# Higher and Equivariant Bundles

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talk via:

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Feza Gürsey Center for Math and Physics

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slides and pointers at: `ncatlab.org/schreiber/show/Higher+and+Equivariant+Bundles`

This talk is
a gentle exposition of
the most basic concept
underlying these articles:

| | |
|---|---|
| *Principal ∞-bundles* | [arXiv:1207.0248/49] |
| *Equivariant Principal ∞-bundles* | [arXiv:2112.13654] |
| *Proper Orbifold Cohomology* | [arXiv:2008.01101] |

following

| | |
|---|---|
| *Diff. Cohomology in a Cohesive ∞-Topos* | [arXiv:1310.7930] |

# Motivation, Overview, Summary and Outlook – in one single slide:

**Generalized Cohomology Theories** $\longleftrightarrow$ **Cohesive Higher Fiber Bundles**

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Here “generalized” subsumes “Whitehead-generalized cohomology” (↔ spectra) but goes further:

| **Cohomology** | ↔ | **Higher Bundles** |
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| non-abelian | ↔ | general fibers |
| twisted | ↔ | associated |
| differential | ↔ | cohesive |
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A major phenomenon/subtlety is that the last two aspects go hand-in-hand:

**Proper $G$-equivariance** corresponds to the **cohesive slice** over $\mathbf{B}G$,

while

**Borel equivariance** corresponds just to the **slice of shapes**.

# Part I – Invitation

Part II – Application

# Part I – Invitation

which walks you from scratch
through just the definition of
equivariant principal 2-bundles
with simple but key examples;

the main *claim* being that this is
the *good definition*™:
transparent, elegant, universal,
generalizable & indeed: practical.

## 2-Groupoids

2-Groupoids are the algebra of 2-dimensional pasting,
such that all composition is associative and invertible:

E.g. homotopy classes of surfaces $\Sigma$ rel boundary paths $\gamma$ in a topological space:

$$\begin{tikzcd}
x_1 \arrow[r, "\gamma"] \arrow[d] & y_1 \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "\Sigma"] & z_1 \arrow[d] \arrow[dl, Rightarrow] \\
x_2 \arrow[r] \arrow[d] & y_2 \arrow[r] \arrow[d] \arrow[dl, Rightarrow] & z_2 \arrow[d] \arrow[dl, Rightarrow] \\
x_3 \arrow[r] & y_3 \arrow[r] & z_3
\end{tikzcd}$$

# 2-Groupoids – Examples.

For $G$ a group, there is its *delooping 1-groupoid* $\mathbf{B}G$:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g_2"] \arrow[d, phantom, "\Vert"] & \\
\bullet \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2"'] & {} & \bullet
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For $A$ an *abelian* group there is the *double delooping 2-groupoid* $\mathbf{B}^2 A = \mathbf{B}\big(\overbrace{\mathbf{B}A}^{\text{“2-group”}}\big)$ :

$$\begin{tikzcd}
\bullet \arrow[rr, "\bullet"] \arrow[dd, "\bullet"'] & {} \arrow[d, Rightarrow, "a_1"'] & \bullet \arrow[dd, "\bullet"] \\
 & {} \arrow[d, Rightarrow, "a_2"] & \\
\bullet \arrow[uurr, "\bullet" description] \arrow[rr, "\bullet"'] & {} & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[rr, "\bullet"] \arrow[d, "\bullet"'] & {} \arrow[d, Rightarrow, "a_1+a_2"] & \bullet \arrow[d, "\bullet"] \\
\bullet \arrow[rr, "\bullet"'] & {} & \bullet
\end{tikzcd}
\qquad\qquad a_i \in A$$

# 2-Groupoids – Examples.

For $G \circlearrowright A$ is a linear action, i.e. by group automorphisms,

there is the delooping 2-groupoid $\mathbf{B}\Big(\underbrace{(\mathbf{B}A) \rtimes G}\Big) \;\simeq\; \big(\mathbf{B}^2 A\big) /\!\!/ G$

of the *semidirect product 2-group*:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"{name=U}] \arrow[r, "g"{name=M, description}] \arrow[r, bend right=60, "g"'{name=D}] & \bullet \arrow[from=U, to=M, Rightarrow, "a_1"] \arrow[from=M, to=D, Rightarrow, "a_2"] & = & \bullet \arrow[r, bend left=60, "g"{name=U2}] \arrow[r, bend right=60, "g"'{name=D2}] & \bullet \arrow[from=U2, to=D2, Rightarrow, "a_1+a_2" description]
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g_1"{name=U1}] \arrow[r, bend right=50, "g_1"'{name=D1}] & \bullet \arrow[from=U1, to=D1, Rightarrow, "a_1"] \arrow[r, bend left=50, "g_2"{name=U2}] \arrow[r, bend right=50, "g_2"'{name=D2}] & \bullet \arrow[from=U2, to=D2, Rightarrow, "a_2"] & = & \bullet \arrow[r, bend left=60, "g_1 \cdot g_2"{name=U3}] \arrow[r, bend right=60, "g_1 \cdot g_2"'{name=D3}] & \bullet \arrow[from=U3, to=D3, Rightarrow, "a_1+g_1(a_2)" description]
\end{tikzcd}$$

# 2-Groupoids – Examples.

This is a special case of the delooping of the *automorphism 2-group* of a group $\Gamma$:

$$\mathbf{B}\Big(\mathrm{Aut}\big(\mathbf{B}\Gamma\big)\Big) \;=\; \mathbf{B}\big(\overbrace{\mathrm{Aut}(\Gamma)/\!\!/\Gamma}\big)$$

$$\begin{tikzcd}[column sep=huge]
(\mathbf{B}\Gamma) & (\mathbf{B}\Gamma) \arrow[l, bend right=40, "\alpha"', ""{name=U, below}] \arrow[l, bend left=40, "\mathrm{Ad}_\gamma \circ \alpha", ""{name=D, above}]
\arrow[Rightarrow, from=U, to=D, "\gamma"]
\end{tikzcd}$$

