11d SuGra from higher to exceptional Cartan geometry

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Abstract

In the search for potential mathematical structures underlying 11-dimensional supergravity with M-theoretic corrections included, two major approaches exist, but had remained unrelated: On the one hand, a widely studied proposal is that M-theory is a kind of Cartan geometry locally modeled on inclusions $K_{d(d)} \hookrightarrow E_{d(d)}$ of maximal compact subgroups of exceptional Lie or Kac-Moody groups in the E-series. On the other hand, the approach to supergravity initiated by D'Auria and Fré may be understood as *higher* (i.e. homotopy theoretic, stacky) super-Cartan geometry locally modeled on the exceptional super Lie cocycles of super Minkowski spacetimes. Here we informally discuss some indication for how the first of these approaches might arise within the second.

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1 Review and Motivation

The fact that the term "M-theory" eventually became attached to a grandiose conjecture [Witten98] tends to overshadow that it was originally coined as "non-committed" shorthand [HoWi95, p. 2] for "membrane theory" [Duff95, Duff99] (directly modeled on the well established term "string theory"), referring to the concrete study of super-membrane sigma models on 11-dimensional supergravity spacetimes [BST87]. It is noteworthy that the latter is a rich topic in itself about which a lot is understood in precise mathematical detail. Seminal mathematical results here include [AtWi01, HoSi05]. Hence close mathematical analysis of M-theory-the-concrete is fruitful in itself, and is plausibly a way to make progress on M-theory-the-grandiose.

Traditional wisdom has it that a key technical problem with M-theory is that membrane sigma-models are understood only classically, not in the quantum version that is expected to be relevant for the M-theory-in-the-grandiose-sense. But actually a little more is true: membrane sigma models – and also the 5-brane sigma models induced by them – are understood in *pre-quantum theory*, in the precise sense in which this term is used in the Kostant-Souriau formalization of quantization via *geometric quantization*. This is a substantial distinction: we have previously shown [FSS13b, Sc15b] how to refine the brane sigma-models further to *higher/local* pre-quantum theory [FRS13a, FRS13b], and in this refined formulation membrane theory already sees a wealth of subtle effects, such as notably the properly globalized BPS groups of brane charges [SaSc15] in generalized twisted differential cohomology [FSS15]. This goes much beyond what genuine classical field theory sees, and arguably probes into M-theory-in-the-grandiose-sense.

As generally in quantization, also pre-quantization involves making choices. Here we discuss:

Claim 1.1. *Exceptional generalized geometry in 11-dimensions is a natural parameterization of the space of choices in the pre-quantization of the fermionic supergravity 4-flux, equivalently of the space of choices in the definition of the M2-brane WZW term on curved superspacetimes.*

More concretely, the statement here is that 11-dimensional supergravity with these pre-quantum membrane and 5-brane effects included is precisely the *higher super Cartan geometry* [Sc15b, Sc15a] which is locally modeled on the exceptional super Lie algebra cocycles on 11-dimensional super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$, regarded as a super Lie algebra (see [Sc13] for a comprehensive account). This is readily seen to be a globalized version of key observations originating in [AuFr82], after understanding the "FDA"s referred to there as the Chevalley-Eilenberg algebras of super L_∞-algebras [SSS09, FSS13b].

1.1 11-dimensional supergravity as super-Cartan geometry

The identification of 11-dimensional supergravity as super-Cartan geometry may be understood in two steps [Sc13].

1. **super-orthogonal structures** A simple but remarkable observation is that the Spin-cover $\mathrm{Spin}(10,1)$ of the Lorentz group is the joint stabilizer of the super Lie bracket and of the M2-brane 4-cocycle μ_4^{M2} on $\mathbb{R}^{10,1|\mathbf{32}}$. A reduction of the super-structure group of an 11-dimensional supermanifolds to this stabilizer group along $\mathrm{Spin}(10,1) \to \mathrm{GL}(\mathbb{R}^{10,1|\mathbf{32}})$ is hence precisely the srtructure needed to equip the tangent bundle with the structure of a bundle of super-Lie algebras equipped with 4-cocycles, whose typical fiber is $(\mathbb{R}^{10,1|\mathbf{32}}, \mu_4^{\mathrm{M2}})$. At the same time such reduction is of course the first-order formulation of pseudo-Riemannian structure, hence of field configurations of graviton and gravitino.

2. **torsion constraint** Requiring such a reduction of the structure group of the supermanifold to be *first-order integrable* means that it is, around each point, equivalent to first infinitesimal order to the canonical reduction on the model space $\mathbb{R}^{10,1|\mathbf{32}}$ (via left translation) means equivalently that the torsion of the reduction structure vanishes. Here in the context of supergeometry this means that the super-torsion vanishes. Now the vanishing of the bosonic component of the super-torsion tensor in 11d is known to be already equivalent to the equations of motion of 11-dimensional supergravity. Requiring also its fermionic component to vanish means to set the "gravitino field strength" to zero, hence means restriction to bosonic avcuua.

These facts exhibit higher supergravity mathematically as a topic in what one might call *parameterized Lie theory*, or in first-order integrable parameterized higher super Lie theory, to be precise, hence higher super Cartan geometry.

Accordingly, every Lie theoretic statement about $(\mathbb{R}^{10,1|\mathbf{32}}, \mu_4)$ is to induce a parameterized analog that impacts on the understanding of 11-dimensional supergravity.

In particular we may ask the 3-form potential C_3 to be chosen such that on each first order infinitesimal neighbourhood it relates to the 4-flux $\overline{\psi}\Gamma_{a_1 a_2}\Psi \wedge e_{a_1} \wedge e_{a_2}$ in a Lie theoretic way.

