Gauss-Manin and Knizhnik-Zamolodchikov connections via abstract homotopy theory

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Abstract

We observe that the construction of Gauss-Manin connections on bundles of fiberwise cohomology groups over homotopy 1-types has a simple explanation in the context of parameterized homotopy theory and generally in higher topos theory. This is so simple that it becomes effectively a tautology when formulated in the corresponding formal language of homotopy type theory: The covering space which exhibits the flat Gauss-Manin connection is simply the fiberwise 0-truncation of the *fiberwise* mapping space (the internal hom-object in the slice) into the corresponding classifying space. This applies at once in the generality of twisted generalized and/or non-abelian cohomology theories, such as twisted K-theory and twisted Cohomotopy. Applied to twisted complex cohomology groups of configuration spaces of points in the plane, it yields the Knizhnik-Zamolodchikov (KZ) connection on bundles of $\widehat{\mathfrak{su}_2}^k$-conformal blocks, and thus the monodromy braid representation characteristic of $\mathfrak{su}_2$-anyons. We close by highlighting that the elegant reflection of this construction in homotopy type theory may provide a proper foundation for hardware-aware topological quantum programming.

Contents

1 Introduction

The following two facts are each "well-known" to their respective experts, but their striking conjunction has been pointed out only recently ([SS22-DBr][SS22-Ord][SS22-TQC]):

Fact 1.1 (**Topological quantum gates via Algebraic topology).**

(1) *Viable future topological quantum computation hardware realizes quantum gates which implement monodromy braid representations on* $\widehat{\mathfrak{su}_2}^k$*-conformal blocks;*

(2) *the "hypergeometric integral construction" identifies these with Gauss-Manin connections on fiberwise twisted cohomology groups of configurations spaces of points in the plane.*

This is striking, because the first item, taken at face value, invokes a fairly long and intricate sequence of constructions from conformal field theory and affine representation theory, while the second item invokes only the most fundamental concepts of algebraic topology: local systems on and cohomology of fibrations of configuration spaces.

This suggests [SS22-TQC] that theoretical reasoning about – and notably the programming and simulation of – topological quantum circuits will profitably proceed *through* the perspective of the second item above.

Concretely, there is already a programming language being developed, called *cohesive homotopy type theory* (cohesive HoTT), for fundamental constructions in differential topology and homotopy theory. Hence Fact 1.1 should imply that the encoding of realistic topological quantum gates in cohesive HoTT is fairly immediate – hence that this functionality is essentially *native* to cohesive HoTT, at least much more immediate than the traditional description of Knizhnik-Zamolodchikov connections on bundles of conformal blocks suggests.

This is what we prove here: we show that once data types for braid groups and for complex Eilenberg-MacLane spaces are given, then the construction of all monodromy braid representations on $\widehat{\mathfrak{su}_2}^k$-conformal blocks is essentially half a line of code. This same statement is made without proof at the end of the announcement article [SS22-TQC]. Here we discuss the proof.

Outline:

§2 builds a bridge between:
1. traditional discussion of Gauss-Manin connections ([Ma58][Gro66][Ka68][KO68][Gri70])
   here specifically for twisted cohomology on fiber bundles ([EFK98, §7.5][Vo03I, Def. 9.13] (e.g. [Ko02, §1.5, 2.1])),
2. abstract homotopy theory
   ([Br65][Qu67][Ad74], for review see [KP97][Ri14], here specifically [FSS20-Cha, §A][SS21-Bun, §3.1])

by proving that, on fiber bundles, these connections are witnessed by the *fiberwise mapping space*-construction known from parameterized point-set topology ([MaSi06], following [Bo70][BB78]);

§3 recalls how such a statement may be understood as a model for certain programs in HoTT ([UFP13]...) and explains the resulting quasi-native construction of Knizhnik-Zamolodchikov connections (e.g. [Ko02, §1.5, 2.1]) in cohesive HoTT.

2 Via parameterized point-set topology

Here we use (just the most basic aspects of) parameterized “point-set” homotopy theory (as laid out in [MaSi06], going back to [Bo70]) in order to show that Gauss-Manin connections in (twisted) generalized cohomology groups on fibers of bundles over 1-types are exhibited by the fiberwise 0th Postnikov stage of the fiberwise mapping space (fiberwise space of sections) into the given classifying space (classifying fibration).

With the relevant notions and results from parameterized point-set homotopy theory in hand, the proof is straightforward, so we use the occasion to briefly introduce and review basics of paramaterized topology as we go along, in order to make the proof reasonably self-contained also for a general mathematical audience.

The construction in itself of the Gauss-Manin connection on fiberwise twisted cohomology groups of locally trivial fiber bundles may be understood without the abstract machinery invoked here; a sketch of such a more low-brow argument is what [EFK98, §7.5] offers. However, it is our abstract re-formulation which provides an elegant handle on Gauss-Manin connections in the language of homotopy type theory – this is discussed in §3 below.

2.1 For generalied and non-abelian cohomology

Given a sufficiently nice fibration $p_{\mathrm{X}} : \mathrm{X} \to \mathrm{B}$, the ordinary complex cohomology groups $H^n(\mathrm{X}_b) := H^n(\mathrm{X}_b; \mathbb{C})$ of its fibers X_b for $b \in B$, form a bundle of abelian groups over B equipped with a flat connection, known as the *Gauss-Manin connection*. Hence a Gauss-Manin connection provides a rule for coherently transporting cohomology classes of spaces X_b as these spaces *vary with a parameter*, the parameter being the point $b \in \mathrm{B}$. Moreover, the flatness of the connection means that the induced transport of cohomology classes depends on parameter paths $\gamma : [0,1] \to \mathrm{B}$ only via their homotopy classes $[\gamma]$ relative to their endpoints. If B is connected, this means equivalently that the Gauss-Manin connection is a homomorphism from the fundamental group of B to the automorphism group of $H^n(X_{b_\bullet}; A)$ at any fixed $b_\bullet$.

$$\begin{tikzcd}[column sep=small]
\text{Category of sets} & \mathrm{Set} & & \overset{\text{Fiberwise cohomology}}{H^n(\mathrm{X}_{b_2})} \arrow[dr, "\sim"] & & \mathrm{X} \arrow[rr, dashed] \arrow[dd, "p_{\mathrm{X}}"', "\text{Fiber bundle}"] & & \overset{\text{Classifying space}}{K(n,\mathbb{C})} \\
\text{Gauss-Manin connection} & & H^n(\mathrm{X}_{b_1}) \arrow[ur, "\sim"] \arrow[rr, "\sim"'] & & H^n(\mathrm{X}_{b_3}) & & & \\
\text{Fundamental path groupoid} & \mathrm{Pth}(\mathrm{B}) \arrow[uu] & & \{b_2\} \arrow[dr, "{[\gamma_{23}]}"] & & \mathrm{B} \arrow[dr, "p_{\mathrm{B}}"'] & & \\
& & \{b_1\} \arrow[ur, "{[\gamma_{12}]}"] \arrow[rr, "{[\gamma_{23} \circ \gamma_{12}]}"'] & & \{b_3\} & & * &
\end{tikzcd} \tag{1}$$

In fact, this applies also to *twisted* cohomology groups, in which case the Knizhnik-Zamolodchikov connection becomes a special case. We come back to this in a moment (§2.2).

