Twisted Cohomotopy implies M5-brane anomaly cancellation

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Abstract

We highlight what seems to be a remaining subtlety in the argument for the cancellation of the total anomaly associated with the M5-brane in M-theory. Then we prove that this subtlety is resolved under the hypothesis that the C-field flux is charge-quantized in the generalized cohomology theory called J-twisted Cohomotopy.

Contents

1 Introduction

Formulating M-theory remains an open problem (e.g. [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19][1][Du19][2]). Even formulating just the field theoretic decoupling limit of the worldvolume theory of M5-branes in M-theory remains an open problem (e.g. [La19, 6.3]). Nevertheless, it is traditionally assumed that enough is known about M-theory in general, and about M5-branes in particular, that it makes sense to check whether field theoretic anomalies (following [AW84][AG85]) on M5-brane worldvolumes cancel against M-theoretic anomaly inflow (following [CH85]) from the bulk spacetime (reviewed in the current context in [Ha05]).

Relevance of anomaly cancellation for M-theory. What from the physics perspective are called *anomalies* is what from the perspective of mathematics are *obstructions* (a point highlighted in [KS04][SSS09]). Hence such a cancellation of the total M5-brane anomaly, if properly identified, is strictly necessary for M-theory to exist: any remaining anomaly is an obstruction against the existence of the theory of which it is an anomaly. But conversely, wherever a putative anomaly in M-theory is found *not* to vanish, by available reasoning, this signifies (with the assumption that M-theory does in fact exist) the presence of a new aspect of the elusive theory that had hitherto been missed: There must then be a new detail in the theory, previously unrecognized, which does imply the cancellation of the remaining anomaly, after all.

[1] [Wi19] at 21:15: "I actually believe that string/M-theory is on the right track toward a deeper explanation. But at a very fundamental level it's not well understood. And I'm not even confident that we have a good concept of what sort of thing is missing or where to find it."

[2] [Du19] at 17:04: "The problem we face is that we have a patchwork understanding of M-theory, like a quilt. We understand this corner and that corner, but what's lacking is the overarching big picture. Directly or indirectly, my research hopes to explain what M-theory really is. We don't know what it is."

For this reason a careful mathematical analysis of anomaly cancellation in M-theory is in order. The tacit assumption that the proverbial magic of M-theory will take care of all cancellations anyway, freeing us from the burden of patient rigorous checks, would work only if the actual formulation of M-theory were known. Since it is not known, the situation is the reverse: A carefully deduced failure of anomalies to cancel provides a hint as to the actual formulation of the elusive theory.

Historical background on M5-brane anomaly cancellation. Indeed, the original computation of the total M5-brane anomaly in [Wi96, 5] found the total anomaly *not* to vanish; and highlighted that the issue remains an open problem ("somewhat puzzling" [Wi96, p. 35]). In reaction, several authors argued for several fixes, but, it seems, without convincing success (see [FHMM98, p.2] for pointers). Finally, [FHMM98, 3] argued that there is a previously neglected summand in the bulk anomaly inflow which needs to be taken into account (the top right term in diagram (5) below). That correction to the bulk anomaly inflow term has since become accepted (e.g., in [BBMN18, (5)]) as the solution to the M5-brane anomaly cancellation. The authors of [BBMN19, A.4-5] recently recall the argument of [FHMM98] in streamlined form. Nonetheless, these arguments remain non-rigorous even by physics standards, due to a lack of actual formulation of M-theory. This is clearly acknowledged and highlighted by one of those authors, in [Ha05, p. 46].[3]

Remaining issue. In this note we point out, in §2 below, that there does still remain one issue with the currently accepted anomaly cancellation argument [FHMM98, 3][BBMN19, A.4-5] in itself. This is a simple observation: these authors made an *Ansatz* (see (6) below) for the C-field configuration ([FHMM98, (2.3)][BBMN19, (A.18)]) which is not the most general admissible under the given assumptions (as also noticed in [Mo15, (3.12)]). Entering their anomaly cancellation argument instead with a general C-field configuration leaves one anomaly contribution uncancelled, shown on the bottom right of (5) below.

Resolution by Hypothesis H. We prove in §3 that this previously neglected remaining anomaly term does in fact vanish, hence that the anomaly cancellation argument of [Wi96, 5][FHMM98, 3][BBMN19, A.4-5] is completed, if one assumes a hypothesis about the proper nature of the C-field in M-theory [Sa13] which in [FSS19b] we called *Hypothesis H*, recalled in §3 below. This hypothesis says that the M-theory C-field is charge-quantized in the generalized cohomology theory called *J-twisted Cohomotopy*. We have previously demonstrated that this hypothesis implies a wealth of further anomaly cancellation conditions [FSS19b][FSS19c] [SS19a][SS19b] and other effects [SS19c] expected in M-theory (exposition in [Sc20]).

