

# M-theory from the superpoint

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Figure:  $\mathbb{R}^{0|1}$

# Prologue



Figure:  $\mathbb{R}^{0|1}$

$\mathbb{R}^{0|1}$  has a single odd coordinate  $\theta$ , and  $\theta^2 = 0$ , so a power series terminates immediately:

$$f(\theta) = f(0) + f'(0)\theta.$$

In essence, this means we should regard  $\theta$  as infinitesimal. Thus  $\mathbb{R}^{0|1}$  is a single point with an infinitesimal neighborhood, as depicted above.

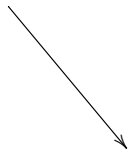
# Prologue

We will peer into the superpoint using *homotopy theory*.

Inside, we will find all the super Minkowski spacetimes of string theory and M-theory, going up to dimension 11.

Then we will find the strings,  $Dp$ -branes and M-branes themselves, thanks to the brane bouquet of Fiorenza, Sati and Schreiber.

$$\mathbb{R}^{10,1|32}$$



$$\mathbb{R}^{9,1|16+16} \leftarrow \mathbb{R}^{9,1|16} \rightarrow \mathbb{R}^{9,1|16+\overline{16}}$$



$$\mathbb{R}^{5,1|8+8} \leftarrow \mathbb{R}^{5,1|8} \rightarrow \mathbb{R}^{5,1|8+\overline{8}}$$



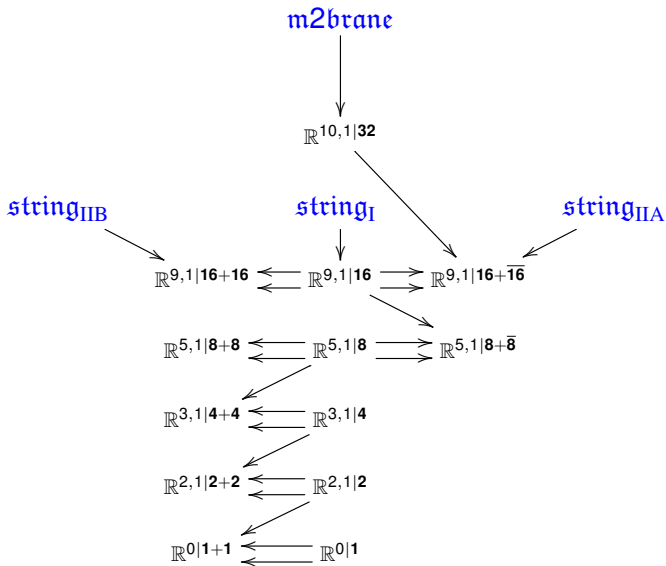
$$\mathbb{R}^{3,1|4+4} \leftarrow \mathbb{R}^{3,1|4}$$

