Globalizing Parallel 2-Transport

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December 6, 2024

some diagrams shown during a talk at:

Multi-Parameter Signatures

sites.google.com/view/2dsignatures

NTNU Trondheim, Dec 2024

an event in algebraic data science
investigating the use of
surface-transport as a machine-learnable feature
of 2d-parameterized data.

Based on:

Early discussion of global 2-transport:
Sc05: hep-th/0509163

Further reading:

Relating local data to differential connection 2-forms:
SW11: arXiv:0802.0663

The full Čech 2-cohomology theory:
SW13: arXiv:0808.1923

More high-powered discussion of higher principal bundles:
SS23: arXiv:2112.13654.

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CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

Myself, had started to think about non-abelian surface transport in my PhD thesis, defended in 2005:

arXiv > hep-th > arXiv:hep-th/0509163

High Energy Physics - Theory

[Submitted on 21 Sep 2005]

From Loop Space Mechanics to Nonabelian Strings

Urs Schreiber

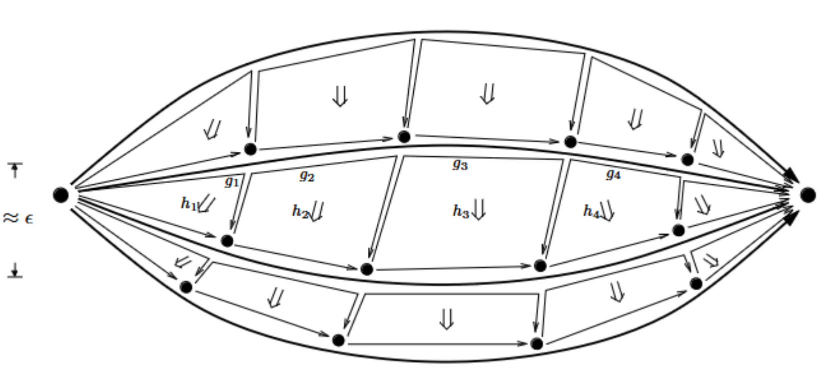
Lifting supersymmetric quantum mechanics to loop space yields the superstring. A particle charged under a fiber bundle thereby turns into a string charged under a 2-bundle, or gerbe. This stringification is nothing but categorification. We look at supersymmetric quantum mechanics on loop space and demonstrate how deformations here give rise to superstring background fields and boundary states, and, when generalized, to local nonabelian connections on loop space. In order to get a global description of these connections we introduce and study categorified global holonomy in the form of 2-bundles with 2-holonomy. We show how these relate to nonabelian gerbes and go beyond by obtaining global nonabelian surface holonomy, thus providing a class of action functionals for nonabelian strings. The examination of the differential formulation, which is adapted to the study of nonabelian p-form gauge theories, gives rise to generalized nonabelian Deligne hypercohomology. The (possible) relation of this to strings in Kalb-Ramond backgrounds, to M2/M5-brane systems, to spinning strings and to the derived category description of D-branes is discussed. In particular, there is a 2-group related to the String-group which should be the right structure 2-group for the global description of spinning strings.

11.4 2-Holonomy in Terms of Local p-Forms

In this subsection the proof of the central proposition 11.2 (p. 260) is sketched in a way that is supposed to clearly point out the underlying mechanisms in a concise way. Several technicalities that these proofs rely on are then discussed in detail in §11.5 (p. 268).

11.4.1 Definition on Single Overlaps

Consider any bigon Σ in a patch U_i, i.e. a 2-morphism in $\mathcal{P}_2(U_i)$ (def. 11.13), and consider a local 2-holonomy functor $\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to \mathcal{G}$ (def. 11.15). Since hol_i is a functor, the 2-group 2-morphism which it associates to Σ can be computed by dividing Σ into many small sub-bigons, evaluating hol_i on each of these and composing the result in $\mathcal{G}$. This is illustrated in the following sketchy figure.



This was motivated from high energy physics (HEP).

Motivations and Perspectives.