Outlook. Since *Hypothesis H* gives rigorous mathematical meaning to the M-theoretic nature of the C-field, our derivation in §3 is a rigorous mathematical proof of the vanishing of the remaining anomaly term (5) from this hypothesis and, as such, completes the argument of [Wi96, 5][FHMM98, 3][BBMN19, A.4-5]. We do not claim to make the rest of that argument rigorous. In order to do so one will need, beyond a rigorous definition of the M-theory C-field by *Hypothesis H*, also a rigorous definition of the M5-brane coupled to this C-field. We have presented results going towards that goal in [FSS19d], but more needs to be done [FSS20a].

Acknowledgement. We thank Domenico Fiorenza for collaboration on the material discussed in §3.

[3] [Ha05, p. 46]: "[...] the solution is not so clear. [The established procedure of anomaly cancellation] will not work for the M5-brane. [...] something new is required. What this something new is, is not a priori obvious. [...] [This is] a daunting task. To my knowledge no serious attempts have been made to study the problem. [...] [The proposal of [FHMM98]] probably should not be viewed as a final understanding of the problem. One would eventually hope for a microscopic formulation of M-theory which makes some of the manipulations [proposed in [FHMM98]] appear more natural."

2 The issue

The geometry under consideration. We are dealing with

(i) families of
(ii) C-field configurations on
(iii) 11-dimensional spacetimes
(iv) sourced by magnetic 5-branes
(v) of unit charge 1.

We now say what this means precisely: First, **(i)** with **(iii)** means that

$$
X \;:=\; \overbrace{X^{11}}^{\text{spacetime manifold}} \times \overbrace{U}^{\text{parameter manifold}}
$$

is the product of an 11-dimensional manifold (spacetime) with any parameter manifold U of any dimension, while **(ii)** means that we consider a closed differential 4-form on X:

$$
\underbrace{G_4}_{\substack{\text{family of} \\ \text{C-field flux densities}}} \in \Omega^4_{\mathrm{cl}}(X) \quad \Longrightarrow \quad \underset{s \in U}{\forall} \Big(\underbrace{G_4^{(s)}}_{\substack{\text{C-field flux density} \\ \text{at parameter } s}} \in \Omega^4_{\mathrm{cl}}(X^{11}) \Big),
$$

which is hence, in particular, a U-parametrized family of differential 4-forms on X^{11}.[4] Moreover, **(iv)** means, just as in Dirac's argument for magnetic 0-branes (e.g. [Fr11, 16.4e]), that X^{11} is the complement of a 5-brane worldvolume, hence that X is an S^4-bundle as shown on the left of (1).

$$
\begin{tikzcd}
\underset{\substack{\text{unit sphere} \\ \text{around} \\ \text{M5-brane}}}{} \; S^4 \arrow[r] & \overset{\substack{\text{spacetime} \\ \text{(families)}}}{X} \arrow[d, "\pi", "\substack{\text{4-sphere} \\ \text{fibration}}"'] & \overset{\substack{\text{C-field} \\ \text{4-flux density} \\ \text{(in families)}}}{[G_4]} \arrow[d, mapsto, "\substack{\text{total flux} \\ \text{through } S^4}"] & \in & \overset{\substack{\text{de Rham} \\ \text{cohomology}}}{H^{\bullet+4}_{\mathrm{dR}}(X)} \arrow[d, "\int_{S^4}"', "\text{fiber integration}"] & \simeq & \overset{\substack{\text{real} \\ \text{cohomology}}}{H^{\bullet+4}(X,\mathbb{R})} \arrow[d, "\pi_*"] \\
& \underbrace{U}_{\substack{\text{parameter} \\ \text{manifold}}} \times \underbrace{(0,\infty)}_{\substack{\text{radial} \\ \text{distance} \\ \text{from brane}}} \times \underbrace{Q_{\mathrm{M5}}}_{\substack{\text{M5-brane} \\ \text{worldvolume} \\ \text{(families)}}} = B & \underbrace{1}_{\substack{\text{single M5} \\ (\Leftrightarrow \text{ abelian 2-form field})}} & \in & H^{\bullet}_{\mathrm{dR}}(B) & \simeq & H^{\bullet}(B,\mathbb{R})
\end{tikzcd}
\tag{1}
$$