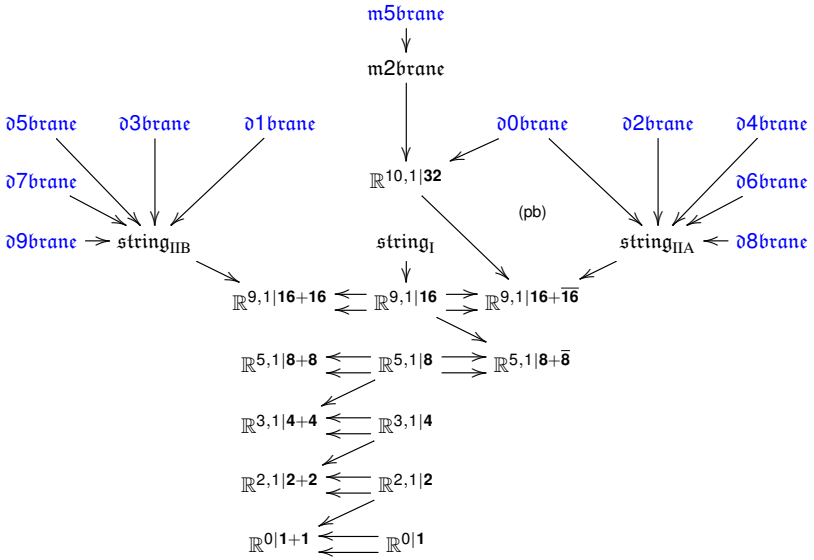


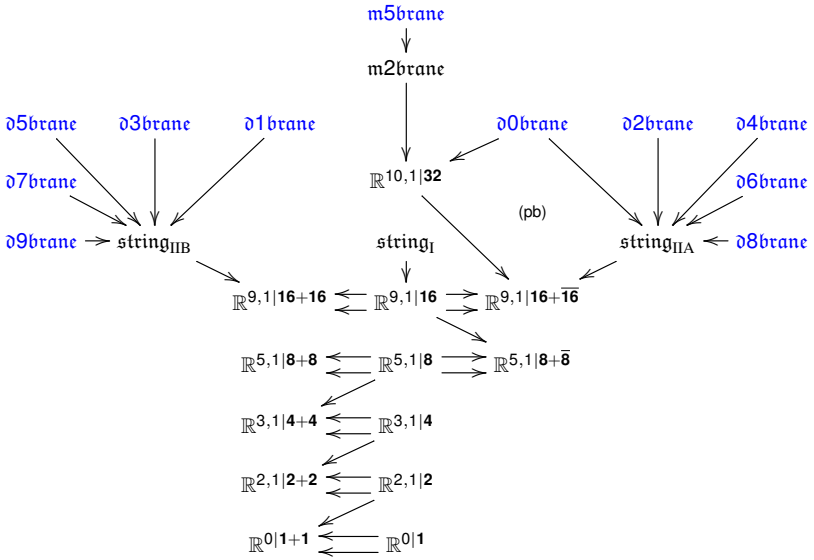
$$\mathbb{R}^{2,1|2+2} \leftarrow \mathbb{R}^{2,1|2}$$



$$\mathbb{R}^{0,1|1+1} \leftarrow \mathbb{R}^{0,1|1}$$







The brane bouquet.



# M-theory

In the mid-1990s, confronted with mounting evidence, the string theory community understood they must study extended objects of dimension  $> 1$ .

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Witten christened this topic

## **M-theory**

The M arguably stands for “membrane”.

# M-theory

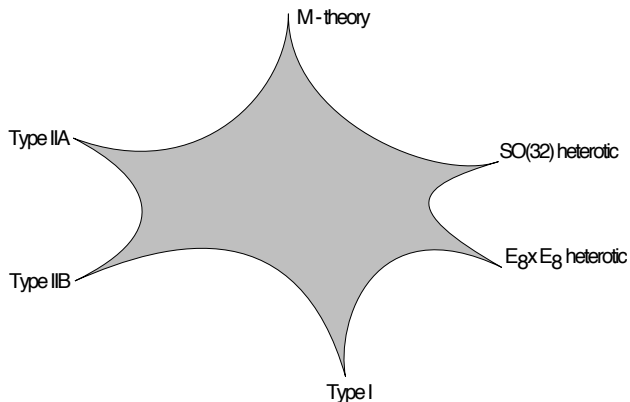


Figure: Polchinski's schematic.

In this highly schematic picture, M-theory unites the five 10d string theories (and 11d supergravity, not shown).

# M-theory

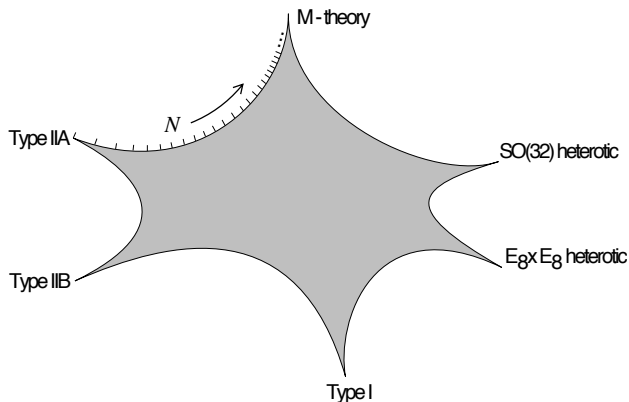


Figure: Polchinski's schematic.

Most directly, M-theory is a limit of type IIA string theory which “grows an extra dimension”.

# M-theory

10d spacetime becomes 11d:

type IIA string theory  $M^{10} \rightsquigarrow N^{11}$  M-theory.

Infinitesimally, 10d Minkowski spacetime becomes 11d:

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But everything in sight is supersymmetric, so it is more correct to pass between the appropriate ‘super Minkowski spacetimes’:

$$\mathbb{R}^{9,1|16+\overline{16}} \rightsquigarrow \mathbb{R}^{10,1|32}$$

We will see this is mathematically natural and beautiful: it is a central extension!

