# Fundamental weight systems are quantum states

David Corfield, Hisham Sati, Urs Schreiber

May 10, 2021

**Abstract**

Weight systems on chord diagrams play a central role in knot theory and Chern-Simons theory; and more recently in stringy quantum gravity. We highlight that the noncommutative algebra of horizontal chord diagrams is canonically a star-algebra, and ask which weight systems are positive with respect to this structure; hence we ask: Which weight systems are quantum states, if horizontal chord diagrams are quantum observables? We observe that the fundamental $\mathfrak{gl}(n)$-weight systems on horizontal chord diagrams with $N$ strands may be identified with the Cayley distance kernel at inverse temperature $\beta = \ln(n)$ on the symmetric group on $N$ elements. In contrast to related kernels like the Mallows kernel, the positivity of the Cayley distance kernel had remained open. We characterize its phases of indefinite, semi-definite and definite positivity, in dependence of the inverse temperature $\beta$; and we prove that the Cayley distance kernel is positive (semi-)definite at $\beta = \ln(n)$ for all $n = 1, 2, 3, \cdots$. In particular, this proves that all fundamental $\mathfrak{gl}(n)$-weight systems are quantum states, and hence so are all their convex combinations. We close with briefly recalling how, under our "Hypothesis H", this result impacts on the identification of bound states of multiple M5-branes.

## Contents

**1 Introduction** **2**

**2 Weights, states and kernels** **3**
- 2.1 Weight systems on chord diagrams . . . 3
- 2.2 Quantum states on quantum observable algebras . . . 6
- 2.3 Cayley distance kernels on symmetric groups . . . 8

**3 Positivity of the Cayley distance kernel** **11**
- 3.1 Indefinite phases . . . 12
- 3.2 Semi-definite phases . . . 14
- 3.3 Definite phases . . . 16

**4 Weight systems as quantum states** **17**

# 1 Introduction

In investigations of problems in string/M-theory ([SS19b], surveyed below in §4), we encountered the following question at the interface of quantum topology and quantum probability theory:

**Question 1.1.** *Which weight systems on horizontal chord diagrams (Def. 2.2) are quantum states (Def. 2.15) with respect to the star-operation of reversal of strands (Prop. 2.9)?*

It is known that all weight systems are *generated*, in a sense ([BN96, Cor. 2.6], review in [SS19b, §3.4]), from *Lie algebra weight systems* $w_{(\mathfrak{g},\mathbf{V})}$ induced by a metric Lie module $(\mathfrak{g},\mathbf{V})$ ([BN95, §2.4], review in [CDM11, §6][SS19b, §3.3]), and in fact from just those with $\mathfrak{g} := \mathfrak{gl}(n)$ and $\mathbf{V} := \mathbf{n}$ the fundamental representation (in the sense of [BN96, Fact 7]). Here we prove that all these *fundamental weight systems* (Def. 2.4) are quantum states:

**Theorem 1.2.** *The fundamental $\mathfrak{gl}(n)$-weight systems $w_{(\mathfrak{gl}(n),\mathbf{n})}$ for $n \in \mathbb{N}_+$ are quantum states on horizontal chord diagrams on $N$ strands, for all $N \in \mathbb{N}_+$; hence so are the mixtures (Ex. 2.16) of their operator images (Ex. 2.17).*

This turns out to be a consequence of the following more general theorem in geometric group theory:

It is well-known that the value of a fundamental Lie algebra weight system $w_{(\mathfrak{gl}(n),\mathbf{n})}$ depends only on the permutation induced (5) by a horizontal chord diagram ([BN96, Prop. 2.1], review in §2.1 below). More concretely, we highlight (Prop. 2.25) that this is the special value at inverse temperature $\beta = \ln(n)$ of the *Cayley distance kernel* (Def. 2.23) on permutations of $N$ elements:

$$\underset{\substack{\text{fundamental}\\ \text{weight system}}}{w_{(\mathfrak{gl}(n),\mathbf{n})}} \xleftrightarrow{\text{Prop. 2.25}} \underset{\substack{\text{Cayley distance kernel at}\\ \text{log-integral inverse temperature}}}{\left[e^{-\beta\cdot d_C}\right] \text{ for } \beta = \ln(n)}$$

Positive (semi-)definite *kernels* are of interest notably in geometric group theory (e.g. [DK18, §2.11]) and in machine learning (e.g. [HSS08][MV17]). While related kernels have recently been proven to be positive-definite [JV18], the archetypical Cayley distance kernel was known to become indefinite at sufficiently low inverse temperature $\beta$, while its general behavior with $\beta$ had remained unknown (Rem. 2.24). Here we prove:

**Theorem 1.3** (Phases of the Cayley distance kernel)**.** *The Cayley distance kernel $e^{-\beta\cdot d_C}$ on the symmetric group on $N$ elements is:*

$$\left\{\begin{array}{lll} \textit{indefinite} & \textit{for} & e^{\beta} \in [1, N-1] \setminus \{1,2,\cdots,N-1\} \\ \textit{positive semi-definite} & \textit{for} & e^{\beta} \in \{1,2,\cdots,N-1\} \\ \textit{positive definite} & \textit{for} & \left\{\begin{array}{l} e^{\beta} \in \{N, N+1, N+2, \cdots\} \\ e^{\beta} > \frac{N-1}{\sqrt[N]{2}-1} \end{array}\right. \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \Rightarrow \begin{array}{l}\textit{fundamental weight systems}\\ w_{(\mathfrak{gl}(n),\mathbf{n})} \textit{ are quantum states}\end{array}$$

*Proof.* This is the content of Prop. 3.9, Prop. 3.13, Prop. 3.16, and Prop. 3.17 below. A slick proof for just the cases $\beta = \ln(n)$, $n \in \mathbb{N}_+$, follows, alternatively, by invoking Schur-Weyl duality; this is shown in Prop. 3.15. □

For illustration, the blue graph in Figure 1 shows, vertically, the value of the smallest eigenvalue (rescaled by $e^{3\beta}$, for visibility) of the Cayley distance kernel on the symmetric group $\mathrm{Sym}(4)$, in dependence of the exponentiated inverse temperature $e^{\beta}$ (running horizontally). This means: where the graph is negative/zero/positive, the Cayley distance kernel is indefinite/positive semi-definite/positive definite, respectively. See also [BN21] for more such computer algebra analysis of the situation.

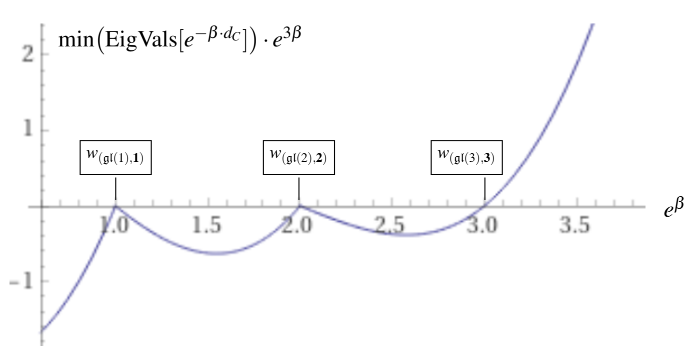


**Figure 1.** Smallest eigenvalue of the Cayley distance kernel on $\mathrm{Sym}(4)$.

**Acknowledgements.** We thank Abdelmalek Abdesselam, Dror Bar-Natan, Carlo Collari and Jean-Philippe Vert for comments and discussion.

# 2 Weights, states and kernels

We briefly recall relevant definitions and facts, and then point out some close relations between the following topics:
- §2.1 – Weight systems on chord diagrams.
- §2.2 – Quantum states on quantum observable algebras.
- §2.3 – Cayley distance kernels on symmetric groups.

**Notation 2.1.** Throughout we consider the following parameters:

**(i)** $N \in \mathbb{N}_+$ the number of strands in horizontal chord diagrams (Def. 2.2), equivalently: the number of elements on which permutations act;

**(ii)** $n \in \mathbb{N}_+$ labels the fundamental weight systems $w_{(\mathfrak{gl}(n),\mathbf{n})}$ (Def. 2.4);

**(iii)** $\beta \in \mathbb{R}_{\geq 0}$ an inverse temperature parameter (Def. 2.23), often specialized to $\beta = \ln(n)$ (Prop. 2.25).

The ground field is the complex numbers $\mathbb{C}$, in which the complex conjugate of $z$ is denoted $\bar{z}$.

## 2.1 Weight systems on chord diagrams

**Definition 2.2** (Horizontal chord diagrams and weight systems ([BN96][CDM11, §5.11][SS19b, §3.1])).
**(i)** The *monoid of horizontal chord diagrams* is the free monoid on the set of pairs of distinct strands

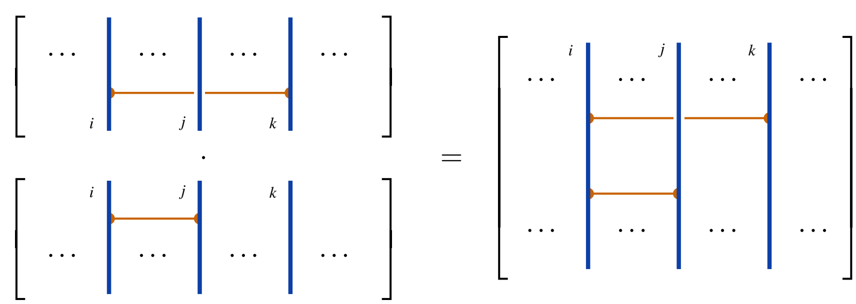
$$\mathscr{D}^{\mathrm{pb}}_N := \mathrm{FreeMonoid}\Big(\big\{(ij) \,|\, 1 \leq i < j \leq N\big\}\Big), \tag{1}$$

where the generator $(ij)$ is called the *chord connecting the ith and jth strand*. Hence a general horizontal chord diagram is a finite list of chords

$$D = (i_1 j_1)(i_2 j_2)\cdots(i_d j_d), \tag{2}$$

possibly empty, and the product operation "·" on horizontal chord diagrams is concatenation of these lists, the neutral element being given by the empty list. For example:

$$(ik)\cdot(ij) = (ik)(ij) \tag{3}$$



**(ii)** The map that sends the chord $(ij)$ to the permutation transposing the $i$th and $j$th of $N$ ordered elements

$$t_{ij} := \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & N-1 & N \\ 1 & 2 & \cdots & j & \cdots & i & \cdots & N-1 & N \end{pmatrix} \tag{4}$$

extends uniquely to a monoid homomorphism from horizontal chord diagrams (1) to the symmetric group on $N$ elements:

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{pb}}_N & \xrightarrow{\quad\mathrm{perm}\quad} & \mathrm{Sym}(N) \\ (i_1 j_1)\cdots(i_d j_d) & \longmapsto & t_{i_1 j_1} \circ \cdots \circ t_{i_d j_d} \end{array} \tag{5}$$

