Twisted Cohomotopy implies M5-brane anomaly cancellation

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Abstract

We highlight what seems to be a remaining subtlety in the argument for the cancellation of the total anomaly associated with the M5-brane in M-theory. Then we prove that this subtlety is resolved under the hypothesis that the C-field flux is charge-quantized in the generalized cohomology theory called J-twisted Cohomotopy.

Contents

1 Introduction

Formulating M-theory remains an open problem (e.g. [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Moo14, 12][CP18, p. 2][Wi19, @21:15][Du19, @17:04]). Even formulating just the field-theoretic decoupling limit of the worldvolume theory of M5-branes in M-theory remains an open problem (e.g. [La19, 6.3]). Nevertheless, it is traditionally assumed that enough is known about M-theory in general, and about M5-branes in particular, that it makes sense to check whether field theoretic anomalies (following [AW84][AG85]) on M5-brane worldvolumes cancel against M-theoretic anomaly inflow (following [CH85]) from the bulk spacetime (reviewed in the current context in [Ha05]).

Relevance of anomaly cancellation for M-theory. What from the physics perspective are called *anomalies* is what from the perspective of mathematics are *obstructions* (a point highlighted in [KS04][SSS09]). Hence such a cancellation of the total M5-brane anomaly, if properly identified, is strictly necessary for M-theory to exist: any remaining anomaly is an obstruction against the existence of the theory of which it is an anomaly. But conversely, wherever a putative anomaly in M-theory is found *not* to vanish, by available reasoning, this signifies (with the assumption that M-theory does in fact exist) the presence of a new aspect of the elusive

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theory that had hitherto been missed: There must then be a new detail in the theory, previously unrecognized, which does imply the cancellation of the remaining anomaly, after all.

For this reason a careful mathematical analysis of anomaly cancellation in M-theory is in order. The tacit assumption that the proverbial magic of M-theory will take care of all cancellations anyway, freeing us from the burden of patient rigorous checks, would work only if the actual formulation of M-theory were known. Since it is not known, the situation is the reverse: A carefully deduced failure of anomalies to cancel provides a hint as to the actual formulation of the elusive theory.

Historical background on M5-brane anomaly cancellation. Indeed, the original computation of the total M5-brane anomaly in [Wi96b, 5] found the total anomaly *not* to vanish; and highlighted that the issue remains an open problem ("somewhat puzzling" [Wi96b, p. 35]). In reaction, several authors argued for several fixes, but, it seems, without convincing success (see [FHMM98, p.2] for pointers). Finally, [FHMM98, 3] argued that there is a previously neglected summand in the bulk anomaly inflow which needs to be taken into account (the top right term in diagram (5) below). That correction to the bulk anomaly inflow term has since become accepted (e.g., in [BBMN18, (5)]) as the solution to the M5-brane anomaly cancellation. The authors of [BBMN19a, A.4-5] recently recall the argument of [FHMM98] in streamlined form. Nonetheless, these arguments remain non-rigorous even by physics standards, due to a lack of actual formulation of M-theory. This is clearly acknowledged and highlighted by one of these authors, in [Ha05, p. 46].[1]

Remaining issue. In this note we point out, in §2 below, that there does still remain one issue with the currently accepted anomaly cancellation argument [FHMM98, 3][BBMN19a, A.4-5] in itself. This is a simple observation: these authors made an *Ansatz* (see (6) below) for the C-field configuration ([FHMM98, (2.3)][BBMN19a, (A.18)]) which is seemingly not the most general admissible under the given assumptions (as also noticed in [Mon15, (3.12)][BBMN19b, (3.16)][BBM20, (2.34)]). Entering their anomaly cancellation argument instead with a general C-field configuration seemingly leaves one anomaly contribution uncancelled, shown on the bottom right of (5) below.

Resolution by Hypothesis H. We prove in §3 that this previously neglected remaining anomaly term does in fact vanish, hence that the anomaly cancellation argument of [Wi96b, §5][FHMM98, §3][BBMN19a, A.4-5] is completed, if one assumes a hypothesis [Sa13, §2.5] about the proper nature of the C-field in M-theory which in [FSS19b][FSS19c][SS19a][SS19b][SS21] we called *Hypothesis H*, recalled in §3.2 below. This hypothesis says that the M-theory C-field is charge-quantized (21) in the generalized cohomology theory called *J-twisted Cohomotopy* (24). We have previously demonstrated that this hypothesis implies a wealth of further anomaly cancellation conditions [FSS19b][FSS19c][SS19a][BSS19] and other effects [SS19b][SS21] expected in M-theory (exposition in [Sc20]).

Outlook. Since *Hypothesis H* gives rigorous mathematical meaning to the M-theoretic nature of the C-field, our derivation in §3 is a rigorous mathematical proof of the vanishing of the remaining anomaly term (5) from this hypothesis and, as such, completes the argument of [Wi96b, §5][FHMM98, §3][BBMN19a, A.4-5]. We do not claim to make the rest of that argument rigorous. In order to do so one will need also a rigorous definition of the M5-brane coupled to this C-field. We have presented results going towards that goal in [FSS19d][FSS20a][FSS20b][FSS20c][FSS19c], but more needs to be done.

Acknowledgement. We thank Domenico Fiorenza for very useful discussions. We thank an anonymous referee for alerting us of [BBMN19b], whose Section 4.1 overlaps with the discussion in our §3.1; see Remark 6.

[1] [Ha05, p. 46]: "[...] the solution is not so clear. [The established procedure of anomaly cancellation] will not work for the M5-brane. [...] something new is required. What this something new is, is not a priori obvious. [...] [This is] a daunting task. To my knowledge no serious attempts have been made to study the problem. [...] [The proposal of [FHMM98]] probably should not be viewed as a final understanding of the problem. One would eventually hope for a microscopic formulation of M-theory which makes some of the manipulations [proposed in [FHMM98]] appear more natural."

2 The issue

The geometry under consideration. We are dealing with (for background see e.g. [Du99][MiSc06]):

(i) families of
(ii) C-field configurations on
(iii) 11-dimensional spacetimes
(iv) sourced by magnetic 5-branes
(v) of unit charge.

We now say what this means precisely: First, **(i)** with **(iii)** means that

$$X \;:=\; \overbrace{X^{11}}^{\text{spacetime manifold}} \times \overbrace{U}^{\text{parameter manifold}}$$

is the product of an 11-dimensional manifold (spacetime) with a parameter manifold U of any dimension, while **(ii)** means that we consider a closed differential 4-form on X:

$$\underbrace{G_4}_{\substack{\text{family of}\\ \text{C-field flux densities}}} \in \Omega^4_{\mathrm{cl}}(X) \quad\Longrightarrow\quad \underset{s\in U}{\forall}\Big(\underbrace{G_4^{(s)}}_{\substack{\text{C-field flux density}\\ \text{at parameter } s}} \in \Omega^4_{\mathrm{cl}}(X^{11})\Big)\,,$$

which hence is, in particular, a U-parametrized family of differential 4-forms on X^{11}.[2] Moreover, **(iv)** means, just as in Dirac's argument for magnetic 0-branes (e.g. [Al85, §2]), that X^{11} is the complement of a 5-brane worldvolume, hence that X is an orthogonal S^4-fiber bundle (see Def. 13) as shown on the left of (1).

$$\begin{array}{ccccccc}
 & & \text{spacetime (families)} & & \text{C-field 4-flux density (in families)} & \text{de Rham cohomology} & \text{real cohomology} \\
\text{unit sphere around M5-brane} & & S^4 \longrightarrow X & & [G_4] \in H^{\bullet+4}_{\mathrm{dR}}(X) & \simeq & H^{\bullet+4}(X;\mathbb{R}) \\
 & \text{4-sphere fiber bundle} & \Big\downarrow S(p) & \text{total flux through } S^4 & \Big\downarrow & \int_{S^4}\ \text{fiber integration} & \Big\downarrow S(p)_* \\
U \times (0,\infty) \times Q_{\mathrm{M5}} = Q & & & 1 \in & H^{\bullet}_{\mathrm{dR}}(Q) & \simeq & H^{\bullet}(Q;\mathbb{R}) \\
\text{parameter manifold} & \text{radial distance from brane} & \text{M5-brane worldvolume (families)} & \text{single M5 } (\Leftrightarrow \text{abelian 2-form field}) & & &
\end{array} \tag{1}$$