$\alpha \in \mathrm{Aut}(\Gamma)$

$\gamma \in \Gamma$

$\Gamma \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\Gamma)$

$$\begin{tikzcd}[column sep=huge]
\bullet & \bullet \arrow[l, bend right=40, "\alpha_1"', ""{name=U1, below}] \arrow[l, bend left=40, "\mathrm{Ad}_\gamma \circ \alpha_1", ""{name=D1, above}] & \bullet \arrow[l, bend right=40, "\alpha_2"', ""{name=U2, below}] \arrow[l, bend left=40, "\mathrm{Ad}_{\gamma_2} \circ \alpha_2", ""{name=D2, above}] & = & \bullet & \bullet \arrow[l, bend right=40, "\alpha_1 \circ \alpha_2"', ""{name=U3, below}] \arrow[l, bend left=40, "\mathrm{Ad}_{\gamma_1} \circ \alpha_1 \circ \mathrm{Ad}_{\gamma_2} \circ \alpha_2", ""{name=D3, above}]
\arrow[Rightarrow, from=U1, to=D1, "\gamma_1"]
\arrow[Rightarrow, from=U2, to=D2, "\gamma_2"]
\arrow[Rightarrow, from=U3, to=D3, "\gamma_1 \cdot \alpha_1(\gamma_2)"]
\end{tikzcd}$$

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\bullet & & \bullet \arrow[ll, bend right, "\alpha_1"'] \arrow[ll, bend left, "\mathrm{Ad}_\gamma \circ \alpha_1"] \arrow[ll, phantom, "\Downarrow \gamma_1"] & & \bullet \arrow[ll, bend right, "\alpha_2"'] \arrow[ll, bend left, "\mathrm{Ad}_{\gamma_2} \circ \alpha_2"] \arrow[ll, phantom, "\Downarrow \gamma_2"] & = & \bullet & & \bullet \arrow[ll, bend right, "\alpha_1 \circ \alpha_2"'] \arrow[ll, bend left, "\mathrm{Ad}_{\gamma_1} \circ \alpha_1 \circ \mathrm{Ad}_{\gamma_2} \circ \alpha_2"] \arrow[ll, phantom, "\overset{\Vert}{\underset{\Downarrow}{\gamma_1 \cdot \alpha_1(\gamma_2)}}"]
\end{tikzcd}$$

NB: Always need to choose whether actions are right- or left-actions,
hence whether group multiplication is opposite or aligned to arrow composition.
Before long we want *structure groups* to act *from the left*
and *equivariance groups* to act *from the right*.

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**(1)** $(\mathbf{B}A) \rtimes G$ is a non-abelian 2-group iff $G$ is a non-abelian group;

**(2)** its delooping sits in this fiber sequence:

$$\begin{tikzcd}
 & \mathbf{B}\Big((\mathbf{B}A) \rtimes G\Big) \arrow[d, phantom, "||"] & \\
\mathbf{B}^2 A \arrow[r, "\mathrm{fib}(p)"] & (\mathbf{B}^2 A)/\!\!/G \arrow[r, "p", "\in \mathrm{KanFib}"'] & \mathbf{B}G
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\bullet"{name=U1}] \arrow[r, bend right=50, "\bullet"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "a"] & \bullet & \mapsto & \bullet \arrow[r, bend left=50, "\mathrm{e}"{name=U2}] \arrow[r, bend right=50, "\mathrm{e}"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "a"] & \bullet & & \\
 & & & \bullet \arrow[r, bend left=50, "g"{name=U3}] \arrow[r, bend right=50, "g"'{name=D3}] \arrow[Rightarrow, from=U3, to=D3, "a"] & \bullet & \mapsto & \bullet \arrow[r, "g"] & \bullet
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# 2-Groupoids – 2-Functors.

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E.g.: if $\mathbb{Z} \curvearrowleft \mathbb{Z}_2$ by sign inversion, and $G \xrightarrow{\sigma} \mathbb{Z}_2$ a homomorphism then
**2nd group cohomology** of $G$ with coefficients in $G_\sigma \curvearrowright \mathbb{Z}$ is 2-functors:

$$\begin{tikzcd}
\mathbf{B}G \arrow[rr, dashed, "\text{2-functor = 2-cocycle}"] \arrow[dr, "\mathbf{B}\sigma" description] & & (\mathbf{B}^2 A) /\!\!/ \mathbb{Z}_2 \arrow[dl] \\
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\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[rr, "g_2"] \arrow[dr, dash, shift left, no head, double] & & \bullet \arrow[dd, "g_3"] \\
& {} \arrow[dr, double, no head] & \\
\bullet \arrow[uu, "g_1"] \arrow[uurr, "g_1 \cdot g_2" description] \arrow[rr, "g_1 \cdot g_2 \cdot g_3"'] & & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma(g_2)"] \arrow[dr, Rightarrow, "c(g_1, g_2)"'] & & \bullet \arrow[dd, "\sigma(g_3)"] \\
& {} \arrow[dr, Rightarrow, "c(g_1 \cdot g_2, g_3)"] & \\
\bullet \arrow[uu, "\sigma(g_1)"] \arrow[uurr, "\sigma(g_1 \cdot g_2)" description] \arrow[rr, "\sigma(g_1 \cdot g_2 \cdot g_3)"] & & \bullet
\end{tikzcd}$$

$\vert\vert\vert$ **associativity** $\mapsto$ **cocycle condition** $\vert\vert\vert$

$$\begin{tikzcd}
\bullet \arrow[rr, "g_2"] \arrow[ddrr, "g_2 \cdot g_3" description] & & \bullet \arrow[dd, "g_3"] \arrow[dl, double, no head] \\
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& {} \arrow[dl, Rightarrow, "c(g_1, g_2 \cdot g_3)"'] & \\
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recall that this 1-morphism acts on $c(g_2, g_3)$!

