1 Introduction and survey

Recall the content of the
K-theory conjecture in string theory: *The collection of NS-flux and RR-flux differential forms in string theory are subject to* charge quantization in twisted K-theory, *in that they are but the Chern-character image of a cocycle in twisted K-theory, and it is this cocycle in (differential) twisted K-theory which is the full incarnation of the NS&RR-fields in string theory.*

In direct analogy to this, we introduce:
Hypothesis H in M-Theory: *The C-field 4-flux and 7-flux forms in M-theory are subject to* charge quantization in J-twisted Cohomotopy cohomology theory *in that they are but the non-abelian Chern character image of a cocycle in J-twisted Cohomotopy theory, and it is this cocycle in (differential) J-twisted Cohomotopy theory which is the full incarnation of the C-field in M-theory .*

In support of Hypothesis H, we here prove that it implies the following phenomena, expected for M2-brane backgrounds in M-theory on 8-manifolds:

Cohomotopy theory	expression		**M-theory**
compatible twisting on 4- & 7-Cohomotopy theory	$W_7[TX] = 0$	(13)	DMW anomaly cancellation condition
any cocycle in J-twisted 7-Cohomotopy	$\mathrm{Spin}(7)$-structure g	(14)	$\geq$ 1/8 BPS M2-brane background
any cocycle in compatibly twisted 4&7-Cohomotopy	$\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$-structure τ	(15)	4/8 BPS M2-brane background
Chern character of rationally twisted 4-Cohomotopy	$d\,G_4 = 0$ $d\,G_7 = -\tfrac{1}{2}G_4\wedge G_4 + L_8$	(19)	C-field Bianchi identity with generic higher curvature correction
Chern character of compatibly rationally twisted 4&7-Cohomotopy	$d\,\widetilde{G}_4 = 0$ $d\,G_7 = -\tfrac{1}{2}\big(\widetilde{G}_4 - \tfrac{1}{4}P_4\big)\wedge\widetilde{G}_4 + K_8$	(20)	shifted C-field Bianchi identity with generic higher curvature correction
Chern character 4-form of Sp(2)-twisted 4-Cohomotopy	$\widetilde{G}_4 \;=\; G_4 + \tfrac{1}{4}p_1(\nabla)$	(21)	C-field shift by background charge
	$[\widetilde{G}_4] \;\in\; H^4(X^8,\mathbb{Z})$	(22)	shifted C-field flux quantization
	$\mathrm{Sq}^2\big([\widetilde{G}_4]\big) \;=\; 0$	(23)	integral equation of motion
Chern character 7-form of compatibly Sp(2)-twisted 4&7-Cohomotopy	$\widetilde{G}_7 \;=\; G_7 + \tfrac{1}{2}H_3\wedge\widetilde{G}_4$	(26)	Page charge
	$d\,\tilde{G}_7 \;=\; -\tfrac{1}{2}\chi_8(\nabla)$	(27)	conservation of Page charge
	$2\int_{S^7} i^*\widetilde{G}_7 \;\in\; \mathbb{Z}$	(28)	level quantization of Hopf-WZ term
integrated Chern character of compatibly Sp(2)-twisted 4&7-Cohomotopy	$N_{M2} \;=\; -I_8$	(32)	C-field tadpole cancellation

We now survey these statements informally. Full details, proofs and references are in [FSS19b][FSS19c]. For background and motivation see [FSS19a]; for equivariant Cohomotopy and M-theory orbifolds see [SS19a][SS19b].

Generalized abelian cohomology. Before we start, briefly a word on "generalized" cohomology theories, recalling some basics, but in a broader perspective:

The *ordinary cohomology groups* $X \mapsto H^\bullet(X, \mathbb{Z})$ famously satisfy a list of nice properties, called the *Eilenberg-Steenrod axioms*. Dropping just one of these axioms (the *dimension axiom*) yields a larger class of possible abelian group assignments $X \mapsto E^\bullet(X)$, often called *generalized cohomology theories*. One example are the *complex topological K-theory groups* $X \mapsto \mathrm{KU}^\bullet(X)$.

By the *Brown representability theorem*, every generalized cohomology theory in this sense has a *classifying space* E_n for each degree, such that the n-th cohomology group is equivalently the set of homotopy classes of maps into this space:

$$\begin{array}{c}\text{generalized abelian}\\ \text{cohomology theory}\end{array} \quad E^n(X) \;\; \overset{\begin{array}{c}\text{Brown's}\\ \text{representability}\\ \text{theorem}\end{array}}{\simeq} \;\; \left\{ X \xrightarrow[\text{= cocycle in } E\text{-theory}]{\text{continuous function}} E_n \right\}_{\big/ \sim_{\text{homotopy}}} \quad \begin{array}{l}\text{for } (E_n)_{n\in\mathbb{N}} \text{ with } E_n \simeq \Omega E_{n+1}\\ \text{a } \textit{spectrum} \text{ of classifying spaces}\end{array} \quad . \tag{1}$$

(Here and in the following, a dashed arrow indicates a map representing a cocycle that is free to choose, as opposed to solid arrows indicating fixed structure maps.)

For example, ordinary cohomology theory has as classifying spaces the *Eilenberg-MacLane spaces* $K(\mathbb{Z}, n)$, while complex topological K-theory in degree 1 is classified by the space underlying the stable unitary group.

For generalized cohomology theories in this sense of Eilenberg-Steenrod, Brown's representability theorem translates the *suspension axiom* into the statement that the classifying spaces E_n in (1) are loop spaces of each other, $E_n \simeq \Omega E_{n+1}$, and thus organize into a sequence of classifying spaces $(E_n)_{n\in\mathbb{N}}$ called a *spectrum*. The fact that each space in a spectrum is thereby an infinite loop space makes it behave like a homotopical *abelian* group (since higher-dimensional loops may be homotoped and hence commuted around each other, by the Eckmann-Hilton argument).

Generalized non-abelian cohomology. But not all cohomology theories are abelian! The classical example, for G any non-abelian Lie group, is the *first non-abelian cohomology* $X \mapsto H^1\big(X, G\big)$, defined on any manifold X as the first Cech cohomology of X with coefficients in the sheaf of G-valued functions.

Nevertheless, this non-abelian cohomology theory also has a classifying space, called BG, and in terms of this it is given exactly as the abelian generalized cohomology theories in (1):

$$\begin{array}{c}\text{degree-1 non-abelian}\\ \text{cohomology theory}\end{array} \quad H^1(X, G) \;\; \overset{\begin{array}{c}\text{principal bundle}\\ \text{theory}\end{array}}{\simeq} \;\; \left\{ X \xrightarrow[\text{= cocycle}]{\text{continuous function}} BG \right\}_{\big/ \sim_{\text{homotopy}}} \quad . \tag{2}$$

Hence the joint generalization of a) generalized abelian cohomology theory (1) and b) non-abelian 1-cohomology theories (2) are assignments of homotopy classes of maps into *any* coefficient space A

$$\begin{array}{c}\text{non-abelian generalized}\\ \text{cohomology theory}\end{array} \quad H(X, A) \;\; := \;\; \left\{ X \xrightarrow[\text{= cocycle}]{\text{continuous function}} A \right\}_{\big/ \sim_{\text{homotopy}}} \quad . \tag{3}$$

All this may naturally be further generalized from topological spaces to higher stacks. In the literature of this broader context the perspective of non-abelian generalized cohomology is more familiar. But it applies to the topological situation as the easiest special case, and this is the case we are concerned with for the present purpose.