One of the most immediate questions that one may ask of a pair $(\mathfrak{g}, \mu)$ (such as $(\mathbb{R}^{10,1|\mathbf{32}}, \mu_4^{\mathrm{M2}})$) consisting of a Lie algebra and a cocycle is: what are Lie algebra extensions $p_* : \hat{\mathfrak{g}} \to \mathfrak{g}$ such that $p^*\mu$ trivializes in cohomology? We will see below in 2 that every such an extension produces a parameterization of local choices of WZW terms for μ by linear splittings of p_*.

If μ here is a 2-cocycle, then there is a well known universal answer to this question: the universal $\hat{\mathfrak{g}}$ for which this is true is the Lie algebra central extension that is induced by μ_2 via the classical formula.

If μ however is a $(p+2)$-cocycle, then the situation is more subtle, as then it depends on whether one regards (super) Lie algebras as forming a 1-category, or whether one understands them as sitting inside the ∞-category of (super-)L_∞-algebras. In the latter case the universal $\hat{\mathfrak{g}}$ is the Lie $(p+1)$-algebra that is the homotopy fiber of μ_{p+2} [FSS13b]. This is what appears in the higher super Cartan geometry mentioned above.

But one may also constrain $\hat{\mathfrak{g}}$ to remain a (super-)Lie 1-algebra, hence a plain (super-)Lie algebra, while still requiring that μ_{p+2} trivializes on it.

Such $\hat{\mathfrak{g}}_1$ for the case of $(\mathfrak{g}, \mu) = (\mathbb{R}^{10,1|\mathbf{32}}, \mu_4^{\mathrm{M2}})$ happen to have already been found in the literature [AuFr82, BAIPV04]: There is (at least) a 1-parameter family of such, and for all members of the family the underlying bosonic vector space (body) is

$$(\widehat{\mathbb{R}^{10,1}})_{\mathrm{bosonic}} \simeq \mathbb{R}^{10,1} \oplus \wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^*$$

Following [H97] we may equivalently express this in terms of purely spatial components by applying Poincaré duality to obtain

$$\widehat{\mathbb{R}^{10,1|\mathbf{32}}}_{\mathrm{bosonic}} \simeq_{\mathrm{lin}} \mathbb{R}^{10,1} \oplus \underbrace{\wedge^2(\mathbb{R}^{10})^*}_{\text{M2-brane}} \oplus \underbrace{\wedge^9\ \mathbb{R}^{10}}_{\text{M9-brane}} \oplus \underbrace{\wedge^5(\mathbb{R}^{10})^*}_{\text{M5-brane}} \oplus \underbrace{\wedge^6\ \mathbb{R}^{10}}_{\text{KK-monopole}} .$$

Considering this in turn for a splitting $\mathbb{R}^{10,1} \simeq \mathbb{R}^{3,1} \oplus \mathbb{R}^7$ adapted to a KK-compactification to 4d gives

$$\widehat{\mathbb{R}^{10,1|\mathbf{32}}}_{\mathrm{bosonic}} \simeq_{\mathrm{lin}} \left(\mathbb{R}^7 \oplus \wedge^2(\mathbb{R}^7)^* \oplus \wedge^5(\mathbb{R}^7)^* \oplus \wedge^6\mathbb{R}^7\right) \oplus \cdots ,$$

where the ellipses indicates summands that involve a tensor factor of $\mathbb{R}^{3,1}$ or $(\mathbb{R}^{3,1})^*$. This last expression is the typical fiber of what later came to be known, independently, as the *exceptional tangent bundle* of the fiber space for 11d SuGra compactified to 4d [Hull07, section 4.4] [PaWa08, section 2].

The observation that motivates the formulation of 11-dimensional supergravity based on such exceptional tangent bundle is the following:

1. A choice of fiberwise identification of an exceptional tangent bundle with a direct product form as above is a choice of reduction of structure groups along $K_{d(d)} \hookrightarrow E_{d(d)}$.
2. Part of the action of $\mathfrak{e}_d$ on the exceptional tangent bundle may be identified with an action of 3-forms, and these are naturally locally identified with the background field for the M2-brane sigma-model.

Therefore the idea is that the field content of 11d-sugra with pre-quantized M2-brane background field $\hat{C}_3$ should equivalently be locally a $K_{d(d)} \hookrightarrow E_{d(d)}$ structure subject to a global twist by the 4-class underlying the $\hat{C}_3$ field, probably for $d = 11$. Indeed, $\mathbb{R}^{10,1} \oplus \wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^*$ is isomorphic the level-2 truncation of the l_1-representation of E_{11} [We04, around (5.2)] (see also e.g. [We11, (2.17)]).

This connection between the super Lie algebra obtained as above in [AuFr82, BAIPV04] and the typical fiber of an exceptional tangent bundle as in [Hull07, PaWa08] seems not to have been explored in print before, except for a remark in [Vau06, p. 14].

2 The M2-WZW term and the exceptional tangent bundle

We discuss here in more detail how the exceptional tangent bundle may be systematically discovered from locally parameterizing the space of κ-symmetry WZW ferms for the M2-brane. In this section we consider mostly just a single (exceptional) tangent space. Below in 3 we look at the globalization of this story over curved superspacetimes.

It is a famous fact [BST87] that

- a) the equations of motion of 11-dimensional supergravity imply that the bilinear fermionic component $G_4^{\psi\psi}$ of the super-4-form flux on 11-dimesnional spacetime X is a *definite form* (in terminology borrowed from that of G_2-manifolds), which in each tangent space is $\mathrm{Spin}(10,1)$-equivalent to the left-invariant super 4-form $\overline{\psi} \wedge \Gamma^{a_1 a_2} \wedge \psi \wedge e_a \wedge e_b$ on super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$.
- b) $G_4^{\theta\theta}$ is the curvature 4-form of the κ-symmetry WZW term for the M2-brane sigma-model with target space the give 11d superspacetime.

