-cone quantum observables on the topological charge sectors may be understood to be the homology of a based loop space of the $\mathcal{A}$-cocycle space on Y:

$$\mathrm{QObs}_\bullet \;\equiv\; H_\bullet\big(\Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A});\,\mathbb{C}\big)\,, \tag{120}$$

equipped with the *Pontrjagin product*

$$\begin{tikzcd}
\mathrm{QObs}_\bullet \otimes \mathrm{QObs}_\bullet \arrow[rr, "(-)\cdot(-)"] \arrow[d, equal] & & \mathrm{QObs}_\bullet \arrow[d, equal] \\
H_\bullet\big(\Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A})\big) \otimes H_\bullet\big(\Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A})\big) \arrow[r, phantom, "\simeq"', "\substack{\text{Künneth}\\ \text{isomorphism}}" below] & H_\bullet\Big(\Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A}) \times \Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A})\Big) \arrow[r, "\substack{\text{push along}\\ \text{loop concatenation}}"'] & H_\bullet\big(\Omega\,\mathrm{Maps}^{\ast/}(Y,\,\mathcal{A})\big)
\end{tikzcd} \tag{121}$$

and with the anti-involution

$$(-)^\ast \;:\; \mathrm{Obs}_\bullet \longrightarrow \mathrm{Obs}_\bullet \tag{122}$$

given by push-forward of homology along reversal of loops followed by complex conjugation (reflecting discrete light cone time reversal).

Remark 4.21 (Nature of the topological observables).
(i) In (120) the classifying space $\mathcal{A}$ is (just) the "topological realization" (the "shape", see [SS21, §3.3]) of the full higher moduli stack of on-shell flux quantized higher gauge fields, the latter being a homotopy fiber product of $\mathcal{A}$ with the classifying sheaf $\Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}}$ of on-shell flux densities (as explained in [SS24b, §3.3][SS24a][GSS24a] with technical details in [FSS23, §9]).
(ii) This means that the observables in (120) do not resolve the actual higher gauge field configurations but only their topological soliton sectors, whence they are "topological observables" only, which is what we are interested in here.

(iii) Moreover, where in the traditional path-integral picture the non-commutative product operation on quantum observables reflects their successive temporal ordering, the Pontrjagin product (121) orders by windings of observed configurations along the S^1-circle fiber, which hence plays the role of the boosted circle fiber in discrete light cone quantization, cf. [SS23d, p. 8].

Therefore the sector of the quantum states that suffice to take expectation values of these topological quantum observables are, for short, the *topological quantum states*:

Topological quantum states. Given a star-algebra of quantum observables, the corresponding quantum states are embodied by the expectation values that they induce, which are linear forms ρ on observables subject to (1.) reality, (2.) semi-positivity, and (3.) normalization (e.g. [Mey95, §I.1.1][Wa10, §7][La17, Def. 2.4], exposition in [Gl11, p. 11]):

$$\mathrm{QStates}_\bullet \;:=\; \left\{ \rho \,:\, \mathrm{Obs}_\bullet \xrightarrow[\text{linear}]{} \mathbb{C} \;\middle|\; \underset{\mathcal{O}\in\mathrm{Obs}_\bullet}{\forall} \Big(\underbrace{\rho(\mathcal{O}^*) \,=\, \rho(\mathcal{O})^*}_{\text{reality}},\; \underbrace{\rho(\mathcal{O}^*\cdot\mathcal{O}) \,\geq 0 \,\in \mathbb{R}\hookrightarrow\mathbb{C}}_{\text{(semi-)positivity}} \Big),\; \underbrace{\rho(1) = 1}_{\text{normalization}} \right\}. \tag{123}$$

This subsumes all *mixed* states ("density matrices"). Among them the *pure* states (those which form a Hilbert space of states) are characterized as not being convex combinations of other states.

