Homotopy Theory for Topological Quantum Computing

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On a novel application of low-dim AlgTop to a key open problem in Quantum Technology.

digest of:

Engineering of Anyons on M5-Probes
Srní lecture notes (2025)
[`arXiv:2501.17927`]

talk at

ICMS25, AUS Sharjah, Feb 2025

Synopsis

topological quantum computing
on "FQH-platforms" aims to manipulate
exotic quantum states of magnetic flux

largely escapes traditional tools of physics
but novel approach shows it admits
description via classifying spaces

⇒ theorems in low-dim AlgTop
provide otherwise elusive analysis
of **tech-relevant quantum effects**

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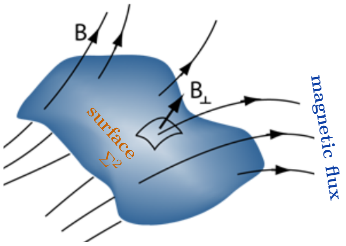
We are concerned with algebro-topological phenomena arising when magnetic flux penetrates a semi-conducting surface Σ^2.

The "gauge group" of the electromagnetic field is $G \equiv \mathrm{U}(1)$ and *ordinarily* such flux is classified by maps to $B\mathrm{U}(1) \simeq \mathbb{C}P^\infty$.

Precisely, when quantum-effects are being resolved, then:

Theorem [7] (Yang-Mills flux quantum observables): *For ordinary gauge fields on a spacetime* $\simeq \mathbb{R}^{1,1} \times \Sigma^2$ *the* ***quantum observables of field flux*** *through* Σ^2 *form the group-convolution* C^**-algebra* $\underbrace{\mathbb{C}\Big[C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]}_{\text{quantum flux observables}}$ for $\Lambda \subset \mathfrak{g}$ an Ad-invariant lattice.



G	Lie group ("gauge group")
$\mathfrak{g}$	its Lie algebra

Commercial-value quantum computing will require **robust** quantum observables, insensitive to local fluctuations, only depending on **topological sectors** of field configurations.

$$\underbrace{\mathbb{C}\Big[C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]}_{\text{all quantum flux observables}} \xleftarrow{[-]^*} \underbrace{\mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]}_{\text{robust topological observables}}$$

$C^\infty(\text{-},\text{-})$	manifold of smooth functions
$(\text{-}) \ltimes (\text{-})$	semidirect product via adjoint
$\mathbb{C}[-]$	group convolution C^*-algebra
$\pi_0(-)$	path-connected components

Proposition [7] (topological sector observables): The topological flux quantum observables form the homology Pontrjagin algebra of maps from space to classifying space. (shown now assuming $\Lambda = 0$, for simplicity):

$$\overset{\text{topological flux quantum observables}}{\mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G\big)\Big]} \;\simeq\; \mathbb{C}\Big[\pi_0\, \mathrm{Maps}\big(\Sigma^2,\, G\big)\Big]$$

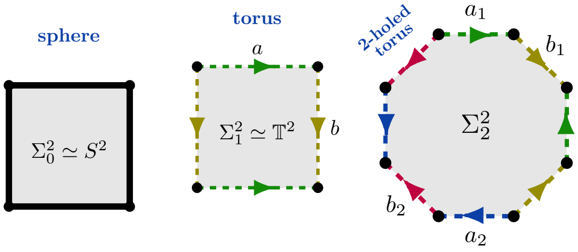
$$\simeq\; \underbrace{\mathbb{C}\Big[\pi_1\, \mathrm{Maps}\big(\Sigma^2,\, BG\big)\Big]}_{\substack{\text{group algebra of fundamental group}\\ \text{of maps to classifying space}}} \;\simeq\; \underbrace{H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}},\, BG\big);\, \mathbb{C}\Big)}_{\substack{\text{homology Pontrjagin algebra of}\\ \text{soliton moduli space}}}$$

soliton on X	=	topological field configuration that vanishes at the ends of X
	⇒	classified by *pointed* map $X_{\cup\{\infty\}} \to BG$ from one-point compactification