Traditionally, Gauss-Manin connections are constructed algebraically. Here we work entirely homotopy-theoretically and instead make use of the fact that twisted ordinary cohomology of a topological space of CW-type is *representable*, in that for $n \in \mathbb{N}$ there exists a topological space $\mathrm{K}(n,\mathbb{C})$ (the nth *Eilenberg-MacLane space*) such that cohomology is identified with the connected components of the *mapping space* into it:

$$H^n(\mathrm{X}_b) \simeq \pi_0 \mathrm{Map}\big(\mathrm{X}_b, \mathrm{K}(\mathbb{C}, n)\big) . \tag{2}$$

Mapping spaces. For mapping spaces to work well we may assume without practical restriction that we work in the category kTopSpc of compactly generated space (k-spaces, see [SS21-Bun, Nota. 1.0.15] for pointers). Here, for $\mathrm{X}_b \in \mathrm{kTopSpc}$, we have an adjunction

$$\begin{tikzcd}
\mathrm{kTopSpc} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathrm{Map}(\mathrm{X}_b, -)"'] & \mathrm{kTopSpc} \arrow[l, bend right, "\mathrm{X}_b \times (-)"']
\end{tikzcd} , \tag{3}$$

meaning that there is an *exponential law* for k-topological spaces, namely a natural bijection of the form

$$\mathrm{kTopSpc}\big(\mathrm{X}, \mathrm{Map}(\mathrm{Y}, \mathrm{Z})\big) \simeq \mathrm{kTopSpc}(\mathrm{X} \times \mathrm{Y}, \mathrm{Z}) .$$

Generalized cohomology. The following construction of the Gauss-Manin connection over fiber bundles relies only on the existence of such a classifying space, as in (2), but not on its concrete nature. This means that the construction applies also to “generalized cohomology theories”. If, for instance, $E^n(-)$ is a Whitehead-generalized cohomology theory, such as topological K-theory, elliptic cohomology or cobordism cohomology, then there exists a *spectrum* of classifying spaces $\{E_n\}_{n\in\mathbb{N}}$ such that

$$E^n(X_b) \;\simeq\; \pi_0\mathrm{Map}(\mathrm{X}_b,\, E_n)\,.$$

Regarded the other way around, for *any* topological space $\mathrm{A} \in \mathrm{kTopSpc}$ we may regard

$$A^0(X_b) \;:=\; \pi_0\mathrm{Map}(\mathrm{X}_b,\, \mathrm{A}) \tag{4}$$

as *non-abelian generalized cohomology* with coefficients in A. For example, if $\mathrm{A} = BG$ is the classifying space of a discrete or compact Lie group G, then

$$BG^0(X_b) \simeq H^1(X_b; G)$$

is equivalently the traditional non-abelian cohomology in degree 1 with coefficients in G, which classifies G-principal bundles. Or if $\mathrm{A} = S^n \subset \mathbb{R}^{n+1}$ is the topological n-sphere, then

$$(S^n)^0(X_b) \simeq \pi^n(X_b)$$

is unstable *Cohomotopy*.

The fiberwise mapping space. The key fact now is that in *parameterized homotopy theory* ([CJ98][BM21][MaSi06]...) the mapping space construction (3) generalizes to *slices* if the base space B is (compactly generated and) Hausdorff, which we assume from now on:

$$B \in \mathrm{kHaus} \hookrightarrow \mathrm{kTopSpc}\,. \tag{5}$$

Here the *slice category* $\mathrm{kTopSpc}_{/\mathrm{B}}$ is the category whose objects $(\mathrm{X}, p_{\mathrm{X}})$ are k-topological space X equipped with a continuous map $p_{\mathrm{X}} : \mathrm{X} \to \mathrm{B}$ and whose morphisms $(\mathrm{X}, p_{\mathrm{X}}) \to (\mathrm{Y}, p_{\mathrm{Y}})$ are compatible maps $\mathrm{X} \to \mathrm{Y}$, hence:

$$\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}}\big((\mathrm{X},p_{\mathrm{X}}),\,(\mathrm{Y},p_{\mathrm{Y}})\big) \;=\; \mathrm{kTopSpc}\big(\mathrm{X},\,\mathrm{Y}\big) \underset{\mathrm{kTopSpc}(\mathrm{X},\mathrm{B})}{\times} \{p_{\mathrm{X}}\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{kTopSpc}\big(\mathrm{X},\,\mathrm{Y}\big) \arrow[d, "p_{\mathrm{Y}}\circ(-)"] \\
\{p_{\mathrm{X}}\} \arrow[r, hook] & \mathrm{kTopSpc}\big(\mathrm{X},\,\mathrm{B}\big).
\end{tikzcd} \tag{6}$$

If we understand X as an object in the *slice category* $\mathrm{kTopSpc}_{/\mathrm{B}}$ over B via p_{X} (1) and if we denote by $p_{\mathrm{B}}^{*}\mathrm{A}$ the trivial fiber bundle over B with fiber A regarded in the slice category, then their *fiberwise mapping space* is a topological space which is itself fibered over B, such that the fibers of the fiberwise mapping space are the ordinary mapping spaces (3) on the fibers:

$$\begin{tikzcd}
\overset{\text{Ordinary mapping space on fiber}}{\mathrm{Map}\big(\mathrm{X}_b,\,\mathrm{A}\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \overset{\text{Fiberwise mapping space (itself a topological space over B)}}{\mathrm{Map}\big((\mathrm{X},p_{\mathrm{X}}),\,p_{\mathrm{B}}^{*}\mathrm{A}\big).} \arrow[d] \\
* \arrow[r, hook, "b"] & \underset{\text{Parameter space}}{\mathrm{B}}
\end{tikzcd} \tag{7}$$

The right choice of topology on the total fiberwise mapping space is subtle[1] and the result that such a topology exists (see [MaSi06, §1.3.7-§1.3.9], following [BB78, Thm. 3.5]) may be regarded as the engine which drives our quick re-construction of Gauss-Manin connections from the point of view of point-set topology. Here the right topology is that which ensures the sliced analog of the adjunction (3)

$$\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}} \arrow[r, phantom, "\perp"] \arrow[r, leftarrow, shift left=2, "(\mathrm{X},p_{\mathrm{X}})\times(-)"] & \mathrm{kTopSpc}_{/\mathrm{B}}\,, \arrow[l, leftarrow, shift left=2, "\mathrm{Map}((\mathrm{X},p_{\mathrm{X}}),-)"]
\end{tikzcd} \tag{8}$$

hence, equivalently, the *exponential law* in the slice [BB78, Thm. 3.5][MaSi06, (1.3.9)]: a natural bijection of the form

$$\mathrm{kTopSpc}_{/B}\Big((\mathrm{X},p_{\mathrm{X}}),\overset{\text{Fiberwise mapping space}}{\mathrm{Map}\big((\mathrm{Y},p_{\mathrm{Y}}),\,(\mathrm{Z},p_{\mathrm{Z}})\big)}\Big) \;\simeq\; \mathrm{kTopSpc}_{/B}\Big(\overset{\text{Fiberwise product space}}{(\mathrm{X},p_{\mathrm{X}})\times(\mathrm{Y},p_{\mathrm{Y}})},(\mathrm{Z},p_{\mathrm{Z}})\Big), \tag{9}$$

where now the product on the right is that in the slice, hence is the *fiber product* of k-topological spaces:

$$(\mathrm{X},p_{\mathrm{X}})\times(\mathrm{Y},p_{\mathrm{Y}}) \;\simeq\; \big(\mathrm{X}\times_{\mathrm{B}}\mathrm{Y},\, p_{\mathrm{X}}\circ\mathrm{pr}_{\mathrm{X}} = p_{\mathrm{Y}}\circ\mathrm{pr}_{\mathrm{Y}}\big)\,.$$

