# Exposition of Higher Gauge Theory and Nonabelian Differential Cohomology

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**Abstract**

The classical notion of *principal connections* is fundamental in mathematics (Lie theory, Chern-Weil theory, Cartan geometry) and in quantum physics (gauge theory, Dyson series, Berry phases).

This survey reviews motivations and constructions for *higher*-structure enhancements of this notion that are finally finding attention (categorified symmetries, higher dimensional holonomy, higher gauge fields), amplifying that there are two nominally different higher generalizations in use, modeled alternatively on:

**(i)** Chern-Weil theory of connection forms,
**(ii)** Chern-Dold theory of character forms.

In the "ordinary" abelian case, these perspectives coincide and are well-studied (Deligne cohomology, Cheeger-Simons characters, ordinary differential cohomology), but crucial applications require their non-abelian generalizations which have received less attention.

We motivate and survey both directions of nonabelian higher connection theory, with reference to our models of

**(i)** Čech cocycles for differential characteristic classes ([41][13], which underlies our original take on stringy gauge fields and branes) and
**(ii)** the character map on non-abelian cohomology ([16], which underlies our current take via non-abelian flux-quantization).

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# 1 Introduction and Overview

Our topic is — as we explain and make precise in a moment — nothing less than a unification of

| **infinitesimal analysis** | and | **spatial analysis,** |
|---|---|---|

both going back to Leibniz, the former famously so and the latter (Leibniz's *analysis situs*) revived by Poincaré [36] to become the modern

| **differential geometry** | and | **algebraic topology,** |
|---|---|---|

which we may usefully think of as the

| **local** | and | **global** |
|---|---|---|

aspects of the subject of non-discrete mathematics: continuous, cohesive and smooth space.

This already makes plausible the relevance to physics — though the intimacy of the relevance to *fundamental physics* has been a source of wonder in the past [1] and appears to only be strengthened by the further developments that we aim to review here. In short the above dichotomy in physics is [40] fundamentally that between

| **flux** | and | **charge.** |
|---|---|---|

More technically, these two aspects are respectively exhibited by imposing

| **differential equations** | and | **simplicial identities.** |
|---|---|---|

More precisely, we will explain that for our purposes this means to consider:

$$\begin{tikzcd} & \textbf{Differential generalized nonabelian cohomology} \arrow[dl, "\text{flux}"'] \arrow[dr, "\text{charge}"] & \\ \textbf{Higher Lie-algebra valued differential forms satisfying Maurer-Cartan PDEs} & \text{and} & \textbf{Cocycles in generalized nonabelian cohomology} \end{tikzcd}$$

In full detail, we will explain that this means to consider the *homotopy fiber product* of the corresponding "moduli stacks", which in §2 will be expressed by the following symbols:

$$\begin{tikzcd} & \mathbf{H}^1(-;\,\Omega\mathcal{A})_{\mathrm{diff}} \arrow[dl, "\text{flux}"'] \arrow[dr, "\text{charge}"] & \\ \Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{flat}} \arrow[dr, "\eta^{\int}", "\text{shape unit}"'] & & \mathbf{H}^1(-;\,\Omega\mathcal{A}) \arrow[dl, "\mathrm{ch}"', "\text{character map}"] \\ & \int\big(\Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}}\big) & \end{tikzcd}$$

The elements of this homotopy fiber product are the higher analogs of the familiar **connections** on (or "in") fiber bundles. Seen this way, connections are — by a happy coincidence of terminology — literally what connects local differential geometry with global algebraic topology.

In fact, this is how connections were historically motivated in classical Cartan-Chern-Weil theory (cf. the history recalled in [16, Rem. 8.1] [2]): as an intermediate means to the end of computing characteristic classes via local differential data – this we recall on the following pages.

[1] C. N. Yang said [52], about discovering that nuclear gauge fields "are" principal connections (cf. [51, Tbl. 1]): "*What could be more mysterious, what could be more awe-inspiring, than to find that the basic structure of the physical world is intimately tied to deep mathematical concepts, concepts which were developed out of considerations rooted only in logic and in the beauty of form.*"

[2] The numbering in [16] we refer to is that of the published version, which differs from the numbering in the preprint version; see [`ncatlab.org/schreiber/show/The+Character+Map+in+Non-Abelian+Cohomology`].

We set the scene by briefly recalling some classical constructions in streamlined form.

**Ordinary Principal Bundles.** For $G$ a Hausdorff-topological group, a *principal $G$-bundle* $P$ over a base space $X$ is a map $P \to X$ such that there exists an open cover $C := \coprod_i U_i \overset{\iota}{\twoheadrightarrow} X$ over which $P$ is identified with the trivial fibration $C \times G$ in a way that the fibers are identified by $G$-valued transition functions $g : C \times_X C \to G$ on double overlaps of charts, $C \times_X C = \coprod\limits_{i,j} U_i \cap U_j$:

**A map $p$ is a principal $G$-bundle...**

$$
\begin{tikzcd}
P \arrow[d, "p"] & C \times G \arrow[l, "t"'] \arrow[d, "\mathrm{pr}_C"] \arrow[dl, phantom, "\text{(pb)}"] \\
X & C \arrow[l, two heads, "\iota"]
\end{tikzcd}
$$

**...if its pull-back to an open cover $\iota$ is the trivial $G$-fibration such that...**

$$
\begin{tikzcd}
\{x_i\} \times G \arrow[rr, "\cdot\, g_{ij}(x)"] \arrow[dr, "t_i(x)"'] & & \{x_j\} \times G \arrow[dl, "t_J(x)"] \\
& P_x &
\end{tikzcd}
$$

**...the induced transition functions act by group multiplication.**

These transition functions clearly satisfy on triple overlaps $C \times_X C \times_X C$ the *Čech cocycle* condition

$$
g_{ij} \cdot g_{jk} \;=\; g_{ik}
\qquad
\begin{tikzcd}
& \{x_j\} \times G \arrow[d, "t_j(x)"] \arrow[ddr, "g_{jk}(x)"] & \\
& P_x & \\
\{x_i\} \times G \arrow[uur, "g_{ij}(x)"] \arrow[ur, "t_i(x)"] \arrow[rr, "g_{ik}(x)"] & & \{x_k\} \times G \arrow[ul, "t_k(x)"']
\end{tikzcd}
$$

and transform under a principal bundle isomorphism $P \xrightarrow{\phi} P'$ by the *Čech coboundary* relation

$$
g'_{ij} \;=\; h_i^{-1} \cdot g_{ij} \cdot h_j
\qquad
\begin{tikzcd}
\{x_i\} \times G \arrow[rrr, "h_i(x)"] \arrow[dr, "t_i(x)"] \arrow[dd, "g_{ij}(x)"'] & & & \{x_i\} \times G \arrow[dl, "t'_i(x)"'] \arrow[dd, "g'_{ij}(x)"] \\
& P_x \arrow[r, "\phi_x"] & P'_x & \\
\{x_j\} \times G \arrow[rrr, "h_j(x)"] \arrow[ur, "t_j(x)"'] & & & \{x_j\} \times G \arrow[ul, "t'_j(x)"] \, ,
\end{tikzcd}
$$

whence the isomorphism classes of principal bundles map to the *Čech cohomology* of the base space:

$$
\mathrm{Prncpl}G\mathrm{Bndl}(X)_{/\sim} \xrightarrow{\;\;\sim\;\;} H^1\big(X;\, G\big)\,.
$$

As indicated, this map is in fact a bijection (for well-behaved $X$, such as smooth manifolds), as one finds effectively by reading the above construction in reverse. The outer parts of these diagrams then also show that if we write

(i) $\mathbf{B}G$ for the topological groupoid with a single object and $G$ worth of morphisms,

(ii) $\widehat{X}$ for the topological groupoid with $C$ as its space of objects and $C \times_X C$ as its space of morphisms, for $C \xrightarrow{\iota} X$ a *good* open cover (over which every $G$-bundle trivializes),

then the groupoid of principal $G$-bundles is identified with the groupoid of continuous functors $g : \widehat{X} \to \mathbf{B}G$ with continuous natural transformations between these:

$$
\begin{tikzcd}
\text{groupoid of principal } G\text{-bundles} & \mathrm{Prncpl}G\mathrm{Bndl}(X) \arrow[r, "\sim"] \arrow[d, two heads] & \mathrm{Fnct}\big(\widehat{X},\, \mathbf{B}G\big) \arrow[d, two heads] & \text{groupoid of Čech co-cycles \& co-boundaries} \\
\text{isomorphism classes of principal } G\text{-bundles} & \mathrm{Prncpl}G\mathrm{Bndl}(X)_{/\sim} \arrow[r, "\sim"] & H^1(X;\, G) & \text{Čech cohomology set}
\end{tikzcd}
$$

**Ordinary Nonabelian Cohomology.** A deeper but classical theorem says (cf. [SS25b, Thm. 4.1.13]) that this situation is preserved by the "topological realization" of topological groupoids to topological spaces

$$
|-| : \;\; \mathrm{TopGrpd} \longrightarrow \mathrm{TopSpc}
$$

under which a smooth functor $g : \widehat{X} \to \mathbf{B}G$ becomes a continuous map $|g| : |\widehat{X}| \to |\mathbf{B}G|$ from $|\widehat{X}| \simeq X$ to the *classifying space* $BG := |\mathbf{B}G|$ – which still represents isomorphism classes of principal $G$-bundles:

$$
\underset{\text{bundles}}{\text{principal}} \quad \boxed{\mathrm{Prncpl}G\mathrm{Bndl}(X)_{/\sim} \xrightarrow[\sim]{\text{Čech}} \pi_0\,\mathrm{Fnct}\big(\widehat{X},\, \mathbf{B}G\big)} \xrightarrow[\sim]{|-|} \boxed{\pi_0\,\mathrm{Maps}(X,\, BG) \;=\; H^1(X;\, G)} \quad \text{ordinary non-abelian cohomology}
$$

**Ordinary Abelian Cohomology.** In the special case that $G \equiv A$ an *abelian* group, one readily sees that there is a fiber-wise $A$-tensor product of principal $A$-bundles

$$\begin{array}{ccc} \mathrm{Prnc}A\mathrm{Bndl}(X)_{/\sim} \times \mathrm{Prnc}A\mathrm{Bndl}(X)_{/\sim} & \longrightarrow & \mathrm{Prnc}A\mathrm{Bndl}(X)_{/\sim} \\ \big([P],\,[P']\big) & \longmapsto & \big[P \times_A P'\big] \end{array}$$

given by [3]

$$(P \times_A P')_x \;:=\; \big\{(p,p') \,\in\, P_x \times P'_x\big\}\big/\big((a\cdot p,\, p') \sim (p,\, a^{-1}\cdot p')\big)\,,$$

which makes the isomorphism classes naturally form an abelian group.