- *in classical HEP*

  classical background connection ("gauge field") is the physical datum to be understood

  probed by paths ("Wilson loops")

  or surfaces ("Wilson surfaces")

  that are chosen as prescribed observables

  Theorem: Connections are fully characterized by their path holonomy.(e.g. [SW09])

  Analogously for 2-connections and 2-holonomy. (e.g. [SW13])

- *in quantum HEP*

  one imagines "averaging" (path integral) over all background fields

  and now the averaged parallel transport characterizes the path!

  cf. knot invariants via Chern-Simons theory

- *in data science* (I gather) it is currently

  the background connection that is chosen by hand

  in fact to be the most trivial/tautological one,

  and information about the path/surface to be extracted

In HEP instead of using the free Lie algebra, one would use $\mathfrak{u}(N)$ for arbitrary/large N.

In HEP one cares also about connections **on non-trivial bundles**: "instantons", "solitons".

This is relevant when the target space X of the paths/surfaces $[0,1]^n \to X$ is no longer just Euclidean space $\mathbb{R}^d$.

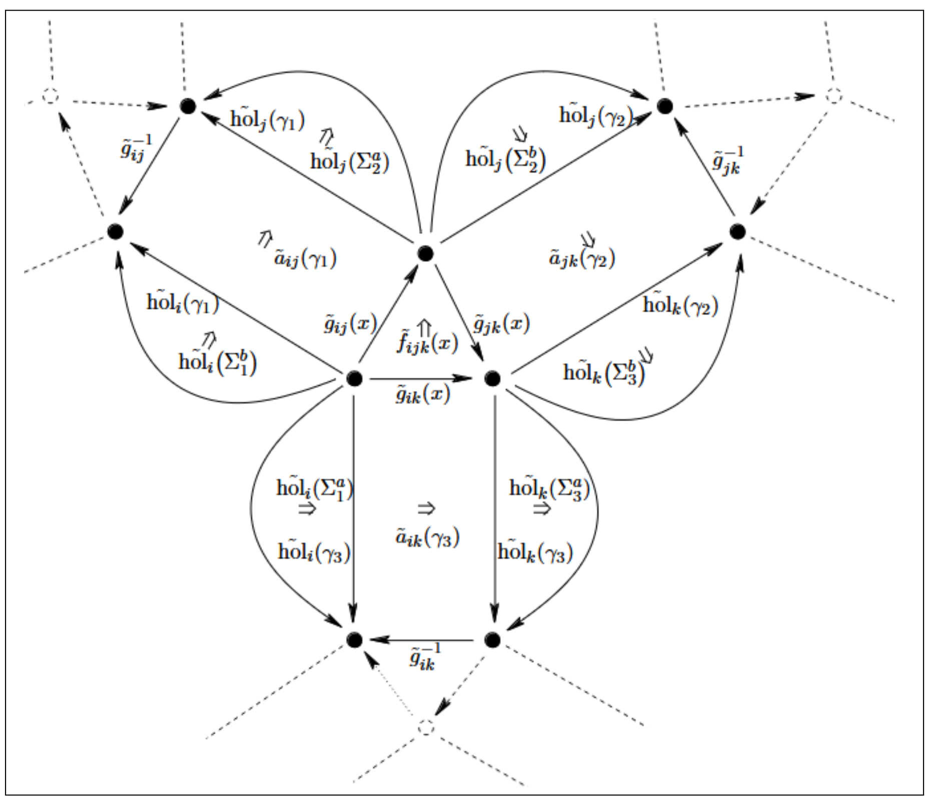
In data science this case may remain underappreciated, but consider for instance:

$$X := \left\{ \begin{array}{c} \text{points on earth} \\ \text{\& wind direction} \end{array} \right\}$$

$$= S(TS^2) \simeq \mathbb{R}P^3 \simeq S^3/\mathbb{Z}_2$$

In this case the local surface holonomy needs to be accompanied by *Čech cocycle data*

Indicated on the right, now to be explained.