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[1] “*The point-set topology of parametrized spaces is surprisingly subtle. Parametrized mapping spaces are especially delicate.*” [MaSi06, p. 15]

Of course, the unit for this product is the identity map on the base space:

$$(\mathrm{X}, p_{\mathrm{X}}) \times (\mathrm{B}, \mathrm{id}_{\mathrm{B}}) \;\simeq\; (\mathrm{X}, p_{\mathrm{X}})\,. \tag{10}$$

This exponential law in slices implies a wealth of useful structure:

Proposition 2.1 (Base change adjoint triple). *For any map between base spaces $f : \mathrm{B} \to \mathrm{B}'$, there is a base change adjoint triple*

$$\begin{tikzcd} \mathrm{kTopSpc}_{/\mathrm{B}} \arrow[r, shift left=3ex, "f_!"] \arrow[r, phantom, shift left=1.5ex, "\perp"] \arrow[r, phantom, shift right=1.5ex, "\perp"] \arrow[r, shift right=3ex, "f_*"'] & \mathrm{kTopSpc}_{/\mathrm{B}'}\,, \arrow[l, "f^*" description] \end{tikzcd} \tag{11}$$

where f^* denotes the pullback operation (formed in kTopSpc*), its the left adjoint $f_!$ is given by postcomposition with f, and the right adjoint f_* is given by the following pullback construction:*

$$\begin{tikzcd} f_*(\mathrm{X}, p_{\mathrm{X}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{pb})"] & \mathrm{Map}\big((\mathrm{B}, f),\, (\mathrm{X}, f \circ p_{\mathrm{X}})\big) \arrow[d] \\ (\mathrm{B}', \mathrm{id}_{\mathrm{B}'}) \arrow[r, "\widetilde{\mathrm{id}}"'] & \mathrm{Map}\big((\mathrm{B}, f),\, (\mathrm{B}, f)\big)\,. \end{tikzcd} \tag{12}$$

Proof. For the left adjoint $(f_! \dashv f^*)$ the required hom-isomorphism is immediate from the universal property of the pullback:

$$\begin{tikzcd} \mathrm{X} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{pb})"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] \\ & \mathrm{B} \arrow[r, "f"'] & \mathrm{B}'\,. \end{tikzcd}$$

For the right adjoint, the required hom-isomorphism is obtained as the following sequence of natural isomorphisms:

$$\begin{aligned} & \mathrm{kTopSpc}_{/\mathrm{B}'}\big((U, p_{\mathrm{U}}),\, f_*(\mathrm{X}, p_{\mathrm{X}})\big) \\ \simeq\; & \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U, p_{\mathrm{U}}),\, \mathrm{Map}\Big((\mathrm{B}, f),\, (\mathrm{X}, f \circ p_{\mathrm{X}})\Big) \underset{\mathrm{Map}\left((\mathrm{B}, f), (\mathrm{B}, f)\right)}{\times} \{\widetilde{\mathrm{id}}\}\Big) && \text{by (12)} \\ \simeq\; & \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U, p_{\mathrm{U}}),\, \mathrm{Map}\big((\mathrm{B}, f),\, (\mathrm{X}, f \circ p_{\mathrm{X}})\big)\Big) \underset{\mathrm{Map}\left((U, p_{\mathrm{U}}), \mathrm{Map}\left((\mathrm{B}, f), (\mathrm{B}, f)\right)\right)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (6)} \\ \simeq\; & \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U, p_{\mathrm{U}}) \times (\mathrm{B}, f),\, (\mathrm{X}, f \circ p_{\mathrm{X}})\Big) \underset{\mathrm{kTopSpc}_{/\mathrm{B}'}\left((U, p_{\mathrm{U}}) \times (\mathrm{B}, f), (\mathrm{B}, f)\right)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (9)} \\ \simeq\; & \Big(\mathrm{kTopSpc}(f^*\mathrm{U}, \mathrm{X}) \underset{\mathrm{kTopSpc}(f^*\mathrm{U}, \mathrm{B}')}{\times} *\Big) \underset{\left(\mathrm{kTopSpc}(f^*\mathrm{U}, \mathrm{B}) \underset{\mathrm{kTopSpc}(f^*\mathrm{U}, \mathrm{B}')}{\times} *\right)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (6)} \\ \simeq\; & \mathrm{Map}\big(f^*\mathrm{U}, \mathrm{X}\big) \underset{\mathrm{Map}(f^*\mathrm{U}, \mathrm{B})}{\times} * && \text{by (13)} \\ \simeq\; & \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*(\mathrm{U}, p_{\mathrm{U}}),\, (\mathrm{X}, p_{\mathrm{X}})\big) && \text{by (6)}\,. \end{aligned}$$

Here the penultimate step is observing that the fiber products (limits) may be interchanged: Instead of computing the horizontal fiber product of the vertical fiber product in the following diagram, we may first compute the horizontal fiber products (shown on the right, again by (6))

$$\begin{tikzcd} \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{X}\big) \arrow[r, "p_{\mathrm{X}} \circ (-)"] \arrow[d, "f \circ p_{\mathrm{X}} \circ (-)"'] & \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}\big) \arrow[d, "f \circ (-)"] & * \arrow[l] \arrow[d] & & \mathrm{kTopSpc}_{/\mathrm{B}}\big(f * (\mathrm{U}, p_{\mathrm{U}}),\, (\mathrm{X}, p_{\mathrm{X}})\big) \arrow[d] \\ \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}'\big) \arrow[r, "\mathrm{id}"] & \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}'\big) & * \arrow[l] & & * \\ * \arrow[u] \arrow[r] & * \arrow[u] & * \arrow[l] \arrow[u] & & * \arrow[u] \end{tikzcd} \tag{13}$$

and then the evident remaining vertical fiber product is as claimed. □

Example 2.2 (Space of sections). When $\mathrm{B}' = *$ in (11), the right base change (12) constructs *spaces of sections*:

$$\Gamma_{\mathrm{X}_b}(-) \;\simeq\; (p_{\mathrm{X}_b})_* \,:\, \mathrm{kTopSpc}_{/\mathrm{X}_b} \longrightarrow \mathrm{kTopSpc}\,. \tag{14}$$

Generally, one may understand the right base change as forming *fiberwise spaces of sections*.

