Higher Topos Theory in Physics

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Recall that a *category* is a class of *objects* X, Y, ... (eg. sets, vector spaces, manifolds, groups, ...) with prescribed sets $\mathrm{Hom}(X, Y)$ of (homo-)*morphisms* between them, regarded abstractly as maps $f : X \to Y$ and ultimately defined by their composition law,

$$(\text{-}) \circ (\text{-}) : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$$
$$X \xrightarrow{f} Y \xrightarrow{g} Z \ , \quad X \xrightarrow{g \circ f} Z \tag{1}$$

which is required to be associative and unital. The point of category theory is to reason about (functors, natural transformations and then) *dualities* in the form of adjunctions (e.g. [Schreiber 2018]).

The most basic understanding of these notions should be sufficient to appreciate this entry. An introduction to categories aimed at mathematical physicists may be found in [Geroch 1985], for further basic exposition we recommend [Awodey 2006].

Make place for mathematical physics. A *topos* is a category inside of which the core machinery of mathematics can *take place* (whence the name $\tau\acute{o}\pi o\varsigma$, for "place", plural: $\tau\acute{o}\pi o\iota$). For instance, the usual category of sets is a topos, and most of the rigorous mathematics of the 20th century (tacitly) takes place in this default topos of sets. But there are other toposes. Traditional expositions highlight the *petit* toposes of sheaves (see below) on the open subsets of a fixed topological space or on the affine patches of a scheme — but these traditional examples are *not* instructive for the main application of higher topos theory in physics: Instead, practicing mathematical physicists are mostly working (mainly unknowingly) inside *gros* toposes of sheaves on a category whose objects are *all* the *probe spaces* on which a given notion of geometry is modeled. We discuss this by way of the key examples:

	Cartesian space		infinitesimal halo		super space		higher morphism		orbi-singularity		neg-dim sphere
probe	$\mathbb{R}^n$	$\times$	$\mathbb{D}^m_k$	$\times$	$\mathbb{R}^{0\vert q}$	$\times$	Δ^r	$\times$	$\underset{\curlyvee}{G}$	$\times$	$\mathbb{S}^{-d}$
geometry	differential topology		differential geometry		super geometry		homotopy theory		proper equivariance		stable homotopy
physics	fields		variations		fermions		gauge symmetry		orbi-singularities		quantum

The local algebraic coordinate manipulations used so successfully but often informally in physics may be regarded as defining generalized coordinate charts which serve as "probes" for the actual global geometric spaces in which physics takes place. The theory of (higher, *gros*) toposes may be understood as making this precise: Probes form a (higher) *site* and global spaces form the (higher) *sheaf topos* on such a site.

Probing space. Where traditional smooth manifolds are sets equipped with a smooth structure which is *locally diffeomorphic* to Cartesian spaces $\mathbb{R}^n$, we may more generally ask only that a *smooth set* X be whatever may consistently be *probed* by *plotting out* Cartesian spaces inside it – the idea being that such a plot is a smooth map "$\mathbb{R}^n \to X$", only that at this point of bootstrapping X into existence we are yet to say what this means. But first, for consistency the system of sets of n-dimensional plots

$$\underset{\text{space}}{X} \ : \ \underset{\text{probe}}{\mathbb{R}^n} \ \mapsto \ \underset{\text{set of plots}}{\mathrm{Plt}(\mathbb{R}^n, X)}$$

of X (the latter to be defined thereby) should, clearly, satisfy the following consistency conditions:

(1.) precomposition of plots (pre-sheaf condition) For $\phi \in \mathrm{Plt}(\mathbb{R}^n, X)$ and smooth $f : \mathbb{R}^{n'} \to \mathbb{R}^n$, the would-be composition "$\mathbb{R}^{n'} \xrightarrow{f} \mathbb{R}^n \xrightarrow{\phi} X$" should exist as $\phi \circ f \in \mathrm{Plt}(\mathbb{R}^{n'}, X)$, such that $\phi \circ \mathrm{id} = \phi$ and $(\phi \circ f) \circ f' = \phi \circ (f \circ f')$. In category theoretical language this says that $\mathrm{Plt}(-, X)$ is a *presheaf of sets* on the category CrtSp of smooth Cartesian spaces.

(2.) gluing of plots (sheaf condition) Given an open cover $\big\{U_i \overset{\iota_i}{\hookrightarrow} \mathbb{R}^n\big\}_{i \in I}$ which is *differentiably good* — meaning that the patches U_i and their non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are all diffeomorphic to $\mathbb{R}^n$ — those I-tuples of plots ϕ_i by the U_i which coincide on all overlaps $U_i \cap U_j$ should be in natural bijection with the global plots by the full $\mathbb{R}^n$:

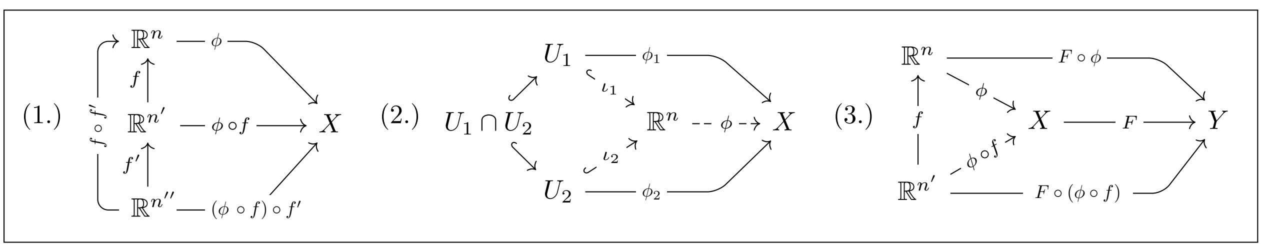
$$\begin{array}{ccc} \mathrm{Plt}(\mathbb{R}^n, X) & \xrightarrow{\sim} & \Big\{ \big(\phi_i \in \mathrm{Plt}(U_i, X)\big)_{i \in I} \,\Big\vert\, \underset{U_i \cap U_j}{\forall} \phi_i = \phi_j \Big\} \\ \phi & \longmapsto & \big(\phi \circ \iota_i\big)_{i \in I} \end{array}$$

In topos-theoretic language, this says that the presheaf $\mathrm{Plt}(-, X)$ of plots of X must be a *sheaf* with respect to the *coverage* (aka: *Grothendieck pre-topology*) of differentiably good open covers on the *site* CartSp of smooth Cartesian spaces.

It just remains to similarly characterize the smooth maps between such smooth sets:

(3.) postcomposition of plots (sheaf morphism) A *smooth map* $F : X \to Y$ between such smooth sets should be whatever consistently takes plots $\phi \in \mathrm{Plt}(\mathbb{R}^n, X)$ to their would-be composites "$\mathbb{R}^n \xrightarrow{\phi} X \xrightarrow{F} Y$" being plots $F \circ \phi \in \mathrm{Plt}(\mathbb{R}^n, Y)$, consistency requiring that for smooth $f : \mathbb{R}^{n'} \to \mathbb{R}^n$ we have $(F \circ \phi) \circ f = F \circ (\phi \circ f)$. In category/topos-theoretic laguage this says that smooth maps F are the *morphisms of sheaves* over the site CartSp.