Finally, **(v)** means that the corresponding fiber integration (1) of G_4 over the 4-sphere fibers is unity

$$
\pi_*[G_4] \;=\; 1 \;\in\; H^0(B,\mathbb{R}) \tag{2}
$$

as shown on the right of (1). The general solution to (2) is the sum of half the Euler class of the S^4-fibration (e.g. [BT82, 11][BC97, (2.3)]) with any *basic* class (by exactness of the Gysin sequence, e.g. [BT82, 14.33]), namely one pulled back from the base of the fibration:

[4] The inclined reader may think of the 4-flux data G_4 as being a value at stage U of the *mapping stack* $\mathbf{Fields}(X^{11}) := [X^{11}, \underline{\Omega}^4]$, and as the anomaly polynomials (5) as being (classes of) differential forms on this mapping stack. While this is the correct point of view (exposition in [FSS13]) here we will not further dwell on it.

$$
\begin{tikzcd}
 & {[\mathrm{vol}_{S^4}]} & & \in H^4(S^4) \\
\underset{\substack{\text{general}\\ \text{4-flux density}\\ \text{with unit flux}\\ \text{through } S^4}}{[G_4]} \; = & \underset{\substack{\text{Euler class of}\\ S^4\text{-fibration}}}{\tfrac{1}{2}\chi_4} \arrow[u, mapsto] \; + & \overset{\substack{\text{basic component:}\\ \text{pulled back from}\\ \text{base of } S^4\text{-fibration}}}{\pi^*\left[G_4^{\mathrm{basic}}\right]} & \in H^4(X) \arrow[u, "i_x^*"] \\
 & & \underset{\substack{\text{4-class on base}\\ \text{of } S^4\text{-fibration}}}{\left[G_4^{\mathrm{basic}}\right]} \arrow[u, mapsto] & \in H^4(B) \arrow[u, "\pi^*"] \arrow[uu, bend right, swap, "0"]
\end{tikzcd}
\tag{3}
$$

Remark 1 (The 1/2BPS M5 configuration and its generalization)**.** The local model of the situation (1) is the trivial S^4-fibration of the near horizon geometry of the smooth 1/2-BPS black M5-brane solution of 11-dimensional supergravity ([GT93], reviewed in [AFHS98, 2.1.2]), restricted to the Poincaré patch of 7-dimensional anti de-Sitter spacetime:

$$
\begin{tikzcd}
S^4 \arrow[r] & \mathrm{AdS}_7^{\mathrm{Poin}} \times S^4 \arrow[d, "\pi = \mathrm{pr}_1"] & G_4 = \mathrm{vol}_{S^4} \\
\underset{\substack{\text{M5-brane}\\ \text{worldvolume}}}{\mathbb{R}^{5,1}} \times \underset{\substack{\text{radial}\\ \text{distance}}}{(0,\infty)} \underset{\mathrm{diff}}{\simeq} & \underset{\substack{\text{Poincaré chart of}\\ \text{anti-de Sitter spacetime}}}{\mathrm{AdS}_7^{\mathrm{Poin}}} &
\end{tikzcd}
\tag{4}
$$

So the point of (1) is to generalize the situation away from this highly symmetric 1/2-BPS configuration (4) to more general 5-brane configurations. While few to no black M5-brane solutions to 11d supergravity beyond (4) are known explicitly, only their topological structure matters for the discussion of anomaly cancellation; and that topological structure is (essentially by definition) what is expressed by (1).

Remark 2 (G_4 is singular on the M5-brane locus)**.** Condition (2) implies (immediately so by the Poincaré Lemma, since G_4 is closed) that the flux density G_4 can *not* be extended to the locus of the M5-brane itself, which is (or would be) at the center $r = 0 \in [0,\infty)$ of the punctured ball $S^4 \times (0,\infty)$ in (1). Instead it must have/would have a singularity at $r = 0$, as is manifest also from the basic example (4). Parts of the literature gloss over this subtlety; and the point made in [FHMM98, p. 4-5] was to argue that this is the source of the missing anomaly cancellation of [Wi96]. To handle the singularity mathematically, these authors declared[5] to multiply G_4 by a smooth radial cutoff function, thus rendering it no longer closed [FHMM98, (2.3), (3.4)] but, mathematically, extendable to the brane locus. Luckily, the key computation [FHMM98, (3.3)], recalled in (5) below, applies just as well if instead one leaves G_4 intact but removes the singular locus from spacetime, just as usual in supergravity (4).