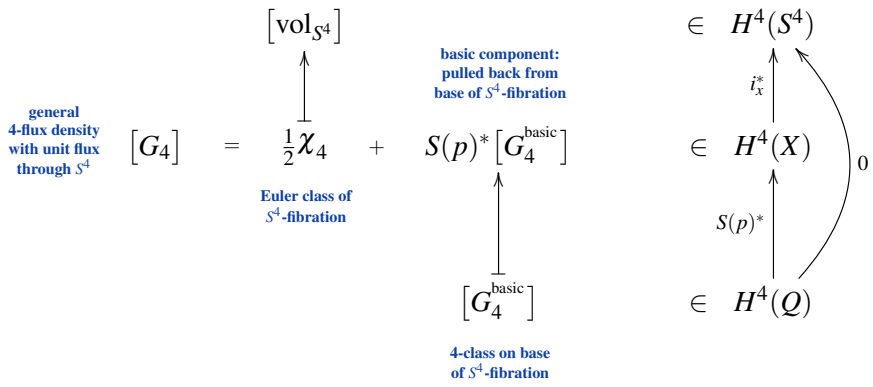
Finally, **(v)** means that the corresponding fiber integration (1) of G_4 over the 4-sphere fibers is unity[3]

$$S(p)_*[G_4] \;=\; 1 \;\in\; H^0(Q;\mathbb{R}) \tag{2}$$

as shown on the right of (1). The general solution to (2) is the sum of half the Euler class of the S^4-fibration (e.g. [BT82, §11][BC97, (2.3)]) with any *basic* class (by exactness of the Gysin sequence, e.g. [BT82, 14.33]), namely one pulled back from the base of the fibration:

[2] The inclined reader may think of the 4-flux data G_4 as being a value at stage U of the *mapping stack* $\mathbf{Fields}(X^{11}) := [X^{11}, \underline{\Omega}^4]$ into the sheaf of differential 4-forms, and of the anomaly polynomials (5) as being (classes of) differential forms on this mapping stack. While this is the correct point of view (exposition in [FSS13]), we will not further dwell on it here.

[3] Our derivations in §3 immediately apply generally to any integer charge $S(p)_*[G_4] \in \mathbb{N}$ (2). But for $N \geq 2$ even the nature of the higher gauge field on the M5-brane(s) remains open (see [FSS20a] for pointers and for a resolution for $N = 2$) and it seems premature to extrapolate the existing computations of worldvolume anomalies to this case (compare [HMM98, below (2.4)]).



(3)

Remark 1 (The $1/2$BPS M5 configuration and its generalization)**.** The local model of the situation (1) is the trivial S^4-fibration of the near horizon geometry of the smooth $1/2$-BPS black M5-brane solution of 11-dimensional supergravity ([GT93], reviewed in [AFHS98, §2.1.2]), restricted to the Poincaré patch of 7-dimensional anti de-Sitter spacetime:

$$\begin{array}{ccccc} S^4 & \longrightarrow & \mathrm{AdS}_7^{\mathrm{Poin}} \times S^4 & & G_4 = \mathrm{vol}_{S^4} \\ & & \Big\downarrow{\scriptstyle S(p)=\mathrm{pr}_1} & & \\ \mathbb{R}^{5,1} \times (0,\infty) & \underset{\mathrm{diff}}{\simeq} & \mathrm{AdS}_7^{\mathrm{Poin}} & & \end{array} \tag{4}$$

M5-brane worldvolume; radial distance; Poincaré chart of anti-de Sitter spacetime

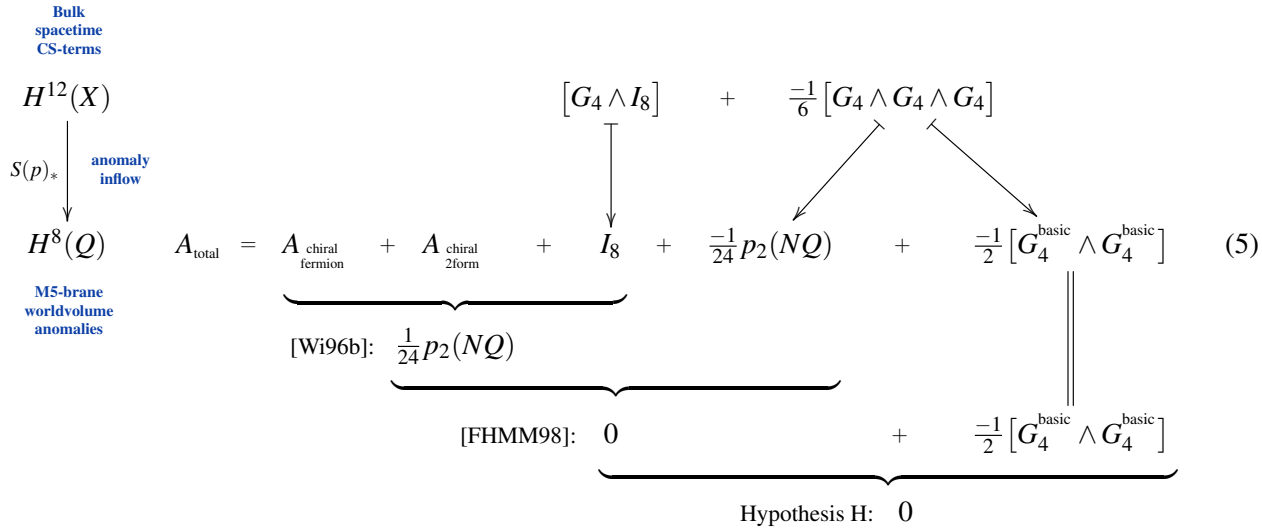
So the point of (1) is to generalize the situation away from this highly symmetric $1/2$-BPS configuration (4) to more general 5-brane configurations. While few to no black M5-brane solutions to 11d supergravity beyond (4) are known explicitly, only their topological structure matters for the discussion of anomaly cancellation; and that topological structure is (essentially by definition) what is expressed by (1).

Remark 2 (G_4 is singular on the M5-brane locus)**.** Condition (2) implies (immediately so by the Poincaré Lemma, since G_4 is closed) that the flux density G_4 can *not* be extended to the locus of the M5-brane itself, which is (or would be) at the center $r = 0 \in [0,\infty)$ of the punctured ball $S^4 \times (0,\infty)$ in (1). Instead it must/would have a singularity at $r = 0$, as is manifest also from the basic example (4). Parts of the literature gloss over this subtlety; and the point made in [FHMM98, p. 4-5] was to argue that this is the source of the missing anomaly cancellation of [Wi96b]. To handle the singularity mathematically, these authors declared[4] to multiply G_4 by a smooth radial cutoff function, thus rendering it no longer closed [FHMM98, (2.3), (3.4)] but, mathematically, extendable to the brane locus. Luckily, the key computation [FHMM98, (3.3)], recalled in (5) below, applies just as well if instead one leaves G_4 intact but removes the singular locus from spacetime, just as usual in supergravity (4).

Remark 3 (Focus on real cohomology)**.** We focus here entirely on the anomaly polynomials in real cohomology, hence ignoring all torsion contributions (which become visible in integral cohomology) as well as all "global" anomaly contributions (which become visible in differential cohomology, see [FSS20d] for discussion of all these notions of cohomology and their relations). Because, while vanishing of the anomaly in real cohomology is not sufficient for full anomaly cancellation (which must happen in differential integral cohomology), it is the *necessary* first step. No argument about torsion of global contributions to the M5 anomaly (which, of course, one will eventually want to address) can affect the proof of anomaly cancellation at the rational/real approximation; and as long as subtleties do remain here, it behooves us to first focus on these. Therefore, we often abbreviate $H^\bullet(-) := H^\bullet(-;\mathbb{R})$, here and in the following.

[4] [FHMM98, p. 4]: "We leave to the future the very interesting question of the relation of this approach to that based on a direct study of solutions to supergravity."