In short, these three consistency conditions say equivalently that the following diagrams commute:



This is all fairly self-evident, and yet it means exactly that smooth sets form the *sheaf topos* on the *site* CrtSp over the *base topos* of sets:

$$\begin{aligned} &\mathrm{SmthSet} \\ &:= \underset{\text{with gluing condition}}{\mathrm{Sh}(\mathrm{CrtSp}, \mathrm{Set})} \hookrightarrow \underset{\text{without gluing condition}}{\mathrm{PSh}(\mathrm{CrtSp}, \mathrm{Set})}\,. \end{aligned} \tag{2}$$

For example, a Cartesian space $\mathbb{R}^n$ and generally a smooth manifold M is seen as a smooth set by taking its plots to be the ordinary smooth functions into it:

$$\begin{array}{l} \mathrm{CrtSp} \hookrightarrow \mathrm{SmoothMfd} \hookrightarrow \mathrm{SmthSet} \\ \qquad\qquad M \longmapsto \mathrm{Plt}(-, M) := C^\infty(-, M)\,. \end{array} \tag{3}$$

Under this **Yoneda embedding** we now *do* have a notion of smooth maps $\mathbb{R}^n \to X$ from a Cartesian space into any smooth set. Consistency of the above "bootstrap"-definition now demands that the prescribed plots naturally coincide with these actual smooth maps. This crucial self-consistency demand on our bootstrap-definition of smooth sets happens to be satisfied due to the general fact of category theory famous as the **Yoneda lemma**:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{SmthSet}}\big(\mathbb{R}^n,\ X\big) & \xrightarrow{\sim} & \mathrm{Plt}(\mathbb{R}^n, X) \\ F & \mapsto & F \circ \mathrm{id}_{\mathbb{R}^n}\,. \end{array} \tag{4}$$

In particular the embedding (3) of ordinary smooth manifolds into our generalized context is *fully faithful.*

Another basic result of topos theory gives that a smooth set is determined already by the *germs* of its plots, i.e. their equivalence classes under restriction to small neighbourhoods of any point (say $0 \in \mathbb{R}^n$):

$$\begin{aligned} &\mathrm{PltGrm}(\mathbb{R}^n, X) \\ &:= \mathrm{Plt}(\mathbb{R}^n, X) \big/ \Big(\phi \sim \phi' \text{ iff } \underset{\substack{\mathbb{R}^n \overset{\iota}{\hookrightarrow} \mathbb{R}^n) \\ \mathrm{opn} \\ 0 \in \iota(\mathbb{R}^n)}}{\exists} \underbrace{(\phi \circ \iota = \phi' \circ \iota)}_{\text{germ relation}}\Big)\,. \end{aligned} \tag{5}$$

Namely, maps $F : X \to Y$ in $\mathrm{PSh}(\mathrm{CrtSp}, \mathrm{Set})$ restrict to germs of plots

$$F\circ : \mathrm{PltGrm}(\mathbb{R}^n,\, X) \to \mathrm{PltGrm}(\mathbb{R}^n,\, X') \qquad (6)$$

and the *localization* L at (forcing the invertibility of) the *local isomorphisms*

$$\underset{\text{local isomorphism}}{X \overset{\text{liso}}{\longrightarrow} Y} \;\Leftrightarrow\; \underset{n}{\forall}\; \mathrm{PltGrm}\big(\mathbb{R}^n, X\big) \underset{\text{isomorphism (bijection)}}{\overset{\text{iso}}{\longrightarrow}} \mathrm{PltGrm}\big(\mathbb{R}^n, Y\big)$$

yields an equivalent category:

$$\mathrm{SmthSet} \;\simeq\; L^{\mathrm{liso}}\mathrm{PSh}\big(\mathrm{CrtSp}, \mathrm{Set}\big)\,. \qquad (7)$$

This reflects the intuition that arbitrarily small probes (of any dimension) are sufficient for exploring the smooth structure of a smooth set.

Toposes as categories of probe-able spaces. In conclusion so far, it is remarkable how the fundamental concepts of topos theory naturally align with physical intuition of "space" as whatever is witnessed by "probing" it (cf. the notion of *probe branes* in string theory, whose worldvolumes play much the role of the above plots, now for probing "D-geometry").

In fact, that is already the definition: A *topos* (here short for: *Grothendieck topos*, as usual) is a category $\mathcal{T}$ for which:

(1.) there exists a category S consisting of a set (instead of a proper class) of "probe"-objects, which is a *site* in that it is

(2.) equipped with a consistent notion (*coverage* or *Grothendieck pre-topology*) of what it means for any such probe object to be *covered* by other probes,

(3.) such that $\mathcal{T}$ is equivalently the category of objects consistently probeable by these probe objects, in immediate generalization of the example (2), called the category of sheaves on S:

$$\underset{\text{(Grothendieck-) topos}}{\mathcal{T}} \;\simeq\; \underset{\text{category of sheaves on site } S.}{\mathrm{Sh}(S, \mathrm{Set})} \qquad (8)$$

For further introduction to sheaf toposes see e.g. [Schreiber 2018] [Schapira 2023] and

Where field spaces take place. As a first example of the power of topos theory in physics, notice that much of the subtlety of *field theory* in contrast to "point mechanics" goes back to the fact that physical field configurations Φ are (smooth) *maps* from a (spacetime) manifold X to some coefficient space F

$$\underset{\text{physical field}}{\Phi} \;:\; \underset{\text{spacetime}}{X} \longrightarrow \underset{\text{field values}}{F}\,, \qquad (9)$$

so that spaces of field histories are *mapping spaces* (generally: spaces of sections of F-fiber bundles over X). These are at best infinite-dimensional Fréchet manifolds (under the unrealistic assumption that X is compact) and in general fall entirely outside the scope of traditional differential geometry.

In contrast, the topos of smooth sets – like every topos – is *cartesian closed*, meaning that

(1.) Cartesian products exist, immediately so by (4):

$$\mathrm{Plt}\big(\mathbb{R}^n, X \times Y\big) \;:=\; \mathrm{Plt}\big(\mathbb{R}^n, X\big) \times \mathrm{Plt}\big(\mathbb{R}^n, Y\big)\,,$$

(2.) there is guaranteed to be a smooth set

$$\mathrm{Fields} := \mathrm{Maps}(X, F)$$

of smooth maps (between any two smooth sets X, F) such that there are natural bijections of maps

$$\underset{\text{map to mapping space}}{U \to \mathrm{Maps}(X, F)} \quad\leftrightarrow\quad \underset{\text{map on product space}}{U \times X \to F}\,.$$

Concretely, plotting out a probe U inside the space of maps from $X \to F$ — which one may think of as X-parameterized elements of F — should just be a U-parameterized family of such maps, hence a $U \times X$-parameterized family of elements of F:

$$\begin{array}{ccc} \mathrm{Plt}\big(U,\, \mathrm{Maps}(X, F)\big) & := & \mathrm{Hom}\big(U \times X,\, F\big) \\ \begin{array}{ccc} U & \longrightarrow & \mathrm{Fields} \\ u & \mapsto & \Phi_u \end{array} & \leftrightarrow & \begin{array}{c} U \times X \longrightarrow F \\ (u, x) \mapsto \Phi_u(x)\,. \end{array} \end{array} \qquad (10)$$

This tautological and intuitively transparent prescription *defines* the mapping space $\mathrm{Maps}(X, F)$ as a smooth set, and yet subsumes all the traditional definitions whenever they happen to be applicable:

Namely, also infinite-dimensional Fréchet manifolds are faithfully included among smooth sets — via the analogous formula (3), as are *diffeological spaces* (11), cf. *Figure 1.* — and when the mapping space exists in these subcategories it agrees with (10).

