

# Higher and Equivariant Bundles

Urs Schreiber on joint work with Hisham Sati

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& Center for Quantum and Topological Systems

New York University, Abu Dhabi

talk via:

Higher Structures Seminar @

Feza Gürsey Center for Math and Physics

Istanbul, 8 Feb 2022

slides and pointers at: [ncatlab.org/schreiber/show/Higher+and+Equivariant+Bundles](https://ncatlab.org/schreiber/show/Higher+and+Equivariant+Bundles)

This talk is  
a gentle exposition of  
the most basic concept  
underlying these articles:

<i>Principal <math>\infty</math>-bundles</i>	[arXiv:1207.0248/49]
<i>Equivariant Principal <math>\infty</math>-bundles</i>	[arXiv:2112.13654]
<i>Proper Orbifold Cohomology</i>	[arXiv:2008.01101]

following

<i>Diff. Cohomology in a Cohesive <math>\infty</math>-Topos</i>	[arXiv:1310.7930]
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# Motivation, Overview, Summary and Outlook – in one single slide:

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**Generalized Cohomology Theories**  $\leftrightarrow$  **Cohesive Higher Fiber Bundles**

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<b>Cohomology</b>	$\leftrightarrow$	<b>Higher Bundles</b>
non-abelian	$\leftrightarrow$	general fibers
twisted	$\leftrightarrow$	associated
differential	$\leftrightarrow$	cohesive
$G$ -equivariant	$\leftrightarrow$	sliced over <b>BG</b>

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A major phenomenon/subtlety is that the last two aspects go hand-in-hand:

**Proper  $G$ -equivariance** corresponds to the **cohesive slice** over **BG**,  
while

**Borel equivariance** corresponds just to the **slice of shapes**.

# Part I – Invitation

Part II – Application

# Part I – Invitation

which walks you from scratch  
through just the definition of  
equivariant principal 2-bundles  
with simple but key examples;

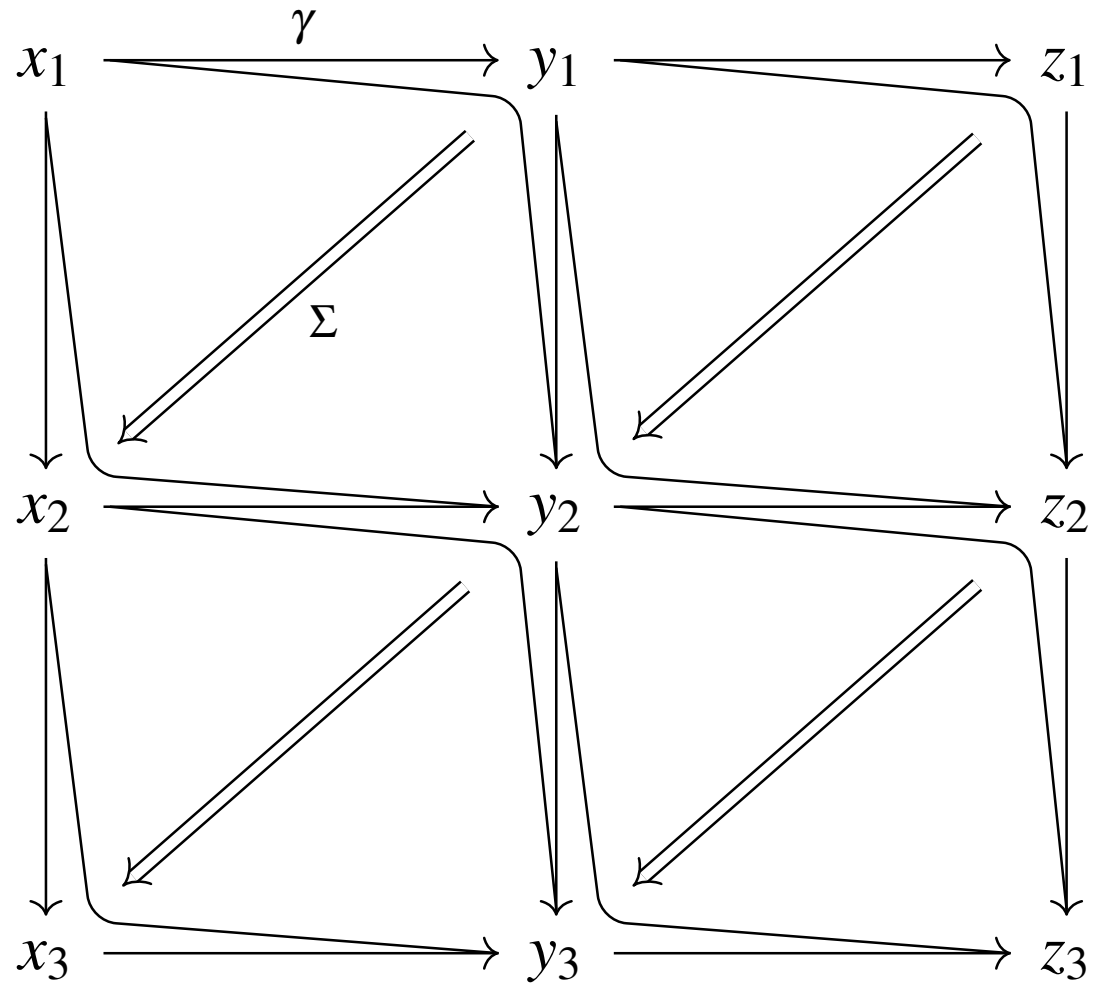
the main *claim* being that this is  
the *good definition*<sup>TM</sup>:

transparent, elegant, universal,  
generalizable & indeed: practical.

# 2-Groupoids

2-Groupoids are the algebra of 2-dimensional pasting, such that all composition is associative and invertible:

E.g. homotopy classes of surfaces  $\Sigma$  rel boundary paths  $\gamma$  in a topological space:

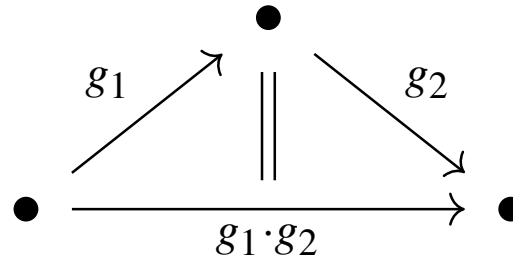




## 2-Groupoids – Examples.

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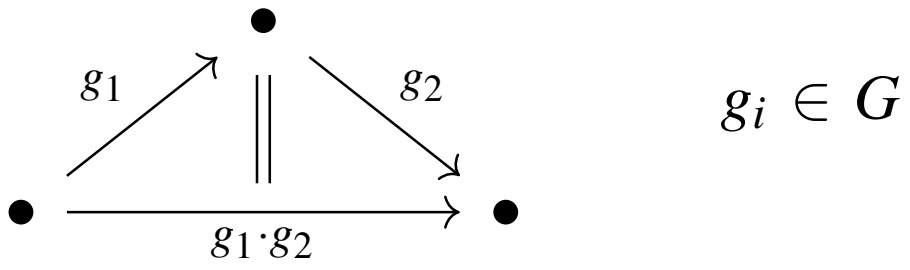
For  $G$  a group, there is its *delooping 1-groupoid*  $\mathbf{B}G$ :



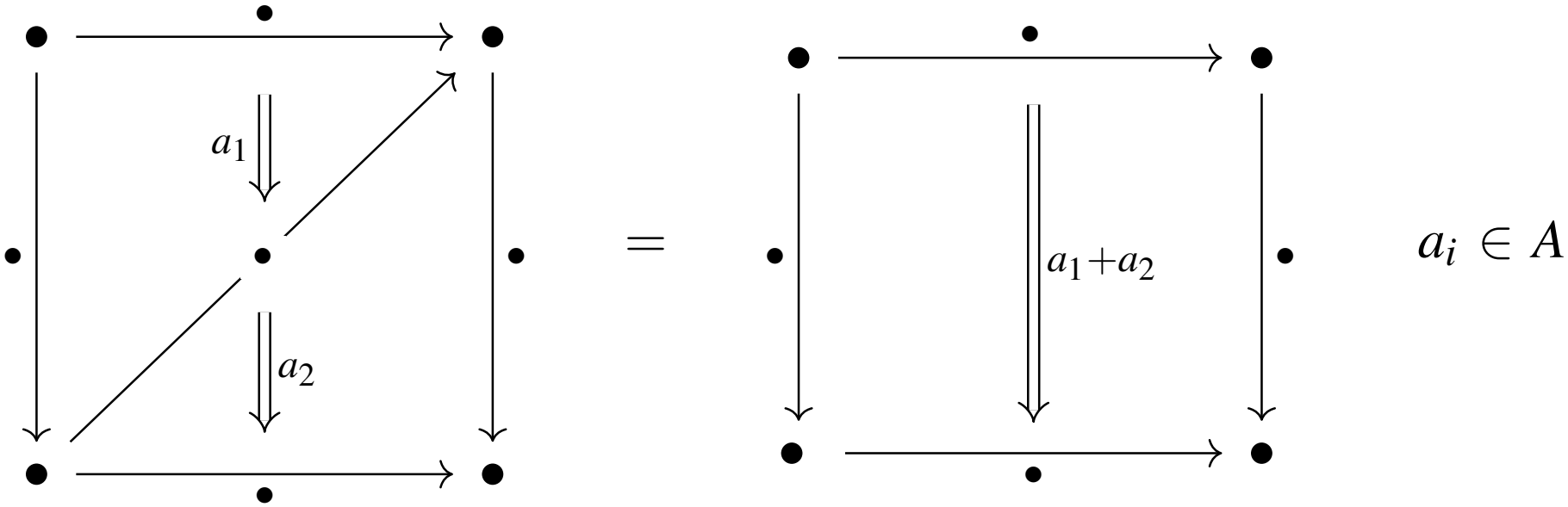
$$g_i \in G$$

# 2-Groupoids – Examples.

For  $G$  a group, there is its *delooping 1-groupoid*  $\mathbf{B}G$ :



For  $A$  an *abelian* group there is the *double delooping 2-groupoid*  $\mathbf{B}^2A = \mathbf{B}(\overbrace{\mathbf{B}A}^{\text{“2-group”}})$  :