The anomaly polynomials. The cohomology classes contributing to the total M5-brane anomaly in the situation (1) are given in the literature as follows:



(5)

We discuss the various items in (5):

(i) The term I_8 (17) is the "one-loop polynomial" [DLM95][VW95], while the terms $A_{\text{chiral}}_{\text{fermion}}$ and $A_{\text{chiral}}_{\text{2form}}$ are the plain anomalies [Wi96b, (5.1), (5.4)] of the chiral fermion and of the abelian chiral (i.e., with self-dual curvature) 2-form field in 6d QFT. These were expected in [Wi96b] to cancel against the influx of I_8, but found there ([Wi96b, (5.7)]) to cancel only up to a remaining term $\frac{1}{24}p_2(NQ)$, where NQ denotes the normal bundle to the M5-brane locus in spacetime.

(ii) The Chern-Simons term $-\frac{1}{6}G_4 \wedge G_4 \wedge G_4$ of 11-dimensional supergravity was argued in [FHMM98, §3] [BBMN19a, A.4-5] to contribute to the anomaly influx from the bulk. Then a formula due to [BC97, Lem 2.1] shows that this gives rise to the previously missing summand of $\frac{-1}{24}p_2(NQ)$. However, these authors consider an Ansatz for the C-field configuration [FHMM98, (2.3), (3.4)][BBMN19a, (2.4)] which amounts to assuming

$$\left[G_4^{\text{basic}}\right] \overset{!}{=} 0 \tag{6}$$

in (3). If this assumption is not made, then the bulk Chern-Simons term in addition contributes an influx term $\frac{-1}{2}\left[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\right]$ (bottom right of (5)), whose vanishing remains to be discussed.

(iii) That the Ansatz (6) remained unjustified was pointed out in [FSS19a, (19)] and then in [BBMN19b, (3.16) & App. C] (where the basic component is denoted γ_4, see also [BBM20, (2.34)]). Previously in [Mon15, (3.12)] the term G_4^{basic} was assumed to be non-vanishing, in general, and as a resolution it is was suggested [Mon15, (3.7)] that the traditional expression from [Wi96b, (5.7)] for the self-dual field anomaly $A_{\text{chiral}}_{\text{2form}}$ in real cohomology is wrong, in that it gets corrected by just the seemingly missing summand $\frac{+1}{2}\left[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\right]$. Unfortunately, we are unable to verify this derivation. Fortunately, it makes no difference:

(iv) We prove in §3 that the seemingly restrictive Ansatz (6) is *implied* as soon as the dual G_7-flux satisfies a Bianchi identity of a widely expected form (Theorem 4 below); in particular, if it satisfies the Bianchi identity that is implied by Hypothesis H (Thm. 9 below). In this way, Hypothesis H enforces the vanishing of the problematic remaining anomaly term by itself:

$$\text{Hypothesis H} \quad \Rightarrow \quad \left[G_4^{\text{basic}}\right] = 0 \quad \Rightarrow \quad \left[G_4^{\text{basic}} \wedge G_4^{\text{basic}}\right] = 0 \qquad \text{in situation 1}\,.$$

This means, according to (5), that the total M5-brane anomaly is finally cancelled.

3 A resolution

We now prove that *Hypothesis H* implies, in the situation (1), the vanishing of the problematic basic term $\big[G_4^{\text{basic}}\big]$ in (3), thus implying the vanishing of the total M5-brane anomaly according to (5). We proceed in two steps:

(1) In §3.1, we observe a general mechanism that applies as soon as the Bianchi identity $dG_7 = -\frac{1}{2}G_4 \wedge G_4$ holds with any correction by Pontrjagin classes (which is traditionally not guaranteed, see Rem. 7).

(2) In §3.2, we discuss how Hypothesis H implements this mechanism.

Both steps rely on facts about tangent structure on sphere bundles, whose proofs we relegate to appendix A.

3.1 For generic G_7-Bianchi identity

Theorem 4 (Vanishing of basic component)**.** *Given a black M5-brane background* (1) *with C-field flux* G_4 (3) *satisfying a Bianchi identity of the form*

$$d\,G_7 \;=\; -\tfrac{1}{2}G_4 \wedge G_4 \;+\; P\big(p_1(\nabla^{TX}), p_2(\nabla^{TX})\big) \quad \in \Omega^8_{\mathrm{dR}}(X) \tag{7}$$

for P any *polynomial of Pontrjagin forms, then the basic component of* $[G_4]$ (3) *vanishes:*

$$\big[G_4^{\text{basic}}\big] \;=\; 0 \;\in\; H^4(B;\mathbb{R})\,. \tag{8}$$

Proof. The key point is that all Pontrjagin forms on a manifold that is an orthogonal spherical fibration are basic forms, by Prop. 22. This means with (7) that also the cup-square of the class of the 4-flux is basic:

$$[G_4]^2 \;=\; S(p)^*\Big(P\big(p_1(NQ), p_2(NQ)\big)\Big) \;\in\; H^8(X;\mathbb{R})\,. \tag{9}$$

Consider then the fiber integration

$$S(p)_* \;:\; H^\bullet(X;\mathbb{R}) \longrightarrow H^{\bullet-4}(Q;\mathbb{R}) \tag{10}$$

along the fibers of $S(p)$, as in (1). By [BC97, Lemma 2.1], the fiber integration of the odd cup power χ_4^{2k+1} of the Euler class $\chi_4 \in H^4(X;\mathbb{R})$ of the fibration $S(p)$ are proportional to cup powers of the second Pontrjagin class of NQ:

$$S(p)_*(\chi_4^{2k+1}) = 2\big(p_2(NQ)\big)^k \;\in\; H^{8k}(Q)\,, \tag{11}$$

while the fiber integration of the even cup powers of the Euler class vanishes for all $k \in \mathbb{N}$:

$$S(p)_*(\chi_4^{2k}) = 0 \;\in\; H^{8k-4}(Q)\,. \tag{12}$$

Notice also the *projection formula* (e.g. [BT82, Prop. 6.16][FSS18, (2)])

$$S(p)_*\Big(\big(S(p)^*\alpha\big)\wedge\beta\Big) \;=\; \alpha \wedge S(p)_*\beta\,, \tag{13}$$

which in particular implies that the fiber integral of basic forms vanishes:

$$S(p)_*S(p)^*\alpha \;=\; S(p)_*\Big(S(p)^*\alpha \wedge 1\Big) \;=\; \alpha \wedge S(p)_*1 \;=\; 0\,. \tag{14}$$

Therefore, from (9) and by repeated use of formulas (11 - 14) we get:

$$\begin{aligned}
0 &= S(p)_*S(p)^*P\big(p_1(NQ), p_2(NQ)\big)\\
&= \tfrac{1}{2}S(p)_*[G_4\wedge G_4]\\
&= \tfrac{1}{2}S(p)_*\Big(\big(\tfrac{1}{2}\chi_4 + S(p)^*[G_4^{\text{basic}}]\big)\wedge\big(\tfrac{1}{2}\chi + S(p)^*[G_4^{\text{basic}}]\big)\Big)\\
&= \tfrac{1}{2}S(p)_*(\chi_4^2) + S(p)_*\big(\tfrac{1}{2}\chi_4 \wedge S(p)^*[G_4^{\text{basic}}]\big) + \tfrac{1}{2}S(p)_*\big(S(p)^*[G_4^{\text{basic}}]\big)^2\\
&= \tfrac{1}{2}S(p)_*\big(\chi_4^2\big) + S(p)_*\big(\tfrac{1}{2}\chi_4\big)\wedge[G_4^{\text{basic}}] + \tfrac{1}{2}S(p)_*S(p)^*[G_4^{\text{basic}}\wedge G_4^{\text{basic}}]\\
&= [G_4^{\text{basic}}]\,.
\end{aligned} \tag{15}$$

□

Remark 5 (M5-Brane anomaly cancellation for traditional G_7-Bianchi identity)**.** A Bianchi identity for the (Hodge) dual flux density $G_7 := *_{11} G_4$ of the form (7) is traditionally considered as ([DLM95, (1.1)][DFM03, (7.2)])

$$dG_7 \;=\; -\tfrac{1}{2}G_4^2 \;+\; I_8\big(p_1(\nabla^{TX}), p_2(\nabla^{TX})\big) \tag{16}$$

for the polynomial P being just the I_8-term in (5):

$$I_8(p_1,p_2) \;:=\; \tfrac{1}{48}\big(p_2 - \tfrac{1}{4}p_1^2\big)\,. \tag{17}$$

Under this traditional assumption, Theorem 4 implies the condition (6) and hence the vanishing of the remaining M5-brane anomaly in (5).