Under the classification of principal $A$-bundles by $A$-cohomology, this means that the classifying space $BA$ (may be chosen such that it) carries topological group structure itself!, so that the construction iterates:

$$A \;\in\; \mathrm{AbGrp}(\mathrm{TopSpc}) \qquad \text{yields} \qquad \left\{ \begin{array}{l} B^2A \;:=\; B(BA) \\ B^3A \;:=\; B\big(B(BA)\big) \\ \;\;\vdots \\ B^{1+n} \;:=\; B\big(B^nA\big)\,. \end{array}\right.$$

For a *discrete* abelian group $A$ (such as $A = \mathbb{Z}$) these higher-order classifying spaces are also denoted "$K(A,n)$" and called "Eilenberg-MacLane spaces":

$$A \,\in\, \mathrm{AbGrp}(\mathrm{Set}) \qquad \text{yields} \qquad K(A,n) \;:=\; B^nA\,.$$

But this means that for abelian group coefficients $A$ there is *higher-degree $A$-cohomology* appearing as a special case of non-abelian cohomology, as follows:

$$H^{1+n}\big(X;\,A\big) \;:=\; H^1\big(X;\,B^nA\big) \;:=\; \pi_0\,\mathrm{Maps}\big(X,\,B^{1+n}A\big)\,.$$

Of course, when $A$ is discrete then there is also *Čech cohomology* and *singular cohomology* with coefficients in $A$. A classical theorem says that (on our smooth manifold domain $X$) all these notions of ordinary cohomology agree, hence that

*Ordinary $A$-cohomology has classifying spaces $B^nA \,=\, K(A,n)$.*

**Ordinary characteristic classes.** Thereby we obtain an immediate means to *approximate non-abelian cohomology by abelian cohomology*: Every map of classifying spaces $c : BG \longrightarrow B^nA$, hence every *universal characteristic class*

$$[c] \;\in\; H^n\big(BG;\,A\big)\,,$$

induces a *cohomology operation* from non-abelian to abelian cohomology, simply by composition of classifying maps:

$$\begin{array}{cccccc} & H^1\big(X;\,G\big) \;=\; \pi_0\,\mathrm{Maps}\big(X,\,BG\big) & \longrightarrow & \pi_0\,\mathrm{Maps}\big(X,\,B^nA\big) \;=\; H^n\big(X;\,A\big) & \\ \substack{\text{nonabelian}\\ \text{cohomology class}} & [\phi] \;:=\; \big[X \xrightarrow{\phi} BG\big] & \longmapsto & \big[X \xrightarrow{\phi} BG \xrightarrow{c} B^nA\big] \;=:\; c[\phi] & \substack{\text{characteristic}\\ \text{class}} \end{array}$$

For example, for the unitary and orthogonal Lie groups there are universal characteristic classes for $n \in \mathbb{Z}$

$$\substack{\text{universal}\\ \text{Chern classes}} \;\; c_n \;\in\; H^{2n}\big(\mathrm{BU}(d);\,\mathbb{Z}\big)\,, \qquad p_n \;\in\; H^{4n}\big(\mathrm{BO}(d);\,\mathbb{Z}\big) \;\; \substack{\text{universal}\\ \text{Pontrjagin classes}}$$

(non-trivial for sufficiently small $n$ and large $d$) and since, for instance, the frame bundle $\mathrm{Fr}_X$ of a Riemannian manifold $X$ is a principal $\mathrm{O}(d)$-bundle, there are induced Pontrjagin classes of $X$, etc.:

$$\substack{\text{Pontrjagin class of}\\ \text{frame bundle of manifold}} \;\; p_n[X] \;:=\; p_n[TX] \;:=\; p_n[\mathrm{Fr}_X] \;:=\; \big[X \xrightarrow{\vdash \mathrm{Fr}_X} \mathrm{BO}(d) \xrightarrow{p_n} B^{4n}\mathbb{Z}\big] \;\in\; H^{4n}(X;\,\mathbb{Z})\,.$$

We see that

*In terms of classifying spaces,*
*cohomology and characteristic classes*
*become conceptually nicely transparent.*

Of course, what remains to discuss are methods for actually computing these classes. One such method (among many, but pivotal in its way) is to equip the principal bundles with *connections* and then compute these topological characteristic classes by means of differential geometry. This is what we discuss next.

[3] This tensor product of fibrations exists also for non-abelian $G$, but the commutativity of $A$ is needed for principality, namely for the resulting transition functions to again by given by group multiplication.

**Ordinary Character Map.** One more notion of cohomology available on smooth manifolds $X$ is *de Rham cohomology* $H^n_{\mathrm{dR}}(X)$, which we may understand – non-traditionally but equivalently [16, Prop. 6.4] – as equivalence classes of closed differential forms modulo *concordance* of closed forms:

$$H^n_{\mathrm{dR}}(X) \;=\; \Omega^n_{\mathrm{dR}}(X)_{\mathrm{clsd}}\Big/\Big(\omega^{(0)} \sim \omega^{(1)} \quad \text{iff } \exists\, \widehat{\omega} \in \Omega^n_{\mathrm{dR}}([0,1]\times X)_{\mathrm{clsd}} \text{ s.t. } \widehat{\omega}_{|0} = \omega^{(0)},\; \widehat{\omega}_{|1} = \omega^{(1)}\Big)\,.$$

By the de Rham isomorphism, this, too, coincides with a special case of our general notion of cohomology, namely with abelian cohomology for coefficients the real numbers $\mathbb{R}$ (regarded as a discrete topological group).

Combined with the cohomology operation induced by extension of scalars $\mathbb{Z} \hookrightarrow \mathbb{R}$, hence by the induced $B^n\mathbb{Z} \to B^n\mathbb{R}$, this gives a map from integral to de Rham cohomology, which we call the *ordinary character map*

$$\begin{tikzcd}
H^n(X;\,\mathbb{Z}) \arrow[r, "(\mathbb{Z}\hookrightarrow\mathbb{R})_*", "\text{extension of scalars}"'] \arrow[rr, bend right, "\mathrm{ch}^{\mathbb{Z}}", "\text{ordinary character map}"'] & H^n(X;\,\mathbb{R}) \arrow[r, "\sim", "\text{de Rham isomorphism}"'] & H^n_{\mathrm{dR}}(X)\,.
\end{tikzcd}$$

Beware that this character map is not in general injective, in fact it forgets exactly the *torsion subgroups* of the integral cohomology group. Nevertheless — or rather: therefore! — the ordinary character map provides the *first approximation* to integral cohomology, which is generally more readily computed than the full integral cohomology.

In consequence, when applied to integral characteristic classes then the ordinary character gives a useful first approximation to ordinary non-abelian cohomology. Specifically for unitary principal bundles we obtain a sequence of *Chern-de Rham classes*:

$$\begin{tikzcd}
\mathrm{PrncUBndl}(X)_{/\sim} \arrow[r, "\sim"] \arrow[rrr, bend right, "\text{Chern-de Rham classes}"'] & H^1\big(X;\,\mathrm{U}\big) \arrow[r, "c_\bullet", "\text{Chern classes}"'] & \prod_k H^{2k}\big(X;\,\mathbb{Z}\big) \arrow[r, "\mathrm{ch}^{\mathbb{Z}}", "\text{ordinary character}"'] & \prod_k H^{2k}_{\mathrm{dR}}(X)
\end{tikzcd}$$

But since de Rham cohomology is a local differential-geometric notion, the question arises:

*Is it possible to construct the Chern-de Rham classes*
*directly by differential geometry on principal bundles*
*without going through these topological constructions?*

Connections are the answer to this question.

**Ordinary connections.** For $G$ a Lie group and $\mathfrak{g}$ its Lie algebra, we denote the (functor assigning) flat $\mathfrak{g}$-valued differential forms by

$$\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{g}\big)_{\mathrm{flat}} \;:=\; \Big\{ A \in \Omega^1_{\mathrm{dR}}(-;\,\mathfrak{g}) \;\Big|\; \mathrm{d}A + \tfrac{1}{2}[A \wedge A] \;=\; 0 \Big\}\,.$$

There is a *universal* such flat form, called the *Maurer-Cartan form* $\theta \in \Omega^1_{\mathrm{dR}}(G;\,\mathfrak{g})_{\mathrm{flat}}$, in that the flat forms on any Cartesian space $\mathbb{R}^n$, $n \in \mathbb{N}$, are the pullbacks of the MC-form along smooth maps $\phi : \mathbb{R}^n \to G$, unique up to rigid translation along the group:

$$\begin{aligned}
C^\infty\big(\mathbb{R}^n,\, G\big)/G &\xrightarrow{\;\sim\;} \Omega^1_{\mathrm{dR}}\big(\mathbb{R}^n;\,\mathfrak{g}\big)_{\mathrm{flat}} \\
\big[\mathbb{R}^n \xrightarrow{f} G\big] \quad &\longmapsto \qquad f^*\theta\,.
\end{aligned}$$

Hence on a trivial $G$-bundle $P = X \times G$ we have a flat form $A := \mathrm{pr}_G^*\theta$ which restricts on each fiber to the MC-form. On a non-trivial principal $G$-bundle $P$ we still find a $\mathfrak{g}$-valued differential form $A \in \Omega^1_{\mathrm{dR}}(P;\,\mathfrak{g})$ that restricts on each fiber to the MC-form, but it may not itself be flat anymore, the failure being its *curvature*

$$F_A \;:=\; \mathrm{d}A + \tfrac{1}{2}[A \wedge A] \;\in\; \Omega^2_{\mathrm{dR}}(P;\,\mathfrak{g})\,,$$

which is therefore a measure for the non-triviality of $P$, at least if we also demand that it is a *horizontal form* in that it vanishes on vectors tangent to the fibers – in this case $A$ is called a *principal connection*.