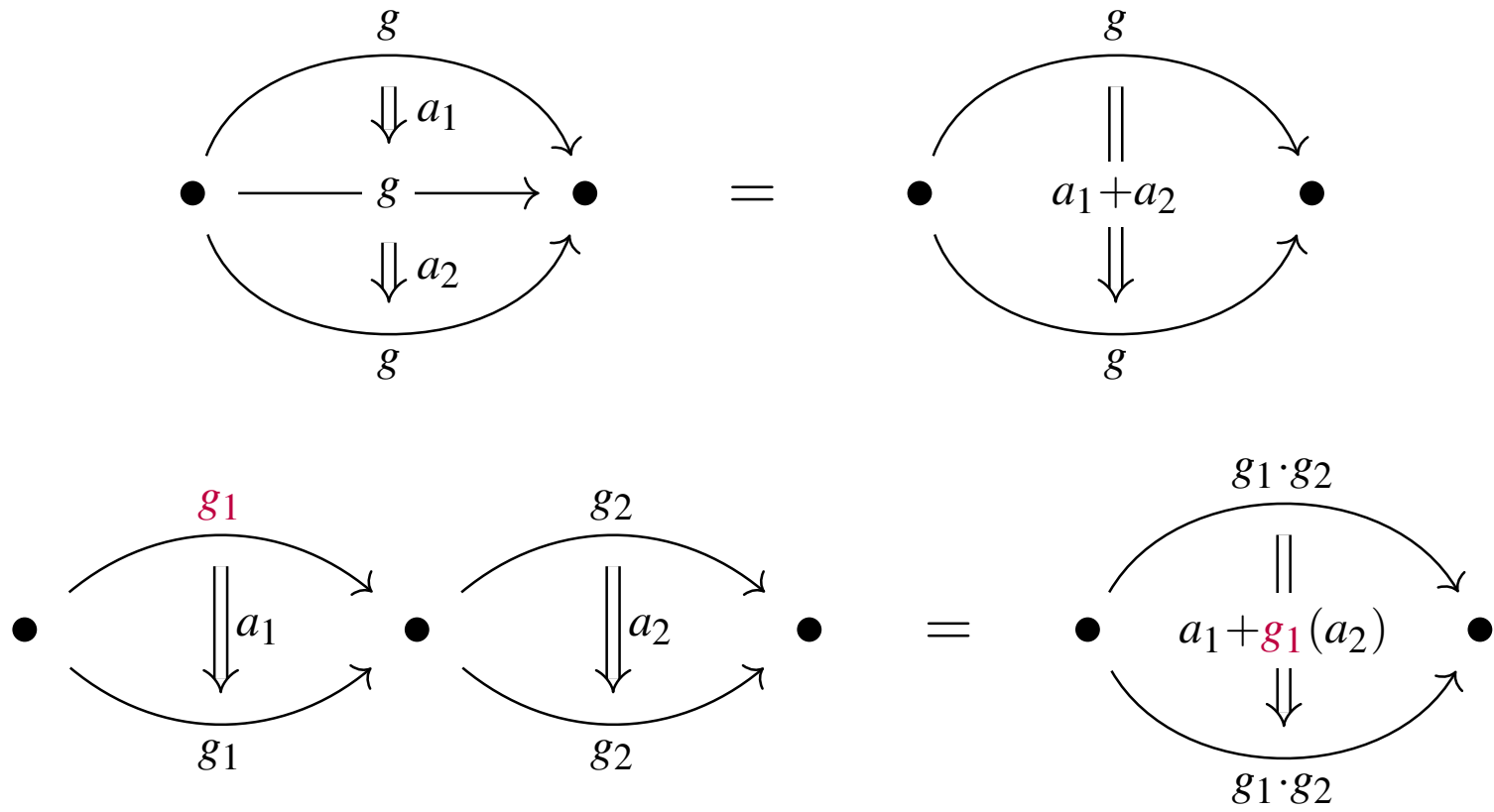


# 2-Groupoids – Examples.

For  $G \curvearrowright A$  is a linear action, i.e. by group automorphisms,

there is the delooping 2-groupoid  $\mathbf{B}(\underbrace{(\mathbf{B}A) \rtimes G}) \simeq (\mathbf{B}^2A) // G$

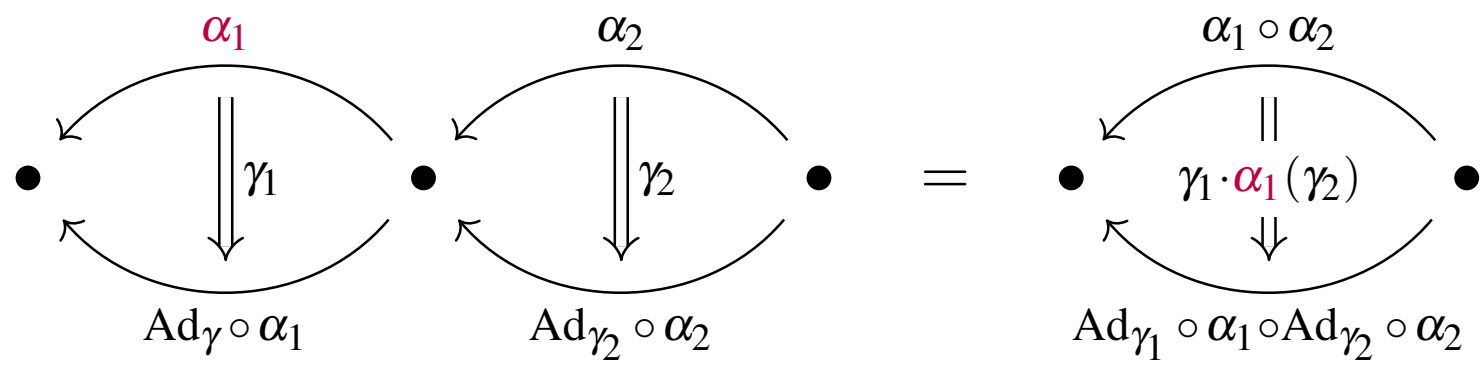
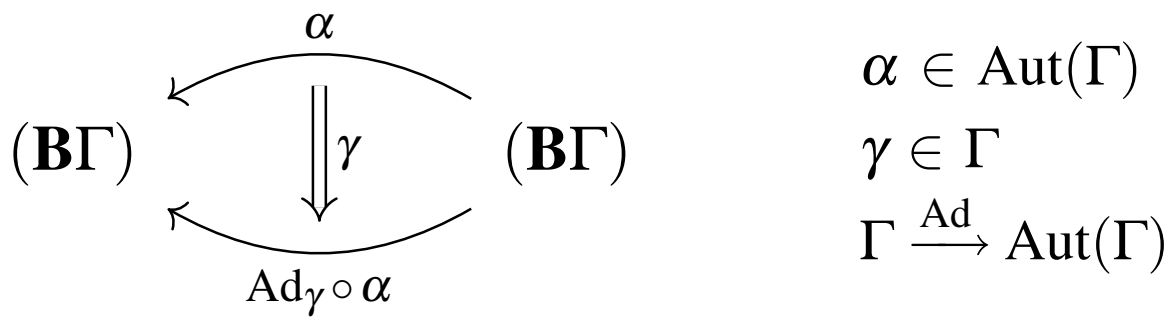
of the *semidirect product 2-group*:



# 2-Groupoids – Examples.

This is a special case of the delooping of the *automorphism 2-group* of a group  $\Gamma$ :

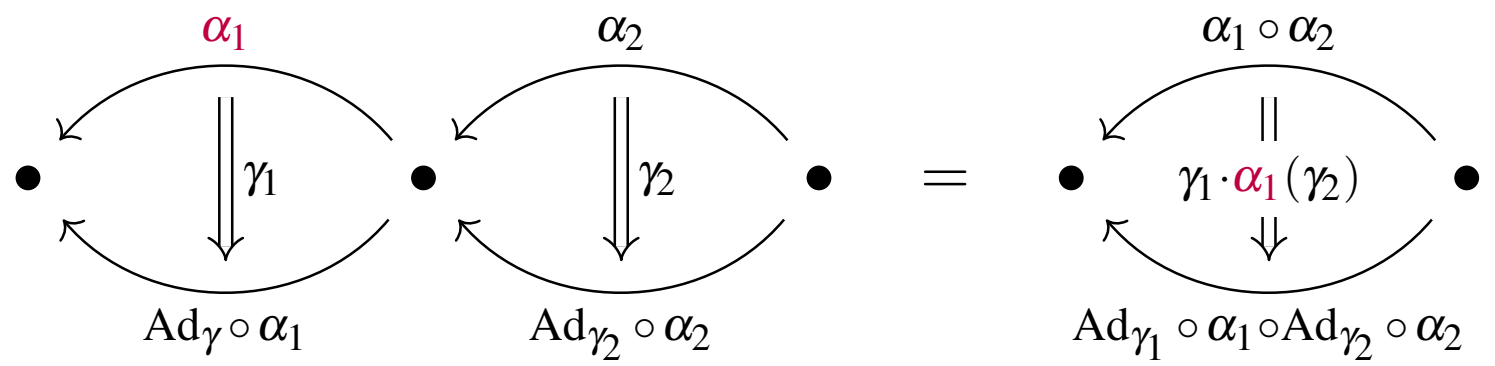
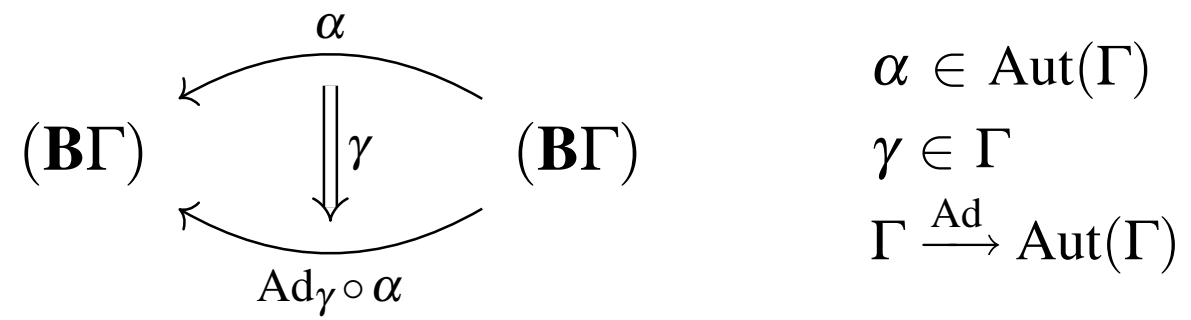
$$\mathbf{B}\left(\text{Aut}(\mathbf{B}\Gamma)\right) = \mathbf{B}\left(\overbrace{\text{Aut}(\Gamma) // \Gamma}\right)$$



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NB: Always need to choose whether actions are right- or left-actions, hence whether group multiplication is opposite or aligned to arrow composition. Before long we want *structure groups* to act *from the left* and *equivariance groups* to act *from the right*.

## 2-Groupoids – Examples.

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Notice:

(1)  $(\mathbf{BA}) \rtimes G$  is a non-abelian 2-group iff  $G$  is a non-abelian group;

# 2-Groupoids – Examples.

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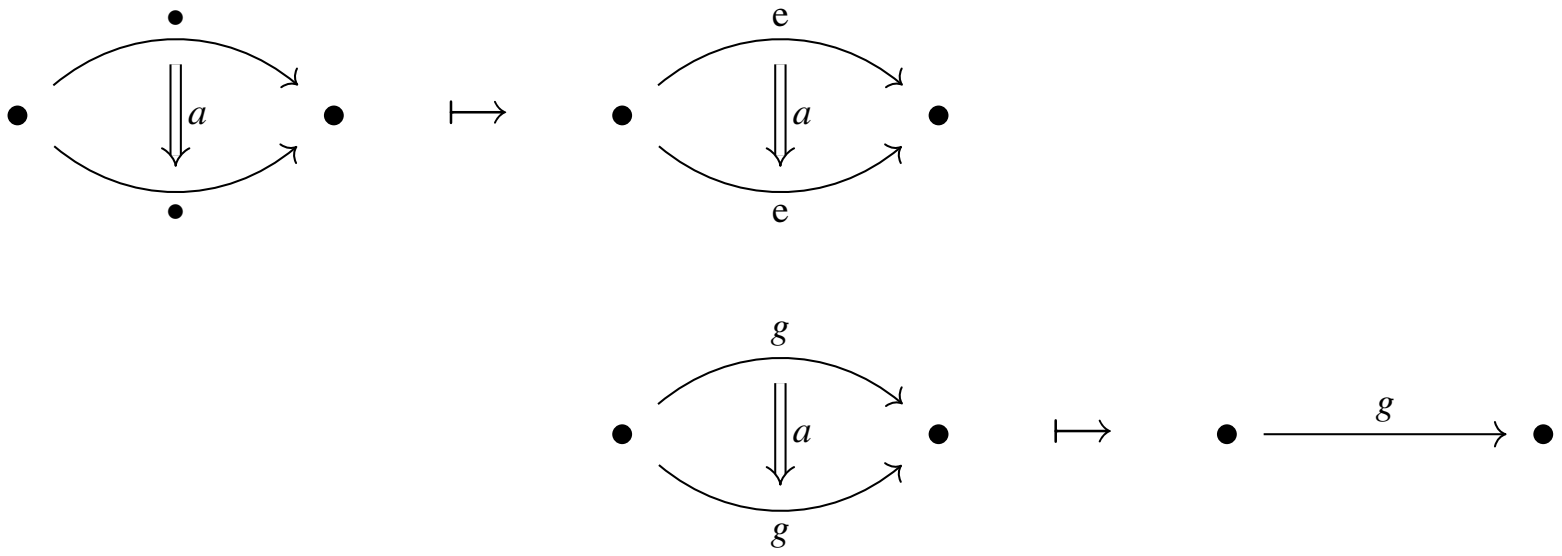
Notice:

(1)  $(\mathbf{B}A) \rtimes G$  is a non-abelian 2-group iff  $G$  is a non-abelian group;

(2) its delooping sits in this fiber sequence:

$$\mathbf{B}^2 A \xrightarrow{\text{fib}(p)} \mathbf{B}((\mathbf{B}A) \rtimes G) \xrightarrow[p \in \text{KanFib}]{p} \mathbf{B}G$$