# Super Minkowski spacetime

- ▶  $\mathbb{R}^{d-1,1|\mathbf{N}}$  is the ‘super version’ of  $\mathbb{R}^{d-1,1}$ .
- ▶ Which is  $\mathbb{R}^d$  with the metric
$$\eta(u, v) = -u^0 v^0 + u^1 v^1 + \dots + u^{d-1} v^{d-1}.$$



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$$\eta(u, v) = -u^0 v^0 + u^1 v^1 + \dots + u^{d-1} v^{d-1}.$$
- ▶  $\mathbb{R}^{d-1,1|\mathbf{N}}$  is a **super Lie algebra**.
- ▶ Meaning it is a super vector space:

$$\mathbb{R}_{\text{even}}^{d-1,1|\mathbf{N}} = \mathbb{R}^{d-1,1}, \quad \mathbb{R}_{\text{odd}}^{d-1,1|\mathbf{N}} = \mathbf{N}$$

- ▶ Equipped with a Lie bracket:

$$[-, -]: \mathbb{R}^{d-1,1|\mathbf{N}} \otimes \mathbb{R}^{d-1,1|\mathbf{N}} \rightarrow \mathbb{R}^{d-1,1|\mathbf{N}}.$$

# Super Minkowski spacetime

This structure is dictated by representation theory.

- ▶  $\text{Spin}(d - 1, 1)$  is the double cover of the connected Lorentz group  $\text{SO}_0(d - 1, 1)$ .
- ▶  $\mathbb{R}^{d-1,1}$  is a representation of  $\text{Spin}(d - 1, 1)$ .
- ▶  $\mathbf{N}$  is a choice of a real spinor representation of  $\text{Spin}(d - 1, 1)$ .
- ▶ The bracket is a choice of a  $\text{Spin}(d - 1, 1)$ -equivariant map.

# Super Minkowski spacetime

Concretely, the bracket on  $\mathbb{R}^{d-1,1|\mathbf{N}}$  is:

- ▶ The only nonzero part of the bracket is the spinor-to-vector pairing:

$$[-, -]: \mathbf{N} \otimes \mathbf{N} \rightarrow \mathbb{R}^{d-1,1}.$$

- ▶ If  $\mathbf{N}$  is irreducible, this map is unique up to rescaling. If  $\mathbf{N}$  is reducible, there is more choice involved.
- ▶ Physicists write this bracket using gamma matrices:

$$[Q_\alpha, Q_\beta] = -2\Gamma_{\alpha\beta}^\mu P_\mu.$$

and call it an “anticommutator”, because  $Q_\alpha$  and  $Q_\beta$  are odd.

# Central extensions

Remember that, physically:

- ▶ Type IIA string theory lives on  $\mathbb{R}^{9,1|16+\overline{16}}$ .
- ▶ M-theory lives on  $\mathbb{R}^{10,1|32}$ .
- ▶ The M-theory hypothesis gives a physical process such that

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## Question

What is this process mathematically?

# Central extensions

It's a **central extension**!

Given

- ▶  $\mathfrak{g}$  a super Lie algebra,
- ▶  $\omega: \Lambda^2 \mathfrak{g} \rightarrow \mathbb{R}$  a 2-cocycle, meaning:

$$\omega([X, Y], Z) \pm \omega([Y, Z], X) \pm \omega([Z, X], Y) = 0,$$

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$$\omega([X, Y], Z) \pm \omega([Y, Z], X) \pm \omega([Z, X], Y) = 0,$$

we can form the central extension:

$$\mathfrak{g}_\omega = \mathfrak{g} \oplus \mathbb{R}c,$$

with one extra generator  $c$ , even and central, and modified Lie bracket:

$$[X, Y]_\omega = [X, Y] + \omega(X, Y)c.$$

# Central extensions

In particular:

- ▶  $\mathbb{R}^{10,1|32}$  is a central extension of  $\mathbb{R}^{9,1|16+\overline{16}}$ .
- ▶ The 2-cocycle is

$$\omega = d\theta^\alpha \wedge \Gamma_{\alpha\beta}^{01\dots 9} d\theta^\beta,$$

where  $\Gamma^{01\dots 9} = \Gamma^0 \Gamma^1 \dots \Gamma^9$ , and  $(x^\mu, \theta^\alpha)$  are the even and odd coordinates on  $\mathbb{R}^{9,1|16+\overline{16}}$ .



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Note that this really is a 2-cocycle:

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Note that this really is a 2-cocycle:

- ▶ It is left-invariant (as a form on the super Lie group).
- ▶  $d\omega = 0$ , by the naive calculation.

