Topological Quantum Gates on FCIs with Bounded Quantum Geometry

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Abstract

Two-band 2D fractional Chern insulators (FCIs) have recently been shown, in principle, to admit unusual anyonic order, associated with Bloch Hopfions localized in momentum space. Here we show that bounding the integrated quantum metric along one primitive Brillouin-zone cycle may refine this topological order to $|C|$ distinct braiding observables, cyclically permuted by lattice translations, for Chern numbers $|C| > 1$. The cyclic structure entails mildly non-abelian braiding phases and otherwise matches the spectral flow exhibited by the FCI's hybrid Wannier states, with the metric bound equivalently a bound on the states' real-space spread.

We give a classification of quantum gates in principle implementable on this topological order; among them we find the non-Clifford T-gate and the non-Clifford entangling CCZ-gate. This points to novel pathways to fault-tolerant quantum processes within the known orbital structure of FCIs, potentially informing experimental searches towards hardware for future topological quantum computers.

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1 Motivation

The Prospect of FCI Topological Order. Fractional Chern insulators (FCIs, [TMW11; Sun+11; Neu+11], cf. the review in [Neu+15]) are potentially topologically ordered [Wen95] crystalline topological quantum materials [GS18], which realize a momentum-space analog of fractional quantum Hall liquids (FQH, [Sto99]). The latter provide firmly established experimental realizations of abelian *anyons* [Gol23], whose characteristic fractional statistics have been observed by independent groups in GaAs heterostructures and in graphene [Bar+20; Nak+20; Wer+25].

While analogous (dual, cf. [SS26b, Fig. 3]), FCIs, as recently realized in twisted $MoTe_2$ and rhombohedral graphene [Par+23; Lu+24], do not require a strong external magnetic field and have the prospect of operating at less extreme temperatures [TMW11] than FQH liquids. These properties make FCIs a potentially realistic platform for future quantum hardware for topological quantum computing devices ([Kit03; Fre+03; Nay+08], cf. [Urt23])—provided suitable anyonic excitations in FCIs can be identified and controlled. This remains to be seen experimentally,[1] but crucially it already remains to be properly understood theoretically.

New Route to Nonabelian Topological Order. In particular, universal topological quantum gates require universal *nonabelian* anyons ([FLW02], cf. [Bra06; MSS24]) whose actual experimental realization has remained open, even theoretically.[2]

Recently, by revisiting theoretical foundations (briefly recalled in § 2.2 below), we have observed [SS26b; SS25b] that 2D 2-band FCIs may potentially exhibit a non-traditional but theoretically well-motivated form of topological order associated with anyonic solitons

[1] Recent optical experiments provide encouraging evidence for fractional charges in FCI states [Li+26], but their braiding statistics, if any, remain to be observed.

[2] The much discussed FQH $\nu = 5/2$ state, while theoretically expected to feature non-universal Ising anyons, appears unstable and remains experimentally elusive [Pan+99; TY26, p. 1]. Note that recent claims of nonabelian anyon experiments (such as [Lo+26]) concern their quantum *simulation* on noisy non-topological hardware, cf. [Xu+24, p. 1470; GKP26].

localized in the crystal's momentum space, namely in the form of *Hopfion* structures [WZ83] in the Bloch Hamiltonian, cf. (19) below.

This derivation [SS26b], however, relied on the idealized assumption that the system's topological monodromy is evaluated within the entire space of mathematically admissible Bloch Hamiltonians, while realistic systems will be subject to bounds.

Here we consider the more physically realistic situation where a bound is imposed on the system's *quantum metric* (recalled in § 2.1 below) integrated along one primitive Brillouin-zone cycle. Our main result (in § 3) is that this bound actually enhances the Hopfionic topological order to potentially enable the realization of topological quantum gates such as the T-gate.

Remark 1.1. The question of identifying microscopic (lattice) models potentially realizing the parameter monodromy, which we classify here on more abstract grounds, is beyond the scope of this letter. We suggest reading our abstract classification result as mathematically establishing novel theoretical possibilities for anyonic topological order which may be worth the attention of model builders and experimentalists.

In particular, we see that this enhanced topological order goes along with a cyclic layering of FQH phases which is of just the same form as expected when translating hybrid FCI Wannier states into Laughlin states for FQH systems. We first recall this layering in § 2.1.

2 Methods

2.1 FCI Quantum Geometry

We briefly recall how *quantum geometry* (cf. [Kol+17]) controls the localization of band-projected x-position eigenstates in 2D crystalline materials, and how these same states exhibit an effective cyclic layer of QH states, an observation due to [Qi11, §I; BQ12, §I].

Quantum Metric. The *quantum metric* [PV80] of a smooth family of pure quantum states (hence of rays in a given Hilbert space $\mathcal{H}$)

$$\begin{aligned} \Sigma &\longrightarrow P\mathcal{H} \\ \sigma &\mapsto |\psi_\sigma\rangle\langle\psi_\sigma|\,, \qquad \langle\psi_\sigma\,|\,\psi_\sigma\rangle \,=\, 1\,, \end{aligned} \tag{1}$$

is the pullback (to Σ) of the Fubini–Study metric on the complex projective space $P\mathcal{H}$, measuring the infinitesimal distinguishability of neighboring states.