Higher principal bundles. This way, the classical statement (2) of principal bundle theory finds the following elegant homotopy-theoretic generalization:

For every *connected* space A, its based loop space $G := \Omega A$ is a higher homotopical group under concatenation of loops (an "∞-group"). Moreover, A itself is equivalently the classifying space for that higher group:

$$\overset{\begin{array}{c}\textit{Every}\\ \text{connected}\\ \text{space...}\end{array}}{A} \;\; \simeq \;\; \overset{\begin{array}{c}\text{...is the}\\ \text{classifying}\\ \text{space...}\end{array}}{B} \; \underset{\begin{array}{c}\text{...of its}\\ \text{loop group.}\end{array}}{\overbrace{\Omega A}^{G}} \tag{4}$$

in that non-abelian G-cohomology in degree 1 classifies higher homotopical G-principal bundles:

$$
H(X,BG) \;=\; \overset{\substack{\text{non-abelian}\\ G\text{-cohomology}}}{H^1(X,G)} \xrightarrow{\;\;\simeq\;\;} \overset{\substack{\text{higher homotopical}\\ G\text{-principal bundles}}}{G\mathrm{Bundles}(X)_{/\sim}}
$$

$$
\Big[X \xrightarrow{\overset{\text{cocycle}}{c}} BG\Big] \;\mapsto\; \left[\begin{tikzcd}
\overset{\substack{G\text{-principal bundle}\\ \text{classified by } c}}{P} \arrow[r] \arrow[d, "c^*(p_{BG})"'] \arrow[dr, phantom, "\substack{\text{homotopy}\\ \text{pullback}}"] & \overset{\substack{\text{universal}\\ G\text{-principal}\\ \text{bundle}}}{G/\!\!/G} \arrow[d, "p_{BG}"] \\
X \arrow[r, "\underset{\substack{\text{classifying map}\\ \text{for } P}}{c}"'] & BG
\end{tikzcd}\right]
\tag{5}
$$

Cohomotopy cohomology theory. The primordial example of a non-abelian generalized cohomology theory (3) is *Cohomotopy cohomology theory*, denoted $\pi^\bullet$. By definition, its classifying spaces are simply the n-spheres S^n:

$$
\substack{\text{Cohomotopy}\\ \text{cohomology theory}} \qquad \pi^n(X) \;:=\; \left\{ X \overset{\text{continuous function}}{\underset{=\text{ cocycle}}{\dashrightarrow}} S^n \right\}_{/\sim_{\text{homotopy}}} .
\tag{6}
$$

Since the $(n \geq 1)$-spheres are connected, the equivalence (4) applies and says that Cohomotopy theory is equivalently non-abelian 1-cohomology for the loop groups of spheres $G := \Omega S^n$:

$$
\underset{\substack{\text{Cohomotopy}\\ \text{theory}}}{\pi^n(X)} \;\simeq\; \underset{\substack{\text{non-abelian 1-cohomology}\\ \text{for sphere loop group}}}{H^1(X,\Omega S^n)} .
$$

A whole range of classical theorems in differential topology all revolve around characterizations of Cohomotopy sets, even if this is not often fully brought out in the terminology.

Evaluated on spaces which are themselves spheres, Cohomotopy cohomology theory evaluates to the (unstable!) *homotopy groups of spheres*, the “vanishing point” of algebraic topology:

$$
\substack{n\text{-cohomotopy groups}\\ \text{of } k\text{-sphere}} \qquad \pi^n\big(S^k\big) \;\simeq\; \Big\{ S^k \dashrightarrow S^n \Big\}_{/\sim} \simeq\; \pi_k\big(S^n\big) \qquad \substack{k\text{-homotopy groups}\\ \text{of } n\text{-sphere}} .
$$

The quaternionic Hopf fibration. A notable example, for the following purpose, of a class in the Cohomotopy group of spheres, is given by the *quaternionic Hopf fibration*

$$
\begin{tikzcd}
S^7 \arrow[rrrrr, bend left, "\overset{\text{quaternionic Hopf fibration}}{h_{\mathbb{H}}}"] & \simeq & \underset{\substack{\text{unit sphere}\\ \text{in quaternionic}\\ \text{2-space}}}{S(\mathbb{H}^2)} \arrow[r, "(q_1,q_2)\mapsto[q_1:q_2]"'] & \underset{\substack{\text{quaternionic}\\ \text{projective}\\ \text{1-space}}}{\mathbb{H}P^1} & \simeq & S^4
\end{tikzcd} ,
\tag{7}
$$

which represents a generator of the non-torsion subgroup in the 4-Cohomotopy of the 7-sphere, as shown on the left here:

$$
\begin{tikzcd}
\substack{\text{quaternionic}\\ \text{Hopf fibration}} & {[S^7 \xrightarrow{h_{\mathbb{H}}} S^4]} \arrow[d, mapsto] & \in & \overset{\substack{\text{non-abelian/unstable}\\ \text{Cohomotopy group}}}{\pi^4(S^7)} \arrow[r, "\overset{\text{stabilization}}{\Sigma^\infty}"] \arrow[d, phantom, "\wr"] & \overset{\substack{\text{abelian/stable}\\ \text{Cohomotopy group}}}{\mathbb{S}^4(S^7)} \arrow[d, phantom, "\wr"] & \ni & \Sigma^\infty[S^7 \xrightarrow{h_{\mathbb{H}}} S^4] \arrow[d, mapsto] & \substack{\text{stabilized}\\ \text{quaternionic}\\ \text{Hopf fibration}} \\
\substack{\text{non-torsion}\\ \text{generator}} & (1,0) & \in & \mathbb{Z}\times\mathbb{Z}_{12} \arrow[r, "(n,a)\mapsto(n \bmod 24)"'] & \mathbb{Z}_{24} & \ni & 1 & \substack{\text{torsion}\\ \text{generator}}
\end{tikzcd}
\tag{8}
$$

Shown on the right is the abelian approximation to non-abelian Cohomotopy cohomology theory, called *stable Cohomotopy theory* and represented, via (1), by the *sphere spectrum* $\mathbb{S}$ (whose component spaces are the infinite-loop space completions of the n-spheres: $\mathbb{S}_n \simeq \Omega^\infty \Sigma^\infty S^n$). Crucially, in this approximation the quaternionic Hopf fibration becomes a torsion generator:

Non-abelian 4-Cohomotopy witnesses integer cohomology groups not only in degree 4, but also in degree 7 – but when seen in the abelian/stable approximation this “extra degree” fades away and leaves only a torsion shadow behind.

