HIGHER-DIMENSIONAL ANYONS
VIA HIGHER COHOMOTOPY

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Abstract. We highlight that integer Heisenberg groups at level 2 underlie topological quantum phenomena: their group algebras coincide with the algebras of quantum observables of abelian anyons in fractional quantum Hall (FQH) systems on closed surfaces. Decades ago, these groups were shown to arise as the fundamental groups of the space of maps from the surface to the 2-sphere — which has recently been understood as reflecting an effective FQH flux quantization in 2-Cohomotopy. Here we streamline and generalize this theorem using the homotopy theory of H-groups: We show that for $k \in \{1, 2, 4\}$, the non-torsion part of $\pi_1 \, \mathrm{Map}\big((S^{2k-1})^2, S^{2k}\big)$ is an integer Heisenberg group of level 2, where we identify this level with 2 divided by the Hopf invariant of the generator of $\pi_{4k-1}(S^{2k})$. In particular, this result implies the existence of higher-dimensional analogs of FQH anyons in the cohomotopical completion of 11D supergravity ("Hypothesis H").

Contents

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1. Introduction

Here we prove a curious result in elementary homotopy theory (on the latter cf. [Whi78]) with a striking relation to contemporary questions in quantum materials research, specifically in fractional quantum Hall systems (on the latter cf. [Sto99; Ton16; nLa26d]) relevant for questions in topological quantum computing (for which cf. [Fre+03; Nay+08; Sta20; SV25a]).

1.1. **Background.**

1.1.1. *The original theorem.* Back in 1974, Hansen [Han74] investigated the fundamental groups of the space of maps $\mathrm{Map}(-,-)$ (see (49)) from the torus $T^2 = (S^1)^2$ to the 2-sphere S^2. He found them — in the connected component $\mathrm{Map}_n(-,-)$ of winding number $n \in \mathbb{Z}$ — to be central extensions of $\mathbb{Z}^2$ by the cyclic group $\mathbb{Z}_{/2n}$:

$$1 \to \mathbb{Z}_{/2n} \longrightarrow \pi_1 \, \mathrm{Map}_n\big(T^2, S^2\big) \overset{\longleftarrow}{\longrightarrow} \mathbb{Z}^2 \to 1 \,. \tag{1}$$

Since such central extensions are classified by the cohomology group

$$H^2_{\mathrm{grp}}(\mathbb{Z}^2; \mathbb{Z}_{/n}) \simeq H^2(T^2; \mathbb{Z}_{/n}) \simeq \mathbb{Z}_{/n} \,, \tag{2}$$

Hansen's result determined these fundamental groups up to a *level* $\ell_n \in \mathbb{Z}_{/2n}$. The groups arising this way (often considered only for unit level $\ell = 1$) are also known as *integer Heisenberg groups* (cf. [nLa26b] and § 2.1 below).

In 1980, this problem was picked up by Larmore & Thomas, who could show [LT80, Thm. 1] that the level in (1) is in fact equal to $\ell = 2$, in all components. In 2001, another proof of this fact was given by one of the authors [Kal01, Prop. 1.5, Cor. 6.14]. [1]

1.1.2. *A hint of quantum physics.* We may make the following observations about this result:

While the abstract group theory literature typically considers the integer Heisenberg groups at level $\ell = 1$, the integer Heisenberg groups are subgroups of *actual* Heisenberg groups $\mathrm{Heis}_3(\mathbb{R}, h)$ — the hallmark structures of quantum mechanics (cf. [Ros04, p. 7]) at *Planck constant* $h \in \mathbb{R}$ (cf. Def. 2.1 below) to which Heisenberg's name is referring here — exactly at this level $\ell = 2$. Here the cocycle that classifies the central extension is the restriction to the integers of the canonical *symplectic form* ω on $\mathbb{R}^2$:

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\quad\omega\quad} & \mathbb{R} \\ \big((q,p),(q',p')\big) & \longmapsto & qp' - pq'. \end{array} \tag{3}$$

It is the two summands on the right of (3) that, when restricting ω to a $\mathbb{Z}_{/n}$-valued group 2-cocycle on $\mathbb{Z}^2$, cause its class to be twice that of the generating class in (2).

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[1] These authors considered more generally the mapping space out of any closed oriented surface of genus g (Ex. 4.2). But this turns out to be a fairly straightforward generalization of the toroidal case, $g = 1$, which we recover as Ex. 4.6 in § 4 (where the statement is generalized further to higher dimensions). Focus on the case of the torus ($g = 1$) is also motivated by the physics application (further discussed in § 5.1) where the experimental realization of FQH anyons becomes unfeasible for higher g, and where FQAH systems necessarily have $g = 1$ (the *Brillouin torus* of crystal momenta).

This quantum-mechanically natural level $\ell = 2$ for integer Heisenberg groups is the case we consider here by default, whence the result of [LT80; Kal01] reads in our notation (cf. Def. 2.2 and Thm. 3.5 below):

$$\pi_1 \, \mathrm{Map}_n(T^2,\, S^2) \;\simeq\; \mathrm{Heis}_3(\mathbb{Z}, 2n) \subset \mathrm{Heis}_3(\mathbb{R}, 2n). \tag{4}$$

In § 3 we give a new proof of this result (4), shorter and more transparent than the previous arguments (by Thm. 3.3 below, invoking the classical theory of Samelson brackets, cf. § 2.2), and generalizing the statement to higher dimensions.

1.1.3. *Relation to FQH Anyons.* It is worth further expanding on this striking appearance of quantum mechanical structures in algebraic topology and homotopy theory (cf. [SS25d; SS25f]):

We recall that the group C^*-algebra of the ordinary Heisenberg group $\mathrm{Heis}_3(\mathbb{R}, h)$ is essentially the *Weyl algebra of quantum observables* on a 1-dimensional quantum system (cf. [Der06, (3)]) — such as the famous but idealized "particle on the line" (or more realistically: a Josephson junction between superconductors in the *Transmon regime*, where q is the practically continuous phase difference of superconducting order parameters across the junction).

Similarly, the group algebra of the level $\ell = 2$ integer Heisenberg group $\mathrm{Heis}_3(\mathbb{Z}, h)$ is essentially the algebra of observables on a quantum system whose *canonical coordinates* q and *canonical momenta* p are constrained to discrete values.

This is notably the case for quantum observables on *anyons* in *fractional quantum Hall systems* (FQH) on a torus (cf. [WN90, (4.9); Ien92, (4.14); Ton16, (5.28)]). These FQH systems are 2-dimensional electron gases in a transverse magnetic field that is sufficiently strong and yet so fine-tuned that there is an exact integer (generally: rational) multiple of magnetic flux quanta per electron (the inverse *filling fraction*). On such a backdrop, the FQH anyons are elementary vortices in the electron gas associated with surplus magnetic flux quanta on top of the exact filling fraction. Here it is the *flux quantization* of magnetic flux (now in the sense of *discretization*, cf. [SS25e]) that makes the relevant Heisenberg group discrete.

What the discrete Heisenberg group here expresses is an intrinsic quantum mechanical *braiding phase* ζ picked up by the quantum state $|\psi\rangle$ of the system as the anyons are moved around each other (cf. Fig. 1).

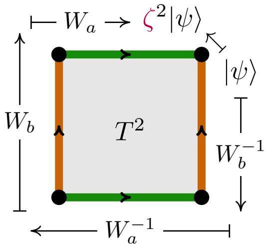

FIGURE 1. Illustrating the non-trivial group commutator $[W_a, W_b] = \zeta^2$ (11) in the integer Heisenberg group $\mathrm{Heis}_3(\mathbb{Z}, h)$ at level $\ell = 2$ (Def. 2.2). With the group elements understood as observables on FQH quantum systems (cf. § 5.1), the central element ζ is represented on their quantum states $|\psi\rangle$ by multiplication with a complex root of unity known as an *anyon braiding phase* (cf. [Ton16, (5.28); SS26b, Fig. 4]).

Remarkably, the FQH anyon braiding phase ζ has been experimentally observed (for the moment not in toroidal but in planar electron gases, though) in recent years by various groups (starting with [Nak+20; Bar+20], for further developments see [Vei+24; Gho+25]). This is of considerable technological interest, since it is the first and currently only observed case of fundamental anyonic *topological quantum* phenomena (cf. [Sta20; Sim23]). These are plausibly necessary (cf. [Das22]) for the future construction of *quantum computers* (cf. [BG25]) of useful scale, namely for the engineering of *topological quantum gates* (cf. [Fre+03; MSS24, §3; SV25a]) which would be *intrinsically* protected against the decohering noise that jeopardizes all quantum circuits.

However, even from a theoretical standpoint, there have remained open questions about the nature of FQH anyons (cf. [SS25f, §A.1]). The remarkable relation (4) to homotopy theory may help shed some light on these:

1.1.4. *Flux Quantization in Cohomotopy.* The above situation § 1.1.3 points to a curious physical interpretation [SS25f] of the result (4):

If we understand $S^2 \hookrightarrow B\mathrm{U}(1)$ as a deformation (namely the 3-skeleton) of the usual classifying space $B\mathrm{U}(1) \simeq B^2\mathbb{Z}$ for magnetic charge, then (4) says that anyons in FQH systems may be understood as surplus magnetic flux quanta whose interaction with the 2D electron gas effectively deforms their *flux quantization law* [SS25e, §3] from ordinary cohomology $H^n(-;\mathbb{Z})$ to *Cohomotopy* $\pi^n(-)$ (cf. [Hu59, §VII; FSS23, Ex. 2.7]) in degree $n = 2$:

$$\begin{aligned} H^n(-;\mathbb{Z}) &\simeq \pi_0 \,\mathrm{Map}(-, B^n\mathbb{Z}) \\ \pi^n(-) &\equiv \pi_0 \,\mathrm{Map}(-, S^n)\,. \end{aligned}$$

This suggests that FQH anyons may generally be understood as surplus magnetic flux quantized in 2-Cohomotopy (“Hypothesis h”, [SS26a; SS26c; SS25f]) which provides a new algebro-topological theory for the effective behavior of FQH anyons. For example, it predicts [SS25f, Fig. D & §3.8] that superconducting islands in FQH materials may support *non-abelian* defect anyons, the realization of which is the holy grail of research on topological quantum materials.

In fact, flux quantization in Cohomotopy was first recognized as a physical possibility in higher-dimensional quantum field theory, specifically in 11-dimensional supergravity, whose “C-field” flux admits quantization in 4-Cohomotopy (“Hypothesis H”, [Sat18, §2.5; FSS20; FSS21; SS23b]). We expand on these matters at the end in § 5.

1.2. **Overview and Results.** This motivates us to ask whether an analogue of (4) may hold in higher dimensions. After briefly establishing the relevant context in § 2, our main result in § 3 is to show (Thm. 3.5) that, indeed:

$$\pi_1 \,\mathrm{Map}_{[n]}\big((S^3)^2,\, S^4\big) \simeq \mathrm{Heis}_3(\mathbb{Z}, 0) \times \mathbb{Z}_{/12}\,, \tag{5}$$

as well as

$$\pi_1 \,\mathrm{Map}_{[n]}\big((S^7)^2,\, S^8\big) \simeq \mathrm{Heis}_3(\mathbb{Z}, 0) \times \mathbb{Z}_{/120} \tag{6}$$

(where in both cases the components are labeled by $[n] \in \mathbb{Z}_{/2}$).