# 2-Groupoids with smooth structure.

A *smooth 2-groupoid* $\mathscr{X}$ is given by a rule

which to each chart $\mathbb{R}^n$, $n \in \mathbb{N}$ assigns the plain 2-groupoid $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$

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So $\mathrm{Probe}(*, \mathscr{X}) = \mathrm{Probe}(\mathbb{R}^0, \mathscr{X})$ is the underlying 2-groupoid
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Grothendieck (1965): “functorial geometry”

common jargon: “pre-2-stacks on the site of Cartesian spaces”

## 2-Groupoids with smooth structure – Examples.

If X is a smooth manifold, then as a smooth 2-groupoid it's this assignment:

$$\mathrm{X} \quad : \quad \mathbb{R}^n \quad \mapsto \quad \mathrm{Probe}(\mathbb{R}^n, \mathrm{X}) := C^\infty(\mathbb{R}^n, \mathrm{X})$$

$$\begin{tikzcd}
 & \mathbf{x} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\|"] & \\
\mathbf{x} \arrow[rr, equal] & & \mathbf{x}
\end{tikzcd}$$

$$\mathbf{x} \in C^\infty(\mathbb{R}^n, \mathrm{X})$$

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---

If $\Gamma$ a *Lie* group, then
the sets of smooth functions $C^\infty(\mathbb{R}^n, \Gamma)$ are plain groups,
and the *smooth delooping groupoid* $\mathbf{B}\Gamma$ is:

$$\mathbf{B}\Gamma \quad : \quad \mathbb{R}^n \quad \mapsto \quad \mathrm{Probe}(\mathbb{R}^n, \mathbf{B}\Gamma) := \mathbf{B}\big(C^\infty(\mathbb{R}^n, \Gamma)\big)$$

$$\begin{tikzcd} & \bullet \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\Vert"] & \\ \bullet \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} & \bullet \end{tikzcd} \qquad \gamma_i \in C^\infty(\mathbb{R}^n, \Gamma)$$

## 2-Groupoids with smooth structure – As smooth homotopy types.

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A smooth 2-functor $\mathscr{X} \xrightarrow{f} \mathscr{Y}$ is called:

| | | |
|---|---|---|
| PrjFib | *projective fibration* | iff for each $\mathbb{R}^n$,<br>every $k+1$-morphism in $\mathrm{Probe}(\mathbb{R}^n, \mathscr{Y})$ that starts<br>at $k$-morphisms which come from $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$<br>lifts compatibly to a $k+1$-morphism in $\mathrm{Probe}(\mathbb{R}^n, \mathscr{X})$ |
| LWEq | *local weak equivalence* | iff for every $\mathbb{R}^n$<br>there exists an open ball $0 \in \mathbb{D}^n_\varepsilon \overset{i}{\hookrightarrow} \mathbb{R}^n$ such<br>that $\mathrm{Probe}(\mathbb{R}^n, f)_{\vert i}$ is a weak homotopy equivalence<br>namely an iso on the evident homotopy groups |
| PrjCof | *projective cofibration* | if (Dugger's sufficient condition):<br>for all $k$, the spaces of $k$-morphisms are<br>disjoint unions of charts $\mathbb{R}^n$ (for any $n$-s) |

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Fact/Def.: *Maps* $\phi$ of 2-stacks and their *homotopy fibers* are *modeled* by pullbacks of this form:

(because 2-stackification is an ∞-lex reflection)

$$\begin{tikzcd}
 & \mathrm{HoFib}_y(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \widehat{*} \arrow[d, "\in \mathrm{PrjFib}"] & * \arrow[l, "\text{fib. resolution}"', "\in \mathrm{LWEq}"] \arrow[dl, bend left, "y"] \\
\varnothing \arrow[r, "\text{cof. domain}", "\in \mathrm{PrjCof}"'] & \widehat{\mathcal{X}} \arrow[r, "\phi"] & \mathcal{Y} \arrow[r, "\in \mathrm{PrjFib}"'] & *
\end{tikzcd}$$

# 2-Groupoids with smooth structure – Homotopy fiber sequences.

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[r, hook] & \underset{\substack{\text{locally trivial}\\ \text{circle-extension}}}{\Gamma} \arrow[r, two heads] & \Gamma/\mathrm{U}_1 \; \text{smooth group} & & & & \\
& & \Gamma /\!\!/ \mathrm{U}_1 \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\in \mathrm{PrjFib}"'] & \mathbf{B}\mathrm{U}_1 \arrow[r] & \mathbf{B}\Gamma \arrow[r] & \mathbf{B}(\Gamma/\mathrm{U}_1) & \\
& & & & & \mathbf{B}\Gamma /\!\!/ \mathbf{B}\mathrm{U}_1 \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r, "\substack{\text{2-cocyle}\\ \text{classifying}\\ \text{the extension}}"'] & \mathbf{B}^2\mathrm{U}_1
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$$\begin{tikzcd}
c \arrow[d, equal] \arrow[r, phantom, "\mapsto"] & \gamma \arrow[d, equal] \arrow[r, phantom, "\mapsto"] & {[\gamma]} \arrow[d, equal] & & & & \\
c & \gamma & {[\gamma]} & & & & \\
& & \gamma \arrow[u, mapsto] \arrow[d, "c"'] \arrow[r, phantom, "\mapsto"] & \bullet \arrow[d, "c"'] \arrow[r, phantom, "\mapsto"] & \bullet \arrow[d, "\gamma"'] \arrow[r, phantom, "\mapsto"] & \bullet \arrow[d, "{[\gamma]}"'] & \\
& & c\cdot\gamma & \bullet & \bullet & \bullet & \\
& & & & & \bullet \arrow[u, mapsto] \arrow[d, bend right=50, "\gamma"'] \arrow[d, bend left=50, "c\cdot\gamma"] \arrow[d, phantom, "\overset{c}{\Longrightarrow}"] \arrow[r, phantom, "\mapsto"] & \bullet \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\overset{c}{\Longrightarrow}"] \\
& & & & & \bullet & \bullet
\end{tikzcd}$$