Cartan calculus then shows that all ad-*invariant polynomials* $\langle -, \cdots, - \rangle \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathfrak{g}}$ evaluated on the curvature 2-form are in fact closed *basic forms* in that they are pulled back from closed forms on the base manifold:

$$\begin{tikzcd}
\substack{\text{MC-form} \\ \text{on } G\text{-fiber}} & G \arrow[r, "\theta"] \arrow[d, "i_x"] & \Omega^1_{\mathrm{dR}}(-;\,\mathfrak{g})_{\mathrm{flat}} \arrow[d, hook] \\
\substack{\text{connection form on} \\ \text{principal } G\text{-bundle}} \arrow[u, "\substack{\text{restricts on} \\ \text{each fiber to}}"] \arrow[d, "\substack{\text{curvature invariants} \\ \text{descend to}}"'] & P \arrow[r, "A"] \arrow[d, "p"] & \Omega^1_{\mathrm{dR}}(-;\,\mathfrak{g}) \arrow[d, "\langle F^{\wedge k}_{(-)} \rangle"] \\
\substack{\text{characteristic form} \\ \text{on base manifold}} & X \arrow[r, "\langle F_A^{\wedge k} \rangle"] & \Omega^{2k}_{\mathrm{dR}}(-)_{\mathrm{clsd}}
\end{tikzcd}$$

Thus: *Connections on principal bundles extract de Rham classes on their base space measuring their non-triviality.* That these de Rham classes indeed give the above Chern-de Rham classes is the content of the *Chern-Weil theorem*.

With the last diagram we have tacitly introduced a non-classical tool for differential topology: The category of:

**Smooth sets.** In order to conceive of "smooth spaces" $\mathcal{X}$ more general but as useful as smooth manifolds, consider that whatever $\mathcal{X}$ may be, we should be able to *probe* it by Cartesian spaces $\mathbb{R}^n$ in that we know what counts as an $\mathbb{R}^n$-*plot* of $\mathcal{X}$, namely as a smooth map $\mathbb{R}^n \to \mathcal{X}$. For the information about these plots to be consistent, they should functorially precompose with ordinary smooth maps $\mathbb{R}^{n'} \to \mathbb{R}^n$ and compatible probes by open balls covering $\mathbb{R}^n$ should uniquely glue to a single probe by all of $\mathbb{R}^n$. But this makes (the system of plots of) $\mathcal{X}$ a *sheaf of sets* on the category $\mathrm{CartSp} \subset \mathrm{SmthMfd}$, with respect to the Grothendieck pre-topology of good open covers, whence we say that [25][44]

$$\mathrm{SmthSet} \;:=\; \mathrm{Sh}(\mathrm{CartSp};\, \mathrm{Set})\,.$$

For example, smooth manifolds $X$ are smooth sets via the ordinary smooth functions $\mathbb{R}^n \to X$. But there are now also *classifying spaces of differential forms* among smooth sets, whose sets of $\mathbb{R}^n$-plots are *defined* to be the sets of smooth forms on $\mathbb{R}^n$. Then a Yoneda argument shows that a map of smooth sets

$$X \xrightarrow{\;\vdash \omega\;} \Omega^{2k}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \qquad \text{corresponds to} \qquad \omega \,\in\, \Omega^{2k}_{\mathrm{dR}}(X)_{\mathrm{clsd}}$$

and precomposition of such maps

$$P \xrightarrow{\;p\;} X \xrightarrow{\;\vdash \omega\;} \Omega^{2k}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \qquad \text{corresponds to} \qquad p^*\omega \,\in\, \Omega^{2k}_{\mathrm{dR}}(P)_{\mathrm{clsd}} \;\; \text{etc.}$$

**The role of principal connections.** We have thus found that ordinary connections complete the following commuting diagram

$$\begin{tikzcd}[column sep=small]
 & \underset{\text{principal connections}}{\mathrm{PrncUConn}(X)_{/\sim}} \arrow[dl, "\substack{\text{curvature}\\\text{invariants}}"', "\langle F^{\wedge\bullet}_{(-)}\rangle"] \arrow[dr, "\substack{\text{class of}\\\text{underlying}\\\text{bundle}}"] & \\
\underset{\text{differential forms}}{\prod_k \Omega^{2k}_{\mathrm{dR}}(X)_{\mathrm{clsd}}} \arrow[dr, "\substack{\text{de Rham}\\\text{classes}}"'] & & \underset{\text{nonabelian cohomology}}{H^1(X;\mathrm{U})} \arrow[dl, "\substack{\text{Chern-de Rham}\\\text{classes}}"] \\
 & \underset{\text{de Rham cohomology}}{\prod_k H^{2k}_{\mathrm{dR}}(X)} &
\end{tikzcd}$$

In this sense principal connections connect ordinary non-abelian cohomology to differential form data. But does this *characterize* principal connections?

In order to approach the answer to this question we need to retain more gauge information: Instead of just the above diagram of *isomorphism classes* of connections and of *cohomology classes* of their underlying bundles etc. we need to look at the groupoids that these structures form, and generally the *smooth higher groupoids* that they form.

**Smooth 2-Groupoids.** We assume now that the reader knows 2-groupoids or can pretend to. We try to use suggestive notation. For instance, with $A$ a discrete abelian group, we indicate its double delooping 2-groupoid by:

$$\mathbf{B}^2 A \;:=\; \left\{
\begin{tikzcd}
\ast \arrow[r, "\ast"] \arrow[dr, phantom, "\Downarrow a_1"] & \ast \arrow[d, "\ast"] \\
\ast \arrow[u, "\ast"] \arrow[ur, "\ast" description] \arrow[r, "\ast"'] \arrow[ur, phantom, "\Downarrow a_2"', shift right=2] & \ast
\end{tikzcd}
\;\equiv\;
\begin{tikzcd}
\ast \arrow[r, "\ast"] \arrow[dr, "\ast" description] \arrow[dr, phantom, "\Downarrow a_4"', shift right=2] & \ast \arrow[d, "\ast"] \arrow[dl, phantom, "\Downarrow a_3"] \\
\ast \arrow[u, "\ast"] \arrow[r, "\ast"'] & \ast
\end{tikzcd}
\;\middle|\;
\begin{array}{l}
a_i \;\in\; A \\
a_1 + a_2 \;=\; a_3 + a_4
\end{array}
\right\}.$$

Just as with smooth sets, we take *smooth 2-groupoids* to be the systems of 2-groupoids (of plots) indexed by (probe) Cartesian spaces

$$\mathrm{SmthGrpd}_2 \;\;:=\;\; \mathrm{Sh}(\mathrm{CartSp},\, \mathrm{Grpd}_2)\,.$$

For example, with $C := \coprod_i U_i \overset{\iota}{\twoheadrightarrow} X$ a (good) open cover, its *Čech 2-groupoid* $\widehat{X}$ is the smooth 2-groupoid given as follows:

$$\big\{\mathbb{R}^n \to \widehat{X}\big\} \;\;:=\;\; \left\{
\begin{tikzcd}
\mathbf{x}_j \arrow[r, "\mathbf{x}_{jk}"] \arrow[dr, phantom, "\Downarrow \mathbf{x}_{ijk}"] & \mathbf{x}_k \arrow[d, "\mathbf{x}_{kl}"] \\
\mathbf{x}_i \arrow[u, "\mathbf{x}_{ij}"] \arrow[ur, "\mathbf{x}_{ik}" description] \arrow[r, "\mathbf{x}_{il}"'] \arrow[ur, phantom, "\Downarrow \mathbf{x}_{ikl}"', shift right=2] & \mathbf{x}_l
\end{tikzcd}
\;\equiv\;
\begin{tikzcd}
\mathbf{x}_j \arrow[r, "\mathbf{x}_{jk}"] \arrow[dr, "\mathbf{x}_{jl}" description] \arrow[dr, phantom, "\Downarrow \mathbf{x}_{ijl}"', shift right=2] & \mathbf{x}_k \arrow[d, "\mathbf{x}_{kl}"] \arrow[dl, phantom, "\Downarrow \mathbf{x}_{jkl}"] \\
\mathbf{x}_i \arrow[u, "\mathbf{x}_{ij}"] \arrow[r, "\mathbf{x}_{il}"'] & \mathbf{x}_l
\end{tikzcd}
\;\middle|\;
\begin{tikzcd}
 & U_i \\
\mathbb{R}^n \arrow[ur, bend left, "\mathbf{x}_i"] \arrow[r, "\mathbf{x}_{ij}"] \arrow[dr, bend right, "\mathbf{x}_j"'] & U_i \cap U_j \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
 & U_j
\end{tikzcd}
\;\;\text{etc.}
\right\}$$

It is now manifest that maps $\widehat{X} \to \mathbf{B}^2 A$ are Čech 2-cocycles and that their homotopies are Čech coboundaries:

$$\big\{\widehat{X} \longrightarrow \mathbf{B}^2 A\big\}_{/\sim} \;\;\simeq\;\; H^2(X;\, A)\,.$$

First to see these notions at work for abelian cohomology:

**Ordinary differential 2-cohomology.** We thus have an evident map between the *cocycle 2-groupoids* of integral and real cohomology, respectively, but we need to find an analogous 2-groupoidal refinement of the de Rham isomorphism in order to map differential forms into the latter:

$$\begin{tikzcd} & \mathbf{B}^2\mathbb{Z} \arrow[d, "\substack{\text{extension}\\ \text{of scalars}}"] \\ \Omega^2_{\mathrm{dR}}(-)_{\mathrm{clsd}} \arrow[r, dotted, "\substack{\text{would-be}\\ \text{de Rham map}}", "?"'] & \mathbf{B}^2\mathbb{R}^\flat \end{tikzcd}$$

This works by finding different but *equivalent* 2-groupoids: A map of smooth 2-groupoids is an equivalence if on all 2-groupoids of plots it induces isomorphisms on homotopy groups of plain 2-groupoids:

$$f : \mathcal{X} \xrightarrow{\sim} \mathcal{Y} \qquad \text{if} \qquad \underset{n\in\mathbb{N}}{\forall}\Big(\pi_\bullet\big\{\mathbb{R}^n \to \mathcal{X}\big\} \underset{\sim}{\xrightarrow{f_*}} \pi_\bullet\big\{\mathbb{R}^n \to \mathcal{Y}\big\}\Big).$$