Proposition 2.3 (Cartesian Frobenius reciprocity). *For $f : \mathrm{B} \to \mathrm{B}'$ a map of base spaces* (5)*, we have a natural isomorphism*

$$f_!\big((\mathrm{X}, p_{\mathrm{X}}) \times f^*(\mathrm{Y}, p_{\mathrm{Y}})\big) \;\simeq\; \big(f_!(\mathrm{X}, p_{\mathrm{X}})\big) \times (\mathrm{Y}, p_{\mathrm{Y}})\,, \tag{15}$$

where $(f_! \dashv f^*)$ is the left base change adjunction (11)*.*

Proof. This follows by the pasting law, which says that the following pullback squares in kTopSpc agree:

$$\begin{tikzcd}
\mathrm{X} \times_{\mathrm{B}} f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] & \simeq & \mathrm{X} \times_{\mathrm{B}'} \mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Y} \arrow[d] \\
\mathrm{X} \arrow[r, "p_{\mathrm{X}}"'] & \mathrm{B} \arrow[r, "f"'] & \mathrm{B}' & & \mathrm{X} \arrow[r, "f \circ p_{\mathrm{X}}"'] & \mathrm{B}'.
\end{tikzcd}$$

□

In generalization of (7), we have:

Proposition 2.4 (Base change is closed functor). *The pullback (base change) of a fiberwise mapping space along any continuous map of base spaces* (5) *$f : B' \longrightarrow B$ is the fiberwise mapping space of the pullbacks of the arguments:*

$$f^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big) \;\simeq\; \mathrm{Map}\big(f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\big)\,. \tag{16}$$

Proof. Apply the Yoneda lemma over $\mathrm{kTopSpc}^{\mathrm{op}}_{/B}$ to the following sequence of natural isomorphisms:

$$\begin{aligned}
&\mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}),\, f^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(f_!(\mathrm{U}, p_{\mathrm{U}}),\, \mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) && \text{by (11)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(\big(\big(f_!(\mathrm{U}, p_{\mathrm{U}})\big) \times (\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (9)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(f_!\big((\mathrm{U}, p_{\mathrm{U}}) \times f^*(\mathrm{X}, p_{\mathrm{X}})\big),\, (\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (15)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}) \times f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (11)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}),\, \mathrm{Map}\big(f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) && \text{by (9)}\,.
\end{aligned}$$

□

In the case at hand this has the consequence that

Proposition 2.5 (Fiberwise mapping space out of fiber bundle is fiber bundle). *If $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ is a fiber bundle with local trivialization*

$$\phi : \mathrm{U} \underset{\mathrm{opn}}{\twoheadrightarrow} \mathrm{B}\,, \qquad \textit{such that} \qquad \phi^*(\mathrm{X}, p_{\mathrm{X}}) \simeq p_{\mathrm{U}}^*\mathrm{X}_0 \;:=\; \mathrm{U} \times \mathrm{X}_0\,, \tag{17}$$

then the fiberwise mapping space is a fiber bundle locally trivial with respect to the same open cover:

$$\begin{aligned}
\phi^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*A\big) &\;\simeq\; \mathrm{Map}\big(\phi^*(\mathrm{X}, p_{\mathrm{X}}),\, \phi^*p_{\mathrm{B}}^*\mathrm{A}\big) && \text{by (16)} \\
&\;\simeq\; \mathrm{Map}\big(p_{\mathrm{U}}^*\mathrm{X}_0,\, p_{\mathrm{U}}^*\mathrm{A}\big) && \text{by (17)} \\
&\;\simeq\; p_{\mathrm{U}}^*\mathrm{Map}\big(\mathrm{X}_0,\, \mathrm{A}\big) && \text{by (16).}
\end{aligned} \tag{18}$$

Fiberwise homotopy theory. Moreover, the topology on the fiberwise mapping space (7) is also "homotopy correct" in that its map to the base B is an h-fibration as soon as p_{X} is an h-fibration (by [Bo70, §6.1], see also [MaSi06, Prop. 1.3.11]), which is the case for B a metrizable space and p_{X} a fiber bundle. This implies that the fibers of the fiberwise mapping space are in fact homotopy fibers, and that forming path ∞-groupoids (singular simplicial complexes),

$$\begin{tikzcd}
\mathrm{kTopSpc} \arrow[r, shift right, "\mathrm{Pth}"'] \arrow[r, phantom, "\perp"] & \Delta\mathrm{Set} \arrow[l, shift right, "|-|"']
\end{tikzcd}$$

respects this property:

$$\begin{tikzcd}
\mathrm{Pth}\,\mathrm{Map}\big(\mathrm{X}_b,\, \mathrm{A}\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{pb})"] & \mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^{*}\mathrm{A}\big). \arrow[d, "\in \mathrm{KanFib}"] \\
* \arrow[r, "b"'] & \mathrm{Pth}(\mathrm{B})
\end{tikzcd} \tag{19}$$

Lemma 2.6 (Fiberwise truncation preserves homotopy fiber products). *For any Kan complex $S \in \Delta\mathrm{Set}_{\mathrm{fib}}$, the operation of fiberwise Postnikov truncation*

$$\pi_{0/S} \,:\, \Delta\mathrm{Set}_{/S} \longrightarrow \Delta\mathrm{Set}_{/S}$$

preserves homotopy products in the slice, hence homotopy fiber products over S.

Proof. This follows for instance with [Lu09, Lem. 6.5.1.2], using that the Kan-model structure on ΔSet presents the ∞-topos of ∞-groupoids. ☐

But iff B is 1-truncated, then any point inclusion is 0-truncated as an object in the slice, its homotopy fiber being the fundamental group of B at that point:

$$\begin{tikzcd}
\overset{\text{discrete}}{\underset{\text{set}}{}}\ \pi_1(\mathrm{B}, b) \arrow[r, "\mathrm{hfib}(b)"] & * \arrow[d, "b"] \\
 & \mathrm{Pth}(\mathrm{B}),
\end{tikzcd}
\qquad \pi_{0/\mathrm{Pth}(\mathrm{B})}(*, b) \;\simeq\; (*, b);$$

Therefore, applying Lemma 2.6 to the diagram (19), regarded as exhibiting a homotopy fiber product over $\mathrm{Pth}(\mathrm{B})$, we obtain a homotopy pullback diagram as shown on the left here:

$$\begin{tikzcd}
\overset{\substack{A\text{-cohomology} \\ \text{of fiber}}}{A^0(\mathrm{X}_b)} \simeq \pi_0\Big(\mathrm{PthMap}\big(\mathrm{X}_b,\, \mathrm{A}\big)\Big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{hpb})"] & \overset{\text{Fiberwise 0-truncation of (path } \infty\text{-groipoid of) fiberwise mapping space}}{\pi_{0/\mathrm{Pth}(\mathrm{B})}\Big(\mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^{*}\mathrm{A}\big)\Big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{hpb})"] & \mathrm{Set}^{*/} \arrow[d, "\substack{\text{Covering space} \\ \text{classifier}}"] \\
\{b\} \arrow[r] & \underset{\substack{\text{Fundamental} \\ \text{groupoid of} \\ \text{base space}}}{\mathrm{Pth}(\mathrm{B})} \arrow[r, "\nabla^{\mathrm{GM}}_{X,A}", "\text{Gauss-Manin connection}"'] & \mathrm{Set}.
\end{tikzcd} \tag{20}$$

The left square shows that the fiberwise 0-truncation of the fiberwise mapping space is a fibration over the fundamental groupoid of B, whose (homotopy) fibers are the generalized cohomology sets (4) of the fiber space X_b. The homotopy pullback shown on the right follows by:

Lemma 2.7 (Univalent universe of sets). *Any homotopy fibration of sets, as in the middle of* (20) *is classified by – i.e. is the homotopy pullback along – an essentially unique map $\nabla^{\mathrm{GM}}_{X,A}$ to the covering space classifier, as shown in the square on the right of* (20)*.*