$\parallel$   
 $(\mathbf{B}^2 A) // G$



## **2-Groupoids – 2-Functors.**

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E.g.: if  $\mathbb{Z} \curvearrowright \mathbb{Z}_2$  by sign inversion, and  $G \xrightarrow{\sigma} \mathbb{Z}_2$  a homomorphism then

**2nd group cohomology** of  $G$  with coefficients in  $G \curvearrowright \mathbb{Z}$  is 2-functors:

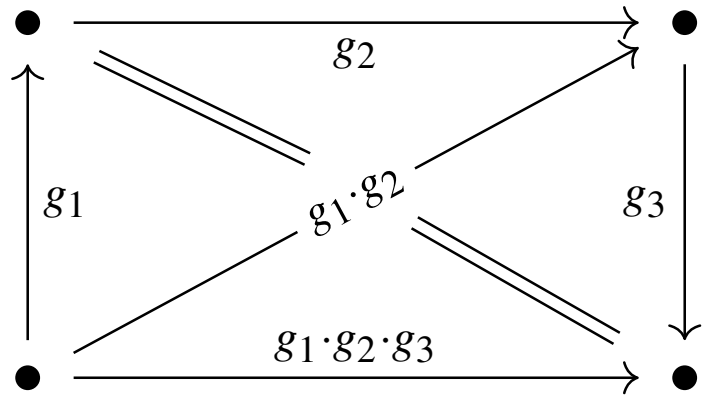
$$\begin{array}{ccc} \mathbf{B}G & \overset{\text{2-functor} = \text{2-cocycle}}{\dashrightarrow} & (\mathbf{B}^2 A) // \mathbb{Z}_2 \\ & \searrow \mathbf{B}\sigma & \swarrow \\ & \mathbf{B}\mathbb{Z}_2 & \end{array}$$

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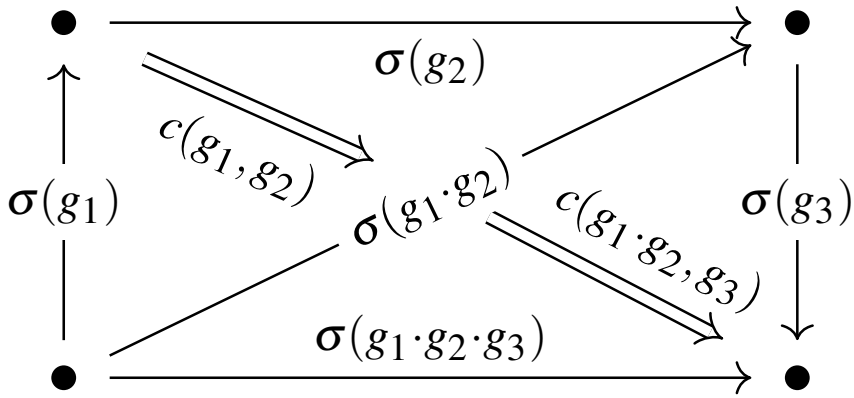
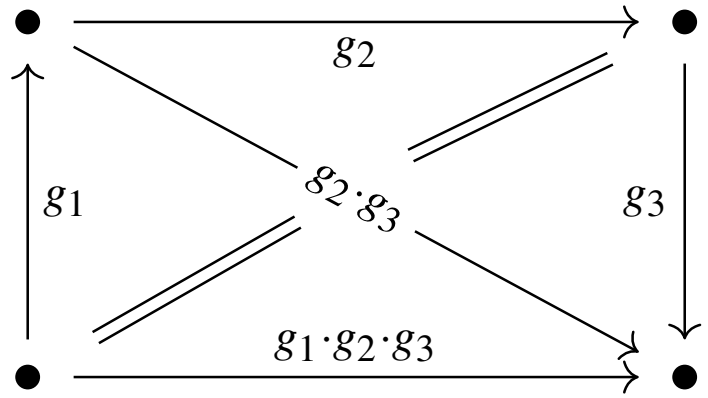
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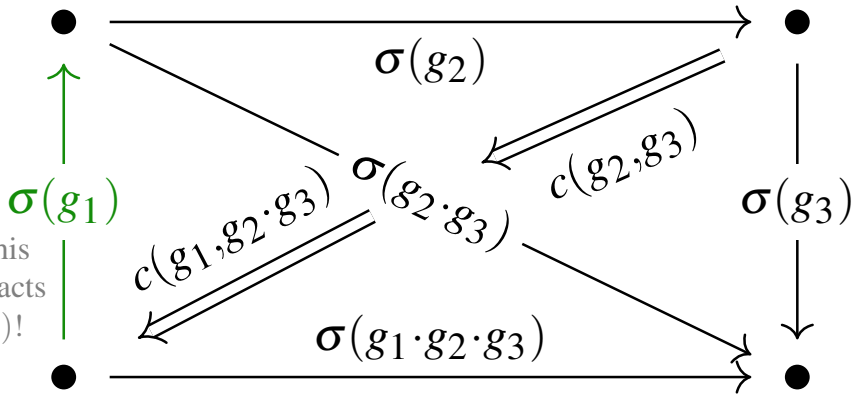
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 \end{array}$$



||| associativity



cocycle condition |||



recall that this 1-morphism acts on  $c(g_2, g_3)$ !

mapsto

## 2-Groupoids with smooth structure.

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A *smooth 2-groupoid*  $\mathcal{X}$  is given by a rule

which to each chart  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$  assigns the plain 2-groupoid  $\text{Probe}(\mathbb{R}^n, \mathcal{X})$

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So  $\text{Probe}(*, \mathcal{X}) = \text{Probe}(\mathbb{R}^0, \mathcal{X})$  is the underlying 2-groupoid

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Grothendieck (1965): “functorial geometry”

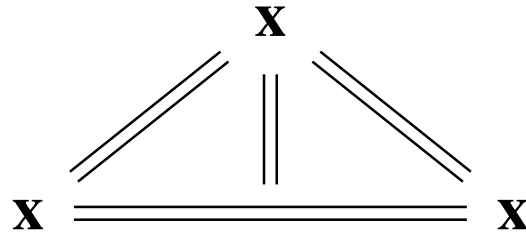
common jargon: “pre-2-stacks on the site of Cartesian spaces”

## 2-Groupoids with smooth structure – Examples.

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If  $X$  is a smooth manifold, then as a smooth 2-groupoid it's this assignment:

$$\mathbf{X} : \mathbb{R}^n \mapsto \text{Probe}(\mathbb{R}^n, \mathbf{X}) := C^\infty(\mathbb{R}^n, \mathbf{X})$$



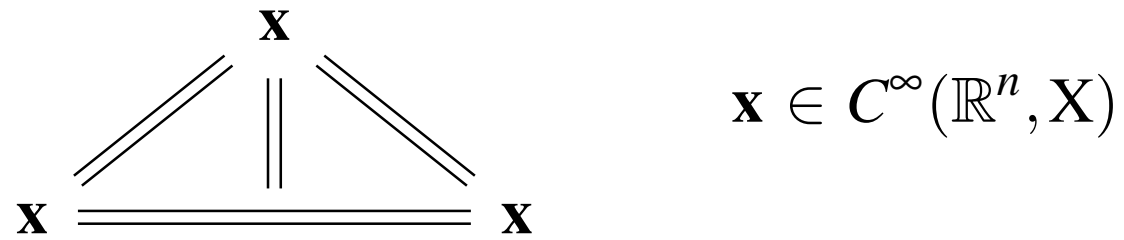
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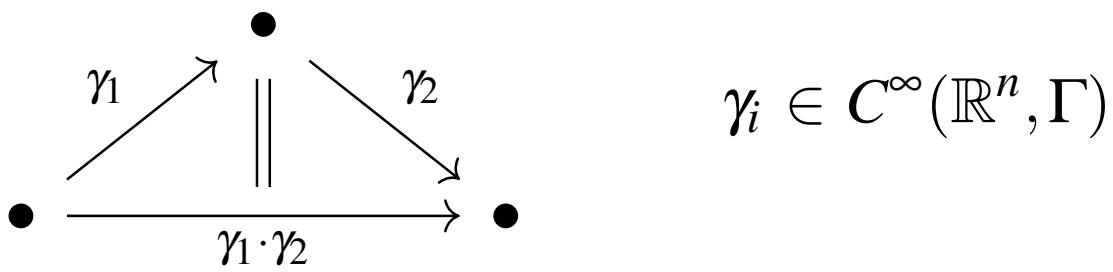
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If  $\Gamma$  a *Lie* group, then the sets of smooth functions  $C^\infty(\mathbb{R}^n, \Gamma)$  are plain groups, and the *smooth delooping groupoid*  $\mathbf{B}\Gamma$  is:

$$\mathbf{B}\Gamma : \mathbb{R}^n \mapsto \text{Probe}(\mathbb{R}^n, \mathbf{B}\Gamma) := \mathbf{B}(C^\infty(\mathbb{R}^n, \Gamma))$$





## **2-Groupoids with smooth structure – As smooth homotopy types.**

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A smooth 2-functor  $\mathcal{X} \xrightarrow{f} \mathcal{Y}$  is called:

PrjFib	<i>projective fibration</i>	iff for each $\mathbb{R}^n$ , every $k + 1$ -morphism in $\text{Probe}(\mathbb{R}^n, \mathcal{Y})$ that starts at $k$ -morphisms which come from $\text{Probe}(\mathbb{R}^n, \mathcal{X})$ lifts compatibly to a $k + 1$ -morphism in $\text{Probe}(\mathbb{R}^n, \mathcal{X})$
LWEq	<i>local weak equivalence</i>	iff for every $\mathbb{R}^n$ there exists an open ball $0 \in \mathbb{D}_\varepsilon^n \xrightarrow{i} \mathbb{R}^n$ such that $\text{Probe}(\mathbb{R}^n, f) _i$ is a weak homotopy equivalence namely an iso on the evident homotopy groups
PrjCof	<i>projective cofibration</i>	if (Dugger's sufficient condition): for all $k$ , the spaces of $k$ -morphisms are disjoint unions of charts $\mathbb{R}^n$ (for any $n$ -s)

# 2-Groupoids with smooth structure – As smooth homotopy types.

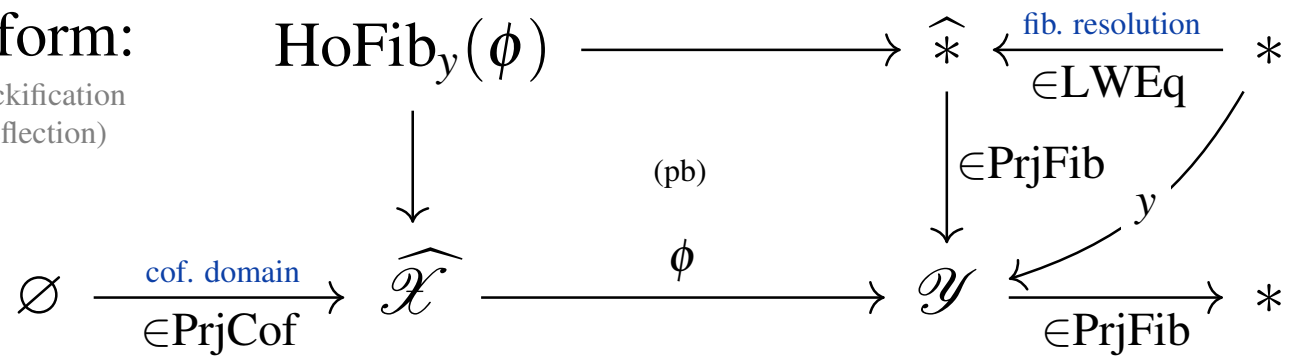
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Fact/Def.: Maps  $\phi$  of 2-stacks and their *homotopy fibers* are modeled by pullbacks of this form:

(because 2-stackification is an  $\infty$ -lex reflection)



# 2-Groupoids with smooth structure – Homotopy fiber sequences.

---

$$\begin{array}{ccccccc}
 U_1 & \hookrightarrow & \Gamma & \twoheadrightarrow & \Gamma/U_1 & \text{smooth group} & \\
 & & \text{locally trivial} & & & & \\
 & & \text{circle-extension} & & & & \\
 & & & & \uparrow \in \text{LWEq} & & \\
 & & \Gamma // U_1 & \xrightarrow{\in \text{PrjFib}} & \mathbf{BU}_1 & \longrightarrow & \mathbf{B}\Gamma & \longrightarrow & \mathbf{B}(\Gamma/U_1) \\
 & & & & & & \uparrow \in \text{LWEq} & & \\
 & & & & & & \mathbf{B}\Gamma // \mathbf{BU}_1 & \xrightarrow{\text{2-cocyle}} & \mathbf{B}^2U_1 \\
 & & & & & & & \text{classifying} & \\
 & & & & & & & \text{the extension} & 
 \end{array}$$

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classifying  
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$$\begin{array}{ccc} c & \gamma & [\gamma] \\ \parallel & \parallel & \parallel \\ c & \gamma & [\gamma] \end{array}$$

$\uparrow$

$$\begin{array}{ccccccc} \gamma & & \bullet & & \bullet & & \bullet \\ c \downarrow & \mapsto & c \downarrow & \mapsto & \gamma \downarrow & \mapsto & [\gamma] \downarrow \\ c \cdot \gamma & & \bullet & & \bullet & & \bullet \\ & & & & & & \uparrow \end{array}$$

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## 2-Groupoids with smooth structure – Homotopy fiber sequences.