For more on smooth sets in field theory see [Schreiber 2017] [Giotopoulos & Sati 2023].

Where anomaly polynomials take place. Contrary to tradition in differential geometry, smooth sets are defined "operationally" and *not* as sets equipped with extra structure, though this case is subsumed: A smooth set X is *concrete* (as a sheaf on CrtSp) – also called a *diffeological space* [Iglesias-Zemmour 2013] – if there exists a plain set X_s such that the plots of X are natural subsets of the maps of plain sets into X_s:

$$\mathrm{Plt}(\mathbb{R}^n, X) \hookrightarrow \mathrm{Hom}_{\mathrm{Set}}\big(\mathbb{R}^n,\, X_s\big)\,. \qquad (11)$$

For example, smooth manifolds (3) are among concrete smooth sets, but also (Delta-generated) topological spaces $\mathrm{X} \in \mathrm{DTopSp}$ are *faithfully* subsumed, via

$$\begin{array}{ccc} \mathrm{DTopSp} \hookrightarrow \mathrm{DifflgSp} \hookrightarrow \mathrm{SmthSet} \\ \mathrm{X} \quad \mapsto \quad \mathrm{Plt}(-,\mathrm{X}) := C^0(-,\mathrm{X})\,, \end{array} \qquad (12)$$

cf. [Sati & Schreiber 2023, Prop. 3.3.19].

Among *non*-concrete smooth sets are the "smooth classifying spaces" $\mathbf{\Omega}^p_{\mathrm{dR}}$, of differential p-forms:

$$\mathrm{Plt}\big(\mathbb{R}^n,\, \mathbf{\Omega}^p_{\mathrm{dR}}\big) \;:=\; \Omega^p_{\mathrm{dR}}(\mathbb{R}^n) \quad \substack{\text{set of smooth} \\ \text{differential } p\text{-forms} \\ \text{on Cartesian space.}}$$

These are classifying in that smooth maps from a smooth manifold X into them are in natural bijection with smooth differential forms on X:

$$\mathrm{Hom}\big(X,\, \mathbf{\Omega}^p_{\mathrm{dR}}\big) \;\simeq\; \Omega^p_{\mathrm{dR}}(X)\,.$$

Remarkably, if X is an n-manifold, then the smooth mapping space (10) into, say, $\mathbf{\Omega}^{n+2}_{\mathrm{dR}}$ is still non-trivial: It contains differential forms that appear only *in families* parameterized by some manifold U:

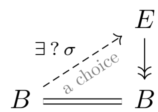
$$U \to \mathrm{Maps}(X, \mathbf{\Omega}^p_{\mathrm{dR}}) \;\leftrightarrow\; \omega_U \,\in\, \Omega^{n+2}_{\mathrm{dR}}(X \times U)\,.$$

In field theory this is the case for (Green-Schwarz-type) "anomaly polynomials" $I_{n+2} = j^{\mathrm{el}}_{n-k} \wedge j^{\mathrm{mag}}_{k+2}$, whose mathematical home, traditionally left vague, is the following diagram of smooth sets (the integral assuming compact support, as usual):

$$\begin{array}{ccccc} \multicolumn{5}{c}{\text{curvature of anomaly line bundle on field space}} \\ \mathrm{Map}(X^n, F) & \xrightarrow{I_{n+2}} & \mathrm{Map}(X^n, \mathbf{\Omega}^{n+2}_{\mathrm{dR}}) & \xrightarrow{\int_{X^n}} & \mathbf{\Omega}^2_{\mathrm{dR}} \\ \overset{\text{probe}}{(U \times X^n \xrightarrow{\Phi} F)} & \mapsto & I_{n+2}(\Phi) & \mapsto & \int_{X^n} I_{n+2}(\Phi) \\ \substack{U\text{-parameterized} \\ \text{family of fields}} & & \substack{\text{anomaly polynomial} \\ \text{in } \Omega^{n+2}(X^n \times U)} & & \substack{\text{local anomaly} \\ \text{in } \Omega^2(U)} \end{array}$$

Constructivism and instanton sectors. A fundamental result of ("elementary") topos theory is that mathematical definitions and theorems may be transported from plain sets to any other topos – if only they are *constructive*, meaning essentially that they do not invoke the usual "axiom of choice". This is not mysterious but another example of physical intuition aligning with topos theory:

Namely, in the default topos of sets, the axiom of choice says equivalently that every surjective map $E \twoheadrightarrow B$ ("epimorphism") admits a *choice* $b \mapsto \sigma(b) \in E_b$ of elements in the fiber E_b over each point $b \in B$ in the base, forming a commuting diagram of this form:



Whatever one may think of this axiom in the case of plain sets, it is clearly *un*justified for smooth sets, where a surjection as above is a *smooth bundle*, such as a fiber bundle or a principal bundle. Even if we assume (as one usually does) that we can choose elements $\sigma(b)$ in each fiber of the bundle separately, for a non-trivial principal bundle there is in general no way to make theses choices *smoothly* (or even continuously) to arrange into a smooth map $\sigma : B \to E$.

Therefore the failure of the axiom of choice in the topos of smooth sets is, in a sense, the *reason* why in physics one sees crucial phenomena like flux quantization, soliton/instanton sectors or fermionic anomalies – all of which reflect the existence of non-trivial fiber bundles (for more see [Sati & Schreiber 2024]).

Hence reasoning in mathematical physics is naturally reflected in constructive mathematics, and toposes are essentially the possible *models* of such constructive or *physical* reasoning. There are many topos models relevant for the discussion of physics:

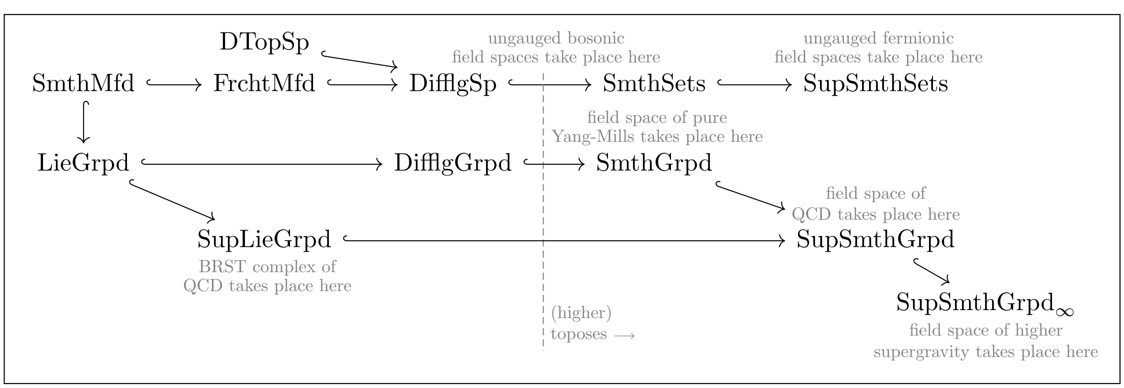


Figure 1. Part of the system of categories of generalized spaces needed in mathematical physics.