Example: $\mathbb{C}\big[\pi_0\, \mathrm{Maps}\big(\Sigma^2_g,\, \mathrm{U}(1)\big)\big] \;\simeq\; \mathbb{C}\big[H^1(\Sigma^2_g;\, \mathbb{Z})\big] \;\simeq\; \mathbb{C}\big[\mathbb{Z}^{2g}\big]$

Σ^2_g orientable surface of genus$=g$

Effective flux of "fractional quantum Hall systems" (FQH). But, at very low temperature, experiment suggests instead of $\mathbb{Z}^{2g}$ its 2nd integer Heisenberg extension $\widehat{\mathbb{Z}^{2g}}$ being the observables of an "**effective Chern-Simons field**", where the center $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}^{2g}}$ observes an **anyon braiding phase**.



Question: Is there classifying space $\mathcal{A}$ for this effective CS field?

Answer: Yes! The 2-sphere $S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq B\mathrm{U}(1)$

Theorem [2][4]: The cofiber presentation of the surface

$$S^1 \xrightarrow{\prod_i [a_i, b_i]} \bigvee_g (S^1_a \vee S^1_b) \longrightarrow \Sigma^2_g \longrightarrow S^2$$

induces short exact sequence exhibiting the Heisenberg extension:

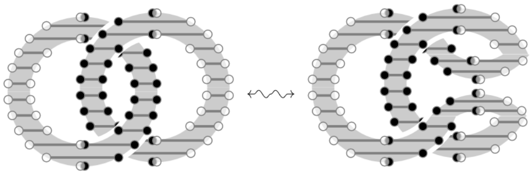
$$1 \to \underbrace{\pi_1 \mathrm{Maps}\big(S^2,\, S^2\big)}_{\mathbb{Z}} \to \underbrace{\pi_1 \mathrm{Maps}\big(\Sigma^2_g,\, S^2\big)}_{\widehat{\mathbb{Z}^{2g}}} \to \underbrace{\pi_1 \mathrm{Maps}^*\big(\bigvee_{2g} S^1,\, S^2\big)}_{\mathbb{Z}^{2g}} \to 1$$

$$\widehat{\mathbb{Z}^{2g}} \;:=\; \left\{ \begin{array}{c} \big(\vec{a}, \vec{b}, n\big) \;\in\; \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z} \\[1em] \big(\vec{a}, \vec{b}, n\big) \cdot \big(\vec{a}', \vec{b}', n'\big) \;= \\ \big(\vec{a} + \vec{a}',\, \vec{b} + \vec{b}',\, n + n' + \underbrace{\vec{a} \cdot \vec{b}' - \vec{a}' \cdot \vec{b}}_{\substack{\text{twice the unit}\\ \text{central extension}}}\big) \end{array} \right\}$$

Question: Can we identify the center $\mathbb{Z}$ as arising from braiding?

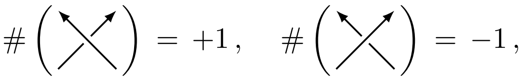
Answer: Yes!

Theorem [8]: $\mathrm{Maps}^*(S^2, S^2)$ is configurations of charged strings such that $\Omega\mathrm{Maps}^*(S^2, S^2)$ is framed links subject to cobordism, $\pi_1\mathrm{Maps}^*(S^2, S^2)$ generated from framed unknot with 1 braiding



$$\begin{array}{ccc} \Omega\mathrm{Maps}^*\big(S^2,\, S^2\big) & \xrightarrow{\quad [-] \quad} & \pi_3(S^2) \simeq \mathbb{Z} \\ L & \longmapsto & \#L \\ \text{framed link} & & \text{linking + framing number} \end{array}$$

is CS observable ("Wilson loop")