In any case, composition with the quaternionic Hopf fibration is a transformation that translates classes in degree-7 Cohomotopy to classes in degree-4 Cohomotopy:

$$\begin{tikzcd}
 & S^7 \arrow[d, "h_{\mathbb{H}}"] & \text{7-Cohomotopy} & \pi^7(X) \arrow[d, "(h_{\mathbb{H}})_*"] \\
X \arrow[ur, dashed, "c"] \arrow[r, dashed, "(h_{\mathbb{H}})_*(c)"'] & S^4 & \text{4-Cohomotopy} & \pi^4(X)
\end{tikzcd}
\quad \text{reflects into}
\tag{9}$$

Twisted non-abelian generalized cohomology. Regarding generalized cohomology theory as homotopy theory of classifying spaces (3) makes transparent the concept of *twistings* in cohomology theory: Instead of mapping into a fixed classifying spaces, a *twisted cocycle* maps into a varying classifying space that may twist and turn as one moves on the domain space. In other words: A *twisting* τ of A-cohomology theory on some X is a bundle over X with typical fiber A, and a τ-twisted cocycle is a *section* of that bundle, as shown on the left in the first line of the following:

$$\begin{aligned}
\underset{\substack{\tau\text{-twisted} \\ \text{non-abelian generalized} \\ A\text{-cohomology theory}}}{A^\tau(X)} \;:=\; & \left\{
\begin{tikzcd}
 & \overset{A\text{-fiber bundle}}{P} \arrow[r] \arrow[d, "p"] & \overset{\substack{\text{universal} \\ A\text{-fiber bundle}}}{A /\!\!/ \mathrm{Aut}(A)} \arrow[d] \\
X \arrow[ur, dashed, bend left, "\substack{\text{continuous section} \\ = \text{twisted cocycle}}"] \arrow[r, equal] & X \arrow[r, "\tau"', "\substack{\text{classifying map} \\ \text{for } P}"' {yshift=-12pt}] & B\mathrm{Aut}(A)
\end{tikzcd}
\right\}_{\big/ \sim_{\frac{\text{homotopy}}{B\mathrm{Aut}(A)}}} . \\
\simeq\; & \left\{
\begin{tikzcd}
X \arrow[rr, dashed, "\text{continuous function}"] \arrow[dr, "\text{twist } \tau"', ""{name=T}] & & A /\!\!/ \mathrm{Aut}(A) \arrow[dl] \\
 & B\mathrm{Aut}(A) &
\arrow[Rightarrow, from=1-2, to=T, "\text{homotopy}"]
\end{tikzcd}
\right\}_{\big/ \sim_{\frac{\text{homotopy}}{B\mathrm{Aut}(A)}}}
\end{aligned}
\tag{10}$$

Here the equivalent formulation shown in the second line follows because A-fiber bundles are themselves classified by nonabelian $\mathrm{Aut}(A)$-cohomology, as shown on the right of the first line (due to (5)).

Twisted Cohomotopy theory. For the example (6) of Cohomotopy cohomology theory in degree $d-1$ there is a canonical twisting on Riemannian d-manifolds, given by the unit sphere bundle in the orthogonal tangent bundle:

$$\underset{\text{Cohomotopy theory}}{\overset{\text{J-twisted}}{}} \quad \pi^{TX^d}(X^d) := \left\{ \begin{tikzcd} & \overset{\text{tangent unit sphere bundle}}{S(TX^d)} \arrow[r] \arrow[d, "p"] & \overset{\text{universal tangent unit sphere bundle}}{S^{d-1} /\!\!/ \mathrm{O}(d)} \arrow[d] \\ X \arrow[ur, dashed, "\substack{\text{continuous section} \\ = \text{twisted cocycle}}"] \arrow[r, equal] & X \arrow[r, "TX^d"', "\substack{\text{classifying map of} \\ \text{tangent/frame bundle}}"' near end] & B\mathrm{O}(d) \end{tikzcd} \right\}_{/\sim_{\frac{\text{homotopy}}{B\mathrm{O}(d)}}} . \tag{11}$$

$$\simeq \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\text{continuous function}"] \arrow[dr, "\substack{TX^d \\ \text{twist}}"'] \arrow[dr, phantom, shift left=4, "\Downarrow \text{homotopy}"] & & S^{d-1} /\!\!/ \mathrm{O}(d) \arrow[dl] \\ & B\mathrm{O}(d) & \end{tikzcd} \right\}_{/\sim_{\frac{\text{homotopy}}{B\mathrm{O}(d)}}}$$

Since the canonical morphism $\mathrm{O}(d) \longrightarrow \mathrm{Aut}(S^{d-1})$ is known as the *J-homomorphism*, we may call this *J-twisted Cohomotopy theory*, for short.

Compatibly J-twisted Cohomotopy in degrees 4 & 7. In view of (9) it is natural to ask for the maximal subgroup $G \subset \mathrm{O}(8)$ for which the quaternionic Hopf fibration is equivariant, so that its homotopy quotient $h_{\mathbb{H}} /\!\!/ G$ exists and serves as a map of *G-twisted* Cohomotopy theories (11) from degree 7 and 4.

This subgroup turns out to be the central product of the quaternion unitary groups $\mathrm{Sp}(n)$ for $n = 1, 2$:

$$\underset{\substack{\text{central product of} \\ \text{quaternion-unitary groups}}}{\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)} \subset \mathrm{O}(8) \quad \text{is maximal subgroup s.t.} \quad \begin{tikzcd} & S^7 /\!\!/ \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \arrow[dl] \arrow[dd, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)"', "\substack{\text{universally twisted} \\ \text{quaternionic Hopf fibration}}"] \\ B\big(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)\big) & \\ & S^4 /\!\!/ \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \arrow[ul] \end{tikzcd} \tag{12}$$

In other words, J-twisted Cohomotopy (11) exists compatibly in degrees 4 & 7 precisely on those 8-manifolds which carry topological $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-structure, i.e., whose structure group of the tangent bundle is equipped with a reduction along $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \hookrightarrow \mathrm{O}(8)$. This reduction is equivalent to a factorization of the classifying map as shown on the left below, with some cohomological consequences shown on the right:

$$\begin{tikzcd} X^8 \arrow[d, "\substack{\text{tangent} \\ \text{bundle}} \; TX^8"'] \arrow[dr, dashed, "\tau"] \arrow[dr, phantom, shift right=3, "\Swarrow \substack{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \\ \text{-structure}}"] & \\ \underset{\substack{\text{classifying space of} \\ \text{orthogonal structure}}}{B\mathrm{O}(8)} & \underset{\substack{\text{classifying space of} \\ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\text{-twists}}}{B\big(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)\big)} \arrow[l] \end{tikzcd} \quad \Rightarrow \quad \left\{ \begin{array}{l} \tfrac{1}{24} \underset{\text{Euler class}}{\chi_8} = I_8 := \tfrac{1}{48}\big(\underset{\text{Pontrjagin classes}}{p_2 - \tfrac{1}{4}(p_1)^2}\big) \\ \Big(H^2(X^8, \mathbb{Z}_2) = 0\Big) \Rightarrow \big(\underset{\substack{\text{Stiefel-Whitney} \\ \text{class}}}{w_6 = 0}\big) \Rightarrow \big(\underset{\substack{\text{integral} \\ \text{Stiefel-Whitney} \\ \text{class}}}{W_7 = 0}\big) \end{array} \right. \tag{13}$$

