Twisted Cohomotopy implies level quantization of the full 6d Wess-Zumino term of the M5-brane

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Abstract

The full 6d Hopf-Wess-Zumino term in the action functional for the M5-brane is anomalous as traditionally defined. What has been missing is a condition implying the higher analogue of level quantization familiar from the 2d Wess-Zumino term. We prove that the anomaly cancellation condition is implied by the hypothesis that the C-field is charge-quantized in twisted Cohomotopy theory. The proof follows by a twisted/parametrized generalization of the Hopf invariant, after identifying the full 6d Wess-Zumino term with a twisted homotopy Whitehead integral formula, which we establish.

Contents

1 Introduction and results

The expected but elusive quantum theory of M5-branes in M-theory (see [Duf99, §3][HSS18, §2]) has come to be widely regarded as a core open problem in string theory, already in its decoupling limit of an expected 6-dimensional superconformal quantum field theory (see [Mo12][HR18]). Most attempts to understand at least aspects of this theory have been based on analogies (such as with the known M2-brane theory) and consistency checks (such as from implications of the expected superconformal structure). But a systematic derivation of the theory from deeper principles has not been possible, since these deeper principles must be those of the ambient M-theory, whose formulation is itself a wide open problem ([Du96, §6][Mo14, §12][Wi19, @21:15][Du19, @17:14]).

Recently in [FSS19a], following [Sa13], we motivated, from rigorous analysis of the super homotopy theory of super p-branes initiated in [FSS13b], a hypothesis about the mathematical foundations of microscopic M-theory:

Hypothesis H. *The M-theory C-field is charge-quantized in J-twisted Cohomotopy theory* ([FSS20c], Def. 4.1).

We proved in [FSS19b][SS19][SS20][FSS20b] that this hypothesis implies a list of subtle consistency conditions that had informally been argued to be necessary for M-theory to exist. This suggests that *Hypothesis H* could indeed be a correct assumption about the mathematical principles underlying microscopic M-theory. If this is the case, further aspects of M-theory must be systematically derivable, by rigorous mathematical deduction.

Here we prove that *Hypothesis H* implies global consistency of the full Hopf-Wess-Zumino-type term that appears in the Green-Schwarz-type action functional of the M5-brane. This used to be an open problem:

The open problem. The full 6d Hopf-Wess-Zumino-term (Hopf-WZ-term) of the single M5-brane (see (3) below for multiple M5-branes), originally proposed in [Ah96, p. 10] and fully established by [BLNPST97, (1)] (reviewed in detail below in §2) is a functional of fields on a 6d *worldvolume* manifold Σ^6 that may be expressed in terms of auxiliary extended fields on a cobounding *extended worldvolume* manifold $\widehat{\Sigma}^7$, as follows (full generality and details are presented in Def. 2.6, Remark 2.2 below, notation is summarized in Table A):

$$\underset{\text{Hopf-WZ term}}{\widehat{S}^{1\mathrm{M5}}_{\mathrm{WZ}}} = 2\widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}} := 2\int_{\widehat{\Sigma}^7} \Big(\tfrac{1}{2}\widehat{H}_3 \wedge \widehat{f}^* \widetilde{G}_4 + \widehat{f}^* G_7 \Big) \quad (1)$$

$$\exp\Big(2\pi i \big(\widehat{S}^{1\mathrm{M5}}_{\mathrm{WZ}} \big) \Big) \in U(1) \quad (2)$$

$\widehat{\Sigma}^7$	Extended worldvolume
$\widehat{f}$	Extended sigma-model field
$\widehat{H}_3$	Extended worldvolume higher gauge field
$\widetilde{G}_4$	Shifted background C-field flux
G_7	Dual background C-field flux

The open problem is to show that this expression (1) is actually well-defined, in that it is independent of the choice of extensions, or at least independent up to integer shifts, so that at least the exponentiated Wess-Zumino action functional (2) is well-defined.

Partial solution in the literature. A suggestive partial solution to this problem was proposed in [In00], by

- **(i)** assuming that G_4 is not only the form datum underlying a topological cocycle in rational Cohomotopy, [1] but even that of an actual smooth function c_{smth} to the smooth 4-sphere [In00, (5.3)];
- **(ii)** focusing on the first summand [In00, (2.4)] and disregarding the second summand in (1), leaving its understanding for later [In00, top of p. 16].

With these simplifications imposed, expression (1) reduces on oriented difference manifolds $\widetilde{\Sigma}^7 := \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$ (6) to the classical Whitehead integral formula [Wh47] (see [BT82, Prop. 17.22]) for the *Hopf invariant* $\mathrm{HI}(c_{\mathrm{smth}} \circ \widetilde{f})$ of maps to the 4-sphere. Since the Hopf invariant is an integer by its homotopy-theoretic definition (recalled as Def. 4.4 below), [In00] suggests that (2) is satisfied and thus refers to the first summand in (1) as the *Hopf-Wess-Zumino term*, a terminology that was used for other sigma-models before [WZ83][TN89], and which has become widely adopted for the M5 since (e.g. [KS03, §3.2][HN11, (2)][Ar18, (4.13)]). But, since assumption **(i)** is not supposed to be generally satisfied, so that disregarding the second term **(ii)** is not generally possible, this is only a partial solution, and the full problem of showing general consistency of (1) by demonstrating (2) had remained open.

Solution by homotopy periods in Cohomotopy. We observe here that the full Hopf-WZ-term (1), including the previously neglected summand $\widehat{f}^* G_7$, has the form of a secondary characteristic class descending from the intersection pairing, originally called a "functional cup product" by Steenrod [St49] and more recently discussed under the name *homotopy period* in [SW08, Ex. 1.9]. That these *homotopy Whitehead integrals* are the proper homotopy-theoretic formulation of the original Whitehead integral formula [Wh47] for the Hopf invariant, was remarked already by Haefliger [Ha78, p. 17]. For the analogous lower-dimensional case of maps from the 3-sphere to the 2-sphere, this had been worked out in [GM81, §14.5].

Our **first main result** here (Theorem 3.4 below) is a transparent proof that the full 6d Hopf-WZ-term (1) (including both summands) is a homotopy period/homotopy Whitehead integral in this sense, which reduces to the Whitehead integral formula for the Hopf invariant in the respective special cases (Remark 4.7 below). In fact, we prove a more general twisted version of the homotopy Whitehead integral, which incorporates also the topological twists that account for the half-integral shift by $\frac{1}{4}p_1$ demanded by flux quantization of the background C-field (Remark 2.2 below) thus generalizing the 6d Hopf-WZ term (1) to curved backgrounds (Def. 2.6 below).

This shows, in particular, that the two summands in (1) can not be invariantly separated, and hence that it is really the full term (1) which deserves to be called the *Hopf-Wess-Zumino term*. Thereby the puzzlement expressed in [In00, top of p. 8] is resolved: The first summand of (1) by itself does not actually qualify as a Wess-Zumino

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[1] This is an after the fact statement, in that we recast it that way in our formulation. In [In00], the 4-sphere was part of spacetime, whereas in our Cohomotopy formulation it serves as a classifying space that receives maps out of spacetime; and we view the formulation in [In00] as a special case where part of spacetime is identified with that classifying space. The sphere as a classifying space for Cohomotopy cohomology theory generalizes the Eilenberg-MacLane spaces that classify ordinary cohomology. See cite[§2]FSS20b for these general concepts, and see [GS20] for a detailed comparison.

term, since it is not (the pullback of) a cocycle. The full term *is* a cocycle, and in fact a cocycle in integral cohomology if *Hypothesis H* is satisfied, by the proof of our second main result:

Our **second main result** (Theorem 4.8 below) shows that under *Hypothesis H* the 6d Wess-Zumino term (1) is generally integral, even in its topologically twisted generalization. This topologically twisted/parametrized generalization of the Hopf invariant thus establishes (2) and hence proves in generality that the 6d Hopf-Wess-Zumino term of the M5-brane is well-defined (namely integral, level-quantized).

Consequences. We briefly highlight some consequences of and conclusions drawn from this result:

1. Level quantization. A key argument of [In00, (2.8)] was that the mathematical incarnation of N coinciding M5-branes is in the bare Hopf-WZ term (1) $\widehat{S}_{\mathrm{WZ}}^{\mathrm{M5}} = \int \frac{1}{2} H_3 \wedge \widehat{f}^* G_4 + \cdots$ being multiplied by $N(N+1)$, at least in its first summand. Since, by our result here, the two summands cannot be invariantly separated, this means that the full term has to be multiplied this way, hence that for N coincident M5-branes the expression (1) generalizes to

$$\widehat{S}_{\mathrm{WZ}}^{N\mathrm{M5}} := N(N+1) \int\limits_{\widehat{\Sigma}^7} \Big(\tfrac{1}{2} \widehat{H}_3 \wedge \widehat{f}^* \widetilde{G}_4 \;+\; \widehat{f}^* G_7 \Big) \qquad \boxed{N \;\|\; \text{Number of coincident M5-branes}} \tag{3}$$

with the factor of 2 in (1) being the case of $N = 1$. Since $N(N+1)$ is even for all N, the condition that (2) is well-defined up to an integral shift (by Theorems 3.4 and 4.8) implies that

$$\exp\Big(2\pi i \big(\widehat{S}_{\mathrm{WZ}}^{N\mathrm{M5}} \big) \Big) \;\in\; U(1) \tag{4}$$

is also well-defined, for all N. Thus the factor $N(N+1)$ plays the role of the *level* of the 6d Wess-Zumino term of the M5-brane; and its even integral form is the *level quantization* for the 6d Hopf-Wess-Zumino term of the M5-brane, in analogy with integral levels of ordinary Wess-Zumino terms [Wi83].

2. Dimensional generalization and the Hopf invariant one theorem. The full 6d Wess-Zumino term of the M5-brane (1) is evidently the special case $k = 1$ of a sequence of Wess-Zumino terms $S_{\mathrm{WZ}}^{1B(4k+1)}$ that exist for all $k \in \mathbb{N}$ on higher gauged p-brane sigma-model fields with $p = 4k+1$, hence the notation $B(4k+1)$. It is precisely those worldvolume dimensions that admit self-dual higher gauge fields. For trivial topological twist τ in (34), the proof of Theorem 3.4 generalizes verbatim to this infinite hierarchy, simply by generalizing the degree of the generator ω_4 in (34) to $2(k+1)$ and the degree of the generator ω_7 to $4k+3$. Similarly, Prop. 4.6 generalizes verbatim and shows that for all $k \in \mathbb{N}$ the anomaly functionals $\widetilde{S}_{\mathrm{WZ}}^{1B(4k+1)}$ (Def. 2.9) of these Wess-Zumino terms compute, in the absence of topological twists and under *Hypothesis H*, the Hopf invariant of the composite of the brane's sigma-model field with the cocycle of the background field in Cohomotopy.

It is interesting to note that, from this perspective, we may take the classical *Hopf invariant one theorem* [Ad60] to say that if the oriented difference of extended worldvolumes is the $(4k+1)$-sphere $\widetilde{\Sigma}^{4k+1} = S^{4k+1}$, then for almost all values of $k \in \mathbb{N}$ the anomaly functional $\widetilde{S}_{\mathrm{WZ}}^{1B(4k+1)}$ (Def. 2.9) is an even integer, in that the only values of k for which it may take odd integer values are precisely those that correspond to branes which actually appear in string/M-theory:

$k =$	0	1	2
$(4k+1)$-brane	string	five-brane	nine-brane

Hypothesis H* with the *Hopf invariant one theorem singles out the worldvolume dimensions $p+1 \in \{2, 6, 10\}$ among p-branes admitting self-dual higher gauge fields, as those whose Wess-Zumino anomaly functional $\widetilde{S}_{\mathrm{WZ}}^{1B(4k+1)}$ is integrally *in*divisible.

3. Unifying role of the quaternionic Hopf fibration. It is noteworthy that the proofs of our main results (Theorem 3.4 and Theorem 4.8) proceed entirely by characterizing lifts in Cohomotopy through the quaternionic Hopf fibration, observing that it is such lifts which reflect, under *Hypothesis H*, the higher gauge field H_3 on the worldvolume of the M5-brane [FSS19b, Prop. 3.20]. This tightly connects the discussion of the 6d Wess-Zumino term here to the analogous cohomotopical discussion of its supersymmetric completion in [FSS15][FSS19c] and to the anomaly cancellation conditions on the background fields in [FSS19b][SS20], all rigorously derived from first principles; and thus suggests that a complete derivation of the elusive quantum M5-brane may exist guided by *Hypothesis H*.

4. Outlook – Refinement to differential Cohomotopy. It is well known (see [FSS12b], review in [FSS13a]) that the definition of Wess-Zumino- and Chern-Simons-terms by field extensions over a cobounding manifold, while an elegant method when it applies, is not the most general definition of these terms. In cases where such field extensions do not exist, the WZ- and CS-terms may still exist, now defined as hypervolume holonomies of cocycles in a *differential* cohomology theory (see [FSS20c, §4.3]). For the ordinary WZ- and CS-term this differential cohomology theory is differential ordinary cohomology, represented equivalently as Cheeger-Simons differential characters or as Deligne cohomology or as bundle gerbes with connections, or as $B^nU(1)$-principal connections.

But for the case of the 6dWZW term of the M5-branes, our results here show that the appropriate differential cohomology theory that generalizes the construction by field extension presented here must be a differential refinement of Cohomotopy cohomology theory. We had constructed one version of such a *differential Cohomotopy cohomology theory* in [FSS15, §4][FSS20c, §5.3], further discussed in [GS20, §3]. Ultimately one should use this to generalize the results we present here to situations where extensions of fields over cobounding manifolds may not exist.

Outline. In §2 we make precise the 6d Hopf-Wess-Zumino term and its anomaly, including topological twisting. In §3 we establish that the full WZ term is a homotopy period/homotopy Whitehead integral. In §4 we prove that *Hypothesis H* implies that the full 6d Hopf-Wess-Zumino term is well-defined.