Moreover, it really does give  $\mathbb{R}^{10,1|32}$ , by the usual “yoga” of gamma matrices.

# Central extensions

## Notation

Every central extension comes with a projection map:

$$\mathfrak{g}_\omega \rightarrow \mathfrak{g}$$

that sets  $c$  to zero; we will often write this map to indicate central extension. For example:

$$\mathbb{R}^{10,1|32} \rightarrow \mathbb{R}^{9,1|16+\overline{16}}.$$

# The superpoint

This prompts a number of questions.

## Question

What singles out the 2-cocycle

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Are any other dimensions of spacetime due to central extension?

## Answer

All of them! This is our main result.

# The superpoint

At the extreme end, we could start with the superpoint  $\mathbb{R}^{0|1}$ , and study its central extensions.

## Definition

The **superpoint**  $\mathbb{R}^{0|1}$  is the super vector space consisting of  $\mathbb{R}$  in odd degree:

$$\mathbb{R}_{\text{even}}^{0|1} = 0, \quad \mathbb{R}_{\text{odd}}^{0|1} = \mathbb{R}.$$



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$$\mathbb{R}_{\text{even}}^{0|1} = 0, \quad \mathbb{R}_{\text{odd}}^{0|1} = \mathbb{R}.$$

- ▶ It has no Lie bracket;
- ▶ It has no metric;
- ▶ It has no spin structure.

We will discover all structure through central extension.

# The superpoint

$\mathbb{R}^{0|1}$  exactly one 2-cocycle:

$$d\theta \wedge d\theta$$

Extending by this 2-cocycle gives  $\mathbb{R}^{1|1}$ , the superline, the worldline of the superparticle:

$$\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}.$$

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Can we find more dimensions?

# Maximal invariant central extensions

This is a game with two moves:

- ▶ We can extend by all 2-cocycles satisfying a suitable invariance condition.
- ▶ We can double the number of spinors.

This will lead us from the superpoint up to 11 dimensions and beyond.

## Maximal invariant central extensions

We want that “suitable invariance condition” to be invariance under  $\text{Spin}(d - 1, 1)$ , but we don't have a metric!

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**Proposition (H.–Schreiber, folklore)**

For a super Minkowski spacetime  $\mathbb{R}^{d-1,1|\mathbf{N}}$ , its connected automorphism group is:

$$\text{Aut}_0(\mathbb{R}^{d-1,1|\mathbf{N}}) \simeq \mathbb{R}^+ \times \text{Spin}(d - 1, 1) \times \text{R-group}$$

where the R-group acts trivially on  $\mathbb{R}^{d-1,1}$ .

Thus, we can recover the group  $\text{Spin}(d - 1, 1)$  by considering the automorphisms of the Lie bracket alone.

# The dimensional ladder

## Dimension 3

First, we will double the number of fermionic dimensions:

$$\mathbb{R}^{0|2}$$

We will write this operation as follows:

$$\mathbb{R}^{0|2} \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \mathbb{R}^{0|1}$$

Now,  $\mathbb{R}^{0|2}$  has two odd generators,  $\theta_1$  and  $\theta_2$ , and there are three 2-cocycles:

$$d\theta_1 \wedge d\theta_1, \quad d\theta_1 \wedge d\theta_2, \quad d\theta_2 \wedge d\theta_2.$$

Extending by all three we get:

$$\mathbb{R}^{3|2} \longrightarrow \mathbb{R}^{0|2}.$$

# The dimensional ladder

Dimension 3

Now something remarkable happens: a metric appears!

$$\text{Aut}_0(\mathbb{R}^{3|2}) = \mathbb{R}^+ \times \text{Spin}(2, 1).$$

Thanks to this metric, we can look for  $\text{Spin}(2, 1)$ -invariant 2-cocycles on  $\mathbb{R}^{2,1|2}$ . There are none, because the only  $\text{Spin}(2, 1)$ -invariant map:

$$\mathbf{2} \otimes \mathbf{2} \rightarrow \mathbb{R}$$

is antisymmetric.



# The dimensional ladder

Dimension 4

Double the number of spinors again:

$$\mathbb{R}^{2,1|2+2} \longleftarrow \mathbb{R}^{2,1|2}$$

There is precisely one  $\text{Spin}(2, 1)$ -invariant 2-cocycle, and extending by this gives:

$$\mathbb{R}^{3,1|4} \longrightarrow \mathbb{R}^{2,1|2+2}$$

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$$\mathbb{R}^{3,1|4} \longrightarrow \mathbb{R}^{2,1|2+2}$$

Again, the metric is not a choice:

$$\text{Aut}_0(\mathbb{R}^{3,1|4}) = \mathbb{R}^+ \times \text{Spin}(3, 1) \times U(1).$$

$U(1)$  is the R-group.

There are no further  $\text{Spin}(3, 1)$ -invariant 2-cocycles.

# The dimensional ladder

Dimension 6

Double the number of spinors again:

$$\mathbb{R}^{3,1|4+4} \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \mathbb{R}^{3,1|4}$$

Now there are two  $\text{Spin}(3, 1)$ -invariant 2-cocycles.

$$\mathbb{R}^{5,1|8} \longrightarrow \mathbb{R}^{3,1|4+4} .$$

# The dimensional ladder

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Again, the metric is not a choice:

$$\text{Aut}_0(\mathbb{R}^{5,1|8}) = \mathbb{R}^+ \times \text{Spin}(5, 1) \times \text{Sp}(1).$$

$\text{Sp}(1)$  is the R-group.

There are no further  $\text{Spin}(5, 1)$ -invariant 2-cocycles.

# The dimensional ladder

## Dimension 10

Now we have a choice of two different ways to double the spinors, a type IIA and type IIB:

$$\mathbb{R}^{5,1|\mathbf{8}+\bar{\mathbf{8}}} \leftarrow \mathbb{R}^{5,1|\mathbf{8}}$$

and

$$\mathbb{R}^{5,1|\mathbf{8}+\mathbf{8}} \leftarrow \mathbb{R}^{5,1|\mathbf{8}}$$

There are no  $\text{Spin}(5, 1)$ -invariant 2-cocycles in type IIB, but on type IIA there are four:

$$\mathbb{R}^{9,1|\mathbf{16}} \longrightarrow \mathbb{R}^{5,1|\mathbf{8}+\bar{\mathbf{8}}}.$$

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Again, the metric is not a choice:

$$\text{Aut}_0(\mathbb{R}^{9,1|\mathbf{16}}) = \mathbb{R}^+ \times \text{Spin}(9, 1).$$

There are no further  $\text{Spin}(9, 1)$ -invariant 2-cocycles.

# The dimensional ladder

## Dimension 11

Again, we have a choice of two different ways to double the spinors, a type IIA and type IIB:

$$\mathbb{R}^{9,1|16+\overline{16}} \begin{array}{l} \longleftarrow \\ \longleftarrow \end{array} \mathbb{R}^{9,1|16}$$

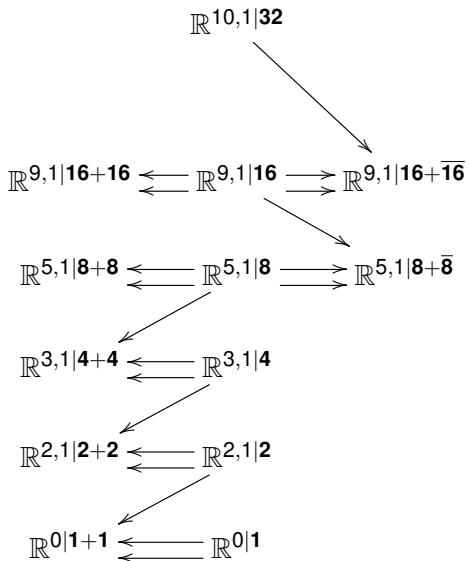
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There are no  $\text{Spin}(9, 1)$ -invariant 2-cocycles in type IIB, but on type IIA there is one, the one we started with:

$$\mathbb{R}^{10,1|32} \longrightarrow \mathbb{R}^{9,1|16+\overline{16}}.$$

# Theorem (H.–Schreiber)





# The brane scan

We have seen that 2-cocycles give central extensions.

Fact

The 2nd **Chevalley–Eilenberg cohomology group**

$$H^2(\mathfrak{g})$$

classifies central extensions of  $\mathfrak{g}$ .

# The brane scan

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## Fact

The 2nd **Chevalley–Eilenberg cohomology group**

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classifies central extensions of  $\mathfrak{g}$ .

## Question

What do higher degree cocycles in  $H^\bullet(\mathfrak{g})$  classify?

## Answer (Physics)

Invariant  $(p+2)$ -cocycles on  $\mathbb{R}^{d-1,1|\mathbf{N}}$  classify some of the  $p$ -branes.

## Answer (Mathematics)

Higher degree cocycles classify extensions to  $L_\infty$ -algebras.