J-Twisted Cohomotopy and Topological *G*-Structure. For every topological coset space realization of an n-sphere, there is a canonical homotopy equivalence between the corresponding classifying spaces for a) twisted Cohomotopy and b) topological G-structure, as follows:

$$\overset{\substack{\text{coset space structure} \\ \text{on topological } n\text{-sphere}}}{S^n \underset{\text{homeo}}{\simeq} G/H} \quad \Rightarrow \quad \overset{\substack{G\text{-twisted Cohomotopy /} \\ \text{topological } H\text{-structure}}}{S^n /\!\!/ G \underset{\text{htpy}}{\simeq} BH} .$$

In particular, on spin 8-manifolds we have the following equivalences between a) J-twisted Cohomotopy cocycles (11) and b) topological G-structures:

$$\begin{array}{c} S^7 /\!\!/ \mathrm{Spin}(8) \\ \simeq \; B\,\mathrm{Spin}(7) \end{array} \quad \Rightarrow \quad \left\{ \begin{tikzcd}[row sep=large] & \overset{\substack{\text{classifying space} \\ \text{for J-twisted} \\ \text{Cohomotopy theory}}}{S^7 /\!\!/ \mathrm{Spin}(8)} \arrow[d] \\ X^8 \arrow[ur, dashed, "\overset{\substack{\text{cocycle in} \\ \text{J-twisted Cohomotopy}}}{c}"] \arrow[r, "\underset{\substack{\text{tangent} \\ \text{spin structure}}}{TX^8}"'] \arrow[ur, phantom, shift left=6, "\Downarrow \text{homotopy}"'] & B\mathrm{Spin}(8) \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd}[row sep=large] & \overset{\substack{\text{classifying space} \\ \text{for topological} \\ \mathrm{Spin}(7)\text{-structure}}}{B\mathrm{Spin}(7)} \arrow[d, "Bi"] \\ X^8 \arrow[ur, dashed, "\overset{\substack{\text{topological} \\ \mathrm{Spin}(7)\text{-structure}}}{g}"] \arrow[r, "\underset{\substack{\text{tangent} \\ \text{spin structure}}}{TX^8}"'] \arrow[ur, phantom, shift left=6, "\Downarrow \text{homotopy}"'] & B\mathrm{Spin}(8) \end{tikzcd} \right\} \tag{14}$$

and

$$\begin{array}{c} S^7 /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \\ \simeq \; B\,\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \end{array} \quad \Rightarrow \quad \left\{ \begin{tikzcd}[row sep=large] & \overset{\substack{\text{classifying space} \\ \text{for } \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\text{-twisted} \\ \text{Cohomotopy theory}}}{S^7 /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} \arrow[d] \\ X^8 \arrow[ur, dashed, "\overset{\substack{\text{cocycle in} \\ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\text{-twisted} \\ \text{Cohomotopy theory}}}{c}"] \arrow[r, "\underset{\substack{\text{tangent} \\ \text{spin structure}}}{TX^8}"'] \arrow[ur, phantom, shift left=6, "\Downarrow \text{homotopy}"'] & B\mathrm{Spin}(8) \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd}[row sep=large] & \overset{\substack{\text{classifying space} \\ \text{for topological} \\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\text{-structure}}}{B\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)} \arrow[d, "Bi"] \\ X^8 \arrow[ur, dashed, "\overset{\substack{\text{topological} \\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\text{-structure}}}{g}"] \arrow[r, "\underset{\substack{\text{tangent} \\ \text{spin structure}}}{TX^8}"'] \arrow[ur, phantom, shift left=6, "\Downarrow \text{homotopy}"'] & B\mathrm{Spin}(8) \end{tikzcd} \right\} \tag{15}$$

As the existence of a G-structure is a non-trivial topological condition, so is hence the existence of J-twisted Cohomotopy cocycles. Notice that this is a special effect of twisted non-abelian generalized Cohomology: A non-twisted generalized cohomology theory (abelian or non-abelian) always admits at least one cocycle, namely the trivial or zero-cocycle. But here for non-abelian J-twisted Cohomotopy theory on 8-manifolds, the existence of *any* cocycle is a non-trivial topological condition.

Compatibly $\mathrm{Sp}(2)$-Twisted Cohomotopy in degree 4 & 7. For focus of the discussion, we will now restrict attention to G-structure for the further quaternion-unitary subgroup

$$\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(1)\cdot\mathrm{Sp}(2)$$

of (12). In summary then, due to the $\mathrm{Sp}(2)$-equivariance of the quaternionic Hopf fibration (12) the map (9) from degree-7 to degree-4 Cohomotopy passes to $\mathrm{Sp}(2)$-twisted Cohomotopy:

$$\begin{tikzcd}[column sep=large, row sep=large] & & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, dashed, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)", "\substack{\mathrm{Sp}(2)\text{-twisted} \\ \text{quaternionic Hopf fibration}}"'] \arrow[ddl] \\ X \arrow[urr, dashed, "\overset{\substack{\text{cocycle in} \\ \text{twisted} \\ \text{7-Cohomotopy}}}{c}"] \arrow[rr, dashed, "(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_*(c)"] \arrow[dr, "\underset{\substack{\text{twist, uniformly} \\ \text{in degrees 4 \& 7}}}{\tau}"'] & & \underset{\substack{\text{induced cocycle} \\ \text{in twisted} \\ \text{4-Cohomotopy}}}{S^4 /\!\!/ \mathrm{Sp}(2)} \arrow[dl] \\ & B\mathrm{Sp}(2) & \end{tikzcd}$$

and hence (9) becomes:

$$
\begin{array}{c}\text{Sp(2)-twisted}\\ \text{7-Cohomotopy}\end{array} \quad \pi^{i_7\circ\tau}(X) \;:=\; \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\text{continuous function}"] \arrow[dr, "\text{twist }\;\tau"'] & \arrow[d, phantom, "\Downarrow\,\text{homotopy}"] & S^7/\!\!/\mathrm{Sp}(2) \arrow[dl] \\ & B\mathrm{Sp}(2) & \end{tikzcd} \right\}_{\Big/\sim_{\frac{\text{homotopy}}{B\mathrm{Sp}(2)}}} . \qquad (16)
$$

$$
\begin{tikzcd}
\pi^{i_7\circ\tau}(X) \arrow[d, "\text{reflects into}"', "\left(h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)\right)_*"] \\
\pi^{i_4\circ\tau}(X)
\end{tikzcd}
$$

$$
\begin{array}{c}\text{Sp(2)-twisted}\\ \text{4-Cohomotopy theory}\end{array} \quad \pi^{i_4\circ\tau}(X) \;:=\; \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\text{continuous function}"] \arrow[dr, "\text{twist }\;\tau"'] & \arrow[d, phantom, "\Downarrow\,\text{homotopy}"] & S^4/\!\!/\mathrm{Sp}(2) \arrow[dl] \\ & B\mathrm{Sp}(2) & \end{tikzcd} \right\}_{\Big/\sim_{\frac{\text{homotopy}}{B\mathrm{Sp}(2)}}} . \qquad (17)
$$