The isse is:

On a general manifold X
connection data in general only exists *locally*
namely on an *open cover*

$$C \;:=\; \coprod_{i \in I} U_i \xrightarrow{(\iota_i)_{i \in I}} X \qquad \text{for} \qquad \left\{ \mathbb{R}^n \simeq U_i \underset{\text{opn}}{\overset{\iota_i}{\hookrightarrow}} X \right\}_{i \in I}$$

and so in order to continue parallel transport
from one chart U_i to the next U_j
one needs a *gauge transformation*
relating the connection on U_i with that on U_j
where they overlap, on $U_{ij} \;:=\; U_i \cap U_j$

and this must be consistent:
gauge transforming from U_i to U_j and from there to U_k
must be the same as transforming directly from U_i to U_k

or at least these two gauge transformations
must be *gauge-of-gauge* equivalent on $U_{ijk} := U_i \cap U_j \cap U_k$

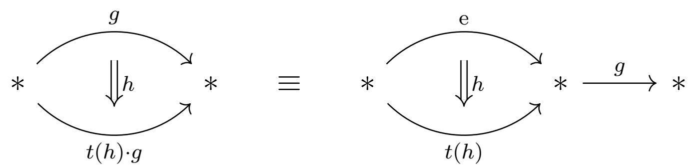
and so on.

Such data subject to such consistency conditions
is called *Čech cocycle data*.

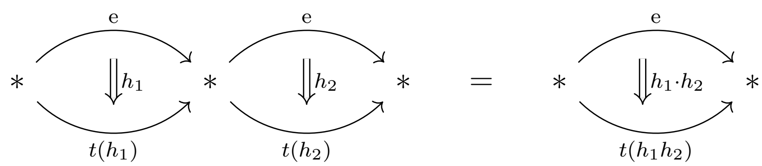
Labelling surface elements by strict 2-groupoids coming from crossed modules of groups means:

Consider a group homomorphisms $H \xrightarrow{t} G$

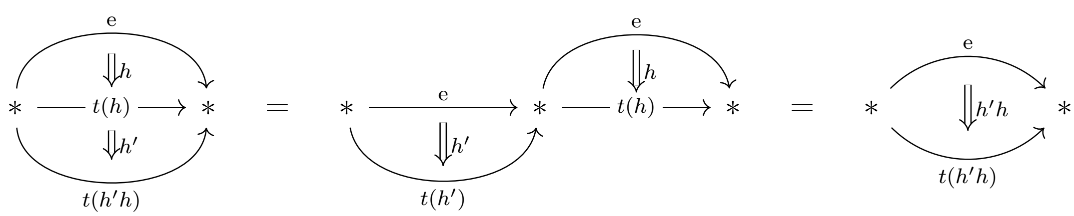
labelling surface elements with pairs $g \in G$ and $h \in H$ like this:



such that

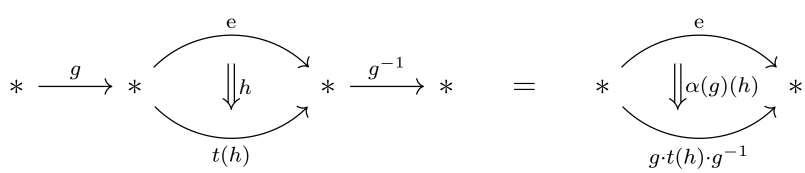


and



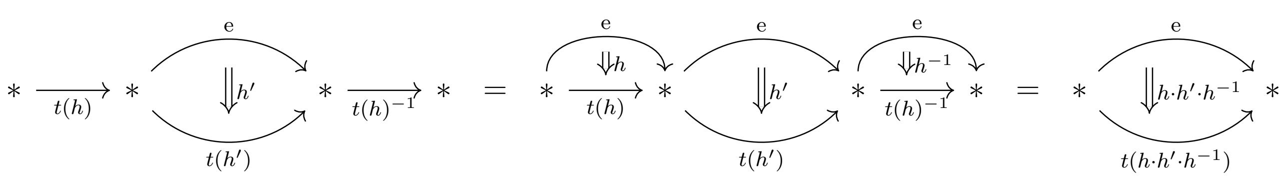
and such that the order of horizontal/vertical composition doesn't matter.