Remarkably, this result therefore implies the rigorous mathematical possibility of higher dimensional FQH-like anyon braiding phases in higher dimensions. We explain in § 5.2 how this is indeed realized in the (physically hypothetical but mathematically well-defined) completion of *11D supergravity* by cohomotopical flux-quantization, where we find 2-dimensional ("2-brane") anyons in 6D, [2] descending from 5-dimensional ("5-brane") anyons in 11D.

Our novel move in proving these higher-dimensional generalizations is the intermediate Thm. 3.3, bringing the classical theory of Samelson products on H-groups to bear on the problem (cf. Prop. 2.8), which also serves to make the proof of the original 2-dimensional result much more transparent.

The proof strategy immediately generalizes to other situations, discussed in § 4.

We close in § 5 by expanding on the potential implications of this result on contemporary questions in the physics and engineering of anyonic quantum materials.

2. Preliminaries

Before we come to the main theorem in § 3, here we briefly set up the context.

2.1. Group theory.

The ordinary *Heisenberg group* owes its name to its role as a cornerstone of basic quantum mechanics (cf. [Ros04, p. 7]), where it reflects the fact that the elementary *phase space* $\mathbb{R}^2$ with its canonical symplectic form (3) becomes "noncommutative":

Definition 2.1 (Ordinary Heisenberg group)**.** For $h \in \mathbb{R}$ ("Planck's constant"), the underlying smooth manifold is

$$\mathrm{Heis}_3(\mathbb{R}, h) \;:=_{\mathrm{SmthMfd}}\; \mathbb{R}^2 \times \mathbb{R}_{/h}\,, \tag{7}$$

with generating elements to be denoted

$$\left.\begin{aligned} W_a^q &:= \big((q,0),[0]\big) \\ W_b^p &:= \big((0,p),[0]\big) \\ \zeta^z &:= \big((0,0),[z]\big) \end{aligned}\right\} \in \mathbb{R}^2 \times \mathbb{R}_{/h}\,,$$

on which the only nontrivial group commutators are

$$[W_a^q \,,\, W_b^p] \;=\; \zeta^{2qp}.$$

Similarly, there are the *higher-dimensional Heisenberg groups*, for $g \in \mathbb{N}_{\geq 1}$,

$$\mathrm{Heis}_{2g+1}(\mathbb{R}, h) \;:=_{\mathrm{SmthMfd}}\; \mathbb{R}^{2g} \times \mathbb{R}_{/h}\,, \tag{8}$$

with generators $\big(W_{a_i}^q, W_{b_i}^p\big)_{i=1}^g$ and ζ^z on which the only nontrivial group commutators are

$$\Big[W_{a_i}^q, W_{b_i}^p\Big] \;=\; \zeta^{2qp}\,, \qquad \text{for } 1 \,\leq\, i \,\leq\, g.$$

Definition 2.2 (Integer Heisenberg group at level $\ell = 2$)**.** For $h \in \mathbb{Z}$, the underlying set is

$$\mathrm{Heis}_3(\mathbb{Z}, h) \;:=_{\mathrm{Set}}\; \mathbb{Z}^2 \times \mathbb{Z}_{/h}\,, \tag{9}$$

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[2] Previous suggestions for anyonic effects in the higher dimensional braiding of 2-branes include [Har07] (which argues also in the context of 11D supergravity) and [Fen+25] (which presents a kind of lattice model for anyonic membranes in 4D).

with generating elements

$$\left.\begin{aligned} W_a &:= \big((1,0),[0]\big) \\ W_b &:= \big((0,1),[0]\big) \\ \zeta &:= \big((0,0),[1]\big) \end{aligned}\right\} \in \mathbb{Z}^2 \times \mathbb{Z}_{/h}\,, \tag{10}$$

on which the only non-trivial group commutator is

$$[W_a, W_b] \,=\, \zeta^2. \tag{11}$$

Similarly, there are the *higher-dimensional integer Heisenberg groups*, for $g \in \mathbb{N}_{\geq 1}$,

$$\mathrm{Heis}_{2g+1}(\mathbb{Z}, h) \;:=_{\mathrm{SmthMfd}}\; \mathbb{Z}^{2g} \times \mathbb{Z}_{/h}\,, \tag{12}$$

with generators $\big(W_{a_i}, W_{b_i}\big)_{i=1}^{g}$ and ζ on which the only nontrivial group commutators are

$$\big[W_{a_i}, W_{b_i}\big] \,=\, \zeta^2\,, \qquad \text{for } 1 \,\leq\, i \,\leq\, g.$$

Remark 2.3 (Other levels). Over the integers, non-isomorphic groups are obtained by instead taking the above group commutator (11) to be

$$[W_a, W_b] \,=\, \zeta^n,$$

for some

$$n \,\in\, \mathbb{Z} \,\simeq\, H^2_{\mathrm{Grp}}(\mathbb{Z}^2; \mathbb{Z})\,,$$

which we call the *level*, since it is the central extension class. Authors in group theory are usually concerned with the case $n = 1$ (cf. [nLa26b]); but for our purpose, the case $n = 2$ is of paramount importance, so we take this to be understood by default (11).

2.2. H-Group theory.

We need the following concepts and results from the theory of homotopy groups (for background see § B) which are classical but not always readily citable (such as the crucial Prop. 2.8 below).

Definition 2.4 (cf. [Whi78, §X.5]). The *Samelson product* $[-,-]_{\mathrm{Sam}}$ on a loop space ΩX (Ex. B.4, and generally on an H-group, Def. B.3, and from there induced on its homotopy groups) is (up to homotopy) the *H-group commutator* descended from the Cartesian product to the smash product:

$$\begin{tikzcd} (\ell_2, \ell_1) \arrow[r, mapsto] & (\ell_2 \star \ell_1) \star (\overline{\ell_2} \star \overline{\ell_1}) \\ (\Omega X) \times (\Omega X) \arrow[r, "{[-,-]}"] \arrow[d] & \Omega X\,. \\ (\Omega X) \wedge (\Omega X) \arrow[ur, "{[-,-]_{\mathrm{Sam}}}"'] & \end{tikzcd} \tag{13}$$

Definition 2.5 (cf. [FHT00, p. 176-7]). For $X \in \mathrm{TopSp}^*$, the *Whitehead* product on its homotopy group (in degrees $n_1, n_2 \in \mathbb{N}_{\geq 1}$),

$$\pi_{n_1}(X) \otimes_{\mathbb{Z}} \pi_{n_2}(X) \xrightarrow{[-,-]_{\mathrm{Wh}}} \pi_{n_1+n_2-1}(X)\,,$$

is given on a pair of representatives

$$[\phi_i \,:\, S^{n_i} \longrightarrow X] \,\in\, \pi_{n_i}(X)\,, \quad i \,\in\, \{1,2\}$$

by the homotopy class of the top composite map of the following diagram:

$$\begin{tikzcd}
S^{n_1+n_2-1} \arrow[r, "f_{n_1,n_2}"] \arrow[d] \arrow[rr, bend left, "{[\phi_1,\phi_2]_{\mathrm{Wh}}}"] & S^{n_1} \vee S^{n_2} \arrow[r, "{(\phi_1,\phi_2)}"] \arrow[d] & X \\
D^{n_1+n_2} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^{n_1} \times S^{n_2} \,, &
\end{tikzcd}$$

where f_{n_1,n_2} is the attaching map for the cell attachment to the wedge sum of spheres that gives the product of spheres, as shown.

Proposition 2.6 (cf. [Whi78, Thm. 7.10 on p. 476])**.** *For* $X \in \mathrm{TopSp}^*$, *the* Whitehead bracket $[-,-]_{\mathrm{Wh}}$ *on* $\pi_{\bullet+1}(X)$ (Def. 2.5) *is given by the Samelson product on* $\pi_\bullet(\Omega X)$ (Def. 2.4) *as:*

$$\widetilde{[\alpha, \alpha_2]}_{\mathrm{Wh}} \;=\; (-1)^{\deg(\alpha_1)} [\widetilde{\alpha}_1,\, \widetilde{\alpha}_2\,]_{\mathrm{Sam}}\,, \tag{14}$$

where $\pi_{\bullet+1}(X) \underset{\sim}{\overset{\widetilde{(-)}}{\longleftrightarrow}} \pi_\bullet(\Omega X)$ *is induced from the hom-isomorphism* (57)*:*

$$\begin{tikzcd}
\pi_0\,\mathrm{Map}^*(S^{n_1},\, \Omega X) \times \pi_0\,\mathrm{Map}^*(S^{n_2},\, \Omega X) \arrow[r, "{[-,-]_{\mathrm{Sam}}}"] \arrow[d, leftrightarrow, "\widetilde{(-)}\times\widetilde{(-)}"] & \pi_0\,\mathrm{Map}^*\big(S^{n_1+n_2},\, \Omega X\big) \arrow[d, leftrightarrow, "\widetilde{(-)}"] \\
\pi_0\,\mathrm{Map}^*(S^{n_1+1},\, X) \times \pi_0\,\mathrm{Map}^*(S^{n_2+1},\, X) \arrow[r, "{\pm[-,-]_{\mathrm{Wh}}}"] & \pi_0\,\mathrm{Map}^*(S^{n_1+n_2+1},\, X).
\end{tikzcd}$$

Lemma 2.7 (cf. [Whi78, Thm. 8.20 on p. 485][3])**.** *The suspension* (54) *of any Whitehead bracket* (14) *vanishes:*

$$\begin{array}{ccc}
\pi_n(S^m) & \xrightarrow{\;\Sigma\;} & \pi_{n+1}(S^{m+1}) \\
{[\alpha,\beta]_{\mathrm{Wh}}} & \longmapsto & 0\,.
\end{array} \tag{15}$$

Proposition 2.8. *For* $k \in \mathbb{N}$, *the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
& (\Omega S^{2k}) \wedge (\Omega S^{2k}) \arrow[dr, "{[-,-]_{\mathrm{Sam}}}"] & \\
S^{4k-2} \arrow[ur, "\widetilde{\mathrm{id}_{2k}} \wedge \widetilde{\mathrm{id}_{2k}}"] \arrow[rr, "2\,\widetilde{h_{\mathbb{K}}}"'] & & \Omega S^{2k}.
\end{tikzcd} \tag{16}$$

Proof. The Whitehead bracket of the generator $[\mathrm{id}_{2k}] := 1 \in \mathbb{Z} \simeq \pi_n(S^{2k})$ with itself has Hopf invariant 2 (cf. [Whi78, Thm. 2.5 on p. 495]):

$$H\big([\mathrm{id}_{2k}, \mathrm{id}_{2k}]_{\mathrm{Wh}}\big) \;=\; 2\,.$$

But for $k \in \{1, 2, 4\}$ the only element in $\pi^{4k-1}(S^{2k})$ of Hopf invariant 2 is twice the class of the corresponding Hopf fibration $h_{\mathbb{K}}$. (That $H(h_{\mathbb{K}}) = 1$ is due to [Hop35]; while torsion elements have vanishing Hopf invariant, due to the homomorphy of $H(-)$, cf. [Whi78, (2.1) on p. 495].) Hence:

$$\begin{aligned}
\big[[\mathrm{id}_2], [\mathrm{id}_2]\big]_{\mathrm{Wh}} &= 2[h_{\mathbb{C}}] \\
\big[[\mathrm{id}_4], [\mathrm{id}_4]\big]_{\mathrm{Wh}} &= 2[h_{\mathbb{H}}] \\
\big[[\mathrm{id}_8], [\mathrm{id}_8]\big]_{\mathrm{Wh}} &= 2[h_{\mathbb{O}}]\,.
\end{aligned} \tag{17}$$

The combination of (14) and (17) proves the claim. ☐

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[3] Beware that [Whi78] writes "E" for the suspension functor, a notation introduced on p. 369 there.