$\#(\;) = +1\,,\quad \#(\;) = -1\,,$

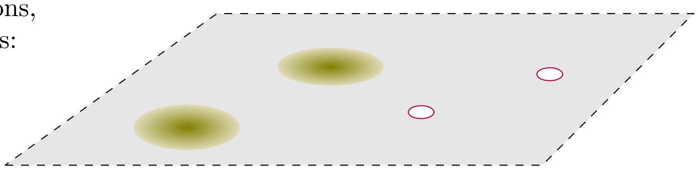
Ergo: Remarkably, topological quantum observables of effective flux in quantum Hall systems is algebro-topologically described by $\left\{ \begin{array}{lr} \text{replacing the classifying space} & B\mathrm{U}(1) \simeq \mathbb{C}P^\infty \\ \text{with its 2-skeleton} & S^2 \simeq \mathbb{C}P^1 \end{array} \right.$

Question 1: Is there rationale for such replacement? **Answer:** Yes [9][10][11]: *Hypothesis H...*

Question 2: Does this new model make novel predictions? **Answer:** Yes – *defect anyons* in FQH-systems.

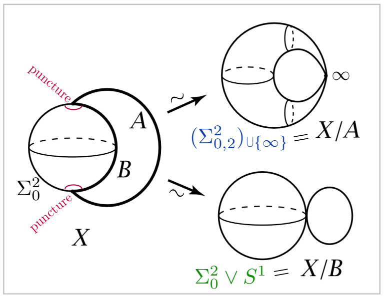
With the classifying space identified for known situations, we find its implications for previously inaccessible cases:

Namely generalize now to *n-punctured* surfaces $\Sigma^2_{g,n}$, reflecting n *defect points* in the semiconductor where the magentic field is *expelled* (type-I superconducting spots).



Proposition. *The observables are, in this generality:*

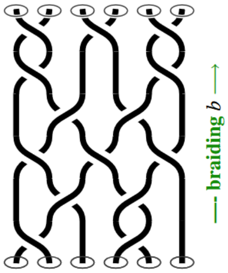
$$\begin{aligned} \mathrm{Obs}_0 \;&\simeq\; \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\Big(\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}},\, S^2 \Big)\Big] \\ &\simeq\; \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\Big(\Sigma^2_g \vee \bigvee_{n-1} S^1,\, S^2 \Big)\Big] \\ &\simeq\; \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\big(\Sigma^2_g,\, S^2 \big) \times \mathbb{Z}^{n-1}\Big] \\ &\underset{g=0}{\simeq}\; \mathbb{C}\big[\mathbb{Z}^n\big] \end{aligned}$$



Topology change due to defects! (cf. [3, p 11]) Different to but not unlike the *genon*-proposal [1].

subject to the **diffeomorph. action** *by:*

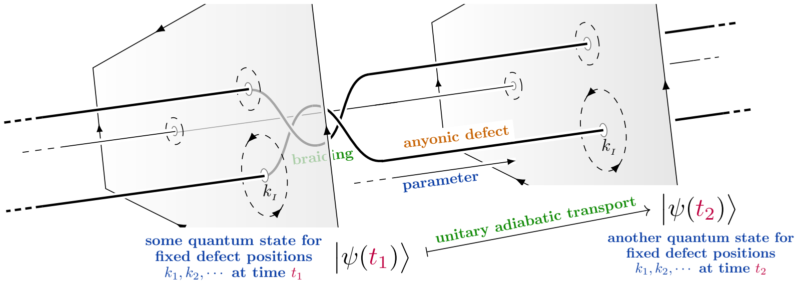
$$1 \to \underset{\text{surface braid group}}{\mathrm{Br}_n(\Sigma^2_g)} \hookrightarrow \underset{\substack{\text{mapping class group}\\ \text{of punctured surface}}}{\pi_0 \mathrm{Homeos}^*_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big)} \twoheadrightarrow \underset{\substack{\text{mapping class group}\\ \text{of plain surface}}}{\mathrm{MCG}(\Sigma^2_g)} \to 1$$