Proof. This may be understood as a simple special case of the general fact that ∞-groupoids form an ∞-topos in which there exists a "small fibration classifier" $\mathrm{Grpd}_\infty^{*/} \to \mathrm{Grpd}_\infty$ ([Lu09, Prop. 3.3.2.7][Ci19, §5.2][KL21]). ☐

Remark 2.8 (Flat connections as functors on the fundamental groupoid). Noticing that $\mathrm{Pth}(\mathrm{B})$ is equivalently the disjoint union over connected components $[b] \in \pi_0(\mathrm{B})$ of delooping groupoids $\mathbf{B}\pi_1(\mathrm{B}, b)$, this map $\nabla^{\mathrm{GM}}_{X,A}$ (20) is over each connected component equivalently a group homomorphism

$$\Omega_b\big(\nabla^{\mathrm{GM}_{X,A}}\big) \,:\, \pi_1(\mathrm{B}, b) \longrightarrow \mathrm{Aut}\big(A^0(\mathrm{X}_b)\big)\,.$$

This is a traditional incarnation of flat connections on a space B (e.g. [De70, §I.1][Di04, Prop. 2.5.1]).

Moreover, from Prop. 2.5 it follows that this local system of sets trivializes over any cover over which p_{X} trivializes, so that it corresponds to a *covering space* which we denote as follows:

$$\underset{b \in \mathrm{B}}{\forall} \qquad \begin{tikzcd}
\overset{\substack{\text{A-cohomology} \\ \text{of fiber}}}{A^0(\mathrm{X}_b)} = \overset{\substack{\text{connected components of} \\ \text{ordinary mapping space on fiber}}}{\pi_0\mathrm{Map}\big(\mathrm{X}_b,\, \mathrm{A}\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle (\mathrm{pb})"] & \overset{\substack{\text{parameterized connected components of} \\ \text{fiberwise mapping space (a covering space over B)}}}{\pi_{0/\mathrm{B}}\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^{*}\mathrm{A}\big)} \arrow[d] \\
* \arrow[r, hook, "b"] & \underset{\text{parameter space}}{\mathrm{B}\,.}
\end{tikzcd} \tag{21}$$

Using this, comparison with [Vo03I, Def. 9.13] readily shows that the flat connection $\nabla^{\mathrm{GM}}_{X,A}$ in (20) is indeed the Gauss-Manin connection. In conclusion, we have shown so far:

Theorem 2.9 (**Gauss-Manin connection in generalized cohomology over fiber bundles via fiberwise mapping spaces).** *If* $\mathrm{B} \in \mathrm{kHaus}$ *is a metrizable homotopy 1-type* ($\pi_{n\geq 2}(\mathrm{B},-)=0$) *and* $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ *is a locally trivial fiber bundle whose typical fiber admits the structure of a CW-complex, then for any* $\mathrm{A} \in \mathrm{kTopSpc}$ *the Gauss-Manin-connection on the A-cohomology sets* (4) *of the fibers* X_b *is exhibited (under Lem. 2.7) by the fiberwise 0-truncation of the fiberwise mapping space* (8) *from* X *into* A*:*

$$\underset{\substack{\text{Gauss-Manin}\\ \text{connection on}\\ \text{A-cohomology}}}{\nabla^{\mathrm{GM}}_{X,A}} \underset{\text{Lem. 2.7}}{\longleftrightarrow} \underset{\text{fiberwise 0-truncation of fiberwise mapping space into A}}{\pi_{0/\mathrm{Pth(B)}}\Big(\mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*\mathrm{A}\big)\Big)}\,.$$

2.2 For twisted generalized cohomology

We generalize the above discussion to the case of fiberwise *twisted* cohomology (Thm. 2.13) and bring out the motivating example of the $\widehat{\mathfrak{su}_2}^k$-Knizhnik-Zamolodchikov equation (Ex. 2.14).

In existing literature this is discussed for the special case that the total space X is equipped with a flat complex line bundle $\mathscr{L}$ classified by a map $\tau : \mathrm{X} \to B\mathrm{U}(1)$. In this case one may consider the τ_b-twisted complex ordinary cohomology of the fibers, namely the cohomology with coefficient in the local system of parallel local sections of $\mathscr{L}$ (e.g. [Vo03II, §5.1.1]):

$$H^{n+\tau_b}(\mathrm{X}_b;\mathbb{C}) = H^n(\mathrm{X}_b;\mathscr{L})\,.$$

At least when p_{X} is a fiber bundle, these twisted cohomology groups again carry a flat Gauss-Manin connection. In the example where $\mathrm{X} = \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{R}^2)$ is a configuration space of points, and p_X the map that forgets the first n of $n+N$ points, then a *hyprgeometric integral construction* identifies this Gauss-Manin-connection on fiberwise twisted complex cohomology with a Knizhnik-Zamolodchikov connection ([EFK98, §7.5]).

The above is the main example of interest for us. However, in [SS22-DBr][SS22-Ord], we explained that it is useful to regard this twisted ordinary cohomology as the home of the twisted Chern characters of twisted *K-theory* groups. For this reason we are interested in speaking of Gauss-Manin connections on bundles of twisted *generalized* cohomology groups.

$$\begin{tikzcd}
\text{Category of sets} & \mathrm{Set} \\
\\
& \mathrm{Pth(B)} \arrow[uu, "\substack{\text{Gauss-Manin}\\ \text{connection}}"'] \\
\text{Fundamental path groupoid}
\end{tikzcd}$$

$$\begin{tikzcd}
& \substack{\text{Fiberwise}\\ \text{twisted cohomology}}\\ A^{\tau_{b_2}}(\mathrm{X}_{b_2}) \arrow[dr, "\sim"] & \\
A^{\tau_{b_1}}(\mathrm{X}_{b_1}) \arrow[ur, "\sim"] \arrow[rr, "\sim"'] & & A^{\tau_{b_3}}(\mathrm{X}_{b_3}) \\
& \{b_2\} \arrow[dr, "{[\gamma_{23}]}"] & \\
\{b_1\} \arrow[ur, "{[\gamma_{12}]}"] \arrow[rr, "{[\gamma_{23}\circ\gamma_{12}]}"'] & & \{b_3\}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{X} \arrow[rr, dashed] \arrow[dr, "\tau"', "\text{global twist}"] \arrow[dd, "p_{\mathrm{X}}"', "\substack{\text{fiber}\\ \text{bundle}}"] & & \mathrm{A}\times_{\mathrm{G}} E\mathrm{G} \arrow[dl, "p_{\mathrm{A}\times_G EG}"] \\
& B\mathrm{G} \arrow[dd, "p_{BG}"] & \text{Classifying fibration} \\
\mathrm{B} \arrow[dr, "p_{\mathrm{B}}"'] & & \\
& * &
\end{tikzcd}$$

Twisted generalized non-abelian cohomology. In generalization of (4), we have:

Definition 2.10 (**Twisted generalized non-abelian cohomology [FSS20-Cha, §2.2][SS20-Orb, Rem. 2.94]).** For

- $\mathrm{G} \in \mathrm{Grp}(\mathrm{kTopSpc})$ a topological group,
- $\mathrm{G} \curvearrowright \mathrm{A} \in \mathrm{G}\,\mathrm{Act}\big(\mathrm{kTopSpc}\big)$ a topological G-space,
  with $\big(\mathrm{A}\times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}}\big) \in \mathrm{kTopSpc}_{/B\mathrm{G}}$ its Borel construction,
- $\tau_b : \mathrm{X}_b \to B\mathrm{G}$ a continuous map,

we say that

$$\begin{aligned}
A^{\tau_b}(\mathrm{X}_b) &:= H^{\tau_b}\big(\mathrm{X}_b;\mathrm{A}\big) \\
&:= \pi_0\Gamma_{\mathrm{X}}\big((\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} E\mathrm{G})\big) \\
&\simeq (p_{B\mathrm{G}})_*\mathrm{Map}\Big(\underbrace{(\mathrm{X}_b,\tau_b)}_{(\tau_b)_!(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b})},\,(\mathrm{A}\times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\Big) \quad \text{by (23)}
\end{aligned} \tag{22}$$

is the τ_b*-twisted A-cohomology* of X_b.