Remark 2.9. We will see that the factor of 2 appearing in (17) is the origin both of the level of the Heisenberg group being 2 (this is the content of our new proof Thm. 3.3), as well as its "Planck constant" being $2n$ when $k = 1$ (by Lem. 3.8 below, which was observed before).

3. The Theorem

We compute here the fundamental groups of the mapping spaces from $(S^{2k-1})^2$ to S^{2k}, for $k \in \{1, 2, 4\}$, using the H-group theory of § 2.2 and find them to essentially be the integer Heisenberg groups of § 2.1.

3.1. **The level two.**

This section contains our main new observation (proof of Thm. 3.3 below). First, we make these basic observations:

Remark 3.1 (Cell structure of the squared sphere).

(i) The canonical CW-complex structure of $(S^{2k-1})^2$ has, besides a 0-cell, two $(2k-1)$-cells and a single $(4k-2)$-cell, attached via the Whitehead product of their identity maps (by Def. 2.5), as shown by the following pushout square on the left:

$$\begin{tikzcd}
S^{4k-3} \arrow[r, "{[\mathrm{id}_{2k-1}, \mathrm{id}_{2k-1}]_{\mathrm{Wh}}}"] \arrow[d, hook] & S^{2k-1} \vee S^{2k-1} \arrow[r] \arrow[d, hook] & * \arrow[d] \\
D^{4k-2} \arrow[r, "i_{4k-2}"] & S^{2k-1} \times S^{2k-1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\mathrm{pr}_c"] \arrow[d, shift right, "\mathrm{pr}_a"'] \arrow[d, shift left, "\mathrm{pr}_b"] & S^{4k-2} \arrow[ul, phantom, "\ulcorner", very near start] \\
 & S^{2k-1} . &
\end{tikzcd} \tag{18}$$

The pushout square on the right forms the smash product of the two $(2k-1)$-cells and thereby exhibits a canonical map denoted pr_c. The two projections onto the $(2k-1)$-sphere factors are shown at the bottom of the above diagram, which we denote by pr_a and pr_b, respectively.

(ii) Note that the following diagram commutes:

$$\begin{tikzcd}
 & (S^{2k-1})^2 \wedge (S^{2k-1})^2 \arrow[dr, "\mathrm{pr}_a \wedge \mathrm{pr}_b"] & \\
(S^{2k-1})^2 \arrow[ur, "\Delta"] \arrow[rr, "\mathrm{pr}_c"] & & S^{4k-2} .
\end{tikzcd} \tag{19}$$

(This follows by inspection; alternatively, it is a special case of Lem. 4.3 below).

(iii) The suspension of these projection maps yields the three coprojections out of the *stable splitting* (using Lem. 2.7 or [Hat02, Prop. 4I.1]) of the squared sphere:

$$\begin{tikzcd}
\Sigma(S^{2k-1})^2 \arrow[r, phantom, "\underset{\mathrm{hmtp}}{\simeq}"] & S^{2k}_a \arrow[d, "\Sigma\mathrm{pr}_a"] \arrow[r, phantom, "\vee"] & S^{2k}_b \arrow[d, "\Sigma\mathrm{pr}_b"] \arrow[r, phantom, "\vee"] & S^{4k-1} \arrow[d, "\Sigma\mathrm{pr}_c"] \\
 & S^{2k}_a & S^{2k}_b & S^{4k-1} .
\end{tikzcd} \tag{20}$$

Notice that this homotopy equivalence does *not* respect the H-cogroup structure on suspensions (from Ex. B.4).

Proposition 3.2. *The connected components of the pointed mapping space are:*

$$\pi_0 \operatorname{Map}^*\big((S^{2k-1})^2,\, S^{2k}\big) \;\simeq\; \begin{cases} \mathbb{Z} & \text{if } k \,=\, 1 \\ \mathbb{Z}_{/2} & \text{if } k \,=\, 2 \\ \mathbb{Z}_{/2} & \text{if } k \,=\, 4\,. \end{cases} \tag{21}$$

Proof. Generally, maps to a $(2k-1)$-connected space like S^{2k} factor through the quotient of the domain by its $(2k-1)$-skeleton. But, by the CW-structure (18), we have:

$$(S^{2k-1})^2 \big/ \mathrm{sk}_{2k-1}(S^{2k-1})^2 \;\simeq\; (S^{2k-1})^2 \big/ (S^{2k-1} \vee S^{2k-1}) \;\simeq\; S^{4k-2}\,,$$

whence:

$$\pi_0 \operatorname{Map}^*\big((S^{2k-1})^2,\, S^{2k}\big) \;\simeq\; \pi_0 \operatorname{Map}^*\big(S^{4k-2},\, S^{2k}\big) \;\equiv\; \pi_{4k-2}(S^{2k})\,.$$

With the standard homotopy group of spheres (62), this proves the claim. □

Our central observation now is the proof of the following:

Theorem 3.3. *For $k \,\in\, \{1,2,4\}$, we have a group isomorphism*

$$\pi_1 \operatorname{Map}^*_0\big((S^{2k-1})^2,\, S^{2k}\big) \;\simeq\; \mathrm{Heis}_3(\mathbb{Z},0) \times \begin{cases} 1 & \text{if } k \,=\, 1 \\ \mathbb{Z}_{/12} & \text{if } k \,=\, 2 \\ \mathbb{Z}_{/120} & \text{if } k \,=\, 4\,. \end{cases} \tag{22}$$

Proof. First, we observe the bijection (22) on underlying sets:

$$\begin{aligned} \pi_1 \operatorname{Map}^*_0\big((S^{2k-1})^2,\, S^{2k}\big) &\simeq \pi_0 \operatorname{Map}^*\big(\Sigma(S^{2k-1})^2,\, S^{2k}\big) && \text{by (67)} \\ &\simeq \pi_0 \operatorname{Map}^*\big(S^{2k} \vee S^{2k} \vee S^{4k-1},\, S^{2k}\big) && \text{by (20)} \\ &\simeq \pi_{2k}(S^{2k})^2 \times \pi_{4k-1}(S^{2k}) && \text{by (59) and (60)} \\ &\simeq \mathbb{Z}^2 \times \mathbb{Z} \times \begin{cases} 1 & \text{if } k \,=\, 1 \\ \mathbb{Z}_{/12} & \text{if } k \,=\, 2 \\ \mathbb{Z}_{/120} & \text{if } k \,=\, 4 \end{cases} && \text{by (62)}\,. \end{aligned}$$

(This is a sequence of bijections of sets, but not of homomorphisms of groups, since the stable splitting used in the second step is not an H-cogroup homomorphism.) Therefore, the candidate Heisenberg group generators (10) are represented by the projectors (20) and the homotopy group generators (62) as:

$$\begin{aligned} W_{a/b} &:= \big[\Sigma(S^{2k-1})^2 \xrightarrow{\Sigma\,\mathrm{pr}_{a/b}} S^{2k} \xrightarrow{\;\mathrm{id}\;} S^{2k}\big] \\ \zeta &:= \big[\Sigma(S^{2k-1})^2 \xrightarrow{\Sigma\,\mathrm{pr}_{c}} S^{4k-1} \xrightarrow{\;h_{\mathbb{K}}\;} S^{2k}\big]\,, \end{aligned} \tag{23}$$

and adjointly as:

$$\begin{aligned} \widetilde{W}_{a/b} &= \big[(S^{2k-1})^2 \xrightarrow{\mathrm{pr}_{a/b}} S^{2k-1} \xrightarrow{\;\widetilde{\mathrm{id}}\;} \Omega S^{2k}\big] \\ \widetilde{\zeta} &= \big[(S^{2k-1})^2 \xrightarrow{\mathrm{pr}_{c}} S^{4k-2} \xrightarrow{\;\widetilde{h_{\mathbb{K}}}\;} \Omega S^{2k}\big]\,, \end{aligned} \tag{24}$$

while the generator of the further torsion group factor is represented as:

$$R \;:=\; \big[\Sigma(S^{2k-1})^2 \xrightarrow{\Sigma\,\mathrm{pr}_{c}} S^{4k-1} \xrightarrow{\;r_{\mathbb{K}}\;} S^{2k}\big]\,.$$

Then, by Rem. B.6 and Def. 2.4, the group commutator $[W_a, W_b]$ is represented by the outer part of the following diagram:

$$\begin{tikzcd}
(S^{2k-1})^2 \arrow[rr, "\widetilde{[W_a, W_b]}"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{pr}_c"] & & \Omega S^{2k} \\
(S^{2k-1})^2 \wedge (S^{2k-1})^2 \arrow[r, "\mathrm{pr}_a \wedge \mathrm{pr}_b"'] \arrow[rr, bend right, "\widetilde{W}_a \wedge \widetilde{W}_b"'] & S^{2k-1} \wedge S^{2k-1} \arrow[r, "\widetilde{\mathrm{id}_{2k}} \wedge \widetilde{\mathrm{id}_{2k}}"'] \arrow[ur, "2\, \widetilde{h_{\mathbb{K}}}"] & (\Omega S^{2k}) \wedge (\Omega S^{2k})\,. \arrow[u, "{[-,-]_{\mathrm{Sam}}}"']
\end{tikzcd} \tag{25}$$

Inside this diagram, we identify (as shown)

(i) the left diagonal map by (19),
(ii) the right diagonal map by (16).

Hence, the top inner triangle in the diagram commutes and proves $[W_a, W_b] = \zeta^2$.

Similarly, $[W_{a/b}, \zeta]$ is represented by the outer part of the following diagram

$$\begin{tikzcd}
(S^{2k-1})^2 \arrow[rr, "\widetilde{[W_{a/b}, \zeta]}"] \arrow[d, "\Delta"'] \arrow[dr, "0"] & & \Omega S^{2k} \\
(S^{2k-1})^2 \wedge (S^{2k-1})^2 \arrow[r, "\mathrm{pr}_{a/b} \wedge \mathrm{pr}_c"'] \arrow[rr, bend right, "\widetilde{W}_{a/b} \wedge \widetilde{\zeta}"'] & S^{2k-1} \wedge S^{4k-2} \arrow[r, "\widetilde{\mathrm{id}_{2k}} \wedge \widetilde{h_{\mathbb{K}}}"'] & (\Omega S^{2k}) \wedge (\Omega S^{2k})\,. \arrow[u, "{[-,-]_{\mathrm{Sam}}}"']
\end{tikzcd} \tag{26}$$

But here the left diagonal map is null-homotopic, as indicated, already by degree reasons. This shows that $[W_{a/b}, \zeta] = \mathrm{e}$, and hence completes the proof. □

This result generalizes to the other connected components by the following argument adapted from [Han74, Prop. 1]:

Lemma 3.4. *The connected components of* $\mathrm{Map}^*\big((S^{2k-1})^2, S^{2k}\big)$ *are all homotopy equivalent.*

Proof. Consider the H-action of the H-group $\mathrm{Map}^*(S^{4k-2}, S^{2k})$ (Ex. B.5) on the mapping space, which is induced by the pinching map (from Ex. B.7):

$$\mathrm{Map}^*\big(S^{4k-2}, S^{2k}\big) \times \mathrm{Map}^*\big((S^{2k-1})^2,\, S^{2k}\big) \xrightarrow{\Theta} \mathrm{Map}^*\big((S^{2k-1})^2,\, S^{2k}\big). \tag{27}$$

Being an H-group action, each $\Theta(f, -)$ is a homotopy auto-equivalence of the mapping space. But then for $[f]$ a generator of $\pi_{4k-2}(S^{2k})$, Prop. 3.2 shows that $\Theta(f, -)$ restricts to a sequence of homotopy equivalences between all its connected components. □

Finally, we adapt these results to the unpointed mapping space:

Theorem 3.5. *The fundamental groups of the unpointed mapping spaces from* $(S^{2k-1})^2$ *to* S^{2k} *are integer Heisenberg groups at level* $\ell = 2$ (Def. 2.2)*, as follows:*

$$\pi_1\, \mathrm{Map}_n\big((S^1)^2,\, S^2\big) \simeq \mathrm{Heis}_3(\mathbb{Z}, 2n)\,, \tag{28a}$$

$$\pi_1\, \mathrm{Map}_{[n]}\big((S^3)^2,\, S^4\big) \simeq \mathrm{Heis}_3(\mathbb{Z}, 0) \times \mathbb{Z}_{/12}\,, \tag{28b}$$

$$\pi_1\, \mathrm{Map}_{[n]}\big((S^7)^2,\, S^8\big) \simeq \mathrm{Heis}_3(\mathbb{Z}, 0) \times \mathbb{Z}_{/120}\,, \tag{28c}$$

for all $n \in \mathbb{Z}$ *and* $[n] \in \mathbb{Z}_{/2}$*, respectively* (21).

Proof. For $k \geq 2$ the homotopy long exact sequence of the basepoint evaluation fibration truncates:

$$\begin{tikzcd}
\mathrm{Map}^*_{[n]}\big((S^{2k-1})^2,\, S^{2k}\big) \arrow[r, "\mathrm{fib}_*"] & \mathrm{Map}_{[n]}\big((S^{2k-1})^2,\, S^{2k}\big) \arrow[r, "\mathrm{ev}_*"] & S^{2k} \\
 & & \overbrace{\pi_2 S^{2k}}^{1} \arrow[dll, rounded corners] \\
\pi_1 \mathrm{Map}^*_{[k]}\big((S^{2k+1})^2,\, S^{2k}\big) \arrow[r, "\sim"] & \pi_1 \mathrm{Map}_{[n]}\big((S^{2k-1})^2,\, S^{2k}\big) \arrow[r, "\mathrm{ev}_*"] & \underbrace{\pi_1 S^{2k}}_{1} ,
\end{tikzcd} \tag{29}$$

whence the claim for $k \geq 2$ follows immediately by Thm. 3.3 and Lem. 3.4.
In the case $k = 1$, the presence of $\pi_2(S^2) \simeq \mathbb{Z}$ obstructs this direct argument, and our Thm. 3.3 and Lem. 3.4 gives (28a) without determining the *Planck constant* to be $2n$. But this value is fixed by Prop. 3.9 below. □

Remark 3.6. Readers familiar with *rational homotopy theory* (cf. [FHT00; FSS23, §III]) may find it instructive to reproduce the statement of Thm. 3.5 at the level of *minimal Sullivan models*: For a X connected nilpotent space of finite rational type, we denote its minimal Sullivan model (over some rational ground field $\mathbb{K}$) by:

$$\mathrm{CE}(\mathfrak{l}X) \in \mathrm{dgCAlg}_{\mathbb{K}} \tag{30}$$

(being the *Chevalley-Eilenberg algebra* CE(−) of the *rational Whitehead bracket* L_∞*-algebra* $\mathfrak{l}X$ of X, cf. [FSS23, Prop. 5.11 & Rem. 5.4]).
The minimal models of positive even-dimensional spheres are (cf. [Men15, §1.2])[4]

$$\mathrm{CE}\big(\mathfrak{l}S^{2k}\big) \simeq \mathbb{K}_{\mathrm{d}}\begin{bmatrix} f \\ h \end{bmatrix} \Big/ \begin{pmatrix} \mathrm{d}\, f &= 0 \\ \mathrm{d}\, h &= \tfrac{1}{2} f \wedge f \end{pmatrix}, \qquad \begin{aligned} \deg(f) &= 2k \\ \deg(h) &= 4k-1\,, \end{aligned}$$

and the generators of the minimal model for the mapping space out of a squared odd-dimensional sphere are (cf. *toroidification* in [SV25b, p. 9; GSS25, p. 35]) given by:

$$\mathrm{CE}\Big(\mathfrak{l}\mathrm{Map}\big((S^{2k-1})^2, S^{2k}\big)\Big) \simeq \mathbb{K}_{\mathrm{d}} \begin{bmatrix} f \\ h \\ \overset{a}{\mathrm{s}}f \\ \overset{b}{\mathrm{s}}f \\ \overset{a}{\mathrm{s}}h \\ \overset{b}{\mathrm{s}}h \\ \overset{ab}{\mathrm{ss}}h \end{bmatrix} \Bigg/ \begin{pmatrix} \mathrm{d}\, f &= 0 \\ \mathrm{d}\, h &= +\tfrac{1}{2} f \wedge f \\ \mathrm{d}\, \overset{a}{\mathrm{s}}f &= 0 \\ \mathrm{d}\, \overset{b}{\mathrm{s}}f &= 0 \\ \mathrm{d}\, \overset{a}{\mathrm{s}}h &= -f \wedge \overset{a}{\mathrm{s}}f \\ \mathrm{d}\, \overset{b}{\mathrm{s}}h &= -f \wedge \overset{b}{\mathrm{s}}f \\ \mathrm{d}\, \overset{ab}{\mathrm{ss}}h &= +\overset{a}{\mathrm{s}}f \wedge \overset{b}{\mathrm{s}}f \end{pmatrix}, \quad \begin{aligned} &\deg\big(\overset{\bullet}{\mathrm{s}}(\text{-})\big) = \\ &\quad \deg(\text{-}) - 2k + 1. \end{aligned}$$

These co-binary differentials dually exhibit the Whitehead brackets over $\mathbb{K}$ (cf. [AA78, Thm. 6.1; FHT00, Prop. 13.16]). In particular, the last line dually is the Whitehead bracket shown above in (25), now over $\mathbb{K}$. For $\mathbb{K} \equiv \mathbb{R}$, this is the nontrivial Lie bracket of the Lie algebra of the ordinary Heisenberg group (Def. 2.1). Of course, the crucial level $\ell = 2$ of this Whitehead bracket, which we established above, cannot be discerned in rational homotopy theory (where all non-vanishing factors correspond to isomorphic models).

---

[4] Here $\mathbb{K}_{\mathrm{d}}[L]$ denotes the free differential graded-commutative algebra on a list L of graded generators, and the quotient divides out the shown differential ideal.

3.2. **Planck's constant.** For completeness, we recall in streamlined form the argument (following [Han74]) for the order $2n$ of the integer Heisenberg group extension (called *Planck's constant* in § 2.1) in the case $k = 1$, (28a) in Thm. 3.5.

Lemma 3.7. *The following forgetful map* (from the fundamental groups of the pointed to that of the unpointed mapping space) *is an isomorphism:*

$$\underbrace{\pi_1 \mathrm{Map}^*\big(S^1 \vee S^1, S^2\big)}_{\mathbb{Z}^2} \xrightarrow{\;\sim\;} \pi_1 \mathrm{Map}\big(S^1 \vee S^1, S^2\big)\,.$$

Proof. The evaluation fiber sequence has a section (landing in the single connected component of $\mathrm{Map}(S^1 \vee S^1, S^2)$), whence the connecting homomorphism vanishes,

$$\begin{tikzcd}
 & {} \arrow[r, two heads] & \pi_2 S^2 \arrow[dll, rounded corners, "0"] \\
\pi_1 \mathrm{Map}^*\big(S^1 \vee S^1, S^2\big) \arrow[r, "\sim"] & \pi_1 \mathrm{Map}\big(S^1 \vee S^1, S^2\big) \arrow[r] & \underbrace{\pi_1 S^2}_{1}\,,
\end{tikzcd}$$

because the previous map is thus split surjective. ☐

Lemma 3.8 ([Hu46, Thm. 5.3(i); Koh60, Lem. 3.9])**.** *For all $n \in \mathbb{Z}$, we have*

$$\pi_1 \mathrm{Map}_n\big(S^2, S^2\big) \;\simeq\; \mathbb{Z}_{/2n}\,.$$

Proof. The evaluation homotopy fiber sequence yields the following homotopy long exact sequence:

$$\begin{tikzcd}
\mathrm{Map}^*_n\big(S^2, S^2\big) \arrow[r] & \mathrm{Map}_n\big(S^2, S^2\big) \arrow[r, "\mathrm{ev}_*"] & S^2 \\
 & & \overbrace{\pi_2\big(S^2\big)}^{\mathbb{Z}} \arrow[dll, rounded corners, "{[[n \cdot \mathrm{id}_2], -]_{\mathrm{Wh}}}"'] \\
\underbrace{\pi_1 \mathrm{Map}^*_n\big(S^2, S^2\big)}_{\pi_3(S^2) \simeq \mathbb{Z}} \arrow[r] & \pi_1 \mathrm{Map}_n\big(S^2, S^2\big) \arrow[r] & \underbrace{\pi_1\big(S^1\big)}_{1}\,,
\end{tikzcd}$$

where the connecting homomorphism is given by the Whitehead bracket with $n \cdot \mathrm{id}_2$, by [Whi46, (3.4)*]. But, due to linearity of the Whitehead bracket and by (17), this means that the connecting homomorphism is given by multiplication with $2n$:

$$\begin{aligned}
1 \,=\, [\mathrm{id}_2] \quad \longmapsto \quad \big[[n \cdot \mathrm{id}_2], [\mathrm{id}_2]\big]_{\mathrm{Wh}} &= n \cdot \big[[\mathrm{id}_2], [\mathrm{id}_2]\big]_{\mathrm{Wh}} \\
&= n \cdot [2 h_{\mathbb{C}}] \\
&= 2n \quad \in \mathbb{Z} \,\simeq\, \pi_3\big(S^2\big)\,.
\end{aligned}$$

From this, the claim follows by exactness of the above sequence. ☐

Proposition 3.9 (following [Han74, Prop. 2])**.** *For $n \in \mathbb{Z}$, we have a short exact sequence of groups of this form:*

$$1 \to \mathbb{Z}_{/2n} \longrightarrow \pi_1 \mathrm{Map}_n\big((S^1)^2,\, S^2\big) \longrightarrow \mathbb{Z}^2 \to 1\,.$$

Proof. First note that the canonical CW-complex structure on $(S^1)^2$ gives rise to a homotopy cofiber sequence of the form

$$S^1 \xrightarrow{[\alpha,\beta]} S^1 \vee S^1 \longrightarrow (S^1)^2 \longrightarrow S^2, \tag{31}$$

where α and β denote a pair of representatives of a pair of generators of $\pi_1(S^1 \vee S^1)$, so that their group commutator $[\alpha, \beta]$ is the *fundamental polygon* of the torus (as seen in Fig. 1).

Consider then the homotopy long exact sequence induced by the homotopy fiber sequence obtained by mapping out of (31):

$$\begin{tikzcd}
 & & \pi_2\mathrm{Map}\big(S^1 \vee S^1, S^2\big) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\underbrace{\pi_1\mathrm{Map}_n\big(S^2, S^2\big)}_{\mathbb{Z}_{/2n}} \arrow[r] & \pi_1\mathrm{Map}_n\big((S^1)^2, S^2\big) \arrow[r] & \underbrace{\pi_1\mathrm{Map}\big(S^1 \vee S^1, S^2\big)}_{\mathbb{Z}^2} \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\pi_0\mathrm{Map}_n\big(S^2, S^2\big)\,, & &
\end{tikzcd}$$

where under the braces we use Lems. 3.7 and 3.8. But the connecting homomorphisms vanish, because by (31) they are given by precomposition with a group commutator but now of representatives of elements of *abelian* groups. Therefore the middle piece shown is short short exact, as claimed. $\square$

4. Generalizations

We may generalize the situation in § 3 from the case of "squared spheres" $(S^{2k-1})^2$ to more general CW-complexes M (such as higher genus surfaces, Ex. 4.2), where the Whitehead product attaching map (18) is allowed to have more arguments and to have multiplicities (Def. 4.1 below). With Lem. 4.3 below established, the proofs in this generality follow the same logic as those in § 3, whence we may be more brief.

Definition 4.1. For $k \in \mathbb{N}_{\geq 1}$ and i, j ranging through some finite set I, let M be the CW-complex given by the following pushout square on the left

$$\begin{tikzcd}
S^{4k-3} \arrow[r, "\phi"] \arrow[d] & \bigvee_i S^{2k-1} \arrow[r] \arrow[d] & \ast \arrow[d] \\
D^{4k-2} \arrow[r] & M \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\mathrm{pr}_c"'] \arrow[d, "\mathrm{pr}_j"] & S^{4k-2} \arrow[ul, phantom, "\ulcorner", very near start] \\
 & S^{2k-1}\,, &
\end{tikzcd} \tag{32}$$

where the attaching map is a product

$$\phi := \prod_{i<j} \big([\iota_i, \iota_j]_{\mathrm{Wh}}\big)^{\lambda_{ij}}, \qquad \lambda_{ij} \in \mathbb{N}\,, \tag{33}$$

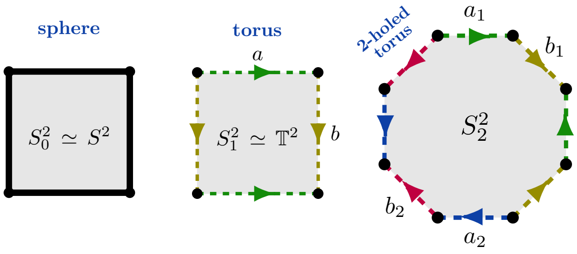
of Whitehead products:

$$\begin{tikzcd}
S^{4k-3} \arrow[r] \arrow[d] \arrow[rr, bend left, "{[\iota_i,\iota_j]_{\mathrm{Wh}}}"] & S^{2k-1} \vee S^{2k-1} \arrow[r, hook, "{(i_i,i_j)}"] \arrow[d] & \bigvee_i S^{2k-1} \\
D^{4k-2} \arrow[r] & S^{2k-1} \times S^{2k-1} \arrow[ul, phantom, "\ulcorner", very near start] \,,
\end{tikzcd}$$

and where the map pr_j is induced by the evident projection onto the jth $(2k-1)$-cell due to the fact that ϕ (33) becomes null-homotopic under this projection:

(34)
$$\begin{tikzcd}
S^{4k-3} \arrow[r, "\phi"] \arrow[d] & \bigvee_i S^{2k-1} \arrow[d] \arrow[dr, bend left, "{(\delta^j_i)_{i\in I}}"] & \\
D^{4k-2} \arrow[r] \arrow[drr, bend right, "*"'] & M \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\mathrm{pr}_j"] & \\
& & S^{2k-1}.
\end{tikzcd}$$

FIGURE 2. Oriented surfaces arise by identifying boundary segments of their *fundamental polygons* such that their 2-cell attaching map is an iterated Whitehead product (35).



Example 4.2. The closed oriented *surface* S^2_g of genus $g \in \mathbb{N}$ is an instance of Def. 4.1 for $k = 1$, where the pushout reflects the classical construction of S^2_g by identifications among the boundary segments of a *fundamental polygon* (cf. [Gib77, Thm. 2.8] and Fig. 2):

(35)
$$\begin{tikzcd}
S^1 \arrow[r, "{[\iota_1,\iota_{g+1}]\cdots[\iota_g,\iota_{2g}]}"] \arrow[d] & \bigvee^{2g} S^1 \arrow[d] \\
D^2 \arrow[r] & S^2_g \arrow[ul, phantom, "\ulcorner", very near start] \,.
\end{tikzcd}$$

The key to generalizing the theorem of § 3 is now the following observation, generalizing (19):

Lemma 4.3. *With M as in Def. 4.1, the following diagrams commute up to homotopy, for $j < j' \in I$:*

(36)
$$\begin{tikzcd}
& M \wedge M \arrow[dr, "\mathrm{pr}_j \wedge \mathrm{pr}_{j'}"] & \\
M \arrow[ur, "\Delta"] \arrow[rr, "\lambda_{jj'} \cdot \mathrm{pr}_c"] & & S^{4k-2}.
\end{tikzcd}$$

Proof. By the *Hopf-Whitney theorem* (cf. [Whi78, (6.19) on p. 244]), it is sufficient to show that both the top and the bottom map in this diagram have the same degree, hence that the degree of the top composite is $\lambda_{jj'}$, in that the pullback of the volume class $1 \in \mathbb{Z} \simeq H^{4k-2}(S^{4k-2};\mathbb{Z})$ is $\lambda_{jj'}$ times the generator of $H^{4k-2}(M;\mathbb{Z})$.

To that end, consider the following diagram:

$$\begin{tikzcd}
S^{4k-3} \arrow[rr, "\lambda_{jj'}"] \arrow[d, "\phi"'] \arrow[drr, "\lambda_{jj'}\cdot[\iota,\iota]_{\mathrm{Wh}}"] & & S^{4k-3} \arrow[d, "{[\iota,\iota]_{\mathrm{Wh}}}"] \\
\bigvee_i S^{2k-1} \arrow[rr, "p_{jj'}"] \arrow[d] & & S^{2k-1} \vee S^{2k-1} \arrow[d] \\
M \arrow[r] \arrow[dr, "\Delta"'] \arrow[dd] & M \times M \arrow[r, "\mathrm{pr}_j \times \mathrm{pr}_{j'}"] \arrow[d] & S^{2k-1} \times S^{2k-1} \arrow[dd] \\
 & M \wedge M \arrow[dr, "\mathrm{pr}_j \wedge \mathrm{pr}_{j'}"] & \\
S^{4k-2} \arrow[rr, "\lambda_{jj'}"] & & S^{4k-2}.
\end{tikzcd}$$

Here:

- **(i)** The vertical parts are the homotopy cofiber sequences (“Puppe sequences”).
- **(ii)** The map $p_{jj'}$ denotes the projection onto the j and j'th wedge summand, whence the top square commutes by definition of ϕ (33).
- **(iii)** The middle square commutes by (34).
- **(iv)** By functoriality of the pushout, the bottom map is the suspension of the top map and as such of the same degree $\lambda_{jj'}$, as shown.

But then the top right triangle in the bottom square commutes by definition, whence the bottom left triangle exhibits the degree of the diagonal map as claimed. ☐

With this in hand, we have an immediate generalization of Thm. 3.3:

Theorem 4.4. *With M as in Def. 4.1, and for $k \in \{1,2,4\}$, we have:*

$$\pi_1 \,\mathrm{Map}^*_0\big(M,\, S^{2k}\big) \simeq \Big\langle (W_i)_{i\in I},\, \zeta \,\Big|\, [W_i, W_j] = \zeta^{2\lambda_{ij}},\, [W_i,\zeta] = \mathrm{e} \Big\rangle \times T\,, \tag{37}$$

where T stands for the same torsion groups as in (22).

Proof. Since the suspension of Whitehead products is null (by Lem. 2.7), it follows from (32) and (33) that we have a *stable splitting* in generalization of (20):

$$\Sigma M \xrightarrow[\sim]{\left((\Sigma \mathrm{pr}_i)_{i\in I},\Sigma \mathrm{pr}_c\right)} \Big(\bigvee_i S^{2k}\Big) \vee S^{4k-1}.$$

This gives the set of generators as claimed in (37).