For instance, a simple check reveals the following equivalences, which allow for the desired inclusion of closed forms:

$$\begin{tikzcd} & \mathbf{B}^2\mathbb{Z} \arrow[d, "\sim"', "\substack{\text{resolve integer}\\ \text{coefficients}}"] \\ \underset{\substack{\text{resulting}\\ \text{fiber product}}}{\mathbf{B}\mathrm{U}(1)_{\mathrm{conn}}} \arrow[r] \arrow[d] & \widehat{\mathbf{B}^2\mathbb{Z}} \arrow[d, "\text{fibration}"', "\substack{\text{resolved exten-}\\ \text{sion of scalars}}"] \\ \Omega^2_{\mathrm{dR}}(-)_{\mathrm{clsd}} \arrow[r, "\substack{\text{desired inclusion}\\ \text{of closed forms}}"'] & \widehat{\mathbf{B}^2\mathbb{R}^\flat} \\ & \mathbf{B}^2\mathbb{R}^\flat \arrow[u, "\sim", "\substack{\text{resolve real}\\ \text{coefficients}}"'] \end{tikzcd}$$

$$\mathbf{B}^2\mathbb{Z} = \left\{ \begin{tikzcd} & \ast \arrow[dr, "\ast"] \arrow[d, phantom, "\Downarrow n"] & \\ \ast \arrow[ur, "\ast"] \arrow[rr, "\ast"'] & {} & \ast \end{tikzcd} \;\middle|\; n \in \mathbb{Z} \right\}$$

$$\widehat{\mathbf{B}^2\mathbb{Z}} := \left\{ \begin{tikzcd} & A + \mathrm{d}\lambda + a \arrow[dr, "(\lambda', a')"] \arrow[d, phantom, "\Downarrow (n,\kappa)"] & \\ A \arrow[ur, "(\lambda, a)"] \arrow[rr, "(\lambda+\lambda'+n-\kappa,\, a+a'+\mathrm{d}\kappa)"'] & {} & \substack{A + \mathrm{d}\lambda + a\\ + \mathrm{d}\lambda' + a'} \end{tikzcd} \;\middle|\; \begin{array}{lcl} A, a & \in & \Omega^1_{\mathrm{dR}}(-) \\ \lambda, \kappa & \in & \Omega^0_{\mathrm{dR}}(-) \\ n & \in & \mathbb{Z} \end{array} \right\}$$

$$\widehat{\mathbf{B}^2\mathbb{R}^\flat} := \left\{ \begin{tikzcd} & F + \mathrm{d}a \arrow[dr, "a'"] \arrow[d, phantom, "\Downarrow \kappa"] & \\ F \arrow[ur, "a"] \arrow[rr, "a+a'+\mathrm{d}\kappa"'] & {} & \substack{F + \mathrm{d}a\\ + \mathrm{d}a'} \end{tikzcd} \;\middle|\; \begin{array}{lcl} F & \in & \Omega^2_{\mathrm{dR}}(-)_{\mathrm{clsd}} \\ a & \in & \Omega^1_{\mathrm{dR}}(-) \\ \kappa & \in & \Omega^0_{\mathrm{dR}}(-) \end{array} \right\}$$

$$\mathbf{B}^2\mathbb{R}^\flat := \left\{ \begin{tikzcd} & \ast \arrow[dr, "\ast"] \arrow[d, phantom, "\Downarrow \kappa"] & \\ \ast \arrow[ur, "\ast"] \arrow[rr, "\ast"'] & {} & \ast \end{tikzcd} \;\middle|\; \kappa \in \mathbb{R} \right\}$$

In fact, this also "resolves" the above map to a *fibration*, implying that the homotopy fiber product is given by the ordinary fiber product $\mathbf{B}\mathrm{U}(1)_{\mathrm{conn}}$, which one sees is just the restriction of $\widehat{\mathbf{B}^2\mathbb{Z}}$ to $a, \kappa = 0$:

$$\mathbf{B}\mathrm{U}(1)_{\mathrm{conn}} = \left\{ \begin{tikzcd} & A + \mathrm{d}\lambda \arrow[dr, "\lambda'"] \arrow[d, phantom, "\Downarrow n"] & \\ A \arrow[ur, "\lambda"] \arrow[rr, "\lambda+\lambda'+n"'] & {} & \substack{A + \mathrm{d}\lambda\\ + \mathrm{d}\lambda'} \end{tikzcd} \;\middle|\; \begin{array}{lcl} A & \in & \Omega^1_{\mathrm{dR}}(-) \\ \lambda & \in & \Omega^0_{\mathrm{dR}}(-) \\ n & \in & \mathbb{Z} \end{array} \right\} \xleftrightarrow{\;\sim\;} \left\{ \begin{tikzcd} & A + \mathrm{d}\lambda \arrow[dr, "{[\lambda']}"] \arrow[d, phantom, "\Vert"] & \\ A \arrow[ur, "{[\lambda]}"] \arrow[rr, "{[\lambda+\lambda']}"'] & {} & \substack{A + \mathrm{d}\lambda\\ + \mathrm{d}\lambda'} \end{tikzcd} \;\middle|\; \begin{array}{lcl} A & \in & \Omega^1_{\mathrm{dR}}(-) \\ {[\lambda]} & \in & C^\infty(\text{-}, \mathrm{U}(1)) \end{array} \right\}$$

But this is evidently the coefficient object for U(1)-connections, in that:

$$\Big\{ \widehat{X} \xrightarrow{(A,\lambda)} \mathbf{B}\mathrm{U}(1)_{\mathrm{conn}} \Big\} \;\simeq\; \left\{ \begin{array}{c} \text{Connection 1-forms } A_i \text{ on each } U_i \text{, related} \\ \text{by gauge transformations } \lambda_{ij} \text{ on } U_i \cap U_j \\ \text{satisfying charge quantization on } U_i \cap U_j \cap U_k \end{array} \right\} \;\simeq\; \mathrm{PrncU(1)Conn}(X)\,.$$

In this sense of a homotopy fiber product, principal U(1)-connections are *exactly* the connection between integral 2-cohomology and closed differential 2-forms.

**Revisiting ordinary connections.** It is straightforward to *write down* the smooth groupoid $\mathbf{B}G_{\mathrm{conn}}$ which generalizes the previous $\mathbf{B}\mathrm{U}(1)_{\mathrm{conn}}$ to any Lie group $G$ with Lie algebra $\mathfrak{g}$: Namely

$$\mathbf{B}G_{\mathrm{conn}} \;:=\; \left\{ \begin{tikzcd} & g^{-1}(A+\mathrm{d})g \arrow[dr, "g'"] \\ A \arrow[ur, "g"] \arrow[rr, "gg'"'] && \begin{array}{c}(gg')^{-1}A(gg') \\ +\,(gg')^{-1}\mathrm{d}(gg')\end{array} \end{tikzcd} \;:=\; \begin{tikzcd} & \mathrm{Ad}_g(A) + g^\ast\theta \arrow[dr, "g'"] \\ A \arrow[ur, "g"] \arrow[rr, "gg'"'] && \begin{array}{c}\mathrm{Ad}_{g'g}(A) \\ +\,(g'g)^\ast\theta\end{array} \end{tikzcd} \;\middle|\; \begin{array}{ccc} A & \in & \Omega^1_{\mathrm{dR}}(-;\mathfrak{g}) \\ g & \in & \Omega^0_{\mathrm{dR}}(-;G) \end{array} \right\}$$

is evidently such that maps $\widehat{X} \to \mathbf{B}G_{\mathrm{conn}}$ are Čech cocycles for principal $G$-connections, in fact we have an equivalence of groupoids:

$$\left\{ \widehat{X} \to \mathbf{B}G_{\mathrm{conn}} \right\} \;\simeq\; \mathrm{Prnc}G\mathrm{Conn}(X)\,.$$

**However**, for non-abelian $G$, these groupoids are not evidently homotopy fiber products of a de Rham map with a character map anymore, hence they are not evidently the *universal* answer to the problem of constructing differential form representatives of characteristic classes.

On the other hand, $\mathbf{B}G_{\mathrm{conn}}$ in itself may naturally be addressed as the *groupoid of Lie algebra-valued differential forms*, which may sound like a good construction principle in itself.

Therefore, it is at this point that the topic of – certainly the previous literature on — higher connections bifurcates, depending on which of these two aspects of ordinary connections is taken to be the characteristic one:

(i) **Differential Cohomology**:
the homotopy fiber product of a
non-abelian de Rham- and character-map

(ii) **Principal Connections:**
the globalization of higher
Lie-algebra valued differential forms

But because, until recently, the character map was only understood for abelian generalized cohomology or for non-abelian ordinary cohomology, previous approaches have considered either the abelian sector of (i), or low-dimensional instances of (ii):

| **Abelian differential cohomology** | **Low-dimensional principal connections** |
|---|---|
| Previous exposition of generalized differential cohomology focuses on abelian (Whitehead-generalized) cohomology theories with classifying *spectra* of spaces:<br>[10] Bunke 2012, [3] Amabel et al.: 2021<br>and with application to higher gauge theory:<br>[20] Freed 2002, [47] Szabo 2013 | Previous exposition of higher principal connections focuses on low-dimensional examples (2-form or at most 3-form connections):<br>[4] Baez & Huerta 2011<br>[1] Alfonsi 2024<br>[6] Borsten el al. 2024 |

| **Nonabelian differential cohomology** |
|---|
| The combination of *nonabelian differential cohomology* that we survey here has been in the making for a long time, starting with [39][43][42], gaining more shape in [15], but was fully developed only with the construction of the non-abelian character map in [16].<br>The closest to a previous survey is:<br>[40] Sati & Schreiber 2025 |

A central theme in developing this, which remains under-appreciated, is that there are:

**Two different roles for $L_\infty$-algebra valued differential forms.** The joint non-abelian and higher generalization reveals that also the approach (i) crucially involves higher Lie algebra valued differential forms, but here they appear as *curvature forms* (flux densities) instead of as connection forms (gauge potentials)

| | (i) **Differential Cohomology** | (ii) **Principal Connections** |
|---|---|---|
| $L_\infty$-valued forms | curvature forms | connection forms |
| math jargon | Chern-Dold characters | Ehresmann connections |
| physics jargon | flux densities | gauge potentials |
| MC-condition | Bianchi identities<br>(generic) | flatness<br>(non-generic) |

We will try to explain this in the following.

## 2 Nonabelian Differential Cohomology

In §2.1 we highlight the notion of *generalized nonabelian cohomology*,
in §2.2 we explain the all-important *character map* in this generality,
in §2.3 we use this to explain *generalized nonabelian cohomology*.