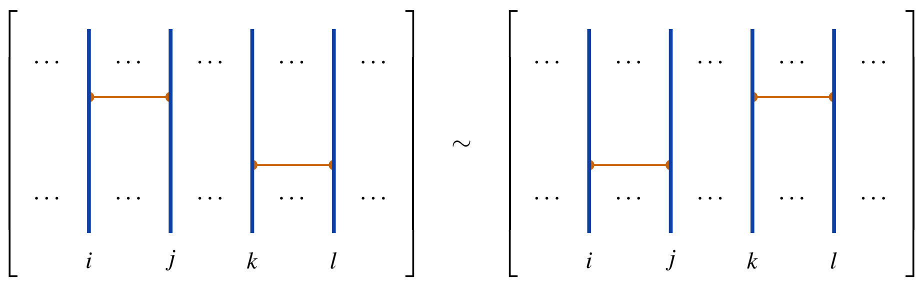
A crucial role in the following discussion is played by the number of cycles in the permutation underlying a chord diagram:

$$\underset{\substack{\text{set of}\\ \text{horizontal chord diagrams}\\ \text{with } N \text{ strands}}}{\mathscr{D}^{\mathrm{pb}}_N} \xrightarrow[\substack{\text{take chord } t_{ij}\\ \text{to transposition } i \leftrightarrow j\\ \text{and consecutive chords to}\\ \text{composition of transpositions}}]{\text{perm}} \underset{\substack{\text{set of}\\ \text{permutations of}\\ N \text{ elements}}}{\mathrm{Sym}(N)} \xrightarrow[\substack{\text{number of}\\ \text{cycles (orbits)}\\ \text{of a permutation}}]{\#\text{cycles}} \{1,\cdots,N\} \subset \mathbb{N} \tag{6}$$

**(iii)** The *algebra of horizontal chord diagrams*

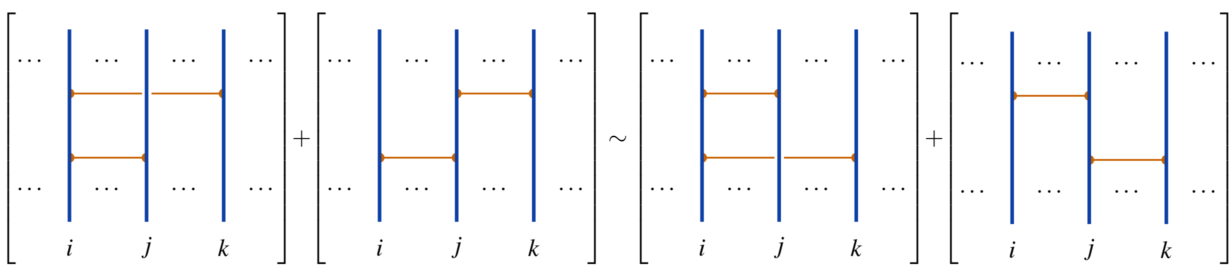
$$\mathscr{A}^{\mathrm{pb}}_N \;:=\; \mathbb{C}[\mathscr{D}^{\mathrm{pb}}_N]/(2\mathrm{T}, 4\mathrm{T}) \tag{7}$$

is the associative unital algebra, graded by number of chords, which is spanned by the monoid of horizontal chord diagrams (1) and then quotiented by the ideal generated by:
**(a)** the *2T relations*:




(8)

**(b)** the *4T relations*:




(9)

**(iv)** The complex vector space of *weight systems on horizontal chord diagrams* is the graded linear dual space

$$\mathscr{W}^{\mathrm{pb}}_N \;:=\; \left(\mathscr{A}^{\mathrm{pb}}_N\right)^*$$

to (7); hence a weight system is a complex-linear map

$$w \;:\; \mathscr{A}^{\mathrm{pb}}_N \longrightarrow \mathbb{C} \tag{10}$$

and is of degree $-d$ if it is supported on chord diagrams of degree $d$.

**Remark 2.3** (Dependence on algebra structure)**.** The definition of weight systems (10) according to Def. 2.2 does not depend on the algebra structure on the space of chord diagrams (7), but the specialization of weight systems to quantum states (Def. 2.12) does.

**Fundamental Lie algebra weight systems.** Recall from [BN96] (reviewed in [SS19b, §3.4]) that the main source of weight systems on horizontal chord diagrams (Def. 2.2) are metric Lie representations $\rho : \mathfrak{g} \otimes \mathbf{V} \to \mathbf{V}$ of metric Lie algebras $\mathfrak{g}$

$$\mathrm{MetricLieModules} \xrightarrow{\;w_{(-)}\;} \mathscr{W}^{\mathrm{pb}}_N\,, \tag{11}$$

where the *Lie algebra weight system* $w_{(\mathfrak{g},\mathbf{V})}$ sends a chord diagram $D \in \mathscr{D}^{\mathrm{pb}}_N$ to (see (18) for illustration): the number obtained by labelling all strands by $V$, all vertices by $\rho$, all chords by $\mathfrak{g}$, then closing all strands to circles using the metric, regarding the result as Penrose notation for a rank-0 tensor in the category of finite-dimensional complex vector spaces and evaluating it as such to a complex number.

**Definition 2.4** (Fundamental $\mathfrak{gl}(n)$-weight system)**.** For $n \in \mathbb{N}_+$, we write

$$w_{(\mathfrak{gl}(n),\mathbf{n})} \;:\; \mathscr{A}^{\mathrm{pb}}_N \longrightarrow \mathbb{C} \tag{12}$$

for the normalized Lie algebra weight system (11) induced by the defining complex $n$-dimensional Lie representation $\mathbf{n}$ of the general complex-linear Lie algebra $\mathfrak{gl}(n)$ equipped with the metric given by the trace in $\mathbf{n} \simeq_{\mathbb{C}} \mathbb{C}^n$.

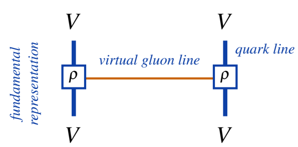
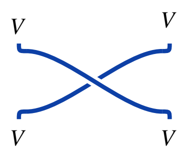
**Example 2.5** (Fundamental metric on $\mathfrak{gl}(2)$)**.** For $n = 2$ and with respect to the complex linear basis of $\mathfrak{gl}(2)$ given by

$$\left\{ x_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, x_+ := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, x_- := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \tag{13}$$

the metric on $\mathfrak{gl}(2)$ according to Def. 2.4 has components

$$\big(g_{ij} := g(x_i, x_j)\big)_{i,j \in \{0,1,+,-\}} \;=\; \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{14}$$

**Lemma 2.6** (Fundamental weights and braiding [BN96, Fact 6])**.** *The fundamental metric on $\mathfrak{gl}(2)$ (Ex. 2.5) has the special property that it makes the value of the fundamental $\mathfrak{gl}(2)$-weight system* (12) *on a single chord be the braiding operation:*

| **Metric Lie algebra** | **Metric contraction of fundamental action tensors** |
|---|---|
| $(\mathfrak{g}, g)$ |  $= g^{ij} x_i \otimes x_j \;\; \in \; \mathrm{End}(V \otimes V)$ |
| $\big(\mathfrak{gl}(2), \mathrm{tr}_{\mathbf{2}}(- \cdot -)\big)$ |  $= \tfrac{1}{2} x_0 \otimes x_0 + \tfrac{1}{2} x_1 \otimes x_1 + x_+ \otimes x_- + x_- \otimes x_+$ |

(15)

*Proof.* By explicit computation in the canonical linear basis (13) with its metric components (14). For example:

$$g^{ij} x_i \otimes x_j \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \underbrace{\tfrac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{=0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

□

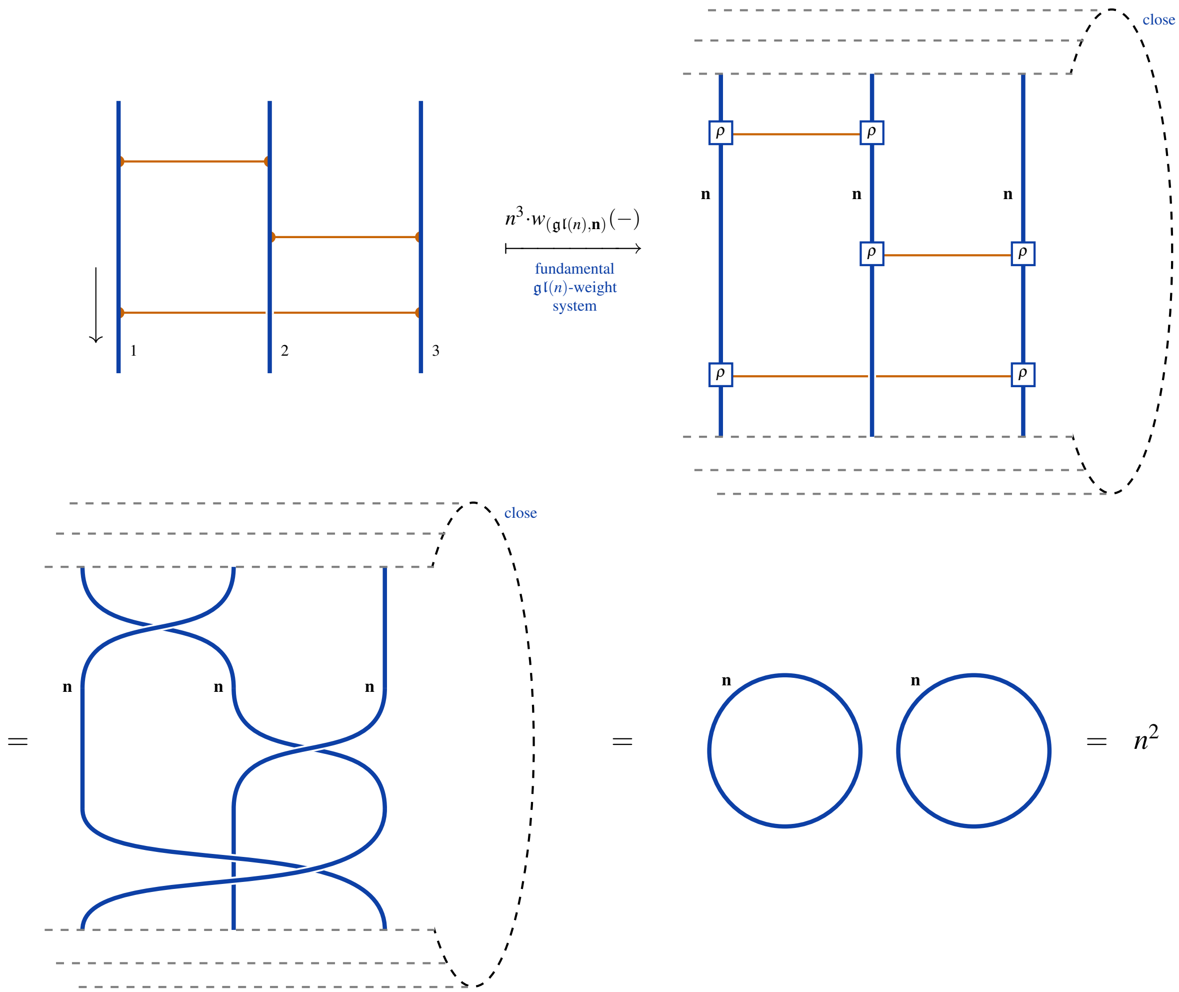
**Corollary 2.7** (Fundamental weight systems in terms of permutation cycles ([BN96, Prop. 2.1]))**.** *The value of the fundamental $\mathfrak{gl}(2)$-weight system $w_{(\mathfrak{gl}(2),\mathbf{2})}$* (12) *on a horizontal chord diagram $D \in \mathscr{A}^{\mathrm{pb}}_N$ equals, up to normalization, 2 taken to the power of the number of cycles* (6) *in the permutation* $\mathrm{perm}(D)$ (5) *corresponding to the chord diagram:*

$$\begin{aligned} w_{(\mathfrak{gl}(2),\mathbf{2})}([D]) \;&=\; \overbrace{2^{-N}}^{\text{normalization}} \cdot 2^{\#\mathrm{cycles}(\mathrm{perm}(D))} \\ &= e^{\ln(2)\cdot\big(\#\mathrm{cycles}(\mathrm{perm}(D)) - N\big)}. \end{aligned} \tag{16}$$