Triality between $\mathrm{Sp}(2)$**-structure and** $\mathrm{Spin}(5)$**-structure.** While the group (12) is abstractly isomorphic to a central product of Spin-groups, the two are *distinct* as subgroups of $\mathrm{Spin}(8)$, and not conjugate to each other. But as subgroups they are turned into each other by the ambient action of triality:

$$
\begin{tikzcd}
& \mathrm{Sp}(2) \arrow[r, hook] & \overset{\begin{array}{c}\text{central product of}\\ \text{quaternion-unitary groups}\end{array}}{\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)} \arrow[r, leftrightarrow, "\simeq"] \arrow[dd, hook] & \overset{\begin{array}{c}\text{central product of}\\ \text{Spin-groups}\end{array}}{\mathrm{Spin}(3)\cdot\mathrm{Spin}(5)} \arrow[dd, hook] & \mathrm{Spin}(5) \arrow[l, hook'] & \\
\text{M2} & & & & & \text{M5} \\
& & \mathrm{Spin}(8) \arrow[r, leftrightarrow, "\simeq", "\underset{\text{triality automorphism}}{\mathrm{tri}}"'] & \mathrm{Spin}(8) & &
\end{tikzcd}
$$

Generalized Chern characters. Since generalized cohomology theory is rich, one needs tools to break it down. The first and foremost of these is the *generalized Chern character* map. This extracts differential form data underlying a cocycle in nonabelian generalized cohomology.

The Chern character is familiar in twisted K-theory, shown in the first half of the following:

$$\begin{tikzcd}
\boxed{\text{torsionful generalized cohomology theory}} \arrow[r, "\text{approximation by generalized Chern character}"] & \boxed{L_\infty\text{-valued de Rham cohomology theory}}
\end{tikzcd}\tag{18}$$

$$\begin{tikzcd}
\boxed{\text{Chern character on ordinary integral cohomology}} & \overset{\substack{\text{ordinary}\\ \text{integral cohomology}}}{H^3(X,\mathbb{Z})} \arrow[r, "\substack{\text{extension of scalars}\\ \text{\& de Rham theorem}}"] & \overset{\substack{\text{de Rham}\\ \text{cohomology}}}{H^3_{\mathrm{dR}}(X)} \\
& \underset{\text{bundle gerbe}}{\tau} \arrow[r, mapsto] & \underset{\text{3-form}}{[H_3]}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\text{Chern character on B-field-twisted K-theory}} & \overset{\substack{\tau\text{-twisted}\\ \text{complex K-theory}}}{\mathrm{KU}^{\tau}(X)} \arrow[r, "\substack{\tau\text{-twisted}\\ \text{Chern character}\\ \mathrm{ch}^{\tau}}"] & \overset{\substack{H_3\text{-twisted}\\ \text{de Rham cohomology}}}{H^{[H_3]}_{\mathrm{dR}}(X)} \\
& \underset{\substack{\text{virtual twisted}\\ \text{vector bundle}}}{V} \arrow[r, mapsto] & \underset{\substack{\text{exponentiated}\\ \text{curvature form}}}{\big[\mathrm{tr}\big(\exp(\mathrm{F})\big)\big]}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\text{Chern character on non-abelian } \mathrm{O}(n)\text{-cohomology}} & \overset{\substack{\text{non-abelian}\\ O(n)\text{-cohomology}}}{H^1\big(X,\mathrm{O}(n)\big)} \arrow[r, "\text{characteristic forms}"] & \overset{\substack{\text{de Rham cohomology tensor}\\ \text{invariant polynomials on } \mathfrak{o}(n)}}{H_{\mathrm{dR}}\big(X\big)\otimes \mathrm{inv}\big(\mathfrak{o}(n)\big)} \\
& \underset{\text{vector bundle}}{\tau} \arrow[r, mapsto] & \tau_{\mathbb{R}} := \Big\{\underset{\substack{\text{Stiefel-Whitney}\\ \text{forms}}}{\big[W_i(\nabla_\tau)\big]}, \underset{\substack{\text{Pontrjagin}\\ \text{forms}}}{\big[p_k(\nabla_\tau)\big]}\Big\}_{i,k}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\text{Chern character on J-twisted } n\text{-Cohomotopy}} & \overset{\substack{\tau\text{-twisted}\\ \text{Cohomotopy theory}}}{\pi^{\tau}\big(X\big)} \arrow[r, "\substack{\text{cohomotopical}\\ \text{Chern character}}"] & \overset{\substack{\tau_{\mathbb{R}}\text{-twisted}\\ \text{rational Cohomotopy theory}}}{\pi^{\tau_{\mathbb{R}}}\big(X\big)_{\mathbb{R}}}
\end{tikzcd}$$

In order to see what the *cohomotopical Chern character* in the last line is, we need some general theory of generalized Chern characters. This is *rational homotopy theory*:

Rational homotopy theory. In the language of homotopy theory, generalized Chern character maps are examples of *rationalization*, whereby the homotopy type of a topological space (here: the classifying space of a generalized cohomology theory) is approximated by tensoring all its homotopy groups with the rational numbers (equivalently: the real numbers), thereby disregarding all torsion subgroups in homotopy groups and in cohomology groups.

$$\begin{tikzcd}
\boxed{\substack{\text{generalized}\\ \text{cohomology theory}}} \arrow[r, "\text{Chern character}"] \arrow[d, leftrightarrow, "\substack{\text{classifying}\\ \text{spaces}}"'] & \boxed{\substack{L_\infty\text{-valued}\\ \text{differential forms}}} \arrow[d, leftrightarrow, "\substack{\text{Sullivan model}\\ \text{construction}}"] \\
\boxed{\substack{\text{full}\\ \text{homotopy theory}}} \arrow[r, "\text{rationalization}"'] & \boxed{\substack{\text{rational}\\ \text{homotopy theory}}}
\end{tikzcd}$$

What makes rational homotopy theory amenable to computations is the existence of *Sullivan models*. These are differential graded-commutative algebras (dgc-algebras) on a finite number of generating elements (spanning the rational homotopy groups) subject to differential relations (enforcing the intended rational cohomology groups). In the supergravity literature Sullivan models are also known as "FDA"s. Here are some basic examples:

	Rational super space	**Loop super L_∞-algebra**	**Chevalley-Eilenberg super dgc-algebras** ("Sullivan models", "FDA"s)
General	X	$\mathfrak{l}X$	$\mathrm{CE}(\mathfrak{l}X)$
Super spacetime	$\mathbb{T}^{d,1\vert\mathbf{N}}$	$\mathbb{R}^{d,1\vert\mathbf{N}}$	$\mathbb{R}\big[\{\psi^\alpha\}_{\alpha=1}^{N}, \{e^a\}_{a=0}^{d}\big] \Big/ \begin{pmatrix} d\,\psi^\alpha & = 0 \\ d\,e^a & = \overline{\psi}\Gamma^a\psi \end{pmatrix}$
Eilenberg-MacLane space	$K(\mathbb{R}, p+2)$ $\simeq_{\mathbb{R}} B^{p+1}S^1$	$\mathbb{R}[p+1]$	$\mathbb{R}\big[c_{p+2}\big] \Big/ \big(d\,c_{p+2} \;=0\big)$
Odd-dimensional sphere	S^{2k+1}	$\mathfrak{l}(S^{2k+1})$	$\mathbb{R}\big[\omega_{2k+1}\big] \Big/ \big(d\,\omega_{2k+1} \;=\; 0\big)$
Even-dimensional sphere	S^{2k}	$\mathfrak{l}(S^{2k})$	$\mathbb{R}\big[\omega_{2k}, \omega_{4k-1}\big] \Big/ \begin{pmatrix} d\,\omega_{2k} & = \; 0 \\ d\,\omega_{4k-1} & = \; -\omega_{2k}\wedge\omega_{2k} \end{pmatrix}$
M2-extended super spacetime	$\widehat{\mathbb{T}^{10,1\vert\mathbf{32}}}$	$\mathfrak{m2brane}$	$\mathbb{R}\big[\{\psi^\alpha\}_{\alpha=1}^{32}, \{e^a\}_{a=0}^{10}, h_3\big] \Big/ \begin{pmatrix} d\,\psi^\alpha = \; 0 \\ d\,e^a \; = \overline{\psi}\Gamma^a\psi \\ d\,h_3 \; = \; \frac{i}{2}(\overline{\psi}\Gamma_{ab}\psi)\wedge e^a \wedge e^b \end{pmatrix}$

Under *Sullivan's theorem* the rational homotopy type of well-behaved spaces are equivalently encoded in their Sullivan model dgc-algebras:

$$\begin{tikzcd}
\mathrm{Spaces}_{/\sim_{\substack{\text{rational} \\ \text{weak homotopy} \\ \text{equivalence}}}} \arrow[r, "\substack{\text{form} \\ \text{loop Lie algebra} \\ \mathfrak{l}}", "\simeq"'] \arrow[rr, bend left, "\substack{\text{form} \\ \text{Sullivan model} \\ \mathrm{CE}(\mathfrak{l}-)}", "\simeq"'] & L_\infty\mathrm{Algebras}_{/\sim_{\substack{\text{quasi-} \\ \text{isomorphism}}}} \arrow[r, "\substack{\text{form} \\ \text{Chevalley-Eilenberg algebra} \\ \mathrm{CE}}", "\simeq"'] & \mathrm{dgcAlgebras}^{\mathrm{op}}_{/\sim_{\substack{\text{quasi-} \\ \text{isomorphism}}}} \\
& & \text{“FDA”s}_{\substack{\text{supergravity} \\ \text{jargon}}}
\end{tikzcd}$$

for spaces & algebras
nilpotent & of finite types

When applying the rational approximation to twisted generalized cohomology theory, the order matters: There are in general more *rational twists* $X \xrightarrow{\tau} B\mathrm{Aut}(A_{\mathbb{R}})$ for *twisted rational cohomology* than there are rationalizations $\tau_{\mathbb{R}}$ of full twists $X \xrightarrow{\tau} B\mathrm{Aut}(A)$ for *rational twisted cohomology*. We consider first the general rational twists:

Rationally twisted rational Cohomotopy. We find that the *rationally twisted rational Cohomotopy* sets in degrees 4 and 7 are equivalently characterized by cohomotopical Chern character forms as follows:

	rational twist	rational twisted Cohomotopy		cohomotopical Chern characters	
7-Cohomotopy	$X \xrightarrow{\tau^7} B\mathrm{Aut}\big(S^7_{\mathbb{R}}\big)$	$\pi^{(\tau^7)}(X)$	$\simeq$	$\left\{ \overset{\text{7-form}}{\widetilde{G}_7} \;\middle\vert\; d\,\widetilde{G}_7 = \overset{\text{characteristic form of twist } \tau^7}{K_8} \right\}_{/\sim}$	(19)
4-Cohomotopy	$X \xrightarrow{\tau^4} B\mathrm{Aut}\big(S^4_{\mathbb{R}}\big)$	$\pi^{(\tau^4)}(X)$	$\simeq$	$\left\{ \overset{\text{4-form \& 7-form}}{(G_4, G_7)} \;\middle\vert\; \begin{array}{l} d\,G_4 = \;\; 0 \\ d\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4 + \underset{\text{characteristic form of twist } \tau^4}{L_8} \end{array} \right\}_{/\sim}$	

Here *all* real 8-classes $[K_8], [L_8] \in H^8(X, \mathbb{R})$ may appear, for *some* rational twist $\tau^{4/7}$. But constraints on these characteristic forms appear when we consider more than rational twisted structure:

Compatibly rationally twisted rational Cohomotopy. We may ask that the rational twists $\tau^{4,7}$ in (19) are related analogously to how the twisted parametrized Hopf fibration (12) relates the full (non-rational) twists, through (16). We find that this happens precisely when the difference of the characteristic 8-classes in (19) is a complete square

$$L_8 \;=\; K_8 + \big(\tfrac{1}{4}P_4\big) \wedge \big(\tfrac{1}{4}P_4\big)$$

and in that case the situation of (19) becomes the following:

	compatible rational twists	rational compatibly twisted Cohomotopy		cohomotopical Chern characters	
7-Cohomotopy	$X \xrightarrow{\tau^7} B\mathrm{Aut}\big(S^7_{\mathbb{R}}\big)$	$\pi^{(\tau^7)}(X)$	$\simeq$	$\left\{ \widetilde{G}_7 \;\middle\vert\; d\,\widetilde{G}_7 = \overset{\text{characteristic form of twist } \tau^7}{K_8} \right\}_{/\sim}$	
	$\boxed{\begin{array}{l} \text{shifted 4-form} \\ \widetilde{G}_4 := G_4 + \tfrac{1}{4}P_4 \\ \widetilde{G}_7 := G_7 + \tfrac{1}{2}H_3 \wedge \widetilde{G}_4 \\ \text{shifted 7-form} \end{array}}$		$\simeq$	$\left\{ \left(\begin{array}{l} H_3, \\ \widetilde{G}_4, G_7 \end{array} \right) \;\middle\vert\; \begin{array}{l} d\,H_3 = \;\; \widetilde{G}_4 - \tfrac{1}{2}P_4 \\ d\,\widetilde{G}_4 = 0 \\ d\,G_7 = -\tfrac{1}{2} dH_3 \wedge \widetilde{G}_4 + K_8 \end{array} \right\}_{/\sim}$	(20)
4-Cohomotopy	$X \xrightarrow{\tau^4} B\mathrm{Aut}\big(S^4_{\mathbb{R}}\big)$	$\pi^{(\tau^4)}(X)$	$\simeq$	$\left\{ (\widetilde{G}_4, G_7) \;\middle\vert\; \begin{array}{l} d\,\widetilde{G}_4 = \;\; 0 \\ d\,G_7 = -\tfrac{1}{2}(\widetilde{G}_4 - \tfrac{1}{2}P_4) \wedge \widetilde{G}_4 + K_8 \end{array} \right\}_{/\sim}$	

Here still *all* real 8-classes and 4-classes $[K_8] \in H^8(X, \mathbb{R})$, $[P_4] \in H^4(X, \mathbb{R})$ may appear, for *some* pair of compatible rational twists.