Therefore the equivariant quantum states (jargon: "generally covariant") on $\Sigma^2_{0,n}$ are representations of the *wreath product of solitonic and defect phases*:

$$\mathbb{Z} \wr \mathrm{Br}_n(\Sigma^2_0) \;=\; \overset{\text{solitonic anyons}}{\mathbb{Z}^n} \rtimes \overset{\text{defect anyons}}{\mathrm{Br}_n(\Sigma^2_0)} \;\twoheadrightarrow\; \mathbb{Z}^n \rtimes \mathrm{Sym}_n$$

Such *braid representations for defects* have not previously been derived for FQH systems –
but are just what is needed for the grand goal of *topological quantum gates*:
programmable unitary transformations of quantum systems,
insensitive to continuous deformations (hence to noise!)



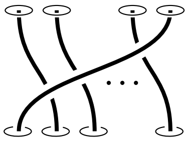
Concretely, the worldvolume domains with n defects retract onto those with a single defect, whence so do the corresponding quantum observables and quantum states:

$$\begin{tikzcd} \mathrm{Obs}_\bullet\Big((\Sigma^2_{0,1})_{\cup\{\infty\}} \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}"] & (\Sigma^2_{0,n})_{\cup\{\infty\}} \arrow[r, two heads] & (\Sigma^2_{0,1})_{\cup\{\infty\}} \Big) \\ \mathbb{Z} \arrow[r, hook] & \mathbb{Z}^n \arrow[r, two heads] & \mathbb{Z}\,, \end{tikzcd}$$

$$\mathcal{H}_{\Sigma^2_{0,n}} \;\simeq\; \mathbb{C}^n \;\simeq\; \mathbb{C} \oplus \begin{ytableau} \; & \; & \; & \none[\cdots] & \; \\ \; \end{ytableau} \;\in\; \mathrm{Sym}_n\mathrm{Rep}$$

This yields what are known as *controlled qdit-rotation gates*, the workhorse of quantum algorithms & the bottleneck for noise-protection, now topologically protected as cylic defect braidings:

$$\text{e.g.:}\quad \ydiagram{2,1} \;\simeq\; \left\{ (213) \mapsto \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}^{Z}, \quad (231) \mapsto \overbrace{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}}^{R_y(2\alpha)} \; \begin{matrix} \text{where} \\ \alpha = 4\pi/3 \end{matrix} \right\},$$



Conclusion & Outlook:
With non-linear flux-quantization laws taken into account in physics, substantial Algebraic Topology reveals previously unrecognized phenomena potentially visible in experiment and relevant for quantum technology. (Potentially a much more fruitful commercial AlgTop-application than TDA!)

Vistas. With this map from AlgTop to quantum effects established, there is opportunity to make AlgTop research inform quantum technology. Concretely:

Open problems in low-dim AlgTop.	**Relevance for quantum system.**
describe the $\mathrm{Br}_n(\Sigma^2_{g>0})$-action on $\pi_1 \mathrm{Maps}^*\big((\Sigma^2_{g>0,n})_{\cup\{\infty\}},\, S^2\big)$	novel exotic topological quantum gates
higher degree homology $H_{\bullet>0}\Big(\Omega\,\mathrm{Maps}^*\big((\Sigma^2_{g,n})_{\cup\{\infty\}},\, S^2\big);\, \mathbb{C}\Big)$	"higher order" effects in topological phases
generalize to allow $b > 0$ boundary components $\Sigma^2_{g,n,b}$	small $b > 0$ is experimentally most accessible case

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If there is time left – let's shift gears:

We have seen that:

topological quantum states $\mathcal{H}_\Sigma$ of solitonic field fluxes with classifying space $\mathcal{A}$ on spacetime domain $\mathbb{R}^{1,1} \times \Sigma$	form representations of π_1 of the soliton moduli space $\mathrm{Fields}_\Sigma \;:=\; \mathrm{Maps}^*\big(\Sigma,\, \mathcal{A}\big) /\!\!/ \mathrm{Aut}^*(\Sigma)$