# 2-Groupoids with smooth structure – Homotopy fiber sequences.

For example, write $\mathrm{U}_n,\ n \in \mathbb{N} \sqcup \{\omega\}$
for the unitary group on a countably-dimensional complex Hilbert space
and regard this as a smooth group by its “continuous diffeology”:

$$\mathrm{U}_n \;:\; \mathbb{R}^k \;\mapsto\; \mathrm{Probe}(\mathbb{R}^k, \mathrm{U}_n) \;:=\; C^0(\mathbb{R}^k, \mathrm{U}_n)$$

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Then we have the following long fiber sequence of smooth 2-groupoids:

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[r, hook] & \mathrm{U}_n \arrow[r, two heads] & \mathrm{PU}_n & & & & \\
& & \mathrm{U}_n /\!\!/ \mathrm{U}_1 \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}\mathrm{U}_1 \arrow[r] & \mathbf{B}\mathrm{U}_n \arrow[r] & \mathbf{B}\mathrm{PU}_n & \\
& & & & & \mathbf{B}\mathrm{U}_n /\!\!/ \mathbf{B}\mathrm{U}_1 \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}^2\mathrm{U}_1
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& & & & & \mathbf{B}\mathrm{U}_n /\!\!/ \mathbf{B}\mathrm{U}_1 \arrow[u, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}^2\mathrm{U}_1
\end{tikzcd}$$

This is all compatible with complex conjugation, so that there is a map like this:

$$\begin{tikzcd}
\mathbf{B}\mathrm{PU}_n /\!\!/ \mathbb{Z}_2 & {} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r] & \mathbf{B}^2\mathrm{U}_1 /\!\!/ \mathbb{Z}_2
\end{tikzcd}$$

## 2-Groupoids with smooth structure – Čech groupoids.

For X a smooth manifold with $\big\{\mathrm{U}_i \hookrightarrow \mathrm{X}\big\}_{i \in I}$ a good open cover, in that

$$\big(\mathbf{x}, (i_1, \cdots, i_n)\big) \in C^\infty\big(\mathbb{R}^m, U_{i_1} \cap \cdots \cap U_{i_n}\big) \qquad \Rightarrow \qquad U_{i_1} \cap \cdots \cap U_{i_n} \underset{\mathrm{diff}}{\simeq} \mathbb{R}^{\mathrm{dim}(\mathrm{X})},$$

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we have the smooth *Čech 2-groupoid*:

**Čech 2-groupoid of good open cover**

$$\begin{tikzcd}
\varnothing \arrow[r, "\in \mathrm{PrjCof}"'] & \widehat{\mathrm{X}} \arrow[rr, "\text{cofibrant resolution}", "\in \mathrm{LWEq}"'] & & \mathrm{X}
\end{tikzcd}$$

**smooth manifold regarded as smooth 2-groupoid**

$$\begin{tikzcd}
& (\mathbf{x}, j) \arrow[dr, "{(\mathbf{x},(j,k))}"] \arrow[d, phantom, "\parallel"] & & & & \mathbf{x} \arrow[dr, equal] \arrow[d, phantom, "\parallel"] & \\
(\mathbf{x}, i) \arrow[ur, "{(\mathbf{x},(i,j))}"] \arrow[rr, "{(\mathbf{x},(i,k))}"'] & {\Downarrow}\, (\mathbf{x},(i,j,k)) & (\mathbf{x}, k) & \longmapsto & \mathbf{x} \arrow[ur, equal] \arrow[rr, equal] & & \mathbf{x}
\end{tikzcd}$$

which is a *projectively cofibrant resolution* of X.

# 2-Groupoids with smooth structure – Čech cocycles.

Smooth 2-functors from such a Čech resolution $\widehat{\mathrm{X}} \to \mathrm{X}$ to the delooping $\mathbf{B}\Gamma$ of a Lie group are *cocycles* in the *Čech cohomology* of X with coefficients in Γ:

$$\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, "\text{smooth functor = Čech cocycle}"] & & \mathbf{B}\Gamma
\end{tikzcd}$$

$$\begin{tikzcd}
& (x,j) \arrow[dr, "{(x,(j,k))}"] \arrow[d, phantom, "{\begin{array}{c}\| \\ (x,(i,j,k)) \\ \Downarrow\end{array}}"] & & & & \bullet \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\|"] & \\
(x,i) \arrow[ur, "{(x,(i,j))}"] \arrow[rr, "{(x,(i,k))}"'] & {} & (x,k) & \mapsto & \bullet \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & {} & \bullet
\end{tikzcd}$$

**Čech relations** **cocycle condition**

# Principal bundles via smooth groupoids – Universal principal bundles.

The inclusion of the unique base point into $\mathbf{B}\Gamma$ has the following *fibrant resolution*:

$$\begin{tikzcd}
{*} \arrow[d, "\in\mathrm{LWEq}"] \\
\mathbf{E}\Gamma \arrow[d, "\in\mathrm{PrjFib}"] \\
\mathbf{B}\Gamma
\end{tikzcd}$$

**universal**
**Γ-principal bundle**
in its stacky incarnation

$$\begin{tikzcd}
 & \gamma\cdot\gamma_1 \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\|"] & \\
\gamma \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1\cdot\gamma_2"'] & {} \arrow[d, mapsto] & \gamma\cdot\gamma_1\cdot\gamma_2 \\
 & \bullet \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\|"] & \\
\bullet \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1\cdot\gamma_2"'] & {} & \bullet
\end{tikzcd}$$

## Principal bundles via smooth groupoids.

The *homotopy fiber* of a 2-functor = Čech cocycle is equivalently
*the principal bundle* P *it classifies*:

$$\begin{tikzcd}
\mathrm{P} \arrow[d] & \overset{\textbf{principal bundle}}{\widehat{\mathrm{P}}} \arrow[l, "\in\mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\Gamma \arrow[d, "\in\mathrm{PrjFib}"] & \ast \arrow[l, "\in\mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in\mathrm{LWEq}"'] \arrow[r, "\textbf{Čech cocycle}"'] & \mathbf{B}\Gamma &
\end{tikzcd}$$

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$$\begin{tikzcd}
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\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r, "\text{Čech cocycle}"'] & \mathbf{B}\Gamma &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & & \gamma_x \cdot \gamma_{ij}(x) \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\|"] & \\
 & & & & & \gamma_x \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & {} & \gamma_x \cdot \gamma_{ik}(x) \\
 & & & & & & {} \arrow[d, mapsto] & \\
 & (x,j) \arrow[dr] \arrow[d, phantom, "\|"] & & & & & \bullet \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\|"] & \\
(x,i) \arrow[ur] \arrow[rr] & {} & (x,k) & {} \arrow[r, mapsto] & {} & \bullet \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & {} & \bullet
\end{tikzcd}$$

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The *homotopy fiber* of a 2-functor = Čech cocycle is equivalently *the principal bundle* P *it classifies*:

$$\begin{tikzcd}
 & \big((x,j),\gamma_x\cdot\gamma_{ij}(x)\big) \arrow[dr] \arrow[d, phantom, "\overset{\|}{\underset{\|}{\text{gluing prescr.}}}" description] & & & & \gamma_x\cdot\gamma_{ij}(x) \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\|" description] & \\
\big((x,i),\gamma_x\big) \arrow[ur] \arrow[rr] & & \big((x,k),\gamma_x\cdot\gamma_{ik}(x)\big) \arrow[rr, mapsto] & & \gamma_x \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & & \gamma_x\cdot\gamma_{ik}(x)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[dd, mapsto] & & & \text{principal bundle} & & & {} \arrow[dd, mapsto] \\
 & \mathrm{P} \arrow[d] & & \widehat{\mathrm{P}} \arrow[ll, "\in\mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\Gamma \arrow[d, "\in\mathrm{PrjFib}"] & * \arrow[l, "\in\mathrm{LWEq}"] \arrow[dl, bend left] & \\
{} & \mathrm{X} & & \widehat{\mathrm{X}} \arrow[ll, "\in\mathrm{LWEq}"] \arrow[r, "\text{Čech cocycle}"'] & \mathbf{B}\Gamma & & {}
\end{tikzcd}$$

$$\begin{tikzcd}
 & (x,j) \arrow[dr] \arrow[d, phantom, "\|" description] & & & & \bullet \arrow[dr, "\gamma_{jk}(x)"] \arrow[d, phantom, "\|" description] & \\
(x,i) \arrow[ur] \arrow[rr] & & (x,k) \arrow[rr, mapsto] & & \bullet \arrow[ur, "\gamma_{ij}(x)"] \arrow[rr, "\gamma_{ik}(x)"'] & & \bullet
\end{tikzcd}$$

## Principal 2-bundles via smooth 2-groupoids.

This neat formulation of ordinary principal bundles immediatly generalizes to give principal 2-bundles:

E.g. for the structure 2-group $\mathrm{Aut}(\mathbf{B}\Gamma)$ these are equivalently Giraud's *non-abelian gerbes*:

$$\begin{tikzcd}
P \arrow[d] & \overset{\textbf{principal 2-bundle}}{\widehat{P}} \arrow[l, "\in \mathrm{LWEq}"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\mathrm{Aut}(\mathbf{B}\Gamma) \arrow[d, "\in \mathrm{PrjFib}"] & * \arrow[l, "\in \mathrm{LWEq}"] \arrow[dl, bend left] \\
X & \widehat{X} \arrow[l, "\in \mathrm{LWEq}"] \arrow[r, "\textbf{Čech cocycle}"'] & \mathbf{B}\mathrm{Aut}(\mathbf{B}\Gamma) &
\end{tikzcd}$$

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 & \text{principal 2-bundle} & & \\
\mathrm{P} \arrow[d] & \widehat{\mathrm{P}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{E}\mathrm{Aut}(\mathbf{B}\Gamma) \arrow[d, "\in \mathrm{PrjFib}"] & * \arrow[l, "\in \mathrm{LWEq}"'] \arrow[dl, bend left] \\
\mathrm{X} & \widehat{\mathrm{X}} \arrow[l, "\in \mathrm{LWEq}"'] \arrow[r, "\text{Čech cocycle}"'] & \mathbf{B}\mathrm{Aut}(\mathbf{B}\Gamma) &
\end{tikzcd}$$

While it's tradition to be esoteric about this simple affair,
here to highlight that this is really about *twisted cohomology*: ⟶

# Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathbb{Z}\ ) \simeq (\mathbf{B}\mathbb{Z}\ ) \rtimes \mathbb{Z}_2$, with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathbb{Z}\ ) \simeq (\mathbf{B}^2\mathbb{Z}\ )/\!/\mathbb{Z}_2$ and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then

**2nd integral cohomology** of X with local coefficients is smooth 2-functors:

$$\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth 2-functor = Čech 2-cocycle}"] \arrow[dr, "\sigma"'] & & (\mathbf{B}^2\mathbb{Z}\ )/\!/\mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