*Generally, the analogous statement is true for the fundamental $\mathfrak{gl}(n)$-weight system (Def. 2.4) for all $n \in \mathbb{N}$, $n \geq 2$:*

$$w_{(\mathfrak{gl}(n),\mathbf{n})}([D]) \;=\; e^{\ln(n)\cdot\big(\#\mathrm{cycles}(\mathrm{perm}(D)) - N\big)}. \tag{17}$$

For example:




(18)

## 2.2 Quantum states on quantum observable algebras

**Quantum observables.** The following Definition 2.8 is often considered for Banach algebras, where it yields the concept of $C^*$*-algebras* (e.g. [Me95, §A4][La17, Def. C.1]. We need the simple specialization to finite-dimensional star-algebras (e.g. [BGQR13, §II]), or rather the evident mild generalization of that to degreewise finite-dimensional graded algebras:

**Definition 2.8** (Star-algebra)**.** A *star-algebra*, for the present purpose, is a degreewise finite-dimensional graded associative algebra $\mathscr{A}$ over the complex numbers, equipped with an involutive anti-linear anti-homomorphism $(-)^*$ (the *star-operation*), hence with a function

$$\mathscr{A} \xrightarrow{(-)^*} \mathscr{A}$$

which satisfies:

| | | | |
|---|---|---|---|
| **(0)** | (degree): | $\deg(A) = \deg(A^*)$ | for all homogeneous $A \in \mathscr{A}$ |
| **(1)** | (anti-linearity): | $\big(a_1 A_1 + a_2 A_2\big)^* = \bar{a}_1 A_1^* + \bar{a}_2 A_2^*$ | |
| **(2)** | (anti-homomorphism): | $\big(A_1 A_2\big)^* = A_2^* A_1^*$ | for all $a_i \in \mathbb{C}$, $A_i \in \mathscr{A}$ |
| **(3)** | (involution): | $\big((A)^*\big)^* = A$ | |

where $\bar{a}_i$ denotes the complex conjugate of $a_i$.

We highlight the following example:

**Proposition 2.9** (Star-structure on horizontal chord diagrams)**.** *The algebra of horizontal chord diagrams* (7) *becomes a complex star-algebra (Def. 2.8) via the star-operation*

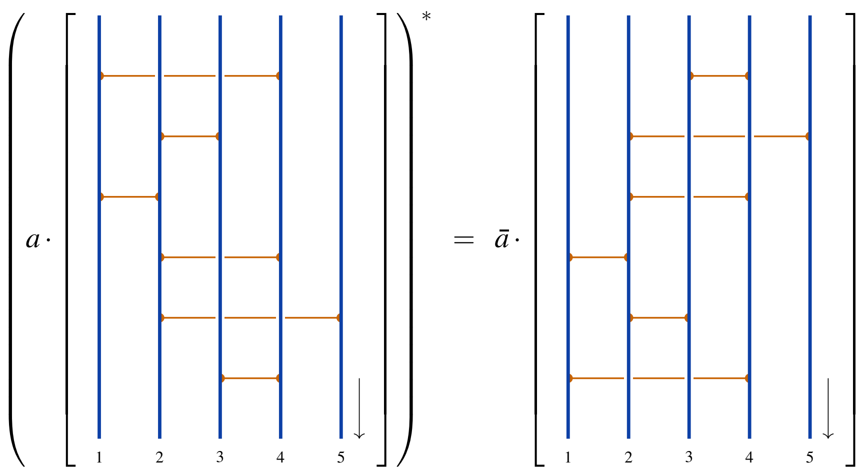
$$\begin{array}{ccc} \mathscr{A}^{\mathrm{pb}}_N & \xrightarrow{\quad(-)^*\quad} & \mathscr{A}^{\mathrm{pb}}_N \\ a_1 \cdot D_1 + a_2 \cdot D_2 & \longmapsto & \bar{a}_1 \cdot D_1^* + \bar{a}_2 \cdot D_2^* \,, \end{array} \tag{19}$$

*where*

$$\mathscr{D}^{\mathrm{pb}}_N \xrightarrow{\;(-)^*\;} \mathscr{D}^{\mathrm{pb}}_N \tag{20}$$

*is the operation that reverses the orientation of strands in a chord diagram* (1)*, hence which reverses the ordering of the corresponding lists* (2)*.*

For example:




*Proof.* The statement evidently holds before quotienting out the 2T-relations (8) and 4T-relations (9) in (7); and the reversal operation manifestly preserves these relations, hence preserves the ideals they generate, hence passes to the quotient.

More abstractly, this star-involution is the involutory antipode of the Hopf algebra structure on the homology of loop spaces ([MM65, p. 262]) under the identification of horizontal chord diagrams with the homology of the loop space of configuration spaces of points ([Koh02, Thm. 4.1]); see around (51) in §4 below for more on this perspective. □

**Proposition 2.10** (Reversed chord diagrams give inverse permutations)**.** *The function* (5) *that sends horizontal chord diagrams to permutations sends reversed chord diagrams* (20) *to inverse permutations:*

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{pb}}_N & \xrightarrow{\;\mathrm{perm}\;} & \mathrm{Sym}(N) \\ {\scriptstyle (-)^*}\downarrow & & \downarrow{\scriptstyle (-)^{-1}} \\ \mathscr{D}^{\mathrm{pb}}_N & \xrightarrow{\;\mathrm{perm}\;} & \mathrm{Sym}(N) \end{array} \qquad \mathrm{perm}(D^*) \;=\; \big(\mathrm{perm}(D)\big)^{-1} \tag{21}$$

*Proof.* This follows immediately from the definition (2) and the fact that any transposition is its own inverse. □

**Example 2.11** (Perm is a star-monoid homomorphism)**.** Since perm is a monoid homomorphism by construction (5) in Def. 2.2, Prop. 2.10 may be read as saying that it is in fact a *star-monoid homomorphism*. In particular, we have:

$$\mathrm{perm}\big(D_1^* \cdot D_2\big) \;=\; \mathrm{perm}(D_1)^{-1} \circ \mathrm{perm}(D_2)\,.$$

**Quantum states.** The following is the standard mathematical formulation of what are often called *mixed states* or *density matrices* in quantum physics, subsuming, as a special case, the traditional pure states that may be identified with elements of a Hilbert space.

**Definition 2.12** (State on a star-algebra. e.g. [Me95, §I.1.1][La17, Def. 2.4])**.** Given a star-algebra $(\mathscr{A},(-)^*)$ (Def. 2.8), a *state* is a complex-linear function

$$\rho \,:\, \mathscr{A} \longrightarrow \mathbb{C} \tag{22}$$

which satisfies:

|  |  |  |  |
|---|---|---|---|
| **(1)** | (positivity): | $\rho\big(A^*A\big) \geq 0 \in \mathbb{R} \subset \mathbb{C}$ | for all $A \in \mathscr{A}$; |
| **(2)** | (normalization): | $\rho(\mathbf{1}) = 1$ | for $\mathbf{1} \in \mathscr{A}$ the algebra unit. |

**Remark 2.13** (S-Matrices – part of the *GNS construction*, e.g. [KR97, Prop. 4.5.1 & p. 270][BGQR13, (II.5)])**.** Given a star-algebra $(\mathscr{A},(-)^*)$ (Def. 2.8), we may identify any linear form $\rho$ (22) on the underlying vector space $\mathscr{A}$ with the following sesquilinear form

$$\begin{array}{ccc} \mathscr{A} \otimes \mathscr{A} & \xrightarrow{\quad \rho((-)^* \cdot (-)) \quad} & \mathscr{C} \\ (A_1, A_2) & \longmapsto & \rho\big(A_1^* \cdot A_2\big)\,. \end{array} \tag{23}$$

Observe that a linear form is a state (Def. 2.12) precisely if its induced sesquilinear form is (normalized to $\rho(\mathbf{1}^* \cdot \mathbf{1}) = 1$ and) positive semi-definite or positive definite:

$$\rho(-) \text{ is a state} \quad \Leftrightarrow \quad \rho\big((-)^* \cdot (-)\big) \text{ is normalized and positive (semi-)definite.}$$

**Remark 2.14** (Positivity)**.** The point of Def. 2.12 is the positivity condition (which might rather deserve to be called *semi-positivity*, by Rem. 2.13; but *positivity* is the established terminology here) while the normalization condition is just that: If $\rho$ is a (semi-)positive linear map with $\rho \neq 0$ then $\frac{1}{\rho(\mathbf{1})} \cdot \rho$ is a state.

**Definition 2.15** (Weight systems that are quantum states)**.** Here we say that a weight system on horizontal chord diagrams (Def. 2.2) *is a quantum state* if it is a state (Def. 2.12) with respect to the canonical star-algebra structure on horizontal chord diagrams from Prop. 2.9.

We will show that the fundamental weight systems (Def. 2.4) are quantum states (Def. 2.15) by regarding them as kernels in geometric group theory, in Prop. 2.25 below. Notice that from any set of quantum states like this, we obtain at least a convex hull of all their operator images as further states:

**Example 2.16** (Convex combinations of quantum states)**.** For $k \in \mathbb{N}_+$ the *mixture* of a $k$-tuple $\big(\rho_i : \mathscr{A} \to \mathbb{C}\big)_{1 \leq i \leq k}$ of quantum states (Def. 2.15) for *probability distribution* $\big(p_i \in \mathbb{R}_{\geq 0}\big)_{1 \leq i \leq k}$, $\sum\limits_i p_i = 1$ is the quantum state given by the convex linear combination $\sum\limits_i p_i \cdot \rho_i \quad \in \mathscr{A}^*$.

**Example 2.17** (Operator-state correspondence)**.** For $\rho : \mathscr{A} \to \mathbb{C}$ any quantum state (Def. 2.12), every non-null observable $O \in \mathscr{A}$, $\rho(O^*O) \neq 0$ induces another state $\rho_O$ given by $\rho_O\big(A\big) \;:=\; \frac{1}{\rho(O^*O)} \cdot \rho\big(O^* \cdot A \cdot O\big)$.