Remark 6. Theorem 4 in conjunction with Remark 5 may be compared to an analogous physics argument in [BBMN19b, §4.1] (which appeared after [FSS19a] and during the writing of the first version of this article; we thank an anonymous referee for pointing this out).

Remark 7 (Further corrections to the G_7-Bianchi identity)**.** The traditional Bianchi identity (16) incorporates only one of several expected corrections to the plain supergravity Bianchi identity ($P = 0$). These M-theoretic *higher curvature corrections* are traditionally investigated via an action principle (e.g. [HT03][HO06][ST17]):

(i) From the action principle one expects further higher derivative contributions to the Bianchi identity (16) ([ST17, (4.11)], following [HT03, around (56)]):

$$dG_7 \;=\; -\tfrac{1}{2}G_4^2 \;+\; I_8(p_1,p_2) \;+\; \overset{\text{further corrections}}{\delta\Delta L}, \tag{18}$$

which locally, on a chart $U \subset X$ where $G_4\big|_U = dC_3^U$ ([ST17, below (3.4)]), are of the form ([ST17, below (4.11)])

$$\delta\Delta L\big|_U \;=\; d\big(\tfrac{\delta}{\delta C_3^U}\cdots\big)\,. \tag{19}$$

This shows at once that:

(a) *locally on U* the correction is exact (which is the case highlighted in [ST17, below (4.11)]), but

(b) *globally on X* it fails to be exact as soon as G_4 is not globally exact (is only the curvature 4-form of a 2-gerbe connection with local connection 3-forms $\{C_3^U\}_{U\in\mathscr{U}}$ on an open cover $\mathscr{U}$ of X, e.g. [FSS12, p. 22]), which is the generic case and the case of interest here, due to (2).

(ii) The action principle, and hence any Bianchi identity derived from it, must moreover incorporate a global shift [Ts04, (4.16)]:

$$S(g,G_4) \;\longmapsto\; S\big(g, G_4 + \tfrac{1}{4}p_1(\nabla^{TX})\big)\,, \tag{20}$$

reflecting the expected [Wi96a, (1.2)][(1.2)][Wi96b] shifted flux quantization (23) of the C-field.

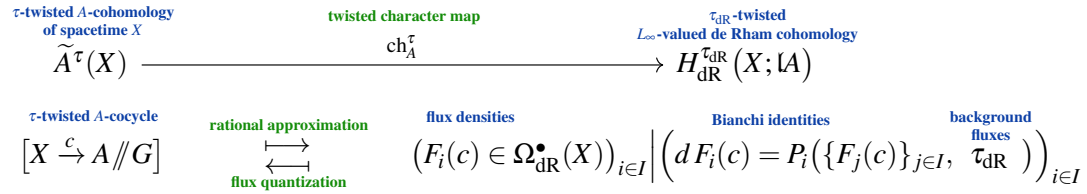
Apart form the general question of whether a classical action principle, of all things, can be the right principle for resolving foundations of M-theory, the complete form of the higher curvature correction in (18) remains open, and its combination with the shift (20) in the action principle seems not to have been discussed yet. Hence, under traditional assumptions, it remains unknown whether the assumption (7) is met in full M-theory.

What has been missing is a principle that fixes Bianchi identities on more fundamental grounds. Such a principle is *cohomological flux quantization* (21), to which we turn now in §3.2.

3.2 Via Hypothesis H

We briefly recall from [FSS19b][FSS20c] the motivation and formulation of *Hypothesis H* on the flux quantization principle for the C-field in M-theory. Then we show (Thm. 9 below) how this Hypothesis implements the above mechanism for cancellation of the remaining M5-anomaly term.

Flux/charge quantization of higher gauge fields. The key idea is that the mathematical nature of any higher gauge field is encoded in a *twisted generalized cohomology theory* $\widetilde{A}^{\tau}(-)$, a notion known as *flux quantization* or *charge quantization* (see [Fr00][Sa10][FSS20d]): A generalized twisted *character map* [FSS20d, §5] approximates cocycles in τ-twisted A-cohomology by flux densities in twisted L_∞-algebra valued de Rham cohomology, namely by differential forms satisfying polynomial differential relations – *Bianchi identities*:

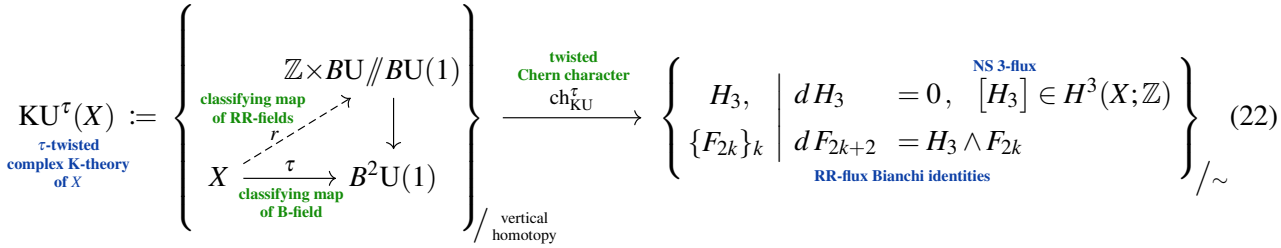


(21)

Flux/charge quantization in A-theory means to demand that the flux densities are in the image of the twisted A-character map (21) of an actual cocycle c in twisted A-cohomology, which then embodies the actual field configuration (its topological sector as shown here, for brevity, and the full field configuration after refinement to *differential* A-cohomology [FSS20d, §4.3]).

The archetypical example is Dirac's flux quantization of the electromagnetic field (e.g. [Al85, §2][Fr00, §2]), which is the demand that the ordinary electromagnetic flux density F_2 (the Faraday tensor) is the character image of a cocycle in ordinary integral degree-2 cohomology $A(-)=H^2(-;\mathbb{Z})$ (hence is the curvature 2-form of a connection on a complex line bundle), which equivalently means that it represents an integral cohomology class $[F_2]\in H^2(X;\mathbb{Z})\to H^2(X;\mathbb{R})$. Here the Bianchi identity obtained from (21) is the simple closure condition $dF_2=0$ ([FSS20d, Ex. 4.10]).

The most famous example is the K-theory conjecture in string theory [MM97][Wi98][Fr00] which states (review in [Wi00][Fr08]) that the B-field and the RR-field fluxes in type II string theory are jointly quantized in twisted [Wi98, §5.3][BM00] (and differential, see [GS19c] for recent rigorous developments) topological K-theory, $A^{\tau}(-)=\mathrm{KU}^{\tau}(-)$. Indeed, the character map (21) in this case takes the following form ([FHT02, §2.5][FSS20d, Prop. 5.5], shown here for type IIA string theory, for definiteness):



(22)

The Bianchi identities on the right of (22) are exactly those expected to be satisfied by the NS B-field flux H_3 and the RR-flux densities F_{2k} in type IIA string theory (see [FSS16b, §4][BMSS19, §1] for details and pointers).