2 The full 6d Hopf-WZ term of the M5-brane

In this section we present a precise definition, paraphrasing from the informal literature, of the 6d Hopf-Wess-Zumino term of the M5-brane, refine it to include topological twists reflecting the shifted quantization condition on the C-field flux, and then prove that the corresponding anomaly functional is a homotopy invariant.

First we state (in Def. 2.4 below) the 6d WZ term for "small" sigma-model fields as found in the original articles [Ah96, p. 11][BLNPST97, (1)], then we consider its globalization via extension to cobounding extended worldvolumes as in [In00, (5.4)] (Def. 2.6 below). Throughout, we include the half-integral shift of G_4 by $\frac{1}{4}p_1$, demanded by the flux quantization of the C-field [Wi97a]; see Remark 2.2 below. Finally, we discuss the corresponding anomaly functional (Def. 2.9 below) and show that it is a homotopy invariant on the space of gauged sigma-model fields (Lemma 2.10).

To be precise, we begin by introducing the relevant ingredients:

Definition 2.1 (Background C-field and higher gauged sigma-model fields)**.**
(i) Let X^8 be a smooth 8-manifold which is connected, simply connected[2] and spin, to be called the *target spacetime*[3].
(ii) Let Σ be a smooth manifold, which is compact and oriented, to be called
(a) the *worldvolume* if it is 6-dimensional $\Sigma := \Sigma^6$ without boundary;

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[2] All results in the following readily generalize to non-connected X, but nothing essential is gained thereby. The assumption that X is simply connected is to allow the use of Sullivan model analysis in §3 and §4 (as in [FSS19b, Rem 2.6][FSS20c, Rem. 3.53]). For this it would be sufficient to assume that X is *nilpotent* [FSS20c, Def. 3.52] in that it has nilpotent fundamental group acting nilpotently on homotopy and homology groups of its universal cover. This assumption should not be necessary, but without it all proofs will become much more involved.

[3] This pertains to M-theory on 8-manifolds, see [FSS19b, Remark 3.1]. We will often just write X for X^8.

(b) the *extended worldvolume* if it is 7-dimensional $\Sigma := \widehat{\Sigma}^7$, with collared boundary

$$\Sigma^6 = \partial\widehat{\Sigma}^7 \xhookrightarrow{(\mathrm{id},0)} \big(\partial\widehat{\Sigma}^7\big) \times [0,1) \hookrightarrow \widehat{\Sigma}^7 \,. \tag{5}$$

(c) the *oriented difference of extended worldvolumes* if it is 7-dimensional $\Sigma := \widetilde{\Sigma}^7$ and arising as the oriented difference

$$\widetilde{\Sigma}^7 \;=\; \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2 \;:=\; \widehat{\Sigma}^7_1 \cup_{\Sigma^6} \big(\widehat{\Sigma}^7_2\big)^{\mathrm{op}} \tag{6}$$

(where $(-)^{\mathrm{op}}$ denotes orientation reversal) of two collared coboundary extension $\widehat{\Sigma}^7_{1,2}$ (5) of the same worldvolume $\partial\widehat{\Sigma}^7_{1,2} = \Sigma^6$; in particular $\widetilde{\Sigma}^7$ has no boundary.

(iii) A *background field* configuration on X^8 is

(a) an affine $\mathrm{Spin}(8)$-connection ∇ on the tangent bundle[4] TX^8;

(b) a pair of differential forms

$$\begin{array}{l} G_4 \in \Omega^4_{\mathrm{dR}}\big(X^8\big) \\ 2G_7 \in \Omega^7_{\mathrm{dR}}\big(X^8\big) \end{array} \quad \text{such that} \quad \begin{array}{l} d\;\; G_4 = 0\,, \\ d\,2G_7 = -G_4 \wedge G_4 + \big(\tfrac{1}{4}p_1(\nabla)\big) \wedge \big(\tfrac{1}{4}p_1(\nabla)\big)\,, \end{array} \tag{7}$$

where the Pontrjagin 4-form (e.g. [KN63, §XII.4])

$$p_1(\nabla) := \langle R_\nabla \wedge R_\nabla \rangle \tag{8}$$

is the value of the curvature 2-form R of ∇ in the normalized Killing form invariant polynomial $\langle -,- \rangle$ on $\mathfrak{so}(8)$. Notice that in terms of the shifted flux form [Wi97a, (1.2)][Wi97b, (1.2)][FSS19b, §3.4][FSS20b]

$$\widetilde{G}_4 \;:=\; G_4 + \tfrac{1}{4}p_1(\nabla) \tag{9}$$

the second condition in (7) [Wi97b, (3.6)][FSS19b, §3.3] equivalently reads

$$d\,2G_7 = -\big(\widetilde{G}_4 \wedge \widetilde{G}_4 - \tfrac{1}{2}p_1(\nabla) \wedge \widetilde{G}_4\big). \tag{10}$$

(iv) A *higher gauged*[5] *sigma-model field* is a pair

$$\big(f, H_3\big) \;=\; \big(\Sigma \xrightarrow{f \text{ smooth}} X,\; dH_3 \;=\; f^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big)\big) \tag{11}$$

consisting of

(a) a smooth function f from the (extended) worldvolume to spacetime,

(b) a smooth differential 3-form H_3 on the (extended) worldvolume, which trivializes the pullback along f of the difference between G_4 from (7) and $\frac{1}{4}p_1(\nabla)$ from (8),

both required to have *sitting instants* on any collared boundary (5), in that in some neighborhood of the boundary they are constant in the direction perpendicular to it [FSS10, Def. 4.2.1];

(v) A *gauge transformation* or *homotopy* between two higher gauged sigma-model fields (11)

$$\big(f_0,\, (H_3)_0\big) \xRightarrow{\big(\eta,\, (H_3)_{[0,1]}\big)} \big(f_1,\, (H_3)_1\big) \tag{12}$$

is a pair consisting of a smooth homotopy η from f_0 to f_1 and a differential 3-form $(H_3)_{[0,1]} \in \Omega^3_{\mathrm{dR}}\big(\Sigma \times [0,1]\big)$ gauging η and restricting to $(H_3)_{0,1}$ at the boundaries of the interval:

$$\begin{tikzcd}
\Sigma \arrow[d, "{(\mathrm{id},0)}"'] \arrow[dr, "f_0"] & \\
\Sigma \times [0,1] \arrow[r, "\eta \text{ smooth}"] & X \\
\Sigma \arrow[u, "{(\mathrm{id},1)}"] \arrow[ur, "f_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\widetilde{H}_3)_0 \\
(H_3)_{[0,1]} \arrow[u, mapsto, "{(\mathrm{id},0)^*}"'] \arrow[d, mapsto, "{(\mathrm{id},1)^*}"] \\
(H_3)_1
\end{tikzcd}
\qquad
d\big((H_3)_{[0,1]}\big) = \eta^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big). \tag{13}$$

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[4] The theorems below hold, as general statements about the 6d WZ term, for ∇ a connection on any Spin bundle. But application to the actual M5-brane system requires ∇ to be a tangent connection on spacetime.

[5] This is the higher analog of abelian gauging of 2d WZW model fields (e.g. [Fo03, (5)]), making the 6d Wess-Zumino term the action functional of a *higher gauged Wess-Zumino model* [FSS13b].

(vi) We write

$$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X) \;:=\; \big\{ (f, H_3) \big\}, \qquad \pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big) \;:=\; \big\{ (f, H_3) \big\}_{\big/\sim_{\mathrm{homotopy}}} \tag{14}$$

for the sets[6] of higher gauged sigma-model fields (11) and of their *homotopy classes* (12), respectively.

Remark 2.2 (Shifted flux quantization corrections to the Hopf-WZ term)**.** The existing literature on the Hopf-WZ term [Ah96, p. 11][BLNPST97, (1)][In00, (2.4)] [KS03, §3.2][HN11, (2)][Ar18, (4.13)] disregards any topological correction terms to the C-field flux proportional to $p_1(\nabla)$, shown in (9), (10). Hence, in comparing to this literature (as in Def. 2.4, Def. 2.6 below), one has to restrict to the special case that $p_1(\nabla) = 0$ (for instance in that $\nabla = 0$, hence that spacetime is assumed to be flat). Beyond this special case, such correction terms are famously thought to be required, by circumstantial arguments provided in [Wi97a][Wi97b]. The result of [FSS19b, §3.3] (recalled as Remark 3.3 below) is that charge quantization of the C-field in J-twisted Cohomotopy theory [FSS20c, §5.3] implies exactly these corrections (10); and the main Theorem 4.8 below says that with these p_1-corrections and this cohomotopical charge quantization, the full Hopf-WZ term is actually guaranteed to be anomaly-free (namely: integral, level-quantized).

As an important example of Def. 2.1, we offer the following:

Lemma 2.3 (The 7-sphere as an extended worldvolume)**.** *In the situation of Def. 2.1, let the oriented difference of extended worldvolumes be the 7-sphere:* $\Sigma := \widetilde{\Sigma}^7 := S^7$. *Then the set* (14) *of homotopy classes of extended gauged sigma-model fields is the set underlying the 7th homotopy group of target spacetime X:*

$$\pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(S^7, X\big)\Big) \;\simeq\; \pi_7(X)\,. \tag{15}$$

Proof. Since homotopy classes of continuous functions between smooth manifolds are given by smooth homotopy classes of smooth functions (e.g. [BT82, Cor. 17.8.1]) it follows that already smooth homotopy classes of ungauged sigma-model fields are in bijection to $\pi_7(X)$ (since the target spacetime X is assumed to be connected there is no dependence on a basepoint). Hence it only remains to show that, for any extended sigma-model field $\widetilde{f}$, there exists at least one gauging $\widetilde{H}_3$ (11) and that, for any two such gaugings $\big(\widetilde{f}, (\widetilde{H}_3)_0\big)$ and $\big(\widetilde{f}, (\widetilde{H}_3)_1\big)$ of the same extended sigma-model field $\widetilde{f}$, there exists a gauged homotopy (12)

$$\big(\widetilde{f}, (\widetilde{H}_3)_0\big) \xRightarrow{\big(\widetilde{\eta}, (\widetilde{H}_3)_{[0,1]}\big)} \big(\widetilde{f}, (\widetilde{H}_3)_1\big)$$

between them. For the existence of the gauging $\widetilde{H}_3$ for a given $\tilde{f}$, we only need to notice that because $H^4_{\mathrm{dR}}(S^7) \cong H^4(S^7;\mathbb{R}) = 0$, we have $f^*[G_4 - \frac{1}{4}p_1(\nabla)] = 0$ and so there exists $\widetilde{H}_3 \in \Omega^3_{\mathrm{dR}}\big(S^7\big)$ such that $d\widetilde{H}_3 = f^*\left(G_4 - \frac{1}{4}p_1(\nabla)\right)$. Similarly, given two gaugings $(\widetilde{H}_3)_0$ and $(\widetilde{H}_3)_1$ of $\tilde{f}$, since $H^3_{\mathrm{dR}}\big(S^7\big) = 0$ and $(\widetilde{H}_3)_1 - (\widetilde{H}_3)_0 \in \Omega^3_{\mathrm{dR}}\big(S^7\big)$ is closed by assumption, there exists

$$\alpha \in \Omega^2_{\mathrm{dR}}\big(S^7\big) \quad \text{ such that } \quad d\alpha = (\widetilde{H}_3)_1 - (\widetilde{H}_3)_0\,.$$

Thus

$$\Big(\widetilde{\eta} : (x,s) \longmapsto \widetilde{f}(x)\,,\; (\widetilde{H}_3)_{[0,1]} \;:=\; (\widetilde{H}_3)_1 + (s-1)\cdot d\alpha + (ds)\wedge\alpha\Big)$$

constitutes a homotopy as required. □

We now consider the 6d Hopf-WZ term in its various incarnations, surveyed in Table A.

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[6] The inclined reader will notice (see [FSS13b] for exposition) that the set $\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)$ is of course the underlying set of global sections of the atlas for the smooth *moduli 2-stack* of higher gauged sigma-model fields on Σ [FSS20c, §4.3], and $\pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big)$ is the set of connected components of the geometric realization of this moduli 2-stack. All of the following discussion lifts to the higher differential geometry of moduli stacks of fields, but for the sake of brevity we will not further consider this here.

$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, X^{11}\big) \xrightarrow{\;S\;} \mathbb{R}$	Hopf-WZ action functional on worldvolume Σ^6	Def. 2.4
$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, X^{11}\big) \xrightarrow{\;\widehat{S}\;} \mathbb{R}$	Extended Hopf-WZ functional on coboundary $\partial\widehat{\Sigma}^7 = \Sigma^6$	Def. 2.6
$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X^{11}\big) \xrightarrow{\;\widetilde{S}\;} \mathbb{R}$	Hopf-WZ anomaly functional on oriented difference $\widetilde{\Sigma}^7 = \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$	Def. 2.9

Table A – Incarnations of the Hopf-WZ term. The 6d Hopf-WZ term functional $S := S^{\mathrm{M5}}_{\mathrm{WZ}}$ is a priori defined on gauged sigma-model fields on Σ^6. Its global definition involves an extension $\widehat{S}$ to extended fields on a coboundary $\widehat{\Sigma}^7$. The difference of any two extensions is the anomaly functional $\widetilde{S}$ on fields on the oriented difference $\widetilde{\Sigma}^7 = \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$.