# Principal 2-bundles via smooth 2-groupoids – Example: Twisted cohomology.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathbb{Z}\ ) \simeq (\mathbf{B}\mathbb{Z}\ ) \rtimes \mathbb{Z}_2$, with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathbb{Z}\ ) \simeq (\mathbf{B}^2\mathbb{Z}\ )/\!\!/\mathbb{Z}_2$ and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then

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& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
(x,j) \arrow[rr] \arrow[dr, phantom, "\Rightarrow"] & & (x,k) \arrow[dd] \\
& \arrow[dr, phantom, "\Rightarrow"] & \\
(x,i) \arrow[uu] \arrow[uurr] \arrow[rr] & & (x,l)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[dr, Rightarrow, "c_{ijk}(x)"'] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
& \sigma_{ik}(x) \arrow[dr, Rightarrow, "c_{ikl}(x)"] & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[uurr] \arrow[rr, "\sigma_{il}(x)"] & & \bullet
\end{tikzcd}$$

||| **Čech relations** $\mapsto$ **cocycle condition** |||

$$\begin{tikzcd}
(x,j) \arrow[rr] \arrow[ddrr] & \arrow[dl, phantom, "\Leftarrow"] & (x,k) \arrow[dd] \\
& \arrow[dl, phantom, "\Leftarrow"] & \\
(x,i) \arrow[uu] \arrow[rr] & & (x,l)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[ddrr] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \arrow[dl, Rightarrow, "c_{jkl}(x)"] \\
& \sigma_{jl}(x) \arrow[dl, Rightarrow, "c_{ijl}(x)"'] & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[rr, "c_{il}(x)"] & & \bullet
\end{tikzcd}$$

recall that this
1-morphism acts
on $c_{jkl}(x)$!

# Principal 2-bundles via smooth 2-groupoids – Example: Jandl gerbes.

For structure 2-group $\mathrm{Aut}(\mathbf{B}\mathrm{U}_1) \simeq (\mathbf{B}\mathrm{U}_1) \rtimes \mathbb{Z}_2$, with $\mathbf{B}\mathrm{Aut}(\mathbf{B}\mathrm{U}_1) \simeq (\mathbf{B}^2\mathrm{U}_1)/\!/\mathbb{Z}_2$
and $\widehat{\mathrm{X}} \xrightarrow{\sigma} \mathbf{B}\mathbb{Z}_2$ a double covering, then
**2nd** $\mathrm{U}_1$**-valued cohomology** of X with local coefficients is smooth 2-functors:

$$\begin{tikzcd}
\widehat{\mathrm{X}} \arrow[rr, dashed, "\text{smooth 2-functor = Čech 2-cocycle}"] \arrow[dr, "\sigma" description] & & (\mathbf{B}^2\mathrm{U}_1)/\!/\mathbb{Z}_2 \arrow[dl] \\
& \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
(x,j) \arrow[rr] & & (x,k) \arrow[dd] \\
& {} & \\
(x,i) \arrow[uu] \arrow[uurr] \arrow[rr] & & (x,l)
\arrow[from=1-1, to=2-2, Rightarrow, shorten <=6mm]
\arrow[from=2-2, to=3-3, Rightarrow, shorten >=6mm]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
& {} & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[uurr, "\sigma_{ik}(x)" description] \arrow[rr, "\sigma_{il}(x)"] & & \bullet
\arrow[from=1-1, to=2-2, Rightarrow, shorten <=6mm, "c_{ijk}(x)"']
\arrow[from=2-2, to=3-3, Rightarrow, shorten >=6mm, "c_{ikl}(x)"]
\end{tikzcd}$$

$|||$ **Čech relations** $\qquad \mapsto \qquad$ **cocycle condition** $|||$

$$\begin{tikzcd}
(x,j) \arrow[rr] \arrow[ddrr] & & (x,k) \arrow[dd] \\
& {} & \\
(x,i) \arrow[uu] \arrow[rr] & & (x,l)
\arrow[from=1-3, to=2-2, Rightarrow, shorten <=6mm]
\arrow[from=2-2, to=3-1, Rightarrow, shorten >=6mm]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, "\sigma_{jk}(x)"'] \arrow[ddrr, "\sigma_{jl}(x)" description] & & \bullet \arrow[dd, "\sigma_{kl}(x)"] \\
& {} & \\
\bullet \arrow[uu, "\sigma_{ij}(x)"] \arrow[rr, "c_{il}(x)"] & & \bullet
\arrow[from=1-3, to=2-2, Rightarrow, shorten <=6mm, "c_{jkl}(x)"]
\arrow[from=2-2, to=3-1, Rightarrow, shorten >=6mm, "c_{ijl}(x)"']
\end{tikzcd}$$

recall that this
1-morphism acts
on $c_{jkl}(x)$!

# Principal 2-bundles via smooth 2-groupoids – Punchline.

So:

Non-abelian 1-cohomology is modulated by 1-stacks $\mathbf{B}\Gamma$,
abelian 2-cohomology is modulated by 2-stacks $\mathbf{B}^2 A$, etc.

Higher fiber/principal bundles are *bundles of such moduli stacks*,
hence are families of moduli stacks that vary over the base space,
hence locally modulate cohomology as before,
but now subject to global twists.

## Principal 2-bundles via smooth 2-groupoids – Equivariance.

Finally, the higher topos of smooth 2-groupoids
has *equivariance* natively built into it: just let domain spaces be groupoids, too.