What has arguably found less attention is that the definition of the M2-brane sigma model with target space a curved superspacetime X is not complete with just this 4-form curvature: the higher WZW term in the M2-brane action functional is locally a choice of form potential C_3 for $G_4^{\psi\psi}$, and globally it is the 3-connection of a 3-bundle (2-gerbe) whose local connection 3-forms are given by these choices of C_3. (Such a 3-bundle with 3-connection is a higher pre-quantization of $G_4^{\psi\psi}$ regarded as a pre-3-plectic form.) One place in the physics literature where the need of this extra information is at least mentioned is [Wi86, page 17].

A systematic study of 11d-supergravity with these pre-quantum corrections coming from the M2 and the M5-brane sigma-models included is in [Sc13], with lecture notes in [Sc15b]. For the moment here we will focus just on the space of local choices, and stay within the realm of traditional differential geometry. We will see that the space of local choices is naturally parameterized by splittings of the 11d exceptional generalized tangent bundle, hence by exceptional generalized metrics.

It is useful to state the problem of parameterizing spaces of form potentials for left-invariant closed forms in generality, to separate its general structure from the intricacies of its application to M2-branes WZW terms. In generality it looks as follows.

2.1 Atiyah sequence for $(p+1)$-form connections

Consider a germ of a Lie group G (hence a "local Lie group" where we consider working on arbitray small contractible neighbourhoods of the neutral element of an actual Lie group and ignore the global topology of the group). Consider furthermore a closed an left-invariant differential $(p+2)$-form

$$\omega \in \Omega^{p+2}_{\mathrm{cl,li}}(G)\,.$$

Since we are working just locally on a germ, by the Poincaré lemma ω is guaranteed to have a potential

$$A \in \Omega^{p+1}(G)$$

in that

$$dA = \omega$$

where of course A may not be left-invariant itself, unless ω comes from a trivial Lie algebra cocycle. But we may force that to happen after passing to an extension:

Assume that there is an extension of (germs of) Lie groups

$$p : \hat{G} \longrightarrow G$$

with the property that pulled back along this extension, ω does become left-invariantly trivial, i.e. such that there is a left invariant potential form

$$\hat{A} \in \Omega^{p+1}_{\mathrm{li}}(\hat{G})$$

such that

$$d\hat{A} = p^*\omega\,.$$

If this may be found, then (at least part of) the space of potentials for ω down on G has a neat parameterization as follows.

Every splitting

$$\sigma : G \longrightarrow \hat{G}$$

of the bundle underlying the extension (i.e. a section of the underlying map of (germs of) smooth manifolds, not required to respect the group structure) gives rise to a potential for ω, namely the pullback $\sigma^*\hat{A}$ of the left-invariant "reference potential" which we assumed to exist on $\hat{G}$:

$$\begin{aligned} d(\sigma^*\hat{A}) &= \sigma^*(d\hat{A}) \\ &= \sigma^*(p^*\omega) \\ &= (p\circ\sigma)^*\omega\,. \\ &= \mathrm{id}^*\omega \\ &= \omega \end{aligned}$$

Notice that by the left-invariance of $\hat{A}$, two sections σ that differ by an action of $\hat{G}$ on itself give rise to the same potential form: For every element $\hat{g} \in \hat{G}$ write $L_{\hat{g}} : \hat{G} \longrightarrow \hat{G}$ for the action on $\hat{G}$ given by left-multiplication. Then

$$\begin{aligned} (L_{\hat{g}}\circ\sigma)^*\hat{A} &\simeq \sigma^*(L_{\hat{g}}^*\hat{A}) \\ &\simeq \sigma^*\hat{A} \end{aligned}\,.$$

This means that the parameterization of potential forms which we found is really the quotient space $\Gamma_G(\hat{G})/\hat{G}$. But this has a nice re-interpretation: this is equivalently the space of *pointed* sections of p (those that send the neutral element of G to the neutral element of $\hat{G}$).

This is useful, because it implies that as we restrict further from germs to infinitesimal neighbourhoods, hence to Lie algebras, then the space of sections becomes the space of *linear* splittings of the Lie algebra extension $\hat{\mathfrak{g}} \longrightarrow \mathfrak{g}$:

$$0 \longrightarrow \ker(p_*) \longrightarrow \hat{\mathfrak{g}} \underset{p_*}{\overset{\sigma_*}{\rightleftarrows}} \mathfrak{g} \longrightarrow 0\,.$$

This is a very familiar situation. An example of this at the level of Lie algebroids is the Atiyah sequence of a principal bundle, whose fiberwise linear splittings correspond to choices of connection 1-forms. Here we see something analogous for connection $(p+1)$-forms.

Notice that on the level of Lie algebras $\hat{A}$ is identified with an element of the Chevalley-Eilenberg dg-algebra $\mathrm{CE}(\hat{\mathfrak{g}})$ such that $d_{\mathrm{dCE}}\hat{A} = (p_*)^*\omega$.

2.2 Realization for M2-brane WZW terms

We may now specify the above general discussion to the case of the M2-brane WZW term.

In this case (as reviewed in [FSS13b])

- $\mathfrak{g} := \mathbb{R}^{10,1|\mathbf{32}}$;
- $\omega := \overline{\psi}\wedge\Gamma^{a_1a_2}\overline{\psi}\wedge e_{a_1}\wedge e_{a_2}$

and so the question is if there exists a suitable super Lie algebra extension $p_* : \hat{\mathfrak{g}} \longrightarrow \mathfrak{g}$ and an element $\hat{A} \in \mathrm{CE}(\hat{\mathfrak{g}})$ such that

$$d_{\mathrm{CE}}\hat{A} = (p_*)^*\overline{\psi}\wedge\Gamma^{a_1a_2}\overline{\psi}\wedge e_{a_1}\wedge e_{a_2}\,.$$

If so, then all pullbacks of $\hat{A}$ along linear splittings of p_* are possible WZW terms for the M2-brane.

This is non-trivial. But it is precisely this problem that was solved already (even if not presented from the perspective used here) in [AuFr82, section 6] and more comprehensively in [BAIPV04].