Definition 2.4 (6d Hopf-WZ term for small sigma-model fields)**.** In the setting of Def. 2.1, let $U \subset X^8$ be a chart (a contractible open subset). For Σ^6 any closed orientable 6-manifold, write $\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) \subset \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, X\big)$ for the subset of those higher gauged sigma-model fields (14) which factor through $U \subset X$ (the "U-small sigma-model fields"). As the de Rham cohomology of U is trivial in positive degree, we may choose local potentials $C_3^U \in \Omega^3_{\mathrm{dR}}\big(U\big)$ for $\iota_U^*(G_4 + \frac14 p_1(\nabla))$ and $2C_6^U \in \Omega^6_{\mathrm{dR}}\big(U\big)$ for $\iota_U^* 2G_7 + C_3^U \wedge \iota_U^*(G_4 - \frac14 p_1(\nabla))$.

$$\begin{tikzcd} & U \arrow[d, hook, "\iota_U"] \\ \Sigma^6 \arrow[ur, dashed, "f_U"] \arrow[r, "f"'] & X^8 \end{tikzcd}$$

$$\begin{aligned} dC_3^U &= \iota_U^*\big(G_4 + \tfrac14 p_1(\nabla)\big) \\ d2C_6^U &= \iota_U^* 2G_7 + C_3^U \wedge \iota_U^*\big(G_4 - \tfrac14 p_1(\nabla)\big) \end{aligned} \tag{16}$$

$$\begin{aligned} dG_4 &= 0 \\ d2G_7 &= -G_4 \wedge G_4 + \big(\tfrac14 p_1(\nabla)\big) \wedge \big(\tfrac14 p_1(\nabla)\big) \end{aligned}$$

Then the *M5 6d Wess-Zumino term action functional* on these small fields is the function

$$\begin{array}{ccc} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) & \xrightarrow{\qquad S^{\mathrm{M5}}_{\mathrm{WZ}} \qquad} & \mathbb{R} \\ \big(f_U, H_3\big) & \longmapsto & S^{\mathrm{M5}}_{\mathrm{WZ}}\big(f_U, H_3\big) := \tfrac12 \int_{\Sigma^6} \big(-H_3 \wedge f_U^* C_3^U + f_U^* 2C_6^U\big). \end{array} \tag{17}$$

Lemma 2.5 (Independence of choices)**.** *The functional* $S^{\mathrm{M5}}_{\mathrm{WZ}}\big(f_U, H_3\big)$ (17) *is indeed well defined, in that it does not depend on the choice of the local potentials* C_3^U *and* C_6^U (16).

Proof. A different choice of local potentials is of the form $(C_3^U + \alpha_3^U, 2C_6^U + 2\alpha_6^U)$, with differentials $d\alpha_U^3 = 0$ and $d2\alpha_6^U = \alpha_3^U \wedge \iota_U^*(G_4 - \frac14 p_1(\nabla))$. As the local chart U is contractible, this implies $\alpha_3^U = d\alpha_2^U$ and $2\alpha_6^U = \alpha_2^U \wedge \iota_U^*\big(G_4 - \frac14 p_1(\nabla)\big) + d\alpha_5^U$. Therefore, we have

$$\begin{aligned} & \int_{\Sigma^6} \Big(-H_3 \wedge f_U^*\big(C_3^U + \alpha_3^U\big) + 2f_U^*\big(C_6^U + \alpha_6^U\big)\Big) - \int_{\Sigma^6} \Big(-H_3 \wedge f_U^*\big(C_3^U\big) + f_U^*\big(2C_6^U\big)\Big) \\ &= \int_{\Sigma^6} -H_3 \wedge d f_U^*\big(\alpha_2^U\big) + f_U^*\Big(\alpha_2^U \wedge \iota_U^*\big(G_4 - \tfrac14 p_1(\nabla)\big)\Big) + d f_U^*\big(\alpha_5^U\big) \\ &= \int_{\Sigma^6} -H_3 \wedge d f_U^*\big(\alpha_2^U\big) + f_U^*\big(\alpha_2^U\big) \wedge f^*\big(G_4 - \tfrac14 p_1(\nabla)\big) + d f_U^*\big(\alpha_5^U\big) \\ &= \int_{\Sigma^6} -H_3 \wedge d f_U^*\big(\alpha_2^U\big) + f_U^*\big(\alpha_2^U\big) \wedge dH_3 + d f_U^*\big(\alpha_5^U\big) \\ &= \int_{\Sigma^6} d\Big(H_3 \wedge f_U^*\big(\alpha_2^U\big) + f_U^*\big(\alpha_5^U\big)\Big) = 0. \end{aligned}$$

□

Now we globalize this definition, following the well-known procedure originally introduced in the 2-dimensional case in [Wi83].

Definition 2.6 (Global 6d Hopf-Wess-Zumino term via extended worldvolumes)**.** In the situation of Def. 2.1, for Σ^6 a given worldvolume, let $\widehat{\Sigma}^7$ be a compact oriented smooth collared cobounding 7-manifold[7] according to (5)

$$\Sigma^6 := \partial \widehat{\Sigma}^7 \,. \tag{18}$$

Then we say that the corresponding *extended action functional* for the 6d Hopf-WZ-term on the closed manifold Σ^6 is the function

$$\begin{array}{ccc} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, X\big) & \xrightarrow{\qquad\qquad \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}} \qquad\qquad} & \mathbb{R} \\ \big(\widehat{f}, \widehat{H}_3\big) & \longmapsto & \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widehat{f}, \widehat{H}_3\big) \;:=\; \tfrac{1}{2}\underset{\widehat{\Sigma}^7}{\int} \Big(\widehat{H}_3 \wedge \widehat{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widehat{f}^* 2G_7\Big) \end{array} \tag{19}$$

on the set of extended gauged sigma-model fields (11). (For flat backgrounds, $\nabla = 0$, this reduces to the (1), see Remark 2.2.)

Lemma 2.7 (Global Hopf-WZ-term restricts to local Hopf-WZ-term)**.** *In the situation of Def. 2.4, consider a worldvolume Σ^6. Then, for every choice of extended worldvolume $\widehat{\Sigma}^7$ (18) the corresponding* extended action functional *$\widehat{S}$ (Def. 2.6) coincides, for any chart $U \subset X$, on U-small extended sigma-model fields $\widehat{f} = \iota_U \circ \widehat{f}_U$* (16) *with the local action functional S (Def. 2.4) evaluated on the boundary values $f := \widehat{f}\,|_{\Sigma^6}$ of the extended fields:*

$$\begin{tikzcd} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, U\big) \arrow[rr, "\widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}"] \arrow[d, "(-)|_{\partial\widehat{\Sigma}^7}"'] & & \mathbb{R} \\ \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) \arrow[urr, "S^{\mathrm{M5}}_{\mathrm{WZ}}"'] & & \end{tikzcd} \qquad \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widehat{f}, \widehat{H}_3\big) \;=\; S^{\mathrm{M5}}_{\mathrm{WZ}}\Big(f := \widehat{f}\,|_{\partial\Sigma^7},\, H_3 := \big(\widehat{H}_3\big)|_{\partial\widehat{\Sigma}^7}\Big)\,.$$

Proof. Observe, with (11) and (16), that

$$\begin{aligned} d\big(-\widehat{H}_3 \wedge \widehat{f}^*_U C^U_3 + \widehat{f}^*_U 2C^U_6\big) \;&=\; -\underbrace{d\widehat{H}_3}_{\widehat{f}^*\left(G_4 - \frac{1}{4}p_1(\nabla)\right)} \wedge\, \widehat{f}^*_U C^U_3 \;+\; \widehat{H}_3 \,\wedge\, \underbrace{f^*_U dC^U_3}_{\widehat{f}^*(G_4 + \frac{1}{4}p_1(\nabla))} \;+\; \underbrace{\widehat{f}^*_U d2C^U_6}_{\substack{\widehat{f}^*(G_4 - \frac{1}{4}p_1(\nabla)) \wedge \widehat{f}^*_U C^U_3 \\ + \widehat{f}^* 2G_7}} \\ &= \widehat{H}_3 \wedge \widehat{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) \;+\; \widehat{f}^* 2G_7 \,. \end{aligned} \tag{20}$$

With this, the claim follows by Stokes' theorem:

$$\begin{aligned} \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widehat{f}, \widehat{H}_3\big) &:= \tfrac{1}{2}\underset{\widehat{\Sigma}^7}{\int} \Big(\widehat{H}_3 \wedge \widehat{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widehat{f}^*_U 2G_7\Big) \\ &= \tfrac{1}{2}\underset{\widehat{\Sigma}^7}{\int} d\big(-\widehat{H}_3 \wedge \widehat{f}^*_U C^U_3 + \widehat{f}^*_U 2C^U_6\big) \\ &= \tfrac{1}{2}\underset{\partial\widehat{\Sigma}^7}{\int} \big(-\widehat{H}_3 \wedge \widehat{f}^*_U C^U_3 + \widehat{f}^*_U 2C^U_6\big)|_{\partial\Sigma^7} \\ &= \tfrac{1}{2}\underset{\Sigma^6}{\int} \big(-H_3 \wedge f^*_U C^U_3 + f^*_U 2C^U_6\big) \\ &=: S^{\mathrm{M5}}_{\mathrm{WZ}}\big(f_U, H_3\big)\,. \end{aligned} \tag{21}$$

□

Example 2.8 (Coboundaries for $\Sigma^6 = S^3 \times S^3$)**.** In the situation of Def. 2.1, consider as worldvolume the product manifold of two 3-spheres (this is considered in [MS15, Example 2] in the non-commutative setting):

$$\Sigma^6 \;=\; S^3 \times S^3 \,.$$

---

[7] This always exists, since the oriented cobordism ring in dimension 6 is trivial and by the collar neighbourhood theorem.

In this case there is a canonical choice of cobounding manifold $\widehat{\Sigma}^7$ (18) given by the Cartesian product of the 4-disk D^4 (the closed 4-dimensional ball) with the 3-sphere, in either order (as in [Sa13]):

$$\widehat{\Sigma}^7_L := D^4 \times S^3 \quad \text{and} \quad \widehat{\Sigma}^7_R := \big(S^3 \times D^4\big)^{\mathrm{op}}. \tag{22}$$

Here we are equipping each of

$$\left.\begin{array}{lcr} S^3 \times S^3, & & \\ D^4 \times S^3 & = & D^4 \ \times \big(\partial D^4\big), \\ S^3 \times D^4 & = & \big(\partial D^4\big) \times \ \ D^4 \ , \end{array}\right\} \subset D^4 \times D^4 \subset \mathbb{R}^8$$

with the orientation induced from the canonical embedding into $\mathbb{R}^8$, which implies, by the odd-dimensionality of S^3, that the boundary of $S^3 \times D^4$ is $\big(S^3 \times S^3\big)^{\mathrm{op}}$ (opposite orientation). This way, with (22) we indeed have

$$\partial \widehat{\Sigma}^7_{L,R} = \Sigma^6 := S^3 \times S^3$$

as oriented manifolds. Observe that the union of one of these coboundaries with the orientation reversal of the other is the 7-sphere (as considered in Lemma 2.3): [8]

$$\begin{aligned} \widetilde{\Sigma}^7 := \widehat{\Sigma}^7_L \cup \big(\widehat{\Sigma}^7_R\big)^{\mathrm{op}} &= D^4 \times \big(\partial D^4\big) \cup \big(\partial D^4\big) \times D^4 \\ &= \partial\big(D^4 \times D^4\big) \\ &\simeq \partial D^8 \\ &= S^7 . \end{aligned} \qquad \begin{tikzcd} S^3 \times S^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(po)}"] & D^4 \times S^3 \arrow[d] \\ S^3 \times D^4 \arrow[r] & S^7 \end{tikzcd} \tag{23}$$

While Def. 2.6 gives global meaning to the local Hopf-WZ term (Def. 2.4), by Lemma 2.7, this potentially comes at the cost that the global definition depends on the choice of coboundary (18). The following definition measures this potential dependency:

Definition 2.9 (Hopf-WZ anomaly functional)**.** In the situation of Def. 2.1, with given worldvolume Σ^6, consider in Def. 2.6 two choices $\widehat{\Sigma}^7_{L,R}$ of collared cobounding extended worldvolumes (5) $\partial\Sigma^7_{L,R} = \Sigma^6$. This makes their oriented difference (6) a smooth closed 7-manifold $\widetilde{\Sigma}^7 := \widehat{\Sigma}^7_L - \widehat{\Sigma}^7_R := \widehat{\Sigma}^7_L \cup_{\Sigma^6} \big(\widehat{\Sigma}^7_R\big)^{\mathrm{op}}$. Then for

$$\begin{tikzcd} \widehat{\Sigma}^7_L \arrow[drr, "\widehat{f}_L"] & & & d\big(\widehat{H}_3\big)_L = \widehat{f}^*_L G_4 \arrow[d, mapsto, "\iota^*_{\partial L}"] \\ \Sigma^6 \arrow[u, hook, "\iota_{\partial L}"] \arrow[d, hook, "\iota_{\partial R}"'] \arrow[rr, "f"] & & X & d H_3 = f^* G_4 \\ \widehat{\Sigma}^7_R \arrow[urr, "\widehat{f}_R"'] & & & d\big(\widehat{H}_3\big)_R = \widehat{f}^*_R G_4 \arrow[u, mapsto, "\iota^*_{\partial R}"] \end{tikzcd}$$

any pair of gauged extended sigma-model fields (11), extending the same ordinary sigma-model field f over the two choices of coboundaries, respectively, we obtain a gauged extended sigma-model field $\big(\widetilde{f}, \widetilde{H}_3\big)$ on the closed 7-manifold $\widetilde{\Sigma}^7$ (6) (which is smooth by the assumption of sitting instants in (11)):

$$\begin{tikzcd} \widehat{\Sigma}^7_L \arrow[d, hook, "\iota_L"] \arrow[drr, "\widehat{f}_L"] & & & d\big(\widehat{H}_3\big)_L = \widehat{f}^*_L G_4 \\ \widetilde{\Sigma}^7 \arrow[rr, "\widetilde{f}"] & & X & d\widetilde{H}_3 = \widetilde{f}^* G_4 \arrow[u, mapsto, "\iota^*_L"'] \arrow[d, mapsto, "\iota^*_R"'] \\ \widehat{\Sigma}^7_R \arrow[u, hook, "\iota_R"] \arrow[urr, "\widehat{f}_R"'] & & & d\big(\widehat{H}_3\big)_R = \widehat{f}^*_R G_4 \end{tikzcd} \tag{24}$$

---

[8] Note that a different manipulation treats these, untraditionally, as manifolds with corners [Sa14][Sa13].