## 2.3 Cayley distance kernels on symmetric groups

**Cayley distance metric on symmetric groups.** The following is at the heart of geometric group theory (e.g. [DK18]).

**Definition 2.18** (Cayley distance (e.g. [Di88, p. 112]))**.** The *Cayley graph* of the symmetric group $\mathrm{Sym}(N)$ is the undirected graph whose vertices are the group elements and which has exactly one edge between any pair of group elements if they differ by composition with a single transposition (4) on the right:

$$\cdots\; \sigma \;\text{———}\; \sigma \circ t_{ij} \;\cdots$$

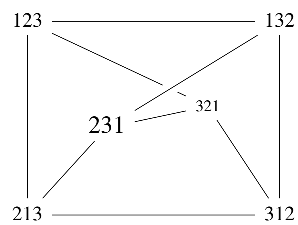
We denote the corresponding *Cayley graph distance function* by

$$d_C \; : \; \mathrm{Sym}(N) \times \mathrm{Sym}(N) \longrightarrow \mathbb{N},$$

hence:

$$d_C(\sigma_1, \sigma_2) \;=\; d_C(e, \sigma_1^{-1} \circ \sigma_2) \;=\; \left\{ \begin{array}{l} \text{minimal number } k \text{ of transpositions} \\ \big\{ t_{i_1 j_1}, \cdots, t_{i_k j_k} \in \mathrm{Sym}(N) \big\} \\ \text{such that} \\ \sigma_1^{-1} \circ \sigma_2 \;=\; t_{i_1 j_1} \circ \cdots \circ t_{i_k j_k} \;. \end{array} \right. \tag{24}$$

**Example 2.19** (Cayley graph of Sym(3))**.** The Cayley graph of the symmetric group on $N = 3$ elements, with edges for arbitrary transpositions, looks as follows:




(25)

Here, e.g., "231" is shorthand for the permutation $\sigma \in \mathrm{Sym}(3)$ with $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. If we order these 6 permutations to a linear basis for $\mathbb{C}[\mathrm{Sym}(N)]$ as follows

$$\big[123, 213, 132, 321, 312, 231\big],$$

then the matrix of Cayley distances (24) between these is

$$\big[d_c\big] \;=\; \left[ \begin{array}{cccccc} 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{array} \right]. \tag{26}$$

For later reference, we record the basic properties of the Cayley distance:

**Lemma 2.20.** *The Cayley distance* (24) *is left invariant:* $\underset{\sigma \in \mathrm{Sym}(N)}{\forall} \; d_C\big(\sigma \circ (-), \sigma \circ (-)\big) \;=\; d_C\big(-, -\big)\,.$

**Lemma 2.21** (Cayley's formula, e.g. [Di88, p. 118])**.** *The Cayley distance function* (24) *equals N minus the number of cycles in the permutation:*

$$d_C(\sigma_1, \sigma_2) \;=\; N - \#\mathrm{cycles}(\sigma_1^{-1} \circ \sigma_2)\,. \tag{27}$$

*Proof.* Since both sides of the equation are invariant under left multiplication (Lemma 2.20), it is sufficient to show the statement for $\sigma_1 = e$, hence for any $\sigma = \sigma_2$. Here, notice first that any cyclic permutation of $k+1$ elements is the product of no fewer than $k$ transpositions:

$$\left( \begin{array}{ccccc} 1 & 2 & 3 & \cdots & k+1 \\ k+1 & 1 & 2 & \cdots & k \end{array} \right) \;=\; \underbrace{t_{k,k-1} \circ \cdots \circ t_{3,2} \circ t_{2,1} \circ t_{1,k+1}}_{k \text{ transpositions}}\,, \tag{28}$$

where we understand that for $k = 0$ the composite on the right is the neutral element. But every permutation $\sigma$ is the composite of such cyclic permutations, one for each of its cycles, with those in different cycles commuting with each other. Since (28) has one transposition fewer than the number of elements, this implies that for every cycle the minimum number of transpositions needed is reduced by one. $\square$

**Lemma 2.22** (Cayley distance is preserved by inclusion of symmetric groups)**.** *The canonical inclusion of symmetric groups*

$$\mathrm{Sym}(N) \xhookrightarrow{\;\; i \;\;} \mathrm{Sym}(N+1)$$

*preserves Cayley distance (Def. 2.18):*

$$\underset{\substack{\sigma_1,\sigma_2 \\ \in \mathrm{Sym}(N)}}{\forall} \;\; d_C(\sigma_1,\sigma_2) \;=\; d_C\big(i(\sigma_1), i(\sigma_2)\big)\,.$$

*In other words, the Cayley distance matrix* $[d_C]$ *of* $\mathrm{Sym}(N)$ *is the principal submatrix of that of* $\mathrm{Sym}(N+1)$ *on the permutations in the image of the inclusion i.*

*Proof.* Observing that $\#\mathrm{cycles}\big(i(\sigma_1)^{-1} \circ i(\sigma_2)\big) = \#\mathrm{cycles}\big(\sigma_1^{-1} \circ \sigma_2\big) + 1$, the claim follows by Cayley's formula (Lemma 2.21). □

**Cayley distance kernels on symmetric groups.** We now consider the corresponding kernels.

**Definition 2.23** (Cayley distance kernel)**.** The *Cayley distance kernel* at *inverse temperature* $\beta \in \mathbb{R}_{\geq 0}$ is the function on pairs of permutations that is given by the exponential of the Cayley distance (Def. 2.18) weighted by $-\beta$:

$$e^{-\beta \cdot d_C(-,-)} \;:\; \mathrm{Sym}(N) \times \mathrm{Sym}(N) \longrightarrow \mathbb{R}\,.$$

Understood as a matrix, we naturally conflate this with its induced sesqui-linear form:

$$\begin{array}{ccc} \mathbb{C}[\mathrm{Sym}(N)] \otimes \mathbb{C}[\mathrm{Sym}(N)] & \xrightarrow{\qquad\qquad \langle -,-\rangle_\beta \qquad\qquad} & \mathbb{C} \\ \Big( \sum\limits_{\sigma_1 \in \mathrm{Sym}(N)} a_{\sigma_1} \cdot \sigma_1, \sum\limits_{\sigma_2 \in \mathrm{Sym}(N)} b_{\sigma_2} \cdot \sigma_2 \Big) & \longmapsto & \sum\limits_{\substack{\sigma_1,\sigma_2 \\ \in \mathrm{Sym}(N)}} \bar{a}_{\sigma_1} \cdot b_{\sigma_2} \cdot e^{-\beta \cdot d_C(\sigma_1,\sigma_2)}\,. \end{array} \tag{29}$$

**Remark 2.24** (Related literature)**.** The Cayley distance kernel (Def. 2.23) is mentioned, for instance, in [FV86, §4][DH92, §4][FV93, p. xx], but has received less attention than related kernels in geometric group theory. Notably the closely related *Mallows kernel* (see, e.g., [Di88, §6B]), which is instead formed from the *Kendall distance* $d_K$ given by the minimum number of *adjacent* permutations, is widely studied and has recently been proven [JV18] to be positive definite, generally. In contrast, the Cayley distance kernel may become indefinite for small $\beta$ (Example 3.1) and its general dependency on $\beta$ had remained unknown.

We now relate Cayley distance kernels to the weight systems from §2.1:

**Proposition 2.25** (Fundamental $\mathfrak{gl}(n)$-weight system is Cayley distance kernel at $\beta = \ln(n)$)**.** *The fundamental* $\mathfrak{gl}(n)$*-weight system* $w_{(\mathfrak{gl}(n),\mathbf{n})}$ *(Def. 2.4), regarded as a sesquilinear form* (23) *on horizontal chord diagrams of N strands, equals the Cayley distance kernel (Def. 2.23) at inverse temperature* $\beta = \ln(n)$ *on the corresponding permutations* (5) *of N elements:*

$$w_{\mathbf{n}}\Big( \Big(\sum_i a_i [D_i]\Big)^* \cdot \Big(\sum_j b_j [D_j]\Big) \Big) \;=\; \sum_{i,j} \bar{a}_i b_j \cdot e^{-\ln(n)\cdot d_C\big(\mathrm{perm}(D_i), \mathrm{perm}(D_j)\big)}\,. \tag{30}$$

*Proof.* We compute as follows:

$$\begin{aligned} w_{\mathbf{n}}\Big( \Big(\sum_i a_i [D_i]\Big)^* \cdot \Big(\sum_j b_j [D_j]\Big) \Big) &= \sum_{i,j} \bar{a}_i b_j \cdot w_{\mathbf{n}}\big(D_i^* \cdot D_j\big) \\ &= \sum_{i,j} \bar{a}_i b_j \cdot e^{\ln(n)\cdot \Big( \#\mathrm{cycles}\big(\mathrm{perm}(D_i^* \cdot D_j)\big) - N \Big)} \\ &= \sum_{i,j} \bar{a}_i b_j \cdot e^{\ln(n)\cdot \Big( \#\mathrm{cycles}\big(\mathrm{perm}(D_i^*)^{-1} \circ \mathrm{perm}(D_j)\big) - N \Big)} \\ &= \sum_{i,j} \bar{a}_i b_j \cdot e^{-\ln(n)\cdot d_C\big(\mathrm{perm}(D_i), \mathrm{perm}(D_j)\big)}\,. \end{aligned} \tag{31}$$

Here the first step is sesqui-linearity, the second step is Cor. 2.7, the third step is Ex. 2.11, and the last step is Lemma 2.21. □

It follows immediately that the fundamental $\mathfrak{gl}(n)$-weight system on $\mathscr{A}_N^{\mathrm{pb}}$ is a quantum state precisely if the Cayley distance kernel on $\mathrm{Sym}(N)$ is positive (semi-)definite at $\beta = \ln(n)$:

$$w_{(\mathfrak{gl}(n),\mathbf{n})} \text{ is a quantum state} \qquad \Leftrightarrow \qquad e^{-\ln(n)\cdot d_C} \text{ is positive (semi-)definite} \,. \tag{32}$$

Therefore we now turn to analyzing the positivity of the Cayley distance kernel.

# 3 Positivity of the Cayley distance kernel

We discuss the (non-/semi-)positivity of the Cayley distance kernel (hence of its lowest eigenvalue) in dependence of the inverse temperature parameter $\beta$.
Throughout, we take *semi-definite* to imply that there is *at least one* vanishing eigenvalue.

To start with, it is instructive to look at the first non-trivial case:

**Example 3.1** (Cayley distance kernel on Sym(3))**.** In the case $N = 3$, with the matrix of Cayley distances given by (26) in Example 2.19, the eigenvalues of the corresponding matrix $\left[e^{-\beta\cdot d_c}\right]$ representing the Cayley distance kernel (29) are readily computed to be

$$\frac{e^{2\beta} \pm 3e^{\beta} + 2}{e^{2\beta}} \quad \text{and} \quad \frac{e^{2\beta} - 1}{e^{2\beta}}\,,$$

where the first two have multiplicity 1, while the last has multiplicity 4. For the given domain of the parameter $\beta \in \mathbb{R}_{\geq 0}$ all these eigenvalues are always positive, except for one which may change sign as $\beta$ varies:

$$e^{2\beta} - 3e^{\beta} + 2 \;\text{ is }\; \begin{cases} = 0 & \text{for} \quad \beta = \ln(1), \\ < 0 & \text{for} \quad \beta \in \big(\ln(1), \ln(2)\big), \\ = 0 & \text{for} \quad \beta = \ln(2), \\ > 0 & \text{for} \quad \beta > \ln(2). \end{cases}$$

It follows for the Cayley distance kernel on $\mathrm{Sym}(3)$ that:

$$\left[e^{-\beta\cdot d_C}\right] \;\text{ is }\; \begin{cases} \text{positive semi-definite} & \text{for} \quad \beta = \ln(1), \\ \text{indefinite} & \text{for} \quad \beta \in \big(\ln(1), \ln(2)\big), \\ \text{positive semi-definite} & \text{for} \quad \beta = \ln(2), \\ \text{positive definite} & \text{for} \quad \beta > \ln(2). \end{cases} \tag{33}$$

In general, the spectrum of the Cayley distance kernel has an explicit expression in terms of the representation theory of the symmetric group:

**Definition 3.2** (Irreducible characters of the symmetric groups)**.** For $\lambda$ a partition of $N \in \mathbb{N}$, hence a weakly decreasing sequence of positive natural numbers that sum to $N$:

$$\lambda \;=\; (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)\,, \qquad \sum_i \lambda_i \;=\; N\,, \qquad \lambda_i \in \mathbb{N}_+\,, \tag{34}$$

we write

$$\chi^{(\lambda)}(-) \;:=\; \mathrm{Tr}\big(\rho^{(\lambda)}(-)\big) \;:\; \mathrm{Sym}(N) \longrightarrow \mathbb{C} \tag{35}$$

for the irreducible character corresponding to the irreducible complex linear representation of $\mathrm{Sym}(N)$

$$\rho^{(\lambda)} \;:\; \mathrm{Sym}(N) \longrightarrow \mathrm{GL}(N, \mathbb{C}) \tag{36}$$

which is the *Specht module* labeled by $\lambda$ (e.g. [Sag01, §2.3]).