Where variational calculus takes place. Informal physics texts often refer to variables whose value is non-vanishing but "so tiny" that their square may be neglected, $\epsilon^2 = 0$. This naive coordinate expressions – which algebraically means that $\epsilon \in \mathbb{R}[\epsilon]/(\epsilon^2)$ – may be understood as describing simple infinitesimal *probe* spaces, from which topos theory immediately provides us with rigorous models of global differential geometry containing actual infinitesimal quantities (historically known as "synthetic differential geometry"):

Namely, a fundamental (if maybe underappreciated) fact of differential geometry is that smooth functions between smooth manifolds are *fully faithfully* reflected in the homomorphisms of real algebras of smooth functions which they induce:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{SmthMfd}}\big(X,\,Y\big) & \simeq & \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}\big(C^\infty(Y),\,C^\infty(X)\big) \\ f & \longleftrightarrow & f^* \end{array}$$

exhibiting a full embedding into the opposite category of commutative real algebras:

$$\begin{array}{ccccc} \mathrm{CrtSp} & \hookrightarrow & \mathrm{SmthMnfd} & \xhookrightarrow{C^\infty(-)} & \mathrm{CAlg}^{\mathrm{op}}_{\mathbb{R}} \\ & & X & \mapsto & C^\infty(X)\,. \end{array}$$

Reading this backwards – just as familiar from algebraic geometry – we may declare *infinitesimally thickened* Cartesian spaces to be that whose algebras of smooth functions, by definition, contain nilpotent monomials, defined as forming this full subcategory:

$$\begin{array}{ccc} \mathrm{ThCrtSp} & \xhookrightarrow{\quad C^\infty(-)\quad} & \mathrm{CAlg}^{\mathrm{op}}_{\mathbb{R}} \\ \mathbb{R}^n \times \mathbb{D}^m_k & \mapsto & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{R}[\epsilon_1, \cdots, \epsilon_m]\big/\big(\epsilon^{k+1}\big) \end{array} \tag{13}$$

which becomes a site via coverings of the form $\big\{U_i \times \mathbb{D}^m_k \xrightarrow{\iota_i \times \mathrm{id}} \mathbb{R}^n \times \mathbb{D}^m_k\big\}_{i \in I}$ for $(\iota_i)_{i\in I}$ a differentiably good open cover as before. Therefore we obtain (8) the topos

$$\mathrm{FrmlSmthSet} \;:=\; \mathrm{Sh}\big(\mathrm{ThCrtSp}, \mathrm{Set}\big) \tag{14}$$

of (synthetic-) *differential smooth sets*. Smooth manifolds X are still faithfully included here, by defining their plots algebraically

$$\begin{aligned} &\mathrm{Plt}\big(\mathbb{R}^n \times \mathbb{D}^m_k, X\big) \\ &:= \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}\Big(C^\infty(X),\, C^\infty(\mathbb{R}^n \times \mathbb{D}^m_k)\Big)\,, \end{aligned} \tag{15}$$

and hence exist now alongside the infinitesimal halos $\mathbb{D}^m_k$. Notice that these contain only a single actual point

$$\iota : * = \mathbb{R}^0 \xrightarrow{\exists !} \mathbb{D}^m_k$$

and yet are larger than that point (have more plots, namely by other infinitesimals).

This topos (15) is a most convenient environment where naïve differential geometric intuition becomes a rigorous reality. E.g., for X a smooth manifold, its tangent bundle $TX \xrightarrow{p} X$ (also a smooth manifold) is just the mapping space out of the infinitesimal interval:

$$\begin{array}{ccc} \mathrm{Map}\big(\mathbb{D}^1_1, X\big) & \simeq & TX \\ \big\downarrow{\scriptstyle \mathrm{Map}(\iota, X)} & & \big\downarrow{\scriptstyle p} \\ \mathrm{Map}(*, X) & \simeq & X\,. \end{array} \tag{16}$$

Proceeding in this spirit, one finds that traditional variational calculus on jet bundles, hence all of classical Lagrangian field theory, takes place in FrmlSmthSet where the notorious technical subtleties are naturally being taken care of.

For more see [Dubuc 1979] [Kock 1986] [Kock & Reyes 1987] [Schreiber & Khavkine 2017].

Where classical fermion fields take place. It is commonplace that operator products of fermionic *quantum* fields satisfy Clifford algebra relations, but it may be less widely appreciated that (therefore) already classical fermion fields are "anti-commuting variables" subject to the algebraic relation

$$\psi\,\psi' \;=\; -\psi'\,\psi\,, \tag{17}$$

which is *necessary* for the usual Dirac-Lagrangian of fermion fields to make algebraic sense

$$L_\psi(x) \;\propto\; \overline{\psi}(x) D \psi(x)\, \mathrm{dvol}\,,$$

since otherwise it would be a total derivative. But beyond such formal algebraic manipulations, what *is* a classical fermion field? Remarkably, the naive algebraic coordinate-manipulations common in physics are again perfectly valid as statements about fermionic *probe* spaces, from where topos theory takes automatic care of providing a good notion of general fermionic (field) spaces:

Concretely, a simple *finite-dimensional Cartesian* fermionic space with n bosonic and q fermionic coordinate functions – a "super-space" $\mathbb{R}^{n|q}$ – ought to be fully characterized by the fact that its "algebra of smooth functions" $C^\infty(\mathbb{R}^{n|q})$ is (defined to be) the $\mathbb{Z}/2$-graded commutative (super-commutative) algebra (we write $\mathrm{sCAlg}_{\mathbb{R}}$ for their category) which is the plain tensor product of the ordinary smooth functions on $\mathbb{R}^n$ with the "odd functions" on $\mathbb{R}^q$, the latter defined to form the Grassmann algebra $\wedge^\bullet(\mathbb{R}^q)^*$ of the linear dual space:

$$C^\infty(\mathbb{R}^{n|q}) := C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \wedge^\bullet(\mathbb{R}^q)^* \in \mathrm{sCAlg}_{\mathbb{R}} . \quad (18)$$

As before for bosonic infinitesimals (13), we may *define* smooth maps of Cartesian super-spaces to *be* the reverse homomorphisms of their super-commutative function algebras:

$$f : \mathbb{R}^{n|q} \to \mathbb{R}^{n'|q'} \;\leftrightarrow\; f^* : C^\infty(\mathbb{R}^{n'|q'}) \to C^\infty(\mathbb{R}^{n|q}) .$$

This means to define the category of smooth Cartesian super-spaces as the full sub-category of the opposite of the category $\mathrm{sCAlg}_{\mathbb{R}}$ of super-commutative algebras on those of the form (18):

$$\begin{array}{ccc} \mathrm{SupCrtSp} & \xhookrightarrow{C^\infty(-)} & \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^{n|q} = \mathbb{R}^n \times \mathbb{R}^{0|q} & \mapsto & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \wedge^\bullet(\mathbb{R}^q)^* . \end{array} \quad (19)$$

Declaring the coverings of Cartesian super-spaces to be of the form $\big\{ U_i \times \mathbb{R}^{0|q} \xrightarrow{\iota_i \times \mathrm{id}} \mathbb{R}^n \times \mathbb{R}^{0|q} \big\}_{i \in I}$ for $(\iota_i)_{i \in I}$ a differentiably good open cover as before, we readily obtain the topos of super smooth sets:

$$\mathrm{SupSmthSet} := \mathrm{Sh}(\mathrm{SupCrtSp}, \mathrm{Set}) . \quad (20)$$