Note that (the expectation value of) a state ρ in (123) is *not required* to preserve the algebra product, those that do are called *multiplicative states*:

$$\rho : \mathrm{Obs}_\bullet \to \mathbb{C} \text{ is multiplicative} \qquad :\Leftrightarrow \qquad \underset{\mathcal{O},\mathcal{O}'\in\mathrm{Obs}_\bullet}{\forall} \; \rho\big(\mathcal{O}\cdot\mathcal{O}'\big) \;=\; \rho\big(\mathcal{O}\big)\,\rho\big(\mathcal{O}'\big)\,. \tag{124}$$

For these we will need the following general fact:

Lemma 4.22 (**Multiplicative states are pure** (e.g. [Zhu93, Ex. 13.3-4][Wa10, Lem. 7.20-21])). *Every multiplicative state* (124) *is pure; and on central observables the multiplicative states coincide with the pure states.*

Quantum states of solitons on holographic open M5-branes. Specializing this to the present case of solitons stuck at the O-planes of holographic open M5-branes wrapped on S^1_A (according to Figure BD) with their B-field flux quantized in equivariant 3-Cohomotopy (97) in the sector of vanishing total charge (due to Rem. 2.23), Prop. (4.5) gives that the topological quantum observables (120) here are:

$$\mathrm{QObs}_\bullet \;\equiv\; H_\bullet\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2);\,\mathbb{C}\big)\,.$$

From the base case of the Hurewicz theorem this means that in degree=0 these topological quantum observables form the space

$$\mathrm{Obs}_0 \;=\; \mathbb{C}\big[\pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)\big]$$

and as such are represented by compactly supported functions

$$\mathcal{O} \;:\; \pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big) \longrightarrow \mathbb{C}\,. \tag{125}$$

Now by Thm. 4.18 these quantum observables detect the total crossing number $\#L$ of the links L which the solitons (of vanishing total charge at the O-plane in the M5-brane) form in their transverse space as discrete light-cone evolution moves them along the along S^1_A. A choice of $\mathbb{C}$-linear basis of the topological quantum observables is hence given by:

$$\mathrm{Obs}_0 \;\simeq\; \Big\langle \mathcal{O}_n \;:\; [L] \;\mapsto\; \delta\big(\#(L), n\big) \Big\rangle_{n\in\mathbb{Z}}\,, \tag{126}$$

in which the Pontrjagin product (121) and star-operatiom (122) is readily found to be

$$\mathcal{O}_n \cdot \mathcal{O}_{n'} \;=\; \mathcal{O}_{n+n'}\,, \qquad \big(\mathcal{O}_n\big)^* \;=\; \mathcal{O}_{-n}\,. \tag{127}$$

Proposition 4.23 (**The pure topological quantum states in degree=0**). *The (expectation values of) pure quantum states* (123) *on* QObs_0 (126) *are precisely the linear maps of the form*

$$\begin{array}{ccc} \mathrm{QObs}_0 & \xrightarrow{\;\;\rho_k\;\;} & \mathbb{C} \\ \mathcal{O}_n & \longmapsto & \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,n\big) \end{array} \tag{128}$$

for any

$$k \in \mathbb{R}\setminus\{0\}\,. \tag{129}$$

Proof. With (127) and by Lem. 4.22, a pure state ρ on the commutative observables Obs_0 restricts to and is fixed by a group homomorphism

$$\rho\big(\mathcal{O}_{n+n'}\big) \;=\; \rho\big(\mathcal{O}_n\cdot\mathcal{O}_{n'}\big) \;=\; \rho\big(\mathcal{O}_n\big)\,\rho\big(\mathcal{O}_{n'}\big)$$

from the additive group of integers to the multiplicative group of non-vanishing (due to the normalization condition) complex numbers, hence:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{C}^\times \\ n \ &\longmapsto \ \rho(\mathcal{O}_n) \;=\; \rho(\mathcal{O}_1)^n\,. \end{aligned} \tag{130}$$

Moreover, using also the reality condition (123) gives that $\rho(\mathcal{O}_1)$ is unitary

$$\rho\big(\mathcal{O}_1\big)^* \underset{(123)}{=} \rho\big(\mathcal{O}_1^*\big) \underset{(127)}{=} \rho\big(\mathcal{O}_{-1}\big) \underset{(130)}{=} \rho\big(\mathcal{O}_1\big)^{-1}$$

and hence of the claimed form (128).