This is remarkable because

such representations are equiv. *vector bundles* $\mathcal{H}_\Sigma$ on Fields_Σ *with flat connections* ∇ a.k.a.: *local systems* on moduli

$$\begin{tikzcd} \overset{\text{quantum states \& observables}}{\big(\mathcal{H}_\Sigma,\, \nabla\big)} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Mod}_{\mathbb{C}}^{\mathbb{C}/} \arrow[d, two heads] \\ \mathrm{Fields}_\Sigma \arrow[r, "\vdash \mathcal{H}_\Sigma"] & \mathrm{Mod}_{\mathbb{C}} \end{tikzcd}$$

with the homotopy type of Fields_Σ understood as an ∞-groupoid, (physics newspeak: generalized symmetry) flat vector bundles are equivalently functors $\vdash \mathcal{H}_\Sigma$ to the groupoid $\mathrm{Mod}_{\mathbb{C}}$

which is the special case of ∞-*local systems* [6]: chain complex-bundles with flat ∞-connection

$$\begin{tikzcd} \text{higher quantum states \& higher observables} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Mod}_{H\mathbb{C}}^{H\mathbb{C}/} \arrow[d, two heads] \\ \mathrm{Fields}_\Sigma \arrow[r, "\vdash \mathcal{H}^\infty_\Sigma"] & \mathrm{Mod}_{H\mathbb{C}} \end{tikzcd}$$

which are equivalently Fields_Σ-*parameterized module spectra* for the E_∞-ring $H\mathbb{C}$ hence $H\mathbb{C}[\Omega \mathrm{Fields}_\Sigma]$-modules

detecting higher structure in the moduli space:

here $H\mathbb{C}$ denotes the *homotopy complex numbers*: the EM-ring spectrum of $\mathbb{C}$

and $H\mathbb{C}[\Omega \mathrm{Fields}_\Sigma]$ is the *homotopy Pontrjagin algebra* whose $\pi_\bullet$ is $\mathrm{Obs}_\bullet$

These objects form the **tangent ∞-topos** $T\mathrm{Grpd}_\infty$ (over $H\mathbb{C}$), which is [5][6]:
(i) the arena of parameterized stable homotopy theory,
(ii.) categorial semantics of a novel quantum programming language

Remarkably, this provides an AlgTop angle on an ill-understood but central physics aspect:

What exactly is ***quantum measurement*** *of anyonic topological order?*

Fact. [5]
Given quantum states $\mathcal{H} \in \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Fields}}$,
- a *quantum measurement basis* is
  - a choice of space W (of "*possible worlds*")
  - a map $W \xrightarrow{i} \mathrm{Fields}$ whose
  - base change is *ambidextrous*:

$$\begin{tikzcd} \mathrm{Mod}_{\mathbb{C}}^{W} \arrow[r, shift left=2, "i_! \simeq i_*"] \arrow[r, phantom, "\perp\;\top"] & \mathrm{Mod}_{\mathbb{C}}^{\mathrm{Fields}} \arrow[l, shift left=2, "i^*"] \end{tikzcd}$$

  - a $V \in \mathrm{Mod}_{\mathbb{C}}^{W}$ which (co)induces $\mathcal{H} \simeq i_* V$,
- the *measurement & collapse operation* is is

  the counit $\;i^*\mathcal{H} \simeq i^* i_* V \xrightarrow{\mathrm{ret}^i_V} V\,.$

Example.
Focusing on $\mathrm{Fields} := * /\!\!/ \pi_0 \mathrm{Homeo}(\Sigma^2_{g,n,b})$, then such measurement bases are given by finite index subgroups of $\pi_0 \mathrm{Homeo}(\Sigma^2_{g,n,b})$.

There is a rich theory of these, potentially of direct relevance for realizing topological quantum computing...

More on these quantum-information theoretic aspects next week in [12]

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