These authors find [AuFr82, (6.2)] [BAIPV04, (28)] that there exists a 1-parameter class of solutions to this problem given by super Lie algebras $\hat{g}$ which are generically fermionic extensions of the M-theory super Lie algebra [To95, H97], and hence whose bosonic body is generically:

$$\hat{\mathfrak{g}}_{\mathrm{bos}} \simeq \mathbb{R}^{10,1} \oplus \wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^*$$

(except for one value of the parameter, at which the $\wedge^5$-summand disappears). Moreover, these authors find a class of Chevalley-Eilenberg 3-forms $\hat{C}$ that trivialize the M2-brane 4-cocycle on this extension. It is given for

$$s \in \mathbb{R} - \{0\}$$

by [AuFr82, (6.1)] [BAIPV04, (28)]

$$\hat{C}(s) := \alpha_{\mathrm{LP}}(s) \underbrace{B^{a_1 b_1} \wedge e_{a_1} \wedge e_{a_2}}_{\hat{C}_{\mathrm{LP}}} + \alpha_{\mathrm{CS}}(s) \underbrace{B^{a_1}{}_{a_2} \wedge B^{a_2}{}_{a_3} \wedge B^{a_3}{}_{a_1}}_{\hat{C}_{\mathrm{CS}}} + \cdots ,$$

where $\{B^{a_1 a_2}\}$ is a basis for the left-invariant 1-forms on the summands $\wedge^2(\mathbb{R}^{10,1})^*$, and where we show only the terms generated by $\{e_a\}$ and $\{B^{a_1 a_2}\}$.

According to [BAIPV04, (30)] we have

- for $s = -3$ then $\alpha_{\mathrm{CS}}(-3) = 0$ and with it the second term above vanishes
- for $s \to 0$ then the bosonic part of $s\hat{C}(s)$ goes to $\hat{C}_{\mathrm{CS}}$

We observe below in section 2.3 that $\hat{C}_{\mathrm{LP}}$ akin to a Liouville-Poincaré form on a cotangent bundle, while $\hat{C}_{\mathrm{CS}}$ is akin to a Chern-Simons form.

In this context it is maybe curious that in the limit $s \to 0$ the M-theory super Lie algebra here becomes a limiting case of $\mathfrak{osp}(1|32)$ [FIO15].

$$e^a := dx^a + \theta\Gamma^a d\theta$$
$$\psi^\alpha = d\theta^\alpha$$

2.3 The M2-Liouville-Poincaré form and 3-form shift symmetry

Proposition 2.1. *Given a bosonic 3-form $C \in \wedge^3(\mathbb{R}^{10,1|\mathbf{32}})^*$ then the linear splitting*

$$\begin{aligned} \mathbb{R}^{10,1} &\xrightarrow{\sigma_*^C} \mathbb{R}^{10,1} \oplus \wedge^2(\mathbb{R}^{10,1})^* \\ v &\longmapsto (v, \iota_v C) \end{aligned}$$

has the property that

$$(\sigma_*^C)^* \hat{C}_{\mathrm{LP}} = C \,.$$

Corollary 2.2. *For $s = -3$ then*

$$(\sigma_*^C)^* \hat{C}(-3) = C \,.$$

In particular, the map from linear splittings to bosonic 3-forms is surjective.

Remark 2.3. The formula for the splitting in prop.2.1 is coincides with the formula that the literature on exceptional generalized geometry postulates to encode the 3-form degrees of freedom [Hull07, (4.2)] [PaWa08, (B.23)].

2.4 Gauge fields and Chern-Simons forms

On the other hand, consider which section would parameterize C via pullback if only the second summand $\hat{C}_{\mathrm{CS}}$ were present in $\hat{C}$, hence the case $s \to 0$. This would most naturally be understood by using the Lorentz metric to make the linear identification

$$\wedge^2(\mathbb{R}^{10,1})^* \xrightarrow{\simeq} \mathfrak{so}(10,1)\,.$$

Notice that this matches the role that $B^a{}_b$ plays in the super Lie algebra $\hat{\mathfrak{g}}$, where it acts on fermions via action with $B_{ab}\Gamma^{ab}$ on the spin representation, i.e. via the matrix representation of $\mathfrak{so}(10,1)$ on the fermions.

With such an identification, then a linear splitting is an $\mathfrak{so}(10,1)$-valued linear 1-form A, and the 3-form that it parameterizes is

$$(\sigma^A_*)^*\hat{C}_{\mathrm{CS}} = \langle A \wedge [A, A]\rangle\,,$$

where $\langle -, -\rangle$ is the invariant bilinear (Killing) form. This is of course the Chern-Simons form for the linear 1-form A regarded as a constant differential 1-form.

Hence we see that for generic value of the parameter s in the possible choices of $\hat{C}$, the 3-form potentials that are parameterized by linear splittings as above are naturally interpreted as having a component proportional to the Chern-Simons form of a nonabelian gauge field.

Now I don't see at the moment how this is more than a curiosity, but it seems suggestive of the following expectations

- such a Chern-Simons component is what one expects to see appear in heterotic Hořava-Witten "comapc-tifications" of the setup;
- in the context of gauged supergravity it is part of the R-symmetry that is being gauged, and from the 11-dimensional perspective that R-symmetry is an isometry of the compactification space, hence is locally a Lorentz transformation;
- the interpretation of the splitting as a 1-form with values in bivectors is also the natural interpretation in the context of Kaluza-Klein reduction of the on fibers with 2-cycles by which the 3-form C is fiber integrated to a space of 1-forms $A^i := \int_{\Sigma^i} C$. For this case, too, it is folklore that the $\{A^o\}$, which a priori are abelian, become gauged under a nonabelian group.

3 From higher geometry to exceptional geometry

We give now the general abstract formulation.

3.1 Definite forms

We discuss formalization of the concept of *definite forms* in the sense in which they traditionally appear for instance in G_2-structure, but pre-quantized to WZW-terms.