# Principal 2-bundles via smooth 2-groupoids – Equivariance.

Finally, the higher topos of smooth 2-groupoids
has *equivariance* natively built into it: just let domain spaces be groupoids, too:

For $\mathrm{X} \circlearrowright G$ a smooth action of a finite group on a smooth manifold.

there exists an
*equivariant good open cover*

$$\begin{tikzcd}
\underset{i\in I}{\coprod} \mathrm{U}_i \arrow[loop above, "G"] \arrow[rr, two heads] & & \mathrm{X} \arrow[loop above, "G"]
\end{tikzcd}$$

and its *equivariant Čech groupoid*:

$$\begin{tikzcd}
\varnothing \arrow[r, "\in \mathrm{PrjCof}"'] & \widehat{\mathrm{X} /\!\!/ G} \arrow[rr, "\text{cofibrant resolution}", "\in \mathrm{LWEq}"'] & & \overset{\text{action groupoid}}{\mathrm{X} /\!\!/ G}
\end{tikzcd}$$

$$\begin{tikzcd}
 & (x,i)\cdot g_1 \arrow[dr, "g_2"] \arrow[dd] \arrow[dl, phantom, "\Swarrow"] & \\
(x,i) \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2" description] \arrow[dd, "{(x,(i,j))}"'] & & (x,i)\cdot g_1 \cdot g_2 \arrow[dd, "{(x,(i,j))\cdot g_1 \cdot g_2}"] \arrow[ddll, Rightarrow] \\
 & (x,j)\cdot g_1 \arrow[dr] \arrow[dl, phantom, "\Swarrow"] & \\
(x,j) \arrow[ur] \arrow[rr, "g_1 \cdot g_2"'] & & (x,j)\cdot g_1 \cdot g_2
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
 & x \cdot g_1 \arrow[dr, "g_2"] \arrow[d, phantom, "\Swarrow"] & \\
x \arrow[ur, "g_1"] \arrow[rr, "g_1 \cdot g_2" description] & {} & x \cdot g_1 \cdot g_2
\end{tikzcd}$$

# Principal 2-bundles via smooth 2-groupoids – Equivariance.

For $\mathrm{X} \circlearrowleft G$ a smooth manifold and $(\Gamma /\!\!/ C) \circlearrowleft G$ a smooth 2-group both equipped with smooth $G$-action, a $G$-equivariant $\Gamma$-principal 2-bundle on $\mathrm{X}$ is modulated by a smooth 2-functor like this:

$$\begin{tikzcd}
\widehat{\mathrm{X} /\!\!/ G} \arrow[rr, dashed] \arrow[dr] & & \mathbf{B}\Gamma /\!\!/ G \arrow[dl] \\
& \mathbf{B}G &
\end{tikzcd}$$

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& \mathbf{B}G &
\end{tikzcd}$$

$$\begin{tikzcd}
& (x,i)\cdot g_1 \arrow[dr, "g_2"] \arrow[dd] \arrow[d, Rightarrow] & \\
(x,i) \arrow[ur, "g_1"] \arrow[rr, "g_1\cdot g_2"] \arrow[dd, "{(x,(i,j))}"'] & {} & (x,i)\cdot g_1\cdot g_2 \arrow[dd, "{(x,(i,j))\cdot g_1\cdot g_2}"] \arrow[ddll, Rightarrow] \\
& (x,j)\cdot g_1 \arrow[dr] \arrow[d, Rightarrow] & \\
(x,j) \arrow[ur] \arrow[rr, "g_1\cdot g_2"'] & {} & (x,j)\cdot g_1\cdot g_2
\end{tikzcd}$$

equivariant Čech relations

$$\mapsto$$

$$\begin{tikzcd}
& \bullet \arrow[dr, "\rho_x(g_2)"] \arrow[dd, dash] \arrow[d, Rightarrow, "c_x(g_1{,}g_2)"'] & \\
\bullet \arrow[ur, "\rho_x(g_1)"] \arrow[rr, "\rho_x(g_1\cdot g_2)"] \arrow[dd, "\gamma_{ij}(x)"'] & {} & \bullet \arrow[dd, "\gamma_{g\cdot i g\cdot j}(g\cdot x)"] \arrow[ddll, Rightarrow] \\
& {} \arrow[dr] \arrow[d, Rightarrow] & \\
\bullet \arrow[ur, dash] \arrow[rr, "\rho_x(g_1\cdot g_2)"'] & {} & \bullet
\end{tikzcd}$$

equivariant 2-cocyle condition

# Principal 2-bundles via smooth 2-groupoids – Equivariant examples.

E.g. an equivariant $\mathrm{PU}_\omega$-bundle

over the point, where $\widehat{* /\!\!/ G} \;=\; \mathbf{B}G\,,$

is a projective $G$-representation:

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{(delooping of)}\\ \phi\text{-twisted extension of}\\ \text{equivariance group}}}{\mathbf{B}G^{\tau}} \arrow[rr, "\text{(anti-)unitary representation}", "\text{of equivariace group}"'] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}(\mathrm{U}_\omega \rtimes \mathbb{Z}_2) \arrow[d] \\
* \arrow[r] & \mathbf{B}G \arrow[rr, "\tau", "\substack{\text{moduli of real-equivariant}\\ \text{projective bundle}}"'] \arrow[dr, "\substack{\text{(anti-)unitarity of}\\ \text{the representation}}\ \mathbf{B}\phi"'] & & \mathbf{B}(\mathrm{PU}_\omega \rtimes \mathbb{Z}_2) \arrow[dl, "\mathbf{B}\mathrm{pr}_2"] \\
& & \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd}$$