Observing that the exponentiated Cayley distance function from the origin

$$\sigma \longmapsto e^{-\beta\cdot d_C(e,\sigma)} = e^{-\beta\cdot N} e^{\beta\cdot\#\mathrm{cycles}(\sigma)}$$

is a class function (manifestly so by Cayley's formula (27) used on the right), the following Prop. 3.3 is the special case of a general character formula for kernels on finite groups [RKHS02, Thm. 1.1][Ka02, Cor. 5.4] (for just the eigenvalues this is due to [DS81, Cor. 3], with a streamlined derivation given in [FG15, Thm. 4.3], generalizing a classical result for abelian groups [Ba79, Cor. 3.2] following [Lo75]):

**Proposition 3.3** (Character formula for spectrum of Cayley distance kernel)**.** *For all $N \in \mathbb{N}_+$ and $\beta \in \mathbb{R}_{\geq 0}$, the eigenvalues of the Cayley distance kernel $\left[e^{-\beta\cdot d_C}\right]$ are*

**(i)** *indexed by the partitions $\lambda$ of $N$* (34)*;*

**(ii)** *given by the formula*

$$\big(\mathrm{EigVals}[e^{-\beta\cdot d_C}]\big)_\lambda = \frac{e^{-\beta\cdot N}}{\chi^{(\lambda)}(e)} \sum_{\sigma\in\mathrm{Sym}(N)} e^{\beta\cdot\#\mathrm{cycles}(\sigma)} \cdot \chi^{(\lambda)}(\sigma), \tag{37}$$

*where $\chi^{(\lambda)}$ is the corresponding irreducible character* (35)*;*

**(iii)** *appearing with multiplicity $\big(\chi^{(\lambda)}(e)\big)^2$ (the square of the dimension of the $\lambda$th irreducible representation);*

**(iv)** *whose corresponding eigenvectors are the (complex conjugated) component functions*

$$\big(\mathrm{EigVects}[e^{-\beta\cdot d_C}]\big)_{\lambda,i,j} = \big(\bar{\rho}^{(\lambda)}_{ij}(\sigma)\big)_{\sigma\in\mathrm{Sym}(N)} \in \mathbb{C}\big(\mathrm{Sym}(N)\big)$$

*of the irreducible representations $\rho^{(\lambda)}$* (36) *for all $1 \leq i, j \leq \chi^{(\lambda)}(e)$.*

**Example 3.4** (Eigenvalues of unit multiplicity)**.** The Cayley distance kernel on $\mathrm{Sym}(N)$ has exactly two eigenvalues of multiplicity 1, namely the homogeneous distribution (corresponding to the trivial irrep of $\mathrm{Sym}(N)$) and the signature-distribution (corresponding to the sign irrep) whose eigenvalues are the sum (necessarily positive) and the signed sum (for $\mathrm{sgn}(\sigma)$ the signature), respectively, over any one row of the Cayley distance kernel matrix:

| $\lambda$ | $\big(\mathrm{EigVects}[e^{-\beta\cdot d_C}]\big)_\lambda$ | $\big(\mathrm{EigVals}[e^{-\beta\cdot d_C}]\big)_\lambda$ |
|---|---|---|
| $(N)$ | $\big(1\big)_{\sigma\in\mathrm{Sym}(N)}$ | $\sum\limits_{\sigma\in\mathrm{Sym}(N)} e^{-\beta d_c(e,\sigma)} > 0$ |
| $(1 \geq \cdots \geq 1)$ | $\big(\mathrm{sgn}(\sigma)\big)_{\sigma\in\mathrm{Sym}(N)}$ | $\sum\limits_{\sigma\in\mathrm{Sym}(N)} \mathrm{sgn}(\sigma)\cdot e^{-\beta d_c(e,\sigma)}$ |

(38)

This follows from Prop. 3.3 since symmetric groups have exactly these two 1-dimensional irreps (e.g. [Sag01, Ex. 2.3.6, 2.3.7][Di88, §7.B(2)]), but in these simple cases the eigenvalues are also readily seen using the left-invariance property of the Cayley distance kernel (Lemma 2.20).

### 3.1 Indefinite phases

**Lemma 3.5** (Cauchy interlace)**.** *Let $N_1 < N_2 \in \mathbb{N}$ and $\beta \in \mathbb{R}_{\geq 0}$.*

**(i)** *If $\left[e^{-\beta\cdot d_C}\right]$ is indefinite on $\mathrm{Sym}(N_1)$ then it is indefinite on $\mathrm{Sym}(N_2)$.*

**(ii)** *If $\left[e^{-\beta\cdot d_C}\right]$ is positive definite on $\mathrm{Sym}(N_2)$ then it is positive definite on $\mathrm{Sym}(N_1)$.*

*Proof.* Since the Cayley distance kernel on $\mathrm{Sym}(N_1)$ is a principal submatrix of that on $\mathrm{Sym}(N_2)$, by Lemma 2.22, this follows from the general fact that for $A_1$ a principal submatrix of a symmetric hermitian matrix $A_2$, their lowest eigenvalues satisfy

$$\min\big(\mathrm{EigVals}(A_2)\big) \leq \min\big(\mathrm{EigVals}(A_1)\big)$$

(a direct consequence of $\min\big(\mathrm{EigVals}(A)\big) = \min\limits_{|v|=1}\langle\bar{v}, Av\rangle$, and a simple case of Cauchy's interlace theorem, e.g. [Hw04]). □

**Proposition 3.6** (Cayley distance kernel indefinite for $0 < \beta < \ln(2)$)**.** *For all N (Nota. 2.1), the Cayley distance kernel ceases to be positive semi-definite as soon as $\beta < \ln(2)$:*

$$0 < \beta < \ln(2) \qquad \Rightarrow \qquad \underset{N \geq 2}{\forall}\; e^{-\beta \cdot d_C} \text{ is indefinite on } \mathrm{Sym}(N)\;.$$

*Proof.* Use Example 3.1 in Lemma 3.5. ☐

To proceed further, we need the following curious result from enumerative combinatorics:

**Lemma 3.7** (First polynomial relation, e.g. [St11, Prop. 1.3.7])**.** *The following holds as an equation of polynomials in $e^\beta$:*

$$\sum_{\sigma \in \mathrm{Sym}(N)} e^{\beta \cdot \#\mathrm{cycles}(\sigma)} \;=\; \prod_{k=0}^{N-1} \big(e^\beta + k\big)\,.$$

*Proof.* Identify permutations with the marked lists underlying their unique representatives in cycle notation for which heads of cycles are the smallest elements in their cycle and cycles are ordered by their heads. Then observe that, in this guise, permutations are manifestly enumerated, starting from the empty such list, by iteratively, over $k = 1, 2, 3, \cdots, N$, including the element $k+1$ into the list, either adjoined to the right of the list if it is the head of a cycle (in which case it contributes a factor $e^\beta$ to $e^{\beta \cdot \#\mathrm{cycles}}$), or else inserted after one of the $k$ elements already in the list (in which case it contributes a factor of 1). ☐

We also need this direct variant of Lemma 3.7:

**Lemma 3.8** (Second polynomial relation)**.** *The following holds as an equation of polynomials in $e^\beta$:*

$$\sum_{\sigma \in \mathrm{Sym}(N)} \mathrm{sgn}(\sigma) \cdot e^{\beta \cdot \#\mathrm{cycles}(\sigma)} \;=\; \prod_{k=0}^{N-1} \big(e^\beta - k\big)\,,$$

*where $\mathrm{sgn}(\sigma)$ denotes the signature of a permutation.*

*Proof.* We compute as follows:

$$\begin{aligned}
\prod_{k=0}^{N-1} \big(e^\beta - k\big) &= (-1)^N \prod_{k=0}^{N-1} \big(-e^\beta + k\big) \\
&= (-1)^N \sum_{\sigma \in \mathrm{Sym}(N)} \big(-e^\beta\big)^{\#\mathrm{cycles}(\sigma)} \\
&= \sum_{\sigma \in \mathrm{Sym}(N)} (-1)^{N + \#\mathrm{cycles}(\sigma)} \cdot e^{\beta \cdot \#\mathrm{cycles}(\sigma)} \\
&= \sum_{\sigma \in \mathrm{Sym}(N)} (-1)^{\#\mathrm{cycles}_{\mathrm{ev.length}}(\sigma)} \cdot e^{\beta \cdot \#\mathrm{cycles}(\sigma)} \\
&= \sum_{\sigma \in \mathrm{Sym}(N)} \mathrm{sgn}(\sigma) \cdot e^{\beta \cdot \#\mathrm{cycles}(\sigma)}\,.
\end{aligned}$$

Here the second step is Lemma 3.7. The fourth step observes that if $N$ is even/odd, then there must be an even/odd number of permutations of odd length, so that the sign that remains is given by the number of permutations of even length. But this is the signature, since the number of transpositions making a cycle is one less than its length (28). ☐

Using this we can improve the characterization of indefiniteness in Proposition 3.6:

**Proposition 3.9.** *The Cayley distance kernel on* $\mathrm{Sym}(N)$ *is indefinite for $e^\beta$ below $N-1$ and non-integer:*

$$e^\beta \;\in\; (0,1) \cup (1,2) \cup \cdots (N-2, N-1) \qquad \Rightarrow \qquad e^{-\beta \cdot d_C} \textit{ is indefinite.} \tag{39}$$

*Proof.* Observe that the Cayley distance kernel has the following eigenvalue

$$\begin{aligned}\big(\mathrm{EigVals}[e^{-\beta \cdot d_C}]\big)_{(1\geq\cdots\geq 1)} &= e^{-\beta\cdot N} \sum_{\sigma\in\mathrm{Sym}(N)} \mathrm{sgn}(\sigma)\cdot e^{\beta\cdot\#\mathrm{cycles}(\sigma)} \\ &= e^{-\beta\cdot N}\prod_{k=0}^{N-1}\big(e^{\beta}-k\big)\,.\end{aligned} \tag{40}$$

Here the first line is Example 3.4 expressed using Cayley's formula (27), and the second step is Lemma 3.8. Hence we find that the Cayley distance kernel always has an eigenvalue of the following sign:

$$\prod_{k=0}^{N-1}\big(e^{\beta}-k\big) \;\text{ is }\; \begin{cases} >0 & \text{for} \quad e^{\beta} > N-1, \\ =0 & \text{for} \quad e^{\beta}\in\{0,1,\cdots,N-1\}, \\ <0 & \text{for} \quad e^{\beta}\in\cdots(N-4,N-3)\cup(N-2,N-1). \end{cases} \tag{41}$$

Here the first two lines are immediate from the form of the polynomial; and with this the third line follows by observing that all roots of the polynomial have unit multiplicity, so that its sign must change whenever $e^{\beta}$ crosses one of its zeros.