Remark 3 (Focus on real cohomology)**.** We focus here entirely on the anomaly polynomials in real cohomology, hence ignoring all torsion contributions (which become visible in integral cohomology) as well as all "global" anomaly contributions (which become visible in differential cohomology). Because, while vanishing of the anomaly in real cohomology is not sufficient for full anomaly cancellation (which must happen in differential integral cohomology) it is the *necessary* first step. No argument about torsion of global contributions to the M5 anomaly (which, of course, one will eventually want to address) can affect the proof of anomaly cancellation at the rational/real approximation; and as long as subtleties do remain here, it behooves us to first focus on these. Therefore we sometimes abbreviate $H^\bullet(-) := H^\bullet(-,\mathbb{R})$, here and in the following.

[5] [FHMM98, p. 4]: "We leave to the future the very interesting question of the relation of this approach to that based on a direct study of solutions to supergravity."

The anomaly polynomials. The cohomology classes contributing to the total M5-brane anomaly in the situation (1) are given in the literature as follows:

$$
\begin{tikzcd}
\substack{\text{Bulk}\\\text{spacetime}\\\text{CS-terms}} \\
H^{12}(X) \arrow[d, "\pi_*"', "\substack{\text{anomaly}\\\text{inflow}}"] & & & & & {[G_4 \wedge I_8]} \arrow[d, mapsto] & + & \tfrac{-1}{6}[G_4 \wedge G_4 \wedge G_4] \arrow[dl, mapsto] \arrow[dr, mapsto] \\
H^8(B) & A_{\mathrm{total}} = & A_{\substack{\text{chiral}\\\text{fermion}}} + & A_{\substack{\text{chiral}\\\text{2form}}} & + & I_8 & + \tfrac{-1}{24}p_2(N) + & & \tfrac{-1}{2}\big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big] \arrow[d, equal] \\
\substack{\text{M5-brane}\\\text{worldvolume}\\\text{anomalies}} & & & & & & & & \tfrac{-1}{2}\big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big]
\end{tikzcd}
\tag{5}
$$

$$
\underbrace{A_{\substack{\text{chiral}\\\text{fermion}}} + A_{\substack{\text{chiral}\\\text{2form}}} + I_8}_{\text{[Wi96]: } \frac{1}{24}p_2(N)}
$$

$$
\underbrace{\tfrac{1}{24}p_2(N) + \tfrac{-1}{24}p_2(N)}_{\text{[FHMM98]: } 0}
$$

$$
\underbrace{0 + \tfrac{-1}{2}\big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big]}_{\text{Hypothesis H: } 0}
$$

We discuss the various items in (5):

(i) The term I_8 is the "one-loop polynomial" [DLM95][VW95], while the terms $A_{\substack{\text{chiral}\\\text{fermion}}}$ and $A_{\substack{\text{chiral}\\\text{2form}}}$ are the plain anomalies [Wi96, (5.1), (5.4)] of the chiral fermion and of the abelian chiral (i.e., with self-dual curvature) 2-form field in 6d QFT. These were expected in [Wi96] to cancel against the influx of I_8, but found there ([Wi96, (5.7)]) to cancel only up to a remaining term $\frac{1}{24}p_2(N)$, where N denotes the normal bundle to the M5-brane locus in spacetime.

(ii) The Chern-Simons term $-\frac{1}{6}G_4 \wedge G_4 \wedge G_4$ of 11-dimensional supergravity was argued in [FHMM98, 3] [BBMN19, A.4-5] to contribute to the anomaly influx from the bulk. Then a formula due to [BC97, Lem 2.1] shows that this gives rise to the previously missing summand of $\frac{-1}{24}p_2(N)$. However, these authors consider an Ansatz for the C-field configuration [FHMM98, (2.3), (3.4)][BBMN19, (2.4)] which amounts to assuming

$$
[G_4^{\text{basic}}] \overset{!}{=} 0 \tag{6}
$$

in (3). If this restrictive assumption is not made, then the bulk Chern-Simons term in addition contributes an influx term $\frac{-1}{2}\big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big]$, which remains uncancelled.