# The brane scan

## The physical answer

The Lie algebra cohomology of  $\mathbb{R}^{d-1,1|\mathbf{N}}$  gives rise to particular  $p$ -branes called **Green–Schwarz  $p$ -branes**.

- ▶ Write a generating set of left-invariant forms:

$$e^\mu = dx^\mu - \theta \Gamma^\mu d\theta, \quad d\theta^\alpha.$$

- ▶ Find the  $\text{Spin}(d-1, 1)$ -invariant combinations:

$$\mu_p = e^{\nu_1} \wedge \cdots \wedge e^{\nu_p} \wedge d\bar{\theta} \Gamma_{\nu_1 \dots \nu_p} d\theta.$$

- ▶ This is  $(p+2)$ -cocycle if and only if it is closed:

$$d\mu_p = 0.$$

- ▶ This happens only for special values of  $d$ ,  $\mathbf{N}$  and  $p$ .

# The brane scan

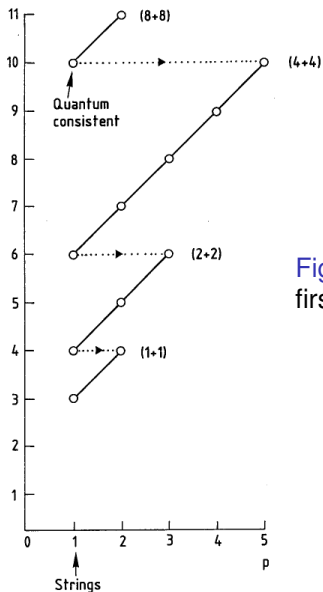


Figure: M. Duff - Supermembranes: the first fifteen weeks, 1988

# The brane scan

This figure is called **the old brane scan**.

It fails to show many examples of branes that would be important later:

- ▶ D-branes and the M5-brane.
- ▶ Black branes from supergravity.
- ▶ Brane intersections.

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Where can we find these? To answer, we use some homotopy theory!

# The brane bouquet

## The mathematical answer

- ▶ The brane scan  $(p + 2)$ -cocycles on  $\mathbb{R}^{d-1,1|\mathbf{N}}$ :

$$\mu_p = e^{\nu_1} \wedge \cdots \wedge e^{\nu_p} \wedge d\bar{\theta} \Gamma_{\nu_1 \dots \nu_p} d\theta.$$

- ▶ Extending by these  $(p + 2)$ -cocycles, we get the **brane scan algebras**:

$$\text{string}_I = \mathbb{R}_{\mu_I}^{9,1|16}, \quad \text{string}_{IIA} = \mathbb{R}_{\mu_{IIA}}^{9,1|16+\overline{16}}, \quad \text{string}_{IIB} = \mathbb{R}_{\mu_{IIB}}^{9,1|16+16},$$

$$\text{m2brane} = \mathbb{R}_{\mu_{M2}}^{10,1|32}.$$

- ▶ Because these are not 2-cocycles, the resulting extensions are not super Lie algebras—they are **super  $L_\infty$ -algebras**.

# The brane bouquet

A super  $L_\infty$ -algebra  $\mathfrak{g}$  is like a Lie algebra, defined on a chain complex of super vector spaces:

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathfrak{g}_n \xleftarrow{\partial} \cdots$$

But the Jacobi identity *does not hold*:

$$[[X, Y], Z] \pm [[Y, Z], X] \pm [[Z, X], Y] \neq 0.$$

Instead, it holds up to *coherent homotopy*: we get infinitely many identities like this:

$$[[X, Y], Z] \pm [[Y, Z], X] \pm [[Z, X], Y] = \partial[X, Y, Z] + [\partial(X \wedge Y \wedge Z)].$$

This says the Jacobi identity holds up to a chain homotopy, given by a trilinear bracket:

$$[-, -, -]: \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g},$$

satisfying its own Jacobi-like identity up to a 4-linear bracket . . .



# The brane bouquet

A super Lie algebra is a super  $L_\infty$ -algebra concentrated in degree 0:

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Given any  $(p+2)$ -cocycle  $\omega: \Lambda^{p+2}\mathfrak{g} \rightarrow \mathbb{R}$ , we can construct an  $L_\infty$ -algebra  $\mathfrak{g}_\omega$  as follows:

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where

- ▶  $\mathfrak{g}$  is in degree 0,  $\mathbb{R}$  is in degree  $p$ .
- ▶  $[-, -]$  is the Lie bracket.
- ▶ The  $(p+2)$ -linear bracket,  $[-, \dots, -] = \omega$ , is the cocycle.
- ▶ All other brackets are 0.