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For example, write  $U_n$ ,  $n \in \mathbb{N} \sqcup \{\omega\}$

for the unitary group on a countably-dimensional complex Hilbert space and regard this as a smooth group by its “continuous diffeology”:

$$U_n : \mathbb{R}^k \mapsto \text{Probe}(\mathbb{R}^k, U_n) := C^0(\mathbb{R}^k, U_n)$$

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Then we have the following long fiber sequence of smooth 2-groupoids:

$$\begin{array}{ccccccc}
 U_1 & \hookrightarrow & U_n & \longrightarrow & \mathbf{P}U_n & & \\
 & & & & \uparrow \in \text{LWEq} & & \\
 & & U_n // U_1 & \longrightarrow & \mathbf{B}U_1 & \longrightarrow & \mathbf{B}U_n & \longrightarrow & \mathbf{B}P U_n \\
 & & & & & & & & \uparrow \in \text{LWEq} \\
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 & & & & & & & & & & & & \uparrow \in \text{LWEq} & \\
 & & & & & & & & & & & & \mathbf{BU}_n // \mathbf{BU}_1 & \longrightarrow & \mathbf{B}^2U_1
 \end{array}$$

This is all compatible with complex conjugation, so that there is a map like this:

$$\mathbf{BPU}_n // \mathbb{Z}_2 \xleftarrow{\in \text{LWEq}} \longrightarrow \mathbf{B}^2U_1 // \mathbb{Z}_2$$



## 2-Groupoids with smooth structure – Čech groupoids.

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For  $X$  a smooth manifold with  $\{U_i \hookrightarrow X\}_{i \in I}$  a good open cover, in that

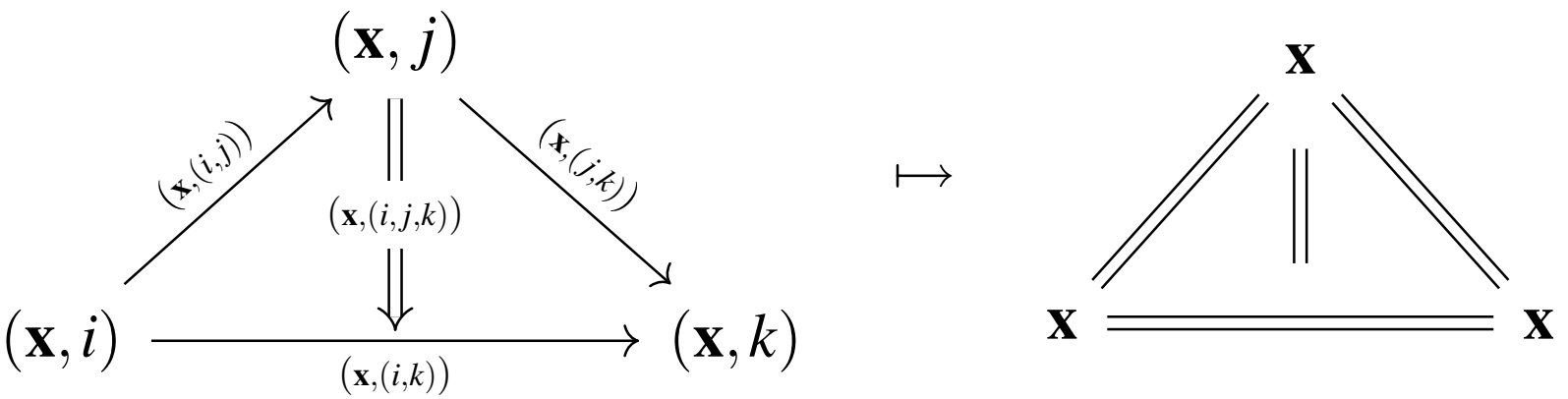
$$\left(\mathbf{x}, (i_1, \dots, i_n)\right) \in C^\infty\left(\mathbb{R}^m, U_{i_1} \cap \dots \cap U_{i_n}\right) \quad \Rightarrow \quad U_{i_1} \cap \dots \cap U_{i_n} \underset{\text{diff}}{\simeq} \mathbb{R}^{\dim(X)},$$

# 2-Groupoids with smooth structure – Čech groupoids.

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we have the smooth Čech 2-groupoid:



which is a *projectively cofibrant resolution* of  $X$ .

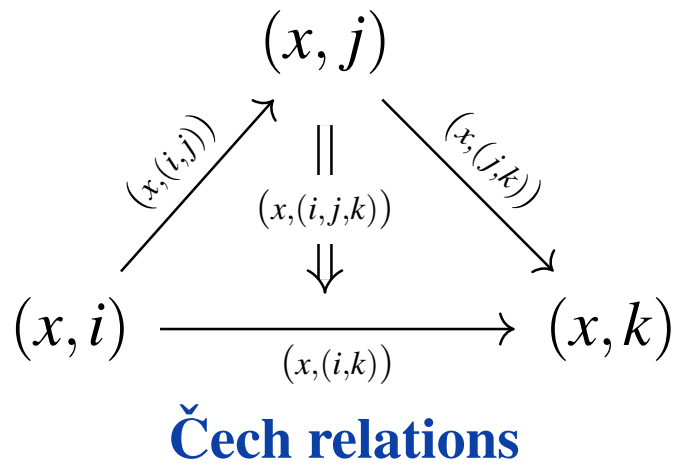
# 2-Groupoids with smooth structure – Čech cocycles.

Smooth 2-functors from such a Čech resolution  $\widehat{X} \rightarrow X$

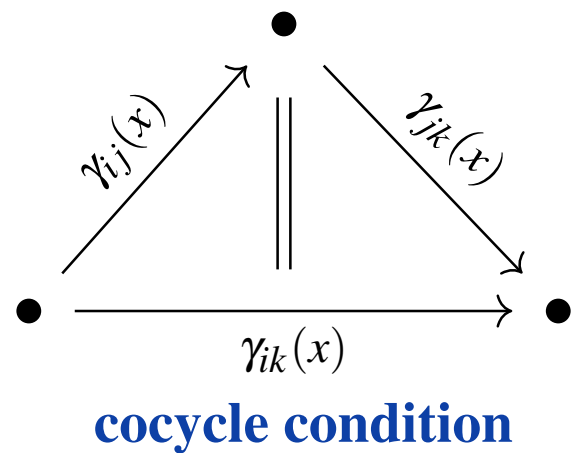
to the delooping  $\mathbf{B}\Gamma$  of a Lie group

are *cocycles* in the Čech cohomology of  $X$  with coefficients in  $\Gamma$ :

$$\widehat{X} \xrightarrow{\text{smooth functor} = \check{\text{Cech cocycle}}} \mathbf{B}\Gamma$$

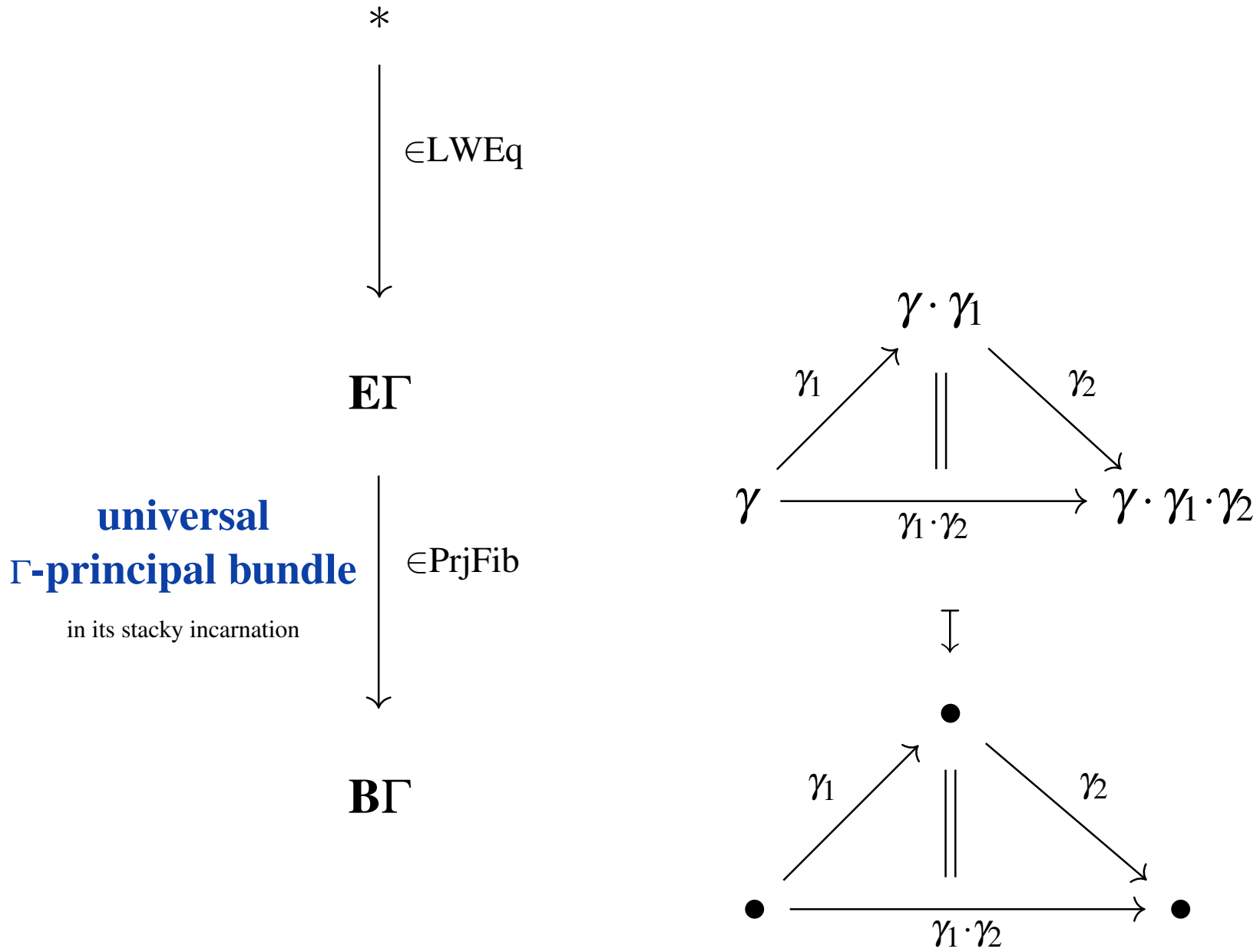


$\mapsto$



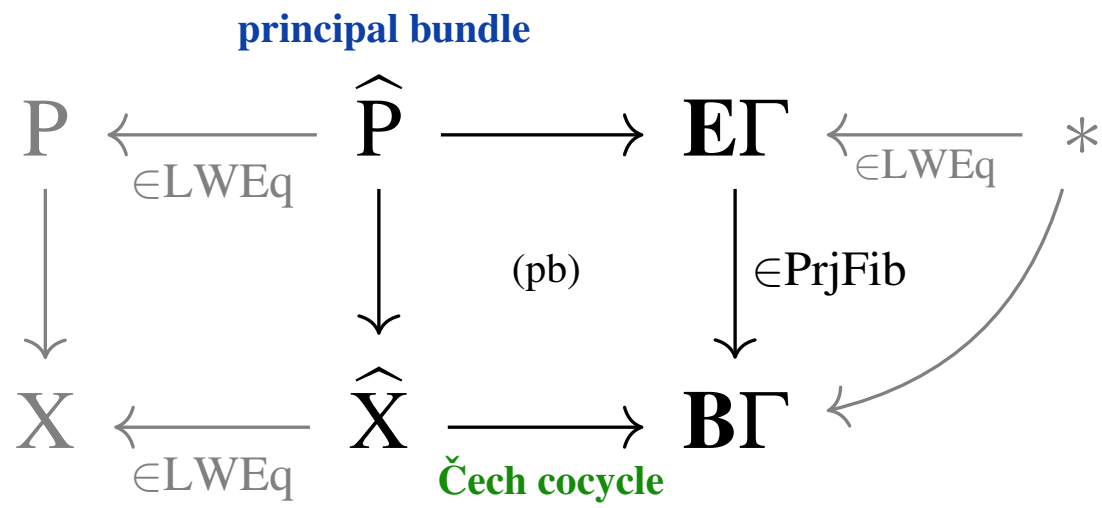
# Principal bundles via smooth groupoids – Universal principal bundles.