Throughout, let $\mathbb{G}$ be a braided cohesive ∞-group, def. **??**, equipped with a Hodge filtration, def. **??**, and write $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ for the corresponding differential coefficient object, def. **??**.

Definition 3.1. Given

1. a $\mathbb{G}$-principal connection $\nabla^F : F \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$, def. **??**;
2. an F-fiber bundle $E \to X$, def. **??**;

then a *definite parameterization* of ∇ by E is a $\mathbb{G}$-principal connection on the total space of the fiber bundle

$$\nabla^E : E \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}} ,$$

such that the equivariant differential concretification $\mathrm{conc}//\mathbf{Aut}(F) \circ \sigma(\nabla^E)$, prop. **??**, of the section $\sigma(\nabla^E)$ corresponding to ∇^E under prop. **??**

$$\begin{array}{ccccc} & [F, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]//\mathbf{Aut}(F) & \xrightarrow{\mathrm{conc}//\mathbf{Aut}(F)} & \mathbb{G}\mathbf{Conn}(F)//\mathbf{Aut}(F) \\ {}^{\sigma(\nabla^E)}\nearrow & & \searrow \quad \swarrow & \\ X & \longrightarrow & \mathbf{BAut}(F) & \end{array}$$

is definite, def. **??**, on $* \xrightarrow{\vdash \nabla^F} [F, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \xrightarrow{\mathrm{conc}} \mathbb{G}\mathbf{Conn}(X)$.

Proposition 3.2. *There is a canonical ∞-functor from definite parameterizations of ∇^F over $E \to X$, def. 3.1, to lifts, def.* **??**, *of the structure group of E (via prop.* **??***) through the quantomorphism ∞-group extension, def.* **??**

$$\begin{array}{ccc} & & \mathbf{BQuantMorph}(\nabla^F) \\ & {}^{\mathbf{g}}\nearrow & \downarrow \\ X & \longrightarrow & \mathbf{BAut}(F) \end{array} .$$

Specifically if the structure ∞-group of E has already been reduced along some $G \to \mathbf{HamSympl}(\nabla^F)$, then there is a canonical ∞-functor from definite parameterizations to lifts to $\mathbf{Heis}_G(\mathbf{L}_{\mathrm{WZW}})$-structures

$$\begin{array}{ccc} & & \mathbf{BHeis}_G(\nabla^F) \\ & {}^{\mathbf{g}}\nearrow & \downarrow \\ X & \longrightarrow & \mathbf{BAut}(F) \end{array} .$$

In particular for a definite parameterization on $E \to X$ to exist it is necessary that E admits a lift to $\mathbf{QuantMorph}(\nabla^F)$*-structure.*

Proof. By prop. **??** and prop. **??**. □

Corollary 3.3 (obstruction to definite parameterizations)**.** *With $E \to X$ and ∇^F as in def. 3.2, assume that the structure group of E is reduced along $\mathbf{HamSympl}(\nabla^F) \hookrightarrow \mathbf{Aut}(F)$, def.* **??**. *Then an obstruction for a definite parameterization, def. 3.1, of ∇^F over $E \to X$ to exist is the obstruction class $[\mathbf{P}_\nabla(E)]$ of def.* **??**.

We now consider definite parmeterizations of WZW terms over infinitesimal disk bundles, which are induced from WZW terms on the base space.

Definition 3.4. Let V be a framed object, def. **??** and $\nabla^{\mathbb{D}^V} : \mathbb{D}^V \longrightarrow \mathbf{B}\mathbb{G}_{\text{conn}}$ a $\mathbb{G}$-principal connection, def. **??** on its infinitesimal disk, def. **??**. Then for X a V-manifold, def. **??**, a $\mathbb{G}$-principal connection $\nabla^X : X \longrightarrow \mathbf{B}\mathbb{G}_{\text{conn}}$ on X is a *definite globalization* of $\nabla^{\mathbb{D}^V}$ over X if its pullback $\nabla^{T^k X}$ to the infinitesimal disk bundle along the horizontal map in def. **??**

$$\nabla^{T^k X} : T^k X \xrightarrow{\text{ev}} X \xrightarrow{\nabla^X} \mathbf{B}\mathbb{G}_{\text{conn}}$$

is a definite parameterization of $\nabla^{\mathbb{D}^V}$ over $T^k X$ in the sense of def. 3.1.

Proposition 3.5. *There is a canonical functor from definite globalizations of ∇ over X, def. 3.4, to* $\mathbf{QuantMorph}(\nabla^{\mathbb{D}^V})$*-structures on X, i.e. to G-structures on X, def.* **??**, *for G the quantomorphism group of $\nabla^{\mathbb{D}^V}$, def.* **??**.

Proof. The defining construction $\nabla^X \mapsto \nabla^{T^k X}$ is clearly functorial, being given by precomposition. Then prop. 3.2 gives a functor sending the $\nabla^{T^k X}$ further to $\mathbf{Stab}_{\mathrm{GL}(V)}(\nabla^{\mathbb{D}^V})$-structures on X. By prop. **??** these are equivalently $\mathbf{QuantMorph}(\nabla^{\mathbb{D}^V})$-structures. □

Corollary 3.6 (obstruction to definite globalization)**.** *An obstruction for a definite globalization of $\nabla^{\mathbb{D}^V}$ over X to exist is the obstruction class*

$$\mathbf{P}_{\nabla^{\mathbb{D}^V}}(X) := \mathbf{P}_{\nabla^{\mathbb{D}^V}}(T^k X)$$

of def. **??**.

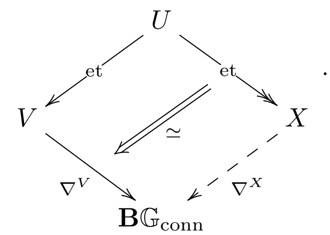
Proof. By prop. 3.5 and corollary 3.3. □

Definition 3.7. We call a definite globalization as in def. 3.4, *infinitesimally integrable* if the $\mathbf{QuantMorph}(\nabla^{\mathbb{D}^V})$-structure corresponding to it under prop. 3.5 is infinitesimally integrable according to def. **??**.