For example, given a smooth spinor bundle $\mathcal{S} \to X$, it becomes a super smooth set $\Pi\mathcal{S}$ by:

$$\mathrm{Plt}\big(\mathbb{R}^{n|q}, \Pi\mathcal{S}\big) := \big\{ \phi : \mathbb{R}^n \to X \big\} \times \wedge^q \Gamma\big(\phi^* \mathcal{S}\big) , \quad (21)$$

from which one finds that spinor fields are the odd points in the super smooth function set out of $\Pi\mathcal{S}$:

$$\begin{array}{lcl} \Phi_{\mathrm{evn}} : \mathbb{R}^{0|0} \to \mathrm{Map}(\Pi\mathcal{S}, \mathbb{R}) & \leftrightarrow & \phi \in C^\infty(X) \\ \Phi_{\mathrm{odd}} : \mathbb{R}^{0|1} \to \mathrm{Map}(\Pi\mathcal{S}, \mathbb{R}) & \leftrightarrow & \psi \in \Gamma(\mathcal{S}^*) . \end{array}$$

But under the following faithful embedding of super-commutative algebras into commutative algebras of super sets, the super smooth function set (21) is identified with the Grassmann algebra on spinor fields:

$$\mathrm{sCAlg}(\mathrm{Set})_{\mathbb{R}} \hookrightarrow \mathrm{CAlg}(\mathrm{SupSmthSet})_{\underline{\mathbb{R}}}$$

in that

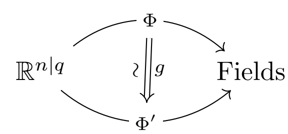
$$\begin{array}{ccl} A & \mapsto & \underline{A} : \mathbb{R}^{n|q} \mapsto \big(A \otimes C^\infty(\mathbb{R}^{n|q})\big)_{\mathrm{evn}} \\ \Gamma\big(\wedge^\bullet \mathcal{S}^*\big) & \mapsto & \underline{\Gamma\big(\wedge^\bullet \mathcal{S}^*\big)} \simeq \mathrm{Map}\big(\Pi\mathcal{S}, \mathbb{R}\big) . \end{array}$$

This exhibits the anticommutation relation (18). Cf. [Konechny & Schwarz 1998] [Sachse 2008].

Where discrete gauge fields take place. We may read the *gauge principle* in physics as saying that in gauge field spaces the very notion of equality is relaxed: Two (plots/families (10) of) gauge fields

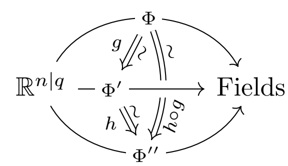
$$\Phi, \Phi' : \mathbb{R}^{n|q} \longrightarrow \mathrm{Fields}$$

may be nominally distinct and yet identified via gauge transformations:



(22)

that may be composed



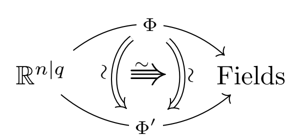
(23)

and reversed

$$g : \Phi \Rightarrow \Phi' \quad \text{implies} \quad \left\{ \begin{array}{l} g^{-1} : \Phi' \Rightarrow \Phi, \\ g^{-1} \circ g = \mathrm{id}_{\Phi} \\ g \circ g^{-1} = \mathrm{id}_{\Phi'} . \end{array} \right.$$

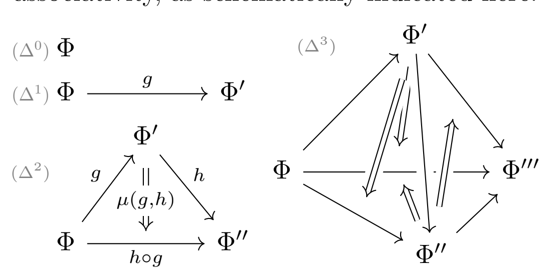
This means that the plots of gauge field spaces no longer form plain sets, but *groupoids*: sets of objects with invertible maps (gauge transformations) making them a category (1).

Furthermore, for *higher* gauge fields, two such gauge transformations, in turn, may be nominally distinct and yet identified by "gauge-of-gauge transformations"



(24)

satisfying a 2-dimensional analog of composition and associativity, as schematically indicated here:



(25)

and so on to ever higher order gauge transformations, now making the plots form *higher groupoids*.

Traditional physics literature describes this phenomenon mostly infinitesimally, where it is captured by the homological algebra of *BRST complexes*: Here the infinitesimal gauge transformations appear as "ghost fields" and the infinitesimal higher-gauge transformations as "ghost-of-ghost fields".

The topos theory for going beyond the infinitesimal higher gauge transformations is again via probes: We detect the nature of a higher gauge groupoid $\mathcal{X}$ by recording the system of sets $\mathrm{Plt}(\Delta^n, \mathcal{X})$ of n-dimensional higher gauge transformations of the shape indicated for low n in (25) – called the *n-simplices* Δ^n. Equipped with the fairly evident cell-preserving maps, these n-simplices form a site (with trivial coverage) denoted Δ (the "simplex category"), so that higher groupoids should be found in the corresponding (pre-)sheaf topos of "simplicial sets",

$$\mathrm{Sh}(\Delta, \mathrm{Set})_{\mathrm{Kan}} \hookrightarrow \mathrm{Sh}(\Delta, \mathrm{Set}) ,$$

as those simplicial sets all whose higher gauge transformations have composites, if suitably consecutive, in a manner that is suitably associative, unital and invertible up to further gauge transformations. All this turns out to be neatly encoded by the *Kan condition*, which demands simply that whenever we find (probe) in a simplicial set $\mathcal{X}$ the boundary of an n-simplex *except* the kth $(n-1)$-face – called a *horn* $\Lambda^n_k \hookrightarrow \Delta^n$ – then we may also find the missing face and the interior n-morphism:

$$\mathcal{X} \in \mathrm{Sh}(\Delta, \mathrm{Set})_{\mathrm{Kan}} \quad \Leftrightarrow \quad \begin{array}{ccc} \forall\ \Lambda^n_k & \longrightarrow & \mathcal{X} \\ \downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n . & & \end{array}$$

Some important examples:

(1.) Given a *discrete* group G, there is a groupoid

$$\mathbf{B}G \;=\; \text{“}\left\{ \begin{array}{c} \overset{g}{\circlearrowright} \\ \mathrm{pt} \end{array} \;\middle|\; g \in G \right\}\text{”} \tag{26}$$

with a single Δ^0-plot pt, one Δ^1-plot $\mathrm{pt} \xrightarrow{g} \mathrm{pt}$ for each group element, Δ^2-plots given by pairs of group elements and witnessing the group operation

$$\mathrm{Plt}(\Delta^2, \mathbf{B}G) = \left\{ \begin{array}{ccc} & \mathrm{pt} & \\ {}^{g_1}\nearrow & \Downarrow \exists ! & \searrow^{g_2} \\ \mathrm{pt} & \underset{g_2 \cdot g_1}{\longrightarrow} & \mathrm{pt} \end{array} \;\middle|\; \begin{array}{c} (g_1, g_2) \\ \in G \times G \end{array} \right\} , \tag{27}$$

and generally $\mathrm{Plt}(\Delta^n, \mathbf{B}G) = G^{\times^n}$. This groupoid appears as the field fiber F of G-Dijkgraaf-Witten theory.