This implies that



for some group homomorphism $\alpha : \; G \longrightarrow \mathrm{Aut}(H)$ such that $t\big(\alpha(g)(h)\big) \; = \; g \cdot t(h) \cdot g^{-1}$

and that



hence that $\alpha\big(t(h)\big)(h') \; = \; h \cdot h' \cdot h^{-1}$

The boxed items make a **crossed module of groups**.

First consider the ordinary case of

Parallel 1-Transport along paths.

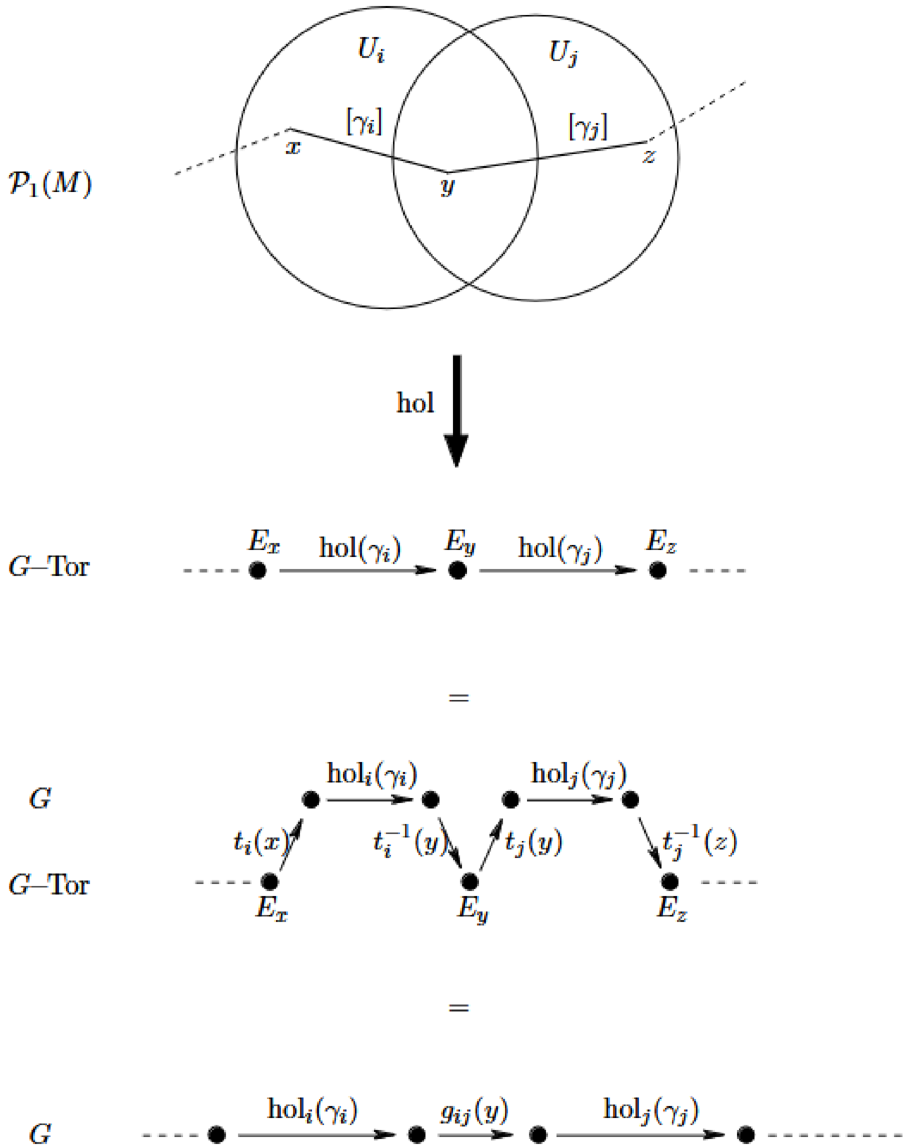
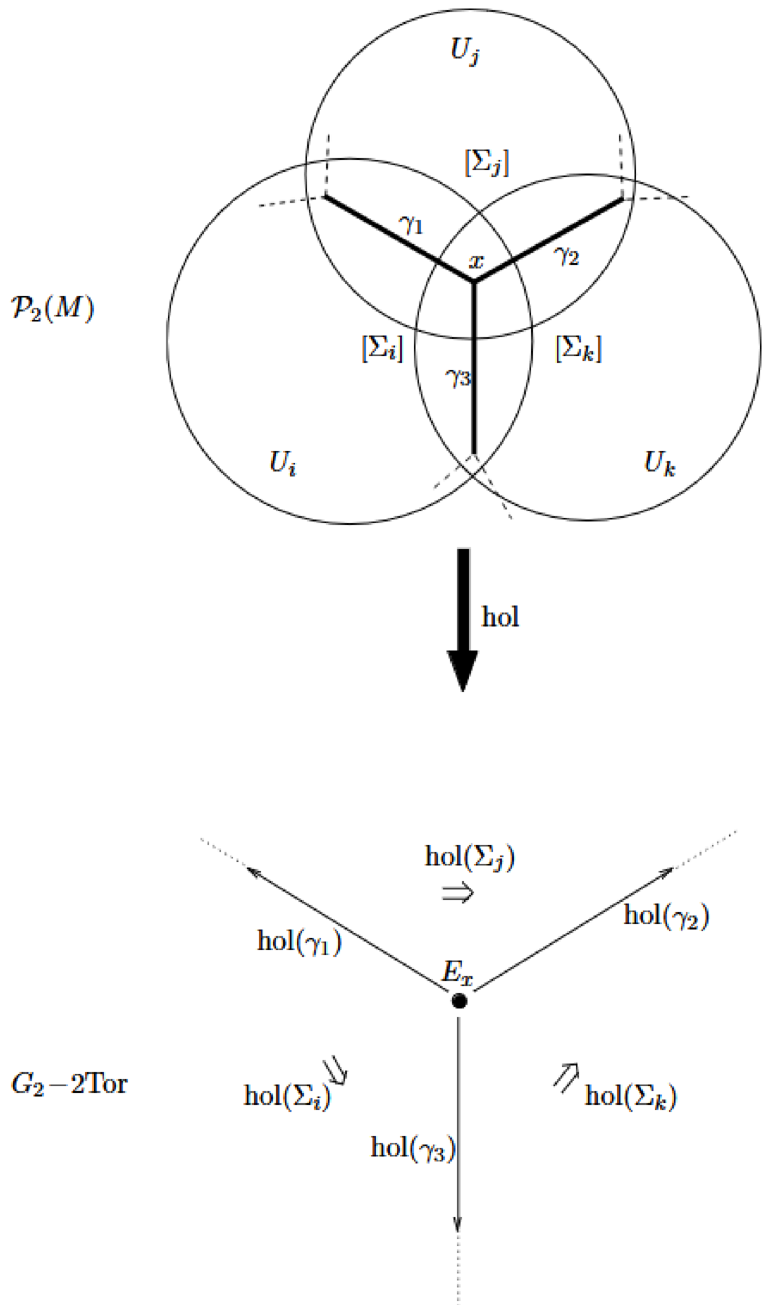


Figure 11: Global (1-)holonomy in terms of (1-)torsor (1-)morphisms. The functor hol associates torsor morphisms between fibers to paths in the base manifold. Using trivializations t_i on patches U_i these torsor morphisms can be identified with elements of the structure group. The step $t_i^{-1} \circ t_j$ from one trivialization to another one on double overlaps U_{ij} gives rise to multiplication by the transition function g_{ij}.

To generalize this to

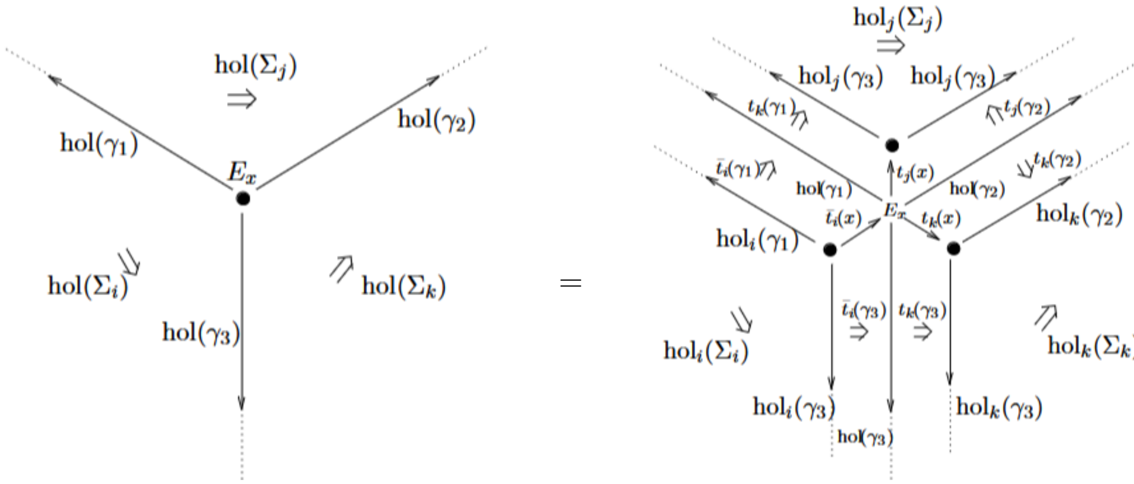
2-Transport along surfaces.

12.4.2.2 Surface Holonomy in Terms of Local Trivializations. Now consider the application of hol to any 2-morphism $[\Sigma] \in \mathcal{P}_2(M)$ that does not necessarily sit in a single patch.

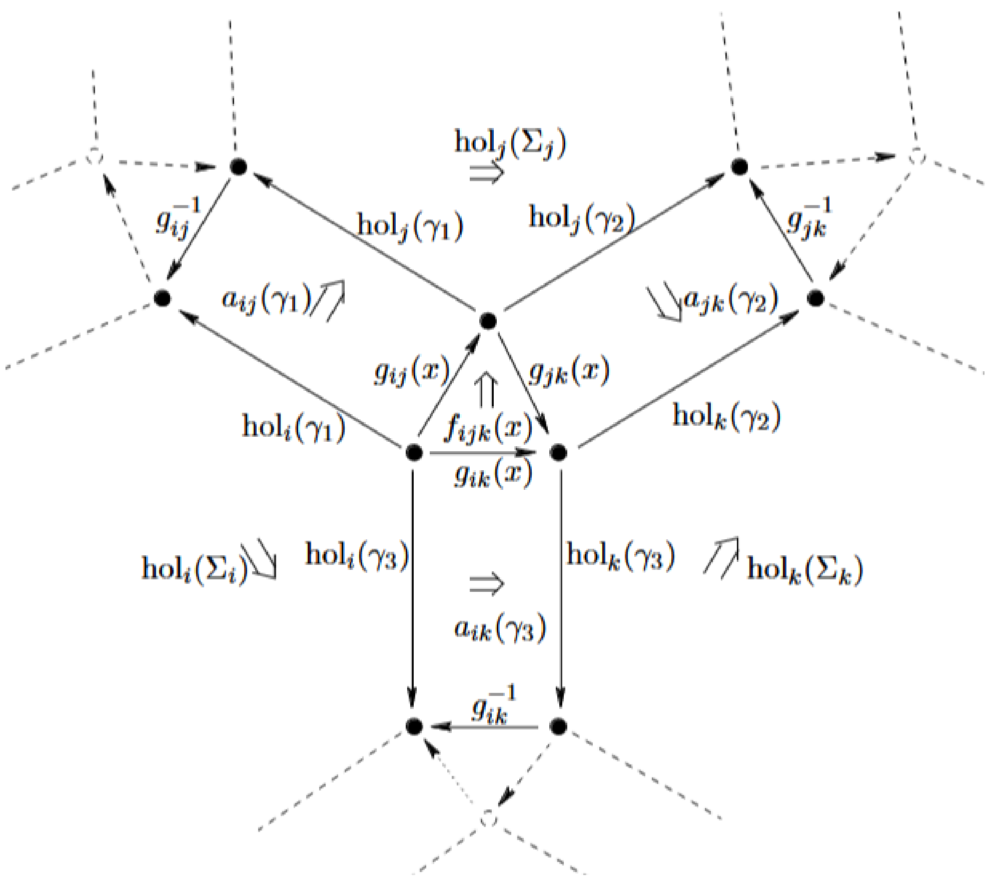


We can decompose $[\Sigma]$ into several 2-morphisms that all do sit inside a single patch of the covering. Their images under hol can be regarded as the "bottom" (leftmost surface) of the tincan diagram (12.11). Since this tincan diagram 2-commutes, we can replace each hol(Σ_i) by the the respective tincan with its leftmost surface cut out. This is shown in the following figure.

Now

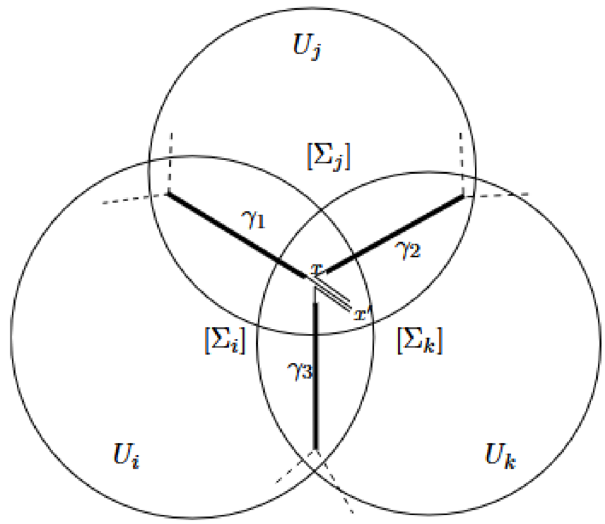


and hence the 2-holonomy is given by

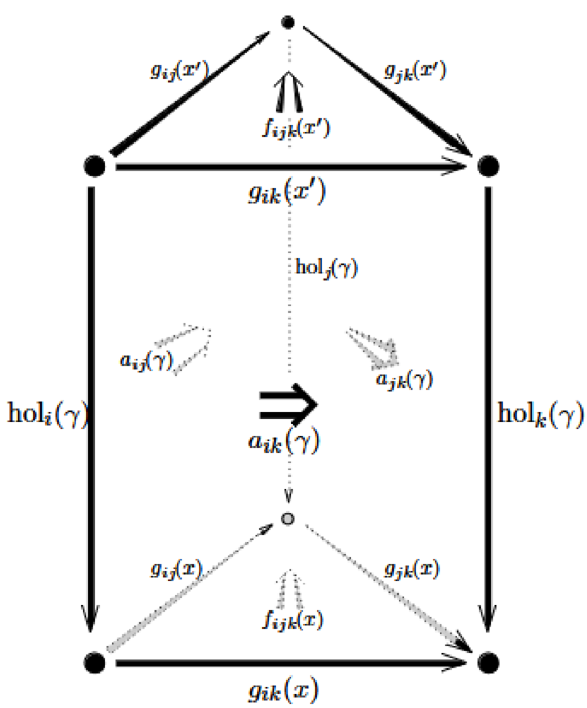


Consistency conditions. For this 2-homology not to depend on the choice of where to insert the gauge transformations, the following conditions must be met:

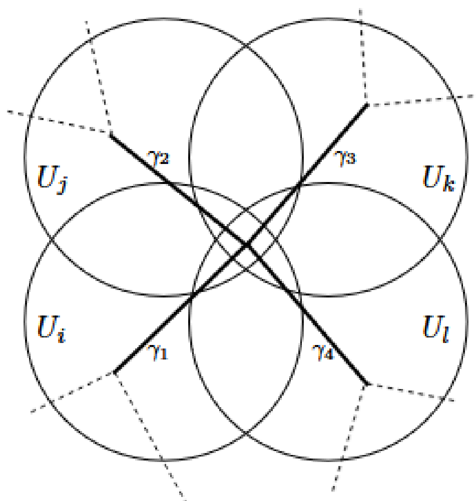
One comes from the condition that degenerate surfaces do not contribute. Consider moving $x \in U_{ijk}$ to $x' \in U_{ijk}$ by extending all of γ_1, γ_2, γ_3 by the *same* path $x \to x'$.

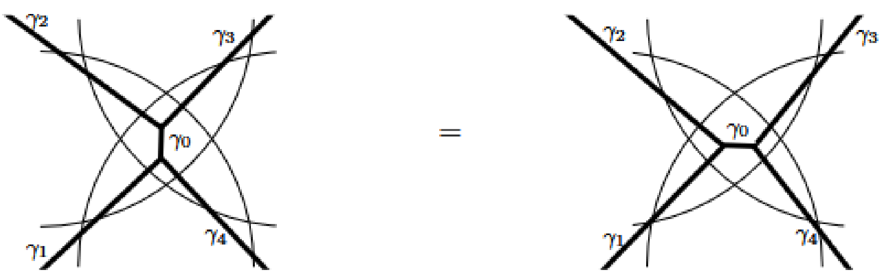


This removes $f_{ijk}(x)$ in the above diagram and replaces it by a diagram of the above form around x' but with all Σ_i vanishing. Since this must not contribute, this diagram has to equal the 2-morphism $f_{ijk}(x)$ that was replaced. In other words, the a_{ij} must be such that the following diagram 2-commutes:



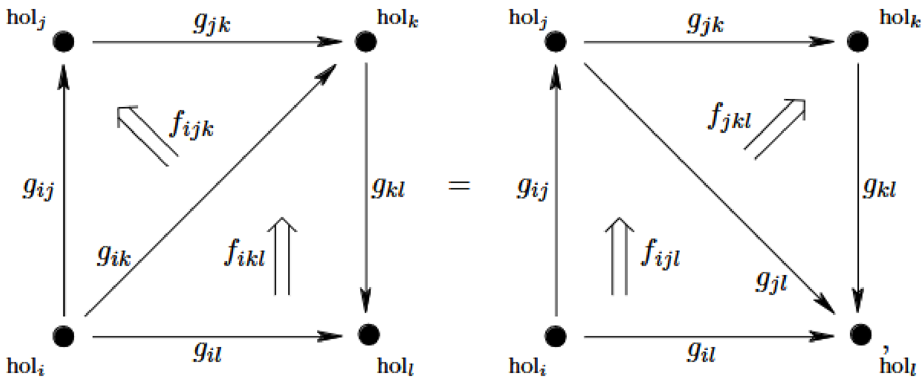
The other consistency condition is obtained by considering vertices at which more than three edges meet. Whenever this is the case, we can insert constant paths until only trivalent vertices are left. But these constant paths can be inserted in more than one way. Four a 4-valent vertex this is indicated by the following figure.



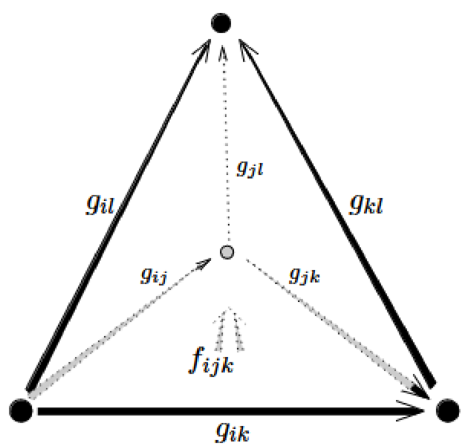


Here γ_0 denotes a constant path, which has been drawn with a spatial extension just for convenience. It is a well known theorem (see for instance [84]) that all triangulations can be obtained from any given one by a series of moves of two types, one of which is the one from one of the lower two pictures to the other. The other is the "bubble move" which was called the "left and right unit law" in §11.2.4 (p.251).

Invariance of hol under this move is expressed by the equation



which is equivalent to the 2-commutativity of this tetrahedron:



This is the tetrahedron transition law on quadruple overlaps known from equation (11.15) (p. 260) and from §11.2.4 (p.251).

This is the nonabelian Čech 2-cocycle condition for the **principal 2-bundle** underlying the 2-connection.