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E.g. an equivariant $\mathrm{PU}_\omega$-bundle

over the point, where $\widehat{\ast /\!\!/ G} \;=\; \mathbf{B}G\,,$

is a projective $G$-representation:

$$\begin{tikzcd}
 & \text{(delooping of)} & & & \\[-20pt]
 & \text{$\phi$-twisted extension of} & & & \\[-20pt]
 & \text{of equivariance group} & & & \\[-10pt]
\mathrm{U}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}G^\tau \arrow[rr, "\text{(anti-)unitary representation}", "\text{of equivariace group}"'] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}\big(\mathrm{U}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_2\big) \arrow[d] & \\
\ast \arrow[r] & \mathbf{B}G \arrow[rr, "\tau", "\text{moduli of real-equivariant projective graded bundle}"'] \arrow[dr, "\mathbf{B}\phi", "\text{(anti-)unitarity of the representation}"'] & & \mathbf{B}\big(\mathrm{PU}^{\mathrm{gr}}_\omega \rtimes \mathbb{Z}_2\big) \arrow[dl, "\mathbf{B}\mathrm{pr}_2"] \arrow[dr, "\mathbf{B}c\ \ \text{sign of grading}"] & \\
 & & \mathbf{B}\mathbb{Z}_2 & & \mathbf{B}\mathbb{Z}_2
\end{tikzcd}$$

# Principal 2-bundles via smooth 2-groupoids – Equivariant examples.

E.g. an equivariant $\mathrm{PU}_\omega$-bundle

over the point, where $\widehat{\ast /\!/ G} \;=\; \mathbf{B}G\,,$

is a projective $G$-representation:

$$\begin{tikzcd}[column sep=large]
 & \overset{\substack{\text{(delooping of)}\\ ct =: \phi\text{-twisted extension of}\\ \text{quantum symmetry group}}}{\mathbf{B}G^{\tau}} & & & \\[-2em]
\mathrm{U}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbf{B}G^{\tau} \arrow[rr, "\text{linear realization of}", "\text{quantum symmetry}"'] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathbf{B}\big(\mathrm{U}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2\big) \arrow[d] & \\
\ast \arrow[r] & \mathbf{B}G \arrow[rr, "\tau", "\substack{\text{moduli of real-equivariant}\\ \text{projective graded bundle}}"'] \arrow[dr, "\mathbf{B}ct", "\substack{\text{(anti-)unitarity of}\\ \text{charge+time reversal}}"'] & & \mathbf{B}\big(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2\big) \arrow[dl, "\mathbf{B}\mathrm{pr}_2"] \arrow[dr, "\mathbf{B}c\quad \substack{\text{sign of}\\ \text{charge conjugation}}"] & \\
 & & \mathbf{B}\mathbb{Z}_2 & & \mathbf{B}\mathbb{Z}_2
\end{tikzcd}$$

This happens to encode all about *quantum symmetries of gapped systems*
(cf. Freed & Moore 2013 , good review in Thiang 2018, §4, ).

## Part I – Invitation

# Part II – Application

# Part II – Application

which provides a brief outlook on
how the above technology gives
a transparent construction of
twisted equivariant KR-theory.

# Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

The “smooth” (namely continuous-diffeological) group $\mathrm{PU}^{\mathrm{gr}}_{\omega}$
canonically acts on
the “smooth” space Fred of Fredholm operators on a $\mathbb{Z}_2$-graded Hilbert space.

Sections of the corresponding *associated equivariant bundles* are cocycles
for *twisted equivariant Real K-theory* (generalizing Pavlov 2014, §3.19):

$$\underset{\text{KR-cohomology}}{\overset{\text{twisted equivariant}}{\mathrm{KR}^{\tau}_{G}(\mathrm{X})}} \;:=\; \overset{\text{connected components of shape of sliced mapping stack}}{\tau_0 \textstyle\int \mathrm{Map}\Big( (\mathrm{X} \sslash G, \tau), \, \mathrm{Fred} \sslash (\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2) \Big)_{\mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2)}}$$

$$= \left\{ \begin{tikzcd} & & \mathrm{Fred} \sslash (\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2) \arrow[d, "\substack{\text{stacky universal}\\ \text{Fredholm bundle}}"] \\ \overset{\substack{\text{Real good}\\ \text{orbifold}}}{\widehat{\mathrm{X} \sslash G}} \arrow[urr, dashed, "\substack{\text{cocycle in}\\ \text{twisted equivariant Real K-theory}}"] \arrow[rr, "\tau", "\substack{\text{modulation of stable Real-equivariant}\\ \text{projective graded bundle}}"'] \arrow[dr, "\text{Real structure}"'] & & \mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2) \arrow[dl] \\ & \mathbf{B}\mathbb{Z}_2 & \end{tikzcd} \right\} \Big/ \sim_{\mathrm{hmtp}}$$

# Twisted equivariant KR-theory – Outlook.

This transparent formulation serves to reveal that there is
more quantum physics encoded in twisted equivariant KR-theory
than has previously bee uncovered.

To be discussed in:

H. Sati, & U. S.: *Anyonic Defect Branes in Twisted equivariant K-Theory*

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$$\overset{\text{twisted equivariant KR-cohomology}}{\mathrm{KR}^{\tau}_{G}(\mathrm{X})} \;:=\; \overset{\text{connected components of shape of sliced mapping stack}}{\tau_0 \int \mathrm{Map}\Big( \big(\mathrm{X} /\!\!/ G, \tau\big), \mathrm{Fred} /\!\!/ \big(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2\big)\Big)_{\mathbf{B}(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2)}}$$

$$= \left\{ \begin{tikzcd}
 & & \mathrm{Fred} /\!\!/ \big(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2\big) \arrow[d, "\text{stacky universal Fredholm bundle}"] \\
\overset{\text{Real good orbifold}}{\widehat{\mathrm{X} /\!\!/ G}} \arrow[urr, dashed, "\text{cocycle in twisted equivariant Real K-theory}"] \arrow[rr, "\tau", "\text{modulation of stable Real-equivariant projective graded bundle}"'] \arrow[dr, "\text{Real structure}"'] & & \mathbf{B}\big(\mathrm{PU}^{\mathrm{gr}}_{\omega} \rtimes \mathbb{Z}_2\big) \arrow[dl] \\
 & \mathbf{B}\mathbb{Z}_2 &
\end{tikzcd} \right\} \Big/ \sim_{\mathrm{hmtp}}$$