It just remains to observe that every map of the form (128) really is (the expectation value of) a quantum state (123), which follows readily. □

Remark 4.24 (**Pure topological quantum states as wave-functions).** Being linear forms on 0-homology $\mathrm{Obs}_0 \equiv H_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)$, the pure topological quantum states (128) are naturally identified with 0-cocycles in $H^0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)$ and as such are functions on our soliton configuration space of the form

$$\begin{aligned} \Omega\,\mathrm{Conf}^I(\mathbb{R}^2) \twoheadrightarrow \pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big) &\longrightarrow \mathbb{C} \\ L \qquad\qquad\qquad &\longmapsto \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,\#(L)\big) \end{aligned} \tag{131}$$

in that their evaluation on a 0-chain representing the homology class $\mathcal{O}_n$ — namely on any (framed, oriented) link L with total crossing number n — is $\exp\big(\tfrac{2\pi\mathrm{i}}{k}\,\#(L)\big) \;=\; \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,n\big)$.

This is remarkable because it coincides with the known form of quantum states/observables of abelian Chern-Simons theory:

Remark 4.25 (**Identification with quantum observables of** $\mathrm{U}(1)$**-CS theory).**
(i) For Chern-Simons theory with abelian gauge group $\mathrm{U}(1)$ it is widely understood by appeal to path-integral arguments ([Wi89, p. 363][FK89, p. 169] following [Pol88]) that

- the quantum states of the gauge field are labeled by a *level* [12] $k \in \mathbb{R}\setminus\{0\}$,
- the quantum observables are labeled by framed links L,
  often considered as equipped with labels (charges) q_i on their ith connected component L_i

and the expectation value of these observables in these states is the charge-weighted exponentiated framing- and linking numbers (Def. 4.10) as follows ([Wi89, pp. 363], cf. review e.g. in [MPW19, (5.1)]):

$$\mathrm{W}_k(L) \;=\; \exp\bigg(\tfrac{2\pi\mathrm{i}}{k}\Big(\textstyle\sum_i q_i^2\,\mathrm{frm}(L_i) + \sum_{i,j} q_i q_j\,\mathrm{lnk}(L_i,L_j)\Big)\bigg)\,. \tag{132}$$

(ii) However, with the charges q_i being integers, we may equivalently replace a q_i-charged component L_i with q_i unit-charged disjoint copies of L_i, and hence assume without loss of generality that $\forall_i\ q_i = 1$. With this we may observe that the Chern-Simons expectation values (132) coincide exactly with our pure topological quantum states (131):

$$W_k(L) \;=\; \exp\bigg(\tfrac{2\pi\mathrm{i}}{k}\Big(\sum_i \mathrm{frm}(L_i) + \sum_{i,j}\mathrm{lnk}(L_i,L_j)\Big)\bigg) \;=\; \exp\Big(\tfrac{2\pi\mathrm{i}}{k}\#(L)\Big).$$

In conclusion, we have established the following:

Fact. *With flux quantization on holographic M5-branes taken to be in 3-Cohomotopy* (Hypothesis H, §3)*, the pure topological quantum states* (Prop. 4.23, Rem. 4.24) *in degree=0 of B-field solitons stuck on O-planes in open holographic M5-branes wrapped on* S^1_A (Figure B) *are exactly those of abelian Chern-Simons theory* (Rem. 4.25).

Note that this is crucially a consequence of:

- the result that total B-field charge vanishes on holographic M5-banes (§2.4, Rem. 2.23);
- the choice of flux quantization in Cohomotopy of the higher gauge B-field on M5-branes (Hypothesis H, §3) which is admissible due to the super-space analysis of §2 (based on [GSS24b]);
- the appearance of framed links by analysis of the resulting soliton moduli space in §4.1.

---

[12] Notice that the level quantization which for non-abelian compact gauge groups forces the level k to be an integer does not apply in the abelian case considered here (cf. e.g. [FK89, p. 169]) so that the level may indeed be any non-zero real number, just as in (129).