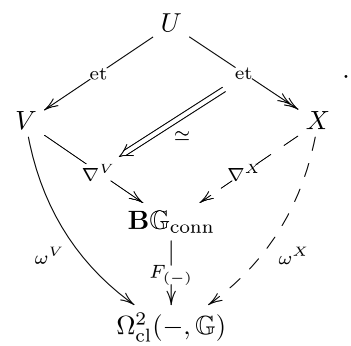
So far the obstructions in corollary 3.3 and corollary **??** are such that their vanishing is necessary but possibly not sufficient for the existence of a definite globalization. This is because, by construction, they obstruct precisely the existence of the differential concretification of the section corresponding to a global principal connection, but not necessarily the existence of that section itself, before differential concretification. That is to say, when these obstructions vanish then a definite and diffentially concrete section of the $\mathbb{G}\mathbf{Conn}(\mathbb{D}^V)$-fiber bundle associated to the frame bundle is guarateed to exist, but the above results do not guarantee yet, that this concrete section comes from an un-concrete section obtained by restricting a global $\mathbb{G}$-principal connection to all infinitesimal disks. We need to refine the obstruction information in order to guarantee this.

To this end, we now consider fully integrable definite globalization, i.e. such that do not only coincide with the prescribed prequantum geometry on infinitesimal disks as in def. 3.7, but do so on an entire V-cover, def. **??**.

Definition 3.8. Given a V-manifold with V-cover $V \leftarrow U \rightarrow X$ and given a $\mathbb{G}$-principal connection $\nabla^V : V \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$, def. **??**, an *integrable definite globalization* of ∇^V over X is a $\mathbb{G}$-principal connection on X $\nabla^X : X \dashrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ such that there is a homotopy



Remark 3.9. The notion in def. 3.8 is the pre-quantization, def. **??**, of the integrable globalization of just the curvature ω^V of the connection:



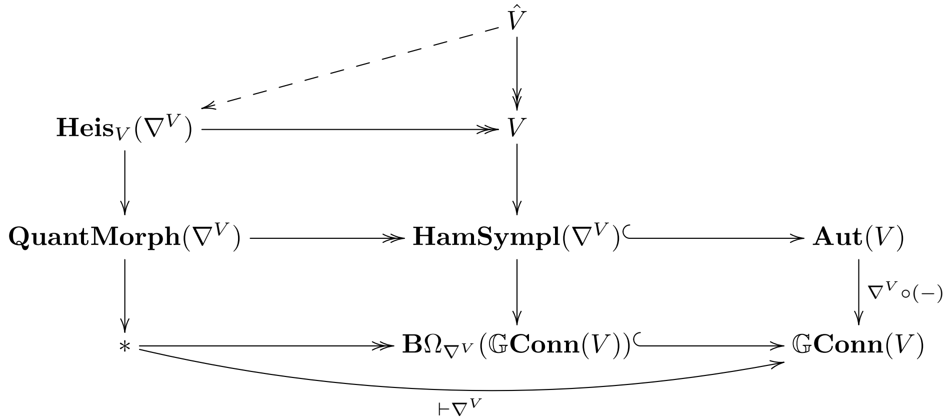
Example 3.10. Given an integrable globalization as in def. 3.8, forget the connection and consider just the maps modulating the underlying $\mathbb{G}$-principal bundles $P^V \to V$ and $P^X \to X$, respectively. Then base-chaning the correspondence diagram along the point inclusion $* \to \mathbf{B}\mathbb{G}$ and using that both local diffeomorphisms as well as 1-epimorphisms are stable under pullback, it follows that P^X is a P^V-manifold.

Definition 3.11. Let V be an object equipped with the structure of a differential cohesive group, def. **??**. We say that a $\mathbb{G}$-principal connection, def. **??** $\nabla^V : V \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ is *equivariant* if the left ∞-action of V on itself, def. **??**, is Hamiltonian in that it factors

$$V \longrightarrow \mathbf{HamSympl}(\nabla^V) \longrightarrow \mathbf{Aut}(V)$$

through the object underlying the Hamiltonian symplectomorphism ∞-group, def. **??**, of ∇^V.

Remark 3.12. The condition in def. 3.11 means that there exists a cover $\hat{V}$ of V over which the left V-action on itself factors through the Heisenberg group, def. **??**, of ∇^V, hence that we have the dashed morphism in the following diagram (from the proof of theorem **??**):



Notice that we do not require the dashed morphism to respect group structure.

For instance for ∇^V the canonical prequantum bundle on a symplectic vector space (V, ω), then, by the discussion in **??**, $\mathbf{Heis}_V(\nabla^V)$ is the traditional Heisenberg group extension $U(1) \to \mathrm{Heis}(V, \omega) \to V$. While as a group extension this does not split, as a map of underlying spaces is the trivial $U(1)$-principal bundle over V and hence does split and admit a dashed section as above, even with $\hat{V} = V$.

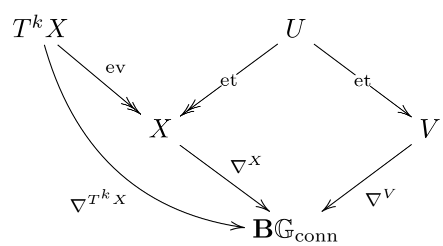
Now the total left part of the diagram says that restricted along $\hat{V} \to V$ the operation $V \overset{\nabla^V \circ (-)}{\longrightarrow} \mathbb{G}\mathbf{Conn}(V)$ of (left-)translating the connection over V is cohesively gauge equivalent to the trivial action, hence that the translation may be gauged away. This is the refinement of the curvature form ω^V being genuinely left invariant over all of V.