(2.) For a topological space $\mathrm{X} \in \mathrm{Top}$, its singular simplicial complex is Kan, representing the *higher path groupoid* $\int \mathrm{X}$ (also "fundamental"- or "Poincaré"-groupoid):

$$\begin{aligned} &\mathrm{Plt}\big(\Delta^n, \textstyle\int \mathrm{X}\big) \\ &:= \mathrm{Hom}_{\mathrm{Top}}\Big(\big\{\vec{x} \in (\mathbb{R}_{\geq 0})^{n+1} \,\big|\, \textstyle\sum_i x_i = 1\big\},\, \mathrm{X}\Big) . \end{aligned} \tag{28}$$

(3.) Given a chain-complex in non-negative degrees

$$V_\bullet = \big[\ \cdots \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0\ \big] \tag{29}$$

it becomes a Kan-simplicial set HV by

$$\mathrm{Plt}\big(\Delta^n, HV\big) := \mathrm{Hom}_{\mathrm{Ch}_\bullet}\Big(N_\bullet \mathbb{Z}[\Delta^n],\, V_\bullet\Big) , \tag{30}$$

where we probe $V_\bullet$ with the normalized chains complex $N_\bullet(-)$ of cellular singular chains on Δ^n.

In particular, when $V_\bullet = A[n]$ is concentrated on an abelian group in degree n, then

$$\mathbf{B}^n A \;:=\; HA[n] \;=:\; H(A, n) \tag{31}$$

is known as the nth *Eilenberg-MacLane space* of A.

One readily checks that the maps between Kan simplicial sets form themselves Kan simplicial sets

$$\mathcal{X}, \mathcal{Y} \in \mathrm{Sh}(\Delta)_{\mathrm{Kan}} \;\;\Rightarrow\;\; \mathrm{Map}(\mathcal{X}, \mathcal{Y}) \in \mathrm{Sh}(\Delta)_{\mathrm{Kan}} .$$

For example, the maps from $\int \mathrm{X}$ (28) to $\mathbf{B}G$ (26) form the groupoid of gauge fields and their gauge transformations in Dijkgraaf-Witten theory on X:

$$\mathrm{Map}\big(\textstyle\int X, \mathbf{B}G\big) \;\simeq\; \mathrm{Flat}G\mathrm{Bund}_X .$$

Moreover, these higher mapping groupoids have canonical composition operations

$$(\text{-}) \circ (\text{-}) : \mathrm{Map}(\mathcal{X}, \mathcal{Y}) \times \mathrm{Map}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Map}(\mathcal{X}, \mathcal{Z})$$

which are associative and unital. Hence in gauge-theoretic enhancement of the base topos of

- sets
  with sets of maps between them,

we have now

- Kan-simplicial sets
  with Kan-simplicial sets of maps between them

behaving like higher (gauge) groupoids with higher groupoids of maps between them and forming what is called a category *enriched* in simplicial sets, to be denoted:

$$\mathbf{Grpd}_\infty \;:=\; \mathbf{Sh}(\Delta, \mathrm{Set})_{\mathrm{Kan}} \in \mathrm{Sh}(\Delta)\text{-Cat} . \tag{32}$$

This is (one incarnation of) the default *higher topos* of *higher groupoids* or *higher homotopy types*.

In particular, a *gauge equivalence* (math jargon: *homotopy equivalence*) of higher groupoids are maps $\phi : \mathcal{X} \overset{\text{heq}}{\longleftrightarrow} \mathcal{X}' : \phi'$ which are inverses *up to gauge transformations* in that there exists:

$g : \Delta^1 \to \mathrm{Map}(\mathcal{X},\mathcal{X})\,, \quad g' : \Delta^1 \to \mathrm{Map}(\mathcal{X}',\mathcal{X}')$

$$
\begin{array}{ccccc}
 & & \mathcal{X}' & & \\
 & {\scriptstyle \phi}\nearrow & {\scriptstyle g}\Downarrow\wr & \searrow{\scriptstyle \phi'} & \\
\mathcal{X} & & = \!=\!=\!= & & \mathcal{X}
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{X}' & & = \!=\!=\!= & & \mathcal{X}' \\
 & {\scriptstyle \phi'}\searrow & {\scriptstyle g'}\Downarrow\wr & \nearrow{\scriptstyle \phi} & \\
 & & \mathcal{X} & &
\end{array}
\tag{33}
$$

E.g. there are homotopy equivalences exhibiting (31) as a *based loop space* (of basepoint-preserving maps and higher homotopies inside the mapping space)

$$
\mathbf{B}^n A \overset{\text{heq}}{\longrightarrow} \Omega_{\mathrm{pt}} \mathbf{B}^{n+1} A := \mathrm{Map}^{\mathrm{pt}/}\big(\textstyle\int S^1, \mathbf{B}^{n+1} A\big)\,. \tag{34}
$$

One may equivalently understand the Kan-simplicial enrichment in (32) as the universal way of turning classes W of maps that ought to be such homotopy equivalences (33) – but cannot be in an ordinary category – into actual homotopy equivalences in a simplicially-enriched category, a process known as *simplicial localization* $\mathbf{L}^W$:

$$
\mathbf{Grpd}_\infty \underset{\mathrm{DK}}{\simeq} \mathbf{L}^{\text{heq}}\,\mathrm{Sh}(\Delta)_{\mathrm{Kan}}\,. \tag{35}
$$

Where smooth gauge fields take place. Via the paradigm of probes, it is now immediate that higher *(super-)smooth groupoids* $\mathcal{X}$ (faithfully subsuming Lie groupoids and diffeological groupoids) are whatever when probed with $\mathbb{R}^{n|q}$ exhibit a Kan-simplicial set of plots (22):

$$
\mathrm{PSh}\big(\mathrm{SupCrtSp},\, \mathrm{Sh}(\Delta)_{\mathrm{Kan}}\big) \;=\;
\left\{
\begin{array}{r}
\mathcal{X} : \mathrm{SupCrtSp}^{\mathrm{op}} \to \mathrm{Sh}(\Delta)_{\mathrm{Kan}} \\
\mathbb{R}^{n|q} \;\mapsto\; \mathbf{Plt}(\mathbb{R}^{n|q}, \mathcal{X})
\end{array}
\right\}
\tag{36}
$$

In gauge-theoretic enhancement of (7), gauge equivalences between smooth higher groupoids should be *local homotopy equivalences*:

$$
\underset{\substack{\text{local homotopy}\\ \text{equivalence}}}{\mathcal{X} \overset{\text{lheq}}{\longrightarrow} \mathcal{Y}} \Leftrightarrow \underset{n,q}{\forall}\, \underset{\substack{\text{homotopy}\\ \text{equivalence}}}{\mathbf{PltGrm}\big(\mathbb{R}^{n|q}, X\big) \overset{\text{heq}}{\longrightarrow} \mathbf{PltGrm}\big(\mathbb{R}^{n|q}, Y\big)}
$$

and the *higher topos* of (super) *smooth* ∞*-groupoids* (aka ∞-stacks, here incarnated as a simplicially enriched category) is the simplicial localization of (36):

$$
\begin{aligned}
&\mathbf{SupSmthGrpd}_\infty \\
&\quad := \mathbf{L}^{\text{lheq}}\,\mathrm{PSh}\Big(\mathrm{SupCrtSp},\, \mathrm{Sh}(\Delta, \mathrm{Set})_{\mathrm{Kan}}\Big)\,.
\end{aligned}
$$

Important examples:

(1.) For a (super-)Lie group G, the analog of (27) is

$$
\begin{aligned}
&\mathrm{Plt}\Big(\Delta^2,\, \mathbf{Plt}\big(\mathbb{R}^{n|q}, \mathbf{B}G\big)\Big) \\
&= \left\{
\begin{array}{ccc|l}
 & \mathrm{pt} & & \\
{\scriptstyle g_1}\nearrow & \Downarrow\exists ! & \searrow{\scriptstyle g_2} & (g_1, g_2) \\
\mathrm{pt} & \underset{g_2 \cdot g_1}{\longrightarrow} & \mathrm{pt} & \in C^\infty\big(\mathbb{R}^{n|q},\, G\big)^{\times^2}
\end{array}
\right\}
\end{aligned}
\tag{37}
$$

which exhibits the *super smooth groupoid* that deloops the Lie group G.