# The brane bouquet

A super Lie algebra is a super  $L_\infty$ -algebra concentrated in degree 0:

$$\mathfrak{g}_0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots$$

Given any  $(p+2)$ -cocycle  $\omega: \Lambda^{p+2}\mathfrak{g} \rightarrow \mathbb{R}$ , we can construct an  $L_\infty$ -algebra  $\mathfrak{g}_\omega$  as follows:

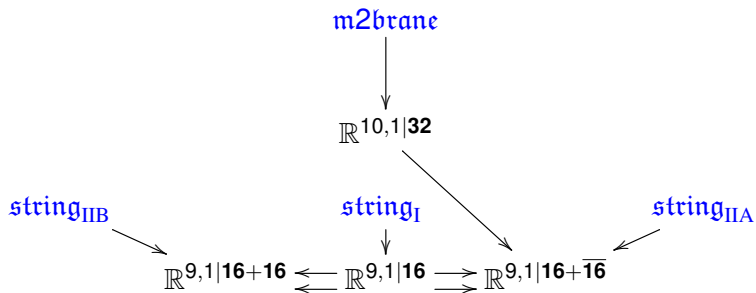
$$\mathfrak{g} \longleftarrow 0 \longleftarrow \dots \longleftarrow \mathbb{R}$$

where

- ▶  $\mathfrak{g}$  is in degree 0,  $\mathbb{R}$  is in degree  $p$ .
- ▶  $[-, -]$  is the Lie bracket.
- ▶ The  $(p+2)$ -linear bracket,  $[-, \dots, -] = \omega$ , is the cocycle.
- ▶ All other brackets are 0.

In homotopy theory, this operation is called ‘taking the homotopy fiber’ of  $\omega$ .

# The brane bouquet



# The brane bouquet

Thanks to  $\text{string}_I$ ,  $\text{string}_{\text{IIA}}$ ,  $\text{string}_{\text{IIB}}$  and  $\text{m2brane}$ , we can find some of the branes missing from the brane scan.

## Fact

The left-invariant forms on  $\mathfrak{g}_\omega$  are generated by the left-invariant forms on  $\mathfrak{g}$  with one additional  $(p+1)$ -form  $b$  such that  $db = \omega$ .

For example:

- ▶ On  $\text{string}_{\text{IIA}} = \mathbb{R}^{\mu_{\text{IIA}}|^{9,1|16+\overline{16}}}$ , the left-invariant forms are
- ▶ from  $\mathbb{R}^{9,1|16+\overline{16}}$ :

$$e^\nu = dx^\nu - \theta \Gamma^\nu d\theta, \quad d\theta^\alpha$$

- ▶ and a 2-form  $F$  such that

$$dF = \mu_{\text{IIA}}.$$

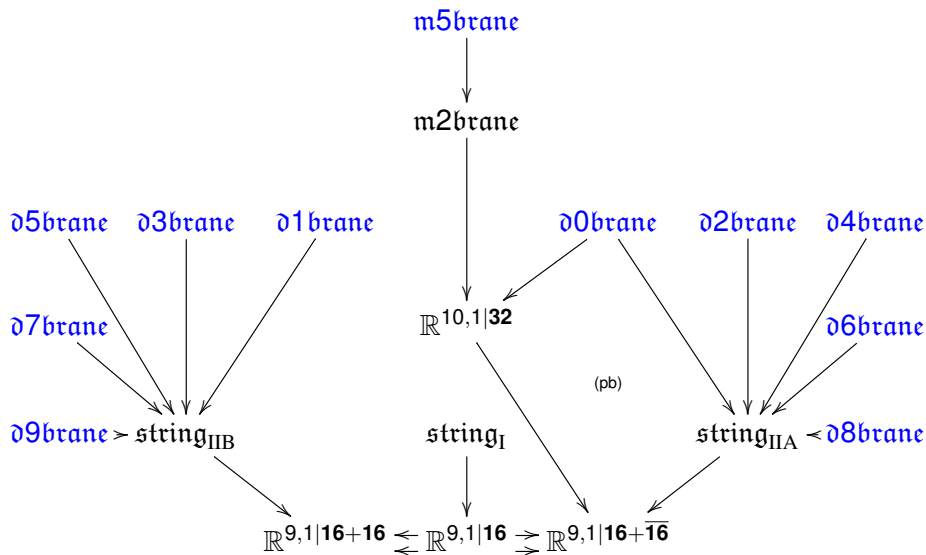
# The brane bouquet

Thanks to  $F$ , there are new cocycles on  $\text{string}_{\text{IIA}}$ .

$$\mu_{Dp} = \sum_{k=0}^{(p+2)/2} c_k^p e^{\nu_1} \wedge \dots \wedge e^{\nu_{p-2k}} \wedge d\bar{\theta} \wedge \Gamma_{\nu_1 \dots \nu_{p-2k}} d\theta \wedge F \wedge \dots \wedge F.$$

- ▶  $c_k^p$  are some coefficients chosen to make  $d\mu_{Dp} = 0$ .
- ▶ With some theoretical machinery due to Fiorenza–Sati–Schreiber, we can turn this cocycle into the  $Dp$ -brane action.
- ▶ Similarly, we can find a cocycle for the M5-brane on  $\text{m2brane}$ .

# The brane bouquet





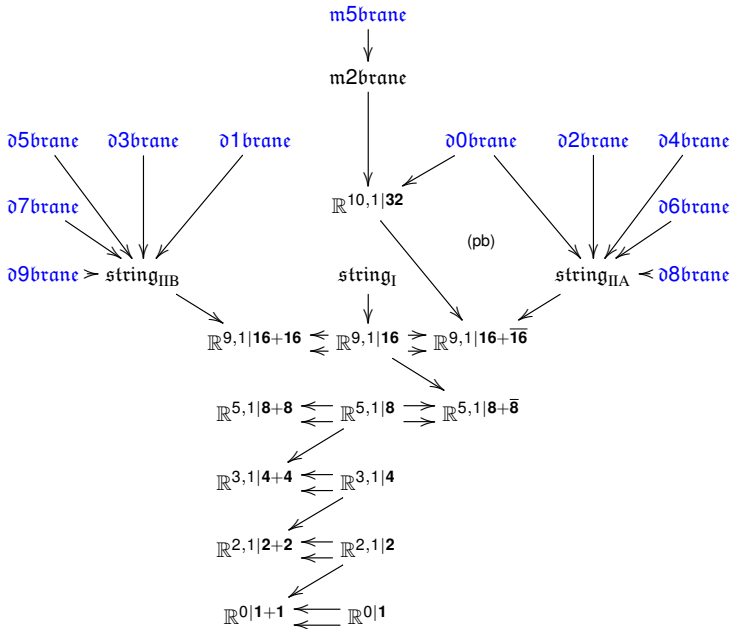




Figure:  $\mathbb{R}^{0|1}$

**MANY THANKS**

## References I

The use of  $L_\infty$ -algebras in physics originates with the work of D'Auria and Fré, who call them 'free differential algebras'.

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The connection between Lie algebra cohomology and Green–Schwarz  $p$ -brane actions is due to de Azcárraga and Townsend:

- ▶ J. A. de Azcárraga and P. K. Townsend, Superspace geometry and the classification of supersymmetric extended objects, *Phys. Rev. Lett.* **62** (1989), pp. 2579–2582.

## References II

The discovery that the WZW terms for  $Dp$ -branes and the M5-branes live on the 'extended superspacetimes'  $string_{IIA}$ ,  $string_{IIB}$  and  $m2brane$  appears in two articles. The case of the type IIA  $Dp$ -branes and the M5-brane is in:

- ▶ C. Chryssomalakos, J. de Azcárraga, J. Izquierdo, and C. Pérez Bueno, The geometry of branes and extended superspaces, *Nucl. Phys. B* **567** (2000), pp. 293-330, arXiv:hep-th/9904137.

while the type IIB  $Dp$ -branes are in section 2 of:

- ▶ M. Sakaguchi, IIB-branes and new spacetime superalgebras, *JHEP* 04 (2000), pp. 019, arXiv:hep-th/9909143.

## References III

Later, Fiorenza, Sati and Schreiber placed this into the context of the homotopy theory of  $L_\infty$ -algebras, discovering the brane bouquet:

- ▶ D. Fiorenza, H. Sati, U. Schreiber, Super Lie  $n$ -algebra extensions, higher WZW models, and super  $p$ -branes with tensor multiplet fields, *Intern. J. Geom. Meth. Mod. Phys.* **12** (2015), 1550018 (35 pages). arXiv:1308.5264.

Finally, Schreiber and I derived the brane bouquet from the superpoint:

- ▶ J. Huerta and U. Schreiber, M-theory from the superpoint. To appear in *Lett. Math. Phys.* arXiv:1702.01774