In terms of this, the difference between the two extended action functionals (Def. 2.6) corresponding to the two choices of coboundaries may be expressed as a single integral over $\widetilde{\Sigma}^7$:

$$
\begin{aligned}
\widetilde{S}\big(\widetilde{f},\widetilde{H}_3\big) &:= \widehat{S}\big(\widehat{f}_L,(\widehat{H}_3)_L\big) - \widehat{S}\big(\widehat{f}_R,(\widehat{H}_3)_R\big) \\
&= \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7} \big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\big).
\end{aligned}
\tag{25}
$$

We call expression (25) the *anomaly functional* of the 6d Wess-Zumino term.

Lemma 2.10 (Hopf-WZ anomaly functional is homotopy invariant)**.** *In the situation of Def. 2.1, let* $\Sigma := \widetilde{\Sigma}^7$ *be a closed 7-manifold. Then the Hopf-WZ anomaly functional* (25) *is well-defined on the set* (14) *of homotopy-classes of higher gauged sigma-model fields:*

$$
\begin{tikzcd}
\pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X\big)\Big) \arrow[r, "\widetilde{S}"] & \mathbb{R} \\
\big[\widetilde{f},\widetilde{H}_3\big] \arrow[r, mapsto] & \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7}\Big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\Big)
\end{tikzcd}
\tag{26}
$$

in that the integral on the right is independent of the choice of representative $\big(\widetilde{f},\widetilde{H}_3\big)$ *in its homotopy class.*

Proof. Given a gauge transformation/homotopy (12) between two extended gauged sigma-model fields

$$
\big(\widetilde{f}_0,(\widetilde{H}_3)_0\big) \xRightarrow{\big(\widetilde{\eta},(\widetilde{H}_3)_{[0,1]}\big)} \big(\widetilde{f}_1,(\widetilde{H}_3)_1\big)
$$

we need to show that $\widetilde{S}\big(\big[\widetilde{f}_1,(\widetilde{H}_3)_1\big]\big) = \widetilde{S}\big(\big[\widetilde{f}_0,(\widetilde{H}_3)_0\big]\big)$. With the data (13) and using Stokes' theorem we directly compute as follows:

$$
\begin{aligned}
\widetilde{S}\Big(\big[\widetilde{f}_1,(\widetilde{H}_3)_1\big]\Big) - \widetilde{S}\Big(\big[\widetilde{f}_0,(\widetilde{H}_3)_0\big]\Big) &= \tfrac{1}{2} \int\limits_{\partial\left(\widetilde{\Sigma}^7\times[0,1]\right)} \Big(\big(\widetilde{H}_3\big)_{[0,1]} \wedge \widetilde{\eta}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{\eta}^* 2G_7\Big) \\
&= \tfrac{1}{2} \int\limits_{\widetilde{\Sigma}^7\times[0,1]} d\Big(\big(\widetilde{H}_3\big)_{[0,1]} \wedge \widetilde{\eta}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{\eta}^* 2G_7\Big) \\
&= \tfrac{1}{2} \int\limits_{\widetilde{\Sigma}^7\times[0,1]} \Big(\underbrace{d\Big(\big(\widetilde{H}_3\big)_{[0,1]}\Big)}_{\widetilde{\eta}^*\left(G_4 - \frac{1}{4}p_1(\nabla)\right)} \wedge \widetilde{\eta}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{\eta}^* d2G_7\Big) \\
&= \tfrac{1}{2} \int\limits_{\widetilde{\Sigma}^7\times[0,1]} \widetilde{\eta}^*\Big(\underbrace{\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + d2G_7}_{=0}\Big) \\
&= 0\,,
\end{aligned}
$$

where in the last step, under the brace, we used the condition (7). ☐

3 The full M5 Hopf-WZ anomaly is a homotopy Whitehead integral

We first recall from [FSS19b][FSS20c, §5.3] how the background C-field $(G_4, 2G_7)$ is a cocycle in twisted rational Cohomotopy; this is Remark 3.3 below. Then we prove, in Theorem 3.4, that the Hopf-WZ anomaly functional from §2 is equivalently a lift in rational Cohomotopy through the equivariant quaternionic Hopf fibration, hence is in particular a homotopy invariant of both the gauged sigma-model fields and the background fields in Cohomotopy. Further below, in §4, this allows us to identify the anomaly functional as a twisted/parametrized generalization of a homotopy Whitehead integral.

Notions from rational homotopy theory. In the following, we make free use of Sullivan model dgc-algebras in rational homotopy theory (i.e., what in supergravity are called "FDA"s [FSS13b][FSS19a]); see [Su77][BG76] for the original accounts, [Hes06] for introduction, [GM81] for a standard textbook account, and see [FSS16][FSS17][FSS19a][FSS20c] for review streamlined towards our application. As in these references, for X a simply-connected topological space of finite rational type, we write $\mathrm{CE}(\mathfrak{l}X)$ for its minimal Sullivan model differential graded-commutative algebra (dgc-algebra), indicating that this is the Chevalley-Eilenberg algebra of the minimal Whitehead L_∞-algebra $\mathfrak{l}X$ corresponding to the loop group of X (see [FSS20c, Prop. 3.64]). For making Sullivan models explicit, we display the list of differential relations on each generator, thereby declaring what the generators are (see [FSS20c, (99)]), as shown in the following examples.

Example 3.1 (Quaternionic Hopf fibration)**.** We denote the minimal relative Sullivan model for the plain quaternionic Hopf fibration $h_{\mathbb{H}}$ as follows (see [FSS19b, Lemma 3.18]):

$$\begin{tikzcd}
 & S^7 \arrow[dd, "h_{\mathbb{H}}"] & & \mathrm{CE}\big(\mathfrak{l}S^7\big) \arrow[r, equal] & \big(d\,\omega_7 \;=\; 0\big) \\
\text{quaternionic Hopf fibration} & & \text{minimal relative Sullivan model (``FDA'')} & & \\
 & S^4 & & \mathrm{CE}\big(\mathfrak{l}S^4\big) \arrow[uu, hook, "\mathrm{CE}(\mathfrak{l}h_{\mathbb{H}})"'] \arrow[r, equal] & \left(\begin{array}{lcl} d\,\omega_4 & = & 0 \\ d\,\omega_7 & = & -\omega_4\wedge\omega_4 \end{array}\right) \arrow[uu, "\begin{smallmatrix} 0 & \omega_7 \\ \rotatebox{90}{$\mapsto$} & \rotatebox{90}{$\mapsto$} \\ \omega_4 & \omega_7 \end{smallmatrix}"']
\end{tikzcd} \tag{27}$$

topological homotopy theory

dgc-algebraic rational homotopy theory

Example 3.2 (Classifying space of Fivebrane-extended Spin-group)**.** The minimal relative Sullivan model for the homotopy fiber $B\widehat{\mathrm{Sp}(2)}$ of the classifying map for the Euler 8-class χ_8 on the classifying space for the quaternionc unitary group $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$ (see [FSS20b, §A]) is as follows (using [FSS19b, (97)][FSS20c, Lem. 4.24]):

$$\begin{tikzcd}
B\widehat{\mathrm{Sp}(2)} \arrow[r, "\simeq"] \arrow[d, "\mathrm{hofib}(\chi_8)"'] & B\widehat{\mathrm{Spin}(5)} \arrow[d, "\mathrm{hofib}\left(-\frac{1}{4}p_2+\left(\frac{1}{4}p_1\right)^2\right)"'] & \text{homotopy fiber trivializing obstructing 8-class} \\
B\mathrm{Sp}(2) \arrow[r, "\mathrm{tri}", "\simeq"'] \arrow[rr, bend right, "\chi_8"'] & B\mathrm{Spin}(5) \arrow[r, "-\frac{1}{4}p_2+\left(\frac{1}{4}p_1\right)^2"] & K(\mathbb{Z},8)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \left(\begin{array}{c} d\,\theta_7 = \chi_8 \\ d\,\chi_8 = 0 \\ d\,\frac{1}{2}p_1 = 0 \end{array}\right) & \left(\begin{array}{l} d\,\theta_7 = -\frac{1}{4}p_2+\left(\frac{1}{4}p_1\right)^2 \\ d\,p_2 = 0 \\ d\,\frac{1}{2}p_1 = 0 \end{array}\right) \arrow[l, "\simeq"'] & \\
\text{minimal relative Sullivan model} & \left(\begin{array}{c} d\,\chi_8 = 0 \\ d\,\frac{1}{2}p_1 = 0 \end{array}\right) \arrow[u, hook] & \left(\begin{array}{c} d\,p_2 = 0 \\ d\,\frac{1}{2}p_1 = 0 \end{array}\right) \arrow[u, hook] \arrow[l, "\mathrm{tri}^*", "\simeq"'] & \big(d\,c_8 = 0\big) . \arrow[l, "-\frac{1}{4}p_2+\left(\frac{1}{4}p_1\right)^2 \;\mapsfrom\; c_8"'] \arrow[ll, bend left, "\chi_8 \;\mapsfrom\; c_8"]
\end{tikzcd} \tag{28}$$

Notice that the higher extension $\widehat{\mathrm{Sp}(2)}$ in (28) is a version of the *Fivebrane 6-group*, and tangential $\widehat{\mathrm{Sp}(2)}$-structure trivializing this 8-class is a kind of *Fivebrane structure* according to [SSS09][SSS12]: a higher analog of *String structure* (trivializing $\frac{1}{2}p_1$), which itself is a higher analog of Spin-structure (trivializating w_2). In higher analogy to how Spin-structure is the topological condition on target spacetime needed for anomaly cancellation of the spinning particle [1], and String-structure is the topological condition for anomaly cancellation of the (heterotic) string (i.e. for the Green-Schwarz mechanism, see [SSS12][FSS20a][FSS20b]), so Fivebrane structure is meant to be the topological condition needed for anomaly cancellation of the (heterotic) five-brane. That this is the case for $\widehat{\mathrm{Sp}(2)}$-structure, as concerns the Hopf-WZ term of the M5-brane, is brought out by our main Theorems 3.4 and 4.8 below, see Remark 4.3 below.

Rationalization over the real numbers. In order to have a *smooth* non-abelian de Rham theorem ([FSS20c, Thm. 3.87]) involving the real de Rham dg-algebras $\Omega^\bullet_{\mathrm{dR}}(-)$ of smooth differential forms, we take the rational base field to be $\mathbb{R}$ instead of $\mathbb{Q}$ (as in [GM81][FSS20c, Rem. 3.51]), so that our "rational homotopy groups" are actually "real homotopy groups" $\pi(X) \otimes_{\mathbb{Z}} \mathbb{R}$; which makes no essential difference (by [BG76, Lem. 11.7]). Accordingly, for X a simply-connected topological space, we write

$$X \xrightarrow[\text{rationalization}]{\eta^{\mathbb{R}}_X} L_{\mathbb{R}}X \tag{29}$$

for its rationalization (localization over the real numbers, see [FSS20c, Def. 3.55]).

Example 3.3 (Background C-field is cocycle in rational twisted Cohomotopy)**.** The minimal Sullivan model ("FDA") of the 4-sphere is free on generators ω_4 and ω_7 (in degrees 4 and 7, respectively), subject to differential relations shown on the bottom of (27). This means ([Sa13, §2.5], see also [FSS16, §2][FSS19a, (59)][FSS20c, Exml. 3.81]) that the background C-field data (7) in the case that $p_1(\nabla) = 0$ (8), is equivalently a flat L_∞-algebra valued differential form [FSS20c, Def. 3.77] with values in the Whitehead L_∞-algebra $\mathfrak{l}S^4$ [FSS20c, Exmpl. 3.68], namely a dg-algebra homomorphism from $\mathrm{CE}\big(\mathfrak{l}S^4\big)$ to the de Rham algebra of X:

$$\begin{tikzcd}
X \arrow[r, "{(G_4, 2G_7)}"] & L_{\mathbb{R}} S^4 \arrow[rr, leftrightarrow, "\substack{\text{non-abelian de Rham theorem} \\ \text{[FSS20c, Thm. 3.87]}}"] & & \Omega^\bullet_{\mathrm{dR}}(X) & \left(\begin{aligned} d\,\omega_4 &= 0 \\ d\,\omega_7 &= -\omega_4 \wedge \omega_4 \end{aligned}\right) \arrow[l, "\substack{G_4 \mapsfrom \omega_4 \\ 2G_7 \mapsfrom \omega_7}"']
\end{tikzcd} \tag{30}$$

cocycle in rational 4-Cohomotopy **dg-algebra homomorphism**

More generally [FSS19b, Prop. 3.20], consider $X = X^8$ a spin 8-manifold for M-theory compactified on 8-manifolds [FSS19b, Rem. 3.1], hence such that:

(i) the tangent bundle of X^8 is equipped with tangential $\mathrm{Sp}(2) \to \mathrm{Spin}(8)$-structure τ (reflecting M2-brane background, see [FSS19b, p. 8 & §2.3]) with compatible connection ∇ (see [FSS20c, Def. 5.25]):

$$\begin{tikzcd}
X^8 \arrow[rr, "\substack{\text{tangential} \\ \mathrm{Sp}(2)\text{-structure} \\ \tau}"] \arrow[dr, "\substack{TX^8 \\ \text{classifying map} \\ \text{of spin-structure}}"'] & & B\mathrm{Sp}(2) \arrow[dl] \arrow[from=1-3, to=2-1, Rightarrow, shorten <=4em, shorten >=4em] \\
& B\mathrm{Spin}(8) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}\big(X^8\big) \arrow[rr, "\substack{\chi_8(\nabla) \mapsfrom \chi_8 \\ \frac{1}{2}p_1(\nabla) \mapsfrom \frac{1}{2}p_1}"] & & \left(\begin{aligned} d\,\chi_8 &= 0 \\ d\,\tfrac{1}{2}p_1 &= 0 \end{aligned}\right) \\
& \left(\begin{aligned} d\,p_2 &= 0 \\ d\,p_1 &= 0 \end{aligned}\right) \arrow[ul, "\substack{p_2(\nabla) \mapsfrom p_2 \\ p_1(\nabla) \mapsfrom p_1}"] \arrow[ur, "\substack{p_2 \mapsto 4\left(\left(\frac{1}{4}p_1\right)^2 - \chi_8\right) \\ p_1 \mapsto p_1}"'] &
\end{tikzcd} \tag{31}$$

