Homotopy axiomatic cohesion

Some basic ideas

Urs Schreiber

December 6, 2011

Special thanks to

- Mike Shulman
- Dave Carchedi
- Richard Williamson

Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

Outline

Geometry

Internal language

Homotopy theory

Cohesive ∞-toposes

Goal:

Pair

geometry $+$ homotopy theory

axiomatically.

Motivation:

Gauge theory in physics:

- fields are smooth functions on spacetime;
- gauge transformations are "smooth homotopies";
- gauge-of-gauge transformation are "higher smooth homotopies".

Axiomatization unifies varying details, such as replacing smooth geometry by supergeometry.

I

Geometry: Cohesive toposes

In geometry, toposes play two different roles

1. as generalized topological spaces;
2. as *collections* of geometric spaces.

Axiomatic cohesion characterizes toposes $\mathbf{H}$ of the second kind.
(Lawvere)

First axiom – H is local:

There exist adjoint functors

$$\mathbf{H} \; \begin{array}{c} \xleftarrow{\quad \mathrm{Disc} \quad} \\ \xrightarrow{\quad \Gamma \quad} \\ \xleftarrow{\quad \mathrm{coDisc} \quad} \end{array} \; \mathrm{Set} \,,$$

Interpretation:

$$\mathbf{H} \; \begin{array}{c} \xleftarrow{\text{discrete cohesive structure}} \\ \xrightarrow{\text{underlying point set}} \\ \xleftarrow[\text{codiscrete cohesive structure}]{} \end{array} \; \mathrm{Set}$$

Second axiom – H is strongly connected:

$$\mathbf{H}\;\begin{array}{c} \xrightarrow{\;\exists\;\Pi\;} \\ \xleftarrow{\;\mathrm{Disc}\;} \\ \xrightarrow{\;\Gamma\;} \\ \xleftarrow{\;\mathrm{coDisc}\;} \end{array}\;\mathrm{Set}\;,$$

Π preserves finite products.

Interpretation:

$$\mathbf{H}(X, \mathrm{Disc} S) \simeq \mathrm{Set}(\Pi X, S)$$

says: Π sends any object to its set of *connected components*.

Consequence

The space $* \in \mathbf{H}$ underlying $\mathbf{H}$ looks like a fat point:

- strong connectedness: $\Pi * = *$;
- locality: if $X \to *$ is epi, then it has a section.

So $* \in \mathbf{H}$

- is connected
- "looks contractible";

Consequence

Since the generalized space underlying $X \in \mathbf{H}$ is the slice topos

$$\mathbf{H}_{/X} \xrightarrow{\text{étale}} \mathbf{H}_{/*} \,,$$

every $X \in \mathbf{H}$ is a space *locally modeled* on the fat point $\mathbf{H}_{/*}$:

the *abstract cohesive blob*.

Examples

- $\mathbf{H} = \mathrm{Sh}(\mathrm{TopologicalManifolds})$;
- $\mathbf{H} = \mathrm{Sh}(\mathrm{SmoothManifolds})$,
  contains generalized smooth spaces such as *diffeological spaces*.
- $\mathbf{H} = \mathrm{CahierTopos}$,
  a model for synthetic differential geometry;

II
Internal language

We may reflect cohesion structure back into $\mathbf{H}$.

$$(\mathbf{\Pi} \dashv \flat \dashv \sharp) : \ \mathbf{H} \underset{\Gamma}{\overset{\Pi}{\underset{\longleftarrow \mathrm{Disc} \hookleftarrow}{\longrightarrow}}} \mathrm{Set} \underset{\mathrm{coDisc}}{\overset{\mathrm{Disc}}{\underset{\longleftarrow \Gamma \longrightarrow}{\hookrightarrow}}} \mathbf{H}$$

and express these endofunctors in the *internal language* of $\mathbf{H}$.

Internal Type Theory language of a topos **H**

Dictionary

type	$\vdash X : \mathrm{Type}$	object $X \in \mathbf{H}$
term	$\vdash x : X$	morphism $x : * \to X$
dependent type	$x : X \vdash A(x) : \mathrm{Type}$	morphism $A \to X$
type of propositions	$\vdash \mathrm{Prop} : \mathrm{Type}$	subobject classifier Ω
proposition	$x : X \vdash \phi(x) : \mathrm{Prop}$	monomorphism $A \hookrightarrow X$ $\simeq$ morphism $\phi : X \to \Omega$
dependent sum	$\vdash \exists x, A(x)$	$A \in \mathbf{H}$
dependent product	$\vdash \forall x, A(x)$	object of sections $\Gamma_X(A)$