### 2.1 Nonabelian Cohomology

**Cohomology via classifying spaces.** It is a classical and yet possibly undervalued fact that *reasonable cohomology theories have classifying spaces* (and more generally *classifying stacks*). To quickly recall (more details and pointers in [16, §2] ):

– **Ordinary cohomology.** This begins with the observation that (reduced) ordinary singular cohomology, with coefficients in a discrete abelian group $A$, is classified in degree $n$ by Eilenberg-MacLane spaces $K(A,n)$ – in that on well-behaved topological spaces $X$, notably on smooth manifolds, there are natural isomorphisms between the ordinary cohomology groups and the connected components of the respective (pointed) mapping spaces:

$$H^n(X;\,A) \;\simeq\; \pi_0\,\mathrm{Maps}\big(X,\,K(A,n)\big)\,, \qquad \widetilde{H}^n(X;\,A) \;\simeq\; \pi_0\,\mathrm{Maps}^{*/}\big(X,\,K(A,n)\big)\,. \tag{1}$$

This equivalence makes manifest the characteristic properties of cohomology: homotopy invariance, exactness and wedge property, since these are now immediately implied by general abstract properties of mapping spaces.

Moreover, these EM-spaces are in fact loop spaces of each other, via weak homotopy equivalences

$$\sigma_n \;:\; K(A,n) \xrightarrow{\;\sim\;} \Omega K(A,n+1) \tag{2}$$

that thereby represent the *suspension isomorphisms* between ordinary cohomology groups, as follows:

$$\widetilde{H}^n(X;A) \underset{(1)}{\simeq} \mathrm{Maps}^{*/}\big(X,\,K(A,n)\big) \underset{(2)}{\xrightarrow{(\sigma_n)_*}} \mathrm{Maps}^{*/}\big(X,\,\Omega K(A,n+1)\big) \underset{\text{adjunction}}{\simeq} \mathrm{Maps}^{*/}\big(\Sigma X,\,K(A,n+1)\big) \underset{(1)}{\simeq} \widetilde{H}^{n+1}\big(\Sigma X;\,A\big)\,.$$

– **Ordinary non-abelian cohomology.** Note here that it is the loop space property (2), and hence the corresponding suspension isomorphism, which reflect the fact that the coefficient $A$ has been assumed to be an *abelian* group: For a non-abelian group $G$, an Eilenberg-MacLane space $K(G,1) \simeq BG$ still exists, but is *not a loop space.*

While the suspension isomorphism is thus lost for non-abelian coefficients, the assignment

$$X \;\longmapsto\; H^1(X;\,G) \;:=\; \pi_0\,\mathrm{Maps}(X,\,BG) \;\;\in\; \mathrm{Set}^{*/} \tag{3}$$

still satisfies homotopy invariance, exactness and wedge property, just by the general properties of mapping spaces, and hence has all the characteristic properties of ordinary cohomology – except for its abelian-ness. Accordingly, (3) is known as *non-abelian cohomology*, famous from early applications in Chern-Weil theory.

– **Whitehead-generalized cohomology theory.** But if or as long as we do insist on abelian cohomology groups related by suspension isomorphisms, we may still immediately generalize ordinary cohomology in the form (1), simply by using any other sequence of classifying spaces $(E_n)_{n=0}^{\infty}$, being successive loop spaces of each other as in (2),

$$\sigma_n \;:\; E_n \xrightarrow{\;\sim\;} \Omega E_{n+1}\,,$$

as such called a *sequential* $\Omega$*-spectrum of spaces*, or just a *spectrum*, for short. The Brown representability theorem says that the resulting assignments

$$X \;\mapsto\; E^n(X) \;:=\; \pi_0\,\mathrm{Maps}(X;\,E_n)$$

are equivalently the *generalized cohomology theories* as introduced by Whitehead, including examples such as K-theory, elliptic cohomology and cobordism cohomology.

– **Non-abelian generalized cohomology.** But as we just saw, suspension isomorphisms are to be regarded as *extra* structure on cohomology. Not necessarily requiring them leads to consider *any pointed space* $\mathcal{A}$ (which we may as well assume to be connected) as the classifying space of a non-abelian generalized cohomology theory, defined in evident generalization of (3) simply by

$$H^1(X;\,\Omega\mathcal{A}) \;:=\; \pi_0\,\mathrm{Maps}(X,\,\mathcal{A})\,. \tag{4}$$

Here the notation on the left is suggestive of the fact that any loop space $\Omega\mathcal{A}$ canonically carries the structure of a higher homotopy-coherent group – a groupal $A_\infty$-space or $\infty$-group, for short – whose de-looping is equivalent to the connected component of the original space:

$$\mathcal{A} \;\simeq\; B\,\Omega\mathcal{A}\,. \tag{5}$$

For instance, in the archetypical case where $\mathcal{A} \equiv S^n$ is the $n$-sphere, then the non-abelian generalized cohomology theory that it classifies is known as (unstable) Cohomotopy $\pi^n$

$$\widetilde{H}^1\big(X;\,\Omega S^n\big) \;\;\equiv\;\; \pi_0\,\mathrm{Maps}^{*/}\big(X,\,S^n\big) \;\;\equiv\;\; \pi^n(X)\,, \tag{6}$$

in dual reference to the familar *homotopy* groups

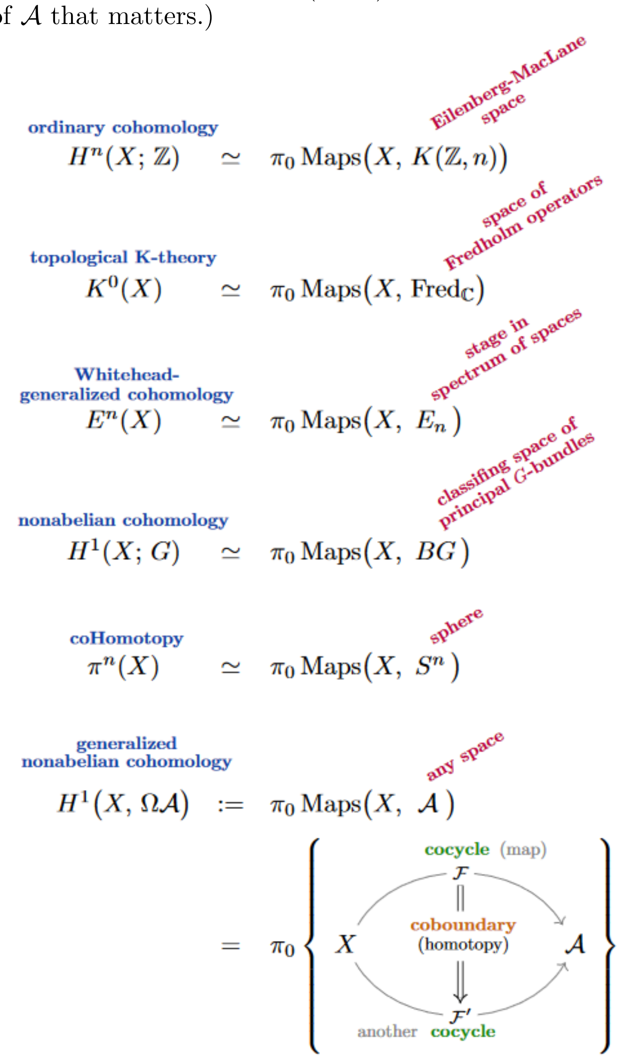
$$\pi_n(X) \;\;\simeq\;\; \pi_0\,\mathrm{Maps}^{*/}\big(S^n,\,X\big)\,.$$

Another example of non-abelian generalized cohomology is unstable topological K-theory [29], whose classifying spaces are taken to be finite stages $\mathrm{U}(n)$ of the sequential colimits which construct the classifying spaces of topological K-theory.

**Classifying spaces for generalized cohomology.** It is a classical fact of algebraic topology — which may have remained somewhat underappreciated in mathematical physics — that reasonable generalized cohomology theories have *classifying spaces* $\mathcal{A}$, in that the sets of cohomology classes assigned to a given domain space (which we take to be a smooth manifold $X^d$) are in natural bijection with the homotopy classes $\pi_0\mathrm{Map}\big(X,\,\mathcal{A}\big)$ of continuous maps from $X$ into $\mathcal{A}$. (Throughout, it is only the homotopy type of $\mathcal{A}$ that matters.)

The archetypical examples are Eilenberg-MacLane spaces like $K(\mathbb{Z}, n)$ which classify ordinary cohomology such as integral cohomology, in any degree $n$. As $n$ ranges, these EM-spaces happen to be loop spaces of each other, up to weak homotopy equivalence: $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n+1)$.
Generalizing from this classical example, one considers Whitehead-generalized cohomology theories which are classified by any sequences of pointed topological spaces $\{E_n\}_{n\in\mathbb{N}}$ equipped with weak homotopy equivalences $E_n \simeq \Omega E_{n+1}$, called a *spectrum of spaces*.
This implies that each $E_n$ is an infinite-loop space, which makes them be "abelian $\infty$-groups", reflecting the fact that the homotopy classes of maps into these spaces indeed have the structure of abelian groups.
Perhaps the most familiar example of such *abelian* generalized cohomology is topological K-theory, whose classifying space $\mathrm{KU}_0$ may be identified with the space of Fredholm operators on an infinite-dimensional separable complex Hilbert space.

While Whitehead-generalized cohomology theory has received so much attention that it is now widely understood as the default or even the exclusive meaning of "generalized cohomology", historically long preceding it is the *non-abelian cohomology* of Chern-Weil theory, classified by the original classifying spaces $BG$ of compact Lie groups $G$.
Unless $G$ happens to be abelian itself, this nonabelian cohomology does not assign abelian cohomology groups, nor even any groups at all, but just pointed cohomology sets. Nevertheless, as the historical name "nonabelian cohomology" clearly indicates, these systems of cohomology sets may usefully be regarded as constituting a kind of cohomology theory, too.




In this vein one may observe [16, §2] that (the homotopy type of) every connected space $\mathcal{A}$ is equivalently the classifying space of an infinity-group $\Omega\mathcal{A}$, namely of its own loop space regarded as an $A_\infty$-space under concatenation of loops), so that homotopy classes of maps into any connected space are examples of an evident generalization of Chern-Weil-style nonabelian cohomology.

A fundamental and historical example of such "truly-generalized" nonabelian cohomology is CoHomotopy, whose classifying spaces are the (homotopy types) of spheres. Notice that "generalized nonabelian cohomology" is really "not necessarily abelian". It subsumes all the other cases: For $E_\bullet$ a spectrum we have: $E^n(X) \simeq H^1(X;\Omega E_n)$

**Developing non-abelian cohomology.** Fundamental, elementary, and compelling as the notion of non-abelian generalized cohomology in (2.1) is, it has long remained underappreciated. For example, none of the original authors [7][37][46] on Cohomotopy (6) address their subject as a cohomology theory, instead the early development revolves around partial fixes for the perceived defect of co-homotopy sets to not in general carry group structure. The situation does not improve with the early development of "non-abelian gerbes", whose original description [27] appears unwieldy.

Explicit acknowledgment of (stacky) non-abelian generalized cohomology in the transparent guise (2.1) appears only in a lecture [48] (possibly following [45]). Two independent developments in 2009 finally put non-abelian generalized cohomology into fruitful context:

- The discovery of non-abelian Poincaré duality [32, §3.8], relating non-abelian cohomology (later made explicit in [33, Def. 6]) of manifolds to "non-abelian homology" in the guise of "factorization homology" (which, in contrast to non-abelian cohomology, takes work to define);
- The observation in theoretical physics [39][43][42] that charge/flux-quantization laws [40] for higher gauge fields are generally in non-abelian cohomology.

## 2.2 The Character Map

**Approximating Homotopy Types by Differential Equations.** Consider spaces $\mathcal{A}$ which are connected, simply connected and of "rational finite type", the latter meaning that the rational homotopy- and cohomology groups are degreewise finite-dimensional.

Consider the $\mathbb{R}$-linear duals to the homotopy groups as cochains for singular rational cohomology supported on these spherical chains. Choosing a graded linear basis $\vec{f} := (f^{(i)})_{i\in I}$, the cochain differential of these basis elements is equal to graded-symmetric polynomials $P^{(i)}$ in these variables, $\mathrm{d}\, f_i \;=\; P^{(i)}\big(\vec{f}\,\big)$, and as such extends uniquely to make the graded-commutative algebra spanned by the $\vec{f}$ into a differential such algebra (dgc-algebra), which we denote as follows [16, Prop. 5.11]:

$$
\overset{\text{minimal Sullivan model}}{\underset{\text{Chevalley-Eilenberg dgc-algebra}}{\mathrm{CE}}\Big(\underset{\substack{\mathbb{R}\text{-Whitehead bracket}\\ L_\infty\text{-algebra}}}{\mathfrak{l}}\big(\underset{\substack{\text{homotopy type}\\ \text{(1-cnct \& }\mathbb{R}\text{-finite)}}}{\mathcal{A}}\big)\Big)}
\;:=\;
\Big(\underset{\substack{\text{Grassmann}\\ \text{algebra}}}{\wedge^\bullet_{\mathbb{R}}}\,\underset{\substack{\text{rationalized}\\ \text{homotopy groups}}}{\big(\pi_\bullet(\Omega\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{R}\big)}{}^{\overset{}{\underset{\substack{\text{degree-wise}\\ \mathbb{R}\text{-dualized}}}{\vee}}},\;\; \underset{\substack{\text{cochain}\\ \text{differential}}}{\mathrm{d}}\Big)
\;\simeq\;
\underset{\substack{\text{freely generated}\\ \text{dgc-algebra}}}{\mathbb{R}_{\mathrm{d}}\big[\vec{f}\,\big]}\underset{\text{modulo}}{/}\underset{\text{differential ideal}}{\Big(\mathrm{d}\, f^{(i)} \;=\; P^{(i)}\big(\vec{f}\,\big)\Big)_{i\in I}}\,.
$$

A classical theorem by Sullivan says that the cochain cohomology of this dgc-algebra coincides with the rational cohomology of $\mathcal{A}$:

$$
\underset{\substack{\text{cochain cohomology}\\ \text{of CE-algebra of}\\ \text{Whitehead } L_\infty\text{-algebra}}}{H^\bullet\big(\mathrm{CE}(\mathfrak{l}\mathcal{A})\big)} \;\simeq\; \underset{\substack{\text{real cohomology}\\ \text{of the space}}}{H^\bullet(\mathcal{A};\,\mathbb{R})}
$$

and that this condition uniquely characterizes its differential. As such, $\mathrm{CE}\big(\mathfrak{l}\mathcal{A}\big)$ is known as the *minimal Sullivan model* of the homotopy type of $\mathcal{A}$

For example, the 2-sphere $\mathcal{A} \equiv S^2$ has non-torsion homotopy groups in degree=2 (generated by the identity map — $f_2$) and in degree=3 (generated by the complex Hopf fibration — $h_3$), but its ordinary cohomology is concentrated in degree (0 and) 2, so that the differential must make $f_2$ a non-exact cocycle while removing both $h_3$ as well as $f_2\, f_2$ from cohomology.
This is accomplished by:

$$
\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \end{bmatrix} \Big/ \left(\begin{array}{lcl} \mathrm{d}\, f_2 &=& 0 \\ \mathrm{d}\, h_3 &=& f_2\, f_2 \end{array}\right).
$$

For the 4-sphere we have the analogous situation with non-torsion homotopy groups in degree=4 (generated by the identity map — $g_4$) and in degree=7 (genrated by the quaternionic Hopf fibration — $g_7$) but with ordinary cohomology concentrated in degree (0 and) 4, so that the differential must remove $g_7$ and $g_4\, g_4$ from cohomology.
This is accomplished by:

$$
\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} g_4 \\ g_7 \end{bmatrix} \Big/ \left(\begin{array}{lcl} \mathrm{d}\, g_4 &=& 0 \\ \mathrm{d}\, g_7 &=& g_4\, g_4 \end{array}\right).
$$

**Rational Nonabelian Cohomology.** The *fundamental theorem of dgc-algebraic rational homotopy theory* (cf. [16, §5]) says that the homotopy theory of such dgc-algebras coincides with the $\mathbb{R}$-*rational homotopy theory* of these spaces: where a map $f : \mathcal{A} \to \mathcal{B}$ is regarded as an equivalence if $f_*$ is an iso on all rationalized homotopy groups.

For us, thinking of the homotopy type $\mathcal{A}$ as the classifying space for generalized cohomology, this defines *rational* nonabelian cohomology [16, §6]:

$$
\underbrace{\underset{\substack{\text{nonabelian}\\ \text{rational cohomology}}}{H^1\big(X;\,\Omega\mathcal{A}^{\mathbb{R}}\big)} \;:=\; \underset{\substack{\text{rational homotopy}\\ \text{classes of maps}}}{\pi_0\,\mathrm{Maps}\big(X^{\mathbb{R}},\,\mathcal{A}^{\mathbb{R}}\big)} \xrightarrow[\substack{\text{fundamental theorem of}\\ \text{dgc-algebraic rational homotopy}}]{\sim} \underset{\substack{\text{dg-algebra homs}\\ \text{modulo concordance}}}{\pi_0\,\mathrm{Hom}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\,\Omega^\bullet_{\mathrm{dR}}(X)\big)} \;=:\; \underset{\substack{\text{nonabelian}\\ \text{de Rham cohomology}}}{H^1_{\mathrm{dR}}\big(X;\,\mathfrak{l}\mathcal{A}\big)}}_{\text{nonabelian de Rham isomorphism}}
$$

**Generalized Character Map.** But this makes immediate the character map in the generality of generalized nonabelian cohomology [16, Def. IV.2]: It is the cohomology operation induced by the the universal comparison map to the rationalization of the classifying space:

$$
\mathcal{A} \xrightarrow[\mathbb{Q}\text{-rationalization}]{\eta^{\mathbb{Q}}_{\mathcal{A}}} \mathcal{A}^{\mathbb{Q}} \xrightarrow[\text{extension of scalars}]{\otimes_{\mathbb{Q}}\mathbb{R}} \mathcal{A}^{\mathbb{R}}
$$

$$
\underbrace{\underset{\substack{\text{generalized}\\ \text{nonabelian}\\ \text{cohomology}}}{H^1\big(X;\,\Omega\mathcal{A}\big)} \xrightarrow{\big((\otimes_{\mathbb{Q}}\mathbb{R})\circ\eta^{\mathbb{Q}}_{\mathcal{A}}\big)_*} H^1\big(X;\,\Omega\mathcal{A}^{\mathbb{R}}\big) \xrightarrow[\sim]{\substack{\text{nonabelian}\\ \text{de Rham}\\ \text{isomorphism}}} \underset{\substack{\text{nonabelian}\\ \text{de Rham}\\ \text{cohomology}}}{H^1_{\mathrm{dR}}\big(X;\,\mathfrak{l}\mathcal{A}\big)}}_{\text{generalized non-abelian character map}}
$$

**Examples of character maps.**

$$
\begin{array}{cccc}
\begin{array}{c}\textbf{Ordinary}\\ \textbf{integral cohomology}\end{array} & H^n(X;\,\mathbb{Z}) \xrightarrow{\;\textbf{de Rham map}\;} H^n_{\mathrm{dR}}(X) \,\simeq\, \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathbb{R}[\omega_n],\, H^\bullet_{\mathrm{dR}}(X)\big) & \begin{array}{c}\textbf{differential forms}\\ \textbf{in degree } n\end{array} \\
\begin{array}{c}\textbf{Traditional}\\ \textbf{nonabelian cohomology}\end{array} & H^1(X;\,G) \xrightarrow{\;\textbf{Chern-Weil homomorphism}\;} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, H^\bullet_{\mathrm{dR}}(X)\big) & \begin{array}{c}\textbf{differential forms for}\\ \mathfrak{g}\textbf{-invariant polynomials}\end{array} \\
\begin{array}{c}\textbf{Topological}\\ \textbf{K-theory}\end{array} & K^0(X) \xrightarrow{\;\textbf{Chern character}\;} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathbb{R}[\omega_0, \omega_2, \omega_4, \cdots],\, H^\bullet_{\mathrm{dR}}(X)\big) & \begin{array}{c}\textbf{differential forms}\\ \textbf{in every even degree}\end{array} \\
\begin{array}{c}\textbf{abelian Whitehead-}\\ \textbf{generalized cohomology}\end{array} & E^n(X) \xrightarrow{\;\textbf{Chern-Dold character}\;} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\wedge^\bullet(\pi_\bullet(E)\otimes_{\mathbb{Z}}\mathbb{R})^\vee,\, H^{\bullet+n}_{\mathrm{dR}}(X)\big) & \begin{array}{c}\textbf{differential forms for}\\ \textbf{rational homotopy groups}\\ \textbf{of the classifying space}\end{array} \\
\begin{array}{c}\textbf{Generalized}\\ \textbf{non-abelian cohomology}\end{array} & H^1\big(X;\,\Omega\mathcal{A}\big) \xrightarrow{\;\substack{\textbf{nonabelian}\\ \textbf{character}}\;} H^1_{\mathrm{dR}}(X;\,\mathfrak{l}A) \,:=\, \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)\big/\sim & \begin{array}{c}\textbf{differential forms with}\\ \textbf{coefficients in}\\ \textbf{Whitehead } L_\infty\textbf{-algebra}\end{array}
\end{array}
\tag{7}
$$