Theorem 3.13. *Given a differentially cohesive group V, given a V-manifold X, def. **??**, given an equivariant connection ∇^V, def. 3.11, then a necessary condition for the existence of an integrable definite globalization ∇^X, of ∇^V over X, def. 3.8, is the existence of a G-structure on X, def. **??**, for $G = \mathbf{QuantMorph}(\mathbf{L}_{\nabla^{\mathbb{D}^V}})$ the quantomorphism group, def. **??**, of the restriction*

$$\nabla^{\mathbb{D}^V} : \mathbb{D}^V \to V \overset{\nabla^V}{\longrightarrow} \mathbf{B}\mathbb{G}_{\mathrm{conn}}$$

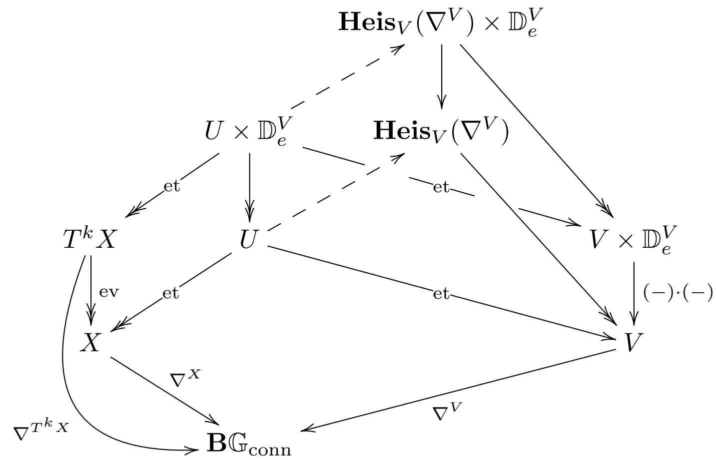
*of ∇^V to the infinitesimal disk, def. **??**, of V, such that moreover this G-structure is integrable, def. **??**, relative to the left-invariant G-structure $\mathbf{g}_{\mathrm{LI}}$ of V, example **??**.*

Proof. Assuming ∇^X exists, consider its pullback to the infinitesimal disk bundle via the horizontal map ev in the defining pullback in def. **??**:



We now find a necessary conditions for $\nabla^{T^k X}$ to exist, which is hence also a necessary condition for ∇^X to exist.

First observe that by prop. **??** the infinitesimal disk bundle of U is both the pullback of that on X as well as of that on V. By prop. **??** the latter is canonically trivialized via left translation such that the map ev restricts over the V-cover to the left action of V on its infinitesimal disk $\mathbb{D}^V_e$ at the neutral element. This means that the above diagram completes to a pasting composite as shown by solid arrows in the following diagram.



Moreover, by the assumptions in def. 3.11 the connection ∇^V is locally invariant under left translation, up to gauge transformation, as discussed in remark 3.12, (possibly after further refining the cover U via the cover $\hat{V}$ of V, which we suppress notationally) so that we get the dashed lifts in the above diagram.

By prop. **??**, $\nabla^{T^k X}$ is equivalently a section σ of the associated $[\mathbb{D}^V_e, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]$-fiber bundle, such that σ is locally on U equivalent to the $((- \times \mathbb{D}^V_e) \dashv [\mathbb{D}^V_e, -]))$-adjunct of

$$U \times \mathbb{D}^V_e \xrightarrow{\mathrm{ev}|_U} U \xrightarrow{\mathrm{et}} V \xrightarrow{\nabla^V} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \ .$$

Under differential concretification $[\mathbb{D}^V_e, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \to \mathbb{G}\mathbf{Conn}(\mathbb{D}^V_e)$ (def. **??**) this implies, via prop. **??**, a section σ_{conc} of the associated $\mathbb{G}\mathbf{Conn}(\mathbb{D}^V_e)$-bundle.

But by the above diagram, the section σ is locally equivalently the adjunct of

$$U \times \mathbb{D}^V_e \xrightarrow{\mathrm{et}} V \times \mathbb{D}^V_e \xrightarrow{(-)\cdot(-)} V \xrightarrow{\nabla^V} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \ ,$$

which in turn is equivalently the adjunct of

$$U \times \mathbb{D}^V_e \dashrightarrow \mathbf{Heis}_V(\nabla^V) \times \mathbb{D}^V_e \longrightarrow V \times \mathbb{D}^V_e \xrightarrow{(-)\cdot(-)} V \xrightarrow{\nabla^V} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \ ,$$

and so σ_{conc} is of the form

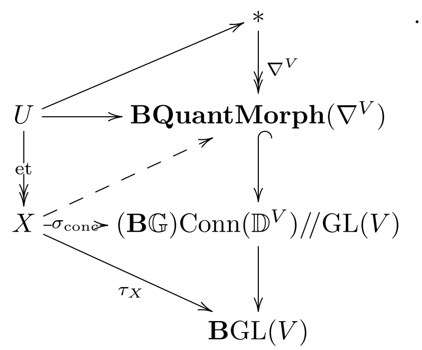
$$U \longrightarrow \mathbf{Heis}_V(\nabla^V) \longrightarrow \mathbb{G}\mathbf{Conn}(\mathbb{D}^V_e) \ .$$

But by the diagram in remark 3.12 this means that σ_{conc} is locally constant, up to equivalence.

