Gauss-Manin and Knizhnik-Zamolodchikov connections via abstract homotopy theory

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Abstract

We observe that the construction of Gauss-Manin connections on bundles of fiberwise cohomology groups has a simple explanation in the context of parameterized homotopy theory and generally in higher topos theory. This is so simple that it becomes effectively a tautology when formulated in the corresponding formal language of homotopy type theory: The covering space which exhibits the flat Gauss-Manin connection is simply the fiberwise 0-truncation of the *fiberwise* mapping space (the internal hom-object in the slice) into the corresponding classifying space. This applies at once in the generality of twisted generalized and/or non-abelian cohomology theories, such as twisted K-theory and twisted Cohomotopy. Applied to twisted complex cohomology groups of configuration spaces of points in the plane, it yields the Knizhnik-Zamolodchikov (KZ) connection on bundles of $\widehat{\mathfrak{su}_2}^k$-conformal blocks, and thus the monodromy braid representation characteristic of $\mathfrak{su}_2$-anyons. We close by highlighting that the elegant reflection of this construction in homotopy type theory may provide a proper foundation for hardware-aware topological quantum programming.

Contents

1 Introduction

The following two facts are each "well-known" to their respective experts, but their striking conjunction has been pointed out only recently ([SS22-DBr][SS22-Ord][SS22-TQC]):

Fact 1.1 (**Topological quantum gates via Algebraic topology).**

(1) *Viable future topological quantum computation hardware realizes quantum gates which implement monodromy braid representations on $\widehat{\mathfrak{su}_2}^k$-conformal blocks;*

(2) *the "hypergeometric integral construction" identifies these with Gauss-Manin connections on fiberwise twisted cohomology groups of configurations spaces of points in the plane.*

This is striking, because the first item, taken at face value, invokes a fairly long and intricate sequence of constructions from conformal field theory and affine representation theory, while the second item invokes only the most fundamental concepts of algebraic topology: local systems on and cohomology of fibrations of configuration spaces.

This suggests [SS22-TQC] that theoretical reasoning about – and notably the programming and simulation of – topological quantum circuits will profitably proceed *through* the perspective of the second item above.

Concretely, there is already a programming language being developed, called *cohesive homotopy type theory* (cohesive HoTT), for fundamental constructions in differential topology and homotopy theory. Hence Fact 1.1 should imply that the encoding of realistic topological quantum gates in cohesive HoTT is fairly immediate – hence that this functionality is essentially *native* to cohesive HoTT, at least much more immediate than the traditional description of Knizhnik-Zamolodchikov connections on bundles of conformal blocks suggests.

This is what we prove here: we show that once data types for braid groups and for complex Eilenberg-MacLane spaces are given, then the construction of all monodromy braid representations on $\widehat{\mathfrak{su}_2}^k$-conformal blocks is essentially half a line of code. This same statement is made without proof at the end of the announcement article [SS22-TQC]. Here we discuss the proof.

Outline:

§2 builds a bridge between:

1. traditional discussion of Gauss-Manin connections ([Ma58][Gro66][Ka68][KO68][Gri70]),
   here specifically for twisted cohomology on fiber bundles ([EFK98, §7.5][Vo03I, Def. 9.13] (e.g. [Ko02, §1.5, 2.1])),
2. abstract homotopy theory
   ([Br65][Qu67][Ad74], for review see [KP97][Ri14], here specifically [FSS20-Cha, §A][SS21-Bun, §3.1])

by proving that, on fiber bundles, these connections are witnessed by the *fiberwise mapping space*-construction known from parameterized point-set topology ([MS06], following [Bo70][BB78]);

§3 recalls how such a statement may be understood as a model for certain programs in HoTT ([UFP13], review in [Sh12]) and explains the resulting quasi-native construction of Knizhnik-Zamolodchikov connections (e.g. [Ko02, §1.5, 2.1]) in cohesive HoTT.

2 Via parameterized point-set topology

Here we use (just the most basic aspects of) parameterized "point-set" homotopy theory (as laid out in [MS06], going back to [Bo70]) in order to show that Gauss-Manin connections in (twisted) generalized cohomology groups on fibers of bundles are exhibited by the fiberwise 0th Postnikov stage of the fiberwise mapping space (fiberwise space of sections) into the given classifying space (classifying fibration).

With the relevant notions and results from parameterized point-set homotopy theory in hand, the proof is straightforward, so we use the occasion to briefly introduce and review basics of paramaterized topology as we go along, in order to make the proof reasonably self-contained also for a general mathematical audience.

The construction in itself of the Gauss-Manin connection on fiberwise twisted cohomology groups of locally trivial fiber bundles may be understood without the abstract machinery invoked here; a sketch of such a more low-brow argument is what [EFK98, §7.5] offers. However, it is our abstract re-formulation which provides an elegant handle on Gauss-Manin connections in the language of homotopy type theory – this is discussed in §3 below.

Below we make repeated used of the *pasting law* (in one direction): In any category with pullbacks, the pullback along a composite morphism is the pasting of the pullbacks along the two factors:

$$
\begin{tikzcd}
(f_2 \circ f_1)^* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "p_X"] \\
B'' \arrow[r, "f_2 \circ f_1"'] & B
\end{tikzcd}
\;\simeq\;
\begin{tikzcd}
f_2^*\big(f_1^* X\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & f_1^* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d] \\
B'' \arrow[r, "f_2"'] & B' \arrow[r, "f_1"'] & B
\end{tikzcd}
\tag{1}
$$

2.1 For generalized and non-abelian cohomology

Given a sufficiently nice fibration $p_{\mathrm{X}} : \mathrm{X} \to \mathrm{B}$, the ordinary complex cohomology groups $H^n(\mathrm{X}_b) := H^n(\mathrm{X}_b; \mathbb{C})$ of its fibers X_b for $b \in B$, form a bundle of abelian groups over B equipped with a flat connection, known as the *Gauss-Manin connection*. Hence a Gauss-Manin connection provides a rule for coherently transporting cohomology classes of spaces X_b as these spaces *vary with a parameter*, the parameter being the point $b \in \mathrm{B}$. Moreover, the flatness of the connection means that the induced transport of cohomology classes depends on parameter paths $\gamma : [0,1] \to \mathrm{B}$ only via their homotopy classes $[\gamma]$ relative to their endpoints. If B is connected, this means equivalently that the Gauss-Manin connection is a homomorphism from the fundamental group of B to the automorphism group of $H^n(X_{b_\bullet}; A)$ at any fixed $b_\bullet$.

$$
\begin{array}{ccc}
\begin{tikzcd}
\underset{\text{of sets}}{\text{Category}}\ \mathrm{Set} \\
\underset{\text{path groupoid}}{\text{Fundamental}}\ \mathrm{Pth}(\mathrm{B}) \arrow[u, "\underset{\text{connection}}{\text{Gauss-Manin}}"]
\end{tikzcd}
&
\begin{tikzcd}
& \overset{\text{Fiberwise cohomology}}{H^n(\mathrm{X}_{b_2})} \arrow[dr, "\sim"] & \\
H^n(\mathrm{X}_{b_1}) \arrow[ur, "\sim"] \arrow[rr, "\sim"'] & & H^n(\mathrm{X}_{b_3}) \\
& \{b_2\} \arrow[dr, "{[\gamma_{23}]}"] & \\
\{b_1\} \arrow[ur, "{[\gamma_{12}]}"] \arrow[rr, "{[\gamma_{23} \circ \gamma_{12}]}"'] & & \{b_3\}
\end{tikzcd}
&
\begin{tikzcd}
\mathrm{X} \arrow[r, dashed] \arrow[d, "p_{\mathrm{X}}"', "\underset{\text{bundle}}{\text{Fiber}}"] & \overset{\text{Classifying space}}{K(n, \mathbb{C})} \\
\mathrm{B} \arrow[dr, "p_{\mathrm{B}}"'] & \\
& *
\end{tikzcd}
\end{array}
\tag{2}
$$

In fact, this applies also to *twisted* cohomology groups, in which case the Knizhnik-Zamolodchikov connection becomes a special case. We come back to this in a moment (§2.2).