In this internal logic,
cohesion incarnates as *modalities* (qualified truth).
Notably $\sharp$ incarnates as a *geometric modality*:

$$\sharp : \mathrm{Prop} \to \mathrm{Prop}$$

defined by

$$\sharp : \; \Omega \xrightarrow{\text{unit}} \sharp\Omega \xrightarrow{\chi_{\sharp \mathrm{true}}} \Omega \; .$$

Sends subobjects $\phi : A \hookrightarrow X$ to the pullback

$$\begin{array}{ccc} & \longrightarrow & \sharp A \\ {\scriptstyle \sharp\phi}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & \sharp X \end{array} \; .$$

Say $\phi(x : X) : \mathrm{Prop}$ is *discretely true*
(true over a discrete space)
if $\sharp\phi$ is true.

For instance in $\mathbf{H} = \mathrm{Sh}(\mathrm{SmthMfd})$

$$\mathrm{isClosed}(\omega : \mathrm{Differential}n\mathrm{Form})$$

is discretely true.

$$\begin{array}{ccc} \Omega^n_{\mathrm{cl}}(-) & & \Omega^n(-) \\ \big\downarrow & \overset{\sharp}{\longmapsto} & \big\downarrow{\scriptstyle \mathrm{id}} \\ \Omega^n(-) & & \Omega^n(-) \end{array}$$

But:

Internal characterization of cohesion requires to lift modalities from propositions to all types

$$\sharp : \mathrm{Type} \to \mathrm{Type}$$

Needed to characterize reflection:

$$\forall X\ Y : \mathrm{Type}, \mathrm{isCodiscrete}(Y) :$$

$$[\sharp X, Y] \xrightarrow{\simeq} [X \to Y] .$$

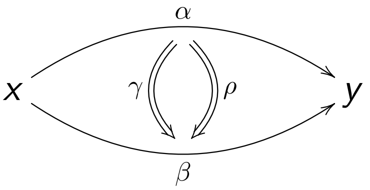
III
Homotopy (type) theory

Identity types. Let $\mathrm{Id}_X(x, y)$ be the type of *proofs* that terms x and y are equal.
Now drop the assumption that there are unique such terms:

there may be different proofs of the same fact, different *paths* from x to y.

Consider paths of paths of paths...

$$\gamma, \rho : \mathrm{Id}_{\mathrm{Id}_X(x,y)}(\alpha, \beta : \mathrm{Id}_X(x, y))$$

"∞-groupoid" : $X \simeq$ $\Bigg\{$ 

$\Bigg\}$

This now has an interpretation (Awodey, Warren) in $\mathbf{H}$ a *model topos* (Rezk, Lurie):

- class of "weak" equivalences $\xrightarrow{\simeq}$;
- class of "bundle morphisms" $\twoheadrightarrow$ = fibrations;
- compatible sSet-enrichment.

Refined dictionary:

type	$\vdash X : \mathrm{Type}$	fibrant object $A \twoheadrightarrow *$
dependent type	$x : X \vdash A(x) : \mathrm{Type}$	fibration $A \twoheadrightarrow X$
identity type	$x\ y : X \vdash \mathrm{Id}_X(x,y) : \mathrm{Type}$	path object $X^{\Delta[1]} \xrightarrow{(s,t)} X \times X$
type of small types	Type	*small object* classifier

IV
Synthesis:
Cohesive ∞-toposes

Using this, the relevant structures on a topos do have internal axiomatization:

- locality;
- connectivity;
- cohesion.

Mike Shulman, *Internalizing the external* (2011)

The resulting structure is automatically *homotopy cohesion*.

Example:

model topos

$$\mathbf{H} := \mathrm{PSh}(\mathrm{SmthMfd}, \mathrm{sSet})$$

with

- weak equivalences are the *stalkwise weak homotopy equivalences*;

Context for higher differential geometry:

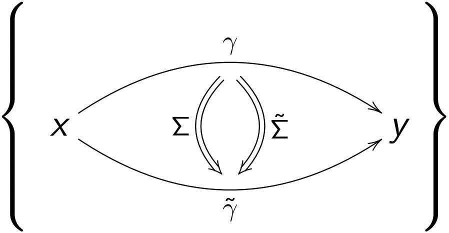
$$\mathbf{H} = \{ \text{ smooth } \infty\text{-groupoids } \}$$

The extra left adjoint

$$\Pi : \mathbf{H} \to \mathrm{sSet}$$

now sends a manifold X to its *path ∞-groupoid*

$\Pi X \simeq$



that consists of actual *geometric paths*, *geometric surfaces*, etc. in X.

Outlook:

Intrinsic notion of
geometric paths
induces
intrinsic notion of "dynamics":

differential cohomology.

End.

Details and further material at

`http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos`