Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds

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To Mike Duff on the occasion of his 70th birthday

Abstract

We consider the hypothesis that the C-field 4-flux and 7-flux forms in M-theory are in the image under the non-abelian Chern character map from the non-abelian generalized cohomology theory called J-twisted Cohomotopy theory. We prove for M2-brane backgrounds in M-theory on 8-manifolds that such charge quantization of the C-field in Cohomotopy theory implies a list of expected anomaly cancellation conditions, including: shifted C-field flux quantization and C-field tadpole cancellation, but also the DMW anomaly cancellation and the C-field's integral equation of motion.

Contents

1 Introduction and survey

We consider the following hypothesis, which we make precise as Def. 3.5 below, based on details developed in §2, see §4 for background, motivation and outlook:

Hypothesis H: *The C-field 4-flux and 7-flux forms in M-theory are subject to* charge quantization in J-twisted Cohomotopy cohomology theory *in that they are in the image of the non-abelian Chern character map from J-twisted Cohomotopy theory.*

In support of Hypothesis H, we prove in §3 that it implies the following phenomena, expected for M2-brane backgrounds in M-theory on 8-manifolds (recalled in Remark 3.1 below):

	Cohomotopy theory	Expression		**M-theory**
§3.2	Compatible twisting on 4- & 7-Cohomotopy theory	$W_7(TX) = 0$	(13)	DMW anomaly cancellation condition [DMW03a][DMW03b, 6]
		$\frac{1}{24}\chi_8(TX) \;=\; I_8(TX) := \frac{1}{48}\big(p_2(TX) - \frac{1}{4}(p_1(TX))^2\big)$	(13)	one-loop anomaly polynomial [DLM95][VW95]
§2.4	Any cocycle in J-twisted 7-Cohomotopy	$\mathrm{Spin}(7)$-structure g	(14)	$\geq 1/8$ BPS M2-brane background [IP88][IPW88][Ts06]
§2.3	Any cocycle in compatibly twisted 4&7-Cohomotopy	$\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$-structure τ	(15)	$4/8$ BPS M2-brane background [MF10, 7.3]
§3.3	Chern character of rationally twisted 4-Cohomotopy	$d\,G_4 = 0$ $d\,G_7 = -\frac{1}{2}G_4 \wedge G_4 + L_8$	(19)	C-field Bianchi identity with generic higher curvature correction [ST16]
§3.3	Chern character of compatibly rationally twisted 4&7-Cohomotopy	$d\,\widetilde{G}_4 = 0$ $d\,G_7 = -\frac{1}{2}\big(\widetilde{G}_4 - \frac{1}{4}P_4\big) \wedge \widetilde{G}_4 + K_8$	(20)	Shifted C-field Bianchi identity with generic higher curvature correction [Ts04]
§3.4	Chern character 4-form of Sp(2)-twisted 4-Cohomotopy	$\widetilde{G}_4 \;=\; G_4 + \frac{1}{4}p_1(\nabla)$	(21)	C-field shift [Wi96a][Wi96b][Ts04]
		$[\widetilde{G}_4] \;\in\; H^4(X,\mathbb{Z})$	(22)	Shifted C-field flux quantization [Wi96a][Wi96b][DFM03][HS05]
§3.5		$(G_4)_0 \;=\; \frac{1}{2}p_1(\nabla)$	(24)	Background charge [Fr09, p. 11][Fr00]
§3.6		$\mathrm{Sq}^3\big([\widetilde{G}_4]\big) \;=\; 0$	(23)	Integral equation of motion [DMW03a][DMW03b, 5]
§3.7	Chern character 7-form of compatibly Sp(2)-twisted 4&7-Cohomotopy	$\widetilde{G}_7 \;=\; G_7 + \frac{1}{2}H_3 \wedge \widetilde{G}_4$	(27)	Page charge [Pa83, (8)][DS91, (43)][Mo05]
		$d\,\widetilde{G}_7 \;=\; -\frac{1}{2}\chi_8(\nabla)$	(28)	
		$2\int_{S^7} i^\ast \widetilde{G}_7 \;\in\; \mathbb{Z}$	(29)	Level quantization of Hopf-WZ term [In00]
§3.8	Integrated Chern character of compatibly Sp(2)-twisted 4&7-Cohomotopy	$N_{M2} \;=\; -I_8$	(33)	C-field tadpole cancellation [SVW96]

Table 1 –*Implications of C-field charge quantization in J-twisted Cohomotopy.*

Organization of the paper.

- In §1 we survey our constructions and results.
- In §2 we introduce twisted Cohomotopy theory, and prove some fundamental facts about it.
- In §3 we use these results to explains and prove the statements in *Table 1*.
- In §4 we comment on background and implications.

Generalized abelian cohomology. Before we start, we briefly say a word on "generalized" cohomology theories, recalling some basics, but in a broader perspective: The *ordinary cohomology groups* $X \mapsto H^\bullet(X,\mathbb{Z})$ famously satisfy a list of nice properties, called the *Eilenberg-Steenrod axioms*. Dropping just one of these axioms (the *dimension axiom*) yields a larger class of possible abelian group assignments $X \mapsto E^\bullet(X)$, often called *generalized cohomology theories*. One example are the *complex topological K-theory groups* $X \mapsto \mathrm{KU}^\bullet(X)$.

By the *Brown representability theorem*, every generalized cohomology theory in this sense has a *classifying space* E_n for each degree, such that the n-th cohomology group is equivalently the set of homotopy classes of maps into this space: [1]

$$\underset{\text{cohomology theory}}{\overset{\text{Generalized abelian}}{}} \quad E^n(X) \;\overset{\substack{\text{Brown's}\\\text{representability}\\\text{theorem}}}{\simeq}\; \left\{ X \xrightarrow[\text{= cocycle in } E\text{-theory}]{\text{continuous function}} E_n \right\}_{\big/\sim_{\text{homotopy}}} . \tag{1}$$

For example, ordinary cohomology theory has as classifying spaces the *Eilenberg-MacLane spaces* $K(\mathbb{Z},n)$, while complex topological K-theory in degree 1 is classified by the space underlying the stable unitary group.

For generalized cohomology theories in this sense of Eilenberg-Steenrod, Brown's representability theorem translates the *suspension axiom* into the statement that the classifying spaces E_n in (1) are loop spaces of each other, $E_n \simeq \Omega E_{n+1}$, and thus organize into a sequence of classifying spaces $(E_n)_{n\in\mathbb{N}}$ called a *spectrum*. The fact that each space in a spectrum is thereby an infinite loop space makes it behave like a homotopical *abelian* group (since higher-dimensional loops may be homotoped and hence commuted around each other, by the Eckmann-Hilton argument).

Generalized non-abelian cohomology. We highlight the fact that not all cohomology theories are abelian. The classical example, for G any non-abelian Lie group, is the *first non-abelian cohomology* $X \mapsto H^1\big(X,G\big)$, defined on any manifold X as the first Čech cohomology of X with coefficients in the sheaf of G-valued functions. Nevertheless, this non-abelian cohomology theory also has a classifying space, called BG, and in terms of this it is given exactly as the abelian generalized cohomology theories in (1):

$$\underset{\text{cohomology theory}}{\overset{\text{Degree-1 non-abelian}}{}} \quad H^1(X,G) \;\overset{\substack{\text{principal bundle}\\\text{theory}}}{\simeq}\; \left\{ X \xrightarrow[\text{= cocycle}]{\text{continuous function}} BG \right\}_{\big/\sim_{\text{homotopy}}} . \tag{2}$$

Hence the joint generalization of generalized abelian cohomology theory (1) and non-abelian 1-cohomology theories (2) are assignments of homotopy classes of maps into *any* coefficient space A

$$\underset{\text{cohomology theory}}{\overset{\text{Non-abelian generalized}}{}} \quad H(X,A) \;:=\; \left\{ X \xrightarrow[\text{= cocycle}]{\text{continuous function}} A \right\}_{\big/\sim_{\text{homotopy}}} . \tag{3}$$

All this may naturally be further generalized from topological spaces to higher stacks. In the literature of this broader context the perspective of non-abelian generalized cohomology is more familiar. But it applies to the topological situation as the easiest special case, and this is the case with which we are concerned for the present purpose.