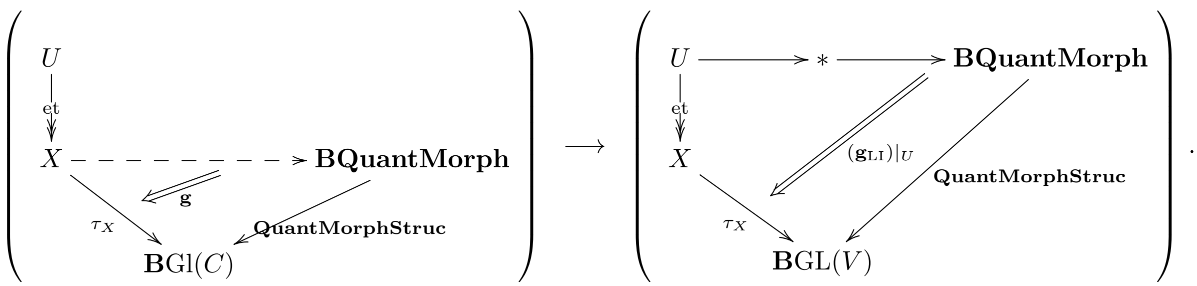
Therefore by prop. **??** and prop. **??** the existence of σ_{conc} is equivalent to a $\mathbf{QuantMorph}(\nabla^{\mathbb{D}^V})$-structure (def. **??**) on X.

Finally, to see that this structure is integrable, def. **??**, notice from the proof of prop. **??** that this

$\mathbf{QuantMorph}(\nabla^{\mathbb{D}^V})$-structure is given by the dashed diagonal lift in



with the left morphism being formally étale by the above construction. Taking this pasting diagram apart, it may be viewed as giving a morphism in the double slice $(\mathbf{H}_{/\mathbf{B}\mathrm{GL}(V)})_{/\mathbf{QuantMorph}(\nabla^{T^k X})\mathbf{Struc}}$



Here the codomain, given by the total pasting diagram, exhibits the constancy of the concretified section σ_{conc} as obtained above. This was obtained from left translation over V with respect to the left invariant framing, prop. **??**, of V, hence the homotopy shown on the right is that exhibits the left invariant G-structure $\mathbf{g}_{\mathrm{LI}}$ of example **??**.

The domain is the structure $\mathbf{g}$ that we constructed by way of the section σ_{conc} and the dashed lift obtained from the homotopy which exhibits this section as constant on U relative to the given trivialization of the frame bundle of U. Finally the morphism itself is the pasting of the diagram for $\mathbf{g}$, pulled back to U, with the top diagonal rectangular part of the original pasting diagram, yielding the diagram for $\mathbf{g}_{\mathrm{LI}}$. Hence this diagram exhibits the integrability according to def. **??**. □

3.2 Generalized geometry

The definition of definite globalizations of principal connections above in 3.1 constrains both the curvature as well as the connection data to be locall equivalent to that of a fixed reference connection. More generally one may ask only the curvature to be definite, and leave the connection data less constrained, hence allow more general pre-quantization of a given closed form data. The extra choices involved in such a globalization turn out to subsume in special case structure that in the literature is known as *generalized geometry* [Hi11, Hull07].

Let $\mathbf{H}$ be an ∞-topos equipped with differential cohesion. Throughout, let $\mathbb{G}$ be a braided cohesive ∞-group in $\mathbf{H}$, def. **??**, equipped with a Hodge filtration, def. **??**, and write $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ for the corresponding differential coefficient object, def. **??**.

The following definition accordingly relaxes def. 3.4.

Definition 3.14. Let

$$\begin{array}{ccc} \hat{V} & \xrightarrow{\hat{\nabla}} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \\ {\scriptstyle p}\downarrow & & \\ V & & \end{array}$$

be a group extension p, def. **??**, equipped with a a $\mathbb{G}$-principal ∞-connection $\hat{\nabla}$.

Then for X a V-manifold, def. **??**, a $\mathbb{G}$-principal connection $\nabla^X : X \longrightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ on X is a *p-definite globalization* of $\nabla^{\mathbb{D}^V}$ over X if its pullback $\nabla^{T^k X}$ to the infinitesimal disk bundle along the horizontal map in def. **??**

$$\nabla^{T^k X} : T^k X \xrightarrow{\mathrm{ev}} X \xrightarrow{\nabla^X} \mathbf{B}\mathbb{G}_{\mathrm{conn}}$$

is a p-definite parameterization of $\nabla^{\mathbb{D}^{\hat{V}}}$ over $T^k X$ in that for the corresponding section

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{conc}(\sigma(\nabla^{T^k X}))} & \mathbb{G}\mathbf{Conn}(\mathbb{D}^V)/\mathrm{GL}(V) \\ & {\scriptstyle \tau_X}\searrow \quad \swarrow & \\ & \mathbf{B}\mathrm{GL}(V) & \end{array}$$

there exists a cover $U \twoheadrightarrow X$, a map $\hat{g} : U \longrightarrow \Gamma_{\mathbb{D}^V}(\mathbb{D}^{\hat{V}})$ and a homotopy filling the following diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{g}} & \Gamma_{\mathbb{D}^V}(\mathbb{D}^{\hat{V}})/\mathrm{GL}(V) \\ \Big\downarrow & \swArrow & \Big\downarrow{\scriptstyle \simeq} \\ & & (\Gamma_{\mathbb{D}^V}(\mathbb{D}^{\hat{V}}) \times *)/\mathrm{GL}(V) \\ & & \Big\downarrow{\scriptstyle (\mathrm{id}, \hat{\nabla}^{\mathbb{D}^{\hat{V}}})/\mathrm{GL}(V)} \\ & & (\Gamma_{\mathbb{D}^V}(\mathbb{D}^{\hat{V}}) \times \mathbb{G}\mathbf{Conn}(\mathbb{D}^{\hat{V}}))/\mathrm{GL}(V) \\ & & \Big\downarrow{\scriptstyle \mathrm{ev}} \\ X & \xrightarrow{\mathrm{conc}(\sigma(\nabla^{T^k X}))} & \mathbb{G}\mathbf{Conn}(\mathbb{D}^V)/\mathrm{GL}(V) \\ & {\scriptstyle \tau_X}\searrow \quad \swarrow & \\ & \mathbf{B}\mathrm{GL}(V) & \end{array} .$$

One choice of such data we say is a $(p, \hat{\nabla})$*-generalized geometry* on X.

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