(iii) That the Ansatz (6) remained unjustified was acknowledged in [Mo15, (3.12)]. There it is suggested [Mo15, (3.7)] that the traditional expression from [Wi96, (5.7)] for the self-dual field anomaly $A_{\substack{\text{chiral}\\\text{2form}}}$ in real cohomology is wrong, in that it gets corrected by just the missing summand $\frac{-1}{2}\big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big]$. Unfortunately, we are unable to verify this derivation. Luckily, assuming Hypothesis H it makes no difference:

(iv) Indeed, we prove in §3 that, assuming with Hypothesis H the M-theory C-field to be charge-quantized in J-twisted Cohomotopy theory, the restrictive Ansatz (6) is *implied* (Prop. 5 below). In this way Hypothesis H enforces vanishing of the problematic remaining anomaly term by itself:

$$
\text{Hypothesis H} \quad \Rightarrow \quad \big[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\big] = 0 \qquad \text{in situation 1}.
$$

This means, according to (5), that the total M5-brane anomaly is finally cancelled.

3 A resolution

We now prove (Prop. 5 below) that *Hypothesis H* implies, in the situation (1), the vanishing of the problematic basic term $\big[G_4^{\text{basic}}\big]$ in (3), thus implying the vanishing of the total M5-brane anomaly according to (5).

We use results from [FSS19b]. The following recalls the key concept:

Definition 4 (J-twisted Cohomotopy cohomology theory)**.** Given a smooth manifold X equipped with topological $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-structure, a *cocycle in J-twisted Cohomotopy* is a continuous section c of the 4-spherical fibration associated to the tangent bundle TX, and its class in the *J-twisted Cohomotopy set* is its homotopy class relative X:

$$
\underset{\substack{\text{J-twisted}\\ \text{4-Cohomotopy}\\ \text{of } X}}{\pi^{4(TX)}(X)} \;:=\; \left\{
\begin{tikzcd}
 & S^4(TX) \arrow[r, "\substack{\text{4-spherical fibration}\\ \text{associated with}\\ \text{tangent bundle}}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r, "\substack{\text{universal}\\ \text{4-spherical fibration}\\ \text{compatible with Hopf fibration}}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \mathrm{SO}(5) \arrow[d, "\substack{\text{universal}\\ \text{4-spherical fibration}}"'] \\
X \arrow[ur, dashed, "c", "\substack{\text{cocycle in}\\ \text{J-twisted}\\ \text{Cohomotopy}}"'] \arrow[r, equal] & X \arrow[r, "TX"', "\substack{\text{topological structure}\\ \text{on tangent bundle}}"] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r] & B\mathrm{SO}(5)
\end{tikzcd}
\right\}\Big/\sim_{\text{homotopy}} \tag{7}
$$

The *cohomotopical twisted Chern character* on J-twisted Cohomotopy is the rationalization map

$$
\begin{array}{ccccc}
\mathrm{ch} \;:\; \pi^{4(TX)}(X) & \longrightarrow & \pi^{4(TX)}_{\mathbb{Q}}(X) & \longrightarrow & \big(\Omega^4(X)\times\Omega^7(X)\big)_{/\sim} \\
{[c]} & \longmapsto & [c]_{\mathbb{Q}} & \longmapsto & \big[(G_4, 2G_7)\big]
\end{array} \tag{8}
$$

taking values in classes of pairs consisting of a smooth differential 4-form and a smooth 7-form on X.

Hypothesis H: *The M-Theory C-field is charge-quantized in J-twisted Cohomotopy theory* (7)*, hence the C-field flux densities G_4, G_7 are in the image of the cohomotopical twisted Chern character* (8)*.*

Application to M5-brane backgrounds. Given a solitonic M5-brane background X as in (1), let the 4-spherical fibration be associated to a $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-structure $NQ_{\mathrm{M5}}\cdot\mathscr{T}$. We write τ for the canonically associated 4-Cohomotopy twist (7), according to the following homotopy-commutative diagram, using notation from [FSS19b, 2.3]:

$$
\begin{tikzcd}
 & & & S^4 /\!\!/ \mathrm{SO}(4) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \mathrm{SO}(5) \arrow[d] \\
X \arrow[urrrr, dashed, bend left, "\substack{\text{cocycle in}\\ \text{twisted}\\ \text{4-Cohomotopy}}"] \arrow[rrrr, bend left, "\substack{\text{twist of}\\ \text{4-Cohomotopy}\\ \tau}"] \arrow[rrr, bend left, "\widehat{NQ_{\mathrm{M5}}}"] \arrow[r] \arrow[d, "\pi"', "\substack{\text{4-spherical}\\ \text{fibration}}"] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \mathrm{SO}(5) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & B\mathrm{SO}(4) \arrow[r, "B\iota"] \arrow[d, "B\iota"] & B\mathrm{SO}(5) \\
B \arrow[r, "NQ_{\mathrm{M5}}\cdot\mathscr{T}"] \arrow[rrr, bend right, "NQ_{\mathrm{M5}}"', "\substack{\text{classifying map for}\\ \text{normal bundle to M5 worldvolume}}"'] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r, "B\mathrm{pr}_5"] & B\mathrm{SO}(5) \arrow[r, equal] & B\mathrm{SO}(5)
\end{tikzcd} \tag{9}
$$