Higher principal bundles. This way, the classical statement (2) of principal bundle theory finds the following elegant homotopy-theoretic generalization. For every *connected* space A, its based loop space $G := \Omega A$ is a higher

[1] Here and in the following, a dashed arrow indicates a map representing a cocycle that can be freely choosen, as opposed to solid arrows indicating fixed structure maps.

homotopical group under concatenation of loops (an "∞-group"). Moreover, A itself is equivalently the classifying space for that higher group:

$$\underset{\text{\textit{Every} connected space...}}{A} \;\simeq\; \underset{\text{...is the classifying space...}}{B}\, \underset{\text{...of its loop group.}}{\overbrace{\Omega A}^{G}} \tag{4}$$

in that non-abelian G-cohomology in degree 1 classifies higher homotopical G-principal bundles:

$$\begin{array}{ccc}
H(X,BG) = \underset{\text{non-abelian } G\text{-cohomology}}{H^1(X,G)} & \xrightarrow{\simeq} & \underset{\text{higher homotopical } G\text{-principal bundles}}{G\mathrm{Bundles}(X)_{/\sim}} \\
\left[X \xrightarrow[\text{cocycle}]{c} BG\right] & \longmapsto & \left[\begin{array}{ccc}
\underset{G\text{-principal bundle classified by } c}{P} & \longrightarrow & \underset{\text{universal } G\text{-principal bundle}}{G /\!\!/ G} \\
{\scriptstyle c^*(p_{BG})}\downarrow & \text{homotopy pullback} & \downarrow{\scriptstyle p_{BG}} \\
X & \xrightarrow[\text{classifying map for } P]{c} & BG
\end{array}\right]
\end{array} \tag{5}$$

Cohomotopy cohomology theory. The primordial example of a non-abelian generalized cohomology theory (3) is *Cohomotopy cohomology theory*, denoted $\pi^\bullet$. By definition, its classifying spaces are simply the n-spheres S^n:

$$\underset{\text{Cohomotopy cohomology theory}}{\pi^n(X)} := \left\{ X \xrightarrow[\text{= cocycle}]{\text{continuous function}} S^n \right\}_{/\sim_{\text{homotopy}}}. \tag{6}$$

Since the $(n \geq 1)$-spheres are connected, the equivalence (4) applies and says that Cohomotopy theory is equivalently non-abelian 1-cohomology for the loop groups of spheres $G := \Omega S^n$:

$$\underset{\text{Cohomotopy theory}}{\pi^n(X)} \;\simeq\; \underset{\text{non-abelian 1-cohomology for sphere loop group}}{H^1(X, \Omega S^n)}.$$

Evaluated on spaces which are themselves spheres, Cohomotopy cohomology theory gives the (unstable!) *homotopy groups of spheres*, the "vanishing point" of algebraic topology:

$$\underset{n\text{-cohomotopy groups of } k\text{-sphere}}{\pi^n(S^k)} \;\simeq\; \left\{ S^k \dashrightarrow S^n \right\}_{/\sim} \simeq \underset{k\text{-homotopy groups of } n\text{-sphere}}{\pi_k(S^n)}$$

A whole range of classical theorems in differential topology all revolve around characterizations of Cohomotopy sets, even if this is not often fully brought out in the terminology.

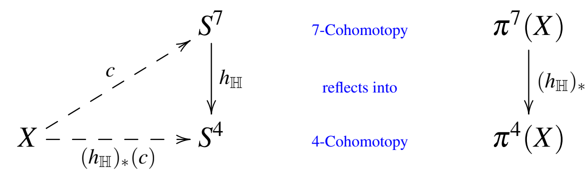
The quaternionic Hopf fibration. A notable example, for the following purpose, of a class in the Cohomotopy group of spheres, is given by the *quaternionic Hopf fibration*

$$S^7 \simeq \underset{\text{unit sphere in quaternionic 2-space}}{S(\mathbb{H}^2)} \xrightarrow{(q_1,q_2) \mapsto [q_1 : q_2]} \underset{\text{quaternionic projective 1-space}}{\mathbb{H}P^1} \simeq S^4, \qquad \overset{\text{quaternionic Hopf fibration}}{h_{\mathbb{H}}} : S^7 \longrightarrow S^4 \tag{7}$$

which represents a generator of the non-torsion subgroup in the 4-Cohomotopy of the 7-sphere, as shown on the left here:

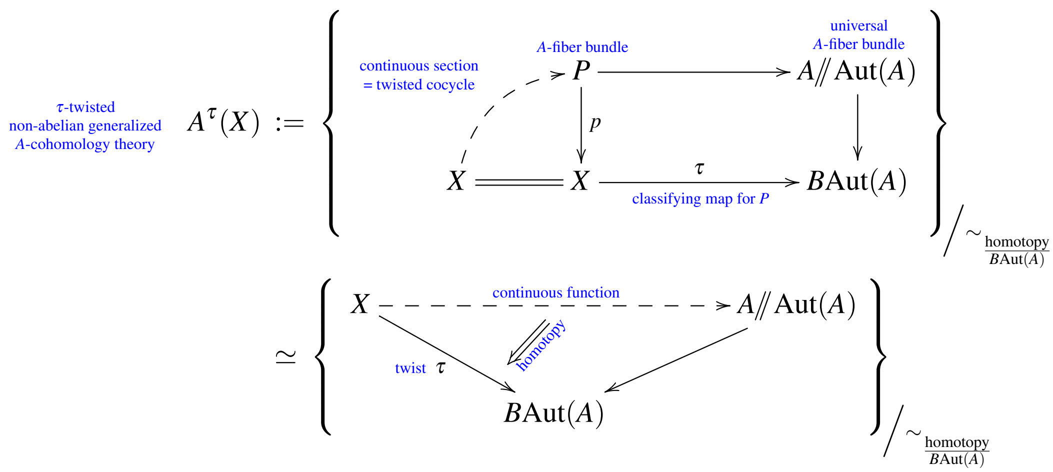
$$\begin{array}{ccccccc}
\text{quaternionic Hopf fibration} & [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] \in & \underset{\text{non-abelian/unstable Cohomotopy group}}{\pi^4(S^7)} & \xrightarrow[\text{stabilization}]{\Sigma^\infty} & \underset{\text{abelian/stable Cohomotopy group}}{\mathbb{S}^4(S^7)} & \ni \Sigma^\infty[S^7 \xrightarrow{h_{\mathbb{H}}} S^4] & \text{stabilized quaternionic Hopf fibration} \\
& \downarrow & \wr\!\wr & & \wr\!\wr & \downarrow & \\
\text{non-torsion generator} & (1,0) \in & \mathbb{Z} \times \mathbb{Z}_{12} & \xrightarrow{(n,a) \mapsto (n \bmod 24)} & \mathbb{Z}_{24} & \ni 1 & \text{torsion generator}
\end{array} \tag{8}$$

Shown on the right is the abelian approximation to non-abelian Cohomotopy cohomology theory, called *stable Cohomotopy theory* and represented, via (1), by the *sphere spectrum* $\mathbb{S}$, whose component spaces are the infinite-loop space completions of the n-spheres: $\mathbb{S}_n \simeq \Omega^\infty \Sigma^\infty S^n$. Crucially, in this approximation, the quaternionic Hopf fibration becomes a torsion generator: non-abelian 4-Cohomotopy witnesses integer cohomology groups not only in degree 4, but also in degree 7; but when seen in the abelian/stable approximation this "extra degree" fades away and leaves only a torsion shadow behind. From the perspective, composition with the quaternionic Hopf fibration can be viewed as a transformation that translates classes in degree-7 Cohomotopy to classes in degree-4 Cohomotopy:



(9)

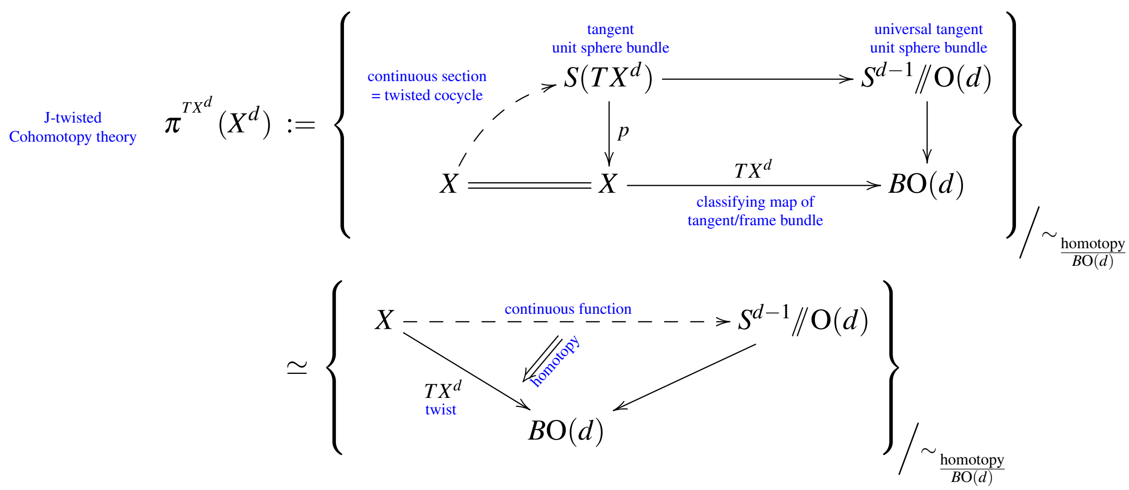
Twisted non-abelian generalized cohomology. Regarding generalized cohomology theory as homotopy theory of classifying spaces (3) makes transparent the concept of *twistings* in cohomology theory: Instead of mapping into a fixed classifying spaces, a *twisted cocycle* maps into a varying classifying space that may twist and turn as one moves in the domain space. In other words, a *twisting* τ of A-cohomology theory on some X is a bundle over X with typical fiber A, and a τ-twisted cocycle is a *section* of that bundle:



(10)

Here the equivalent formulation shown in the second line follows because A-fiber bundles are themselves classified by nonabelian $\mathrm{Aut}(A)$-cohomology, as shown on the right of the first line (due to (5)).

Twisted Cohomotopy theory. For the example (6) of Cohomotopy cohomology theory in degree $d-1$ there is a canonical twisting on Riemannian d-manifolds, given by the unit sphere bundle in the orthogonal tangent bundle:

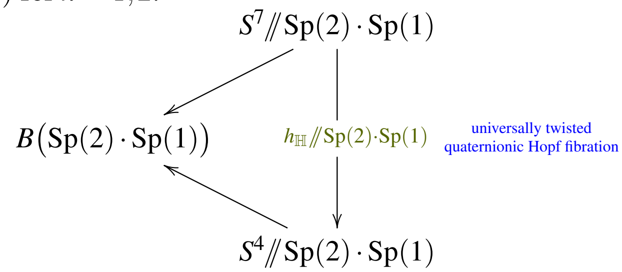


(11)

Since the canonical morphism $\mathrm{O}(d) \longrightarrow \mathrm{Aut}(S^{d-1})$ is known as the *J-homomorphism*, we may call this *J-twisted Cohomotopy theory*, for short.

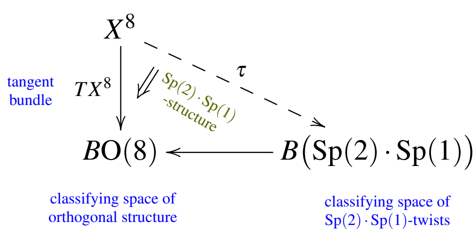
Compatibly J-twisted Cohomotopy in degrees 4 & 7. In view of (9) it is natural to ask for the maximal subgroup $G \subset \mathrm{O}(8)$ for which the quaternionic Hopf fibration is equivariant, so that its homotopy quotient $h_{\mathbb{H}} /\!\!/ G$ exists and serves as a map of *G-twisted* Cohomotopy theories (11) from degree 7 and 4. This subgroup turns out to be the central product of the quaternion unitary groups $\mathrm{Sp}(n)$ for $n = 1, 2$:

$\underbrace{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)}_{\text{central product of quaternion-unitary groups}} \subset \mathrm{O}(8)$ is maximal subgroup s.t.



(12)

In other words, J-twisted Cohomotopy (11) exists compatibly in degrees 4 & 7 precisely on those 8-manifolds which carry topological $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-structure, i.e., whose structure group of the tangent bundle is equipped with a reduction along $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \hookrightarrow \mathrm{O}(8)$. This reduction is equivalent to a factorization of the classifying map as shown on the left below, with some cohomological consequences shown on the right:



$$\Longrightarrow \left\{ \begin{array}{l} \underbrace{\tfrac{1}{24}\chi_8}_{\text{Euler class}} = I_8 := \tfrac{1}{48}\big(\underbrace{p_2 - \tfrac{1}{4}(p_1)^2}_{\text{Pontrjagin classes}}\big) \\ \big(H^2(X^8,\mathbb{Z}_2) = 0\big) \;\Rightarrow\; (\underbrace{w_6}_{\text{Stiefel-Whitney class}} = 0) \;\Rightarrow\; (\underbrace{W_7}_{\text{integral Stiefel-Whitney class}} = 0) \end{array} \right. \qquad (13)$$

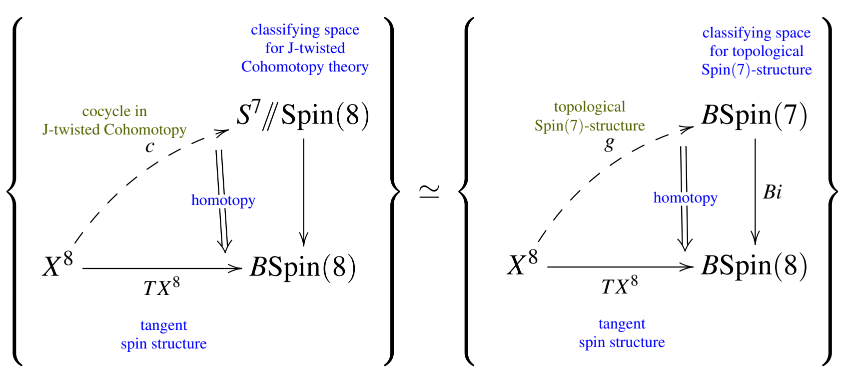
J-Twisted Cohomotopy and Topological G-Structure. For every topological coset space realization G/H of an n-sphere, there is a canonical homotopy equivalence between the classifying spaces for G-twisted Cohomotopy and for topological H-structure (i.e., reduction of the structure group to H), as follows:

$$\overbrace{S^n \underset{\text{homeo}}{\simeq} G/H}^{\text{coset space structure on topological } n\text{-sphere}} \quad \Rightarrow \quad \overbrace{S^n /\!\!/ G \underset{\text{htpy}}{\simeq} BH}^{G\text{-twisted Cohomotopy / topological } H\text{-structure}} .$$

(One may think of this as "moving G from numerator on the right to denominator on the left".)