To see that their group commutators are as claimed, consider the direct analogue of diagram (25), where now the left inner triangle is given by Lem. 4.3:

$$\begin{tikzcd}
M \arrow[rr, "\widetilde{[W_j, W_{j'}]}"] \arrow[d, "\Delta"'] \arrow[dr, "\lambda_{jj'}\cdot \mathrm{pr}_c"] & & \Omega S^{2k} \\
M \wedge M \arrow[r, "\mathrm{pr}_j \wedge \mathrm{pr}_{j'}"'] \arrow[rr, bend right, "\widetilde{W}_j \wedge \widetilde{W}_{j'}"'] & S^{2k-1} \wedge S^{2k-1} \arrow[r, "\widetilde{\mathrm{id}_{2k}} \wedge \widetilde{\mathrm{id}_{2k}}"'] \arrow[ur, "2\,\widetilde{h_{\mathbb{K}}}"] & (\Omega S^{2k}) \wedge (\Omega S^{2k})\,. \arrow[u, "{[-,-]_{\mathrm{Sam}}}"']
\end{tikzcd} \tag{38}$$

Similarly, diagram (26) generalizes immediately, which completes the proof. ☐

It is furthermore straightforward now to pass from Thm. 4.4 to the generalization of Thm. 3.5:

Theorem 4.5. *For M as in Def. 4.1, we have:*

$$\pi_1\,\mathrm{Map}_n\big(M,\,S^2\big) \simeq \Big\langle (W_i)_{i\in I},\,\zeta \,\Big|\, [W_i,W_i] = \zeta^{2\lambda_{ij}},\,[W_i,\zeta] = \mathrm{e}\,,\zeta^{2n} = \mathrm{e}\Big\rangle\,, \tag{39a}$$

$$\pi_1\,\mathrm{Map}_{[n]}\big(M,\,S^4\big) \simeq \Big\langle (W_i)_{i\in I},\,\zeta \,\Big|\, [W_i,W_i] = \zeta^{2\lambda_{ij}},\,[W_i,\zeta] = \mathrm{e}\Big\rangle \times \mathbb{Z}_{/12}\,, \tag{39b}$$

$$\pi_1\,\mathrm{Map}_{[n]}\big(M,\,S^8\big) \simeq \Big\langle (W_i)_{i\in I},\,\zeta \,\Big|\, [W_i,W_i] = \zeta^{2\lambda_{ij}},\,[W_i,\zeta] = \mathrm{e}\Big\rangle \times \mathbb{Z}_{/120}\,. \tag{39c}$$

Proof. With Thm. 4.4 in hand, the remaining argument is verbatim that of the proof of Thm. 3.5, subject just to substituting M for $(S^{2k-1})^2$. □

In particular:

Example 4.6. For $k = 1$ and in the case that $M = S^2_g$ is the closed oriented surface of genus g (Ex. 4.2), Thm. 4.5 gives the higher-dimensional integer Heisenberg groups (12):

$$\pi_1\,\mathrm{Map}_n\big(S^2_g,\,S^2\big) \simeq \mathrm{Heis}_{2g+1}(\mathbb{Z};2n)\,. \tag{40}$$

This is the generality of the situation originally discussed by [Han74; LT80, Thm. 1; Kal01, Prop. 1.5].

5. APPLICATIONS

Here we expand (following up on the indications in § 1.1.3) on how the result of § 3 may be understood as saying that the *quantum observables* of (abelian) *anyons* on the torus (Fig. 1) are equivalently the group algebra of the 2-Cohomotopy of the suspended torus, and in fact also of the 4-Cohomotopy of the suspended squared 3-sphere and of the 8-Cohomotopy of the suspended squared 7-sphere.

By itself, this is just a mathematical fact (Thm. 3.5, 28a). But we recall here how there is physical significance in *magnetic flux quantized in Cohomotopy* [SS25e] which, if physically realized ("Hypothesis h" [SS25f], recalled below in § 5.1) renders this mathematical fact, instead of a coincidence, one instance of a novel algebro-topological effective theory of anyons — which makes predictions, potentially relevant for contemporary materials research, not captured by previous theories.

This appears to be a remarkable new opportunity for (low-dimensional) algebraic topology and homotopy theory to interact with cutting-edge experimental and industry-relevant research (quantum materials, topological quantum computing), potentially of impact comparable to what had been hoped would happen with topological data analysis (TDA, where methods of algebraic topology and homotopy theory are hoped to identify hidden structure in complex data sets).

Moreover, we explain (in § 5.2 below, following [SS25c]) how our generalized result of Thm. 3.5 establishes that quantum observables of (abelian) FQH anyons (of the kind that have been observed in experiment in recent years) arise as topological observables in a completion of *11-dimensional supergravity* (11D SuGra) by flux quantization in 4-Cohomotopy ("Hypothesis H", now with a capital "H").

As before, by itself this is just a mathematical fact (Thm. 3.5, 28b), surprising as it may sound. It does, however, call to mind the much-discussed "holographic" relation between 11D SuGra and condensed matter theory [Her+07; Zaa+15; HLS18;

nLa26a] and may be understood as a novel instance of a general idea of *gauge/gravity duality* whereby strongly-coupled quantum systems (for which there is little traditional theory available) are thought to be mapped onto more tractable dynamics of *branes* fluctuating in auxiliary higher-dimensional auxiliary spacetimes.

While this approach has seen immense activity in the last couple of decades, it is notorious for its dearth of precise definitions and of provable theorems. Our result may open the door to a new form of mathematically substantiated interaction between completed quantum 11D SuGra ("M-theory" [Duf99]) and topological quantum materials (following [SS23a; SS25b; SS25j; SS25d]).

5.1. **Ordinary FQH Anyons via 2-Cohomotopy.** In view of our theorems in §§ 3 and 4, here we briefly review the understanding of FQH anyons [SS26a; SS25f] and of FQAH anyons [SS26b; SS25g] via (magnetic or Berry-)flux quantization in 2-Cohomotopy (further surveyed in [SS25a]).

5.1.1. *Ordinary Magnetic Flux.* Envision a thin slab of material of the form of a closed oriented surface S^2_g (Ex. 4.2) of some thickness $2\epsilon \in \mathbb{R}_{>0}$ and penetrated transversally by a magnetic field. The phenomenon of *Dirac charge quantization* (cf. [Alv85; Fra11, §16.4e]) entails (cf. [SS24, §3.1; SS25c, Cor. 2.3]) that the topological sectors of solitonic magnetic flux configurations through this material form the set of homotopy classes of maps out of its one-point compactification $(-)_{\cup\{\infty\}}$ (enforcing the vanishing of solitonic flux at infinity) to the classifying space of the ("gauge") group U(1):

$$\begin{aligned}\text{Magnetic flux sectors} \;&=\; \pi_0\,\mathrm{Map}^*\Big(\big(S^2_g \times (-\epsilon,+\epsilon)\big)_{\cup\{\infty\}}, B\mathrm{U}(1)\Big)\\ &\simeq\; \pi_0\,\mathrm{Map}^*_0\Big((S^2_g)_+ \wedge S^1, B\mathrm{U}(1)\Big) \qquad (41)\\ &\simeq\; \pi_1\,\mathrm{Map}_0\Big(S^2_g, B\mathrm{U}(1)\Big)\,.\end{aligned}$$

This is a fundamental group of a mapping space much as we have been discussing in § 4, only that here the coefficient is $B\mathrm{U}(1) \simeq \mathbb{C}P^\infty$ instead of $S^2 \simeq \mathbb{C}P^1$. With this "stable" coefficient it is immediate to compute the fundamental group to be:

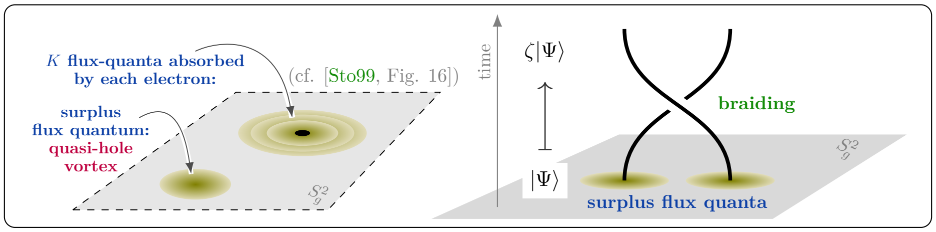
$$\begin{aligned}\cdots \;&\simeq\; \pi_0\,\mathrm{Map}_0\Big(S^2_g, \Omega B\mathrm{U}(1)\Big)\\ &\simeq\; \pi_0\,\mathrm{Map}_0\Big(S^2_g, B\mathbb{Z}\Big)\\ &\simeq\; H^1\Big(S^2_g;\mathbb{Z}\Big)\\ &\simeq\; \mathbb{Z}^{2g}\,.\end{aligned}$$

This is the abelian base group of which $\mathrm{Heis}_{2g+1}(\mathbb{Z})$ (Def. 2.2) is a nonabelian extension. The topological quantum observables on these magnetic flux sectors form the group algebra of $\mathbb{Z}^2$ (by [SS25i, §1; SS25c, (132)]).

5.1.2. *Anyons as Surplus FQH Flux.* Envision next that our slab of material is filled with a very cold electron gas and penetrated by a magnetic field so strong and so fine-tuned that there are exactly some K magnetic flux quanta per electron. This is called a *fractional quantum Hall system* at *filling fraction* $1/K$ (cf. [Sto99; Ton16]).

On such a backdrop, every surplus magnetic flux quantum appears as the lack of $1/K$th of an electron and as such is called a fractional *quasi-hole*. It is these *quasi-holes* that behave as anyons, in that under their movement around each other the quantum state of the entire system picks up a complex *braiding phase* ζ (cf. Fig. 3).

Figure 3. The anyons of fractional quantum Hall systems are vortices in the 2D electon gas induced by surplus magnetic flux quanta on top of an exact rational *filling fraction* of K flux quanta per electron. Under each *braiding* of their worldlines the quantum state $|\psi\rangle$ transforms by multiplication with a *braiding phase* $\zeta = \exp(\pi \mathrm{i}/K)$.



This means that the interaction with the electron gas makes surplus magnetic flux in an FQH system *effectively* behave differently than predicted by the ordinary Dirac charge quantization of § 5.1.1. Therefore it stands to reason that the effective FQH flux is quantized (in the general sense of *flux quantization* [SS25e]) instead by a deformation of the usual classifying space $B\mathrm{U}(1)$. But our Ex. 4.6 suggests that this deformation must be its 3-skeleton

$$S^2 \simeq \mathbb{C}P^1 \xhookrightarrow{\;i\;} \mathbb{C}P^\infty \simeq B\mathrm{U}(1)\,, \tag{42}$$

because if we substitute that for $B\mathrm{U}(1)$ in the above computation (41), then the experimentally observed braiding phase ζ does appear (by Thm. 4.5, 39a) as expected (cf. [Ton16, (5.28)]):

$$\begin{aligned} \text{FQH surplus flux sectors} &\equiv \pi_1\,\mathrm{Map}^*_n\big(S^2_g, S^2\big) \\ &\simeq \mathrm{Heis}_{2g+1}(\mathbb{Z}; 2n)\,. \end{aligned} \tag{43}$$

This match hence suggests the hypothesis (called "Hypothesis h" in [SS25f]) that 2-Cohomotopy is the correct global flux quantization law for effective anyonic FQH surplus flux quanta.

Apart from neatly reproducing the quantum observables (43) of solitonic FQH flux, this *Hypothesis h* predicts experimentally relevant phenomena such as notably the possible attachment of nonabelian *defect anyons* to superconducting islands inside the 2D electron gas (cf. [SS25f, Fig. D & §3.8]).

5.1.3. *FQH Anyon Braiding.* To see more concretely the actual braiding of anyons in this description, note that (as indicated in Fig. 4) the *Pontrjagin theorem* (cf. [Bre93, §II.16; SS23b, §3.2]) identifies $\pi_0\,\mathrm{Map}\big(S^2_g, S^2\big)$ with (cobordism classes of) normally framed codim=2 submanifolds of S^2_g, hence with *signed points* in S^2_g, to be identified with the anyon cores seen in Fig. 3.