This shows that the Cayley distance kernel on $\mathrm{Sym}(N)$ has a negative eigenvalue at least on every *second* of the open intervals claimed. But the same argument applies to the kernel on $\mathrm{Sym}(N-1)$, to show that this has a negative eigenvalue on every *other*, remaining, open interval. Since the latter kernel is (by Lemma 2.22) a principal submatrix of the former, Lemma 3.5 implies that the kernel on $\mathrm{Sym}(N)$ has a negative eigenvalue on all the open intervals (39), as claimed. □

## 3.2 Semi-definite phases

For our proof of the exceptional positive semi-definite phases of the Cayley distance kernel in Prop. 3.13 below, we will need Frobenius' character formula for Schur polynomials, recalled as Prop. 3.11 below, and we will need to know that all coefficients of monomials in Schur polynomials are non-negative, which is manifest from the following form of their definition.

**Definition 3.10** (Schur polynomials (e.g. [Sag01, Def. 4.4.1]))**.** For $\lambda$ a partition (34),

**(i)** a *semistandard Young tableau* (ssYT) $T$ of *shape* $|T| = \lambda$ is sequence $T = (T_{i,j})_{1\leq i, 1\leq j\leq \lambda_i}$ of natural numbers $1 \leq T_{i,j} \leq N$ such that $j_1 < j_2 \Rightarrow T_{ij_1} \leq T_{ij_2}$ and $i_1 < i_2 \Rightarrow T_{i_1 j} < T_{i_2 j}$. We write

$$\mathrm{ssYT}_N \;\supset\; \mathrm{ssYT}_N(n) \tag{42}$$

for the set of all ssYT with $N = \sum_i \lambda_i$ entries, and for its subset of those ssYT with labels $T_{i,j} \leq n$.

**(ii)** The *monomial* corresponding to a ssYT in the polynomial ring on a countable number of generators is

$$x^T \;:=\; x^{\#1s(T)} x^{\#2s(T)} \cdots$$

with $\#1s(T)$ denoting the number of entries of $T$ labeled with the value 1, etc.

**(iii)** The *Schur polynomial* $s_\lambda$ indexed by the partition $\lambda$ is the sum of these monomials over all semistandard Young tableaux $T$ whose shape $|T|$ is $\lambda$:

$$s_\lambda\big(x_1,x_2,\cdots,x_N\big) \;=\; \sum_{\substack{T\in\mathrm{ssYT}_N \\ |T|=\lambda}} x^T\,. \tag{43}$$

**Proposition 3.11** (Character formula for Schur polynomials [Sag01, Thm. 4.6.4])**.** *For $N \in \mathbb{N}_+$ and $\lambda$ a partition of $N$* (34)*, the Schur polynomial $s_\lambda$* (43) *may be expressed as follows:*

$$\begin{aligned} & s_\lambda\big(x_1,x_2,\cdots,x_N\big) \\ & = \tfrac{1}{N!}\sum_{\sigma\in\mathrm{Sym}(N)} \chi^{(\lambda)}(\sigma)\cdot\big(x_1^{\ell_1(\sigma)}+\cdots+x_N^{\ell_1(\sigma)}\big)\big(x_1^{\ell_2(\sigma)}+\cdots+x_N^{\ell_2(\sigma)}\big)\cdots\big(x_1^{\ell_{\#\mathrm{cycles}(\sigma)}(\sigma)}+\cdots+x_N^{\ell_{\#\mathrm{cycles}(\sigma)}(\sigma)}\big)\,, \end{aligned} \tag{44}$$

*where $\chi^{(\lambda)}$ denotes the $\lambda$th irreducible character* (35) *and $\ell_k(\sigma)$ denotes the length of the kth longest cycle of $\sigma$.*

**Lemma 3.12.** *For $e^\beta \in \{1,2,\cdots,N-1\}$ all eigenvalues of the Cayley distance kernel $e^{-\beta\cdot d_C}$ on* $\mathrm{Sym}(N)$ *are non-negative; while for $e^\beta = N$ they are all positive.*

*Proof.* We observe that the eigenvalues $\big(\mathrm{EigVals}[e^{-\beta\cdot d_C}]\big)_\lambda$ (37) at these particular values of $\beta$ appear, up to a positive multiple, as coefficients of monomials in Schur polynomials, which, by Def. 3.10, are manifestly non-negative. Namely for $\lambda$ a partition of $N$ (34) with corresponding Schur polynomial $s_\lambda$ (43) and irreducible character $\chi^{(\lambda)}$ (35) we have the following sequence of equalities, for any $n \in \{1,2,\cdots,N\}$:

$$
\begin{aligned}
s_\lambda\big(\underbrace{x,x,\cdots,x}_{n\text{ args}},\underbrace{0,0,\cdots,0}_{N-n\text{ args}}\big) &= \frac{1}{N!}\sum_{\sigma\in\mathrm{Sym}(N)}\chi^{(\lambda)}(\sigma)(nx^{\ell_1(\sigma)})(nx^{\ell_2(\sigma)})\cdots(nx^{\ell_{\#\mathrm{cycles}(\sigma)}(\sigma)}) \\
&= \left(\frac{1}{N!}\sum_{\sigma\in\mathrm{Sym}(N)}\chi^{(\lambda)}(\sigma)n^{\#\mathrm{cycles}(\sigma)}\right)\cdot x^N \\
&= \underbrace{\left(\frac{\chi^{(\lambda)}(e)}{N!}e^{\ln(n)\cdot N}\big(\mathrm{EigVals}[e^{-\ln(n)\cdot d_C}]\big)_\lambda\right)}_{\in\mathbb{N}.}\cdot x^N\,,
\end{aligned}
$$

where the first step is the character formula for Schur polynomials (Prop. 3.11), while the last step uses the character formula for kernel spectra (Prop. 3.3).

As a result, the expression over the brace is a non-negative integer, whence the eigenvalue must be non-negative, at least. Finally, in the special case that $n = N$, none of the contributing monomials vanishes, and it is clear from (43) that the coefficient over the brace, and hence the eigenvalues, must in fact be positive. □

As a first conclusion of this Lemma, we have:

**Proposition 3.13.** *For $e^\beta \in \{1,2,\cdots,N-1\}$ the Cayley distance kernel $e^{-\beta\cdot d_C}$ on* $\mathrm{Sym}(N)$ *is positive semi-definite.*

*Proof.* By Lemma 3.12, all its eigenvalues at these temperatures are non-negative, while by (41) at least one of them takes the value 0. □

We give a sharpening this conclusion via the following Lemma, which is a consequence of Schur-Weyl duality (e.g. [FH91, §6.1]):

**Lemma 3.14** ([GGK13, Prop. 2.4])**.** *For $n \in \mathbb{N}$, the exponential of $n$ to the number of permutation cycles has an expansion in terms of irreducible characters* (35) *weighted by numbers of semistandard Young tableaux* (42)*:*

$$
n^{\#\mathrm{Cycles}(-)} := \sum_{T\in\mathrm{ssYT}_N(n)}\chi^{|T|}(-)\,.
$$

**Proposition 3.15** (Eigenvalues of Cayley distance kernel at $\beta = \ln(n)$ count Young tableaux)**.** *At $\beta = \ln(n)$, the $\lambda$th eigenvalue of the Cayley distance kernel (according to Prop. 3.3) is proportional to the number of semistandard Young tableaux* (42) *of shape $\lambda$ with entries $\leq n$:*

$$
\mathrm{EigVals}[e^{-\ln(n)\cdot d_C}]_\lambda = \frac{n!}{n^N\cdot\chi^{(\lambda)}(e)}\cdot\#\mathrm{ssYT}_\lambda(n)\,.
$$

*In particular, for the Cayley distance kernel at $\beta = \ln(n)$:*

1. *all eigenvalues are non-negative;*
2. *for $N \leq n-1$ there exists at least one vanishing eigenvalue;*
3. *for $n \geq N$ all eigenvalues are positive.*

*Proof.* We compute as follows:

$$\begin{aligned}\mathrm{EigVals}[e^{-\ln(n)\cdot d_C}]_\lambda &= \tfrac{1}{n^N\cdot\chi^{(\lambda)}(e)} \sum_{\sigma\in\mathrm{Sym}(N)} n^{\#\mathrm{Cycles}(\sigma)}\cdot\bar{\chi}^{(\lambda)}(\sigma)\\ &= \tfrac{1}{n^N\cdot\chi^{(\lambda)}(e)} \sum_{\substack{T\in\\ \mathrm{ssYT}_N(n)}} \sum_{\substack{\sigma\in\\ \mathrm{Sym}(n)}} \chi^{|T|}(\sigma)\cdot\bar{\chi}^{(\lambda)}(\sigma)\\ &= \tfrac{n!}{n^N\cdot\chi^{(\lambda)}(e)} \sum_{\substack{T\in\\ \mathrm{ssYT}_N(n)}} \delta^{|T|,\lambda}.\end{aligned}$$

Here the first line is the character formula (37) from Prop. 3.3, shown under complex conjugation (which does not change the real eigenvalue). The second step inserts Lemma 3.14 and the last step applies Schur orthogonality (e.g. [FH91, Thm. 2.12]).

With this, the final conclusion follows from observing that a partition (Young diagram) admits at least one labelling in $\{1,\cdots,n\}$ to a semistandard Young tableau if and only if it has no more than $n$ rows. □

### 3.3 Definite phases

We have obtained the following statement already in the last clause of Prop. 3.14, using Schur-Weyl duality. Here is a more elementary proof:

**Proposition 3.16.** *For $e^\beta\in\{N,N+1,\cdots\}$ the Cayley distance kernel $e^{-\beta\cdot d_C}$ on $\mathrm{Sym}(N)$ is positive definite.*

*Proof.* By the last clause in Lemma 3.12 the Cayley distance kernel on $\mathrm{Sym}(N+K)$ is positive definite at $e^\beta = N+K$, for all $K\in\mathbb{N}$. Since the Cayley distance kernel on $\mathrm{Sym}(N)$ is a principal submatrix of that on $\mathrm{Sym}(N+K)$ (by Lemma 2.22) the claim follows by Lemma 3.5. □

Finally, we establish existence of a lower bound for the inverse temperature $\beta$ above which the Cayley distance kernel is always positive definite (see Rem. 3.18 for sharper bounds):

**Proposition 3.17** (Lower inverse temperature bound of positive definite phase)**.** *For $e^\beta \geq \frac{N-1}{\sqrt[N]{2}-1}$ the Cayley distance kernel $e^{-\beta\cdot d_C}$ on $\mathrm{Sym}(N)$ is positive definite.*

*Proof.* Using that, by left invariance of the Cayley distance kernel (Lemma 2.20),

- **(i)** its diagonal entries are all equal to $1=e^{-\beta\cdot 0}$,
- **(ii)** the sums over its rows are all equal to each other,

the Gershgorin circle theorem ([Ge31], review in [Va04, §1.1]) says that all its eigenvalues are contained in the single Gershgorin interval

$$\big[1-r(\beta),1+r(\beta)\big]\subset\mathbb{R}, \tag{45}$$

around 1 and of radius equal to the sum of the non-diagonal entries of any row. We compute this *Gershgorin radius* of the Cayley distance kernel as follows:

$$\begin{aligned} r(\beta) &= \sum_{\substack{e\neq\sigma\\ \in\mathrm{Sym}(N)}} e^{-\beta\cdot d_C(e,\sigma)}\\ &= \Bigg(\sum_{\sigma\in\mathrm{Sym}(N)} e^{-\beta\cdot d_C(e,\sigma))}\Bigg)-1\\ &= e^{-\beta\cdot N}\Bigg(\sum_{\sigma\in\mathrm{Sym}(N)} e^{\beta\cdot\#\mathrm{cycles}(\sigma)}\Bigg)-1\\ &= e^{-\beta\cdot N}\Bigg(\prod_{k=0}^{N-1}\big(e^\beta+k\big)\Bigg)-1\,.\end{aligned} \tag{46}$$

Here, after the Gershgorin circle theorem in the first line, the second step adjoins the diagonal element $e^{-\beta \cdot d_C(e,e)} = 1$ to the sum; the third step uses Cayley's formula (Lemma 2.21), and the last step is Lemma 3.7.