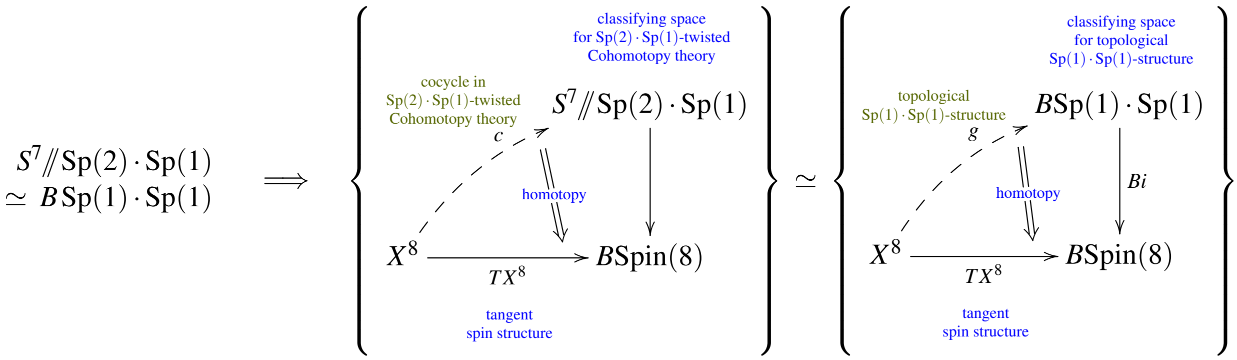
In particular, on Spin 8-manifolds we have the following equivalences between J-twisted Cohomotopy cocycles (11) and topological G-structures:

$$\begin{array}{c} S^7 /\!\!/ \mathrm{Spin}(8) \\ \simeq B\mathrm{Spin}(7) \end{array} \Longrightarrow$$



(14)

and



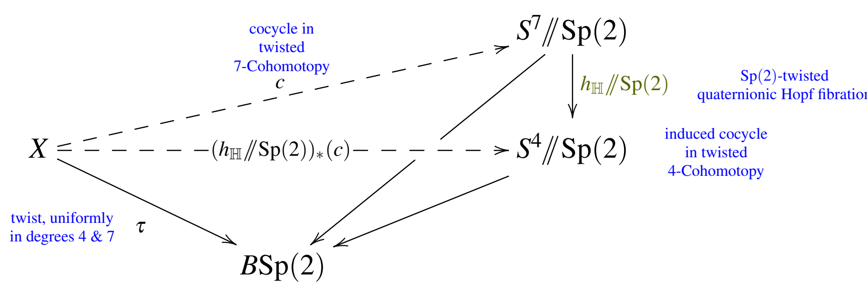
(15)

As the existence of a G-structure is a non-trivial topological condition, so is hence the existence of J-twisted Cohomotopy cocycles. Notice that this is a special effect of twisted non-abelian generalized Cohomology: A non-twisted generalized cohomology theory (abelian or non-abelian) always admits at least one cocycle, namely the trivial or zero-cocycle. But here for non-abelian J-twisted Cohomotopy theory on 8-manifolds, the existence of *any* cocycle is a non-trivial topological condition.

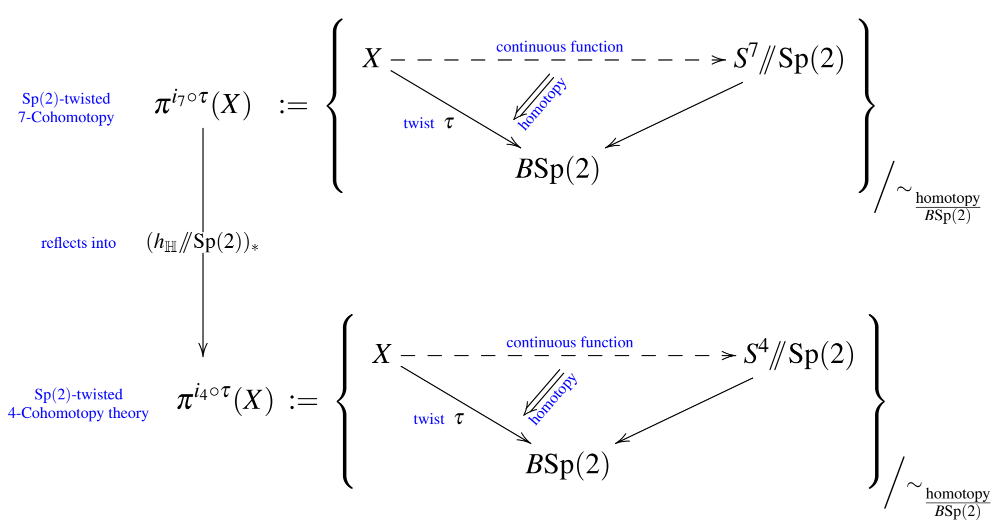
Compatibly $\mathrm{Sp}(2)$-Twisted Cohomotopy in degree 4 & 7. For focus of the discussion, we will now restrict attention to G-structure for the further quaternion-unitary subgroup

$$\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(1)\cdot\mathrm{Sp}(2)$$

in diagram (12). In summary then, due to the $\mathrm{Sp}(2)$-equivariance of the quaternionic Hopf fibration (12), the map (9) from degree-7 to degree-4 Cohomotopy passes to $\mathrm{Sp}(2)$-twisted Cohomotopy:



and hence (9) becomes:



(16)

(17)

Triality between $\mathrm{Sp}(2)$**-structure and** $\mathrm{Spin}(5)$**-structure.** While the group $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ (12) is abstractly isomorphic to the central product of Spin-groups $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$, the two are *distinct* as subgroups of $\mathrm{Spin}(8)$, and not conjugate to each other. But as subgroups they are turned into each other by the ambient action of triality:

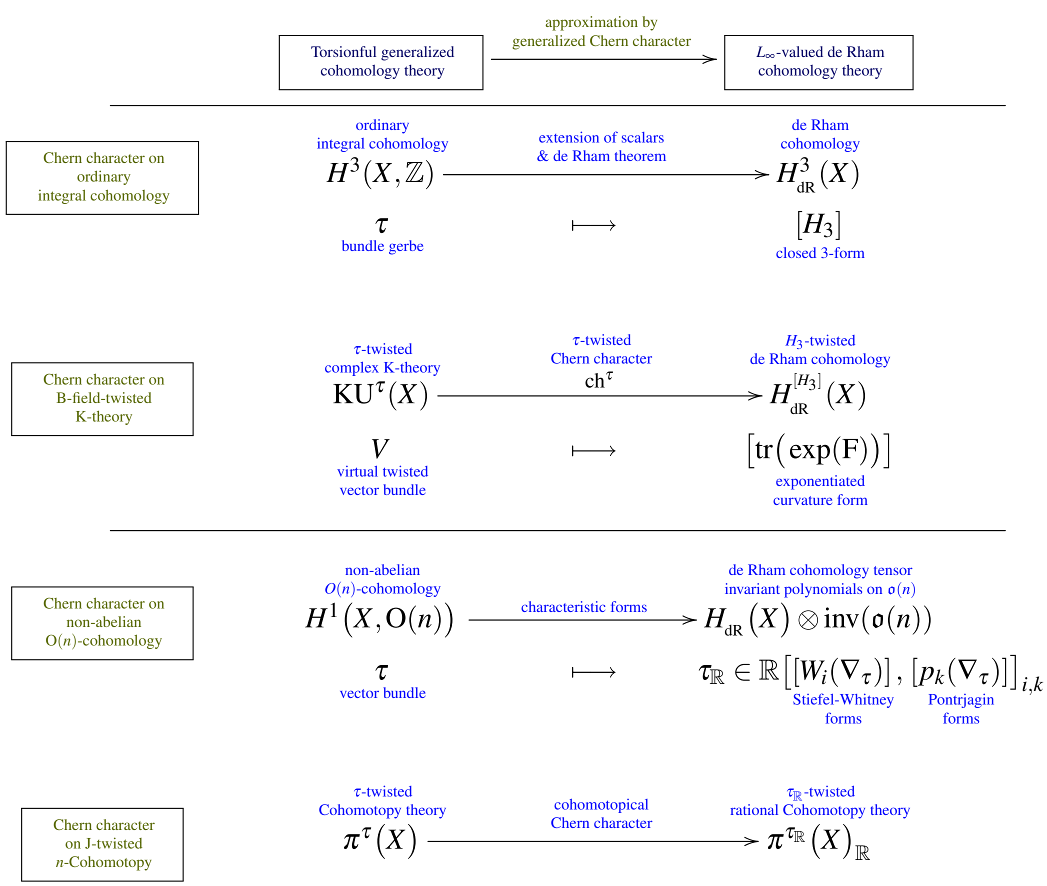
$$\begin{array}{ccccccc} \mathrm{Sp}(2) & \hookrightarrow & \overset{\text{central product of quaternion-unitary groups}}{\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)} & \overset{\simeq}{\longleftrightarrow} & \overset{\text{central product of Spin-groups}}{\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)} & \hookleftarrow & \mathrm{Spin}(5) \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Spin}(8) & \underset{\underset{\text{triality automorphism}}{\mathrm{tri}}}{\overset{\simeq}{\longleftrightarrow}} & \mathrm{Spin}(8) & & \end{array}$$

