Differential cohomology in a cohesive ∞-topos

Talk at Hamburg University

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The work discussed here is available at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

Outline

Classical Chern-Weil theory and its shortcomings

Cohesive homotopy type theory

Applications to Quantum Field Theory

I

Classical Chern-Weil theory and its shortcomings

Classical problem:

Classify (complex) vector bundles in differential geometry.

Vector bundle:

- smooth map of manifolds $E \to X$;
- locally $E|_{U \hookrightarrow X} \simeq U \times V$; for V a vector space;
- glued over $U_1 \cap U_2$ by fiberwise linear maps.

Examples:

- tangent bundle TX of a smooth manifold X;
- line bundle: Hopf fibration $S^3 \otimes_{S^1} \mathbb{C} \to S^2$.

Classification tool:

characteristic cohomology classes

- first Chern class

$$c_1 : \quad \mathrm{VectBund}_{\mathbb{C}}(X)/_{\sim} \longrightarrow H^2(X, \mathbb{Z})$$

$$[E] \longmapsto c_1(E)$$

- second Chern class

$$c_2 : \quad \mathrm{VectBund}_{\mathbb{C}}(X)/_{\sim} \longrightarrow H^4(X, \mathbb{Z})$$

$$[E] \longmapsto c_2(E)$$

- etc.

In differential geometry we want *differential* classes.
First look at c in real cohomology, under

$$\begin{array}{ccc} H^{n+1}(X,\mathbb{Z}) & \longrightarrow & H^{n+1}(X,\mathbb{R}) \\ c(E) & \longmapsto & c(E)_{\mathbb{R}} \end{array},$$

then by the de Rham theorem

$$H^{n+1}(X,\mathbb{R}) \simeq H^{n+1}_{\mathrm{dR}}(X)$$

there c is represented by a closed differential form $F \in \Omega^{n+1}_{\mathrm{cl}}(X)$

$$c(E)_{\mathbb{R}} \sim [F].$$

combine topological and differential information
to differential cohomology $H^{n+1}_{\mathrm{diff}}(X)$

$$
\begin{array}{ccccc}
 & & H^{n+1}_{\mathrm{diff}}(X) & & \\
 & \swarrow & & \searrow^{F} & \\
H^{n+1}(X,\mathbb{Z}) & & & & \Omega^{n+1}_{\mathrm{cl}}(X) \\
 & \searrow & & \swarrow & \\
 & & H^{n+1}(X,\mathbb{R}) \simeq H^{n}_{\mathrm{dR}}(X) & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & \hat{c} & & \\
 & \swarrow & & \searrow & \\
c & & & & F \\
 & \searrow & & \swarrow & \\
 & & c_{\mathbb{R}} = [F] & &
\end{array}
$$

leads to *volume holonomy* over compact n-folds Σ

$$\int_{\Sigma} \; : \quad H^{n+1}_{\mathrm{diff}}(\Sigma) \to U(1)\,.$$

Example:

For $V = \mathbb{C}$: complex line bundles,

- c_1 gives a full classification

$$c_1 : \qquad \mathrm{LineBund}_{\mathbb{C}}(X) \xrightarrow{\simeq} H^2(X, \mathbb{Z})$$

- $H^2_{\mathrm{diff}}(X)$ classifies line bundles with *connection*: *line holonomy*

$$\int_\Sigma \; : \; H^2_{\mathrm{diff}}(\Sigma) \to U(1)$$

Chern-Weil theory:

refine characteristic classes c to differential classes $\hat{c}$ on vector bundles with connection

differential class	$\mathrm{VectBund}_{\mathrm{conn}}(X)/_\sim \xrightarrow{\hat{c}} H^n_{\mathrm{diff}}(X)$
$\downarrow$	$\downarrow \qquad\qquad \downarrow$
class	$\mathrm{VectBund}(X)/_\sim \xrightarrow{c} H^n(X)$

$\hat{c}$ is "secondary characteristic class"

contains information even when $c = 0$

Interpretation in Quantum Field Theory.

- elements in $\mathrm{VectBund}_{\mathrm{conn}}(X)$ are *gauge fields* (electromagnetism, nuclear forces).
- differential class is action functional

$$\exp(iS_c(-)) : \mathrm{VectBund}_{\mathrm{conn}}(\Sigma) \xrightarrow{\hat{c}} H^n_{\mathrm{diff}}(\Sigma) \xrightarrow{\int_\Sigma} U(1)$$

Gives Chern-Simons type QFTs:

- $\int_\Sigma \hat{c}_1 \ : \ A \mapsto \exp(i \int_\Sigma \mathrm{tr} A)$;
- $\int_\Sigma \hat{c}_2 \ : \ A \mapsto \exp(i \int_\Sigma \mathrm{tr}(A \wedge dA) + \frac{2}{3} A \wedge A \wedge A)$
- etc.

Shortcomings

of classical Chern-Weil theory:

1. it is not local – classes instead of cocycles:
   no good obstruction theory;
   no information about locality in QFT;
2. it fails for higher topological structure – higher gauge fields:
   appearing in string theory / supergravity;
3. it is restricted to ordinary geometry:
   sees no supergeometry, infinitesimal geometry, derived differential geometry, etc.

We now discuss these problems
and their solution
in...

II

Cohesive homotopy type theory

1. Locality

Each bundle is on some $U \hookrightarrow X$ equivalent to the trivial bundle $* := U \times V$, therefore

$$\begin{array}{ccc} \mathrm{VectBund}(X)/_\sim & \xrightarrow{\quad c \quad} & H^n(X,\mathbb{Z}) \\ \downarrow{\scriptstyle (-)|_U} & & \downarrow{\scriptstyle (-)|_U} \\ * & \xrightarrow{\text{no local information}} & * \end{array} .$$

But $*$ has nontrivial automorphisms

$$* \xrightarrow{g} * \qquad g \in C^\infty(U, G := \mathrm{Aut}(V))$$

not seen by classical Chern-Weil theory.

These arrange into a *presheaf of groupoids*

$$\mathbf{B}G : U \mapsto \left\{ * \xrightarrow{g \in C^\infty(U,G)} * \right\}$$

Let therefore

$$\mathbf{H} = L_W \operatorname{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{Grpd}),$$

be the 2-category of groupoid presheaves with

$$W = \{\text{stalkwise equivalences}\}$$

formally inverted.

Called the *2-topos of stacks* on smooth manifolds.

Example.

- $\mathbf{H}(U, \mathbf{B}G) = \left\{ * \xrightarrow{g \in C^\infty(U,G)} * \right\}$
- $\pi_0 \mathbf{H}(X, \mathbf{B}G) = \mathrm{VectBund}(X)/_\sim$

2. Higher structure

The nerve

$$N : \mathrm{Grpd} \to \mathrm{sSet}$$

embeds groupoids into simplicial sets.

Simplicial sets model homotopy theory:

- have homotopy groups π_k;
- notion of weak equivalence $f : X \to Y$: isos $\pi_k(X, x) \to \pi_k(Y, f(x))$ on all homotopy groups

Groupoids have only π_0 and π_1 – *homotopy 1-types*.
General simplicial sets: *homotopy types*.

Refine therefore stacks to

$$\mathbf{H} = L_W \, \mathrm{Func}(\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{sSet}) \,,$$

the simplicial category of simplicial presheaves with

$$W = \{\text{stalkwise weak homotopy equivalences}\}$$

formally inverted.

Called the ∞-*topos of* ∞-*stacks* on smooth manifolds.

Example. There is $\mathbf{B}^n U(1) \in \mathbf{H}$ such that

$$\pi_0 \mathbf{H}(X, \mathbf{B}^n U(1)) \simeq H^{n+1}(X, \mathbb{Z}) \,.$$

3. Various geometries

Can build $\mathbf{H}$ over

- smooth manifolds;
- supermanifolds;
- formal (synthetic) manifolds;

In all these cases $\mathbf{H}$ describes

Homotopy types
with differential geometric structure.

Theorem.

These **H** satisfy a simple set of axioms for

"cohesive homotopy types"

(proposed by Lawvere for 0-types).

These axioms imply inside **H** intrinsic

- differential cohomology;
- higher Chern-Weil theory;
- higher Chern-Simons functionals;
- higher geometric prequantization.

This reproduces the traditional notions where they apply, and generalizes them...

III

Applications to Quantum Field Theory

Theorem.

(with Fiorenza, Sati, Stasheff)

- The first Pontryagin class p_1 has a unique smooth refinement

$$\mathbf{B}\mathrm{String} \to \mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3 U(1)$$

  classifying the smooth 3-connected cover $\mathrm{String} \to \mathrm{Spin}$ (outside classical Chern-Weil).

- The second Pontryagin class p_2 has a smooth refinement

$$\mathbf{B}\mathrm{Fivebrane} \to \mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7 U(1)$$

  classifying the smooth 7-connected cover $\mathrm{Fivebrane} \to \mathrm{String}$.

Theorems.

(with Fiorenza, Sati, Stasheff)

- The local Chern-Weil theory of $\mathbf{p}_1$ controls anomaly cancellation of 10d heterotic supergravity.
- The 7-dimensional String-Chern-Simons functor induced by $\mathbf{p_2}$

  $$\exp(iS_{\mathbf{p}_2}) : \mathrm{String2Connections}(\Sigma) \to U(1)$$

  appears in 11-dimensional supergravity after anomaly cancellation.

Analogous statements hold for a wide variety of differential characteristic classes of cohesive homotopy types.

More details and more applications in

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

End.