Along these lines, careful analysis shows [SS26a] that the loop space $\Omega\mathrm{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)$ may be identified with the space of *framed links* L (of anyon worldlines) subject

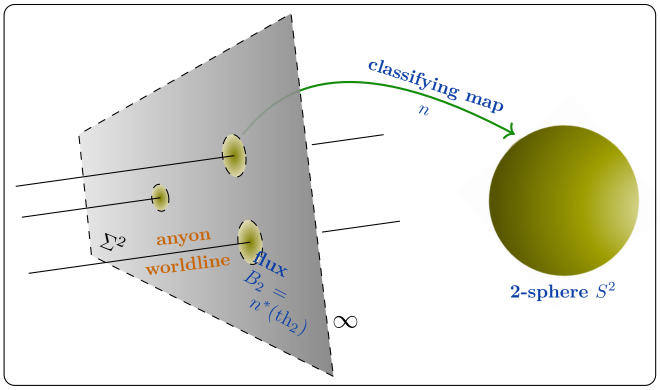


FIGURE 4. The unstable *Pontrjagin theorem* identifies homotopy classes of maps to the n-sphere with cobordism classes of normally framed codim$=n$ submanifolds. We may understand the latter as the cores of flux density quanta obtained as pullback of a *Thom form* th_n of unit weight supported around $0 \in S^n$.

to framed link cobordism, and that under this identification the topological equivalence classes of these loops/worldlines are identified with the *writhe* $\#L$ of the corresponding links, being the *total crossing number* of any of their link diagrams:

$$\begin{tikzcd}
\Omega\,\mathrm{Map}^*\Big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\Big) \arrow[r, two heads] & \pi_1 \mathrm{Map}^*\Big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\Big) \simeq \mathbb{Z} \\
L \arrow[r, mapsto] & \#L\,.
\end{tikzcd} \tag{44}$$

But this says exactly that one power of the *braiding phase* generator ζ is picked up for every crossing of anyon worldlines as seen in Fig. 3 (cf. [SS26a, §3]).

5.1.4. *Identifying FQAH Anyons.* While FQH systems (§§ 5.1.2 and 5.1.3) thereby constitute the first (and currently only) experimentally verified candidate platform for genuine topological quantum hardware, the extremely low temperatures and strong magnetic fields they require obstruct their practical utility as such. It is therefore remarkable that, very recently, an "anomalous" version of fractional quantum Hall systems (FQAH, predicted by [TMW11; Sun+11; Neu+11], further developed in [PRS13; Roy14], recently reviewed in [Ju+24; MSM24; Zha+25]) has been experimentally realized in various materials ([Cai+23; Zen+23; Par+23; Lu+24]):

In these crystalline FQAH systems the role of the magnetic field in "real space" is played instead by an intrinsic property of the "momentum space" of the crystal electrons, called the *Berry curvature* (cf. [Sta20, §2; SS26b, Fig. 3]). Therefore, if anyonic states in FQAH materials could be identified and controlled, this would open the door to practically viable *room temperature* topological quantum hardware.

While traditional theory has arguably remained inconclusive in identifying the nature and signature of potential FQAH anyons, at this point the discussion in § 3 applies, since:

(i) Due to the translational symmetry of crystal lattices, the crystal momentum space is always a torus — known as the *Brillouin torus* $\widehat{T}^d$ (cf. [Thi25, §2.1]), which is 2-dimensional for the effectively 2-dimensional FQAH systems: $\widehat{T}^2 \simeq T^2$.

(ii) For the most prominent FQAH systems with 2 relevant *electron bands*, the space of choices of *valence electron states* at each momentum $[k] \in \widehat{T}^2$ is the space of 1D complex subspaces among the 2D space of valence and conduction electron states, hence is the Grassmannian $\mathbb{C}P^1 \simeq S^2$ (cf. [Ser23, (8.3-4); SS26b, (4); SS25h, Lem. 4.1]).

But this means that the moduli space of these crystal's topological parameters is just the kind of mapping space considered in § 3:

$$\text{Crystal parameter space} \;\simeq\; \mathrm{Map}\big(\widehat{T}^2, \mathbb{C}P^1\big) \;=\; \mathrm{Map}\big(T^2, S^2\big)\,. \tag{45}$$

In particular, the *topological phases* available to the FQAH system constitute the connected components of this space, which is the set of integers known as the *Chern numbers* of the valence bundles (cf. [SS26b, (8)]):

$$\pi_0\,\mathrm{Map}\big(T^2, S^2\big) \xrightarrow[i_*]{\sim} \pi_0\,\mathrm{Map}\big(T^2, B\mathrm{U}(1)\big) \;\simeq\; \mathbb{Z}\,,$$

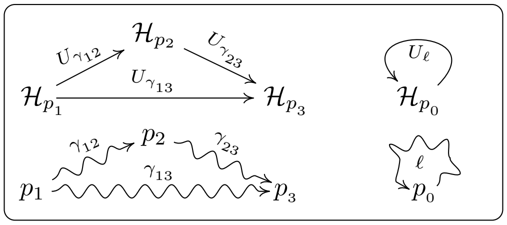
whence one refers to these topological quantum phases as (fractional) *Chern insulators* (cf. [Ser23, §8; Neu+11]).

Furthermore, with [SS26b; SS25g] we may observe now that the *quantum adiabatic theorem* entails (cf. Fig. 5) that anyonic topological order of quantum materials is classified by *local systems* of Hilbert spaces (of gapped ground states) over the system's parameter space, hence here by linear representations of the fundamental groups of the parameter space (45):

$$\underset{\text{in topol. phase with Chern number } C}{\text{Category of anyonic topological orders}} \;=\; \big(\pi_1\,\mathrm{Map}_C\big(T^2, S^2\big)\big)\mathrm{Rep}\,. \tag{46}$$

But this is exactly what Thm. 3.5 (28a) applies to, where it says that these anyonic quantum states are representations of the integer Heisenberg group — just as expected for FQH systems on a torus.

Figure 5. The adiabatic tuning of classical parameters p along paths γ in parameter space induces unitary transformations U_γ between corresponding Hilbert spaces $\mathcal{H}$ of gapped ground states. For topological states these transformations depend only on the homotopy class of γ, exhibiting a *local system* or *flat bundle* of Hilbert spaces over the parameter space. These are equivalently linear representations of the fundamental groups of parameter loops ℓ at each base point, reflecting the *topological order* of the system in any topological phase.



In conclusion, Thm. 3.5 (28a), applied to the adiabatic monodromy (Fig. 5) in the topological parameter space (45) of 2D 2-band (fractional) Chern insulators, implies that the sought-after FQAH anyons are going to appear like the already observed anyons in FQH systems, but now *localized in momentum space*. Further details on experimental signatures predicted by this result are discussed in [SS25g].

5.2. **2-Dimensional Anyons via 4-Cohomotopy.** In the above discussion of FQ(A)H anyons (§ 5.1), the appearance of the 2-sphere and hence of 2-Cohomotopy as a quantization law is tied to the fact that both magnetic flux density and Berry curvature are differential 2-forms. For an analogous interpretation of the 4-sphere coefficient appearing in Thm. 3.5 (28b) and Thm. 4.5 (39b) one needs to understand 4-Cohomotopy as a flux quantization law of a kind of higher gauge field whose magnetic flux density is given by a differential 4-form.

5.2.1. *The C-field in 11D Supergravity.* A higher gauge field (the "C-field") with just such a higher 4-form flux density famously exists in 11-dimensional supergravity theory (11D SuGra, cf. [MS06; FV12; GSS24a, §3]), where its electric Gauß law implies that 4-Cohomotopy is a valid choice ("Hypothesis H", now with capital "H") for its flux quantization.

This means[5] that when the gauge field content of 11D SuGra is globally completed according to Hypothesis H, then the solitonic topological observables on a globally hyperbolic spacetime $X^{1,10} \simeq \mathbb{R}^{1,1} \times X^9$ are spanned by

$$\pi_0 \operatorname{Map}^*\Big(X^9_+ \wedge \mathbb{R}^1_{\cup\{\infty\}}, S^4\Big) \simeq \pi_1 \operatorname{Map}_0\big(X^9, S^4\big)\,. \tag{47}$$

5.2.2. *Near-horizons of intersecting M2/M5-branes.* In order that Thm. 3.5 (28b) applies, we need to look for solutions of 11D SuGra where the spacetime topology contains a factor topologically of the form $S^3 \times S^3$.

Such solutions do exist in the form (cf. [BPS98, §2.2; GMT99])

$$X^{1,10} \simeq \mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}^2 \tag{48}$$

reflecting the near-horizon geometry of suitable M2/M5-brane intersections. (More generally, the 1/2 BPS solutions of 11D SuGra of this form are products of $\mathrm{AdS}_3 \times S^3 \times S^3$ warped over a surface Σ^2, by [DHo+08a; DHo+08b; DHo+09].)

5.2.3. *Anyonic monodromy in 11d SuGra.* If we restrict attention to a bulk causal diamond $D^{1,2} \subset \mathrm{AdS}_3$ and take solitonic flux to be localized along one of the two factors of $\mathbb{R}^2$, the monodromy group (47) becomes

$$\begin{aligned} \pi_1 \operatorname{Map}_0\big(X^9, S^4\big) &\simeq \pi_1 \operatorname{Map}_0\big(D^{1,2} \times \mathbb{R}^1 \times S^3 \times S^3,\, S^4\big) \\ &\simeq \pi_1 \operatorname{Map}_0\big(S^3 \times S^3,\, S^4\big) \\ &\simeq \mathrm{Heis}_3(\mathbb{Z}, 0) \times \mathbb{Z}_{/12}\,, \end{aligned}$$

where in the first step we used that $D^{1,2} \times \mathbb{R}^1$ is contractible, and in the last step we applied our new Thm. 3.5 (28b).

But this means (following [SS25c, §3.3]) that the topological quantum states on cohomotopically quantized C-field flux over such 11D SuGra backgrounds (48) are of just the form (§ 5.1) expected for FQ(A)H anyons on a torus!

This is a novel realization of anyon statistics in quantum 11D SuGra. While the quantum observables are just those of the experimentally observed FQH anyons,

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[5] Generally, if the tangent bundle of the spacetime does not trivialize, the full Hypothesis H demands ([FSS20; FSS21; SS21]) that plain 4-Cohomotopy be replaced by *tangentially twisted 4-Cohomotopy*, represented not by plain maps to the 4-sphere but by sections of 4-spherical fibrations over spacetime. But here we may disregard this further refinement, since the example spacetimes relevant here have trivializable tangent bundles, mainly owing to the fact that so does $S^3 \simeq \mathrm{SU}(2)$ and with it the product space $S^3 \times S^3$.

these anyonic C-field flux quanta are, at face value, higher-dimensional objects, braiding in a higher-dimensional ambient space:[6]

5.2.4. *2-Dimensional anyons.* The analogue of the argument in § 5.1.3, around Fig. 4, via the Pontrjagin theorem shows that the anyonic C-field flux quanta identified above in § 5.2.3 are 2-dimensional inside $S^3 \times S^3$ (hence are $2 + \dim(\mathrm{AdS}_3 \times \mathbb{R}^1) = 6$-dimensional inside $X^{1,10}$, hence are anyonic 5-branes of sorts). Hence, the analogue of (44) says that the "worldvolume" traced out as these higher-dimensional anyons propagate are 3-dimensional links inside $S^3 \times S^3$.