But the M-theoretic lift of these H_3/F_{2k}-Bianchi identities (22) is (see [MaSa03, §4.2][Sa10][FSS16a, §3][BMSS19, §4]) just the G_7-Bianchi identity (7) together with closure and the shifted integrality condition on the G_4-flux:

$$dG_4=0\,,\quad \overbrace{\big[G_4+\tfrac{1}{4}p_1(\nabla^{TX})\big]}^{\widetilde{G}_4}\ \in\ H^4(X;\mathbb{Z})\to H^4(X;\mathbb{R})\,. \tag{23}$$

Therefore, it is natural to ask for a cohomology theory whose twisted character map (21) enforces (7) and (23):

Hypothesis H is the statement [FSS19b, §2][FSS19c][SS19a][SS21] (following [Sa13, §2.5][FSS16a], review in [FSS19, §7]) that the cohomology theory for flux/charge quantization (21) of the C-field in M-theory is Borsuk-Spanier *Cohomotopy theory* $A^\tau(-) = \pi^\tau(-)$ [Sp49] in joint degrees 4 (for the M5-brane charge) and 7 (for the M2-brane charge) related by the quaternionic Hopf fibration and twisted by the tangent bundle via the J-homomorphism ("J-twist"):

The classifying space for degree-n Cohomotopy is (the homotopy type of) the n-sphere S^n, and for (orthogonally) twisted n-Cohomotopy it is the homotopy quotient (Borel construction) $S^n /\!\!/ \mathrm{O}(n+1)$ of the canonical action of the orthogonal group $\mathrm{O}(n+1)$ on $S^n \simeq S(\mathbb{R}^{n+1})$ (recalled as Def. 13 below):

$$\pi^\tau(X) \;:=\; \left\{ \begin{array}{ccc} & & S^4/\!\!/\mathrm{O}(5) \\ & \overset{\text{cocycle } c}{\nearrow} & \downarrow \\ X & \xrightarrow[\tau]{} & B\mathrm{O}(5) \end{array} \right\}_{\Big/ \text{vertical homotopy}} \xrightarrow[\text{[FSS20d, §5.3]}]{\substack{\text{twisted cohomotopical character} \\ \mathrm{ch}^\tau_\pi}} \left\{ \begin{array}{l|l} G_4, & d\,G_4 \;=\; 0\,,\; \big[G_4 + \tfrac{1}{4}p_1(\nabla^\tau)\big] \in H^4(X;\mathbb{Z}) \\ G_7 & d\,G_7 \;=\; -\tfrac{1}{2}G_4 \wedge G_4 + \tfrac{1}{8}p_2(\nabla^\tau) \end{array} \right\}_{/\sim} \tag{24}$$

($\pi^\tau(X)$: τ-twisted 4-Cohomotopy of X)

On the right of (24) we are showing the form of the image of the character map (21) specified to orthogonally twisted 4-Cohomotopy (due to [FSS19b, Prop. 2.5 & 3.13], for more see [FSS20d, §5.3]). Both the $\frac{1}{4}p_1$-shifted integral flux quantization on G_4 (23) is implied from charge-quantization in twisted Cohomotopy, as well as the general form of the G_7-Bianchi identity (7). It just remains to relate the Pontrjagin classes of the twisting bundle τ to the tangent bundle:

The condition that the twist τ be compatible in degrees 7 and 4, along the quaternionic Hopf fibration $h_{\mathbb{H}}$ singles out ([FSS19b, Prop. 2.20]) the quaternionic central product subgroup $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \subset \mathrm{Spin}(8) \to \mathrm{O}(8)$; and demanding that it, moreover, be compatible with factorization through the Atiyah-Penrose twistor fibration $t_{\mathbb{H}}$ (which corresponds [FSS20c][SS20c] to charge-quantization in *heterotic* M-theory) singles out ([FSS20c, Prop. 2.2]) the further subgroup $\mathrm{Sp}(2) \subset \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$:

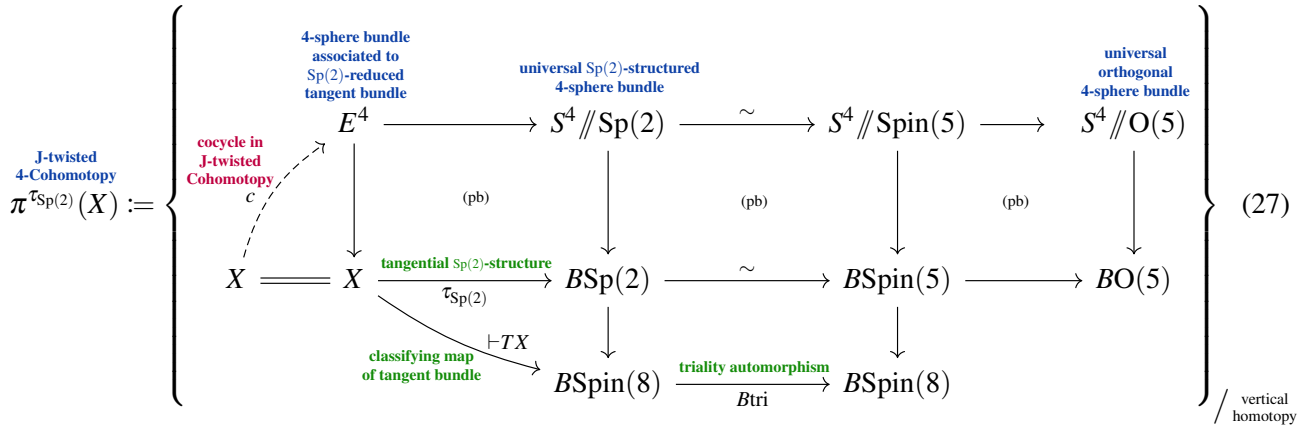
$$\overset{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}{S^7} \xrightarrow[\substack{\text{quaternionic} \\ \text{Hopf fibration}}]{h_{\mathbb{H}}} \overset{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}{S^4}, \qquad \overset{\mathrm{Sp}(2)}{S^7} \xrightarrow[\substack{\text{complex} \\ \text{Hopf fibration}}]{h_{\mathbb{C}}} \overset{\mathrm{Sp}(2)}{\mathbb{C}P^3} \xrightarrow[\substack{\text{Atiyah-Penrose} \\ \text{twistor fibration}}]{t_{\mathbb{H}}} \overset{\mathrm{Sp}(2)}{S^4}. \tag{25}$$

A key subtlety here is that the quaternionic unitary group $\mathrm{Sp}(2)$ and the spin-group $\mathrm{Spin}(5)$ are isomorphic as abstract Lie groups, but not as subgroups of $\mathrm{Spin}(8) \to \mathrm{O}(8)$ (nor are they conjugate subgroups); instead ([FSS19b, Prop. 2.17]) as subgroups they are mapped to each other under the triality automorphism on the ambient $\mathrm{Spin}(8)$-group:

$$\begin{array}{ccc} \mathrm{Sp}(2) & \xrightarrow{\sim} & \mathrm{Spin}(5) \\ {\scriptstyle i_{\mathrm{Sp}(2)}}\downarrow & \text{triality automorphism} & \downarrow{\scriptstyle i_{\mathrm{Spin}(5)}} \\ \mathrm{Spin}(8) & \xrightarrow[\sim]{\mathrm{tri}} & \mathrm{Spin}(8)\,, \end{array} \qquad \begin{array}{ccc} B\mathrm{Sp}(2) & \xrightarrow{\sim} & B\mathrm{Spin}(5) \\ {\scriptstyle Bi_{\mathrm{Sp}(2)}}\downarrow & \text{delooped triality automorphism} & \downarrow{\scriptstyle Bi_{\mathrm{Spin}(5)}} \\ B\mathrm{Spin}(8) & \xrightarrow[\sim]{B\mathrm{tri}} & B\mathrm{Spin}(8) \\ p_1 & \longleftrightarrow & p_1 \\ \big(\tfrac{1}{4}p_1\big)^2 - 24\cdot I_8 & \longleftrightarrow & \tfrac{1}{4}p_2 \end{array} \tag{26}$$

As shown on the bottom right of (26), this triality automorphism, does not affect the first Pontrjagin class, but does induce a nontrivial transformation of Pontrjagin classes in degree 8 ([FSS19b, Lem. 2.19]). Therefore, as we consider tangential $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-structure on spacetime to unify M2/M5-brane charge quantization in J-twisted Cohomotopy, and in fact tangential $\mathrm{Sp}(2)$-structure to account for charges in heterotic-theory, we arrive

at J-twisted Cohomotopy theory of the following form [FSS19b, (17)][FSS19c, (43)]:



Under the twisted character map (21) (with (24) and (26)), this implies the following G_7-Bianchi identity [FSS19b, Prop. 3.8][FSS20d, §5.3]:

$$d\,G_7 = -\tfrac{1}{2}\widetilde{G}_4 \wedge \big(\widetilde{G}_4 - \tfrac{1}{2}p_1(\nabla^{TX})\big) - 12\cdot I_8\big(\nabla^{TX}\big)\,, \tag{28}$$

for $\widetilde{G}_4 := G_4 + \frac{1}{4}p_1$ (23).