In the special case when $\beta = \ln(n)$, the last expression in (46) becomes

$$
\begin{aligned}
r\big(\beta = \ln(n)\big) &= \frac{n \cdot (n+1) \cdots (n+N-1)}{n^N} - 1 \\
&\leq \left(\frac{n+N-1}{n}\right)^N - 1 \\
&= \left(1 + \frac{N-1}{n}\right)^N - 1\,.
\end{aligned}
\tag{47}
$$

Therefore, a sufficient condition for the Gershgorin interval (45) *not* to contain negative numbers, and hence for the Cayley distance kernel to be positive definite, is that

$$
r\big(\beta_N = \ln(n_N)\big) \;<\; 1 \qquad \Leftarrow \qquad \left(1 + \frac{N-1}{n_N}\right)^N - 1 \;<\; 1 \qquad \Leftrightarrow \qquad n_N \;>\; \frac{N-1}{\sqrt[N]{2}-1}\,. \tag{48}
$$

This establishes the statement. □

**Remark 3.18** (Improving the lower inverse temperature bound of the positive definite phase)**.** The lower bound $\beta \;>\; \frac{N-1}{\sqrt[N]{2}-1}$ in Prop. 3.17 is rather loose: Computer algebra computations at least for low $N$ indicate (as seen for $N = 4$ in Figure 1) that the Cayley distance kernel becomes positive definite already for

$$
e^\beta > N-1 \qquad \Rightarrow \qquad \big[e^{-\beta \cdot d_C}\big] \text{ is positive definite on } \mathrm{Sym}(N)\,. \tag{49}
$$

With the proof strategy of Prop. 3.17 there are two steps at which the bound may be tightened:
**(i)** The Gershgorin radius may be reduced. We know from Example 3.4 that the only two eigenvectors of unit multiplicity are the homogeneous distribution and the signature distribution, and we know from (38) and (40) & (41) that both are positive already for all $\beta > N-1$. Therefore, it is sufficient to find a lower bound where the remaining eigenvectors with higher multiplicity become positive. But for such eigenvectors, and since the Cayley distance kernel matrix has non-negative real entries, a refinement of the Gershgorin circle theorem applies, which says ([BS17, Thm. 1]) that they are contained already in *half* the Gershgorin radius (46). Using this factor of $1/2$ on the left of (48) turns the $\sqrt[N]{2}$ on the right into a $\sqrt[N]{3}$ and hence yields the following slightly improved lower bound:

$$
\beta \;>\; \frac{N-1}{\sqrt[N]{3}-1} \qquad \Rightarrow \qquad e^{-\beta \cdot d_C} \text{ is positive definite on } \mathrm{Sym}(N)\,.
$$

**(ii)** The estimate $n \cdot (n+1) \cdots (n+N-1) \leq (n+N-1)^N$ in (47) may be improved. This estimate is just the most conveniently expressed, but it is good only for $n \gg N$, hence far from the expected sharp bound at $n = N-1$.

In fact, A. Abdesselam kindly points out to us that a proof of the sharp lower bound (49) follows with Stanley's *hook-content formula* ([St71, Thm. 15.3][Kr98, Thm. 1]) applied to (37). Since this further strengthening of Thm. 1.3, while interesting in its own right, is not needed for our Theorem 1.2, we will not further dwell on it and instead refer the reader to Abdesselam's upcoming work.

# 4 Weight systems as quantum states

We close by briefly explaining the impact of Thm. 1.2 on current questions in string/M-theory theory, following our discussion in [SS19b] to which we refer for full details and further pointers.

**Chord diagram observables from Hypothesis H.** While informal considerations of quantum physics of branes in string theory has proven to be a rich source for mathematical insights in quantum topology, the underlying mathematical formulation of non-perturbative brane physics itself ("M-theory") has remained wide open (see [SS19b, p. 3 & 6] for pointers). Recently we have explored the *Hypothesis H* [Sa13, §2.5][FSS19b][FSS19b][SS19a][SS21] that the proper mathematical formulation of the *C-field* – which is the only field expected in M-theory, besides the the field of (super-)gravity – is as a cocycle in (twisted) *Cohomotopy theory*. We have shown ([SS19b, §2]) how such a hypothesis implies that (the topological sector of) the *phase space* of $N$ probe $D6 \perp D8$-brane intersections (in an ambient flat spacetime) is homotopy-equivalent to the based loop space of the configuration space of $N$ ordered points in $\mathbb{R}^3$:

$$\Omega\Big(\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)\,. \tag{50}$$

This implies ([SS19b, §2.5]) that the higher homotopical *observables* on such brane systems, conceptualized as the homology of the phase space, is (by [Koh02, Thm. 4.1]) nothing but the algebra of horizontal chord diagrams $\mathscr{A}^{\mathrm{pb}}_N$ from Def. 2.2:

$$\begin{array}{ccccccccc}
\substack{\text{weight}\\ \text{systems}} & & \substack{\text{horizontal}\\ \text{chord diagrams}} & & \substack{\text{homology of}\\ \text{loop space of}\\ \text{configuration space of}\\ \text{ordered points in } \mathbb{R}^3} & & \substack{\text{loop space of}\\ \text{configuration space of}\\ \text{ordered points in } \mathbb{R}^3} & & \substack{\text{loop space of}\\ \text{diff. Cohomotopy cocycle space of}\\ \text{spacetime transversal to}\\ \text{intersecting codim 3/codim 1-branes}} \\
\underset{N \in \mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N\big)^* & \longleftrightarrow & \underset{N \in \mathbb{N}}{\bigoplus} \mathscr{A}^{\mathrm{pb}}_N & \simeq & H_\bullet\Big(\underset{N\in\mathbb{N}}{\sqcup}\Omega\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big) & \longleftarrow & \underset{N\in\mathbb{N}}{\sqcup}\Omega\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{R}^3) & \simeq & \Omega\,\boldsymbol{\pi}^4_{\mathrm{diff}}\left(\begin{array}{c}(\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \\ \cup\,(\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}}\end{array}\right) \\
\big\uparrow & & & & \text{quantum observables} & & \text{phase space (topological sector)} & & \\
\Big(\underset{N \in \mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N\big)^*\Big)_{\substack{\text{normalized}\\ \text{\& positive}}} & & & & & & & & \\
\text{quantum states} & & & & & & & &
\end{array} \tag{51}$$

(Here $H_\bullet(-)$ is ordinary homology with complex coefficients, $\boldsymbol{\pi}^4_{\mathrm{diff}}(-)$ is a presheaf of pointed mapping spaces into the 4-sphere [SS19b, §2.3]; $(-)^{\mathrm{cpt}}$ is one-point compactification and $(-)_+$ is disjoint union with a base point.)

**Chord diagrams in stringy quantum physics.** While it was well-known that chord diagrams organize the quantum observables of perturbative Chern-Simons theory (Vassiliev knot invariants, [Bar91][Ko93][BN95][AF96][BNS96]), we observed in [SS19b, §4] that chord diagrams moreover govern several more recent proposals for aspects of intersecting brane physics, including:

- **(i)** the fuzzy/non-commutative geometry of D-brane intersections seen via the non-abelian Dirac-Born-Infeld (DBI) action functional ([RST04, §3.2], review in [MPRS06, §A][MN06, §4])
- **(ii)** several quantum many-body models for brane/bulk holography:
  - **(a)** dimer/bit-thread models for quantum error correction codes ([JGPE19][Ya20], review in [JE21, §4.2]);
  - **(b)** scattering amplitudes in bulk duals of the SYK model ([BNS18][BINT18], review in [Na19]).

This confluence of occurrences of chord diagrams in quantum brane physics (which seems to previously have gone unnoticed; e.g. the authors of [JGPE19][Ya20] refer to chord diagrams as "dimer" or "bit-thread" networks) finds, assuming Hypothesis H, a natural explanation and unification from the result (51) that chord diagrams indeed constitute the fundamental (topological) quantum observables on intersecting quantum brane systems.

**Weight systems as quantum states of branes.** This allows us to proceed further and next ask for a rigorous characterization, assuming Hypothesis H, of possible *quantum states of intersecting brane systems*, by asking for weight systems which are quantum states in the precise sense of Def. 2.15 – and this is our Question 1.1.

**Bound state of 2 M5-branes.** In particular, we may now rigorously ask, assuming Hypothesis H, whether two M5-branes may form a *bound state* of coincident branes – a statement that is widely expected to be true and which is at the heart of some of the deepest conjectures in contemporary string/M-theory, but for which no actual theory existed.

- **(i)** In [SS19b, §4.9] we explained how the would-be bound state of $N^{(\mathrm{M5})}$ coincident M5-branes (specifically: transversal M5-branes in a pp-wave background) should correspond, under (51), to the Lie algebra weight system $w_{(\mathfrak{gl}(2),\mathbf{N}^{(\mathrm{M5})})}$ (11) for the $N^{(\mathrm{M5})}$-dimensional irrep of $\mathfrak{gl}(2)$, whence the would-be bound state of 2 M5-branes corresponds to the *fundamental* weight system $w_{(\mathfrak{gl}(2),\mathbf{2})}$ from Def. 2.4.

**(ii)** That this be a *bound state* of M5-branes – as opposed to an unstable tachyonic "ghost" state – means to ask whether it is positive as a linear functional on observables (Def. 2.12) and hence whether it is a quantum state in the precise sense of Def. 2.15. That this is the case is the result of our Thm. 1.2! – which we thus may think of as a *no-ghost theorem* for bound M5-brane states.

This establishes the result announced in [SS19b, §3.5].

**Bound state of $N^{(\mathrm{M5})}$ M5-branes.** The natural next question to ask is whether $N^{(\mathrm{M5})}$ coincident M5-branes form bound states, in this same sense, for any $N^{(\mathrm{M5})} \geq 2$, hence whether the non-fundamental Lie algebra weight systems $w_{(\mathfrak{gl}(2),\mathbf{N}^{(\mathrm{M5})})}$ are quantum states on chord diagrams for $N^{(\mathrm{M5})} > 2$. We will discuss this question elsewhere [CSS21].

# References


[AF96] D. Altschuler and L. Freidel, *Vassiliev knot invariants and Chern-Simons perturbation theory to all orders*, Commun. Math. Phys. **187** (1997), 261-287, [`arXiv:q-alg/9603010`].