While $\mathrm{Spin}(5)$ on the right is the structure group of normal bundles to M5-branes, acting on fibers of 4-spherical fibrations around 5-branes through its vector representation, $\mathrm{Sp}(2)$ on the left is the structure group of normal bundles to M2-branes, acting on the 7-spherical fibrations around 2-branes via its defining left action on quaternionic 2-space $\mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$ ([MFGM09][MF10]):

$$\overset{\text{left quaternion multiplication}\;\; \mathrm{Sp}(2)\,\circlearrowright}{S\big(\mathbb{H}^2\big)} = S^7 \qquad\qquad S^4 = \overset{\mathrm{Spin}(5)\,\circlearrowleft\;\; \text{vector representation}}{S\big(\mathbb{R}^5\big)}$$

In this article we consider only the M2-case. But all formulas we derive translate to the M5 case via triality.

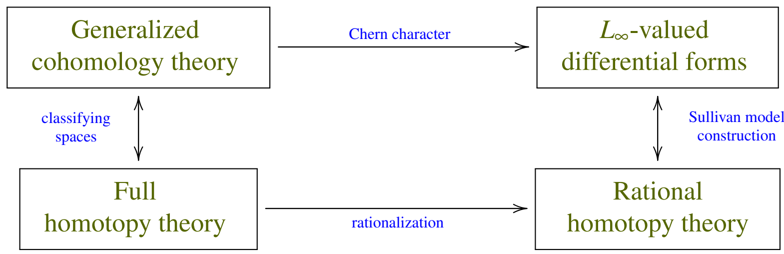
Generalized Chern characters. Since generalized cohomology theory is rich, one needs tools to break it down. The first and foremost of these is the *generalized Chern character* map. This extracts differential form data underlying a cocycle in nonabelian generalized cohomology. The Chern character is familiar in twisted K-theory (see [GS19a][GS19c]), as shown in the top half of the following:



(18)

In order to see what the *cohomotopical Chern character* in the last line is, we need some general theory of generalized Chern characters. This is *rational homotopy theory*:

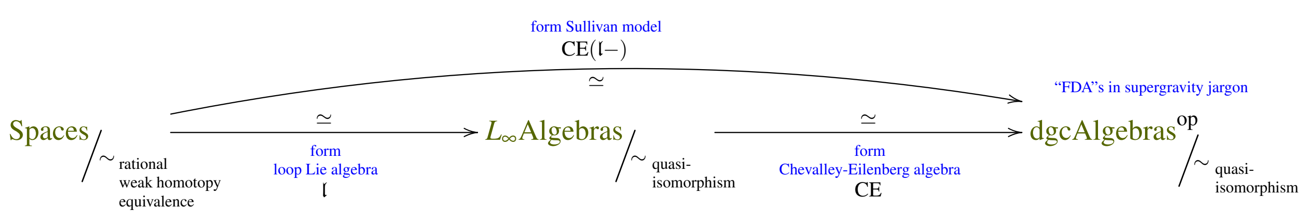
Rational homotopy theory. In the language of homotopy theory, generalized Chern character maps are examples of *rationalization*, whereby the homotopy type of a topological space (here: the classifying space of a generalized cohomology theory) is approximated by tensoring all its homotopy groups with the rational numbers (equivalently: the real numbers), thereby disregarding all torsion subgroups in homotopy groups and in cohomology groups.



What makes rational homotopy theory amenable to computations is the existence of *Sullivan models*. These are differential graded-commutative algebras (dgc-algebras) on a finite number of generating elements (spanning the rational homotopy groups) subject to differential relations (enforcing the intended rational cohomology groups). In the supergravity literature Sullivan models are also known as "FDA"s. Here are some basic examples (see [FSS16b][FSS18a][FSS18b][FSS19a]):

	Rational super space	**Loop super L_∞-algebra**	**Chevalley-Eilenberg super dgc-algebras** ("Sullivan models", "FDA"s)
General	X	$\mathfrak{l}X$	$\mathrm{CE}(\mathfrak{l}X)$
Super spacetime	$\mathbb{T}^{d,1\vert\mathbf{N}}$	$\mathbb{R}^{d,1\vert\mathbf{N}}$	$\mathbb{R}\big[\{\psi^\alpha\}_{\alpha=1}^N,\{e^a\}_{a=0}^d\big]\Big/\begin{pmatrix} d\,\psi^\alpha &= 0 \\ d\,e^a &= \overline{\psi}\Gamma^a\psi\end{pmatrix}$
Eilenberg-MacLane space	$K(\mathbb{R},p+2) \simeq_{\mathbb{R}} B^{p+1}S^1$	$\mathbb{R}[p+1]$	$\mathbb{R}[c_{p+2}]\big/\big(d\,c_{p+2} = 0\big)$
Odd-dimensional sphere	S^{2k+1}	$\mathfrak{l}(S^{2k+1})$	$\mathbb{R}[\omega_{2k+1}]\big/\big(d\,\omega_{2k+1} = 0\big)$
Even-dimensional sphere	S^{2k}	$\mathfrak{l}(S^{2k})$	$\mathbb{R}\big[\omega_{2k},\omega_{4k-1}\big]\Big/\begin{pmatrix} d\,\omega_{2k} &= 0 \\ d\,\omega_{4k-1} &= -\omega_{2k}\wedge\omega_{2k}\end{pmatrix}$
M2-extended super spacetime	$\widehat{\mathbb{T}^{10,1\vert\mathbf{32}}}$	$\mathfrak{m2brane}$	$\mathbb{R}\big[\{\psi^\alpha\}_{\alpha=1}^{32},\{e^a\}_{a=0}^{10},h_3\big]\Big/\begin{pmatrix} d\,\psi^\alpha &= 0 \\ d\,e^a &= \overline{\psi}\Gamma^a\psi \\ d\,h_3 &= \tfrac{i}{2}(\overline{\psi}\Gamma_{ab}\psi)\wedge e^a\wedge e^b\end{pmatrix}$

Under *Sullivan's theorem* the rational homotopy type of well-behaved spaces are equivalently encoded in their Sullivan model dgc-algebras. For spaces and algebras which are nilpotent and of finite type one has:



When applying the rational approximation to twisted generalized cohomology theory, the order matters: There are in general more *rational twists* $X \xrightarrow{\tau} B\mathrm{Aut}(A_{\mathbb{R}})$ for *twisted rational cohomology* than there are rationalizations $\tau_{\mathbb{R}}$ of full twists $X \xrightarrow{\tau} B\mathrm{Aut}(A)$ for *rational twisted cohomology*. [2] We consider first the general rational twists:

Rationally twisted rational Cohomotopy. We find that the *rationally twisted rational Cohomotopy* sets in degrees 4 and 7 are equivalently characterized by cohomotopical Chern character forms as follows:

	rational twist	rational twisted Cohomotopy		cohomotopical Chern characters
7-Cohomotopy	$X \xrightarrow{\tau^7} B\mathrm{Aut}\big(S^7_{\mathbb{R}}\big)$	$\pi^{(\tau^7)}(X)$	$\simeq$	$\Big\{ \underset{\text{7-form}}{\widetilde{G}_7} \;\Big\vert\; d\,\widetilde{G}_7 = \underset{\text{characteristic form of twist } \tau^7}{K_8} \Big\}\Big/_{\sim}$
4-Cohomotopy	$X \xrightarrow{\tau^4} B\mathrm{Aut}\big(S^4_{\mathbb{R}}\big)$	$\pi^{(\tau^4)}(X)$	$\simeq$	$\left\{ \underset{\text{4-form \& 7-form}}{(G_4, G_7)} \;\middle\vert\; \begin{array}{l} d\,G_4 = 0 \\ d\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4 + \underset{\text{characteristic form of twist } \tau^4}{L_8} \end{array} \right\}\Big/_{\sim}$

(19)

Here *all* real 8-classes $[K_8], [L_8] \in H^8(X,\mathbb{R})$ may appear, for *some* rational twists $\tau^{4,7}$. Constraints on these characteristic forms appear when we consider more than rational twisted structure:

Compatibly rationally twisted rational Cohomotopy. We may ask that the rational twists $\tau^{4,7}$ in (19) are related analogously to how the twisted parametrized Hopf fibration (12) relates the full (non-rational) twists, through (16). We find that this happens precisely when the difference of the characteristic 8-classes in (19) is a complete square

$$L_8 \;=\; K_8 + \big(\tfrac{1}{4}P_4\big) \wedge \big(\tfrac{1}{4}P_4\big)$$

and in that case the situation of (19) becomes the following:

	Compatible rational twists	Rational compatibly twisted Cohomotopy		Cohomotopical Chern characters
7-Cohomotopy	$X \xrightarrow{\tau^7} B\mathrm{Aut}\big(S^7_{\mathbb{R}}\big)$	$\pi^{(\tau^7)}(X)$	$\simeq$	$\Big\{ \widetilde{G}_7 \;\Big\vert\; d\,\widetilde{G}_7 = \underset{\text{characteristic form of twist } \tau^7}{K_8} \Big\}\Big/_{\sim}$
	shifted 4-form: $\widetilde{G}_4 := G_4 + \tfrac{1}{4}P_4$; shifted 7-form: $\widetilde{G}_7 := G_7 + \tfrac{1}{2}H_3 \wedge \widetilde{G}_4$		$\simeq$	$\left\{ \begin{pmatrix} H_3, \\ \widetilde{G}_4, G_7 \end{pmatrix} \;\middle\vert\; \begin{array}{l} d\,H_3 = \widetilde{G}_4 - \tfrac{1}{2}P_4 \\ d\,\widetilde{G}_4 = 0 \\ d\,G_7 = -\tfrac{1}{2} dH_3 \wedge \widetilde{G}_4 + K_8 \end{array} \right\}\Big/_{\sim}$
4-Cohomotopy	$X \xrightarrow{\tau^4} B\mathrm{Aut}\big(S^4_{\mathbb{R}}\big)$	$\pi^{(\tau^4)}(X)$	$\simeq$	$\left\{ (\widetilde{G}_4, G_7) \;\middle\vert\; \begin{array}{l} d\,\widetilde{G}_4 = 0 \\ d\,G_7 = -\tfrac{1}{2}(\widetilde{G}_4 - \tfrac{1}{2}P_4) \wedge \widetilde{G}_4 + K_8 \end{array} \right\}\Big/_{\sim}$

(20)

Here still *all* real 8-classes and 4-classes $[K_8] \in H^8(X,\mathbb{R})$, $[P_4] \in H^4(X,\mathbb{R})$ may appear, for *some* pair of compatible rational twists.

[2] This is in contrast with twisting vs. differential refinement where the order does not matter – see [GS19a][GS19b].

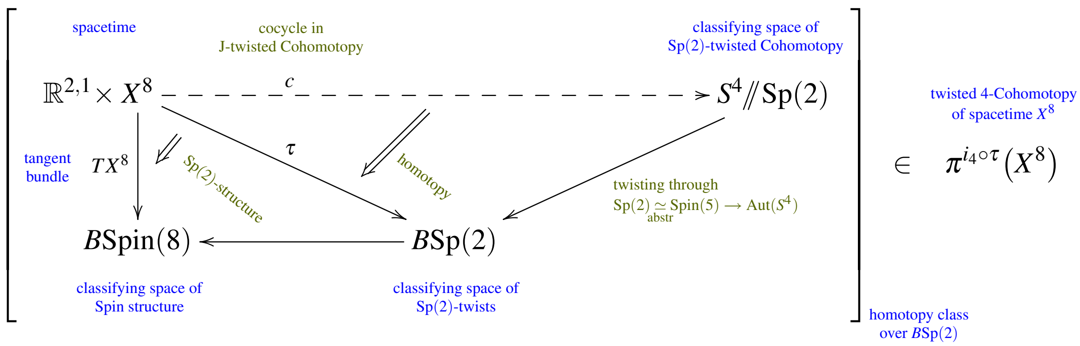
Next we find that these real classes are fixed as we consider full (not just rational) Sp(2)-twists, compatible by the full (not just rational) Sp(2)-twisted quaternionic Hopf fibration (12).

J-Twisted 4-Cohomotopy of Sp(2)-manifolds. Consider a simply-connected Riemannian Spin manifold $\mathbb{R}^{2,1} \times X^8$ with affine connection ∇ and equipped with:

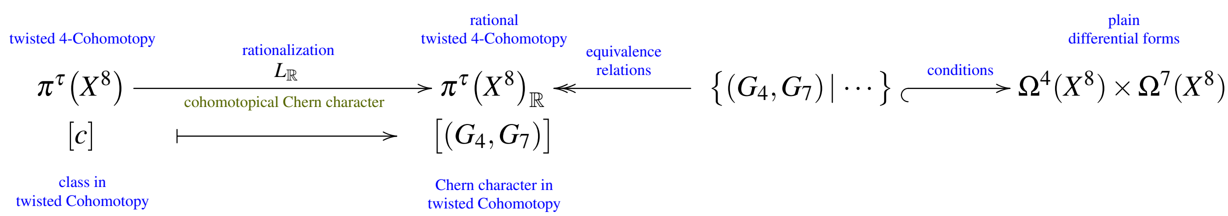
(i) an $\mathrm{Sp}(2)$-structure τ (13);

(ii) a cocycle c in τ-twisted 4-Cohomotopy (17);

hence equipped with a homotopy-commutative diagram of continuous maps as follows:



Then the **4-Cohomotopical Chern character** (18) of $[c]$, hence the differential flux forms (G_4, G_7) underlying $[c]$ by (19), as indicated on the left in the following diagram



satisfy, first of all, this condition:
The **shifted 4-flux** form

$$\underset{\text{naive 4-flux}}{\widetilde{G}_4 \;:=\; G_4} \;+\; \underset{\text{shift by first fractional Pontrjagin form}}{\tfrac{1}{4} p_1(\nabla)} \;\in\; \underset{\text{differential 4-forms}}{\Omega^4\big(X^8\big)} \tag{21}$$

represents an **integral** cohomology class

$$\underset{\text{shifted 4-flux}}{[\widetilde{G}_4]} \;\in\; \underset{\text{integral cohomology}}{H^4\big(X^8, \mathbb{Z}\big)} \xrightarrow{\text{extension of scalars}} \underset{\text{real cohomology}}{H^4\big(X^8, \mathbb{R}\big)} \;\simeq\; \underset{\text{de Rham cohomology}}{H_{\mathrm{dR}}\big(X^8\big)} \tag{22}$$

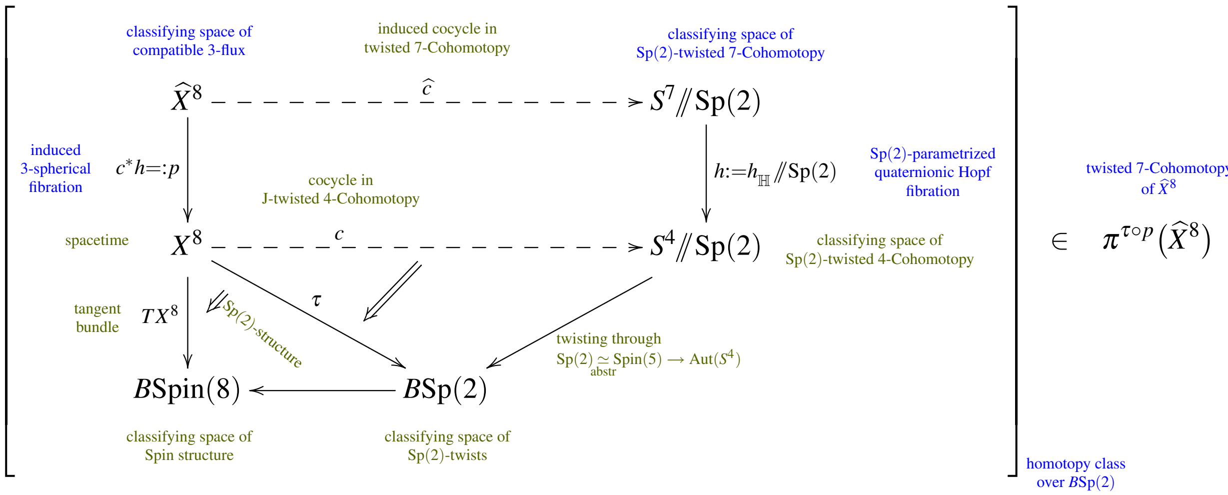
on which the action of the **Steenrod square vanishes**:

$$\overset{\text{Steenrod square of mod-2 reduction of integral shifted 4-flux}}{\mathrm{Sq}^2\big([\widetilde{G}_2]\big) = 0} \qquad \text{hence also} \qquad \overset{\text{Steenrod cube of mod-2 reduction of integral shifted 4-flux}}{\mathrm{Sq}^3\big([\widetilde{G}_2]\big) = 0}\,, \tag{23}$$

and its **background charge** in the case of factorization through $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ is

$$\overset{\text{residual flux of cocucle factoring through } h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)}{(G_4)_0} \;=\; \overset{\text{background charge}}{\tfrac{1}{4} p_1(\nabla)}\,. \tag{24}$$

To see the next condition satisfied by the pair (G_4, G_7), consider the homotopy pullback of the 4-Cohomotopy cocycle c along the Sp(2)-twisted quaternionic Hopf fibration $h_{\mathbb{H}}$ to a cocycle in twisted 7-Cohomotopy on the induced 3-spherical fibration $\widehat{H}^8$ over spacetime:



(25)

Then:

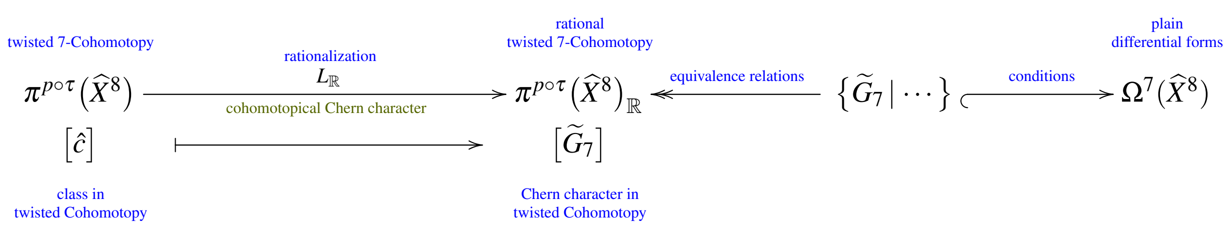
The **pullback 3-spherical fibration** over spacetime

$$\widehat{X}^8 \;:=\; c^*\big(S^7 /\!\!/ \mathrm{Sp}(2)\big)$$

carries a universal 3-flux H_3^{univ} which trivializes the 4-flux relative to its background value

$$d\,H_3^{\mathrm{univ}} \;=\; p^* \widetilde{G}_4 - \tfrac{1}{4} p_1(\nabla)\,. \tag{26}$$

Moreover, the **7-Cohomotopical Chern character** of $[\hat{c}]$, hence the flux forms underlying $[\hat{c}]$ by (20), as indicated on the left in the following diagram



satisfy this condition:

The **shifted 7-flux** form

$$\widetilde{G}_7 \;=\; \underset{\text{naive 7-flux}}{p^* G_7} \;+\; \underbrace{\tfrac{1}{2}\overset{\text{3-flux}}{H_3^{\mathrm{univ}}} \wedge \overset{\text{shifted 4-flux}}{p^* \widetilde{G}_4}}_{\substack{\text{shift by} \\ \text{Hopf-Whitehead term}}} \tag{27}$$

is closed up to the Euler 8-form

$$d\,\widetilde{G}_7 \;=\; -\tfrac{1}{2} p^* \chi_8(\nabla) \tag{28}$$

and **half-integral** on every 7-sphere $S^7 \xrightarrow{i} \widehat{X}^8$:

$$2\int_{S^7} i^* \tilde{G}_7 \;\in\; \mathbb{Z}\,. \tag{29}$$

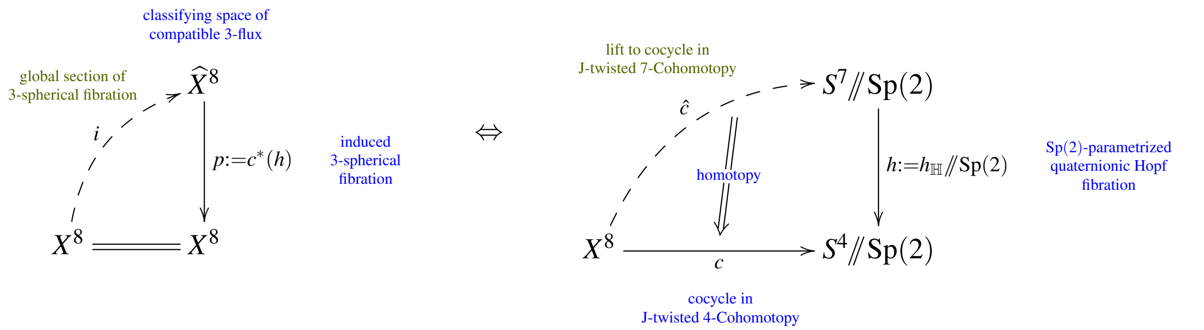
Finally, consider the case when:

(i) Our manifold is the complement in a closed 8-manifold of a finite set of disjoint open balls, i.e. of a tubular neighbourhood $\mathcal{N}$ around a finite set $\{x_1, x_2, \cdots\}$ of points:

$$X^8 \;=\; \overset{\substack{\text{closed} \\ \text{manifold}}}{X^8_{\mathrm{clsd}}} \setminus \underset{\text{around points in } X^8_{\mathrm{clsd}}}{\overset{\substack{\text{tubular} \\ \text{neighbourhood}}}{\mathcal{N}_{\{x_1,x_2,\cdots\}}}} \quad\Rightarrow\quad \overset{\substack{\text{boundary} \\ \text{of } X^8}}{\partial X^8} \;\simeq\; \underset{\{x_1,x_2,\cdots\}}{\bigsqcup} \overset{\substack{\text{sphere} \\ \text{around } x_i}}{S^7} \tag{30}$$

This implies that X^8 is a manifold with boundary a disjoint union of 7-spheres.