Remark 8 (Structure of the Bianchi identity)**.** The Bianchi identity (28) is of the form (16) except for inclusion of the integrality shift (23) and of a relative weight on the I_8-polynomial, corrections that are compatible with the general expectations (Remark 7). Detailed discussion of the consistency/necessity of these particular corrections for all of

(a) C-field tadpole cancellation,

(b) M5 WZ-term level-quantization,

(c) M2-brane Page-charge quantization

is given in [FSS19b, p. 13 & §3.8][FSS19c][SS21, Rem. 4.1]. However, for the application to M5-brane anomaly cancellation, these details are irrelevant. What matters here, by Theorem 4, is that the right hand side of (28) is proportional to $G_4 \wedge G_4$ plus any polynomial in Pontrjagin forms.

M5-Brane anomaly cancellation via Hypothesis H. This allows us to conclude:

Theorem 9. *If the base space Q is parallelizable and the normal bundle NQ_{M5} has $\mathrm{Sp}(2)$-structure then:*

(i) *the ambient black M5-brane spacetime* (1) *$X \to Q$ carries tangential $\mathrm{Sp}(2)$-structure (Def. 11) $\tau_{\mathrm{Sp}(2)}$;*

(ii) *flux-quantization* (21) *of the C-field in τ-twisted 4-Cohomotopy* (27) *enforces – besides the shifted 4-flux quantization* (23) *and the G_7-Bianchi identity* (28) *– the vanishing of the class of $[G_4^{\mathrm{basic}}]$* (6) *and hence of the remaining M5-brane anomaly* (5)*.*

Proof. By the exceptional coset space realization $S^4 \simeq \mathrm{Sp}(2)/\big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)$ (37), Prop. 19 says that the vertical tangent bundle has H-structure, in particular G-structure, for $H \subset G$ being $\mathrm{Sp}(1)\times\mathrm{Sp}(1) \subset \mathrm{Sp}(2) \subset \mathrm{O}(8)$. By Prop. 21 and using the assumption that the tangent bundle of Q is trivializable, this is also the structure on the once-stabilized total tangent bundle, which is claim (i). With this, claim (ii) follows with (28) by Theorem 4. □

Remark 10. The assumption in Theorem 9 are met in the key examples of interest (see [SS19a] [FSS19d][FSS20b] further discussion and pointers):

(i) The assumption that the base space is parallelizable is satisfied for 5-branes wrapped on tori $Q_{\mathrm{M5}} = \mathbb{R}^{5-n,1}\times T^n$ or wrapped on the 3-sphere $Q_{\mathrm{M5}} = \mathbb{R}^{2,1}\times S^3$.

(ii) The assumption that the normal bundle has $\mathrm{Sp}(2)$-structure is satisfied for 5-branes at ADE-singularities, where it even has $\mathrm{Sp}(1)\subset\mathrm{Sp}(2)$-structure.

A Appendix: Tangent structure on sphere bundles

Here we prove some results on (vertical) tangent structures to sphere bundles. The key consequences for the proofs in §3 are:
(i) Prop. 22 (used in the proof of Thm.4 ,
(ii) Prop. 19 (used in the proof of Thm. 9).
The first of these must be well-known to experts, but complete statements/proofs are hard to find in the literature (we give commented pointers to existing references as we proceed). We observe here that (i) follows as a direct corollary of the second statement (ii), which seems to be new. We give a slick homotopy-theoretic proof that neatly ties in with the formulation of *Hypothesis H*.

Homotopy theory. Following [FSS19b][FSS19c][SS21], we make free use of basic notions of homotopy theory ("higher structures"). For mathematical background and pointers see [FSS20d, §A][SS20b, §2]); for exposition in the context of string/M-theory see [JSSW19][FSS19]. This means that all topological spaces in the following are regarded up to weak homotopy equivalence (see [FSS20d, Ex. A.7]), which we denote by an equality sign, e.g. for $S^4/\!\!/\mathrm{Spin}(5) = B\mathrm{Spin}(4)$ in (38) below, where the double slash denotes the homotopy quotient or *Borel construction* for any topological/simplicial group G (see [NSS12b, Prop. 3.73])

$$X/\!\!/G \;=\; H\times_G EG,$$

which subsumes the the classifying space construction $B(-)$ (e.g. [NSS12b, Ex. 3.68])

$$BG \;=\; */\!\!/G \;=\; *\times_G EG \;=\; (EG)/G$$

for *principal G-bundles*, being homotopy pullbacks (e.g. [FSS20d, A.24, A.27]) of the universal G-principal bundle

$$EG \coloneqq G/\!\!/G \to BG$$

([NSS12b, §4.1][SS20b, §2.2], traditional review in [Mit11]):

$$\begin{array}{ccc} P & \longrightarrow & G/\!\!/G = EG \\ {\scriptstyle \text{principal } G\text{-bundle}}\downarrow{\scriptstyle p} & {\scriptstyle \text{homotopy pullback (pb)}} & \downarrow \quad \swarrow {\scriptstyle \text{universal principal } G\text{-bundle}} \\ Q & \xrightarrow[\text{classifying map}]{\vdash P} & BG\,. \end{array} \tag{29}$$

Generally, for any (topological/simplicial) action $G \curvearrowright F$ of G on a typical fiber F, the homotopy quotient serves as the universal G-structured/associated F-fiber bundle ([NSS12a, §4][SS20b, §2.2]):

$$\begin{array}{ccc} E & \longrightarrow & F/\!\!/G = EG\times_G F \\ {\scriptstyle G\text{-structured } F\text{-fiber bundle}}\downarrow{\scriptstyle p} & {\scriptstyle \text{homotopy pullback (pb)}} & \downarrow \quad \swarrow {\scriptstyle \text{universal } G\text{-structured } F\text{-fiber bundle}} \\ Q & \xrightarrow[\text{classifying map}]{\vdash E} & BG\,. \end{array} \tag{30}$$

Here and in the following, we are notationally suppressing the homotopies filling all these diagrams.

***G*-Structures.** Throughout, $n \in \mathbb{N}$ denotes any natural number. All (fiber-)vector spaces and, in particular, all (vertical) tangent spaces are assumed to be finite-dimensional.

Definition 11 (G-Structure)**.** Given a topological group G and a homomorphism $\phi : G \to \mathrm{GL}(d)$, we say that

(i) *G-structure on a real vector bundle $\mathscr{V} \xrightarrow{p} Q$* is a factorization of its classifying map $Q \xrightarrow{\vdash \mathscr{V}} B\mathrm{O}(d)$ through $B\phi : BH \to B\mathrm{GL}(d)$;
(ii) *G-structure on a real smooth manifold M^d* is G-structure on its tangent vector bundle.

Remark 12 (Literature on G-structure)**.** The notion of G-structures as an efficient tool for controlling the geometry of super-string compactifications is discussed in [Lo01][GMPW04][Koe10][Ga11][DDG14][Pr16] [MPT17]. Beware that tradition in Cartan geometry insists that the homomorphism $G \xrightarrow{\phi} \mathrm{GL}(d)$ be injective (e.g. [ČS09, p. 46]). Since this demand excludes common examples of "G-structures" like Spin structures (but also String structures, etc.; and metaplectic structures, etc.) without being necessary for the part of the theory that is relevant here, we do not impose it. In algebraic topology this more general notion is known as $(BG, B\phi)$-*structures* (see [Koc96, §1.4]) or as *tangential structures* [GMTW06, Sec. 5] (observing here that the canonical inclusion $\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$ is the maximal compact subgroup, so that $B\mathrm{O}(d) \xrightarrow{B} \mathrm{GL}(d)$ is a homotopy equivalence). See [SS20b, §4.2] for extensive discussion, comparison and further pointers.

Spherical fibrations.