Next we find that these real classes are fixed as we consider full (not just rational) Sp(2)-twists, compatible by the full (not just rational) Sp(2)-twisted quaternionic Hopf fibration (12).

J-Twisted 4-Cohomotopy of $\mathrm{Sp}(2)$**-manifolds.** Consider a simply-connected Riemannian spin-manifold $\mathbb{R}^{2,1}\times X^8$ with affine connection ∇ and equipped with:

1. an $\mathrm{Sp}(2)$-structure τ (13);
2. a cocycle c in τ-twisted 4-Cohomotopy (17);

hence equipped with a homotopy-commutative diagram of continuous maps as follows:

$$\left[\begin{tikzcd} \text{spacetime} & \text{cocycle in J-twisted Cohomotopy} & \text{classifying space of Sp(2)-twisted Cohomotopy} \\ \mathbb{R}^{2,1}\times X^8 \arrow[rr, dashed, "c"] \arrow[d, "\underset{\text{bundle}}{\overset{\text{tangent}}{TX^8}}"'] \arrow[dr, "\tau"] \arrow[dr, phantom, bend right, "\Leftarrow\ \text{Sp(2)-structure}"'] & \arrow[d, phantom, "\Downarrow\ \text{homotopy}"] & S^4/\!\!/\mathrm{Sp}(2) \arrow[dl, "\substack{\text{twisting through} \\ \mathrm{Sp}(2)\underset{\text{abstr}}{\simeq}\mathrm{Spin}(5)\longrightarrow\mathrm{Aut}(S^4)}"] \\ B\mathrm{Spin}(8) & B\mathrm{Sp}(2) \arrow[l] & \\ \text{classifying space of spin-structure} & \text{classifying space of Sp(2)-twists} & \end{tikzcd}\right]_{\substack{\text{homotopy class} \\ \text{over } B\mathrm{Sp}(2)}} \in \underset{\substack{\text{twisted 4-Cohomotopy} \\ \text{of spacetime } X^8}}{\pi^{i_4\circ\tau}\big(X^8\big)}$$

Then the **4-Cohomotopical Chern character** (18) of $[c]$, hence the differential flux forms underlying $[c]$ by (19)

$$\begin{tikzcd} \overset{\text{twisted 4-Cohomotopy}}{\pi^{\tau}\big(X^8\big)} \arrow[r, "\overset{\text{rationalization}}{L_{\mathbb{R}}}", "\text{cohomotopical Chern character}"'] & \overset{\substack{\text{rational} \\ \text{twisted 4-Cohomotopy}}}{\pi^{\tau}\big(X^8\big)_{\mathbb{R}}} & \blacksquare \arrow[l, "\text{relations}"'] \arrow[r, hook, "\text{conditions}"] & \overset{\substack{\text{plain} \\ \text{differential forms}}}{\Omega^4\big(X^8\big)\times\Omega^7\big(X^8\big)} \\ \underset{\substack{\text{class in} \\ \text{twisted Cohomotopy}}}{[c]} \arrow[r, mapsto] & \underset{\substack{\text{Chern character in} \\ \text{twisted Cohomotopy}}}{\big[(G_4,G_7)\big]} & & \end{tikzcd}$$

satisfy, first of all, this condition:
The **shifted 4-flux** form

$$\widetilde{G}_4 \;:=\; \underset{\substack{\text{naive} \\ \text{4-flux}}}{G_4} \;+\; \underset{\substack{\text{shift by first} \\ \text{fractional} \\ \text{Pontrjagin form}}}{\tfrac{1}{4}p_1(\nabla)} \;\in\; \underset{\substack{\text{differential} \\ \text{4-forms}}}{\Omega^4\big(X^8\big)} \tag{21}$$

is integral

$$\underset{\substack{\text{shifted} \\ \text{4-flux}}}{[\widetilde{G}_4]} \;\in\; \underset{\text{integral cohomology}}{H^4\big(X^8,\mathbb{Z}\big)} \xrightarrow{\text{extension of scalars}} \underset{\text{real cohomology}}{H^4\big(X^8,\mathbb{R}\big)} \;\simeq\; \underset{\text{de Rham cohomology}}{H_{\mathrm{dR}}\big(X^8\big)} \tag{22}$$

and as such its **Steenrod square vanishes**:

$$\overset{\substack{\text{Steenrod square of} \\ \text{mod-2 reduction of} \\ \text{integral shifted 4-flux}}}{\mathrm{Sq}^2\big([\widetilde{G}_2]\big)=0} \qquad \text{hence also} \qquad \overset{\substack{\text{Steenrod cube of} \\ \text{mod-2 reduction of} \\ \text{integral shifted 4-flux}}}{\mathrm{Sq}^3\big([\widetilde{G}_2]\big)=0\,.} \tag{23}$$

To see the next condition, consider the homotopy pullback of the 4-Cohomotopy cocycle c along the $\mathrm{Sp}(2)$-twisted quaternionic Hopf fibration $h_{\mathbb{H}}$ to a cocycle in twisted 7-Cohomotopy on the induced 3-spherical fibration $\widehat{H}^8$ over spacetime:

$$\left[\begin{tikzcd}[ampersand replacement=\&]
\& \text{classifying space of compatible 3-flux} \& \text{induced cocycle in twisted 7-Cohomotopy} \& \text{classifying space of Sp(2)-twisted 7-Cohomotopy} \& \\
\text{induced 3-spherical fibration} \& \widehat{X}^8 \arrow[rr, dashed, "\widehat{c}"] \arrow[d, "c^*h =: p"'] \& \text{cocycle in J-twisted 4-Cohomotopy} \& S^7/\!\!/\mathrm{Sp}(2) \arrow[d, "h := h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)"] \& \text{Sp(2)-parametrized quaternionic Hopf fibration} \\
\text{spacetime} \& X^8 \arrow[rr, dashed, "c"] \arrow[d, "TX^8"'] \arrow[dr, "\tau"] \arrow[dr, phantom, bend left, "\Swarrow"] \arrow[d, phantom, shift left=6, "\Swarrow \text{Sp(2)-structure}"] \& \& S^4/\!\!/\mathrm{Sp}(2) \arrow[dl] \& \text{classifying space of Sp(2)-twisted 4-Cohomotopy} \\
\text{tangent bundle} \& B\mathrm{Spin}(8) \& B\mathrm{Sp}(2) \arrow[l] \& \text{twisting through } \mathrm{Sp}(2) \underset{\text{abstr}}{\simeq} \mathrm{Spin}(5) \longrightarrow \mathrm{Aut}(S^4) \& \\
\& \text{classifying space of spin-structure} \& \text{classifying space of Sp(2)-twists} \& \&
\end{tikzcd}\right]_{\substack{\text{homotopy class} \\ \text{over } B\mathrm{Sp}(2)}} \;\in\; \underset{\substack{\text{twisted 7-Cohomotopy} \\ \text{of } \widehat{X}^8}}{\pi^{\tau\circ p}\big(\widehat{X}^8\big)} \tag{24}$$