Similar "d-dimensional anyons" have previously been discussed mostly for $d = 1$ as linear representations of *loop braid groups* (cf. [nLa26c]) potentially realizable in 3-dimensional quantum materials. The role of $d = 2$-dimensional anyons in 6 ambient dimensions remains to be understood.

APPENDIX A. SOME TOPOLOGY

For reference in the main text and to establish our notation, we briefly recall some notions and facts from basic topology.

We work in the category of compactly generated topological spaces, to be denoted TopSp, and TopSp* for the pointed version. For $X, Y \in \mathrm{TopSp}$ we write

$$\mathrm{Map}(X, Y) \in \mathrm{TopSp} \tag{49}$$

for their *mapping space*, and for $(X, x), (Y, y) \in \mathrm{TopSp}^*$ we write

$$\mathrm{Map}^*(X, Y) \subset \mathrm{Map}(X, Y) \in \mathrm{TopSp} \tag{50}$$

for their *pointed mapping space*.

For example, the circle is naturally a pointed space when realized as

$$S^1 \simeq \frac{[0,1]}{\{0,1\}}$$

and the *based loop space* of a pointed space is

$$\Omega X := \mathrm{Map}^*(S^1, X)\,. \tag{51}$$

Given a pair of pointed spaces $(X, x), (Y, y) \in \mathrm{TopSp}^*$, their *smash product* is

$$X \wedge Y := \frac{X \times Y}{X \times \{y\} \cup \{x\} \times Y}\,. \tag{52}$$

Notably

$$S^{n_1} \wedge S^{n_2} \simeq S^{n_1 + n_2}, \tag{53}$$

and generally

$$\Sigma X := S^1 \wedge X \tag{54}$$

is the *suspension* of a pointed space.

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[6] Alternatively, the usual point-like anyons in the cohomotopical guise of § 5.1.2 may be "geometrically engineered" in 11D SuGra by realizing them on M5-probe worldvolumes (which in turn are embedded into 11D SuGra backgrounds, cf. [HS97; GSS24b]), cf. [SS25b; SS26c], exposition in [SS25j; SS25d].

While the smash product is not Cartesian, it does have a natural diagonal map induced by the Cartesian diagonal, which we thus denote by the same symbol:

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "\Delta"'] & & X \wedge X\,. \\ & X \times X \arrow[ur, two heads] & \end{tikzcd} \tag{55}$$

The *hom-adjunction* is the natural homeomorphism which regards a map of two arguments as a function of the first with values in maps of the second argument:

$$\mathrm{Map}^*(X \wedge Y,\, Z) \xleftrightarrow[\sim]{\widetilde{(-)}} \mathrm{Map}^*\big(X,\, \mathrm{Map}^*(Y, Z)\big). \tag{56}$$

Notably the suspension operation (54) is left-adjoint to forming based loop spaces (51):

$$\mathrm{Map}^*(\Sigma X,\, Y) \xrightarrow[\sim]{\widetilde{(-)}} \mathrm{Map}^*(X,\, \Omega Y). \tag{57}$$

Moreover, since the coproduct of pointed spaces is the *wedge sum*

$$X \vee Y \;:=\; \frac{X \sqcup Y}{\{x, y\}}. \tag{58}$$

we have

$$\mathrm{Map}^*(X \vee Y,\, Z) \;\simeq\; \mathrm{Map}^*(X, Z) \times \mathrm{Map}^*(Y, Z). \tag{59}$$

The *homotopy groups* of $(X, x) \,\in\, \mathrm{TopSp}^*$ are

$$\pi_n(X, x) \;\simeq\; \pi_0 \,\mathrm{Map}^*(S^n, X)\,. \tag{60}$$

Appendix B. Some Homotopy Theory

For reference in the main text and to establish our notation, we briefly recall some notions and facts from basic homotopy theory.

A *homotopy* is a path in a mapping space. The *homotopy classes* of maps are the connected components of the mapping space (50):

Definition B.1. The *pointed homotopy category* $\mathrm{Ho}(\mathrm{TopSp}^*)$ has as objects the pointed topological spaces which admit CW-complex structure, and as hom-sets the connected components of mapping spaces between these:

$$\mathrm{Ho}(\mathrm{TopSp}^*)(X, Y) \;\simeq\; \pi_0 \,\mathrm{Map}(X, Y)\,.$$

In particular, the *homotopy groups* of a pointed space X are

$$\pi_n(X) \;\equiv\; \pi_0 \,\mathrm{Map}^*(S^n,\, X). \tag{61}$$

Example B.2. Some well-known unstable homotopy groups of spheres in low degrees:

$$\begin{array}{lll} \begin{array}{ll} \forall_{k < n \in \mathbb{N}} & \pi_k(S^n) \;\simeq\; 0\,, \\ \forall_{n \in \mathbb{N}_{\geq 1}} & \pi_n(S^n) \;\simeq\; \underbrace{\mathbb{Z}}_{\langle [\mathrm{id}] \rangle}\,, \end{array} & \begin{array}{l} \pi_3(S^2) \;\simeq\; \underbrace{\mathbb{Z}}_{\langle [h_{\mathbb{C}}] \rangle} \\ \pi_7(S^4) \;\simeq\; \underbrace{\mathbb{Z}}_{\langle [h_{\mathbb{H}}] \rangle} \times \underbrace{\mathbb{Z}_{/12}}_{\langle r_{\mathbb{H}} \rangle} \\ \pi_{15}(S^8) \;\simeq\; \underbrace{\mathbb{Z}}_{\langle [h_{\mathbb{O}}] \rangle} \times \underbrace{\mathbb{Z}_{/120}}_{\langle r_{\mathbb{O}} \rangle}\,, \end{array} & \begin{array}{l} \pi_6(S^4) \;\simeq\; \mathbb{Z}_{/2} \\ \pi_{14}(S^8) \;\simeq\; \mathbb{Z}_{/2}\,, \end{array} \end{array} \tag{62}$$

where $[h_{\mathbb{K}}]$ denotes the homotopy class of the $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$-Hopf fibration $h_{\mathbb{K}}$ (cf. [SS25h, §3.2.3]), and $r_{\mathbb{K}}$ represents an unstable *remainder* class.

Here the group structure on the homotopy groups is induced by the *H-cogroup structure* on the spheres (cf. Rem. B.6 below):

Definition B.3 (cf. [Ark11, §2.2])**.** An *H-group* is a group internal to the pointed homotopy category (Def. B.1), hence a pointed topological space (with CW-structure) equipped with a binary operation and with inverses which satisfy the group axioms up to (unspecified) homotopy.

Dually, an *H-cogroup* is a group object internal to the opposite of the pointed homotopy category.

Example B.4. For $(X, x) \in \mathrm{TopSp}^*$:

(i) The loop space ΩX (51) is an H-group (Def. B.3) with binary operation given by concatenation of loops,

$$\begin{array}{ccc} \Omega X \times \Omega X & \xrightarrow{\quad\quad\star\quad\quad} & \Omega X \\ (\ell_1, \ell_2) & \longmapsto & \left(s \mapsto \begin{cases} \ell_1(2s) & \text{if } s \leq 1/2 \\ \ell_1(2s-1) & \text{if } s \geq 1/2 \end{cases} \right), \end{array} \tag{63}$$

and inverses given by reversal of loops:

$$\begin{array}{ccc} \Omega X & \xrightarrow{\overline{(-)}} & \Omega X \\ \ell & \longmapsto & \big(s \mapsto \ell(1-s) \big). \end{array} \tag{64}$$

(ii) The suspension ΣX (54) is an H-cogroup with binary cooperation

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\quad\quad\quad} & \Sigma X \vee \Sigma X \\ (s, x) & \longmapsto & \begin{cases} \big((2s, x), *\big) & \text{if } s \leq 1/2 \\ \big(*, (2s-1)\big) & \text{if } s \geq 1/2\,, \end{cases} \end{array} \tag{65}$$

and coinverses given by

$$\begin{array}{ccc} \Sigma X & \longrightarrow & \Sigma X \\ (s, x) & \longmapsto & (1-s, x)\,. \end{array} \tag{66}$$

Here (co)associativity is witnessed by "thin" homotopies which exhibit reparameterizations of concatenated loops.

Example B.5. More generally, the mapping spaces (57) out of a suspension or into a loop space inherit H-group structure by pointwise group operation, hence where the binary operation on a pair of maps $f, g : X \longrightarrow \Omega S$ is given by the following composite with the smash diagonal (55) on the left and loop concatenation (63) on the right:

$$\begin{tikzcd} X \arrow[r, "f \star g"] \arrow[d, "\Delta"'] & \Omega A \\ X \times X \arrow[r, "f \times g"] & \Omega A \times \Omega A\,. \arrow[u, "\star"'] \end{tikzcd}$$

Remark B.6. Under passage to homotopy classes of maps, H-(co)group structure induces genuine group structure: For $(X, x), (A, a) \in \mathrm{TopSp}^*$ the H-group structure on the mapping space (Ex. B.5) induces actual group structures

$$\begin{aligned} \pi_0 \operatorname{Map}^*(X, \Omega A) &\simeq \pi_0 \operatorname{Map}^*(\Sigma X, A) \\ &\simeq \pi_1 \operatorname{Map}^*(X, A) \quad \in \quad \mathrm{Grp}\,. \end{aligned} \tag{67}$$

In particular, the group structure on homotopy groups (61) in positive degree,

$$\pi_{d+1}(A) \simeq \pi_0 \operatorname{Map}^*(\Sigma S^d, A)\,,$$

arises this way.

Further in this vein:

Example B.7. For X^d a CW-complex of dimension $d \geq 1$, and $S^{d-1} \hookrightarrow D^d \hookrightarrow X$ an embedding of the boundary of a d-ball into the interior of a d-cell of X^d, then the *pinch map* ϕ given by the pushout

(68)
$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d, hook] & * \arrow[d] \\ X^d \arrow[r, "\phi"] \arrow[ur, phantom, "\ulcorner", near end] & X^d \vee S^d \end{tikzcd}$$

exhibits an "H-coaction" (a coaction up to unspecified homotopies) of the H-cogroup $S^d \simeq \Sigma S^{d-1}$ (from Ex. B.4). For any $A \in \mathrm{TopSp}^*$ this entails an actual action of the homotopy group $\pi_d(A) \simeq \pi_0 \mathrm{Map}^*(S^d, A)$ (from Rem. B.6) on the homotopy classes of maps from X^d to A:

$$\pi_d(A) \times \pi_0 \operatorname{Map}^*(X^d, A) \xrightarrow{\sim} \pi_0 \operatorname{Map}^*(X^d \vee S^d, A) \xrightarrow{\phi^*} \pi_0 \operatorname{Map}^*(X^d, A)\,. \tag{69}$$

This is used in the proof of Lem. 3.4 above.

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