Traditionally, Gauss-Manin connections are constructed algebraically. Here we work entirely homotopy-theoretically and instead make use of the fact that twisted ordinary cohomology of a topological space of CW-type is *representable*, in that for $n \in \mathbb{N}$ there exists a topological space $\mathrm{K}(n, \mathbb{C})$ (the nth *Eilenberg-MacLane space*) such that cohomology is identified with the connected components of the *mapping space* into it:

$$
H^n(\mathrm{X}_b) \;\simeq\; \pi_0 \mathrm{Map}\big(\mathrm{X}_b, \mathrm{K}(\mathbb{C}, n)\big)\,. \tag{3}
$$

Mapping spaces. For mapping spaces to work well we may assume without practical restriction that we work in the category kTopSpc of compactly generated space (k-spaces, see [SS21-Bun, Nota. 1.0.15] for pointers). Here, for $\mathrm{X}_b \in \mathrm{kTopSpc}$, we have an adjunction

$$
\begin{tikzcd}
\mathrm{kTopSpc} \arrow[r, shift right, "\mathrm{Map}(\mathrm{X}_b, -)"'] \arrow[r, phantom, "\perp"] & \mathrm{kTopSpc} \arrow[l, shift right, "\mathrm{X}_b \times (-)"']
\end{tikzcd}\,, \tag{4}
$$

meaning that there is an *exponential law* for k-topological spaces, namely a natural bijection of the form

$$\mathrm{kTopSpc}\big(\mathrm{X}, \mathrm{Map}(\mathrm{Y}, \mathrm{Z})\big) \;\simeq\; \mathrm{kTopSpc}(\mathrm{X} \times \mathrm{Y}, \mathrm{Z})\,.$$

Generalized cohomology. The following construction of the Gauss-Manin connection over fiber bundles relies only on the existence of such a classifying space, as in (3), but not on its concrete nature. This means that the construction applies also to "generalized cohomology theories".

- If, for instance, $E^n(-)$ is a Whitehead-generalized cohomology theory, such as topological K-theory, elliptic cohomology or cobordism cohomology, then there exists a *spectrum* of classifying spaces $\{E_n\}_{n\in\mathbb{N}}$ such that
$$E^n(X_b) \;\simeq\; \pi_0 \mathrm{Map}(\mathrm{X}_b, E_n)\,.$$
- Regarded the other way around, for *any* topological space $\mathrm{A} \in \mathrm{kTopSpc}$ we may regard
$$A^0(X_b) \;:=\; \pi_0 \mathrm{Map}(\mathrm{X}_b, \mathrm{A}) \tag{5}$$
as *non-abelian generalized cohomology* with coefficients in A.
- For example, if $\mathrm{A} = BG$ is the classifying space of a discrete or compact Lie group G, then
$$BG^0(X_b) \simeq H^1(X_b; G)$$
is, equivalently, the traditional non-abelian cohomology in degree 1 with coefficients in G, which classifies G-principal bundles.
- Or if $\mathrm{A} = S^n \subset \mathbb{R}^{n+1}$ is the topological n-sphere, then
$$(S^n)^0(X_b) \simeq \pi^n(X_b)$$
is unstable *Cohomotopy*.

The fiberwise mapping space. The key fact now is that in *parameterized homotopy theory* ([CJ98][BM21][MS06]...) the mapping space construction (4) generalizes to *slices* if the base space B is (compactly generated and) Hausdorff, which we assume from now on:

$$B \in \mathrm{kHaus} \hookrightarrow \mathrm{kTopSpc}\,. \tag{6}$$

Here the *slice category* $\mathrm{kTopSpc}_{/\mathrm{B}}$ is the category whose objects $(\mathrm{X}, p_{\mathrm{X}})$ are k-topological space X equipped with a continuous map $p_{\mathrm{X}} : \mathrm{X} \to \mathrm{B}$ and whose morphisms $(\mathrm{X}, p_{\mathrm{X}}) \to (\mathrm{Y}, p_{\mathrm{Y}})$ are compatible maps $\mathrm{X} \to \mathrm{Y}$, hence:

$$\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}}\big((\mathrm{X}, p_{\mathrm{X}}), (\mathrm{Y}, p_{\mathrm{Y}})\big) \;=\; \mathrm{kTopSpc}\big(\mathrm{X}, \mathrm{Y}\big) \underset{\mathrm{kTopSpc}(\mathrm{X},\mathrm{B})}{\times} \{p_{\mathrm{X}}\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{kTopSpc}\big(\mathrm{X}, \mathrm{Y}\big) \arrow[d, "p_{\mathrm{Y}} \circ (-)"] \\
\{p_{\mathrm{X}}\} \arrow[r, hook] & \mathrm{kTopSpc}\big(\mathrm{X}, \mathrm{B}\big).
\end{tikzcd} \tag{7}$$

If we understand X as an object in the *slice category* $\mathrm{kTopSpc}_{/\mathrm{B}}$ over B via p_{X} (2) and if we denote by $p_{\mathrm{B}}^{*}\mathrm{A}$ the trivial fiber bundle over B with fiber A regarded in the slice category, then their *fiberwise mapping space* is a topological space which is itself fibered over B, such that the fibers of the fiberwise mapping space are the ordinary mapping spaces (4) on the fibers:

Ordinary mapping space on fiber　　Fiberwise mapping space (itself a topological space over B)

$$\begin{tikzcd}
\mathrm{Map}\big(\mathrm{X}_b, \mathrm{A}\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}), p_{\mathrm{B}}^{*}\mathrm{A}\big). \arrow[d] \\
* \arrow[r, hook, "b"] & \mathrm{B}
\end{tikzcd} \tag{8}$$

Parameter space

The right choice of topology on the total fiberwise mapping space is subtle[1] and the result that such a topology exists (see [MS06, §1.3.7-§1.3.9], following [BB78, Thm. 3.5]) may be regarded as the engine which drives our quick re-construction of Gauss-Manin connections from the point of view of point-set topology. Here the right topology is that which ensures the sliced analog of the adjunction (4)

$$\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, swap, "\mathrm{Map}((\mathrm{X}, p_{\mathrm{X}}), -)"] & \mathrm{kTopSpc}_{/\mathrm{B}}\,, \arrow[l, bend right, swap, "(\mathrm{X}, p_{\mathrm{X}}) \times (-)"]
\end{tikzcd} \tag{9}$$

[1] "*The point-set topology of parametrized spaces is surprisingly subtle. Parametrized mapping spaces are especially delicate.*" [MS06, p. 15]

hence, equivalently, the *exponential law* in the slice [BB78, Thm. 3.5][MS06, (1.3.9)]: a natural bijection of the form

$$\mathrm{kTopSpc}_{/B}\Big((\mathrm{X},p_{\mathrm{X}}), \overset{\text{Fiberwise mapping space}}{\mathrm{Map}\big((\mathrm{Y},p_{\mathrm{Y}}),\,(\mathrm{Z},p_{\mathrm{Z}})\big)}\Big) \;\simeq\; \mathrm{kTopSpc}_{/B}\Big(\overset{\text{Fiberwise product space}}{(\mathrm{X},p_{\mathrm{X}})\times(\mathrm{Y},p_{\mathrm{Y}})},(\mathrm{Z},p_{\mathrm{Z}})\Big)\,, \tag{10}$$

where now the product on the right is that in the slice, hence is the *fiber product* of k-topological spaces:

$$(\mathrm{X},p_{\mathrm{X}})\times(\mathrm{Y},p_{\mathrm{Y}}) \;\simeq\; \big(\mathrm{X}\times_{\mathrm{B}}\mathrm{Y}, p_{\mathrm{X}}\circ \mathrm{pr}_{\mathrm{X}} = p_{\mathrm{Y}}\circ \mathrm{pr}_{\mathrm{Y}}\big)\,.$$

Of course, the unit for this product is the identity map on the base space:

$$(\mathrm{X},p_{\mathrm{X}})\times(\mathrm{B},\mathrm{id}_{\mathrm{B}}) \;\simeq\; (\mathrm{X},p_{\mathrm{X}})\,. \tag{11}$$

This exponential law in slices implies a wealth of useful structure:

Proposition 2.1 (Base change adjoint triple). *For any map between base spaces $f:\mathrm{B}\to\mathrm{B}'$, there is a base change adjoint triple*

$$\begin{tikzcd} \mathrm{kTopSpc}_{/\mathrm{B}} \arrow[r, shift left=2.5ex, "f_!"] \arrow[r, phantom, shift left=1.2ex, "\perp"] \arrow[r, phantom, shift right=1.2ex, "\perp"] \arrow[r, shift right=2.5ex, "f_*"'] & \mathrm{kTopSpc}_{/\mathrm{B}'}\,, \arrow[l, "f^*" description] \end{tikzcd} \tag{12}$$

where f^* denotes the pullback operation (formed in kTopSpc*), its the left adjoint $f_!$ is given by postcomposition with f, and the right adjoint f_* is given by the following pullback construction:*

$$\begin{tikzcd} f_*(\mathrm{X},p_{\mathrm{X}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Map}\big((\mathrm{B},f),\,(\mathrm{X},f\circ p_{\mathrm{X}})\big) \arrow[d] \\ (\mathrm{B}',\mathrm{id}_{\mathrm{B}'}) \arrow[r, "\widetilde{\mathrm{id}}"'] & \mathrm{Map}\big((\mathrm{B},f),\,(\mathrm{B},f)\big)\,. \end{tikzcd} \tag{13}$$

Proof. For the left adjoint $(f_!\dashv f^*)$ the required hom-isomorphism is immediate from the universal property of the pullback:

$$\begin{tikzcd} \mathrm{X} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Y} \arrow[d, "p_{\mathrm{y}}"] \\ & \mathrm{B} \arrow[r, "f"'] & \mathrm{B}'\,. \end{tikzcd}$$

For the right adjoint, the required hom-isomorphism is obtained as the following sequence of natural isomorphisms:

$$\begin{aligned}
&\mathrm{kTopSpc}_{/\mathrm{B}'}\big((U,p_{\mathrm{U}}),\,f_*(\mathrm{X},p_{\mathrm{X}})\big) \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U,p_{\mathrm{U}}),\,\mathrm{Map}\Big((\mathrm{B},f),\,(\mathrm{X},f\circ p_{\mathrm{X}})\Big) \underset{\mathrm{Map}\big((\mathrm{B},f),(\mathrm{B},f)\big)}{\times} \{\widetilde{\mathrm{id}}\}\Big) && \text{by (13)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U,p_{\mathrm{U}}),\,\mathrm{Map}\big((\mathrm{B},f),\,(\mathrm{X},f\circ p_{\mathrm{X}})\big)\Big) \underset{\mathrm{Map}\big((U,p_{\mathrm{U}}),\mathrm{Map}\big((\mathrm{B},f),(\mathrm{B},f)\big)\big)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (7)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}'}\Big((U,p_{\mathrm{U}})\times(\mathrm{B},f),\,(\mathrm{X},f\circ p_{\mathrm{X}})\Big) \underset{\mathrm{kTopSpc}_{/\mathrm{B}'}\big((U,p_{\mathrm{U}})\times(\mathrm{B},f),(\mathrm{B},f)\big)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (10)} \\
&\simeq\; \Big(\mathrm{kTopSpc}(f^*\mathrm{U},\mathrm{X}) \underset{\mathrm{kTopSpc}(f^*\mathrm{U},\mathrm{B}')}{\times} *\Big) \underset{\Big(\mathrm{kTopSpc}(f^*\mathrm{U},\mathrm{B}) \underset{\mathrm{kTopSpc}(f^*\mathrm{U},\mathrm{B}')}{\times} *\Big)}{\times} \{\widetilde{\mathrm{id}}\} && \text{by (7)} \\
&\simeq\; \mathrm{Map}\big(f^*\mathrm{U},\mathrm{X}\big) \underset{\mathrm{Map}(f^*\mathrm{U},\mathrm{B})}{\times} * && \text{by (14)} \\
&\simeq\; \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*(\mathrm{U},p_{\mathrm{U}}),\,(\mathrm{X},p_{\mathrm{X}})\big) && \text{by (7)}\,.
\end{aligned}$$

Here the penultimate step is observing that the fiber products (limits) may be interchanged: Instead of computing the horizontal fiber product of the vertical fiber product in the following diagram, we may first compute the horizontal fiber products (shown on the right, again by (7))

$$
\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{X}\big) \arrow[r, "p_{\mathrm{X}}\circ(-)"] \arrow[d, "f\circ p_{\mathrm{X}}\circ(-)"'] & \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}\big) \arrow[d, "f\circ(-)"] & * \arrow[l] \arrow[d] \\
\mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}'\big) \arrow[r, "\mathrm{id}"] & \mathrm{kTopSpc}_{/\mathrm{B}'}\big(f^*\mathrm{U}, \mathrm{B}'\big) & * \arrow[l] \\
* \arrow[u] \arrow[r] & * \arrow[u] & * \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{kTopSpc}_{/\mathrm{B}}\big(f*(\mathrm{U}, p_{\mathrm{U}}),\, (\mathrm{X}, p_{\mathrm{X}})\big) \arrow[d] \\
* \\
* \arrow[u]
\end{tikzcd}
\tag{14}
$$

and then the evident remaining vertical fiber product is as claimed. □

Example 2.2 (Space of sections). When $\mathrm{B}' = *$ in (12), the right base change (13) constructs *spaces of sections*:

$$
\Gamma_{\mathrm{X}_b}(-) \;\simeq\; (p_{\mathrm{X}_b})_* \;:\; \mathrm{kTopSpc}_{/\mathrm{X}_b} \longrightarrow \mathrm{kTopSpc}\,. \tag{15}
$$

Generally, one may understand the right base change as forming *fiberwise spaces of sections*.

Proposition 2.3 (Cartesian Frobenius reciprocity). *For $f : \mathrm{B} \to \mathrm{B}'$ a map of base spaces* (6)*, we have a natural isomorphism*

$$
f_!\big((\mathrm{X}, p_{\mathrm{X}}) \times f^*(\mathrm{Y}, p_{\mathrm{Y}})\big) \;\simeq\; \big(f_!(\mathrm{X}, p_{\mathrm{X}})\big) \times (\mathrm{Y}, p_{\mathrm{Y}})\,, \tag{16}
$$

where $(f_! \dashv f^*)$ is the left base change adjunction (12)*.*

Proof. This follows by the pasting law (1), which here says that the following pullback squares in kTopSpc agree:

$$
\begin{tikzcd}
\mathrm{X} \times_{\mathrm{B}} f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & f^*\mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] \\
\mathrm{X} \arrow[r, "p_{\mathrm{X}}"'] & \mathrm{B} \arrow[r, "f"'] & \mathrm{B}'
\end{tikzcd}
\;\simeq\;
\begin{tikzcd}
\mathrm{X} \times_{\mathrm{B}'} \mathrm{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Y} \arrow[d] \\
\mathrm{X} \arrow[r, "f\circ p_{\mathrm{X}}"'] & \mathrm{B}'.
\end{tikzcd}
$$

□

In generalization of (8), we have:

Proposition 2.4 (Base change is closed functor). *The pullback (base change) of a fiberwise mapping space along any continuous map of base spaces* (6) *$f : B' \longrightarrow B$ is the fiberwise mapping space of the pullbacks of the arguments:*

$$
f^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big) \;\simeq\; \mathrm{Map}\big(f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\big)\,. \tag{17}
$$

Proof. Apply the Yoneda Lemma over $\mathrm{kTopSpc}_{/B}^{\mathrm{op}}$ to the following sequence of natural isomorphisms:

$$
\begin{aligned}
& \mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}),\, f^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) \\
& \simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(f_!(\mathrm{U}, p_{\mathrm{U}}),\, \mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) && \text{by (12)} \\
& \simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(\big((f_!(\mathrm{U}, p_{\mathrm{U}})\big) \times (\mathrm{X}, p_{\mathrm{X}}),\, (\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (10)} \\
& \simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big(f_!\big((\mathrm{U}, p_{\mathrm{U}}) \times f^*(\mathrm{X}, p_{\mathrm{X}})\big),\, (\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (16)} \\
& \simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}) \times f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\Big) && \text{by (12)} \\
& \simeq\; \mathrm{kTopSpc}_{/\mathrm{B}}\Big((\mathrm{U}, p_{\mathrm{U}}),\, \mathrm{Map}\big(f^*(\mathrm{X}, p_{\mathrm{X}}),\, f^*(\mathrm{Y}, p_{\mathrm{Y}})\big)\Big) && \text{by (10)}\,.
\end{aligned}
$$

□

For the case at hand, this has the following consequence.

Proposition 2.5 (Fiberwise mapping space out of fiber bundle is fiber bundle). *Let $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ be a fiber bundle with local trivialization*

$$
\phi : \mathrm{U} \underset{\mathrm{opn}}{\twoheadrightarrow} \mathrm{B}\,, \qquad \textit{such that} \qquad \phi^*(\mathrm{X}, p_{\mathrm{X}}) \simeq p_{\mathrm{U}}^*\mathrm{X}_0 \;:=\; \mathrm{U} \times \mathrm{X}_0\,. \tag{18}
$$

Then the fiberwise mapping space is a fiber bundle which is locally trivial with respect to the same open cover:

$$
\begin{aligned}
\phi^*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*A\big) \;&\simeq\; \mathrm{Map}\big(\phi^*(\mathrm{X}, p_{\mathrm{X}}),\, \phi^* p_{\mathrm{B}}^*\mathrm{A}\big) && \text{by (17)} \\
&\simeq\; \mathrm{Map}\big(p_{\mathrm{U}}^*\mathrm{X}_0,\, p_{\mathrm{U}}^*\mathrm{A}\big) && \text{by (18)} \\
&\simeq\; p_{\mathrm{U}}^*\mathrm{Map}\big(\mathrm{X}_0,\, \mathrm{A}\big) && \text{by (17).}
\end{aligned}
\tag{19}
$$

Fiberwise homotopy theory. Moreover, the topology on the fiberwise mapping space (8) is also "homotopy correct" in that its map to the base B is an h-fibration as soon as p_{X} is an h-fibration (by [Bo70, §6.1], see also [MS06, Prop. 1.3.11]), which is the case for B a metrizable space and p_{X} a fiber bundle. This implies that the fibers of the fiberwise mapping space are in fact homotopy fibers, and that forming path ∞-groupoids (singular simplicial complexes),

$$\begin{tikzcd} \mathrm{kTopSpc} \arrow[r, shift right, "\mathrm{Pth}"'] \arrow[r, phantom, "\perp"] & \Delta\mathrm{Set} \arrow[l, shift right, "|-|"'] \end{tikzcd}$$

respects this property:

$$\begin{tikzcd} \mathrm{Pth}\,\mathrm{Map}\big(\mathrm{X}_b,\, \mathrm{A}\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*\mathrm{A}\big). \arrow[d, "\in \mathrm{KanFib}"] \\ * \arrow[r, "b"'] & \mathrm{Pth}(\mathrm{B}) \end{tikzcd} \tag{20}$$

Lemma 2.6 (**Fiberwise truncation is preserved by base change).** *For any map of simplicial sets $f : B' \to B$, the operation of base change (pullack) f^* of Kan fibrations preserves fiberwise Postnikov truncation:*

$$f^* \circ \pi_{0/B} \;\simeq\; \pi_{0/B'} \circ f^* \,.$$

Proof. It is useful to understand this as a special case of a general phenomenon of 0-*truncation* in slice homotopy theories [Lu09, §5.5.6]. Every morphism $p : X \to S$ factors essentially uniquely through $\pi_{0/S}(E)$ as a "0-connected" map followed by a "0-truncated map", and both these classes are preserved by homotopy pullback ([Lu09, Ex. 5.2.8.16 & Lem. 6.5.1.16(6)]). Therefore the claim follows by the pasting law (1), which also holds for homotopy pullbacks:

$$\begin{tikzcd} f^*X \arrow[r] \arrow[d, "\text{0-cnctd}"] \arrow[dr, phantom, "\text{(hpb)}"] & X \arrow[d, "\text{0-cnctd}"] \\ \pi_{0/B'}\big(f^*X\big) \arrow[r] \arrow[d, "\text{0-cnctd}"] \arrow[dr, phantom, "\text{(hpb)}"] & \pi_{0/B}(X) \arrow[d, "\text{0-trnctd}"] \\ B' \arrow[r, "f"'] & B\,. \end{tikzcd} \tag{21}$$

□

Therefore, applying Lemma 2.6 to the diagram (20), we obtain a homotopy pullback diagram as shown on the left here:

$$\begin{tikzcd} \underset{\text{of fiber}}{\overset{A\text{-cohomology}}{A^0(\mathrm{X}_b)}} \simeq \pi_0\Big(\mathrm{PthMap}\big(\mathrm{X}_b,\, \mathrm{A}\big)\Big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \overset{\text{Fiberwise 0-truncation of (path }\infty\text{-groipoid of) fiberwise mapping space}}{\pi_{0/\mathrm{Pth}(\mathrm{B})}\Big(\mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*\mathrm{A}\big)\Big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \mathrm{Set}^{*/} \arrow[d, "\substack{\text{Covering space}\\ \text{classifier}}"] \\ \{b\} \arrow[r] & \underset{\substack{\text{Fundamental}\\ \text{groupoid of}\\ \text{base space}}}{\mathrm{Pth}(\mathrm{B})} \arrow[r, "\nabla^{\mathrm{GM}}_{X,A}", "\text{Gauss-Manin connection}"'] & \mathrm{Set}\,. \end{tikzcd} \tag{22}$$

The left square shows that the fiberwise 0-truncation of the fiberwise mapping space is a fibration over the fundamental groupoid of B, whose (homotopy) fibers are the generalized cohomology sets (5) of the fiber space X_b. The homotopy pullback shown on the right follows by:

Lemma 2.7 (**Univalent universe of sets).** *Any homotopy fibration of sets, as in the middle of* (22)*, is classified by – i.e., is the homotopy pullback along – an essentially unique map $\nabla^{\mathrm{GM}}_{X,A}$ to the covering space classifier, as shown in the square on the right of* (22)*.*

Proof. This may be understood as a simple special case of the general fact that ∞-groupoids form an ∞-topos in which there exists a "small fibration classifier" $\mathrm{Grpd}_\infty^{*/} \longrightarrow \mathrm{Grpd}_\infty$ ([Lu09, Prop. 3.3.2.7][Ci19, §5.2][KL21]). □

Remark 2.8 (**Flat connections as functors on the fundamental groupoid).** Noticing that $\mathrm{Pth}(\mathrm{B})$ is equivalently the disjoint union over connected components $[b] \in \pi_0(\mathrm{B})$ of delooping groupoids $\mathbf{B}\pi_1(\mathrm{B}, b)$, this map $\nabla^{\mathrm{GM}}_{X,A}$ (22) is over each connected component equivalently a group homomorphism

$$\Omega_b\big(\nabla^{\mathrm{GM}_{X,A}}\big) \,:\, \pi_1(\mathrm{B}, b) \longrightarrow \mathrm{Aut}\big(A^0(\mathrm{X}_b)\big)\,.$$

This is a traditional incarnation of flat connections on a space B (e.g. [De70, §I.1][Di04, Prop. 2.5.1]).

Moreover, from Prop. 2.5 it follows that this local system of sets trivializes over any cover over which p_X trivializes, so that it corresponds to a *covering space* which we denote as follows:

$$
\underset{b \in \mathrm{B}}{\forall}
\begin{tikzcd}
\overset{\substack{\text{A-cohomology} \\ \text{of fiber}}}{A^0(\mathrm{X}_b)} = \overset{\substack{\text{Connected components of} \\ \text{ordinary mapping space on fiber}}}{\pi_0 \mathrm{Map}\big(\mathrm{X}_b,\, \mathrm{A}\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{Parameterized connected components of} \\ \text{fiberwise mapping space (a covering space over B)}}}{\pi_{0/\mathrm{B}} \mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^* \mathrm{A}\big)} . \arrow[d] \\
\ast \arrow[r, hook, "b"] & \underset{\text{Parameter space}}{\mathrm{B}}
\end{tikzcd}
\tag{23}
$$

Using this, comparison with [Vo03I, Def. 9.13] readily shows that the flat connection $\nabla^{\mathrm{GM}}_{X,A}$ in (22) is indeed the Gauss-Manin connection. In conclusion, we have shown so far:

Theorem 2.9 (**Gauss-Manin connection in generalized cohomology over fiber bundles via fiberwise mapping spaces).** *Let* $\mathrm{B} \in \mathrm{kHaus}$ *be a metrizable space and* $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ *be a locally trivial fiber bundle whose typical fiber admits the structure of a CW-complex. Then for any* $\mathrm{A} \in \mathrm{kTopSpc}$ *the Gauss-Manin-connection on the A-cohomology sets* (5) *of the fibers* X_b *is exhibited (under Lem. 2.7) by the fiberwise 0-truncation of the fiberwise mapping space* (9) *from* X *into* A*:*

$$
\underset{\substack{\text{Gauss-Manin} \\ \text{connection on} \\ \text{A-cohomology}}}{\nabla^{\mathrm{GM}}_{X,A}} \quad \underset{\text{Lem. 2.7}}{\longleftrightarrow} \quad \underset{\text{Fiberwise 0-truncation of fiberwise mapping space into A}}{\pi_{0/\mathrm{Pth(B)}} \Big(\mathrm{Pth}\,\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^* \mathrm{A}\big) \Big)} .
$$

2.2 For twisted generalized cohomology

We generalize the above discussion to the case of fiberwise *twisted* cohomology (Thm. 2.13) and bring out the motivating example of the $\widehat{\mathfrak{su}_2}^k$-Knizhnik-Zamolodchikov equation (Ex. 2.14).