To see the identification shown in (22), of the space of sections with a right base change of the fiberwise mapping space over the classifying space of twists, apply the Yoneda Lemma to the following sequence of natural bijections:

$$
\begin{aligned}
& \mathrm{kTopSpc}\Big(\mathrm{U},\, (p_{B\mathrm{G}})_* \mathrm{Map}\big((\tau_b)_!(\mathrm{X}_b, \mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\big)\Big) \\
\simeq\; & \mathrm{kTopSpc}_{/B\mathrm{G}}\Big((p_{B\mathrm{G}})^*\mathrm{U},\, \mathrm{Map}\big((\tau_b)_!(\mathrm{X}_b, \mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\big)\Big) && \text{by (11)} \\
\simeq\; & \mathrm{kTopSpc}_{/B\mathrm{G}}\Big(\big((p_{B\mathrm{G}})^*\mathrm{U}\big) \times (\tau_b)_!(\mathrm{X}_b, \mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\Big) && \text{by (9)} \\
\simeq\; & \mathrm{kTopSpc}_{/B\mathrm{G}}\Big((\tau_b)_!\big((p_{\mathrm{X}_b})^*\mathrm{U} \times (\mathrm{X}_b, \mathrm{id}_{\mathrm{X}_b})\big),\, (\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\Big) && \text{by (15)} \\
\simeq\; & \mathrm{kTopSpc}_{/\mathrm{X}_b}\Big((p_{\mathrm{X}_b})^*\mathrm{U} \times (\mathrm{X}_b, \mathrm{id}_{\mathrm{X}_b}),\, (\tau_b)^*(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\Big) && \text{by (11)} \\
\simeq\; & \mathrm{kTopSpc}_{/\mathrm{X}_b}\Big((p_{\mathrm{X}_b})^*\mathrm{U},\, (\tau_b)^*(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\Big) && \text{by (10)} \\
\simeq\; & \mathrm{kTopSpc}\Big(\mathrm{U},\, (p_{\mathrm{X}_b})_*\big((\tau_b)^*(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\big)\Big) && \text{by (11)} \\
\simeq\; & \mathrm{kTopSpc}\Big(\mathrm{U},\, \Gamma_{\mathrm{X}_b}\big((\tau_b)^*(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\big)\Big) && \text{by (14)}\,.
\end{aligned}
\tag{23}
$$

Example 2.11 (Ordinary complex cohomology with coefficients in a local system). For each $n \in \mathbb{N}$ the canonical action of $\mathrm{U}(1) \subset \mathbb{C}^\times$ on $\mathbb{C}$ induces an action on the nth Eilenberg-MacLane space $\mathrm{U}(1) \circlearrowright \mathrm{K}(\mathbb{C}, n)$. For $\tau_b : \mathrm{Pth}(\mathrm{X}_b) \to \mathbf{B}\mathrm{U}(1)$ the classifying map of a flat connection (Rem. 2.8) on a complex line bundle (i.e. on the connected component $[b]$ a group homomorphism $\pi_1(\mathrm{B}, b) \to \mathrm{U}(1)$), the corresponding twisted cohomology according to Def. 2.10 is the traditional cohomology with coefficients in the local system $\mathscr{L}(\tau)$ of parallel sections of this flat connection (e.g. [Vo03I, §5.1.1]):

$$
\mathrm{K}(\mathbb{C}, n)^\tau(\mathrm{X}_b) \;=\; H^{n+\tau}(\mathrm{X}_b; \mathbb{C}) \;\simeq\; H^n\big(\mathrm{X}_b; \mathscr{L}(\tau)\big)\,.
$$

Now given $\mathrm{G} \circlearrowright \mathrm{A}$ as in Def. 2.10, and a fibration $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ as before in §2.1, consider a choice of *global twist*, namely a continuous map

$$
\tau : \mathrm{X} \longrightarrow B\mathrm{G}\,.
$$

Via this twist we may regard X as fibered over the product space $\mathrm{B} \times B\mathrm{G}$, whence its fibers are also still fibered over $B\mathrm{G}$

$$
\big(\mathrm{X}, (p_{\mathrm{X}}, \tau)\big) \;\in\; \mathrm{kTopSpc}_{/\mathrm{B}\times B\mathrm{G}}\,, \qquad \Rightarrow \qquad \underset{b \in \mathrm{B}}{\forall} \quad (i_b \times \mathrm{id}_B \mathrm{G})^*\big(\mathrm{X}, (p_{\mathrm{X}}, \tau)\big) \;=\; (\mathrm{X}_b, \tau_b) \;=\; (\tau_b)_! \mathrm{X}_b \quad \in \mathrm{kTopSpc}_{/B\mathrm{G}}\,.
$$

Forming the fiberwise mapping space (8) in this sense, we obtain the following twisted generalization of (7):

$$
\begin{tikzcd}
\text{Fiberwise mapping space over classifying space of twists} & \text{Fiberwise mapping space (itself a topological space over } \mathrm{B} \times B\mathrm{G}\text{)} \\
\mathrm{Map}\big((\mathrm{X}_b, \tau_b),\, (A \times_{\mathrm{G}} E\mathrm{G}, p_{A\times_{\mathrm{G}}E\mathrm{G}})\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Map}\big(\big(\mathrm{X}, (p_{\mathrm{X}}, \tau)\big),\, p_{\mathrm{B}}^*(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A}\times_{\mathrm{G}}E\mathrm{G}})\big). \arrow[d] \\
\{b\} \times B\mathrm{G} \arrow[r, hook, "i_b \times \mathrm{id}_{B\mathrm{G}}"'] & \mathrm{B} \times B\mathrm{G} \\
\text{Classifying space of twists} & \text{Space of parameters and twists}
\end{tikzcd}
\tag{24}
$$

In order to turn this into a pullback diagram over just B we need the *Beck-Chevalley relation*:

Proposition 2.12 (Cartesian Beck-Chevalley property). *Given a fiber product diagram in* kTopSpc *as shown on the left below, the possible composite base changes* (11) *through the diagram are naturally isomorphic as shown on the right:*

$$
\begin{tikzcd}
& \mathrm{X} \times_{\mathrm{B}} \mathrm{Y} \arrow[dl, "\mathrm{pr}_{\mathrm{X}}"'] \arrow[dr, "\mathrm{pr}_{\mathrm{Y}}"] \arrow[dd, phantom, "\text{(pb)}"] & \\
\mathrm{X} \arrow[dr, "p_{\mathrm{X}}"'] & & \mathrm{Y} \arrow[dl, "p_{\mathrm{Y}}"] \\
& \mathrm{B} &
\end{tikzcd}
\qquad \Rightarrow \qquad
\left\{
\begin{aligned}
(p_{\mathrm{Y}})^* \circ (p_{\mathrm{X}})_! \;&\simeq\; (\mathrm{pr}_{\mathrm{Y}})_! \circ (\mathrm{pr}_{\mathrm{X}})^*\,, \\
(p_{\mathrm{X}})^* \circ (p_{\mathrm{Y}})_* \;&\simeq\; (\mathrm{pr}_{\mathrm{X}})_* \circ (\mathrm{pr}_{\mathrm{Y}})^*\,.
\end{aligned}
\right.
\tag{25}
$$

Proof. The first isomorphism in (25) follows immediately from the pasting law, which for $(\mathrm{E}, p_{\mathrm{E}}) \in \mathrm{kTopSpc}_{/\mathrm{X}}$ gives the following natural identification:

$$\begin{tikzcd}
(p_{\mathrm{Y}})^*(p_{\mathrm{X}})_!\mathrm{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] & & (\mathrm{pr}_{\mathrm{X}})^*E \arrow[r] \arrow[d] \arrow[rr, bend left, "p_{(\mathrm{pr}_{\mathrm{Y}})_!(\mathrm{pr}_{\mathrm{X}})^*\mathrm{E}}"] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{X} \times_{\mathrm{B}} \mathrm{Y} \arrow[r, "\mathrm{pr}_{\mathrm{Y}}"] \arrow[d, "\mathrm{pr}_{\mathrm{X}}"] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] \\
\mathrm{E} \arrow[r, "p_{\mathrm{X}} \circ p_{\mathrm{E}}"'] & \mathrm{B} & \simeq & E \arrow[r, "p_{\mathrm{E}}"'] & \mathrm{X} \arrow[r, "p_{\mathrm{X}}"'] & \mathrm{B}\,.
\end{tikzcd}$$

This implies the second natural isomorphism by adjointness (11) and the Yoneda lemma:

$$\begin{aligned}
\mathrm{kTopSpc}_{/\mathrm{X}}\big((\mathrm{U}, p_{\mathrm{U}}),\, (p_{\mathrm{X}})^*(p_{\mathrm{Y}})_*(\mathrm{E}, p_{\mathrm{E}})\big) &\simeq \mathrm{kTopSpc}_{/\mathrm{B}}\big((p_{\mathrm{X}})_!(\mathrm{U}, p_{\mathrm{U}}),\, (p_{\mathrm{Y}})_*(\mathrm{E}, p_{\mathrm{E}})\big) \\
&\simeq \mathrm{kTopSpc}_{/\mathrm{Y}}\big((p_{\mathrm{Y}})^*(p_{\mathrm{X}})_!(\mathrm{U}, p_{\mathrm{U}}),\, (\mathrm{E}, p_{\mathrm{E}}),\big)
\end{aligned}$$

and similarly for the other side of the isomorphism. ☐

Now considering the Beck-Chevalley relation (25) for the following special case

$$\begin{tikzcd}[column sep=small]
 & B\mathrm{G} \arrow[dl, "\mathrm{p}_{BG}"'] \arrow[dr, "i_B \times \mathrm{id}_{BG}"] & \\
* \arrow[dr, "i_b"'] & \text{(pb)} & \mathrm{B} \times B\mathrm{G} \arrow[dl, "\mathrm{id}_{\mathrm{B}} \times p_{BG}"] \\
 & \mathrm{B} &
\end{tikzcd}
\qquad
(i_b)^* \circ (\mathrm{id}_{\mathrm{B}} \times p_{BG})_* \;\simeq\; (p_{BG})_* \circ (i_b \times \mathrm{id}_{BG})^* \tag{26}$$

implies from (24) the pullback diagram:

$$\begin{tikzcd}
\overset{\text{space of sections over fiber}}{(p_{B\mathrm{G}})_*\mathrm{Map}\big((\mathrm{X}_b, \tau_b),\, (A \times_G EG, P_{A \times_G EG})\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{fiberwise space of sections (itself a topological space over B)}}{(\mathrm{id}_{\mathrm{B}} \times p_{BG})_*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}\big).} \arrow[d] \\
\{b\} \arrow[r, hook, "i_b"'] & \underset{\substack{\text{space of} \\ \text{parameters} \\ \text{and twists}}}{\mathrm{B}}
\end{tikzcd} \tag{27}$$

Since the classifying space $B\mathrm{G}$ – in its construction due to Milgram: $B\mathrm{G} = |N(\mathrm{G} \rightrightarrows *)|$ (recalled e.g. in [SS21-Bun, (2.64)]) – is a CW-complex and hence Serre-cofibrant, the map on the right is still a Serre fibration, so that passing to parameterized connected components works as before in (20) to yield a covering space, generalizing (21), whose fiber over $b \in \mathrm{B}$ is the τ_b-twisted A-cohomology (Def. 2.10) of the fiber X_b:

$$\begin{tikzcd}
\overset{\substack{\tau\text{-twisted} \\ A\text{-cohomology} \\ \text{of fiber}}}{A^\tau(X_b)} \simeq \overset{\text{Connected components of space of sections over fiber}}{\pi_0\Big((p_{B\mathrm{G}})_*\mathrm{Map}\big((\mathrm{X}_b, \tau_b),\, (A \times_G EG, P_{A \times_G EG})\big)\Big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{Fiberwise connected components of} \\ \text{fiberwise space of sections (itself a topological space over B)}}}{\pi_{0/\mathrm{B}}\Big((\mathrm{id}_B \times p_B)_*\mathrm{Map}\big((\mathrm{X}, (p_{\mathrm{X}}, \tau)),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}\big)\Big).} \arrow[d] \\
* \arrow[r, hook, "i_b"'] & \underset{\substack{\text{Space of} \\ \text{parameters} \\ \text{and twists}}}{\mathrm{B}}
\end{tikzcd} \tag{28}$$

As before in the untwisted case, Prop. 2.5 again implies that this covering space trivializes compatibly with any local trivialization of $(\mathrm{X}, p_{\mathrm{X}})$, thus exhibiting its corresponding classifying map $\nabla^{\mathrm{GM}}_{X, G \circlearrowright A}$ (via Lem. 2.7) as the Gauss-Manin connection (cf. the description in [EFK98, §7.5]).