Theorem 5. *Consider an M5-brane geometry* (1) *with unit C-field flux G_4 as in* (3)*. Then under Hypothesis H – hence assuming that G_4 is in the image of the cohomotopical twisted Chern character* (8) *– it follows that the degree four basic class in* (3) *vanishes:*

$$\big[G_4^{\text{basic}}\big] \;=\; 0 \;\in\; H^4(B;\mathbb{R})\,. \tag{10}$$

Proof. From diagram (9) we find that $\tau = B\iota \circ \widehat{NQ_{\text{M5}}} = NQ_{\text{M5}} \circ \pi$, hence that the second Pontrjagin class of τ is

$$p_2(\tau) = \pi^* p_2(NQ_{\text{M5}})\,.$$

With this, [FSS19b, Prop. 2.5 (41)] shows that *Hypothesis H* implies the following property of the squared 4-flux:

$$[G_4 \wedge G_4] \;=\; \tfrac{1}{4}\pi^* p_2(NQ_{\text{M5}}) \;\in\; H^8(X;\mathbb{R})\,. \tag{11}$$

Consider then the fiber integration

$$\pi_* \;:\; H^\bullet(X;\mathbb{R}) \longrightarrow H^{\bullet-4}(B;\mathbb{R}) \tag{12}$$

along the fibers of the given 4-spherical fibration $S^4 \longrightarrow X \xrightarrow{\pi} B$ as in (1) and (9). By [BC97, Lemma 2.1], the fiber integration of the odd cup powers χ^{2k+1} of the Euler class $\chi \in H^4(X;\mathbb{R})$ of the fibration π are proportional to cup powers of the second Pontrjagin class of the SO(5)-principal bundle to which it is associated:

$$\pi_*(\chi^{2k+1}) = 2\big(p_2(NQ_{\text{M5}})\big)^k \;\in\; H^{8k}(B;\mathbb{R})\,, \tag{13}$$

while the fiber integration of the even cup powers of the Euler class vanishes for all $k \in \mathbb{N}$:

$$\pi_*(\chi^{2k}) = 0 \;\in\; H^{8k-4}(B;\mathbb{R})\,. \tag{14}$$

Notice also that, by the projection formula $\pi_*(\pi^*\alpha \wedge \beta) = \alpha \wedge \pi_*\beta$ (e.g. [FSS18, (2)]), one has in particular

$$\pi_*\pi^*\alpha = \pi_*(\pi^*\alpha \wedge 1) = \alpha \wedge \pi_* 1 = 0.$$

Therefore, by repeated use of the projection formula [FSS18, (2)], of equations (13-14) and by the identity (11), we get:

$$\begin{aligned}
0 &= \tfrac{1}{8}\pi_*\pi^* p_2(NQ_{\text{M5}}) \\
&= \tfrac{1}{2}\pi_*[G_4 \wedge G_4] \\
&= \tfrac{1}{2}\pi_*\left((\tfrac{1}{2}\chi + \pi^*[G_4^{\text{basic}}]) \wedge (\tfrac{1}{2}\chi + \pi^*[G_4^{\text{basic}})]\right) \\
&= \tfrac{1}{8}\pi_*(\chi^2) + \pi_*(\tfrac{1}{2}\chi \wedge \pi^*[G_4^{\text{basic}}]) + \tfrac{1}{2}\pi_*(\pi^*[G_4^{\text{basic}}] \wedge \pi^*[G_4^{\text{basic}}]) \\
&= \tfrac{1}{8}\pi_*(\chi^2) + \tfrac{1}{2}\pi_*(\chi) \wedge [G_4^{\text{basic}}] + \tfrac{1}{2}\pi_*\pi^*[G_4^{\text{basic}} \wedge G_4^{\text{basic}}] \\
&= [G_4^{\text{basic}}].
\end{aligned}$$

□

Corollary 6. *Under Hypothesis H the Ansatz* (6) *is implied (Theorem 5) and hence the total M5-brane anomaly according to* (5) *vanishes.*

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