In existing literature this is discussed for the special case that the total space X is equipped with a flat complex line bundle $\mathscr{L}$ classified by a map $\tau : \mathrm{X} \to B\mathrm{U}(1)$. In this case one may consider the τ_b-twisted complex ordinary cohomology of the fibers, namely the cohomology with coefficient in the local system of parallel local sections of $\mathscr{L}$ (e.g. [Vo03II, §5.1.1]):

$$
H^{n+\tau_b}(\mathrm{X}_b; \mathbb{C}) = H^n(\mathrm{X}_b; \mathscr{L}) .
$$

At least when p_X is a fiber bundle, these twisted cohomology groups again carry a flat Gauss-Manin connection. In the example where $\mathrm{X} = \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{R}^2)$ is a configuration space of points, and p_X the map that forgets the first n of $n+N$ points, then a *hyprgeometric integral construction* identifies this Gauss-Manin-connection on fiberwise twisted complex cohomology with a Knizhnik-Zamolodchikov connection ([EFK98, §7.5]).

The above is the main example of interest for us. However, in [SS22-DBr][SS22-Ord], we explained that it is useful to regard this twisted ordinary cohomology as the home of the twisted Chern characters of twisted *K-theory* groups. For this reason we are interested in speaking of Gauss-Manin connections on bundles of twisted *generalized* cohomology groups.

$$
\begin{tikzcd}
\underset{\text{of sets}}{\overset{\text{Category}}{}}\ \mathrm{Set} \\
\underset{\text{path groupoid}}{\overset{\text{Fundamental}}{}}\ \mathrm{Pth(B)} \arrow[u, "\substack{\text{Gauss-Manin} \\ \text{connection}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overset{\substack{\text{Fiberwise} \\ \text{twisted cohomology}}}{A^{\tau_{b_2}}(\mathrm{X}_{b_2})} \arrow[dr, "\sim"] & \\
A^{\tau_{b_1}}(\mathrm{X}_{b_1}) \arrow[ur, "\sim"] \arrow[rr, "\sim"'] & & A^{\tau_{b_3}}(\mathrm{X}_{b_3}) \\
& \{b_2\} \arrow[dr, "{[\gamma_{23}]}"] & \\
\{b_1\} \arrow[ur, "{[\gamma_{12}]}"] \arrow[rr, "{[\gamma_{23} \circ \gamma_{12}]}"'] & & \{b_3\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X} \arrow[rr, dashed, "\text{global twist}"] \arrow[dr, "\tau"'] \arrow[d, "p_{\mathrm{X}}"', "\substack{\text{fiber} \\ \text{bundle}}"] & & \overset{\text{Classifying fibration}}{\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}} \arrow[dl, "p_{\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}}"] \\
\mathrm{B} \arrow[dr, "p_{\mathrm{B}}"'] & B\mathrm{G} \arrow[d, "p_{B\mathrm{G}}"] & \\
& \ast &
\end{tikzcd}
$$

Twisted generalized non-abelian cohomology. In generalization of (5), we have:

Definition 2.10 (**Twisted generalized non-abelian cohomology [FSS20-Cha, §2.2][SS20-Orb, Rem. 2.94]).** For

- $\mathrm{G} \in \mathrm{Grp(kTopSpc)}$ a topological group;
- $\mathrm{G} \circlearrowright \mathrm{A} \in \mathrm{G\,Act}\big(\mathrm{kTopSpc}\big)$ a topological G-space,
- with $\big(\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}, p_{\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}}\big) \in \mathrm{kTopSpc}_{/B\mathrm{G}}$ its Borel construction;
- $\tau_b : \mathrm{X}_b \to B\mathrm{G}$ a continuous map;

we say that

$$\begin{aligned} A^{\tau_b}(\mathrm{X}_b) &:= H^{\tau_b}(\mathrm{X}_b;\mathrm{A}) \\ &:= \pi_0 \Gamma_{\mathrm{X}}\big((\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} EG)\big) \\ &\underset{(25)}{\simeq} (p_{BG})_* \mathrm{Map}\Big(\underbrace{(\mathrm{X}_b,\tau_b)}_{(\tau_b)_!(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b})},\, (\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\Big) \end{aligned} \tag{24}$$

is the τ_b*-twisted A-cohomology* of X_b.

To see the identification shown in (24), of the space of sections with a right base change of the fiberwise mapping space over the classifying space of twists, apply the Yoneda Lemma to the following sequence of natural bijections:

$$\begin{aligned}
&\mathrm{kTopSpc}\Big(\mathrm{U},\, (p_{BG})_*\mathrm{Map}\big((\tau_b)_!(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\big)\Big) && \\
&\simeq \mathrm{kTopSpc}_{/BG}\Big((p_{BG})^*\mathrm{U},\, \mathrm{Map}\big((\tau_b)_!(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\big)\Big) && \text{by (12)} \\
&\simeq \mathrm{kTopSpc}_{/BG}\Big(\big((p_{BG})^*\mathrm{U}\big)\times(\tau_b)_!(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b}),\, (\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\Big) && \text{by (10)} \\
&\simeq \mathrm{kTopSpc}_{/BG}\Big((\tau_b)_!\big((p_{\mathrm{X}_b})^*\mathrm{U}\times(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b})\big),\, (\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\Big) && \text{by (16)} \\
&\simeq \mathrm{kTopSpc}_{/\mathrm{X}_b}\Big((p_{\mathrm{X}_b})^*\mathrm{U}\times(\mathrm{X}_b,\mathrm{id}_{\mathrm{X}_b}),\, (\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\Big) && \text{by (12)} \\
&\simeq \mathrm{kTopSpc}_{/\mathrm{X}_b}\Big((p_{\mathrm{X}_b})^*\mathrm{U},\, (\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\Big) && \text{by (11)} \\
&\simeq \mathrm{kTopSpc}\Big(\mathrm{U},\, (p_{\mathrm{X}_b})_*\big((\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\big)\Big) && \text{by (12)} \\
&\simeq \mathrm{kTopSpc}\Big(\mathrm{U},\, \Gamma_{\mathrm{X}_b}\big((\tau_b)^*(\mathrm{A}\times_{\mathrm{G}} EG, p_{\mathrm{A}\times_{\mathrm{G}}EG})\big)\Big) && \text{by (15)}\,.
\end{aligned} \tag{25}$$

Example 2.11 (Ordinary complex cohomology with coefficients in a local system). For each $n\in\mathbb{N}$, the canonical action of $\mathrm{U}(1)\subset\mathbb{C}^\times$ on $\mathbb{C}$ induces an action on the nth Eilenberg-MacLane space $\mathrm{U}(1)\circlearrowright K(\mathbb{C},n)$. For $\tau_b : \mathrm{Pth}(\mathrm{X}_b)\to \mathbf{B}\mathrm{U}(1)$ the classifying map of a flat connection (Rem. 2.8) on a complex line bundle (i.e., on the connected component $[b]$ a group homomorphism $\pi_1(\mathrm{B},b)\to\mathrm{U}(1)$), the corresponding twisted cohomology according to Def. 2.10 is the traditional cohomology with coefficients in the local system $\mathscr{L}(\tau)$ of parallel sections of this flat connection (e.g. [Vo03I, §5.1.1]):

$$K(\mathbb{C},n)^\tau(\mathrm{X}_b) \;=\; H^{n+\tau}(\mathrm{X}_b;\mathbb{C}) \;\simeq\; H^n\big(\mathrm{X}_b;\mathscr{L}(\tau)\big)\,.$$