(ii) the corresponding Euler 8-form $\chi_8(\nabla)$ trivializes (meaning that the singular M2-brane loci themselves are removed from X^8, see [FSS19b, §2.5])

$$\Theta_7 \in \Omega^7_{\mathrm{dR}}(X) \qquad \text{s.t.} \qquad d\Theta_7 = \chi_8(\nabla) := \mathrm{Pf}(R)\,, \tag{32}$$

which means (by Example 3.2) that the $\mathrm{Sp}(2)$-structure on X further lifts to $\widehat{\mathrm{Sp}(2)}$-structure $\widehat{\tau}$:

$$\begin{tikzcd}
X \arrow[rr, "\widehat{\tau}"] \arrow[dr, "\tau"'] & & B\widehat{\mathrm{Sp}(2)} \arrow[dl, "\mathrm{hofib}(\chi_8)"] \arrow[from=1-3, to=2-1, Rightarrow, shorten <=4em, shorten >=4em] \\
& B\mathrm{Sp}(2) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}\big(X^8\big) & & \left(\begin{aligned} d\,\theta_7 &= \chi_8 \\ d\,\chi_8 &= 0 \\ d\,\tfrac{1}{2}p_1 &= 0 \end{aligned}\right) \arrow[ll, "\substack{\Theta_7 \mapsfrom \theta_7 \\ \chi_8(\nabla) \mapsfrom \chi_8 \\ \frac{1}{2}p_1(\nabla) \mapsfrom \frac{1}{2}p_1}" description] \\
& \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[ul, "\substack{\chi_8(\nabla) \mapsfrom \chi_8 \\ \frac{1}{2}p_1(\nabla) \mapsfrom \frac{1}{2}p_1}"] \arrow[ur, hook] &
\end{tikzcd} \tag{33}$$

Then the general background field data (7) (now including the p_1-terms, Remark 2.2) may be identified with a cocycle in rational τ-*twisted Cohomotopy* (see also [FSS20c, Exmpl. 3.96]):

$$\begin{tikzcd}
X \arrow[rr, "{(G_4, 2G_7)}"] \arrow[dr, "\tau"'] & & L_{\mathbb{R}}\big(S^4 /\!\!/ \mathrm{Sp}(2)\big), \arrow[dl] \arrow[from=1-3, to=2-1, Rightarrow, shorten <=4em, shorten >=4em] \\
& L_{\mathbb{R}} B\mathrm{Sp}(2) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \left(\begin{aligned} d\,\omega_4 &= 0 \\ d\,\omega_7 &= -\omega_4 \wedge \omega_4 + \tfrac{1}{4}p_1 \wedge \tfrac{1}{4}p_1 \\ &\quad - \chi_8 \end{aligned}\right) \arrow[ll, "\substack{G_4 \mapsfrom \omega_4 \\ 2G_7 - \Theta_7 \mapsfrom \omega_7}"'] \\
& \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[ul, "\substack{p_1(\nabla) \mapsfrom p_1 \\ p_2(\nabla) \mapsfrom p_2 \\ \chi_8(\nabla) \mapsfrom \chi_8}"] \arrow[ur, hook, "\substack{p_1 \mapsto p_1 \\ p_2 \mapsto p_2 \\ \chi_8 \mapsto \chi_8}"'] &
\end{tikzcd} \tag{34}$$

cocycle in twisted rational 4-Cohomotopy **relative dg-algebra homomorphism**

Our first main Theorem 3.4 says that not only does rational twisted Cohomotopy naturally encode the background C-field, via Remark 3.3, but that it also naturally encodes the gauging (11) of the M5-brane sigma-model fields (as in [FSS19b, Rem. 3.17]) as well as the anomaly functional of the 6d Hopf-WZ term (Def. 2.9) as a homotopy invariant (Lemma 2.10):

Theorem 3.4 (6d Hopf-WZ anomaly functional is lift through $h_{\mathbb{H}}$)**.** *In the situation of Def. 2.1, consider a closed extended worldvolume $\Sigma := \widetilde{\Sigma}^7$. Then, under the identification of the background field with a cocycle c in rational twisted Cohomotopy, via Remark 3.3, we have:*

(i) *The homotopy classes* (14) *of gaugings $\widetilde{H}_3$* (11) *of an extended sigma-model field $\widetilde{f}$ are in bijection to homotopy classes of homotopy lifts $\widehat{c \circ \widetilde{f}}$ through the quaternionic Hopf fibration $h_{\mathbb{H}}$* (27) *of the composite $c \circ \widetilde{f}$ of $\widetilde{f}$ with the classifying map c* (34) *of the background C-field:*

$$\pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big)_{|_{\widetilde{f}}} \;\simeq\; \left\{ \begin{tikzcd} \widetilde{\Sigma}^7 \arrow[rr, dashed, "\widehat{c\circ\widetilde{f}}"] \arrow[dr, "c\circ\widetilde{f}"'] & \arrow[dl, Rightarrow, shorten=10pt] & L_{\mathbb{R}}\big(S^7/\!\!/\mathrm{Sp}(2)\big) \arrow[dl, "L_{\mathbb{R}}\left(h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)\right)"] \\ & L_{\mathbb{R}}\big(S^4/\!\!/\mathrm{Sp}(2)\big) & \end{tikzcd} \right\}_{\Big/ \sim_{\substack{\mathrm{relative} \\ \mathrm{homotopy}}}} \tag{35}$$

(ii) *Under this bijection* (35)*, twice the anomaly functional (Def. 2.9) equals the correction by the Euler-potential Θ_7* (32) *of the integral*

$$2\widetilde{S}\big(\widetilde{f}, \widetilde{H}_3\big) \;=\; \int\limits_{\widetilde{\Sigma}^7} \Big(\big(\widehat{c \circ \widetilde{f}}\,\big)^*(\omega_7) \;+\; f^*\Theta_7 \Big) \tag{36}$$

of the pullback of the angular cochain $\widetilde{\omega}_7$ on the universal 7-spherical fibration (34) *which is fiberwise the unit volume form on S^7 and which trivializes minus the universal Euler form:*

$$\big\langle \widetilde{\omega}_7, S^7 \big\rangle = 1\,, \quad d\widetilde{\omega}_7 = -\chi_8\,. \tag{37}$$

Proof. By [FSS19b, Lemma 3.19] the dgc-algebra model for the situation is as shown on the right in the following diagram, where the generator $\widetilde{\omega}_7$ in the top right satisfies (37) by [FSS19b, Prop. 2.5 (39)]:

$$\begin{tikzcd} \widetilde{\Sigma}^7 \arrow[rr, dashed, "\widehat{c\circ\widetilde{f}}"] \arrow[dd, "\widetilde{f}"'] & & L_{\mathbb{R}}\big(S^7/\!\!/\mathrm{Sp}(2)\big) \arrow[dd, "L_{\mathbb{R}}\left(h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)\right)"] \arrow[ddll, Rightarrow, shorten=20pt, "\widetilde{H}_3"', "\simeq"] \\ & & \\ X \arrow[rr, "c"'] \arrow[dr, "\tau"'] & & L_{\mathbb{R}}\big(S^4/\!\!/\mathrm{Sp}(2)\big) \arrow[dl] \\ & \mathrm{BSp}(2) & \end{tikzcd} \qquad \begin{tikzcd} \Omega^\bullet_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big) & & \big(d\,\widetilde{\omega}_7 \;=\; -\chi_8\big) \arrow[ll, "\left(2\widetilde{S}(\widetilde{f},\widetilde{H}_3) - \int_{\widetilde{\Sigma}^7} f^*\Theta_7\right)\cdot \mathrm{vol}_{\widetilde{\Sigma}^7} \;\leftarrow\!\shortmid\; \widetilde{\omega}_7"'] \\ & \arrow[ur, Rightarrow, "\eta^*", "\simeq"'] & \left(\begin{aligned} d\,h_3 &= \omega_4 - \tfrac{1}{4}p_1 \\ d\,\omega_4 &= 0 \\ d\,\omega_7 &= -dh_3 \wedge \big(\omega_4 + \tfrac{1}{4}p_1\big) \\ & \quad -\chi_8 \end{aligned}\right) \arrow[ull, "\begin{smallmatrix} \widetilde{H}_3 & \leftarrow\!\shortmid & h_3 \\ \widetilde{f}^*\; G_4 & \leftarrow\!\shortmid & \omega_4 \\ \widetilde{f}^*(2G_7 - \Theta_7) & \leftarrow\!\shortmid & \omega_7 \end{smallmatrix}"'] \arrow[u, "\simeq"', "\begin{smallmatrix} 0 & \frac{1}{4}p_1 & \widetilde{\omega}_7 \\ \uparrow & \uparrow & \uparrow \\ h_3 & \omega_4 & \omega_7 \end{smallmatrix}"] \\ \Omega^\bullet_{\mathrm{dR}}(X) \arrow[uu, "\widetilde{f}^*"] & & \left(\begin{aligned} d\,\omega_4 &= 0 \\ d\,\omega_7 &= -\omega_4 \wedge \omega_4 + \big(\tfrac{1}{4}p_1\big)^2 \\ & \quad -\chi_8 \end{aligned}\right) \arrow[ll, "\begin{smallmatrix} G_4 & \leftarrow\!\shortmid & \omega_4 \\ 2G_7 - \Theta_7 & \leftarrow\!\shortmid & \omega_7 \end{smallmatrix}"'] \arrow[u, hook, "\begin{smallmatrix} \omega_4 & \omega_7 \\ \uparrow & \uparrow \\ \omega_4 & \omega_7 \end{smallmatrix}"'] \\ & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[ul, "\begin{smallmatrix} p_1(\nabla) & \leftarrow\!\shortmid & p_1 \\ p_2(\nabla) & \leftarrow\!\shortmid & p_2 \\ \chi_8(\nabla) & \leftarrow\!\shortmid & \chi_8 \end{smallmatrix}"] \arrow[ur, hook, "\begin{smallmatrix} p_1 & \mapsto & p_1 \\ p_2 & \mapsto & p_2 \\ \chi_8 & \mapsto & \chi_8 \end{smallmatrix}"'] & \end{tikzcd} \tag{38}$$

Here the right vertical morphism is the relative minimal Sullivan model (e.g. [FSS20c, Prop. 3.17]) of the parametrized quaternionic Hopf fibration, which is a cofibration out of a cofibrant object (e.g. [FSS20c, Prop. 3.43]). Since, moreover, every dgc-algebra is projectively fibrant (e.g. [FSS20c, Rem. 3.37]), any homotopy as on the left in (38) is represented by a homotopy η^* as shown on the right (e.g. [FSS20c, Prop. A.16]).

(i) The diagonal morphism on the right of (38) manifestly exhibits a choice of gauging $\widetilde{H}_3$ of $\widetilde{f}$. So to prove the first claim it just remains to see that this establishes a bijection on homotopy classes. Observe that a homotopy of homotopy lifts is now of this form:

$$\begin{tikzcd}[ampersand replacement=\&]
\Omega^\bullet_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big) \& \& \\
\& \& \left(\begin{array}{lcl} d\,h_3 & = & \omega_4 - \tfrac{1}{4}p_1 \\ d\,\omega_4 & = & 0 \\ d\,\omega_7 & = & -dh_3 \wedge \big(\omega_4 + \tfrac{1}{4}p_1\big) - \chi_8 \end{array}\right)
\arrow[ull, bend right=20, "{\begin{array}{ccc} (\widetilde{H}_3)_0 & \mapsfrom & h_3 \\ \widetilde{f}^*\;G_4 & \mapsfrom & \omega_4 \\ \widetilde{f}^*2G_7 - \widetilde{f}^*\Theta_7 & \mapsfrom & \omega_7 \end{array}}"', ""{name=U, below}]
\arrow[ull, bend left=20, "{\begin{array}{ccc} (\widetilde{H}_3)_1 & \mapsfrom & h_3 \\ \widetilde{f}^*\;G_4 & \mapsfrom & \omega_4 \\ \widetilde{f}^*2G_7 - \widetilde{f}^*\Theta_7 & \mapsfrom & \omega_7 \end{array}}", ""{name=D, above}]
\arrow[from=D, to=U, Rightarrow, "\eta^*"] \\
\Omega^\bullet_{\mathrm{dR}}(X) \arrow[uu, "\widetilde{f}^*"] \& \& \left(\begin{array}{lcl} d\,\omega_4 & = & 0 \\ d\,\omega_7 & = & -\omega_4 \wedge \omega_4 + \big(\tfrac{1}{4}p_1\big)^2 - \chi_8 \end{array}\right)
\arrow[ll, "{\begin{array}{ccc} G_4 & \mapsfrom & \omega_4 \\ 2G_7 - \Theta_7 & \mapsfrom & \omega_7 \end{array}}"']
\arrow[u, hook, "{\begin{array}{cc} \omega_4 & \omega_7 \\ \mapsup & \mapsup \\ \omega_4 & \omega_7 \end{array}}"']
\end{tikzcd} . \tag{39}$$

Hence, since path space objects of de Rham dgc-algebras over X are given by de Rham dgc-algebras over $X \times [0,1]$ (e.g. [FSS20c, Lem. 3.88]), this is equivalently (e.g. [FSS20c, Prop. A.16]) a dgc-algebra homomorphism making the following diagram commute under $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$ (where s denotes the canonical coordinate function on $[0,1]$):

$$\begin{tikzcd}[ampersand replacement=\&]
\Omega^\bullet_{\mathrm{dR}}\Big(\widetilde{\Sigma}^7\Big) \& \\
\Omega^\bullet_{\mathrm{dR}}\Big(\widetilde{\Sigma}^7 \times [0,1]\Big)
\arrow[u, "{\begin{array}{cc} 0 & 0 \\ \mapsup & \mapsup \\ s & ds \end{array}}"]
\arrow[d, "{\begin{array}{cc} s & ds \\ \mapsdown & \mapsdown \\ 1 & 0 \end{array}}"'] \&
\left(\begin{array}{lcl} d\,h_3 & = & \omega_4 - \tfrac{1}{4}p_1 \\ d\,\omega_4 & = & 0 \\ d\,\omega_7 & = & -\omega_4 \wedge \omega_4 + \big(\tfrac{1}{4}p_1\big)^2 \\ & & -\chi_8 \end{array}\right)
\arrow[l, "\eta^*" description]
\arrow[ul, "{\begin{array}{ccc} (\widetilde{H}_3)_0 & \mapsfrom & h_3 \\ \widetilde{f}^*\;G_4 & \mapsfrom & \omega_4 \\ \widetilde{f}^*2G_7 - \widetilde{f}^*\Theta_7 & \mapsfrom & \omega_7 \end{array}}"']
\arrow[dl, "{\begin{array}{ccc} (\widetilde{H}_3)_1 & \mapsfrom & h_3 \\ \widetilde{f}^*\;G_4 & \mapsfrom & \omega_4 \\ \widetilde{f}^*2G_7 - \widetilde{f}^*\Theta_7 & \mapsfrom & \omega_7 \end{array}}"] \\
\Omega^\bullet_{\mathrm{dR}}\Big(\tilde{\Sigma}^7\Big) \& \\
(\widetilde{H}_3)_{[0,1]} \& h_3 \arrow[l, mapsto]
\end{tikzcd} . \tag{40}$$

But this homotopy diagram (40) manifestly exhibits the same data and conditions as in (13) for a homotopy of gaugings of a sigma-model field $\widetilde{f}$:

$$\big(\widetilde{f}, (H_3)_0\big) \xRightarrow{\big(\mathrm{id}, (\widetilde{H}_3)_{[0,1]}\big)} \big(\widetilde{f}, (H_3)_1\big) \;,$$

and hence homotopy classes are equivalent to gauge equivalence classes, as claimed.