In slight variation, for $\mathfrak{g}$ denoting the (super-)Lie algebra of G we now also have the smooth groupoid $\mathbf{B}G_{\mathrm{conn}}$ whose plots are $\mathfrak{g}$-valued connection forms A (vector potentials) with their usual gauge transformations:

$$
\begin{aligned}
&\mathrm{Plt}\big(\Delta^2, \mathbf{Plt}(\mathbb{R}^{n|q}, \mathbf{B}G_{\mathrm{conn}})\big) \\
&:= \left\{
\begin{array}{ccc|l}
 & A_1 & & A_i \in \Omega^1(\mathbb{R}^{n|q}) \otimes \mathfrak{g}, \\
{\scriptstyle g_1}\nearrow & \Downarrow\exists ! & \searrow{\scriptstyle g_2} & g_i \in C^\infty(\mathbb{R}^{n|q}, G) \\
A_0 & \underset{g_2 \cdot g_1}{\longrightarrow} & A_2 & A_i = g_i A_{i-1} g_i^{-1} \\
 & & & \qquad + g_i \mathrm{d} g_i^{-1}
\end{array}
\right\}
\end{aligned}
$$

(2.) For $V_\bullet(-)$ a *sheaf* of chain complexes (29) on SupCrtSp, such as the dth *Deligne complex*

$$
\mathrm{Del}^d_\bullet \;:=\; \Big[\mathbb{Z} \hookrightarrow \Omega^0_{\mathrm{dR}}(-) \overset{\mathrm{d}}{\to} \Omega^1_{\mathrm{dR}}(-) \to \cdots \overset{\mathrm{d}}{\to} \Omega^{d-1}_{\mathrm{dR}}(-)\Big]
$$

we get the (super-)smooth version $HV_\bullet(-)$ of the corresponding higher groupoid (30), which we may denote $\quad \mathbf{B}^d \mathrm{U}(1)_{\mathrm{conn}} \;:=\; H\mathrm{Del}^d$

(3.) For X a smooth manifold and any good open cover $\big\{U_i \overset{\iota_i}{\hookrightarrow} X\big\}_{i \in I}$, its *Čech nerve* $\widehat{X}$ has as probes the smooth maps to the U_i and as (higher) gauge transformations the maps into (higher) intersections:

$$
\begin{aligned}
&\mathrm{Plt}\big(\Delta^k, \mathbf{Plt}(\mathbb{R}^{n|q}, \widehat{X})\big) \\
&:= C^\infty\Big(\mathbb{R}^{n|q}, \coprod_{i_1, \cdots, i_k} U_{i_1} \cap \cdots \cap U_{i_k}\Big)\,.
\end{aligned}
\tag{38}
$$

This is locally homotopy equivalent to X:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\text{lheq}} & X \\ \big(\mathbb{R}^{n|q} \to U_{i_1}\big) & \mapsto & \big(\mathbb{R}^{n|q} \to U_{i_1} \hookrightarrow X\big) \end{array}$$

and hence a gauge-equivalent incarnation of X, but it is a "good" (namely *projectively cofibrant*) representative, implying that the mapping spaces out of $\widehat{X}$ into the above classifying objects $\mathbf{B}(-)_{(-)}$ exhaust the gauge equivalence classes of the corresponding maps.

The maps between these objects are (modulate) (higher) gauge fields, classified by higher cohomology:

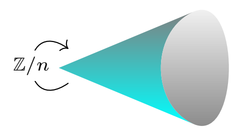
$$\begin{array}{|ccc|} \hline \mathrm{Map}\big(\widehat{X}, \mathbf{B}G_{\mathrm{conn}}\big) & = & \left\{ \begin{array}{c} G\text{-Yang-Mills fields} \\ (A\text{-fields}) \end{array} \right\} \\ \downarrow & & \downarrow \\ \underbrace{\pi_0 \mathrm{Map}\big(\widehat{X}, \mathbf{B}G\big)}_{=H^1(X;G)} & = & \big\{ G\text{-instanton sectors} \big\} \\ \hline \mathrm{Map}\big(\widehat{X}, \mathbf{B}^2\mathrm{U}(1)_{\mathrm{conn}}\big) & = & \big\{ B\text{-fields} \big\} \\ \downarrow & & \downarrow \\ \underbrace{\pi_0 \mathrm{Map}\big(\widehat{X}, \mathbf{B}^2\mathrm{U}(1)\big)}_{=H^3(X;\mathbb{Z})} & = & \left\{ \begin{array}{c} \text{string} \\ \text{charge sectors} \end{array} \right\} \\ \hline \mathrm{Map}\big(\widehat{X}, \mathbf{B}^3\mathrm{U}(1)_{\mathrm{conn}}\big) & = & \big\{ C\text{-fields} \big\} \\ \downarrow & & \downarrow \\ \underbrace{\pi_0 \mathrm{Map}\big(\widehat{X}, \mathbf{B}^2\mathrm{U}(1)\big)}_{=H^4(X;\mathbb{Z})} & = & \left\{ \begin{array}{c} \text{membrane} \\ \text{charge sectors} \end{array} \right\} \\ \hline \end{array} \tag{39}$$

Beyond these examples, consider the countably infinite-dimensional complex Hilbert space $\mathcal{H}$ with its topological space of Fredholm operators $\mathrm{Frd}(\mathcal{H})$, which is a classifying space for topological K-theory. Then a cocycle in *differential* K-theory is a homotopy in $\mathbf{SmthGrpd}_\infty$ of the following form

$$\begin{array}{ccccc} & \overset{\text{RR-flux densities}}{\dashrightarrow} & \prod_k \mathbf{\Omega}^{2k}_{\mathrm{clsd}} & \overset{\text{de Rham map}}{\searrow} & \\ \underset{\text{smth mfd}}{\widehat{X}} & \text{RR-field} & \Downarrow & & \prod_k \mathbf{B}^{2k}\, \mathbb{R}^\flat \\ & \underset{\text{K-cocycle}}{\dashrightarrow} & \int \mathrm{Frd}(\mathcal{H}) & \underset{\text{Chern character}}{\nearrow} & \end{array} \tag{40}$$

Generally, every notion of generalized *differential cohomology* (classifying generalized higher gauge fields) takes place in $\mathbf{SmthGrpd}_\infty$ in an analogous way.

Where singularities take place. While smooth, spaces in physics famously may have singularities, by which we shall mean orbi-singularities: A cone such as the quotient $\mathbb{R}^2/(\mathbb{Z}/n)$ of the plane (by rotation along an angle $2\pi/n$ about the origin) is smooth everywhere except at the tip, where the $\mathbb{Z}/n$-action appears to have "shrunken away" to act only "inside the singular point", appearing much like the groupoid $\mathbf{B}\mathbb{Z}/n$ (26).