The inclusion of the unique base point into  $\mathbf{B}\Gamma$  has the following *fibrant resolution*:



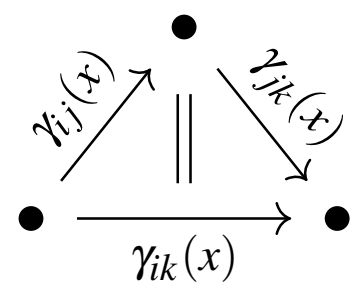
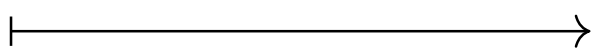
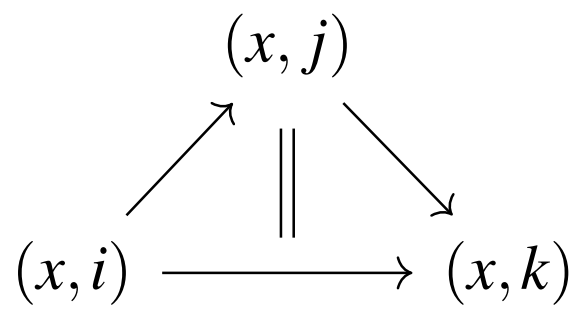
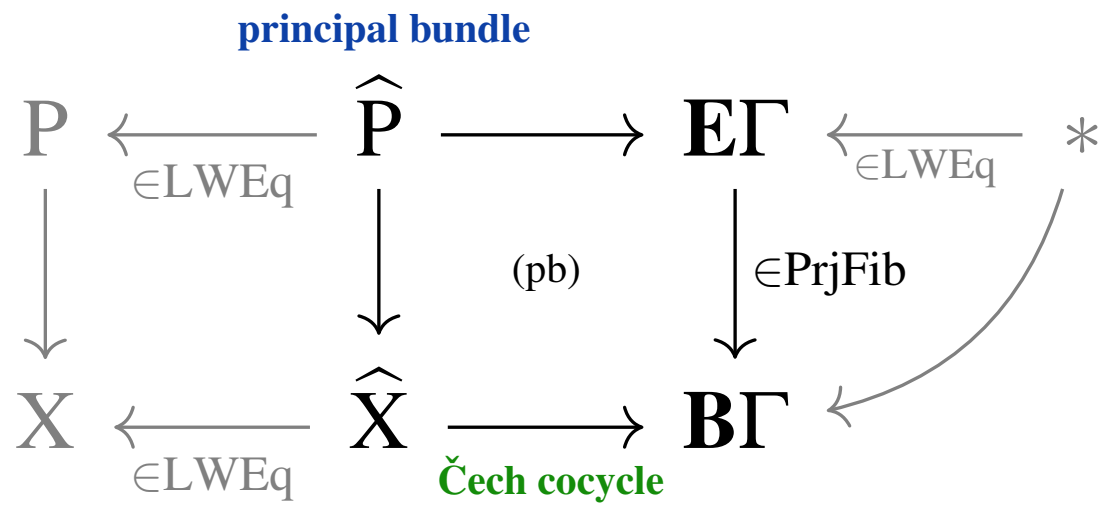
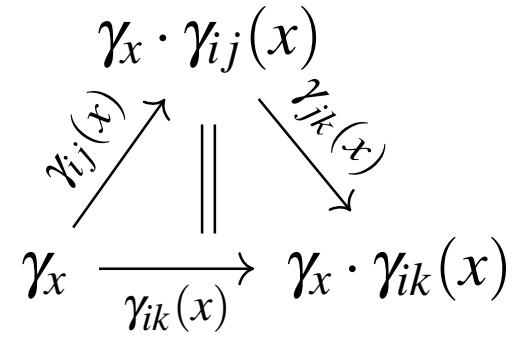
# Principal bundles via smooth groupoids.

The *homotopy fiber* of a 2-functor = Čech cocycle is equivalently *the principal bundle P it classifies*:



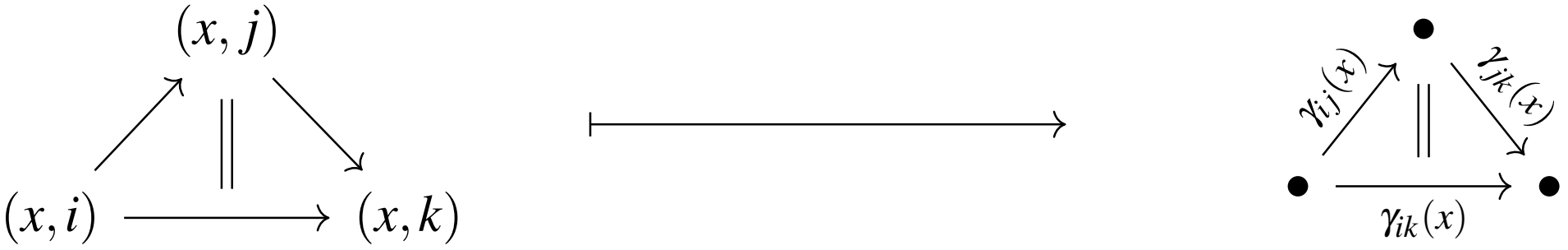
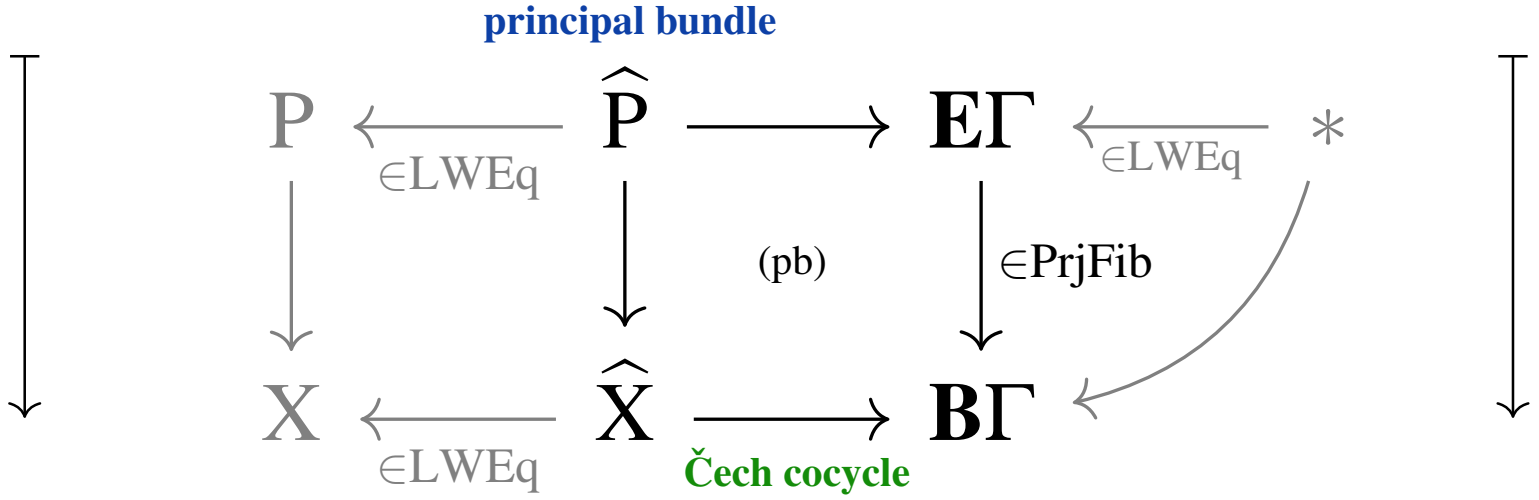
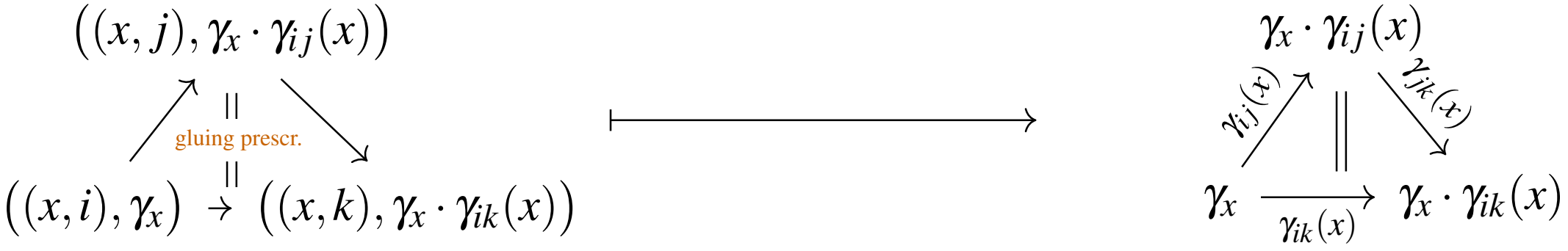
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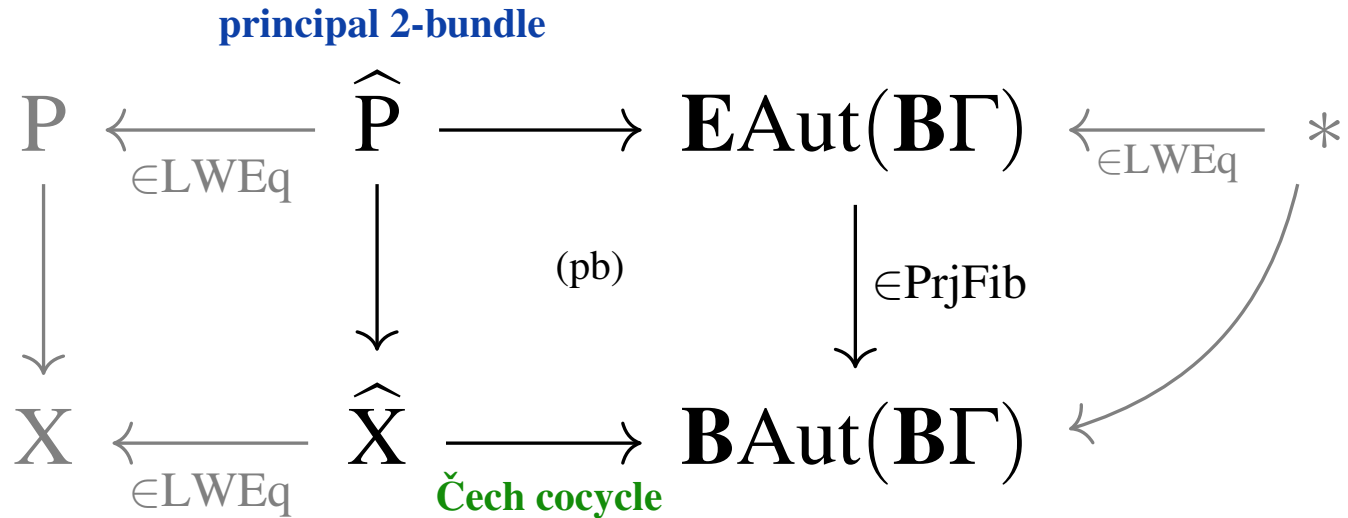