(ii) Consider in the following any 7-form on $\widetilde{\Sigma}^7$ of unit volume:

$$\mathrm{vol}_{\widetilde{\Sigma}^7} \;\in\; \Omega^7_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big) \quad \text{such that} \quad \int\limits_{\widetilde{\Sigma}^7} \mathrm{vol}_{\widetilde{\Sigma}^7} \;=\; 1 \;. \tag{41}$$

Again using the above path space objects, the homotopy η^* on the right in (38) is a dgc-algebra homomorphism that makes the following diagram commute under $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$:

$$\begin{tikzcd}[column sep=huge]
\Omega^\bullet_{\mathrm{dR}}\Big(\widetilde{\Sigma}^7\Big) & \\
\Omega^\bullet_{\mathrm{dR}}\Big(\widetilde{\Sigma}^7\times[0,1]\Big) \arrow[u, "{\begin{array}{cc} 0 & 0 \\ \uparrow & \uparrow \\ s & ds \end{array}}"'] \arrow[d, "{\begin{array}{cc} s & ds \\ \downarrow & \downarrow \\ 1 & 0 \end{array}}"'] & \left(\begin{array}{rcl} d\,h_3 & = & \omega_4-\tfrac{1}{4}p_1 \\ d\,\omega_4 & = & 0 \\ d\,\omega_7 & = & -\omega_4\wedge\omega_4+\big(\tfrac{1}{4}p_1\big)^2 \\ & & -\chi_8 \end{array}\right) \arrow[l, "\eta^*" description] \arrow[ul, "{\begin{array}{cc} \Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7} f^*\Theta_7\Big)\cdot\mathrm{vol}_{\widetilde{\Sigma}^7} & \longmapsfrom\ \omega_7 \\ \tfrac{1}{4}\widetilde{f}^*p_1(\nabla) & \longmapsfrom\ \omega_4 \\ 0 & \longmapsfrom\ h_3 \end{array}}"'] \arrow[dl, "{\begin{array}{cc} \widetilde{H}_3 & \longmapsfrom\ h_3 \\ \widetilde{f}^*\ G_4 & \longmapsfrom\ \omega_4 \\ \widetilde{f}^*2G_7-\widetilde{f}^*\Theta_7 & \longmapsfrom\ \omega_7 \end{array}}"] \\
\Omega^\bullet_{\mathrm{dR}}\Big(S^7\Big) &
\end{tikzcd} \qquad (42)$$

We claim that such an η^* is given by:

$$\begin{array}{lcl}
s\widetilde{H}_3 & \longmapsfrom & h_3 \\
ds\wedge\widetilde{H}_3+s\cdot\widetilde{f}^*\big(G_4\big)+(1-s)\tfrac{1}{4}\widetilde{f}^*\big(p_1(\nabla)\big) & \overset{\eta^*}{\longmapsfrom} & \omega_4 \\
s\cdot\Big(\widetilde{f}^*\big(2G_7\big)-\widetilde{f}^*\big(\Theta_7\big)\Big)+\Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\widetilde{f}^*\big(\Theta_7\big)\Big)\cdot(1-s)\cdot\mathrm{vol}_{\widetilde{\Sigma}^7}+s(1-s)\cdot\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big)+ds\wedge Q_6 & \longmapsfrom & \omega_7
\end{array}$$

where $Q_6\in\Omega^6_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big)$ is any differential form which satisfies

$$dQ_6\;=\;\Big(\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+\widetilde{f}^*\big(2G_7-\Theta_7\big)\Big)-\left(\overbrace{\int\limits_{\widetilde{\Sigma}^7}\Big(\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+\widetilde{f}^*\big(2G_7-\Theta_7\big)\Big)}^{=:2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\tilde{f}^*(\Theta_7)}\right)\cdot\mathrm{vol}_{S^7}\,.$$

This exists by (41) and because cohomology classes of differential forms in top degree on compact connected manifolds are in bijection with the values of their integrals (e.g. [La15, §7.3, Thm. 7.5]).

It is clear that η^* thus defined satisfies the required boundary conditions of a homotopy in (42). Hence it only remains to check that it is indeed a dg-algebra homomorphism, in that it respects the differentials on the generators. This is verified by direct computation:

$$\begin{aligned}
d\eta^*(\omega_7) &= d\left(s\cdot\Big(\widetilde{f}^*\big(2G_7\big)-\widetilde{f}^*\big(\Theta_7\big)\Big)+(1-s)\cdot\Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\tilde{f}^*\Theta_7\Big)\cdot\mathrm{vol}_{S^7}+s(1-s)\cdot\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big)+ds\wedge Q_6\right) \\
&= ds\wedge\Big(\widetilde{f}^*\big(2G_7\big)-\widetilde{f}^*\big(\Theta_7\big)\Big)-ds\wedge\Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\tilde{f}^*\big(\Theta_7\big)\Big)\cdot\mathrm{vol}_{S^7}+ds\wedge\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big) \\
&\quad -2s\cdot ds\wedge\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big)-ds\wedge dQ_6 \\
&= ds\wedge\left(\big(\widetilde{f}^*(2G_7)-\widetilde{f}^*(\Theta_7)\big)+\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big)-\Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\tilde{f}^*\big(\Theta_7\big)\Big)\cdot\mathrm{vol}_{S^7}-dQ_6\right) \\
&\quad -2s\cdot ds\wedge\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4-\tfrac{1}{4}p_1(\nabla)\big) \\
&= ds\wedge\underbrace{\left(\widetilde{f}^*\big(2G_7-\Theta_7\big)+\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)-\Big(2\widetilde{S}-\int_{\widetilde{\Sigma}^7}\tilde{f}^*\Theta_7\Big)\cdot\mathrm{vol}_{S^7}-dQ_6\right)}_{=0} \\
&\quad -2\cdot ds\wedge\widetilde{H}_3\wedge\widetilde{f}^*\big(s\cdot G_4+(1-s)\tfrac{1}{4}p_1(\nabla)\big) \\
&= -\eta^*\big(\omega_4\big)\wedge\eta^*\big(\omega_4\big) \\
&= \eta^*\big(d\omega_7\big).
\end{aligned}$$

Here the crucial non-trivial step is the fourth (third line from below). In the last two steps we used that all 8-forms on $\widetilde{\Sigma}^7$ vanish, so that only the 8-forms on $\widetilde{\Sigma}^7\times[0,1]$ with one factor of ds survive.

The verification on the other two generators is immediate:

$$
\begin{aligned}
d\eta^*(h_3) &= d\big(s\widetilde{H}_3\big) \\
&= ds \wedge \widetilde{H}_3 + s \cdot \Big(\widetilde{f}^*\big(G_4\big) - \tfrac{1}{4}\widetilde{f}^*\big(p_1(\nabla)\big)\Big) \\
&= \eta^*(\omega_4) - \tfrac{1}{4}\widetilde{f}^*\big(p_1(\nabla)\big) \\
&= \eta^*\big(dh_3\big)\,,
\end{aligned}
\qquad
\begin{aligned}
d\eta^*(\omega_4) &= -ds \wedge d\widetilde{H}_3 \\
&\quad + d\Big(s \cdot \widetilde{f}^*\big(G_4\big) + (1-s)\tfrac{1}{4}\widetilde{f}^*\big(p_1(\nabla)\big)\Big) \\
&= 0 \\
&= \eta^*\big(d\omega_4\big)\,.
\end{aligned}
$$

□

4 Hypothesis H implies M5 Hopf-WZ anomaly cancellation

In view of the rational cohomotopical interpretation of background C-field (Remark 3.3) and of the 6d Hopf-WZ anomaly functional (Theorem 3.4) it is natural to hypothesize that the topological sector of the background C-field should be required to be a cocycle in actual twisted Cohomotopy. This non-abelian charge-quantization condition ([FSS20c, §5.3]) is called *Hypothesis H* in [FSS19b]; we recall the precise statement as Def. 4.1 below.

We observe in Prop. 4.6 that, under *Hypothesis H* and in the absence of topological twisting, Theorem 3.4 exhibits the M5 Hopf-WZ anomaly functional as the homotopy Whitehead integral formula (see Remark 4.7 below) for the Hopf invariant (recalled in Def. 4.4 below). This proves the anomaly cancellation (2) for the special case of oriented differences of extended worldvolumes being the 7-sphere and for vanishing topological twist $\frac{1}{4}p_1$ (Remark 2.2). Finally we establish a twisted/parametrized generalization of the integral Hopf invariant in Theorem 4.8, which proves the anomaly cancellation condition (2) generally.

Definition 4.1 (*Hypothesis H* [FSS19b])**.** In the situation of Def. 2.1 we say that:

(i) the background fields $(G_4, 2G_7)$ (7) *satisfy Hypothesis H* if they are classified as in [FSS19b, Def. 3.5] by an actual cocycle c in twisted Cohomotopy [FSS19b, Section 2.1], hence if their classifying map in rational twisted Cohomotopy from Remark 3.3 factors, up to homotopy, through the homotopy quotient $S^4 /\!/ \mathrm{Sp}(2)$ of the 4-sphere canonically acted on by $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$ (see [FSS20b, Prop. 2.1]), followed by the rationalization map (29);

(ii) the (extended or not) higher gauged sigma-model fields $(\widetilde{f}, \widetilde{H}_3)$ (11) *satisfy Hypothesis H* if the corresponding lift (38) through the rationalized parametrized quaternionic Hopf fibration, which classifies them by Theorem 3.4, factors as a lift through the actual parametrized quaternionic Hopf fibration $h_{\mathbb{H}} /\!/ \mathrm{Sp}(2)$ ([FSS19b, Prop. 2.22]):

$$
\begin{tikzcd}
\widetilde{\Sigma}^7 \arrow[rr, dashed, "\widehat{c\circ\widetilde{f}}"] \arrow[dd, "\widetilde{f}"'] \arrow[dd, Rightarrow, shift left=6, "\widetilde{H}_3"] \arrow[rrrr, bend left=15, "\begin{array}{c}\text{rational Hopf-WZ term}\\ 2\widetilde{S}(\widetilde{H}_3, G_4\, 2G_7)\end{array}"] && S^7 /\!/ \mathrm{Sp}(2) \arrow[rr, "\text{rationalization}"] \arrow[dd, "\begin{array}{c}\mathrm{Sp}(2)\text{-parametrized}\\ \text{quaternionic Hopf fibration}\\ h_{\mathbb{H}} /\!/ \mathrm{Sp}(2)\end{array}"] && L_{\mathbb{R}}\big(S^7 /\!/ \mathrm{Sp}(2)\big) \arrow[dd, "L_{\mathbb{R}}(h_{\mathbb{H}} /\!/ \mathrm{Sp}(2))"] \\
& \begin{array}{c}\text{lift to actual}\\ \text{twisted Cohomotopy}\end{array} && \begin{array}{c}\text{rational}\\ \text{twisted Cohomotopy}\end{array} & \\
X \arrow[rr, dashed, "c"] \arrow[dr, "\tau"'] \arrow[rrrr, bend right=15, "\begin{array}{c}(G_4{,}2G_7)\\ \text{cocycle in rational}\\ \text{twisted Cohomotopy}\end{array}"'] && S^4 /\!/ \mathrm{Sp}(2) \arrow[dl] \arrow[rr, "\text{rationalization}"] && L_{\mathbb{R}}\big(S^4 /\!/ \mathrm{Sp}(2)\big) \\
& B\mathrm{Sp}(2) &&&
\end{tikzcd}
\tag{43}
$$

Hopf-WZ term in terms of Fivebrane-extended $\mathrm{Sp}(2)$**-structure** For transparent formulation of the proof of the following integrality theorem (Theorem 4.8 below), it is useful to re-cast the result of Theorem 3.4 in terms of the Fivebrane-extended $\widehat{\mathrm{Sp}(2)}$-structure from Example 3.2:

Definition 4.2 (Quaternionic Hopf fibration parametrized over Fivebrane-extended $\mathrm{Sp}(2)$)**.** Consider the homotopy pullback (e.g. [FSS20c, Def. A.23]) of the $\mathrm{Sp}(2)$-parametrized quaternionic Hopf fibration (43) along the Fivebrane-extension $\widehat{\mathrm{Sp}(2)} \to \mathrm{Sp}(2)$ (Example 3.2). By the pasting law and the homotopy-restriction map on ∞-actions (see [SS20b, Prop. 2.23, 2.85]), we may denote this as follows:

$$\begin{tikzcd}
& S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d, "h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}"'] \arrow[dr, phantom, "\text{(pb)}"] & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] & \\
\begin{array}{c}\widehat{\mathrm{Sp}(2)}\text{-parametrized}\\ \text{quaternionic Hopf fibration}\end{array} & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d, "\widehat{\rho}_{S^4}"'] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[d, "\rho_{S^4}"] & \begin{array}{c}\mathrm{Sp}(2)\text{-parametrized}\\ \text{quaternionic Hopf fibration}\end{array} \\
\begin{array}{c}\text{classifying space for}\\ \text{Fivebrane-extended } \mathrm{Sp}(2)\text{-structure}\end{array} & B\widehat{\mathrm{Sp}(2)} \arrow[r, "\mathrm{hofib}(\chi_8)"'] & B\mathrm{Sp}(2) & \begin{array}{c}\text{classifying space for } \mathrm{Sp}(2)\text{-structure}\\ \text{(BPS M2-brane backgrounds)}\end{array}
\end{tikzcd} \tag{44}$$

Notice, by Example 3.2, tat the minimal relative Sullivan model for the $\widehat{\mathrm{Sp}(2)}$-parametrized quaternionic Hopf fibration on the left of (44) is just like that of the $\mathrm{Sp}(2)$-parametrized Hopf fibration on the right of (44), as given in (38), except that all entries have the generator θ_7 adjoined, with $d\theta_7 = \chi_8$ (universal rational Fivebrane-structure).