[Ba79] L. Babai, *Spectra of Cayley graphs*, J. Combin. Theor. **B27** (1979), 180-189, [`doi:10.1016/0095-8956(79)90079-0`].

[BGQR13] A. P. Balachandran, T. R. Govindarajan, A. R. de Queiroz and A. F. Reyes-Lega, *Algebraic approach to entanglement and entropy*, Phys. Rev. A 88, 022301 (2013) [`arXiv:1301.1300`].

[Bar91] D. Bar-Natan, *Perturbative aspects of the Chern-Simons topological quantum field theory*, PhD thesis, Princeton University, 1991, [`spire:323500`], [`ncatlab.org/nlab/files/BarNatanPerturbativeCS91.pdf`]

[BN95] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** 2 (1995) 423-472, [`doi:10.1016/0040-9383(95)93237-2`].

[BN96] D. Bar-Natan, *Vassiliev and Quantum Invariants of Braids*, The interface of knots and physics (San Francisco, CA, 1995), 129-144, Proc. Sympos. Appl. Math., 51, AMS Short Course Lecture Notes, Amer. Math. Soc., Providence, RI, 1996, [`arXiv:q-alg/9607001`].

[BNS96] D. Bar-Natan and A. Stoimenow, *The Fundamental Theorem of Vassiliev Invariants*, in: *Geometry and Physics*, Lec. Notes in Pure & App. Math. **184**, Marcel Dekker Inc. 1996, [`arXiv:q-alg/9702009`].

[BN21] D. Bar-Natan, *The Cayley distance kernel following* `arXiv://2105.0287` *by Corfield, Sati, and Schreiber* [drorbn.net/AcademicPensieve/2021-05/nb/CayleyDistanceKernel.pdf]

[BS17] I. Bárány and J. Solymosi, *Gershgorin disks for multiple eigenvalues of non-negative matrices*, In: M. Loebl et al. (eds.) *A Journey Through Discrete Mathematics*, Springer, 2017, [`arXiv:1609.07439`] [`doi:10.1007/978-3-319-44479-6_6`].

[BNS18] M. Berkooz, P. Narayan and J. Simón, *Chord diagrams, exact correlators in spin glasses and black hole bulk reconstruction*, J. High Energy Phys. **08** (2018) 192, [`arXiv:1806.04380`]

[BINT18] M. Berkooz, M. Isachenkov, V. Narovlansky, and G. Torrents, *Towards a full solution of the large N double-scaled SYK model*, J. High Energy Phys. **03** (2019) 079, [`arXiv:1811.02584`].

[CDM11] S. Chmutov, S. Duzhin, and J. Mostovoy, *Introduction to Vassiliev knot invariants*, Cambridge University Press, 2012, [`ISBN:9781139107846`], [`arXiv:1103.5628`].

[CSS21] C. Collari, H. Sati, and U. Schreiber, $\mathfrak{gl}(2)$-*Weight systems are quantum states* (in preparation).

[Di88] P. Diaconis, *Group representations in Probability and Statistics*, Institute of Mathematical Statistics Lecture Notes Monogr. Ser. **11** (1988), [`jstor:i397389`], [`jdc.math.uwo.ca/M9140a-2012-summer/Diaconis.pdf`].

[DH92] P. Diaconis and P. Hanlon, *Eigen Analysis for Some Examples of the Metropolis Algorithm*, in D. Richards (ed.) *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications*, Contemporary Mathematics **138**, Amer. Math. Soc., 1992, [`doi:10.1090/conm/138`].

[DS81] P. Diaconis and M. Shahshahani, *Generating a random permutation with random transpositions*, Z. Wahrscheinlichkeitsth. verw. Gebiete **57** (1981) 159-179, [`doi:10.1007/BF00535487`].

[DK18] C. Drutu and M. Kapovich, *Geometric group theory*, Colloquium Publications **63**, Amer. Math. Soc., 2018, [`ISBN:978-1-4704-1104-6`].

[FSS19b] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Comm. Math. Phys. **377** (2020), 1961-2025, [doi:10.1007/s00220-020-03707-2], [arXiv:1904.10207].

[FSS19c] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M5 WZ term level quantization*, Comm. Math. Phys. (2021), [doi:10.1007/s00220-021-03951-0], [arXiv:1906.07417]

[FV86] M. A. Fligner and J. S. Verducci, *Distance Based Ranking Models*, J. Royal Statist. Soc. B **48** 3 (1986) 359-369, [jstor:2345433].

[FV93] M. A. Fligner and J. S. Verducci (eds.), *Probability Models and Statistical Analyses for Ranking Data*, Lecture Notes in Statistics **80**, Springer, 1993, [doi:10.1007/978-1-4612-2738-0]

[FG15] B. Foster-Greenwood and C. Kriloff, *Spectra of Cayley graphs of complex reflection groups*, J. Algebraic Combin. **44** 1 (2016) 33-57, [doi:10.1007/s10801-015-0652-8], [arXiv:1502.07392].

[FH91] W. Fulton and J. Harris, *Representation Theory: a First Course*, Springer 1991 [doi:10.1007/978-1-4612-0979-9].

[Ge31] S. A. Gerschgorin, *Über die Abgrenzung der Eigenwerte einer Matrix*, Bull. l'Acad. Sciences l'URSS **6** (1931), 749-754, [mathnet:izv5235].

[GGK13] A. Gnedin, V. Gorin and S. Kerov, *Block characters of the symmetric groups*, J. Alg. Comb. **38** 1 (2013) 79-101 [arXiv:1108.5044].

[HSS08] T. Hofmann, B. Schölkopf and A. J. Smola, *Kernel methods in machine learning*, Annals of Statistics **36** 3 (2008) 1171-1220 [arXiv:math/0701907].

[Hw04] S.-G. Hwang, *Cauchy's Interlace Theorem for Eigenvalues of Hermitian Matrices*, Amer. Math. Monthly **111** 2 (2004), 157-159, [jstor:4145217].

[JE21] A. Jahn and J. Eisert, *Holographic tensor network models and quantum error correction: A topical review*, [arXiv:2102.02619].

[JGPE19] A. Jahn, M. Gluza, F. Pastawski and J. Eisert, *Majorana dimers and holographic quantum error-correcting code*, Phys. Rev. Research 1, 033079 (2019) [arXiv:1905.03268].

[JV18] Y. Jiao and J.-P. Vert, *The Kendall and Mallows Kernels for Permutations*, IEEE Trans. Pattern Analysis Machine Intelligence (TIPAMI) **40** 7 (2018), 1755-1769, [doi:10.1109/TPAMI.2017.2719680].

[KR97] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras I: Elementary Theory*, Graduate Studies in Mathematics **15**, AMS 1997 [ISBN:978-0-8218-0819-1].

[Ka02] P. Kaski, *Eigenvectors and spectra of Cayley graphs*, Lecture notes, Postgraduate Course in Theoretical Computer Science, Helsinki, 2002, [www.tcs.hut.fi/Studies/T-79.300/2002S/esitelmat/kaski_paper_020506.pdf]

[Koh02] T. Kohno, *Loop spaces of configuration spaces and finite type invariants*, Geom. Topol. Monogr. **4** (2002), 143-160, [arXiv:math/0211056].

[Ko93] M. Kontsevich, *Vassiliev's knot invariants*, Adv. Soviet Math. **16** 2 (1993) [http://pagesperso.ihes.fr/~maxim/TEXTS/VassilievKnot.pdf]

[Kr98] C. Krattenthaler, *An Involution Principle-Free Bijective Proof of Stanley's Hook-Content Formula*, Dis. Math. and Theor. Comp. Sci. **3** (1998) 11–32 [hal:00958904].

[La17] K. Landsman, *Foundations of quantum theory: From classical concepts to Operator algebras*, Springer Open, 2017, [doi:10.1007/978-3-319-51777-3].

[Lo75] L. Lovász, *Spectra of graphs with transitive groups*, Period. Math. Hungar. **6** (1975), 191-195, [doi:10.1007/BF02018821].

[MV17] J. Mairal and J.-P. Vert, *Machine Learning with Kernel Methods*, lecture notes (2017) [http://members.cbio.mines-paristech.fr/~jvert/svn/kernelcourse/course/2021mva].

[MN06] S. McNamara, *Twistor Inspired Methods in Perturbative Field Theory and Fuzzy Funnels*, PhD thesis, Queen Mary, U. of London, 2006, [spire:1351861].

[Me95] P.-A. Meyer, *Quantum Probability for Probabilists*, Lecture Notes in Mathematics **1538**, Springer, 1995, [doi:10.1007/BFb0084701].

[MM65] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Annals of Math. **81** (1965) 211-264 [doi:10.2307/1970615]

[MPRS06] S. McNamara, C. Papageorgakis, S. Ramgoolam, and B. Spence, *Finite N effects on the collapse of fuzzy spheres*, J. High Energy Phys. **0605** (2006), 060, [arXiv:hep-th/0512145].

[Na19] V. Narovlansky, *Towards a Solution of Large N Double-Scaled SYK*, seminar notes, 2019, [ncatlab.org/nlab/files/NarovlanskySYK19.pdf].

[RST04] S. Ramgoolam, B. Spence, and S. Thomas, *Resolving brane collapse with* $1/N$ *corrections in non-Abelian DBI*, Nucl. Phys. **B703** (2004), 236-276, [arXiv:hep-th/0405256].

[RKHS02] D. Rockmore, P. Kostelec, W. Hordijk, and P. F. Stadler, *Fast Fourier Transform for Fitness Landscapes*, Appl. Comput. Harmon. Anal. **12** (2002), 57-76, [doi:10.1006/acha.2001.0346].

[Sag01] B. Sagan, *The symmetric group*, Springer, 2001, [doi:10.1007/978-1-4757-6804-6].

[Sa13] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018), 062304, [doi:10.1063/1.5007185], [arXiv:1310.1060].

[SS19a] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [doi:10.1016/j.geomphys.2020.103775], [arXiv:1909.12277].

[SS19b] H. Sati, U. Schreiber, *Differential Cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams*, [arXiv:1912.10425].

[SS21] H. Sati and U. Schreiber, *M/F-Theory as Mf-Theory*, [arXiv:2103.01877].

[St71] R. Stanley, *Theory and application of plane partitions 2*, Studies in Applied Math. **50** 3 (1971) 259-279 [http://www-math.mit.edu/~rstan/pubs/pubfiles/12-2.pdf].

[St11] R. Stanley, *Enumerative combinatorics Vol. 1*, Cambridge Studies in Advanced Mathematics, Cambridge University Press 2011 [ISBN:9781107602625], [http://www-math.mit.edu/~rstan/ec/ec1.pdf]

[Va04] R. S. Varga, *Gershgorin and His Circles*, Springer, 2004, [doi:10.1007/978-3-642-17798-9].

[Ya20] H. Yan, *Geodesic string condensation from symmetric tensor gauge theory: a unifying framework of holographic toy models*, Phys. Rev. **B102** (2020) 161119, [arXiv:1911.01007].

David Corfield, *Centre for Reasoning, University of Kent, UK.*
d.corfield@kent.ac.uk

Hisham Sati, *Mathematics, Division of Science, New York University Abu Dhabi, UAE.*
hsati@nyu.edu

Urs Schreiber, *Mathematics, Division of Science, New York University Abu Dhabi, UAE; on leave from Czech Academy of Science, Prague.*
us13@nyu.edu