**Generalized Gauge Potentials.** We have seen/recalled that for ordinary principal U(1)-connections there is a single characteristic form $F_2$ (the 1st Chern form) whose closure is witnessed by the existence of a local (meaning: on a cover $C \overset{\iota}{\twoheadrightarrow} X$) coboundary $A$ – which in application to electromagnetism is the *gauge potential*:

$$F_2 \,\in\, \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \xrightarrow{\;\iota^*\;} \Omega^2_{\mathrm{dR}}(C)_{\mathrm{clsd}}\,, \quad A \,\in\, \Omega^1_{\mathrm{dR}}(C)\,, \quad \mathrm{d}\,A \,=\, \iota^* F_2\,.$$

In order to generalize this to higher connections, recall that de Rham coboundaries are equivalently null-concordances:

$$A \;\longmapsto\; \widehat{F_2} \,:=\, t\,F_2 + \mathrm{d}t\,A$$

$$\underset{\substack{\textbf{ordinary}\\ \textbf{de Rham}\\ \textbf{coboundaries}}}{\;}\left\{ A \,\in\, \Omega^1_{\mathrm{dR}}(C) \;\middle|\; \mathrm{d}A \,=\, F_2 \right\} \;\underset{\substack{\textbf{electromagnetic}\\ \textbf{gauge potentials}}}{\longleftrightarrow}\; \left\{ \widehat{F_2} \,\in\, \Omega^2_{\mathrm{dR}}\big([0,1]\times C\big)_{\mathrm{clsd}} \;\middle|\; \begin{array}{l} \widehat{F_2}\big\vert_1 \,=\, F_2 \\ \widehat{F_2}\big\vert_0 \,=\, 0 \end{array} \right\}\underset{\substack{\textbf{ordinary}\\ \textbf{de Rham}\\ \textbf{null-concordances}}}{\;}$$

$$\textstyle\int_{[0,1]} \widehat{F_2} \;=:\; A \;\longleftarrow\; \widehat{F_2}$$

(Where "$t$" denotes the canonical coordinate on the interval $[0,1]$ and we leave implicit the pullbacks of differential forms along the projections $[0,1]\times C \xrightarrow{\mathrm{pr}_2} C \xrightarrow{\iota} X$.)

In the guise of null-concordances gauge potentials make immediate sense also for generalized nonabelian cohomology. For example, gauge potentials in 4-Cohomotopy are found to be of the following form ([26, Prop. 2.48] coinciding with the gauge potentials as known for the duality-symmetric C-field in 11D supergravity):

$$(C_3,\, C_6) \;\longmapsto\; \left\{ \begin{array}{l} \widehat{G}_4 \;:=\; t\,G_4 + \mathrm{d}t\,C_3 \\ \widehat{G}_7 \;:=\; t^2\,G_7 + 2t\mathrm{d}t\,C_6 \end{array} \right.$$

$$\underset{\substack{\mathfrak{l}S^4\textbf{-valued de Rham}\\ \textbf{coboundaries}}}{\left\{ \begin{array}{l} C_3 \,\in\, \Omega^3_{\mathrm{dR}}(C) \\ C_6 \,\in\, \Omega^6_{\mathrm{dR}}(C) \end{array} \;\middle|\; \begin{array}{l} \mathrm{d}\,C_3 \,=\, G_4 \\ \mathrm{d}\,C_6 \,=\, G_7 - \tfrac{1}{2} C_3\, G_4 \end{array} \right\}} \;\underset{\substack{\textbf{C-field}\\ \textbf{gauge}\\ \textbf{potentials}}}{\longleftrightarrow}\; \underset{\substack{\mathfrak{l}S^4\textbf{-valued de Rham}\\ \textbf{null-concordances}}}{\left\{ \begin{array}{l} (\widehat{G}_4,\, \widehat{G}_7) \,\in \\ \Omega^1_{\mathrm{dR}}\big([0,1]\times C;\, \mathfrak{l}S^4\big)_{\mathrm{clsd}} \end{array} \;\middle|\; \begin{array}{l} (\widehat{G}_4, \widehat{G}_7)\big\vert_1 \,=\, (G_4,\, G_7) \\ (\widehat{G}_4, \widehat{G}_7)\big\vert_0 \,=\, 0 \end{array} \right\}}$$

$$\left. \begin{array}{l} C_3 \,:=\, \int_{[0,1]} \widehat{G}_4 \\ C_6 \,:=\, \int_{[0,1]} \Big( \widehat{G}_7 - \tfrac{1}{2}\big(\int_{[0,-]} \widehat{G}_4\big)\widehat{G}_4 \Big) \end{array} \right\} \;\longleftarrow\; (\widehat{G}_4,\, \widehat{G}_7)$$

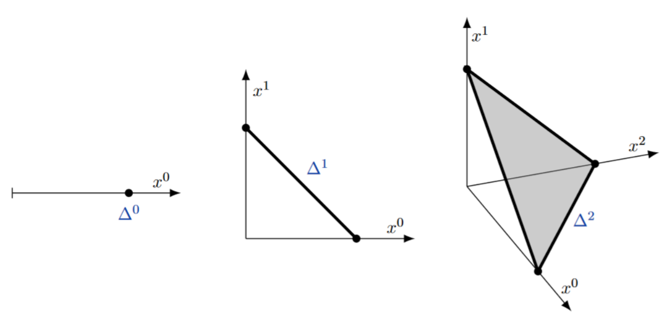
**Generalized Gauge Transformations.** Similarly, gauge transformations of ordinary gauge potentials are equivalently concordances-of-concordances with fixed endpoints. Applying this principle to the above gauge potentials for 4-Cohomotopy yields (still by [26, Prop. 2.48], now reproducing the gauge transformations as known for the duality-symmetric C-field in 11D supergravity):

$$(B_2,\, B_5) \;\longmapsto\; \left\{ \begin{array}{l} \widehat{\widehat{G}}_4 \;:=\; t\,G_4 + \mathrm{d}t\,C_3 + s\,\mathrm{d}t(C_3' - C_3) - \mathrm{d}s\,\mathrm{d}t\,B_2 \\ \widehat{\widehat{G}}_7 \;:=\; t^2\,G_7 + 2t\mathrm{d}t\,C_6 + 2st\mathrm{d}t(C_6' - C_6) - 2\,\mathrm{d}s\,t\mathrm{d}t(B_5 + \tfrac{1}{2}B_2\,C_3) \end{array} \right.$$

$$\underset{\substack{\mathfrak{l}S^4\textbf{-valued}\\ \textbf{gauge transformations}}}{\left\{ \begin{array}{l} B_2 \,\in\, \Omega^2_{\mathrm{dR}}(C) \\ B_5 \,\in\, \Omega^5_{\mathrm{dR}}(C) \end{array} \;\middle|\; \begin{array}{l} \mathrm{d}\,B_2 \,=\, C_3' - C_3 \\ \mathrm{d}\,B_5 \,=\, C_6' - C_6 - \tfrac{1}{2} C_3'\, C_3 \end{array} \right\}} \;\underset{\substack{\textbf{C-field}\\ \textbf{gauge trans-}\\ \textbf{formations}}}{\longleftrightarrow}\; \underset{\substack{\mathfrak{l}S^4\textbf{-valued}\\ \textbf{concordances-of-concordances}}}{\left\{ \begin{array}{l} (\widehat{\widehat{G}}_4,\, \widehat{\widehat{G}}_7) \,\in \\ \Omega^1_{\mathrm{dR}}\big([0,1]^2\times C;\, \mathfrak{l}S^4\big)_{\mathrm{clsd}} \end{array} \;\middle|\; \begin{array}{l} (\widehat{\widehat{G}}_4, \widehat{\widehat{G}}_7)\big\vert_1 \,=\, (\widehat{G}_4',\, \widehat{G}_7') \\ (\widehat{\widehat{G}}_4, \widehat{\widehat{G}}_7)\big\vert_0 \,=\, (\widehat{G}_4,\, \widehat{G}_7) \end{array} \right\}}$$

$$\left. \begin{array}{l} B_2 \,:=\, \int_{s\in[0,1]} \int_{t\in[0,1]} \widehat{G}_4 \\ B_5 \,:=\, \int_{s\in[0,1]} \int_{t\in[0,1]} \Big( \widehat{\widehat{G}}_7 - \tfrac{1}{2}\big(\int_{t'\in[0,-]} \widehat{G}_4\big)\widehat{G}_4 \Big) - \tfrac{1}{2} B_2\, C_3 \end{array} \right\} \;\longleftarrow\; (\widehat{\widehat{G}}_4,\, \widehat{\widehat{G}}_7)$$

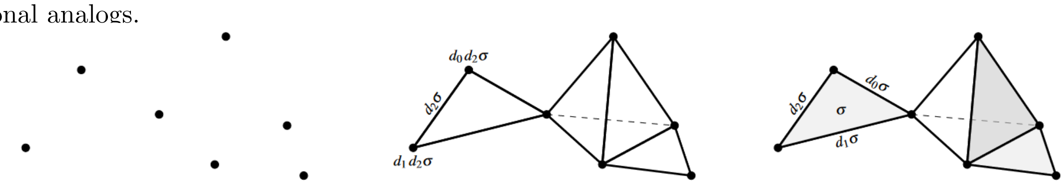
**Generalized Higher Gauge transformations.** And so on: Higher gauge transformations are given by higher order concordances. For subtle technical reasons it is more useful to, equivalently, parameterize these not over the higher cubes $[0,1]^n$ but over higher **simplices** $\Delta^n$ — the higher dimensional continuation of the progression starting with points, lines, triangles, tetrahedra, ... — given by $\Delta^n \;:=\; \Big\{ \vec{x} \,\in\, (\mathbb{R}_{\geq 0})^{n+1} \;\Big|\; \sum_{i=0}^n x^i \,=\, 0 \Big\} \;\subset\; \mathbb{R}^{n+1}$



Therefore the collections of
- flux densities and their
- generalized gauge fields,
- & their gauge transformations
- and higher gauge transformations

(the integrated version of what in physics jargon is the **BRST complex** with higher "ghost fields") constitute:

**Simplicial Sets** (cf. [19][28]) – systems of sets that capture the idea of sets of $(n+1)$-*transformations* between $n$-transformations, for all $n \in \mathbb{N}$, built by successively attaching to each other: edges, triangles, tetrahedra, and their higher dimensional analogs.