# Principal 2-bundles via smooth 2-groupoids.

This neat formulation of ordinary principal bundles immediately generalizes to give principal 2-bundles:

E.g. for the structure 2-group  $\mathbf{Aut}(\mathbf{B}\Gamma)$

these are equivalently Giraud's *non-abelian gerbes*:





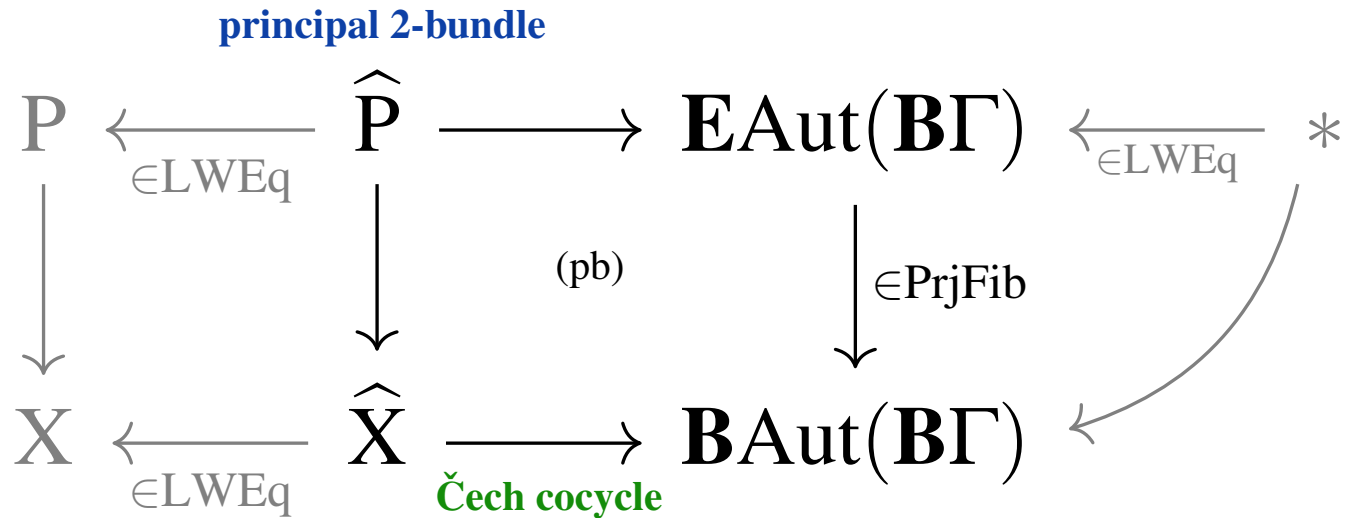
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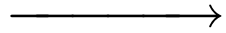
E.g. for the structure 2-group  $\mathbf{Aut}(\mathbf{B}\Gamma)$

these are equivalently Giraud's *non-abelian gerbes*:



While it's tradition to be esoteric about this simple affair,

here to highlight that this is really about *twisted cohomology*:

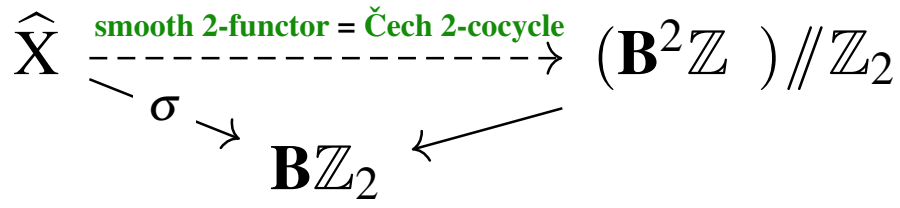


**Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.**

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For structure 2-group  $\mathbf{Aut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}\mathbb{Z}) \rtimes \mathbb{Z}_2$ , with  $\mathbf{BAut}(\mathbf{B}\mathbb{Z}) \simeq (\mathbf{B}^2\mathbb{Z}) // \mathbb{Z}_2$  and  $\widehat{X} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$  a double covering, then

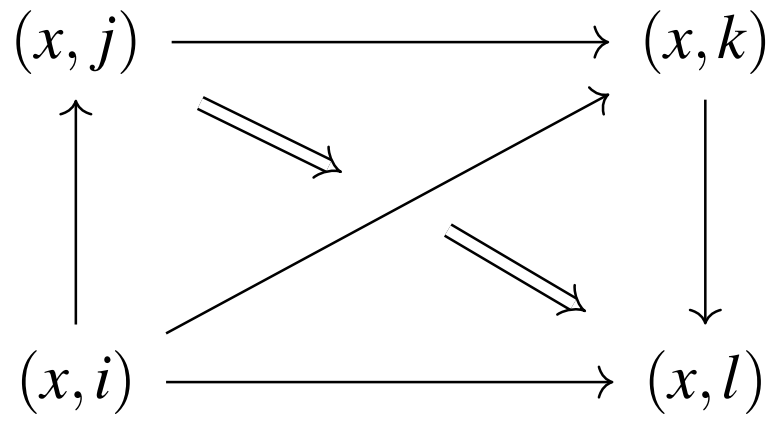
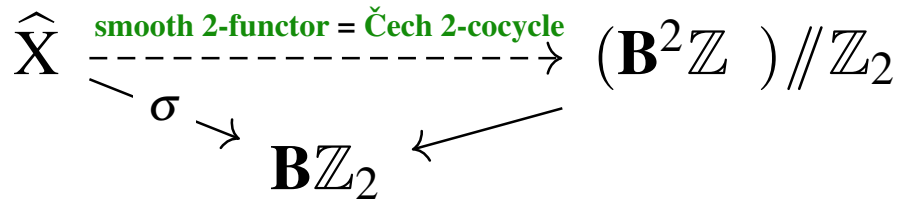
**2nd integral cohomology** of  $X$  with local coefficients is smooth 2-functors:



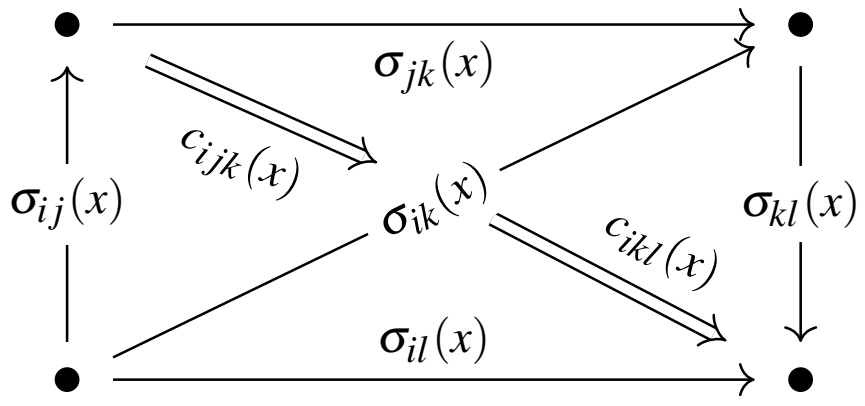
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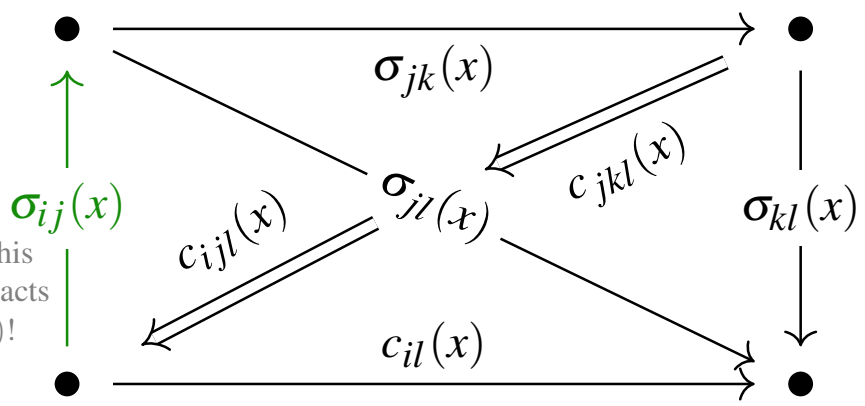
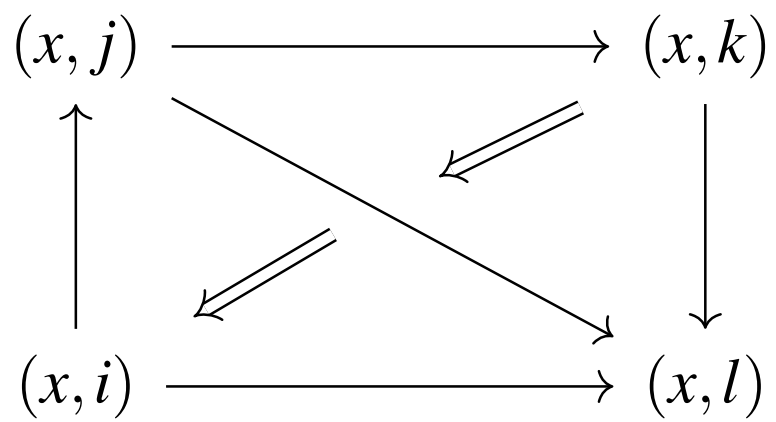
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||| Čech relations



||| cocycle condition |||



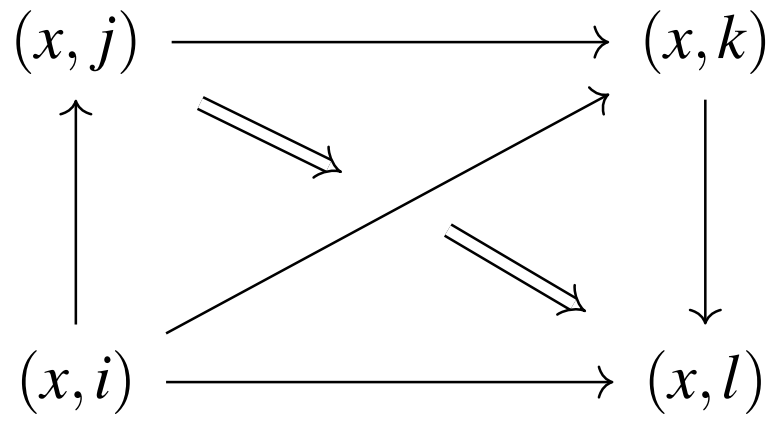
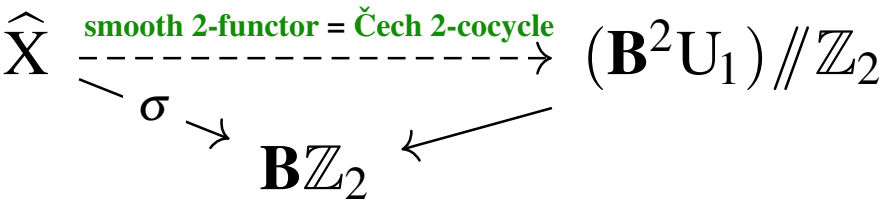
recall that this 1-morphism acts on  $c_{jkl}(x)$ !

mapsto

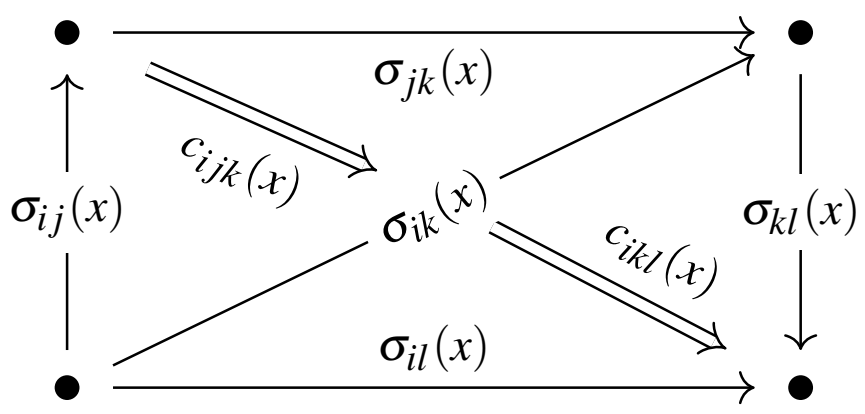
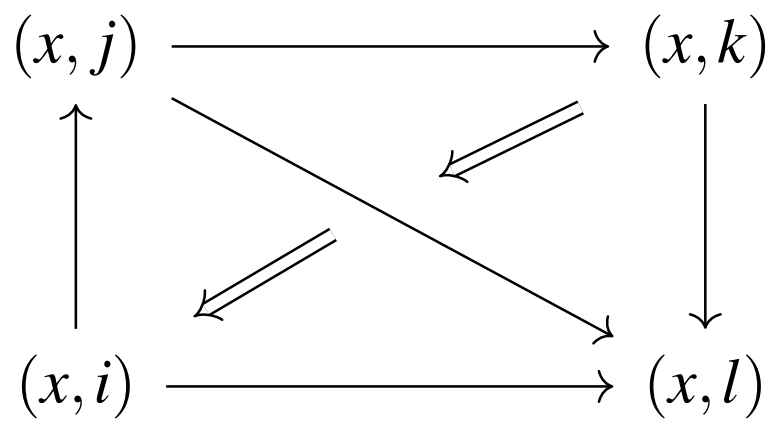
# Principal 2-bundles via smooth 2-groupoids – Example: Jandl gerbes.