Then:
The **pullback 3-spherical fibration** over spacetime

$$\widehat{X}^8 \;:=\; c^*\big(S^7/\!\!/\mathrm{Sp}(2)\big)$$

carries a universal 3-flux H_3^{univ} which trivializes the 4-flux relative to its background value

$$d\,H_3^{\mathrm{univ}} \;=\; p^*\widetilde{G}_4 - \tfrac{1}{4}p_1(\nabla)\,. \tag{25}$$

Moreover, the **7-Cohomotopical Chern character** of $[\hat{c}]$, hence the differential flux forms underlying $[\hat{c}]$ by (20)

$$\begin{tikzcd}
\text{twisted 7-Cohomotopy} \& \& \text{rational twisted 7-Cohomotopy} \& \& \text{plain differential forms} \\
\pi^{p\circ\tau}\big(\widehat{X}^8\big) \arrow[rr, "\overset{\text{rationalization}}{L_{\mathbb{R}}}", "\text{cohomotopical Chern character}"'] \& \& \pi^{p\circ\tau}\big(\widehat{X}^8\big)_{\mathbb{R}} \& \blacksquare \arrow[l, "\text{relations}"'] \arrow[r, hook, "\text{conditions}"] \& \Omega^7\big(\widehat{X}^8\big) \\
\big[\widehat{c}\big] \arrow[rr, mapsto] \& \& \big[\widetilde{G}_7\big] \& \& \\
\text{class in twisted Cohomotopy} \& \& \text{Chern character in twisted Cohomotopy} \& \&
\end{tikzcd}$$

satisfy this condition:
The **shifted 7-flux** form

$$\underset{\text{naive 7-flux}}{\widetilde{G}_7 \;=\; p^*G_7} \;+\; \underbrace{\tfrac{1}{2}\overset{\text{3-flux}}{H_3^{\mathrm{univ}}} \wedge \overset{\substack{\text{shifted} \\ \text{4-flux}}}{p^*\widetilde{G}_4}}_{\substack{\text{shift by} \\ \text{Hopf-Whitehead term}}} \tag{26}$$

is a trivialization of the Euler 8-form

$$d\,\widetilde{G}_7 \;=\; -\tfrac{1}{2}\chi_8(\nabla) \tag{27}$$

and **is half-integral** on every 7-sphere $S^7 \xrightarrow{i} \widehat{X}^8$:

$$2\int_{S^7} i^*\tilde{G}_7 \;\in\; \mathbb{Z}\,. \tag{28}$$

Finally, consider the case that:

1) Our manifold is the complement in a closed 8-manifold of a finite set of disjoint open balls, i.e. of a tubular neighbourhood $\mathcal{N}$ around a finite set $\{x_1, x_2, \cdots\}$ of points:

$$X^8 \;=\; \overset{\text{closed manifold}}{X^8_{\mathrm{clsd}}} \setminus \underset{\text{around points in } X^8_{\mathrm{clsd}}}{\overset{\text{tubular neighbourhood}}{\mathcal{N}_{\{x_1,x_2,\cdots\}}}} \quad\Rightarrow\quad \overset{\text{boundary of } X^8}{\partial X^8} \;\simeq\; \overset{\text{sphere around } x_i}{\underset{\{x_1,x_2,\cdots\}}{\sqcup} S^7} \tag{29}$$

This implies that X^8 is a manifold with boundary a disjoint union of 7-spheres.

2) Such that the corresponding extended spacetime $\widehat{X}^8$ (24) admits a global section; hence, equivalently, such that the given cocycle in twisted 4-Cohomotopy lifts through the quaternionic Hopf fibration to a cocycle in twisted 7-Cohomotopy:

$$\begin{tikzcd}
 & \overset{\text{classifying space of compatible 3-flux}}{\widehat{X}^8} \arrow[d, "p:=c^*(h)"', "\text{induced 3-spherical fibration}"] \\
X^8 \arrow[ur, dashed, "i", "\text{global section of 3-spherical fibration}"'] \arrow[r, equals] & X^8
\end{tikzcd}
\quad\Leftrightarrow\quad
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h:=h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)"', "\text{Sp(2)-parametrized quaternionic Hopf fibration}"] \\
X^8 \arrow[ur, dashed, "\hat{c}", "\text{lift to cocycle in J-twisted 7-Cohomotopy}"'{name=U}] \arrow[r, "c"', "\text{cocycle in J-twisted 4-Cohomotopy}"{name=D}] & S^4 /\!\!/ \mathrm{Sp}(2)
\arrow[Rightarrow, from=U, to=D, "\text{homotopy}"]
\end{tikzcd} \tag{30}$$

Here the choice of points in (29) matters only in so far as a sufficient number of points has to be removed for a lifted cocycle $\hat{c}$ (30) to exist at all.

We observe that:

1) Since the 7-sphere is parallelizable, upon restriction of $\hat{c}$ (30) to the boundary $\partial X^8 \xrightarrow{i} X^8$ (29) the twist vanishes, and we are left with a pair of compatible cocycles in plain Cohomotopy theory as in (9):

$$\begin{tikzcd}
 & S^7 \arrow[d, "h_{\mathbb{H}}", "\text{plain quaternionic Hopf fibration}"'] \\
\underset{\text{boundary 7-spheres}}{\underset{\{x_1,x_2,\cdots\}}{\sqcup} S^7 \simeq \partial X^8} \arrow[ur, dashed, "\hat{c}_{|\partial X^8}", "\text{boundary restriction of twisted 7-Chomotopy cocycle}"'{name=U}] \arrow[r, "(h_{\mathbb{H}})_* \hat{c}_{|\partial X^8}"', "\text{underlying boundary 4-Cohomotopy cocycle}"{name=D}] & S^4
\arrow[Rightarrow, from=U, to=D]
\end{tikzcd}$$

2) By (8), cocycles in stable 7-Cohomotopy have no side-effect in stable 4-Cohomotopy precisely if they are multiples of 24:

$$\text{for } [c_1], [c_2] \;\in\; \overset{\text{7-Cohomotopy}}{\pi^7\big(\partial X^8\big)} \quad \text{we have} \quad \left\{ \begin{array}{ccc} & & \\ \end{array} \right. \Leftrightarrow \begin{array}{rcl} (h_{\mathbb{H}})_*[c_1] & = & (h_{\mathbb{H}})_*[c_2] \;\in\; \overset{\text{stable 4-Cohomotopy}}{\mathbb{S}^4\big(\partial X^8\big)} \\ [c_1] & =_{\mathrm{mod}\; 24} & [c_2] \;\in\; \underset{\text{stable 7-Cohomotopy}}{\mathbb{S}^7\big(\partial X^8\big)} \end{array}$$

This means that the **unit charge** of a lift $\hat{c}$ in (30), as seen by stable Cohomotopy, is 24. In view of (28) this says that the **cohomotopically normalized 7-flux** of X^8 is

$$N_{\mathrm{M2}} \;:=\; \tfrac{1}{12} \int_{X^8} i^* d\widetilde{G}^7 \;=\; \tfrac{1}{12} \int_{\partial X^8} i^* \widetilde{G}^7 \,. \tag{31}$$

Our final result is that
this **equals the I_8-number** (13) of the manifold:

$$N_{\mathrm{M2}} \;=\; -I_8[X^8] \,. \tag{32}$$

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