Remark 4.3 (The Hopf-WZ term in terms of Fivebrane $\widehat{\mathrm{Sp}(2)}$-structure)**.** In terms of the $\widehat{\mathrm{Sp}(2)}$-parametrized quaternionic Hopf fibration from Def. 4.2, the content of Theorem 3.4 becomes the following transparent statement:

By the assumption (32) that the Euler 8-class of X^8 is equipped with a trivialization, hence that we have $\widehat{\mathrm{Sp}(2)}$-structure $\widehat{\tau}$ (33) on X^8, we may pull back the situation in (38) along $B\widehat{\mathrm{Sp}(2)} \xrightarrow{\mathrm{hofib}(\chi_8)} B\mathrm{Sp}(2)$ and regard the twisted Cohomotopy classes c and $\widehat{c \circ \widetilde{f}}$ as having local coefficients in the $\widehat{\mathrm{Sp}(2)}$-parametrized quaternionic Hopf fibration from Def. 4.2.

In this formulation, Theorem 3.4, says equivalently that there is a single rational 7-class on the total space of the universal $\widehat{\mathrm{Sp}(2)}$-parametrized 7-sphere fibration, represented by the sum of the generators (37) and (28):

$$2\widetilde{\mathbf{S}} \;:=\; \overset{\substack{\text{universal}\\ \text{Hopf-WZ term}}}{\widetilde{\omega}_7} \;+\; \overset{\substack{\text{fiberwise}\\ \text{volume form on } S^7}}{} \overset{\substack{\text{universal}\\ \text{Fivebrane structure}}}{\theta_7} \;\; \in \mathrm{CE}\Big(\mathfrak{l}\big(S^7 /\!\!/ \widehat{\mathrm{Sp}(2)}\big)\Big), \qquad \big[\widetilde{\mathbf{S}}\big] \in H^7\Big(S^7 /\!\!/ \widehat{\mathrm{Sp}(2)};\, \mathbb{R}\Big) \tag{45}$$

which is the universal avatar of the Hopf-WZ term (Def. 2.6, under $H^7_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big) \simeq \mathbb{R}$):

$$\begin{tikzcd}
\widetilde{\Sigma}^7 \arrow[r, "\widehat{c \circ \widetilde{f}}"] \arrow[d, "\widetilde{f}"'] & L_{\mathbb{R}}\big(S^7 /\!\!/ \widehat{\mathrm{Sp}(2)}\big) \,, \arrow[d, "L_{\mathbb{R}}\left(h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}\right)"] \arrow[dl, Rightarrow, "\substack{\text{cocycle in rational}\\ \text{twisted 4-Cohomotopy}}"'] \\
X^8 \arrow[r, "c"] \arrow[d, "\substack{\widehat{\tau}\\ \text{Fivebrane}\\ \text{structure}}"'] & L_{\mathbb{R}}\big(S^4 /\!\!/ \widehat{\mathrm{Sp}(2)}\big) \arrow[dl] \arrow[dl, Rightarrow, shift left=4] \\
B\widehat{\mathrm{Sp}(2)} &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}\big(\widetilde{\Sigma}^7\big) & & \left(\begin{array}{l} d\,\theta_7 = \;\;\chi_8 \\ d\,\widetilde{\omega}_7 = -\chi_8 \end{array}\right) \arrow[ll, "\substack{\text{Hopf-WZ term} \qquad\qquad \text{universal}\\ \qquad\qquad\qquad\qquad \text{Hopf-WZ term}\\ 2\widetilde{S}(\widetilde{H}_3, G_4, G_7) \;\mapsfrom\; \left[2\widetilde{\mathbf{S}} := \widetilde{\omega}_7 + \theta_7\right]}"'] \\
\Omega^\bullet_{\mathrm{dR}}\big(X^8\big) \arrow[u, "\widetilde{f}^*"] \arrow[urr, Rightarrow, "\substack{\widetilde{H}_3\\ \text{fluxes}}"'] & & \left(\begin{array}{l} d\,\theta_7 = \chi_8 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + \left(\tfrac{1}{4}p_1\right)^2 \\ \qquad\quad -\chi_8 \\ d\,\omega_4 = 0 \end{array}\right) \arrow[ll, "\substack{2G_7 - \Theta_7 \;\mapsfrom\; \omega_7\\ G_4 \;\mapsfrom\; \omega_4}"'] \arrow[u, two heads] \\
& \left(\begin{array}{l} d\,\theta_7 = \chi_8 \\ d\,\chi_8 = 0 \\ d\,\tfrac{1}{2}p_1 = 0 \end{array}\right) \arrow[ul] \arrow[ur, hook] &
\end{tikzcd} \tag{46}$$

The plain Hopf invariant via homotopy Whitehead integral.

Before analyzing the implications of *Hypothesis H* in the full twisted case, we recall the definition of the plain Hopf invariant (e.g. [MT86, p. 33]) and discuss how this follows from the untwisted Hopf-WZ term:

Definition 4.4 (Hopf invariant)**.** For $k \in \mathbb{N}$ with $k \geq 1$, let

$$S^{4k-1} \xrightarrow{\phi} S^{2k} \tag{47}$$

be a continuous function between higher dimensional spheres, as shown. Then the homotopy cofiber space of ϕ has integral cohomology given by

$$H^p\big(\mathrm{cofib}(\phi), \mathbb{Z}\big) \;\simeq\; \left\{ \begin{array}{ccc} \mathbb{Z} & | & p \in \{2k, 4k\} \\ 0 & | & \text{otherwise} \end{array} \right.$$

Hence, with generators denoted

$$\omega_{2k} \;:=\; \pm 1 \in \mathbb{Z} \;\simeq\; H^{2k}\big(\mathrm{cofib}(\phi), \mathbb{Z}\big)\,, \quad \omega_{4k} \;:=\; \pm 1 \in \mathbb{Z} \;\simeq\; H^{4k}\big(\mathrm{cofib}(\phi), \mathbb{Z}\big),$$

there exists a unique integer

$$\mathrm{HI}(\phi) \;\in\; \mathbb{Z}\,, \qquad \text{s.t.} \qquad \omega_{2k} \cup \omega_{2k} \;=\; \mathrm{HI}(\phi) \cdot \omega_{4k} \tag{48}$$

relating the cup-product square of the first to a multiple of the second. This integer is called the *Hopf invariant* $\mathrm{HI}(\phi)$ of ϕ. It depends on the choice of generators only up to a sign.

We make the following basic observation:

Lemma 4.5 (Recognition of Hopf invariants from Sullivan models)**.** *The unique coefficient in the minimal Sullivan model for a map* ϕ *of spheres as in* (47) *is the Hopf invariant* $\mathrm{HI}(\phi)$ *(Def. 4.4):*

$$\begin{tikzcd}
S^{4k-1} \arrow[rr, "\phi"] & & S^{2k} \\
\big(d\,\omega_{4k-1} = 0\big) & & \left(\begin{array}{lcl} d\,\omega_{4k-1} & = & -\omega_{2k} \wedge \omega_{2k} \\ d\,\omega_{4k} & = & 0 \end{array} \right) \arrow[ll, "\begin{smallmatrix} \mathrm{HI}(\phi) \cdot \omega_{4k-1} \longmapsfrom \omega_{4k-1} \\ 0 \longmapsfrom \omega_{2n} \end{smallmatrix}" description]
\end{tikzcd} . \tag{49}$$

Proof. The homotopy cofiber is represented by the ordinary pushout of topological spaces as shown in the following diagram on the left. This is algebraically represented by the pullback of dgc-algebras as shown on the right:

$$\begin{tikzcd}
& \mathrm{cofib}(\phi) & \\
S^{2k} \arrow[ur] & \text{(po)} & D^{4k} \arrow[ul] \\
& S^{4k-1} \arrow[ul, "\phi"] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \left(\begin{array}{ll} d\,\omega_{2k} & = 0 \\ d\,\omega_{4k-1} & = -\omega_{2k} \wedge \omega_{2k} \\ & \quad + h \cdot \omega_{4k} \\ d\,\omega_{4k} & = 0 \end{array} \right) \arrow[dl, "\begin{smallmatrix} \omega_{2k} & \longmapsfrom & \omega_{2k} \\ \omega_{4k-1} & \longmapsfrom & \omega_{4k-1} \\ 0 & \longmapsfrom & \omega_{4k} \end{smallmatrix}"'] \arrow[dr, "\begin{smallmatrix} \omega_{2k} & \mapsto & 0 \\ \omega_{4k-1} & \mapsto & \omega_{4k-1} \\ \omega_{4k} & \mapsto & \omega_{4k} \end{smallmatrix}"] & \\
\left(\begin{array}{lcl} d\,\omega_{2k} & = & 0 \\ d\,\omega_{4k-1} & = & -\omega_{2k} \wedge \omega_{2k} \end{array} \right) \arrow[dr, "\begin{smallmatrix} \omega_{2k} & \mapsto & 0 \\ \omega_{4k-1} & \mapsto & h \cdot \omega_{4k-1} \end{smallmatrix}"'] & \text{(pb)} & \left(\begin{array}{ll} d\,\omega_{4k-1} & = \omega_{4k} \\ d\,\omega_{4k} & = 0 \end{array} \right) \arrow[dl, "\begin{smallmatrix} \omega_{4k-1} & \longmapsfrom & \omega_{4k-1} \\ 0 & \longmapsfrom & \omega_{4k} \end{smallmatrix}"] \\
& (d\,\omega_{4k-1} = 0) &
\end{tikzcd}$$

One reads off the pullback dgc-algebra at the top by inspection, with the coefficient h as shown, inherited from the Sullivan model for ϕ in the bottom left. By the fact that Sullivan models compute the non-torsion cohomology groups, comparison with (48) shows that $h = \mathrm{HI}(\phi)$. ☐

Using this, we obtain the following corollary of Theorem 3.4:

Proposition 4.6 (Recovering the homotopy Whitehead formula)**.** *In the situation of Def. 2.1, consider the special case when:*

(i) *the background C-field* (7) *satisfies Hypothesis H (Def. 4.1);*

(ii) *the extended worldvolume is the 7-sphere* $\Sigma := \widetilde{\Sigma}^7 := S^7$ *(as in Lemma 2.3 and Example 22);*

(iii) *the* $\mathrm{Spin}(8)$*-bundle over* X *is trivial, as well as the* $\mathrm{Spin}(8)$*-connection* ∇*, and the trivial trivialization* $\Theta_7 = 0$ *of* $\chi_8(\nabla)$ (32) *is chosen.*

Then twice the Hopf-WZ anomaly functional $2\widetilde{S}$ *(Def. 2.9, Lemma 2.10) is equal to the Hopf invariant* $\mathrm{HI}\big(c \circ \widetilde{f}\,\big)$ *(Def. 4.4) of the composite*

$$S^7 \xrightarrow{\;\widetilde{f}\;} X \xrightarrow{\;c\;} S^4$$

of the extended sigma-model field $\widetilde{f}$ (11) *with the (untwisted) Cohomotopy cocycle* c (43) *that classifies the background fields:*

$$2\widetilde{S}\big(\widetilde{f},\widetilde{H}_3\big) \;=\; \int_{S^7} \Big(\widetilde{H}_3 \wedge \widetilde{f}^* G_4 + \widetilde{f}^* 2G_4\Big) \;=\; \mathrm{HI}\big(c \circ \widetilde{f}\,\big) \;\in\; \mathbb{Z}\,. \tag{50}$$

Proof. Under the given assumption, the diagram (38) in Theorem 3.4 reduces to

$$\begin{tikzcd}
\widetilde{\Sigma}^7 = S^7 \arrow[r, "\mathrm{HI}(c\circ\widetilde{f}\,)"] \arrow[d, "\widetilde{f}"'] & S^7 \arrow[d, "h_{\mathbb{H}}"] \arrow[dl, Rightarrow, "\widetilde{H}_3"', "\simeq"] \\
X \arrow[r, "c"'] & S^4
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(S^7) & & \left(d\,\widetilde{\omega}_7 \;=\; 0\right) \arrow[ll, "2\widetilde{S}\cdot \mathrm{vol}_{S^7} \;\mapsfrom\; \widetilde{\omega}_7"'] \\
& \arrow[ur, Rightarrow, "\eta^*", "\simeq"'] & \left(\begin{array}{lcl} d\,h_3 &=& \omega_4 \\ d\,\omega_4 &=& 0 \\ d\,\omega_7 &=& -\omega_4\wedge\omega_4 \end{array}\right) \arrow[ull, "\begin{array}{r} \widetilde{H}_3 \mapsfrom h_3 \\ \widetilde{f}^* G_4 \mapsfrom \omega_4 \\ \widetilde{f}^* 2G_7 \mapsfrom \omega_7 \end{array}"] \arrow[u, "\simeq", "\begin{array}{ccc} 0 & 0 & \widetilde{\omega}_7 \\ \mapsfrom & \mapsfrom & \mapsfrom \\ h_3 & \omega_4 & \omega_7 \end{array}"'] \\
\Omega^\bullet_{\mathrm{dR}}(X) \arrow[uu, "\widetilde{f}^*"] & & \left(\begin{array}{lcl} d\,\omega_4 &=& 0 \\ d\,\omega_7 &=& -\omega_4\wedge\omega_4 \end{array}\right) \arrow[ll, "\begin{array}{c} G_4 \mapsfrom \omega_4 \\ G_7 \mapsfrom \omega_7 \end{array}"'] \arrow[u, hook, "\begin{array}{cc} \omega_4 & \omega_7 \\ \mapsfrom & \mapsfrom \\ \omega_4 & \omega_7 \end{array}"']
\end{tikzcd}
\tag{51}$$

This identifies the top horizontal map with the Hopf invariant, as shown, by Lemma 4.5. ☐

Remark 4.7 (Whitehead integral formulas in the literature)**.** The statement of Prop. 4.6 is essentially that of [Ha78, p. 17] [GM81, §14.5], the integrand being the *functional cup product*-expression of [St49], recalled as a *homotopy period*-expression in [SW08, Exmple 1.9]; but the proof as a special case of Theorem 3.4 is new and more conceptual. In the special case that $G_7 = 0$ (which is given if the classifying map $X \xrightarrow{c} S^4$ (43) is a smooth function) the statement of Prop. 4.6 further reduces to that of the classical *Whitehead integral formula* for the Hopf invariant [Wh47] (see [Ha78][BT82, Prop. 17.22]).