Definition 13 (Orthogonal n-sphere fiber bundle)**.**
(i) We say that an S^n-fiber bundle $p : X \to Q$ is *orthogonal* if it is equivalent to unit sphere bundle $S(p) : S(\mathscr{V}) \to Q$ inside an real vector bundle $p : \mathscr{V} \to Q$ (whose structure group may always be taken to be the orthogonal group).
(ii) This means equivalently that X is associated (30) via a classifying map $Q \xrightarrow{\vdash X} B\mathrm{O}(n+1)$ to the classifying space for the orthogonal group, which fits into a homotopy-pullback diagram of the following form:

$$\begin{array}{ccc} X & \longrightarrow & S^n /\!\!/ \mathrm{O}(n+1) \\ {\scriptstyle \text{orthogonal } n\text{-sphere bundle}}\downarrow{\scriptstyle p} & \text{homotopy pullback (pb)} & \downarrow{\scriptstyle \text{universal orthogonal } n\text{-shere bundle}} \\ Q & \xrightarrow[\text{classifying map}]{\vdash X} & B\mathrm{O}(n+1)\,, \end{array} \tag{31}$$

where on the top right we have the homotopy quotient (the Borel construction) of the n-sphere $S^n \simeq S(\mathbb{R}^{n+1})$ by its canonical action of the orthogonal group.
(iii) More generally, if a topological group G acts continuously on S^n, then we say that an G-*associated* n-sphere fiber bundle $X \to Q$ is one fitting into a homotopy-pullback diagram of this form:

$$\begin{array}{ccc} X & \longrightarrow & S^n /\!\!/ G \\ {\scriptstyle G\text{-structured } n\text{-sphere bundle}}\downarrow{\scriptstyle p} & \text{homotopy pullback (pb)} & \downarrow{\scriptstyle \text{universal } G\text{-structured } n\text{-sphere bundle}} \\ Q & \xrightarrow[\text{classifying map}]{\vdash X} & BG\,, \end{array} \tag{32}$$

Example 14 (Universal orthogonal n-sphere fiber bundle)**.** Denoting the canonical inclusion of orthogonal groups by

$$\begin{array}{ccc} \mathrm{O}(n) & \hookrightarrow & \mathrm{O}(n+1) \\ A & \longmapsto & \mathrm{diag}(1, A)\,, \end{array} \tag{33}$$

the universal example of orthogonal n-sphere bundles (Def. 13) is equivalent to the map Bi_n of classifying spaces induced from (33):

$$\begin{array}{ccccc} S^n & \longrightarrow & S^n /\!\!/ \mathrm{O}(n+1) & = & B\mathrm{O}(n) \\ \downarrow & \text{(pb)} & \downarrow & \swarrow_{Bi_n} & \\ * & \longrightarrow & B\mathrm{O}(n+1) & & \end{array} \tag{34}$$

This example is classical, see for instance [BC97, p. 4]. But it is just a special case of a more general phenomenon that will be useful for our purpose:

Example 15 (Universal G-structured n-sphere fiber bundle)**.** Let

$$H \hookrightarrow G \tag{35}$$

be an inclusion of topological groups. Then the homotopy quotient (Borel construction) of their coset space G/H by its canonical residual left G-action is equivalent to the homotopy type of the classifying space of H ([FSS19b, Lem. 2.7]):

$$\begin{array}{ccccc} G/H & \longrightarrow & (G/H)/\!\!/G = BH & & \\ \downarrow & \text{(pb)} & \downarrow & \swarrow_{Bi} & \\ * & \longrightarrow & BG & & \end{array} \tag{36}$$

Therefore, when the coset space G/H is in fact an n-sphere equipped with G-action, which happens in the following cases ([FSS19b, Rem. 2.9, Prop. 2.23])

Generic

H	$\overset{i}{\hookrightarrow}$	G	G/H
$\mathrm{O}(n)$	$\subset$	$\mathrm{O}(n+1)$	S^n
$\mathrm{SO}(n)$	$\subset$	$\mathrm{SO}(n+1)$	
$\mathrm{Spin}(n)$	$\subset$	$\mathrm{Spin}(n+1)$	
$\mathrm{SU}(n)$	$\subset$	$\mathrm{SU}(n+1)$	S^{2n-1}
$\mathrm{Sp}(n)$	$\subset$	$\mathrm{Sp}(n+1)$	S^{4n-1}

Exceptional

H	$\overset{i}{\hookrightarrow}$	G	G/H
$\mathrm{Sp}(1)\times\mathrm{Sp}(1)$	$\subset$	$\mathrm{Sp}(2)$	S^4
$\mathrm{SU}(3)$	$\subset$	G_2	S^6
G_2	$\subset$	$\mathrm{Spin}(7)$	S^7
$\mathrm{Spin}(7)$	$\subset$	$\mathrm{Spin}(9)$	S^{15}

(37)

then the universal G-associated n-sphere bundle is equivalently the classifying space of H ([FSS19b, Prop. 2.8]):

$$\begin{array}{ccccc} S^n & \longrightarrow & S^n/\!\!/G = BH & & \\ \downarrow & \text{(pb)} & \downarrow & \swarrow_{Bi} & \\ * & \longrightarrow & BG & & \end{array} \tag{38}$$

Vertical tangent bundles to spherical fibrations. We now show how this universal homotopy-theoretic construction of sphere bundles knows everything about their vertical tangent bundles.

Proposition 16 (Classifying map of frame bundle to n-sphere)**.** *Under the identification on the right of* (34)*, the homotopy fiber inclusion* $\mathrm{fib}(Bi_n)$ *of* S^n *into the universal orthogonal* n*-sphere fiber bundle (Example 14) is the classifying map* $\vdash \mathrm{Fr}(S^n)$ *for the orthogonal frame bundle* $\mathrm{Fr}_{\mathrm{O}}(S^n) \to S^n$ *of the* n*-sphere:*

$$\begin{array}{ccc} S^n & \xrightarrow[\vdash \mathrm{Fr}_{\mathrm{O}}(S^n)]{\text{classifying map of orthogonal frame bundle}} & B\mathrm{O}(n) \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle Bi_n} \\ * & \longrightarrow & B\mathrm{O}(n+1) \end{array} \tag{39}$$

Proof. The long homotopy fiber sequence of Bi_n (e.g. [NSS12a, Def. 2.26], following from the pasting law [NSS12a, Prop. 2.23]) shows that the homotopy fiber inclusion of Bi_n classifies an $\mathrm{O}(n)$-principal bundle over the n-sphere whose total space is $\mathrm{O}(n+1)$ with $\mathrm{O}(n)$-action induced by the canonical inclusion i_n (33):

$$\begin{array}{ccccccc} \mathrm{O}(n) & \xrightarrow{i} & \mathrm{O}(n+1) & \longrightarrow & * & & \\ \downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow & & \\ * & \longrightarrow & S^n & \xrightarrow{\mathrm{fib}(Bi_n)} & B\mathrm{O}(n) & & \\ & & \downarrow & \text{(pb)} & \downarrow{\scriptstyle Bi_n} & & \\ & & * & \longrightarrow & B\mathrm{O}(n+1) & & \end{array}$$

Therefore, it is sufficient to observe that we have an isomorphism of $\mathrm{O}(n)$-principal bundles

$$\begin{array}{ccc} \mathrm{O}(n+1) & \xrightarrow{\ \sim\ } & \mathrm{Fr}(S^n) \\ A & \mapsto & \big(A\cdot v_1, \cdots, A\cdot v_n\big) \in \mathrm{Fr}_{A\cdot v_0}(S^n)\,, \end{array}$$

where $v_0, \cdots v_n \in \mathbb{R}^{n+1}$ are the canonical basis vectors and where on the right we regard $S^n = S(\mathbb{R}^{n+1})$ with the induced identification of $T_{A\cdot v_0} S^n \simeq (A\cdot v^0)^\perp \subset \mathbb{R}^{n+1}$. □

More generally:

Proposition 17 (Classifying map of H-frame bundle of H-coset realization of n-sphere)**.** *Given a coset-space realization of the n-sphere $S^n \simeq G/H$ (37) induced from a subgroup inclusion $H \xrightarrow{i} G$ (35) of compact Lie groups, then under the identification on the right of* (38) *the homotopy fiber inclusion* $\mathrm{fib}(Bi)$ *of S^n into the universal G-associated n-sphere fiber bundle (Example 15) is a classifying map for the H-principal bundle on the n-sphere which exhibits its canonical H-structure (Def. 11):*

$$\begin{array}{ccc} S^n & \xrightarrow[\vdash \mathrm{Fr}_H(S^n)]{\text{classifying map of } H\text{-frame bundle}} & BH \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle Bi} \\ * & \longrightarrow & BG \end{array} \tag{40}$$