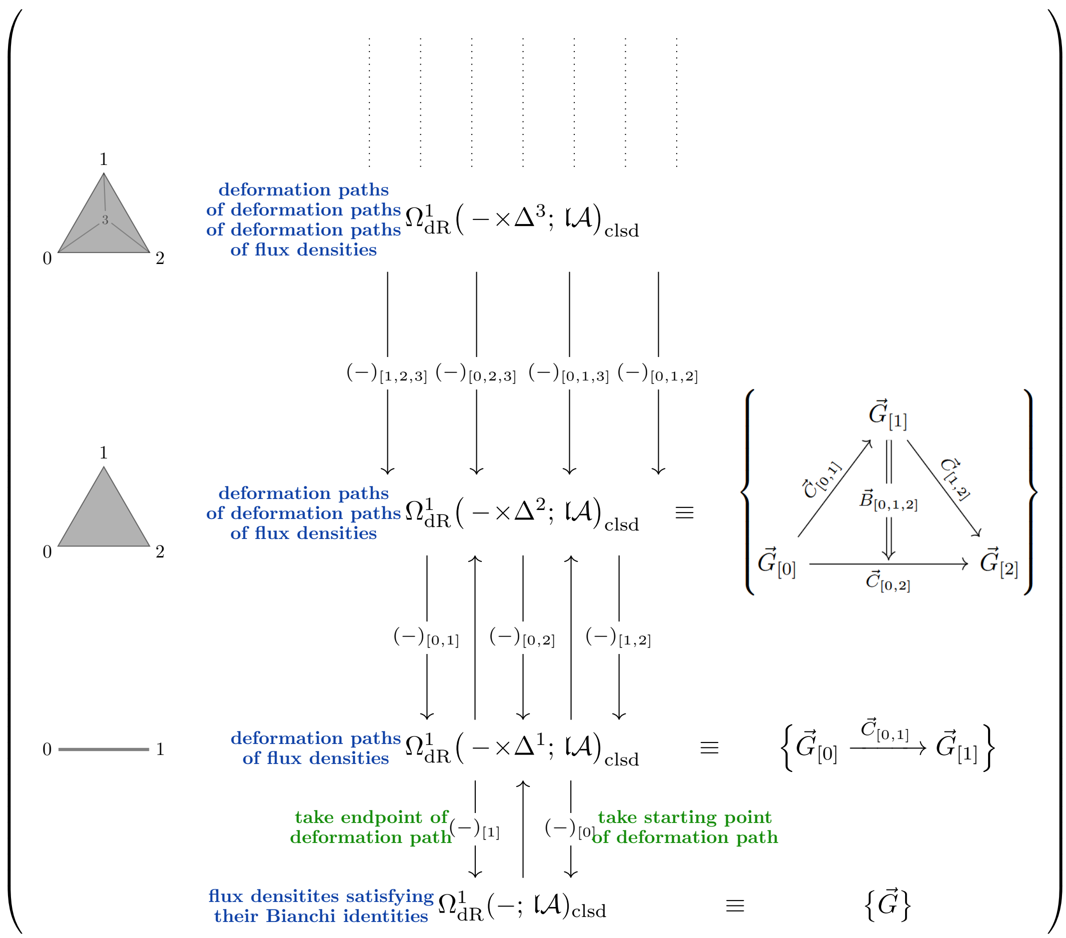

$$
\begin{array}{ll}
\textbf{simplicial set} & \textbf{set of points} \underset{\text{constant edges}}{\overset{\text{endpoints}}{\leftleftarrows\!\rightarrow}} \textbf{set of edges} \underset{\text{thin surfaces}}{\overset{\text{boundary edges}}{\leftleftarrows\!\rightarrow}} \textbf{set of surfaces} \leftleftarrows\!\rightarrow \cdots
\end{array}
$$

$$
\mathcal{X} \quad = \quad \left( X_0 \begin{array}{c} \xleftarrow{\quad d_1 \quad} \\ \xrightarrow{\quad s_0 \quad} \\ \xleftarrow{\quad d_0 \quad} \end{array} X_1 \begin{array}{c} \xleftarrow{\quad d_2 \quad} \\ \xrightarrow{\quad s_0 \quad} \\ \xleftarrow{\quad d_1 \quad} \\ \xrightarrow{\quad s_1 \quad} \\ \xleftarrow{\quad d_0 \quad} \end{array} X_2 \;\vdots\; \cdots \right)
$$

**Moduli of flux deformations.** Or rather, we have a system of such simplicial sets for each base manifold $(-)$ – thus called a *smooth $\infty$-groupoid* or *smooth moduli stack*, which we denote as follows:

$\int \Omega^1_{\mathrm{dR}}\big(-; \mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} =$

Smooth $\infty$-groupoid of moduli/deformations of $\mathfrak{l}\mathcal{A}$-valued flux densities



## 2.3 Differential Cohomology

(...)

Hence $\mathcal{A}$-quantization of flux means first of all that flux densities $\vec{F}$ are to be accompanied by total charges $[\chi]$ such that their $\mathfrak{l}\mathcal{A}$-valued de Rham class $[\vec{F}\,]$ coincides with the character of the total charge:

$$
\begin{tikzcd}
 & & H^1(X;\,\Omega\mathcal{A}) \arrow[d, "\mathrm{ch}^{\mathcal{A}}_X"', "\substack{\text{character}\\ \text{map}}"] & [16\text{, Def. IV.2}] \\
{*} \arrow[r, "\vec{F}", "\text{flux density}"'] \arrow[urr, dashed, bend left, "{[\chi]}", "\text{total charge}"'] \arrow[rr, bend right, "\text{total flux}"'] & \Omega^1_{\mathrm{dR}}(X;\,\mathfrak{l}\mathcal{A}) \arrow[r, ->>] & H^1_{\mathrm{dR}}(X;\,\mathfrak{l}\mathcal{A})\,. &
\end{tikzcd}
\tag{8}
$$

Stated in more detail [40, §3.3], the character map lifts from cohomology classes to *moduli stacks* and $\mathcal{A}$-flux quantization means that non-perturbative gauge field configurations are triples consisting of:

**(i)** *flux densities* $\vec{F} \in \Omega^1_{\mathrm{dR}}\big(X;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}}$ satisfying their pre-metric Bianchi identities;

**(ii)** *local charges* $\chi : X \to \mathcal{A}$ representing classes in $\mathcal{A}$-cohomology;

**(iii)** *deformations* $\widehat{A} : \mathrm{ch}(\chi) \Rightarrow \eta^{\int}(\vec{F}\,)$ of the flux densities into the character fluxes sourced by the local charges.

The last component $\widehat{A}$ turns out to be equivalently the global form of the *gauge potential* which constitutes the actual flux-quantized higher gauge field:

$$
\begin{tikzcd}[column sep=large]
 & & \underset{\substack{\text{Classifying space}\\ \text{of quantized charges}}}{\mathcal{A}} \ \substack{\text{subject to}\\ \mathfrak{l}\mathcal{A}\,\cong\,\mathfrak{l}\mathcal{A}} \arrow[d, "\mathbf{ch}^{\mathcal{A}}_X"', "\substack{\text{differential}\\ \text{character}}"] \quad [16\text{, Def. 9.2}] \\
\underset{\substack{\text{spacetime}\\ \text{manifold}}}{X} \arrow[r, "\vec{F}", "\text{flux density}"'] \arrow[urr, dashed, bend left, "\substack{(\text{Ex. } 2.1)\\ \text{local charge}}", "\chi"', ""{name=U}] & \underset{\substack{\text{Smooth set of}\\ \text{duality-symmetric}\\ \text{flux densities}}}{\Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}}} \arrow[r, "\eta^{\int}", "\substack{\text{up to}\\ \text{deformations}}"'] & \underset{\substack{\text{their deformation}\\ \infty\text{-groupoid}}}{\int\Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}}}
\arrow[from=U, to=2-2, phantom, "\Downarrow\,\widehat{A}\ \text{global gauge potential}"]
\end{tikzcd}
\tag{9}
$$

For example, this procedure (9) recovers [40, Ex. 3.10 & §4.1] the following familiar examples of globally well-defined flux-quantized higher gauge fields:

- *Maxwell field*: global gauge potentials are connections on U(1)-principal bundles, for the choice $\mathcal{A} \equiv B^2\mathbb{Z} \times B^2\mathbb{Q}$

  (as proposed by [12]

  and recast in modern language by [2][9, §7.1][17, §16.4e])

  or rather on electro-magnetic *pairs* of U(1) principal bundles, for the choice $\mathcal{A} = B^2\mathbb{Z} \times B^2\mathbb{Z}$

  (as considered in [18][5, Rem. 2.3][30, Def. 1.16][31, (3)])

- *B-field* in 10d: global gauge potentials are connections on U(1)-bundle gerbes, for the choice $\mathcal{A} \equiv B^3\mathbb{Z} \times B^3\mathbb{Q}$

  (as proposed by [24][22][11], review in [23]),

- *RR-field*: global gauge potentials are cocycles in twisted differential K-theory, for the choice $\mathcal{A} \equiv \mathrm{KU}_0 \,/\!\!/\, B^2\mathbb{Z}$

  (as proposed in various forms by [34][50][35][21][8] and established in full form in [GrS22]);

  and it seamlessly generalizes further to the case of interest here:

- *C-field* in 11d: global gauge potentials are cocycles in (twisted) differential Cohomotopy, for the choice $\mathcal{A} \equiv S^4$

  ("Hypothesis H", proposed in [38, §2.5], checked in [FSS20][14][15] to reproduce the expectations from the M-theory literature, reviewed in [16, §12][40, §4.3]).

(...)

# 3 Higher Chern-Weil Theory

(...)

# 4 Conclusion and Outlook

(...)

# A Background

(...)

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