Our second Theorem 4.8 generalizes this integral situation to arbitrary oriented differences $\widetilde{\Sigma}^7$ of extended worldvolumes and to non-trivial topological twists:

Theorem 4.8 (6d Hopf-WZ anomaly functional is integral)**.** *Let target space $X = X^8$ be an M2-brane background for M-theory on 8-manifolds (Example 3.3), hence a smooth spin 8-manifold equipped with tangential* $\mathrm{Sp}(2)$*-structure* (31) *and with vanishing Euler class* (33)*.*

Then Hypothesis H (Def. 4.1) implies that the general 6d Hopf-Wess-Zumino anomaly functional of the M5-brane (Def. 2.9, Lemma 2.10) takes values in the integers

$$\begin{tikzcd}
\pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X\big)\Big) \arrow[r] \arrow[rr, bend left, "2\widetilde{S}"] & \mathbb{Z} \arrow[r, hook] & \mathbb{R} \\
\big[\widetilde{f}, \widetilde{H}_3\big] \arrow[r, mapsto] & \int\limits_{\widehat{\Sigma}^7} \Big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\Big) &
\end{tikzcd} \tag{52}$$

and hence that the exponentiated action (2) *of the 6d WZ term of the M5-brane (Def. 2.6) is well-defined.*

Proof. Recall that,

- **(a)** by Theorem 3.4, the anomaly term is characterized as a homotopy lift through the rationalization $L_{\mathbb{R}}\big(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big)$ of the parametrized quaternionic Hopf fibration,
- **(b)** by *Hypothesis H* this comes from a lift through the actual $\mathrm{Sp}(2)$-parametrized Hopf fibration $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ (43),
- **c)** by (33) this, in turn, comes from a lift through the $\widehat{\mathrm{Sp}(2)}$-parametrized Hopf fibration $h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}$ (Def. 4.2).

Therefore it is now sufficient to show that the rational class (46) of the universal Hopf-WZ term on the total space of the universal $\widehat{\mathrm{Sp}(2)}$-parametrized 7-sphere fibration (44) is the rational image of an integral cohomology class.

First, consider the Gysin sequence (see e.g. [Sw75, 15.30]) for the universal 4-spherical fibration in the bottom right of (44)

$$S^4 \xrightarrow{\mathrm{hofib}(\rho_{S^4})} S^4 /\!\!/ \mathrm{Sp}(2) \xrightarrow{\rho_{S^4}} B\mathrm{Sp}(2)\ .$$

Observe that the integral cohomology groups of the classifying space (see e.g. [Pi91][Ka06, (12)])

$$H^\bullet\big(B\mathrm{Sp}(2), \mathbb{Z}\big) \;\simeq\; \mathbb{Z}\big[\tfrac{1}{2}p_1, \chi_8\big] \tag{53}$$

are non-torsion groups concentrated in even degrees. Hence the 5-class controlling this Gysin sequence vanishes, and so the long exact sequence breaks up into short exact sequences of this form:

$$0 \longrightarrow H^\bullet\big(B\mathrm{Sp}(2);\mathbb{Z}\big) \xrightarrow{\rho^*_{S^4}} H^\bullet\big(S^4 /\!\!/ \mathrm{Sp}(2);,\mathbb{Z}\big) \xrightarrow{\int_{S^4}} H^{\bullet-4}\big(B\mathrm{Sp}(2);\mathbb{Z}\big) \longrightarrow 0\ . \tag{54}$$

Moreover, since the integral cohomology groups (53) have no torsion, these short exact sequences imply that also

$$H^\bullet\big(S^4 /\!\!/ \mathrm{Sp}(2);\mathbb{Z}\big) \qquad \text{are non-torsion groups.} \tag{55}$$

Now observe, by [FSS19b, Prop. 3.13], that

$$\widetilde{\Gamma}_4 \;:=\; \omega_4 + \tfrac{1}{4}p_1 \;\in\; H^4\big(S^4 /\!\!/ B\mathrm{Sp}(2);\mathbb{Z}\big) \longrightarrow H^4\big(S^4 /\!\!/ B\mathrm{Sp}(2);\mathbb{R}\big) \tag{56}$$

is an integral class, being the universal integral shifted C-field flux density (9). Hence, by (55), the rational trivialization from (38)

$$\begin{aligned} d\omega_7 &= -\omega_4 \wedge \omega_4 + \big(\tfrac{1}{4}p_1\big)^2 - \chi_8 \\ &= -\big(\widetilde{\Gamma}_4 \wedge \widetilde{\Gamma}_4 - \tfrac{1}{2}p_1 \wedge \widetilde{\Gamma}_4\big) - \chi_8 \end{aligned}$$

implies that also the following integral cohomology class vanishes:

$$\Big[\widetilde{\Gamma}_4 \cup \widetilde{\Gamma}_4 - \tfrac{1}{2}p_1 \cup \widetilde{\Gamma}_4 + \chi_8\Big] \;=\; 0 \;\in\; H^8\big(S^4 /\!\!/ \mathrm{Sp}(2);\mathbb{Z}\big)\,. \tag{57}$$

Consider next the integral Gysin sequence corresponding to the 3-spherical fibration which is the parametrized quaternionic Hopf fibration in the top right of (44):

$$S^3 \xrightarrow{\mathrm{hofib}(h_{\mathbb{H}} /\!\!/ G)} S^7 /\!\!/ \mathrm{Sp}(2) \xrightarrow{h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)} S^4 /\!\!/ \mathrm{Sp}(2)\ . \tag{58}$$

Since, rationally, $S^7 \sslash \mathrm{Sp}(2)$ is obtained from $S^4 \sslash \mathrm{Sp}(2)$ by adjoining the relation

$$\begin{aligned} dh_3 &= \omega_4 - \tfrac{1}{4}p_1 \\ &= \widetilde{\Gamma}_4 - \tfrac{1}{2}p_1\,, \end{aligned} \tag{59}$$

by (38), it follows from [FSS19b, Prop. 2.5 (44)] that $\omega_4 - \frac{1}{4}p_1$ is the rational image of the integral Euler class of (58). Consequently,

$$\widetilde{\Gamma}_4 - \tfrac{1}{2}p_1 \;\in\; H^4\big(S^4 \sslash \mathrm{Sp}(2);\, \mathbb{Z}\big) \tag{60}$$

is the integral 4-class controlling the Gysin sequence of (58), which therefore reads:

$$\begin{tikzcd} \cdots \arrow[r] & H^7\big(S^4 \sslash \mathrm{Sp}(2);\mathbb{Z}\big) \arrow[r, "(h_{\mathbb{H}} \sslash \mathrm{Sp}(2))^*"] & H^7\big(S^7 \sslash \mathrm{Sp}(2);\mathbb{Z}\big) \arrow[r, "\int_{S^3}"] & H^4\big(S^4 \sslash \mathrm{Sp}(2);\mathbb{Z}\big) \arrow[r, "\cup(\widetilde{\Gamma}_4 - \frac{1}{2}p_1)"] & H^8\big(S^4 \sslash \mathrm{Sp}(2);\mathbb{Z}\big) \arrow[r] & \cdots \end{tikzcd} \tag{61}$$

Now consider pulling back this situation along the homotopy fiber of χ_8 (28) to yield the sequence of spherical fibrations on the left of (44). After this pullback, the Euler class summand in (57) disappears, and we obtain this vanishing class:

$$\Big[\widetilde{\Gamma}_4 \cup \widetilde{\Gamma}_4 - \tfrac{1}{2}p_1(TX) \cup \widetilde{\Gamma}_4\Big] \;=\; 0 \;\in\; H^8\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};\, \mathbb{Z}\big)\,. \tag{62}$$

With this, the integral Gysin sequence of the 3-spherical fibration in the top left of (44)

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{hofib}(h_{\mathbb{H}} \sslash G)"] & S^7 \sslash \widehat{\mathrm{Sp}(2)} \arrow[r, "h_{\mathbb{H}} \sslash \widehat{\mathrm{Sp}(2)}"] & S^4 \sslash \widehat{\mathrm{Sp}(2)} \end{tikzcd}$$

is seen to be of the following form:

$$\begin{tikzcd} \cdots \arrow[r] & H^7\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{Z}\big) \arrow[r, "(h_{\mathbb{H}} \sslash \widehat{\mathrm{Sp}(2)})^*"] & H^7\big(S^7 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{Z}\big) \arrow[r, "\int_{S^3}"] & H^4\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};,\mathbb{Z}\big) \arrow[r, "\cup\left(\widetilde{\Gamma}_4 - \frac{1}{2}p_1(TX)\right)"] & H^8\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{Z}\big) \arrow[r] & \cdots \\ & & 2\widetilde{\mathbf{S}} \arrow[r, mapsto] & \widetilde{\Gamma}_4 \arrow[r, mapsto] & \underbrace{\widetilde{\Gamma}_4 \cup \widetilde{\Gamma}_4 - \tfrac{1}{2}p_1(TX) \cup \widetilde{\Gamma}_4}_{=0} & \end{tikzcd} \tag{63}$$

Here, in the bottom row, we have observed that the image of $\widetilde{\Gamma}_4$ (56) under forming cup product with the 4-class (60) is just the vanishing class (62), which by exactness of the Gysin sequence implies that there exists an integral 7-class

$$2\widetilde{\mathbf{S}} \;\in\; H^7\big(S^7 \sslash \widehat{\mathrm{Sp}(2)};\, \mathbb{Z}\big)\,, \tag{64}$$

whose integration over the S^3-fibers is $\widetilde{\Gamma}_4$, as shown. Since, by (59) and [FSS19b, Prop. 2.5 (45)], the fiberwise volume form is h_3, this is, rationally, the same fiber integration as that of (46), which by the exactness of the Gysin sequence (63), now with rational coefficients

$$\begin{tikzcd} \cdots \arrow[r] & H^7\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{R}\big) \arrow[r, "(h_{\mathbb{H}} \sslash \widehat{\mathrm{Sp}(2)})^*"] & H^7\big(S^7 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{R}\big) \arrow[r, "\int_{S^3}"] & H^4\big(S^4 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{R}\big) \arrow[r] & \cdots \\ & D \arrow[r, mapsto] & \left(\begin{smallmatrix} h_3 \wedge (\omega_4 + \frac{1}{4}p_1) + \omega_7 + \theta_7 \\ -2\widetilde{\mathbf{S}} \end{smallmatrix}\right) \arrow[r, mapsto] & 0 & \end{tikzcd} \tag{65}$$

implies that the integral class $2\widetilde{\mathbf{S}}$ (64) differs from the rational class (46) by a 7-class D pulled back from $S^4 \sslash \widehat{\mathrm{Sp}(2)}$, as shown. But by (53) and (38) there is no non-trivial 7-class on $S^4 \sslash \widehat{\mathrm{Sp}(2)}$. Hence the equality

$$2\widetilde{\mathbf{S}} \;=\; h_3 \wedge \widetilde{\Gamma}_4 + (\omega_7 + \theta_7)$$

holds, and so the anomaly integrand (46) is indeed the rational image of an integral class (64) and hence has itself integral periods:

$$\begin{tikzcd} 2\widetilde{\mathbf{S}} \arrow[dd, mapsto] & H^7\big(S^7 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{Z}\big) \arrow[r, "(\widehat{c \circ \widetilde{f}})^*"] \arrow[d] & H^7\big(\widetilde{\Sigma}^7;\mathbb{Z}\big) \arrow[r, "\int_{\widetilde{\Sigma}^7}"] \arrow[d] & \mathbb{Z} \arrow[d, hook] \\ & H^7\big(S^7 \sslash \widehat{\mathrm{Sp}(2)};\mathbb{R}\big) \arrow[r, "(\widehat{c \circ \widetilde{f}})^*"] & H^7\big(\widetilde{\Sigma}^7;\mathbb{R}\big) \arrow[r, "\int_{\widetilde{\Sigma}^7}"] & \mathbb{R} \\ \begin{smallmatrix} h_3 \wedge (\omega_4 + \frac{1}{4}p_1) \\ + \omega_7 + \theta_7 \end{smallmatrix} \arrow[rr, mapsto] & & \begin{smallmatrix} \widetilde{H}_3 \wedge \widetilde{f}^* \widetilde{G}_4 \\ + \widetilde{f}^* 2G_7 \end{smallmatrix} \arrow[r, mapsto] & 2\widetilde{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widetilde{f}, \widetilde{H}_3\big) = \widetilde{S}^{1\mathrm{M5}}_{\mathrm{WZ}}\big(\widetilde{f}, \widetilde{H}_3\big). \end{tikzcd} \tag{66}$$

□

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