Remark 4.26 (Comparison to the literature). Specifically the emergence of U(1)-Chern-Simons theory on holographic M5-branes has previously been argued in [MPW19] by inspection of Wilson loops in $D = 5$ super Yang-Mills theory. The realization of non-abelian Chern-Simons knot invariants on suitably wrapped M5-branes has previously been argued in [Wi12][GS12], see also [NO16, §1.1].

It may be noteworthy that in these previous references, going back to [Pol88][Wi89], the all-important framing of links is imposed in an *ad hoc* manner in order to work around an ill-defined expression appearing from the path-integral arguments (going back to [Pol88, p. 326]), whereas in §4.1 we use only well-defined constructions and the framing instead emerges automatically (under Hypothesis H) by careful analysis of the moduli of solitons on M5-branes.

5 Conclusion and outlook

First, in §2, motivated by the holographic realization of otherwise elusive strongly-coupled/correlated quantum systems — namely on probe branes embedded near their own black brane horizon (Figure B); but in view of open questions in its microscopic description, we established, for the first time, the respective super-embedding of holographic M5-branes (Thm. 2.21). Possibly contrary to previous expectations, the computation shows (1.) that the holographic M5-probe (in its background configuration not including fluctuations) must sit exactly at the throat radius (Rem. 2.20) and (2.) that it does not admit non-vanishing B-field flux (Rem. 2.23).

While the second point would render the gauge sector on holographic M5-branes trivial in traditional discussions the problem, the above super-embedding allowed us in §3, following [GSS24b], to consider the completion of the field content on holographic M5-branes by imposing admissible flux quantization laws such as in Cohomotopy theory (Hypothesis H). After such a global completion of the field content there may be non-trivial solitonic sectors even for vanishing B-field flux.

Indeed, after recalling the specific case of solitons on (holographic) "open" M5-branes from [GSS24b, §4.2], we began in §4 by observing that anyonic solitons in the sector of vanishing total charge is just what is envisioned in many discussions of application to topological quantum computation, where computational processed are thought to be represented by oriented links (Rem. 4.1). Now, detailed analysis (in §4.1) of their moduli space, using a previously underappreciated theorem by [Ok05], then showed that the resulting vortex-like solitons on holographic open M5-branes trace out *framed* oriented links in their transverse space (culminating in Prop. 108) topologically classified by their total crossing (framing plus twice the linking) number (Thm. 4.18).

With this we could prove (in §4.2) that the topological quantum states of these worldvolume solitons for the case that the statistical co-variance of all topological quantum observables vanishes (i.e. un-biased quantum states) are exactly those of U(1)-Chern-Simons theory (Prop. 4.23, Rem. 4.25), thus identifying the worldvolume solitons on holographic M5-branes as (abelian) anyons. A similar statement, based on traditional path-integral arguments, has recently been given in [MPW19] (Rem. 4.26).

Outlook. While abelian anyons are not universal for topological quantum gates by themselves, they become so already when combined with quantum measurement gates (see [Pa06][Ll02][Wo10][WP11]). Therefore their experimental realization in the form of manipulatable *solitons* (as found here on holographic M5-branes) would be a major step towards fault-tolerant quantum computation, and a microscopic holographic understanding of their nature should eventually be conducive for overcoming the present impasse in laboratory realizations of anyons.

(Note that we have previously discussed non-abelian anyons arising by cohomotopical flux quantization of, instead, *intersections* of M5-branes – in [SS23b][SS23c] –, but yet without demonstration of the underlying super-embedding.)

In order to obtain a more complete such microscopic holographic understanding of (abelian) anyons, one will need to pass beyond the sector of topological quantum observables considered here (cf. Rem. 4.21) in order to resolve the quantum dynamics also of the local B-field gauge potentials (cf. [GSS24b, §4.1]) and of the fluctuations of the M5-brane worldvolume, hence to fully combine the novel topological sector analysis presented here with the original local microscopic p-brane holography of Duff et al. (p. 2). This combination ought to show how the topological soliton degrees of freedom found here interact with local degrees of freedom – and hence how they may ultimately be manipulated by observers.

We hope to take steps in this direction in subsequent publications.

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