Proof. As before, the long homotopy fiber sequence of Bi (e.g. [NSS12a, Def. 2.26], following from the pasting law [NSS12a, Prop. 2.23]) shows that the homotopy fiber inclusion of Bi classifies an H-principal bundle over the n-sphere whose total space is G with H-action induced by the given subgroup inclusion:

$$\begin{array}{ccccc} H & \xhookrightarrow{i} & G & \longrightarrow & * \\ \downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow \\ * & \longrightarrow & S^n & \to \mathrm{fib}(Bi) \to & BH \\ & & \downarrow & \text{(pb)} & \downarrow{\scriptstyle Bi} \\ & & * & \longrightarrow & BG \end{array}$$

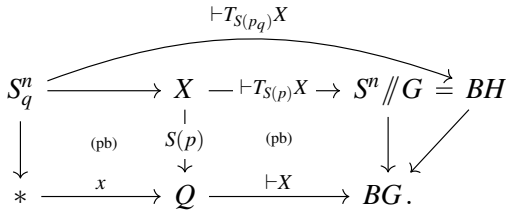
Therefore, it is sufficient to observe that $G \to G/H$ is an H-frame bundle that exhibits H-structure (Def. 11) on the tangent bundle $T(G/H)$. This is basic fact of Cartan geometry, laid out for instance in [ČS09, p. 53]. □

Example 18 (Canonical Spin structure on n-spheres)**.** For the generic coset space realization of the n-sphere from (37), $S^n \simeq \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$, Prop. 17 says that the homotopy fiber inclusion of the map of classifying spaces $B\mathrm{Spin}(n) \xrightarrow{B} \mathrm{Spin}(n+1)$ classifies a $\mathrm{Spin}(n)$-principal bundle of the form $\mathrm{Spin}(n+1) \to S^n$ and that this is a Spin structure (Def. 11) on the n-sphere. A traditional proof of this fact is spelled out in detail in [No15, Thm. A.6.6], see also [Gu88, §2.a].

Proposition 19 (H-Structure on vertical tangent bundle of G-associated sphere bundle)**.** *Given a coset-space realization of the n-sphere $S^n \simeq G/H$ (37) induced from a Lie subgroup inclusion $H \xrightarrow{i} G$ (35), then for a G-associated S^n-fiber bundle $X \xrightarrow{S(p)} Q$ (32):*

- **(i)** *the vertical tangent bundle carries an H-structure (Def. 11);*
- **(ii)** *whose associated G-principal bundle is the pullback along $S(p)$ of that to which X is associated.*

Proof. By the classification (30) of fiber bundles, $S(p)$ sits in a homotopy pullback square as on the right of the following pasting diagram



(41)

Notice that if the top map in the right square classifies H-structure on the vertical tangent bundle, as indicated by its label, then the homotopy-commutativity of the right square is equivalent to claim (ii).

Hence it is sufficient now to prove that the top right map indeed classifies H-structure on the vertical tangent bundle; which then also proves claim (i).

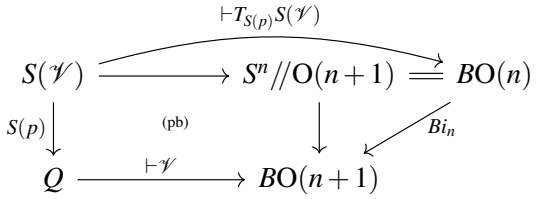
To that end, consider any point $q \in Q$ and write S^n_q for the sphere fiber over it, as shown by the homotopy pullback square on the left of (41). By the pasting law ([NSS12a, Prop. 2.23]) it follows that the full rectangle is a homotopy pullback. Therefore Prop. 17 says that the composite top map in (41) classifies H-structure on the tangent bundle of S^n_q. Since this true for all q, it follows that the H-principal bundle classified by $\vdash T_{S(p)}X$ restricts on each sphere fiber S^n_q to that sphere's tangent H-structure. But this is the defining property of (H-structure on) the vertical tangent bundle of X. □

Corollary 20 (Once-stabilized vertical tangent bundle of orthogonal sphere bundle is basic)**.** *The once-stabilized vertical tangent bundle to an orthogonal sphere bundle $S(p)\colon S(\mathscr{V}) \to Q$ (31) is isomorphic to the pullback of its underlying vector bundle:*

$$\overbrace{T_{S(p)}\big(\underbrace{S(\mathscr{V})}_{\text{orthogonal sphere bundle}}\big)}^{\text{vertical tangent bundle}} \overbrace{\times \mathbb{R}}^{\text{one-step stabiliz.}} \simeq_Q \overbrace{S(p)^*\big(\mathscr{V}\big)}^{\text{pullback of associated vector bundle}}. \tag{42}$$

A traditional proof of this statement is indicated in [Go16, Prop. 1.1.9].

Proof. For the orthogonal subgroup inclusion $\mathrm{O}(n) \xrightarrow{i_n} \mathrm{O}(n+1)$ (33), Prop. 19 gives a homotopy pullback diagram (42) of this form:



(43)

Noticing that postcomposition with Bi_n manifestly corresponds to one-step stabilization of an orthogonal vector bundle, the homotopy-commutativity of this square is exactly the claim to be proven. □

The following is stated without proof as [CrEs03, Fact 3.1], apparently reading between the lines of [Mil56, p. 403].

Corollary 21 (Once-stabilized tangent bundle of orthogonal sphere bundle is basic)**.**
If the base space Q is a smooth manifold, then the once-stabilized tangent bundle of the total space of an orthogonal sphere bundle $S(p) : S(\mathscr{V}) \to X$ (Def. 13) is isomorphic to the pullback along $S(p)$ of the Whitney sum of the tangent bundle of the base with the given vector bundle:

$$T\big(S(\mathscr{V})\big) \times \mathbb{R} \simeq_Q S(p)^*\big(TQ \oplus_Q \mathscr{V}\big). \tag{44}$$

Proof. Consider the following sequence of bundle isomorphisms over the base space Q:

$$\begin{aligned} T\big(S(\mathscr{V})\big)\times\mathbb{R} &\simeq_Q \big(S(p)^*(TQ)\big)\oplus_Q \Big(T_{S(p)}\big(S(\mathscr{V})\big)\times\mathbb{R}\Big) \\ &\simeq_Q \big(S(p)^*(TQ)\big)\oplus_Q \big(S(p)^*\mathscr{V}\big) \\ &\simeq_Q S(p)^*\big(TQ\oplus_Q \mathscr{V}\big)\,. \end{aligned}$$

Here:

(a) The first step is a splitting of the short exact sequence of vector bundles

$$0 \longrightarrow T_{S(p)}S(\mathscr{V}) \longhookrightarrow T\big(S(\mathscr{V})\big) \xrightarrow{dS(p)} S(p)^*(TQ) \longrightarrow 0$$

that defines the vertical tangent bundle $T_{S(p)}S(\mathscr{V})$, and which splits as a special case of the general splitting of short exact sequences of real vector bundles over paracompact Hausdorff base spaces, in particular over smooth manifolds, by forming orthogonal complements with respect to any choice of a continuous fiberwise inner product.

(b) The second step is Corollary 20.

(c) The last step is the distributivity of pullback over Whitney sum of vector bundles. □

In conclusion :

Proposition 22 (Stable characteristic classes on sphere bundles are basic)**.** *Given an orthogonal sphere-fiber bundle $S(\mathscr{V})$,*
(i) *every stable characteristic class – hence every polynomial $P(p_1,p_2,\cdots)$ of Pontrjagin classes p_i – of its vertical tangent bundle is basic, i.e.: pulled back from the base space Q:*

$$P(p_1,p_2,\cdots)\big(T_{S(p)}S(\mathscr{V})\big) \;=\; S(p)^*\underbrace{P(p_1,p_2,\cdots)\big(\mathscr{V}\big)}_{\in H^\bullet(Q)};$$

(ii) *and if the base space Q is a smooth manifold then then analogous statement holds for every stable class of the full tangent bundle*

$$P(p_1,p_2,\cdots)\big(TS(\mathscr{V})\big) \;=\; S(p)^*\underbrace{P(p_1,p_2,\cdots)(TQ\oplus_Q \mathscr{V})}_{\in H^\bullet(Q)};$$

Proof. Since a stable characteristic class of a vector bundle, such as a Pontrjagin class, is one that can be pulled back from any direct sum of that vector bundle with a trivial vector bundle, the first claim follows by Cor. 20 and the second by Cor. 21. □

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