(ii) Such that the corresponding extended spacetime fibration $\widehat{X}^8 \to X^8$ (25) admits a global section; hence, equivalently, such that the given cocycle in twisted 4-Cohomotopy lifts through the quaternionic Hopf fibration to a cocycle in twisted 7-Cohomotopy:

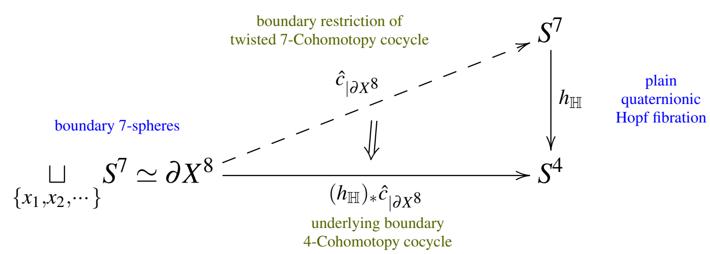


(31)

Here the choice of points in (30) matters only in so far as a sufficient number of points has to be removed for a lifted cocycle $\hat{c}$ (31) to exist at all.

By (26) this lift exhibits a **4-fluxless** background at least at the level of integral cohomology. In order to refine this to 4-fluxlessness at the finer level of (stable) Cohomotopy, we observe the following:

(i) Since the 7-sphere is parallelizable, upon restriction of $\hat{c}$ (31) to the boundary $\partial X^8 \xrightarrow{\;i\;} X^8$ (30) the twist vanishes, and we are left with a pair of compatible cocycles in plain Cohomotopy theory as in (9):



(ii) By (8), cocycles in stable 7-Cohomotopy have no side-effect in stable 4-Cohomotopy, hence remain stably **cohomotopically 4-fluxless** precisely if they are multiples of 24:

$$\text{For } [c_1], [c_2] \in \underbrace{\pi^7\big(\partial X^8\big)}_{\text{7-Cohomotopy}} \quad \text{we have} \quad \left\{ \begin{array}{lll} (h_{\mathbb{H}})_*[c_1] & = & (h_{\mathbb{H}})_*[c_2] \;\in\; \overbrace{\mathbb{S}^4\big(\partial X^8\big)}^{\text{stable 4-Cohomotopy}} \\ \Leftrightarrow & & \\ [c_1] & =_{\text{mod } 24} & [c_2] \;\in\; \underbrace{\mathbb{S}^7\big(\partial X^8\big)}_{\text{stable 7-Cohomotopy}} \end{array} \right.$$

This means that the **unit charge** of a lift $\hat{c}$ in (31), as seen by stable Cohomotopy, is 24. In view of (29) this says that the **cohomotopically normalized 7-flux** of X^8 is

$$N_{\text{M2}} := \tfrac{-1}{12} \int_{X^8} i^* d\widetilde{G}_7 = \tfrac{-1}{12} \int_{\partial X^8} i^* \widetilde{G}_7 \,. \tag{32}$$

Our final result is that:
this **equals the I_8-number** (13) of the manifold:

$$N_{\text{M2}} = I_8[X^8] \,. \tag{33}$$

4 Conclusion

Perturbative string theory has a precise definition via 2d worldsheet SCFT. In contrast, the formulation of its non-perturbative completion to M-theory and of the brane physics this subsumes (see [Du99][BBS06]), remains an open problem (e.g. [HLW97, p. 2][NH98, p. 2][Mo14, 12][CP18, p. 2][Wi19] [6]). The lack of an actual set of fundamental laws of non-perturbative brane physics has recently surfaced in a debate on the extent of validity of the brane uplifts that have been widely discussed for 15 years [DvR18][Ba19, p. 14-22].

Besides the field of gravity, the only other field in M-theory at low-energy is the C-field [CJS78]. A list of cohomological conditions on the C-field, including those shown in Table 1, have been derived as plausible consistency conditions in various expected limiting cases of M-theory (effective field theory limits, decoupling limits etc.) assuming the conjectural string dualities to hold. One imagines that if M-theory exists then thereby it must be consistent, and hence ought to imply all these expected consistency conditions. In order to make this actually happen, the first step in formulating M-theory ought to be the identification of the generalized cohomology theory that charge-quantizes the C-field, just as the first step in formulating a quantum consistent theory of electromagnetism was Dirac's *charge quantization* of the electromagnetic field: as a cocycle in (differential) ordinary cohomology (see [Fr00]),

The string theory literature has mostly regarded the M-theory C-field as a cocycle in ordinary 4-cohomology, with extra constraints imposed on it by hand. A proposal to build at least one of these conditions, the shifted flux quantization condition (§3.4), into the definition of the cohomology theory (making it a "mildly generalized cohomology theory") has been considered in [DFM03][HS05][SSS12][FSS14a]. Another condition, the "integral equation of motion" (§3.6) has been argued in [DMW03a][DMW03b] to be in correspondence with one differential of specific degree in the Atiyah-Hirzebruch spectral sequence for K-theory. In reaction to this state of affairs, it has been suggested [Sa05a][Sa05b][Sa06][Sa10] that the C-field should be regarded as a cocycle in some genuine generalized cohomology theory, such as Cohomotopy theory [Sa13, 2.5]. Indeed, if M-theory is as fundamental to physics as it should be, one may expect the cohomology theory that charge quantizes the C-field to be more fundamental to mathematics than ordinary cohomology with some modifications.

In order to derive what this fundamental generalized cohomology theory actually is, we had initiated a systematic analysis of the bifermionic super p-brane charges from the point of view of super rational homotopy theory [FSS13]; see [FSS19a] for review. We proved in this supergeometric setting, albeit in rational approximation, that the expected charge quantization of the RR-field in twisted K-theory follows from systematic analysis of the D-brane super WZW terms [FSS16a][FSS16b][BSS18]. Then we showed that the exact same logic applies to the super WZW terms of the M-branes [FSS15]. The analysis in this case reveals their cohomology theory to be [FSS15, 3][FSS16a, 2] Cohomotopy cohomology theory in compatible degrees $(4,7)$, related by the quaternionic Hopf fibration; see [FSS19a, 7] for review of this super rational analysis. This proves that if full M-theory retains the super-space structure of its low-energy limit, then the cohomology theory that charge-quantizes the C-field must be such that its rationalization coincides with that of Cohomotopy cohomology theory in degrees $(4,7)$. While there are many different cohomology theories with the same rationalization as Cohomotopy theory, one of these is *minimal* in number of CW-cells: This is Cohomotopy theory itself.

What we have shown in this article is that assuming with *Hypothesis H* that Cohomotopy cohomology theory in compatible degrees $(4,7)$ indeed encodes the charge-quantization of the C-field even beyond the rational approximation, then the list in Table 1 of expected consistency conditions is implied. Further checks of *Hypothesis H* for the case of M-theory orbifolds are presented in [SS19a][SS19b].

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[6] [Wi19] at 21:15: "I actually believe that string/M-theory is on the right track toward a deeper explanation. But at a very fundamental level it's not well understood. And I'm not even confident that we have a good concept of what sort of thing is missing or where to find it."