Cf. [Sati & Schreiber 2020][2021]:

Such an *orbifold* is hence a smooth set which moreover responds to probes by orbi-singularities, whose higher site may be understood to be the full simplicial subcategory on the delooping groupoids of finite groups:

$$\begin{array}{ccc} \mathbf{Snglr} & \hookrightarrow & \mathbf{Grpd}_\infty \\ \overset{G}{\curlyvee} & \mapsto & \mathbf{B}G\,. \end{array}$$

With the gauge equivalences of orbi-singular (smooth) higher groupoids being the singularity-wise equivalences sngeq (notice that the trivial singularity is included as $\mathrm{pt} = \mathbf{B}1 \in \mathbf{Snglrt}$) we obtain the higher topos of *singular* smooth higher groupoids:

$$\begin{aligned} &\mathbf{SnglrSupSmthGrpd}_\infty \\ &:= \mathbf{L}^{\mathrm{sngeq}}\mathbf{PSh}\big(\mathbf{Snglrt}, \mathbf{SupSmthGrpd}_\infty\big) \end{aligned}$$

For $G \circlearrowright X$ a group action in smooth sets, the *homotopy quotient* $X /\!\!/ G$ is

$$\begin{aligned} &\mathrm{Plt}\Big(\Delta^2, \mathbf{Plt}\big(\mathbb{R}^{n|q}, X /\!\!/ G\big)\Big) \\ &:= \left\{ \begin{array}{ccc} & g_1 \cdot \phi & \\ {}^{g_1}\nearrow & \Downarrow \exists ! & \searrow^{g_2} \\ \phi & \xrightarrow[g_2 \cdot g_1]{} & g_2 \cdot g_1 \cdot \phi \end{array} \;\middle|\; \begin{array}{l} \phi \in \mathrm{Plt}\big(\mathbb{R}^{n|q}, X\big) \\ g_i \in \mathrm{Plt}\big(\mathbb{R}^{n|q}, G\big) \end{array} \right\} \end{aligned}$$

and its *orbi-singularization* $\curlyvee(-)$ is given by

$$\begin{aligned} &\mathbf{Plt}\Big(\mathbb{R}^{n|q} \times \overset{G}{\curlyvee}, \curlyvee\big(X /\!\!/ G\big)\Big) \\ &:= \mathbf{Plt}\Big(\mathbb{R}^{n|q}, \mathrm{Map}\big(\mathbf{B}G,\, X /\!\!/ G\big)\Big) \end{aligned}$$

For example, let $G \circlearrowright \mathcal{H}$ be a stable representation of a finite group, then a cocycle in the G-equivariant K-theory of X is a map in $\mathbf{SnglrSmthGrpd}_\infty$ of this form:

$$\curlyvee\big(X /\!\!/ G\big) \xrightarrow{\substack{\text{equivariant}\\ \text{K-cocycle}}} \textstyle\int \curlyvee\big(\mathrm{Frd}(\mathcal{H}) /\!\!/ G\big)\,. \qquad \searrow\; \underset{\curlyvee}{G} \;\swarrow$$

From this one obtains, in analogy with (40), equivariant *differential K-theory*, thought to modulate both RR-fields on orbifold spacetimes as well as ground states of topological phases of crystalline quantum materials (where the orbifold is the Brillouin torus quotiented by the point group of the crystal lattice).

Where quantum physics takes place. Indeed, physics is ultimately quantum, where quantum state spaces are *linear* spaces, varying over classical parameter spaces.

Linearity means *abelian group structure* and in higher gauge theory (homotopy theory), a higher group is the more abelian the more *de-loopings* (34) it admits. Hence fully *linear structure* on a pointed higher groupoid E_0 should be a sequence of ever higher deloopings (called a *spectrum*) exhibited by a sequence of maps

$$E_0 \xrightarrow{\tilde{\sigma}_0} \Omega E_1 \xrightarrow{\Omega(\tilde{\sigma}_1)} \Omega^2 E_2 \xrightarrow{\Omega^2(\tilde{\sigma}_2)} \Omega^3 E_2 \xrightarrow{\Omega^3(\tilde{\sigma}_3)} \cdots\,,$$

which are homotopy equivalences up to suitable gauge equivalence of spectra, called *stable homotopy equivalences* (steq).

Since maps into the looping of a higher groupoid are equivalently homotopies of this form,

$$X \to \Omega_{\mathrm{pt}} Y \qquad \leftrightarrow \qquad \begin{array}{ccc} X & \to & \mathrm{pt} \\ \downarrow & \swArrow & \downarrow \\ \mathrm{pt} & \to & Y \end{array}$$

a parameterized spectrum is a space that may be probed by objects behaving like spheres of negative dimension, forming a higher site of this form:

$$\mathbf{Lin} := \left\{ \begin{array}{ccccccccccccc} & & \mathrm{pt} & = & \mathrm{pt} & = & \mathrm{pt} & = & \cdots \\ & \swarrow & \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & \swarrow & \\ \mathbb{S}^0 & \Downarrow\wr & \mathbb{S}^{-1} & \Downarrow\wr & \mathbb{S}^{-2} & \Downarrow\wr & \mathbb{S}^{-3} & & \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & \\ & & \mathrm{pt} & = & \mathrm{pt} & = & \mathrm{pt} & = & \cdots \end{array} \right\}$$

Quite remarkably, parameterized spectra still form a higher topos, the higher *tangent topos*:

$$\begin{aligned} &\mathbf{LinSnglrSupSmthGrpd}_\infty \\ &:= \mathbf{L}^{\mathrm{steq}}\mathbf{PSh}\big(\mathbf{Lin}, \mathbf{SnglrSupSmthGrpd}_\infty\big)\,. \end{aligned} \tag{41}$$

In this higher topos (41) takes place, for example:
(1.) The non-perturbative ("geometric") quantization of Poisson manifolds, exhibited here by the push-forward of K-module spectra parameterized (via a choice of higher *prequantum line bundle*) over the corresponding symplectic groupoid to its leaf space (cf. [Nuiten 2013] [Schreiber 2014], also [Sati & S. 2023]).
(2.) The circuit logic of quantum information/computing with classical control and dynamic lifting of quantum measurement results, exhibited by the base-change yoga of Real $H\mathbb{C}$-module spectra, among whose "heart" are the finite-dimensional Hilbert spaces (cf. [Sati & Schreiber 2023]).
(3.) The construction of Hilbert spaces of anyonic quantum ground states of topologically ordered crystals, in the guise of $\mathfrak{su}(2)$-conformal blocks wih KZ-connection, exhibited here as sections of equivariant parameterized spectra over smooth configuration spaces of points in the crystal's Brillouin torus orbifold, cf. [Sati & S. 2024] [Myers et al. 2024].

Conclusion. Quite contrary to superficial perception, higher topos theory provides just the mathematical context that physicists are often intuitively but informally assuming anyway. Realizing the higher topos theory yields a wealth of new powerful mathematical tools addressing many of the notoriously subtle issues in mathematical physics and opening the door to rigorous attacks on some of the outstanding open problems of the field.

There clearly remains a large gap between the languages that the communities speak, but this is a historical artefact which is increasingly being bridged.

However, this also means that, despite some history and considerable work, the application of higher topos theory in physics is still in its infancy. Our account here is by necessity a progress report more than the survey of a mature field.

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