For structure 2-group  $\text{Aut}(\mathbf{BU}_1) \simeq (\mathbf{BU}_1) \rtimes \mathbb{Z}_2$ , with  $\mathbf{BAut}(\mathbf{BU}_1) \simeq (\mathbf{B}^2\mathbf{U}_1) // \mathbb{Z}_2$  and  $\widehat{X} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$  a double covering, then

**2nd  $U_1$ -valued cohomology** of  $X$  with local coefficients is smooth 2-functors:

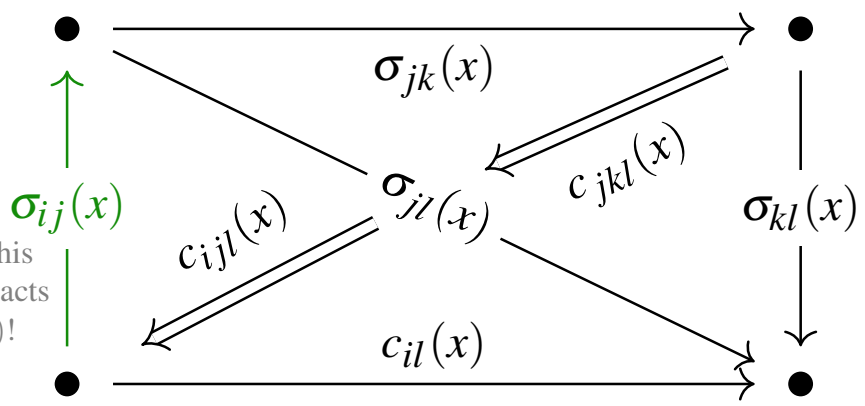


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## Principal 2-bundles via smooth 2-groupoids – Punchline.

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So:

Non-abelian 1-cohomology is modulated by 1-stacks  $\mathbf{B}\Gamma$ ,  
abelian 2-cohomology is modulated by 2-stacks  $\mathbf{B}^2A$ , etc.

Higher fiber/principal bundles are *bundles of such moduli stacks*,  
hence are families of moduli stacks that vary over the base space,  
hence locally modulate cohomology as before,  
but now subject to global twists.

## Principal 2-bundles via smooth 2-groupoids – Equivariance.

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Finally, the higher topos of smooth 2-groupoids

has *equivariance* natively built into it: just let domain spaces be groupoids, too.

# Principal 2-bundles via smooth 2-groupoids – Equivariance.

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has *equivariance* natively built into it: just let domain spaces be groupoids, too:

For  $X \curvearrowright G$  a smooth action of a finite group on a smooth manifold.

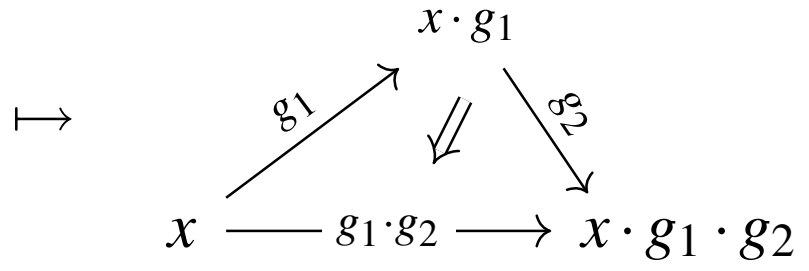
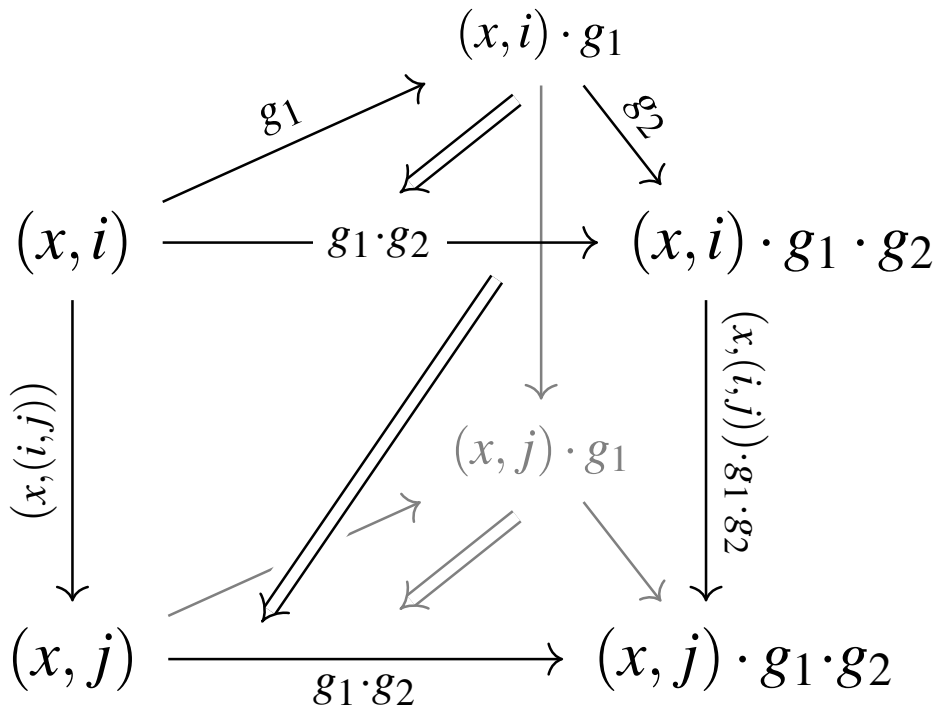
there exists an *equivariant good open cover*

$$\coprod_{i \in I} \overset{G}{U}_i \longrightarrow X$$

and its *equivariant Čech groupoid*:

$$\emptyset \xrightarrow{\in \text{PrjCof}} \widehat{X // G} \xrightarrow[\in \text{LWEq}]{\text{cofibrant resolution}} X // G$$

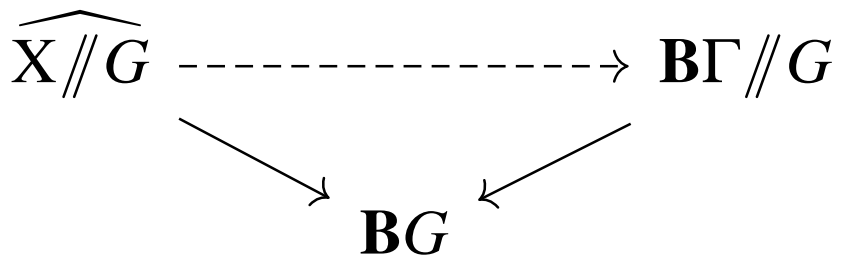
*action groupoid*



# Principal 2-bundles via smooth 2-groupoids – Equivariance.

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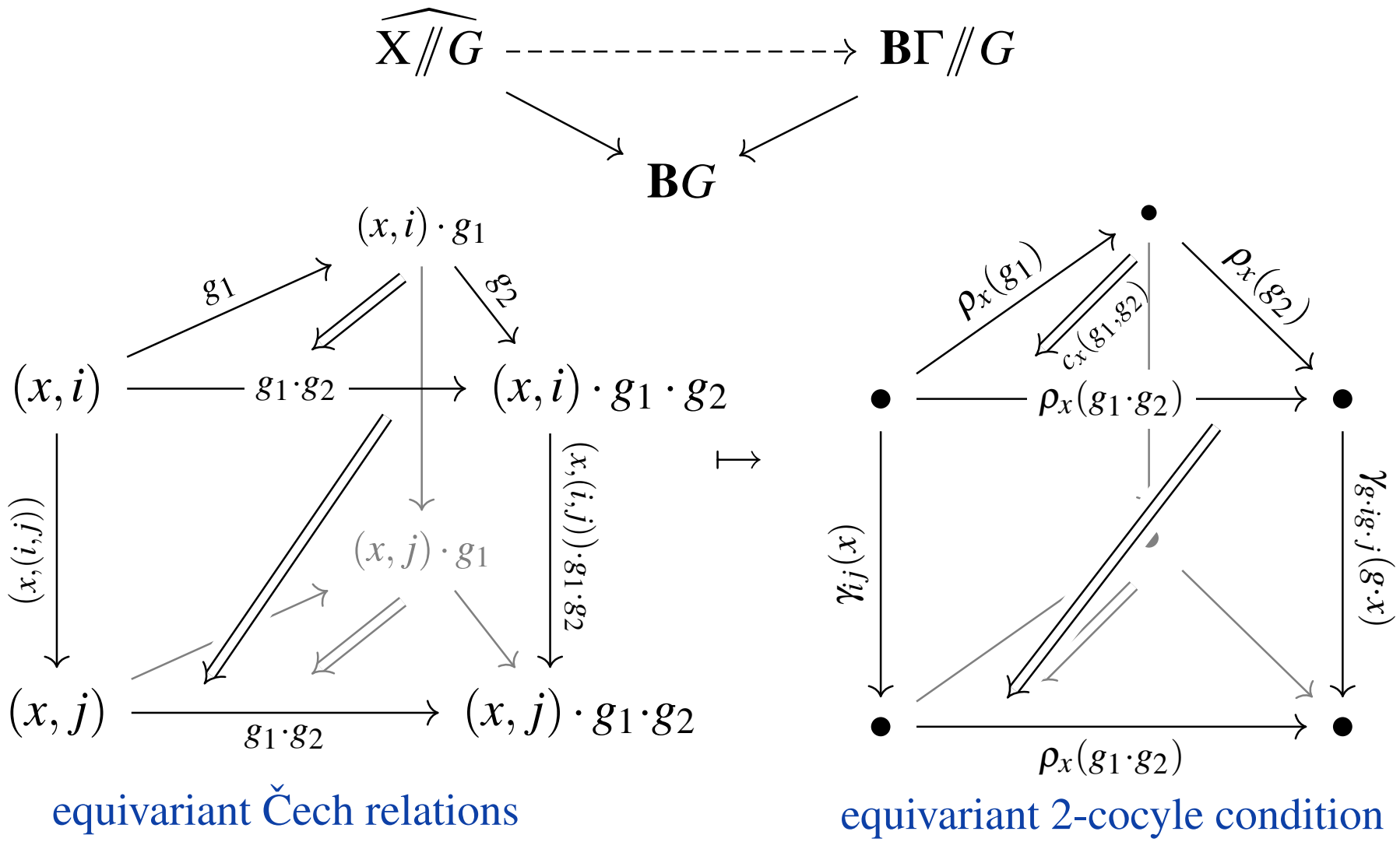
For  $X \curvearrowright G$  a smooth manifold and  $(\Gamma // C) \curvearrowright G$  a smooth 2-group both equipped with smooth  $G$ -action, a  $G$ -equivariant  $\Gamma$ -principal 2-bundle on  $X$  is modulated by a smooth 2-functor like this:





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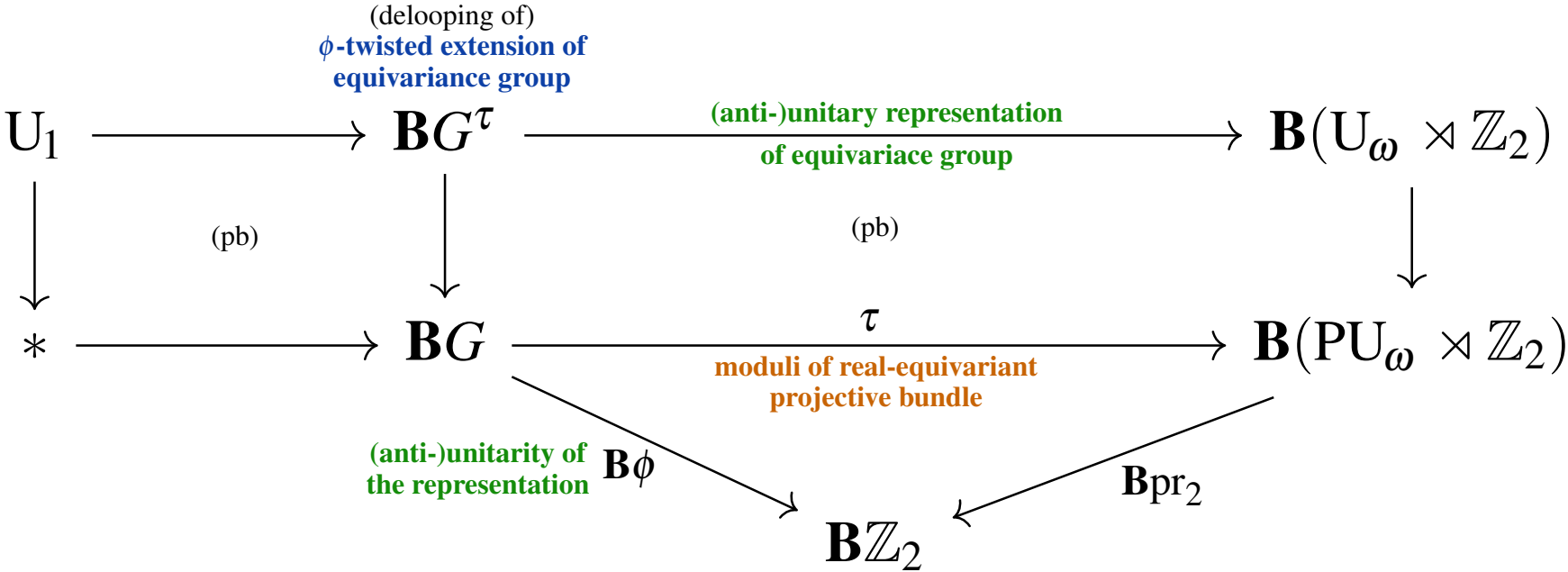


# Principal 2-bundles via smooth 2-groupoids – Equivariant examples.

E.g. an equivariant  $PU_\omega$ -bundle

over the point, where  $* \widehat{\parallel} G = \mathbf{B}G$ ,

is a projective  $G$ -representation:

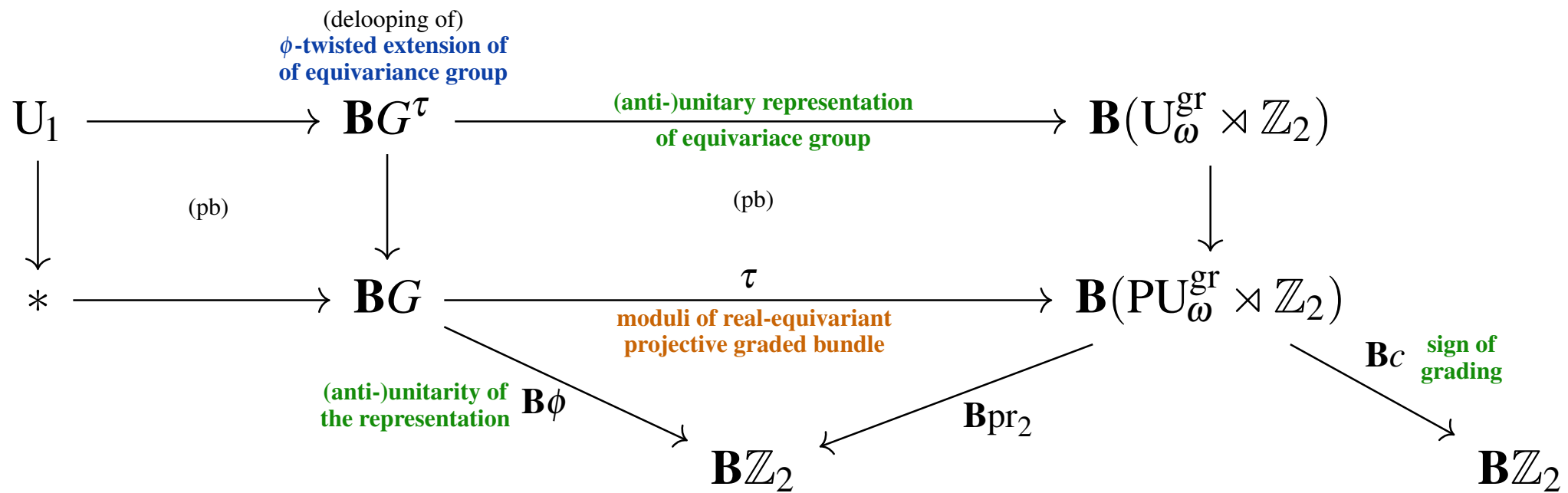


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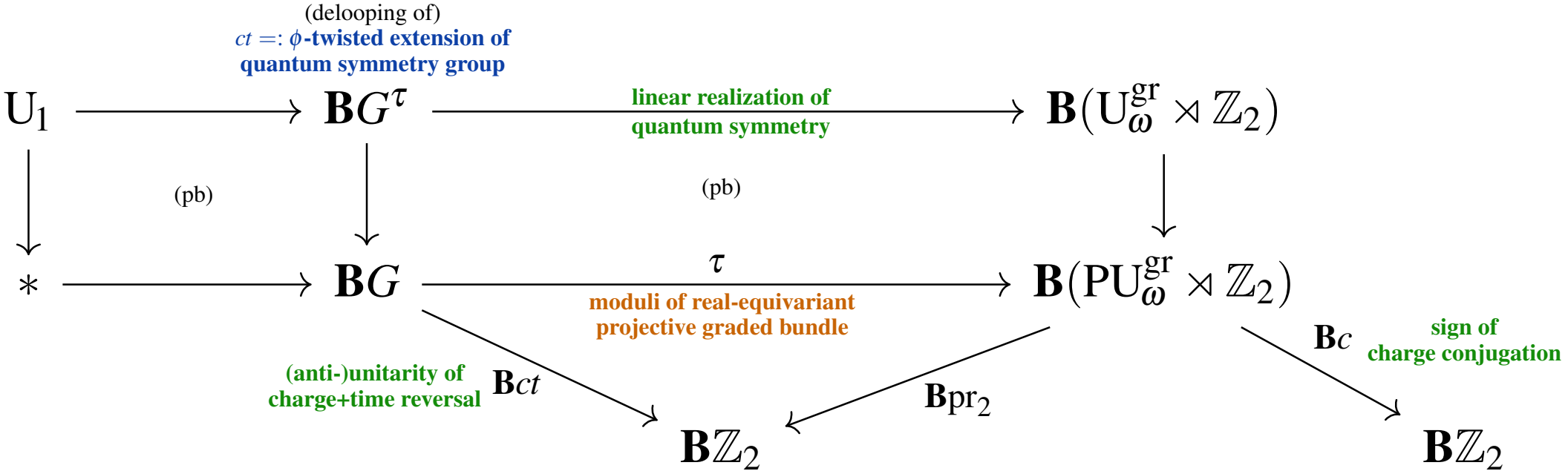


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This happens to encode all about *quantum symmetries of gapped systems* (cf. Freed & Moore 2013 , good review in Thiang 2018, §4, ).

Part I – Invitation

Part II – Application

# Part II – Application

which provides a brief outlook on  
how the above technology gives  
a transparent construction of  
twisted equivariant KR-theory.

# Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

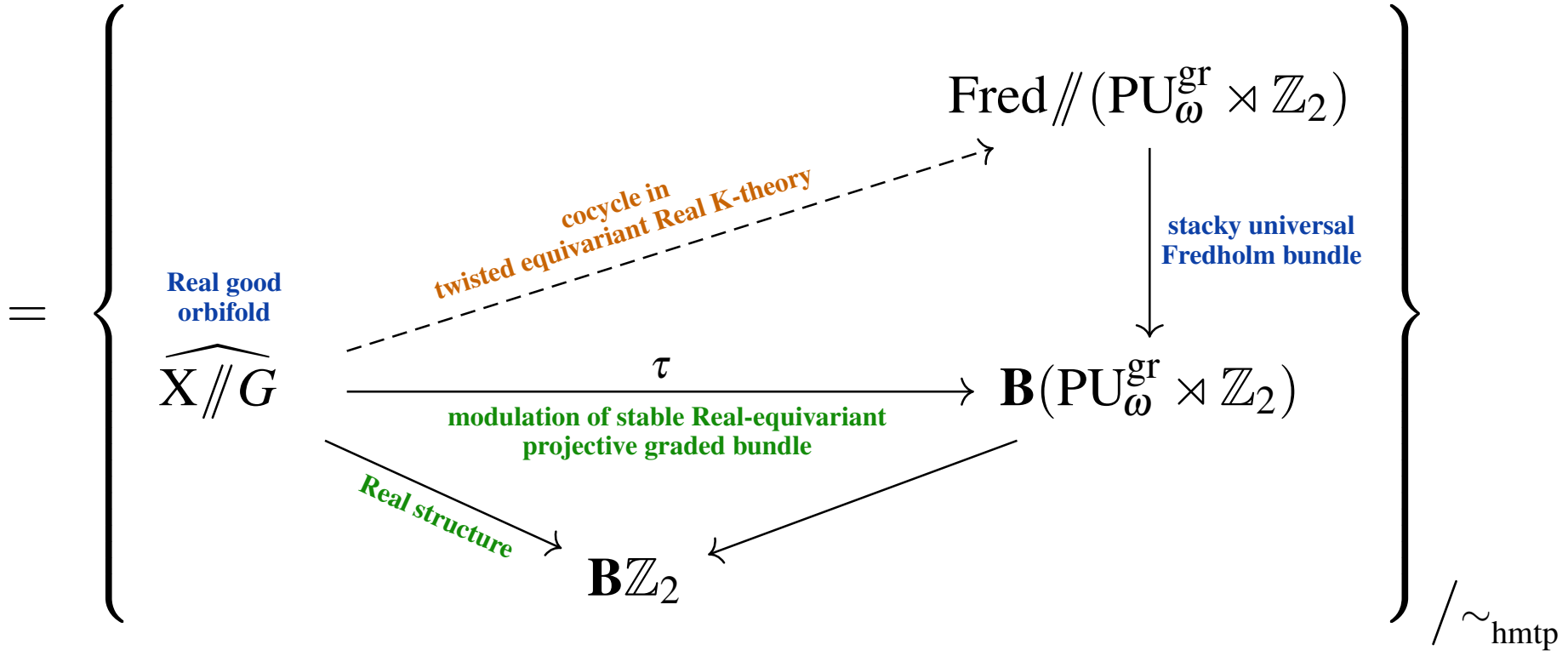
The “smooth” (namely continuous-diffeological) group  $\text{PU}_\omega^{\text{gr}}$  canonically acts on the “smooth” space Fred of Fredholm operators on a  $\mathbb{Z}_2$ -graded Hilbert space.

Sections of the corresponding *associated equivariant bundles* are cocycles for *twisted equivariant Real K-theory* (generalizing Pavlov 2014, §3.19):

twisted equivariant  
KR-cohomology

connected components of shape of sliced mapping stack

$$\text{KR}_G^\tau(X) := \tau_0 \int \text{Map} \left( (X // G, \tau), \text{Fred} // (\text{PU}_\omega^{\text{gr}} \rtimes \mathbb{Z}_2) \right)_{\mathbf{B}(\text{PU}_\omega^{\text{gr}} \rtimes \mathbb{Z}_2)}$$



# Twisted equivariant KR-theory – Outlook.

This transparent formulation serves to reveal that there is more quantum physics encoded in twisted equivariant KR-theory than has previously been uncovered.

To be discussed in:

H. Sati, & U. S.: *Anyonic Defect Branes in Twisted equivariant K-Theory*

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