Now given $\mathrm{G}\circlearrowright\mathrm{A}$, as in Def. 2.10, and a fibration $(\mathrm{X},p_{\mathrm{X}})\in\mathrm{kTopSpc}_{/\mathrm{B}}$, as before in §2.1, consider a choice of *global twist*, namely a continuous map

$$\tau : \mathrm{X}\longrightarrow B\mathrm{G}\,.$$

Via this twist, we may regard X as fibered over the product space $\mathrm{B}\times B\mathrm{G}$, whence its fibers are also still fibered over $B\mathrm{G}$

$$\big(\mathrm{X},(p_{\mathrm{X}},\tau)\big)\;\in\;\mathrm{kTopSpc}_{/\mathrm{B}\times B\mathrm{G}}\,,\qquad\Rightarrow\qquad \underset{b\in\mathrm{B}}{\forall}\;\; (i_b\times\mathrm{id}_B\mathrm{G})^*\big(\mathrm{X},(p_{\mathrm{X}},\tau)\big)\;=\;(\mathrm{X}_b,\tau_b)\;=\;(\tau_b)_!\mathrm{X}_b\;\;\in\mathrm{kTopSpc}_{/B\mathrm{G}}\,.$$

Forming the fiberwise mapping space (9) in this sense, we obtain the following twisted generalization of (8):

$$\begin{tikzcd}
\overset{\text{Fiberwise mapping space over classifying space of twists}}{\mathrm{Map}\big((\mathrm{X}_b,\tau_b),\,(A\times_{\mathrm{G}}EG,p_{A\times_{\mathrm{G}}EG})\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \overset{\text{Fiberwise mapping space (itself a topological space over } \mathrm{B}\times B\mathrm{G})}{\mathrm{Map}\big((\mathrm{X},(p_{\mathrm{X}},\tau)),\,p_{\mathrm{B}}^*(\mathrm{A}\times_{\mathrm{G}}EG,p_{A\times_{\mathrm{G}}EG})\big)} \arrow[d] \\
\underset{\text{Classifying space of twists}}{\{b\}\times B\mathrm{G}} \arrow[r, hook, "i_b\times\mathrm{id}_{B\mathrm{G}}"'] & \underset{\text{Space of parameters and twists}}{\mathrm{B}\times B\mathrm{G}}
\end{tikzcd} \tag{26}$$

In order to turn this into a pullback diagram over just B we need the *Beck-Chevalley relation*:

Proposition 2.12 (Cartesian Beck-Chevalley property). *Given a fiber product diagram in* kTopSpc *as shown on the left below, the possible composite base changes* (12) *through the diagram are naturally isomorphic as shown on the right:*

$$\begin{tikzcd}[column sep=small]
& \mathrm{X}\times_{\mathrm{B}}\mathrm{Y} \arrow[dl, "\mathrm{pr}_{\mathrm{X}}"'] \arrow[dr, "\mathrm{pr}_{\mathrm{Y}}"] \arrow[dd, phantom, "\scriptstyle(\mathrm{pb})"] & \\
\mathrm{X} \arrow[dr, "p_{\mathrm{X}}"'] & & \mathrm{Y} \arrow[dl, "p_{\mathrm{Y}}"] \\
& \mathrm{B} &
\end{tikzcd}
\qquad\Rightarrow\qquad
\left\{\begin{aligned}
(p_{\mathrm{Y}})^*\circ(p_{\mathrm{X}})_! &\;\simeq\; (\mathrm{pr}_{\mathrm{Y}})_!\circ(\mathrm{pr}_{\mathrm{X}})^*\,, \\
(p_{\mathrm{X}})^*\circ(p_{\mathrm{Y}})_* &\;\simeq\; (\mathrm{pr}_{\mathrm{X}})_*\circ(\mathrm{pr}_{\mathrm{Y}})^*\,.
\end{aligned}\right. \tag{27}$$

Proof. The first isomorphism in (27) follows immediately from the pasting law (1), which for $(\mathrm{E}, p_{\mathrm{E}}) \in \mathrm{kTopSpc}_{/\mathrm{X}}$ gives the following natural identification:

$$
\begin{tikzcd}
(p_{\mathrm{Y}})^*(p_{\mathrm{X}})_!\mathrm{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] \\
\mathrm{E} \arrow[r, "p_{\mathrm{X}} \circ p_{\mathrm{E}}"'] & \mathrm{B}
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
(\mathrm{pr}_{\mathrm{X}})^* E \arrow[r] \arrow[d] \arrow[rr, bend left, "p_{(\mathrm{pr}_{\mathrm{Y}})_!(\mathrm{pr}_{\mathrm{X}})^*\mathrm{E}}"] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{X} \times_{\mathrm{B}} \mathrm{Y} \arrow[r, "\mathrm{pr}_{\mathrm{Y}}"] \arrow[d, "\mathrm{pr}_{\mathrm{X}}"] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \mathrm{Y} \arrow[d, "p_{\mathrm{Y}}"] \\
E \arrow[r, "p_{\mathrm{E}}"'] & \mathrm{X} \arrow[r, "p_{\mathrm{X}}"'] & \mathrm{B}\,.
\end{tikzcd}
$$

This implies the second natural isomorphism by adjointness (12) and the Yoneda Lemma:

$$
\begin{aligned}
\mathrm{kTopSpc}_{/\mathrm{X}}\big((\mathrm{U}, p_{\mathrm{U}}),\, (p_{\mathrm{X}})^*(p_{\mathrm{Y}})_*(\mathrm{E}, p_{\mathrm{E}})\big) & \simeq \mathrm{kTopSpc}_{/\mathrm{B}}\big((p_{\mathrm{X}})_!(\mathrm{U}, p_{\mathrm{U}}),\, (p_{\mathrm{Y}})_*(\mathrm{E}, p_{\mathrm{E}})\big) \\
& \simeq \mathrm{kTopSpc}_{/\mathrm{Y}}\big((p_{\mathrm{Y}})^*(p_{\mathrm{X}})_!(\mathrm{U}, p_{\mathrm{U}}),\, (\mathrm{E}, p_{\mathrm{E}})\big)\,,
\end{aligned}
$$

and similarly for the other side of the isomorphism. □

Now considering the Beck-Chevalley relation (27) for the following special case

$$
\begin{tikzcd}
& B\mathrm{G} \arrow[dl, "\mathrm{p}_{BG}"'] \arrow[dr, "i_B \times \mathrm{id}_{BG}"] & \\
* \arrow[dr, "i_b"'] \arrow[rr, phantom, "\scriptstyle(\mathrm{pb})"] & & \mathrm{B} \times B\mathrm{G}\,, \arrow[dl, "\mathrm{id}_{\mathrm{B}} \times p_{BG}"] \\
& \mathrm{B} &
\end{tikzcd}
\qquad\qquad
(i_b)^* \circ \big(\mathrm{id}_{\mathrm{B}} \times p_{BG}\big)_* \;\simeq\; \big(p_{BG}\big)_* \circ \big(i_b \times \mathrm{id}_{BG}\big)^* \tag{28}
$$

implies from (26) the pullback diagram:

$$
\begin{tikzcd}
\overset{\text{Space of sections over fiber}}{(p_{BG})_*\mathrm{Map}\big((\mathrm{X}_b, \tau_b),\, (A \times_G EG, P_{A \times_G EG})\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \overset{\text{Fiberwise space of sections (itself a topological space over B)}}{(\mathrm{id}_{\mathrm{B}} \times p_{BG})_*\mathrm{Map}\big((\mathrm{X}, p_{\mathrm{X}}),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} EG\big).} \arrow[d] \\
\{b\} \arrow[r, hook, "i_b"'] & \underset{\text{Space of parameters and twists}}{\mathrm{B}}
\end{tikzcd}
\tag{29}
$$

Since the classifying space $B\mathrm{G}$ – in its construction due to Milgram: $B\mathrm{G} = |N(\mathrm{G} \rightrightarrows *)|$ (recalled, e.g., in [SS21-Bun, (2.64)]) – is a CW-complex and hence Serre-cofibrant, the map on the right is still a Serre fibration, so that passing to parameterized connected components works as before in (22) to yield a covering space, generalizing (23), whose fiber over $b \in \mathrm{B}$ is the τ_b-twisted A-cohomology (Def. 2.10) of the fiber X_b:

$$
\begin{tikzcd}
\overset{\tau\text{-twisted } A\text{-cohomology of fiber}}{A^{\tau}(X_b)} \simeq \overset{\text{Connected components of space of sections over fiber}}{\pi_0\Big((p_{BG})_*\mathrm{Map}\big((\mathrm{X}_b, \tau_b),\, (A \times_G EG, P_{A \times_G EG})\big)\Big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(\mathrm{pb})"] & \overset{\text{Fiberwise connected components of fiberwise space of sections (itself a topological space over B)}}{\pi_{0/\mathrm{B}}\Big((\mathrm{id}_B \times p_B)_*\mathrm{Map}\big((\mathrm{X}, (p_{\mathrm{X}}, \tau)),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} EG\big)\Big).} \arrow[d] \\
* \arrow[r, hook, "i_b"'] & \underset{\text{Space of parameters and twists}}{\mathrm{B}}
\end{tikzcd}
\tag{30}
$$

As before in the untwisted case, Prop. 2.5 again implies that this covering space trivializes compatibly with any local trivialization of $(\mathrm{X}, p_{\mathrm{X}})$, thus exhibiting its corresponding classifying map $\nabla^{\mathrm{GM}}_{X, G \circlearrowright A}$ (via Lem. 2.7) as the Gauss-Manin connection (cf. the description in [EFK98, §7.5]).

In conclusion, we have now shown the following generalization of Thm. 2.9 to twisted cohomology:

Theorem 2.13 (Gauss-Manin connection in twisted generalized cohomology over fiber bundles via fiberwise mapping spaces). *Let* $\mathrm{B} \in \mathrm{kHaus}$ *be a metrizable space and* $(\mathrm{X}, p_{\mathrm{X}}) \in \mathrm{kTopSpc}_{/\mathrm{B}}$ *be a locally trivial fiber bundle whose typical fiber admits the structure of a CW-complex. Then for any discrete or compact Lie* $\mathrm{G} \in \mathrm{Grp}(\mathrm{kTopSpc})$ *and* $G \circlearrowright \mathrm{A} \in \mathrm{G}\,\mathrm{Act}(\mathrm{kTopSpc})$*, the Gauss-Manin-connection on the twisted A-cohomology sets (24) of the fibers* X_b *is exhibited by the fiberwise 0-truncation of the right base change along* $B\mathrm{G}$ *(29) of the fiberwise mapping space (26) from* X *into the Borel construction* $\mathrm{A} \times_{\mathrm{G}} E\mathrm{G}$*:*

$$
\underset{\substack{\text{Gauss-Manin} \\ \text{connection on} \\ \text{twisted } A\text{-cohomology}}}{\nabla^{\mathrm{GM}}_{X, G \circlearrowright A}}
\quad \underset{\text{Lem. 2.7}}{\longleftrightarrow} \quad
\underset{\text{Fiberwise 0-truncation of right base change along } BG \text{ of fiberwise mapping space into Borel construction on } A}{\pi_{0/\mathrm{Pth}(\mathrm{B})}\Big(\mathrm{Pth}\,(\mathrm{id}_{\mathrm{B}} \times p_{BG})_*\mathrm{Map}\big((\mathrm{X}, (p_{\mathrm{X}}, \tau)),\, p_{\mathrm{B}}^*\mathrm{A} \times_{\mathrm{G}} EG\big)\Big)}\,.
$$

Example 2.14 (**The Knizhnik-Zamolodchikov connection of $\widehat{\mathfrak{su}_2}^k$-conformal blocks).** Consider the situation reviewed in [EFK98, §7] and discussed in [SS22-DBr][SS22-Ord][SS22-TQC], where:

- $\mathrm{B} := \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})$ is the configuration space of N points in the plane;
- $\mathrm{X} := \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{C}) \xrightarrow{(p_N^{n+N},\tau)} \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})$ is fibration which forgets the first n of $n+N$ points in the plane;
- $\mathrm{G} = \mathbb{Z}_\kappa \subset \mathrm{U}(1)$ is a cyclic group, regarded as a subgroup of the circle group;
- $G \circlearrowright \mathrm{A} := \mathbb{Z}_\kappa \circlearrowright \mathrm{K}(\mathbb{C},n)$ is the restricted action on the EM-space from Ex. 2.11.

Then for a suitable choice of global twist

$$\tau : \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{C}) \longrightarrow B\mathbb{Z}_\kappa\,,$$

the Gauss-Manin connection on the fiberwise twisted ordinary cohomology groups (Ex. 2.11)

$$H^n\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big); \mathscr{L}\big(\tau_{(z_I)}\big)\Big)$$

yields (reviewed in [EFK98, §7.5]) the Knizhnik-Zamolodchikov connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$ conformal blocks, for weights determined by the monodromies of τ. This is the result of the *hypergeometric integral construction* of KZ-solutions reviewed (and referenced) in [SS22-DBr][SS22-Ord].

By Theorem 2.13 with Exmaple 2.11, this is realized as equivalently reflected in the fiberwise 0-truncation of (a right base change of) a fiberwise mapping space:

$$\underset{\substack{\text{KZ-connection on}\\ \widehat{\mathfrak{su}_2}^{\kappa-2}\text{-conformal}\\ \text{blocks of degree } n\\ \text{with } N \text{ insertions}}}{\nabla^{\mathrm{KZ}}_{\kappa,N,n}} \underset{\text{Lem. 2.7}}{\longleftrightarrow} \underbrace{\pi_{0/\mathrm{Pth}\,\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})}\Big(\mathrm{Pth}\big(\mathrm{id}_{\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})} \times p_{B\mathbb{Z}_\kappa}\big)_* \mathrm{Map}\Big(\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big), (p_{\mathrm{Conf}},\tau)\Big), p^*_{\mathrm{Conf}}\mathrm{K}(\mathbb{C},n) \times_{\mathbb{Z}_\kappa} E\mathbb{Z}_\kappa\Big)\Big)}_{\text{Fiberwise 0-truncation of right base change along } B\mathbb{Z}_\kappa \text{ of fiberwise mapping space from configuration space into Eilenberg-MacLane fiber bundle}} \tag{31}$$

3 Via dependent homotopy type theory

We briefly explain how theorem 2.13 implies that Gauss-Manin connections on fiber bundles, and in particular the Knizhnik-Zamolodchikov connections of Example 2.14, have a curiously direct formulation in the language of homotopy type theory (HoTT), under its *interpretation* into homotopy theory (reviewed in [Sh12, §3][Ri22]).

The basis of this interetation is the following dictionary, under which the fiberwise homotopy theory recalled in §2.1 becomes essentially tautologous:

Colloquial	**Homotopy-type theory**	**Homotopy type-theory**
object / type	$\mathscr{B} \in \mathbf{H} \simeq \mathbf{H}_{/*}$	$\bullet : * \;\vdash\; \mathscr{B} : \mathrm{Type}$
bundle/fibration / dependent type	$\begin{array}{c}\mathscr{X}\\ \downarrow{\scriptstyle p_{\mathscr{X}}}\\ \mathscr{B}\end{array} \in \mathbf{H}_{/\mathscr{B}}$	$b : \mathscr{B} \;\vdash\; \mathscr{X}(b) : \mathrm{Type}$
fiberwise mapping space / dependent function type	$\mathrm{Map}\big((\mathscr{X}, p_{\mathscr{X}}), (\mathscr{Y}, p_{\mathscr{Y}})\big) \in \mathbf{H}_{/\mathscr{B}}$	$b : \mathscr{B} \;\vdash\; \big(\mathscr{X}(b) \longrightarrow \mathscr{Y}(b)\big) : \mathrm{Type}$
morphism / function	$\begin{tikzcd} \mathscr{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathscr{X}}"'] & & \mathscr{Y} \arrow[dl, "p_{\mathscr{Y}}"] \\ & \mathscr{B} & \end{tikzcd}$	$b : \mathscr{B} \;\vdash\; f : \mathscr{X}(b) \longrightarrow \mathscr{Y}(b)$

map between bases / **map between contexts**	$\mathscr{B}_1 \xrightarrow{f} \mathscr{B}_2$	$\bullet : * \;\vdash\; f : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$
base change / variable substitution	$f_* : \mathbf{H}_{/\mathscr{B}_2} \to \mathbf{H}_{/\mathscr{B}_1}$	$b_1 : \mathscr{B}_1, b_2 : \mathscr{B}_2, \mathscr{X}(b_2) : \mathrm{Type}, \;\vdash\; \mathscr{X}\big(f(b_1)\big) : \mathrm{Type}$
left base change / dependent sum	$f_! : \mathbf{H}_{/\mathscr{B}_1} \to \mathbf{H}_{/\mathscr{B}_2}$	$b_1 : \mathscr{B}_1, b_2 : \mathscr{B}_2, \mathscr{X}(b_1) : \mathrm{Type}, \;\vdash\; \Big(\sum\limits_{f(b_1)=b_2} \mathscr{X}(b_1)\Big) : \mathrm{Type}$
right base change / dependent product	$f_* : \mathbf{H}_{/\mathscr{B}_1} \to \mathbf{H}_{/\mathscr{B}_2}$	$b_1 : \mathscr{B}_1, b_2 : \mathscr{B}_2, \mathscr{X}(b_1) : \mathrm{Type}, \;\vdash\; \Big(\prod\limits_{f(b_1)=b_2} \mathscr{X}(b_1)\Big) : \mathrm{Type}$

Here we write $\mathbf{H} := \mathrm{Grpd}_\infty$ for the ambient ∞-category of plain homotopy theory, and $\mathscr{X} = \int \mathrm{X}$, etc., following notation in [SS20-Orb][SS21-Bun].

Moreover, forming fiberwise connected components (21) is the archetypical *modality* in homotopy type theory (e.g. [UFP13, §7.3][RSS20]):

fiberwise cnctd components / dependent 0-truncation	$\pi_{0/\mathscr{B}}(\mathscr{X}, p_{\mathscr{X}}) \in \mathbf{H}_{/\mathscr{B}}$	$b : \mathscr{B} \;\vdash\; \big[\mathscr{X}(b)\big]_0 : \mathrm{Type}$

Under this translation, our Theorem 2.13 says that the following simple code in HoTT gives Gauss-Manin connections on twisted generalized cohomology:

Gauss-Manin connection	$\pi_{0/\mathscr{B}}\Big((p_{B\mathscr{G}})_* \mathrm{Map}\Big(\big(\mathscr{X}, (p_{\mathscr{X}}, \tau)\big), p^*_{\mathscr{B}}\mathscr{A} /\!\!/ \mathscr{G}\Big)\Big)$	$b : \mathscr{B} \;\vdash\; \Big[\prod\limits_{\tau : B\mathscr{G}} \big(\mathscr{X}(b, \tau) \to \mathscr{A}(\tau)\big)\Big]_0$

Concretely, for the purpose of Ex. 2.14, consider:

1. **The dependent type of points configurations.** The homotopy type of the configuration space of N points in the plane is equivalently that of the classifying space of the braid group $\mathrm{Br}(N)$ on N strands ([FN62, p. 118][FN62, §7], review in [Wi20, pp. 9]):
$$\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \underset{\text{whe}}{\simeq} B\mathrm{Br}(N)\,.$$
But the braid group is finitely generated (by the *Artin presentation* [FN62, §7]), and finitely generated groups together with their classifying spaces are readily defined in HoTT (see [BBCDG21, §6]). Moreover, since the map that forgets

the first n of $n+N$ points corresponds to forgetting the first few Artin generators of the braid group, it is straightforward to define in HoTT the fibration

$$
\begin{array}{ccccc}
\mathscr{X} & \xrightarrow{\qquad\qquad\qquad} & \mathscr{B} & & \scriptsize\begin{array}{c}\text{General homotopy fibration}\\ \text{used here in §3}\end{array} \\
\overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & & \\
\int \mathrm{X} & \xrightarrow{\qquad \int p_{\mathrm{X}} \qquad} & \int \mathrm{B} & & \scriptsize\begin{array}{c}\text{Shape of}\\ \text{general fiber bundle as in §2}\end{array} \\
\overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & & \\
\int \underset{\{1,\cdots,n+N\}}{\mathrm{Conf}}(\mathbb{C}) & \xrightarrow{\qquad\qquad} & \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) & & \scriptsize\begin{array}{c}\text{Fibration of Ex. 2.14}\\ \text{definable in cohesive HoTT}\end{array} \\
\wr\Vert & & \wr\Vert & & \\
B\mathrm{Br}(n+N) & \xrightarrow{\qquad\qquad\qquad} & B\mathrm{Br}(N) & & \scriptsize\begin{array}{c}\text{Finite presentation}\\ \text{definable in plain HoTT}\end{array}
\end{array}
\tag{32}
$$

as a dependent type over $B\mathrm{Br}(N)$. (Here we are writing the *shape modality* "$\int$" to emphasize that this concerns just the bare homotopy type of the configuration spaces, disregarding their differential topology. While this shape operation involves notion and notation of *cohesive* HoTT proper, notice that in order to proceed with this example we may simply take the last line above as the definition of this fibration in plain HoTT.)
While the braid group is useful, this way, for providing the construction of this dependent type, the configuration spaces provide a more conceptually transparent picture of this dependent type, so that we (may and) will *denote* the above dependent type suggestively as follows:

$$
(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad \vdash \quad \int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big) \;:\; \mathrm{Type}
\tag{33}
$$

2. **The dependent type representing twisted cohomology.** The construction of Eilenberg-MacLane spaces like $\mathrm{K}(\mathbb{C},n)$ is well-understood in HoTT (e.g. [LF14]). The definition in HoTT of the canonical action of the cyclic group $\mathbb{Z}_\kappa \subset \mathrm{U}(1)$ on $K(A,n)$ is also straightforward (as in [BvDR18, §4.2]) and yields (e.g. [SS21-Bun, Lem. 2.3.23]) the homotopy type of the Borel construction:

$$
\begin{array}{cccccc}
\mathscr{A} & \longrightarrow & \mathscr{A}/\!\!/\mathscr{G} & \longrightarrow & B\mathscr{G} & \scriptsize\begin{array}{c}\text{General homotopy quotient}\\ \text{used here in §3}\end{array} \\
\overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & \\
\int \mathrm{A} & \longrightarrow & \int\big(\mathrm{A}\times_{\mathrm{G}} E\mathrm{G}\big) & \longrightarrow & B\mathrm{G} & \scriptsize\text{General Borel construction in §2.2} \\
\overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & & \overset{\cdot\cdot}{\Vert} & \\
K(\mathbb{C},n) & \longrightarrow & K(\mathbb{C},n)/\!\!/\mathbb{Z}_\kappa & \longrightarrow & B\mathbb{Z}_\kappa & \scriptsize\begin{array}{c}\text{Borel construction in Ex. 2.14}\\ \text{definable in plain HoTT}\end{array}
\end{array}
$$

The resulting dependent homotopy type we will denote by

$$
\tau : B\mathbb{Z}_\kappa \quad \vdash \quad K(\mathbb{C},n)(\tau)\,.
\tag{34}
$$

3. **The global twist.** Finally, using the generators and relations defining $\mathrm{Br}(n+N)$ and $\mathbb{Z}_\kappa$, it is straightforward to define global twists, being functions of this type:

$$
(z_I)_{I=1}^N \;:\; \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad \vdash \quad \tau \;:\; \int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big) \longrightarrow B\mathbb{Z}_\kappa\,.
$$

With these standard and/or straightforward type constructions in hand, it follows from Thm. 2.13 and Ex. 2.14 that the following line of HoTT code defines – under the interpretation of HoTT code into classical homotop theory – the KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks of degree n over the Riemann sphere with $N+1$ insertions (whose weights are encoded by the monodromies of τ):

KZ-connection on $\widehat{\mathfrak{su}_2}^{\kappa-2}$-conformal blocks	(31)	$(z_I)_{I=1}^N \;:\; \int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \quad \vdash \quad \left[\prod_{t:B\mathbb{Z}_\kappa}\Big(\int \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_I\}_{I=1}^N\big)(\tau) \longrightarrow K(\mathbb{C},n)(\tau)\Big)\right]_0$

(Recall here that $\int \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})$ etc. may be regarded as nothing but suggestive notation for types finitely presented by the Artin braid relations as in (32).)

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