In conclusion, we have now shown the following generalization of Thm. 2.9 to twisted cohomology:

Theorem 2.13 (Gauss-Manin connection in twisted generalized cohomology over fiber bundles via fiberwise mapping spaces). *If* $\mathrm{B} \in \mathrm{kHaus}$ *is a metrizable homotopy 1-type* ($\pi_{n \geq 2}(\mathrm{B}, -) = 0$) *and* $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ *is a locally trivial fiber bundle whose typical fiber admits the structure of a CW-complex, then for any discrete or compact Lie* $\mathrm{G} \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $G \circlearrowright \mathrm{A} \in \mathrm{G}\,\mathrm{Act}(\mathrm{kTopSpc})$*, the Gauss-Manin-connection on the twisted A-cohomology sets* (22) *of the fibers* X_b *is exhibited by* *the fiberwise 0-truncation of the right base change along* $B\mathrm{G}$ (27) *of the fiberwise mapping space* (24) *from* X *into the Borel construction* $\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}$*:*

$$\underset{\substack{\textbf{Gauss-Manin}\\ \textbf{connection on}\\ \textbf{twisted } A\textbf{-cohomology}}}{\nabla^{\mathrm{GM}}_{X, G \circlearrowright A}} \quad \underset{\text{Lem. 2.7}}{\longleftrightarrow} \quad \underset{\textbf{fiberwise 0-truncation of right base change along } B\mathrm{G} \textbf{ of fiberwise mapping space into Borel construction on } \mathrm{A}}{\pi_{0/\mathrm{Pth}(\mathrm{B})}\bigg(\mathrm{Pth}\,(p_{\mathrm{B}\times p_{BG}})_*\mathrm{Map}\Big(\big(\mathrm{X}, (p_{\mathrm{X}}, \tau)\big),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}\Big)\bigg)}\,.$$

Example 2.14 (The Knizhnik-Zamolodchikov connection of $\widehat{\mathfrak{su}_2}^k$-conformal blocks). Consider the situation reviewed in [EFK98, §7] and discussed in [SS22-DBr][SS22-Ord][SS22-TQC], where:

- $\mathrm{B} := \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})$ is the configuration space of N points in the plane;
- $\mathrm{X} := \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{C}) \xrightarrow{(p_N^{n+N}, \tau)} \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})$ is fibration which forgets the first n of $n+N$ points in the plane;
- $\mathrm{G} = \mathbb{Z}_\kappa \subset \mathrm{U}(1)$ is a cyclic group, regarded as a subgroup of the circle group;
- $G \circlearrowright \mathrm{A} := \mathbb{Z}_\kappa \circlearrowright \mathrm{K}(\mathbb{C}, n)$ is the restricted action on the EM-space from Ex. 2.11.

Then for a suitable choice of global twist

$$\tau : \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{C}) \to B\mathbb{Z}_\kappa$$

the Gauss-Manin connection on the fiberwise twisted ordinary cohomology groups (Ex. 2.11)

$$H^n\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{z_I\}_{I=1}^N\big); \mathscr{L}\big(\tau_{(z_I)}\big)\Big)$$

yields (reviewed in [EFK98, §7.5]) the Knizhnik-Zamolodchikov connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$ conformal blocks, for weights determined by the monodromies of τ. This is the result of the *hypergeometric integral construction* of KZ-solutions.

By Theorem 2.13 with Exmaple 2.11, this is realized as equivalently reflected in the fiberwise 0-truncation of (a right base change of) a fiberwise mapping space.

3 Via dependent homotopy type theory

Theorem 2.13 implies that Gauss-Manin connections on fiber bundles over homotopy 1-types, and in particular the Knizhnik-Zamolodchikov connections of Example 2.14, have a curiously direct formulation in the language of homotopy type theory, under its *interpretation* into homotopy theory (reviewed in [Ri22]):

Given dependent types $\mathscr{X}$ and $\mathscr{A}$ which interpret as fibrations of configuration spaces (equivalently: fibrations of delooped braid groups, e.g. [BBCDG21]), and as homotopy quotients of Eilenberg-MacLane spaces (e.g. [FL14]) as in Ex. 2.14, the HoTT syntax for the Gauss-Manin connection in the form (28) according to our Thm. 2.13 is simply this:

$$b : \mathscr{B} \;\;\vdash\;\; \Big| \prod_{\tau : B\mathscr{G}} \big(\mathscr{X}(\tau, b) \to \mathscr{A}\big) \Big|_0$$

We recall how this works. Let $\mathbf{H}$ be an ∞-topos...

Colloquial	**∞-Topos theory**	**Homotopy type theory**
An object	$\mathscr{B} \in \mathbf{H} \simeq \mathbf{H}_{/*}$	$\bullet : * \;\vdash\; \mathscr{B} : \mathrm{Type}$
A bundle	$\begin{array}{c} \mathscr{X} \\ \downarrow{\scriptstyle p_{\mathscr{X}}} \\ \mathscr{B} \end{array} \in \mathbf{H}_{/B}$	$b : \mathscr{B} \;\vdash\; \mathscr{X}(b) : \mathrm{Type}$

Fiberwise mapping space	$\mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big) \in \mathbf{H}_{/\mathscr{B}}$	$b : \mathscr{B} \vdash \big(\mathscr{X}(b) \to \mathscr{A}\big) : \mathrm{Type}$

Now consider $b : * \to \mathscr{B}$. Then:

Fiber of mapping bundle	$\mathbf{H}\Big(U, (b_{(-)})^* \mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big)\Big)$ $\simeq \mathbf{H}\Big((b_{(-)})_! U, \mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big)\Big)$ $\simeq \mathbf{H}\Big(\big((b_{(-)})_! U\big) \times \mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\Big)$ $\simeq \mathbf{H}\Big((b_{(-)})_! \big(U \times \mathscr{X}_{b_\bullet}\big), (p_{\mathscr{B}})^* \mathscr{A}\Big) \qquad (29)$ $\simeq \mathbf{H}\Big(U \times \mathscr{X}_{b_\bullet}, (b_{(-)})^* (p_{\mathscr{B}})^* \mathscr{A}\Big)$ $\simeq \mathbf{H}\Big(U \times \mathscr{X}_{b_\bullet}, \mathscr{A}\Big)$ $\simeq \mathbf{H}\Big(U, \mathrm{Map}(\mathscr{X}_{b_\bullet}, \mathscr{A})\Big)$	$\bullet : * \;\vdash\; \big(\mathscr{X}(b_\bullet) \to \mathscr{A}\big) : \mathrm{Type}$

More generally, if the previous absolute content $*$ is replaced by a nontrivial context $B\mathscr{G}$, with $\mathscr{B}$ depending on it trivially, then we still get

Fiber of twisted mapping bundle	$\mathbf{H}\Big(U, (b_{(-)})^* (p_{\mathscr{C}})_* \mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big)\Big)$ $\underset{(26)}{\simeq} \mathbf{H}\Big(U, (p_{\mathscr{C}})_* (b_{(-)})^* \mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big)\Big)$ $\simeq \mathbf{H}\Big((p_{\mathscr{C}})^* U, (b_{(-)})^* \mathrm{Map}\big(\mathscr{X}, (p_{\mathscr{B}})^* \mathscr{A}\big)\Big)$ $\underset{(29)}{\simeq} \mathbf{H}\Big((p_{\mathscr{C}})^* U, \mathrm{Map}\big(\mathscr{X}_{b_\bullet}, \mathscr{A}\big)\Big)$ $\simeq \mathbf{H}\Big(U, (p_{\mathscr{C}})_* \mathrm{Map}\big(\mathscr{X}_{b_\bullet}, \mathscr{A}\big)\Big)$	$\bullet : * \;\vdash\; \prod_{c : \mathscr{C}} \big(\mathscr{X}(c, b_\bullet) \to \mathscr{A}(c)\big) : \mathrm{Type}$

(...)

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