Specialized to crystalline band theory, the parameters are crystal momenta $\mathbf{k} \in T^d$ varying in the *Brillouin torus* (cf. [Thi25, §2.1] and Fig. 1), and the states are the cell-periodic Bloch eigenstates $|u(\mathbf{k})\rangle$ of an isolated band. For a locally chosen normalized eigenvector $|u(\mathbf{k})\rangle$, with projector $P(\mathbf{k}) = |u(\mathbf{k})\rangle\langle u(\mathbf{k})|$, the quantum metric is

$$g_{ij}(\mathbf{k}) \,=\, \mathrm{Re}\Big\langle \partial_{k_i} u(\mathbf{k}) \Big| \Big(1 - P(\mathbf{k})\Big) \Big| \partial_{k_j} u(\mathbf{k}) \Big\rangle. \tag{2}$$

This expression measures the variation of the band eigenspace across the Brillouin zone.

Hybrid Wannier states. This momentum-space geometry determines a real-space localization scale: For a single isolated band in two dimensions, the maximally localized *hybrid Wannier states* form a family

$$\big|W_n(k_y)\big\rangle, \qquad n \in \mathbb{Z}, \tag{3}$$

where n labels lattice translations along x. These states are localized along x while retaining transverse crystal momentum k_y, and are eigenstates of the band-projected position operator $\widehat{P}\hat{x}\widehat{P}$, where $\widehat{P}$ projects onto the full band [Mar+12, Sec. II.H].

Applying the one-dimensional spread identity of [MV97, Sec. III and App. C] at each fixed k_y identifies the physical position variance of these hybrid Wannier states with the x-average of the quantum metric:

$$\mathrm{Var}_{W_n(k_y)}(\hat{x}) \;=\; \frac{a_x}{2\pi}\int_0^{2\pi/a_x} g_{xx}(k_x,k_y)\,\mathrm{d}k_x \;=:\; \overline{g}_{xx}(k_y), \tag{4}$$

where $a_{x/y}$ denote the lattice periods, cf. Fig. 1. Consequently:

A uniform bound on the x-averaged quantum metric is equivalent to a uniform bound on the physical x-position variance of the hybrid Wannier eigenstates of projected position:

$$\overline{g}_{xx}(k_y) \le \ell^2 \quad \text{for all } k_y \qquad \Leftrightarrow \qquad \mathrm{Var}_{W_n(k_y)}(\hat{x}) \le \ell^2 \quad \text{for all } k_y, n. \tag{5}$$

Imposing the same bound throughout the allowed Hamiltonian deformations ensures that these eigenstates retain a root-mean-square width along x of at most ℓ.

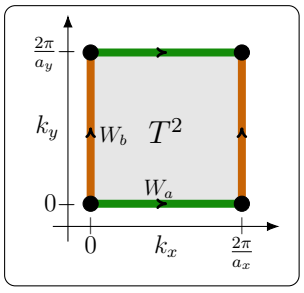


Figure 1: The Brillouin zone of an effectively two-dimensional crystalline system is a torus T^2, depicted here by a fundamental domain with opposite edges identified. The momentum coordinates k_x and k_y have reciprocal periods $2\pi/a_x$ and $2\pi/a_y$, respectively; the displayed aspect ratio is schematic. Below, in § 3, we constrain the k_x-averaged quantum metric $\overline{g}_{xx}(k_y)$ (2) uniformly in k_y, selecting the horizontal cycle as the "small" momentum direction. The associated momentum 1-cycles are W_a for this bounded direction and W_b for the unrestricted vertical direction, cf. Fig. 3.

Cyclic QH Layers of FCI states. A striking insight of Qi and Barkeshli–Qi is that the hybrid Wannier states $\big|W_n(k_y)\big\rangle$ (3) of a single Chern band exhibit an effective multilayer quantum Hall structure [Qi11; BQ12] in that:

(i) the integer index n in (3) labels successive lattice translates:

$$x_n(k_y) \;:=\; \big\langle W_n(k_y)\,\big|\,\hat{x}\,\big|\,W_n(k_y)\big\rangle \;=\; n\,a_x + \overline{x}(k_y), \tag{6}$$

where $\overline{x}(k_y)$ is a common offset determined by the Berry phase around the k_x-cycle;

(ii) for a band of Chern number $C \neq 0$, traversing the transverse Brillouin-zone cycle shifts each center by C lattice cells, so that [Qi11, Eqs. (2)–(3)]

$$\big|W_n\big(k_y + 2\pi/a_y\big)\big\rangle \;=\; \big|W_{n+C}(k_y)\big\rangle\,. \tag{7}$$

Continuation in k_y therefore joins the orbitals into $|C|$ families, distinguished by the residue

$$n \bmod C \;\in\; \mathbb{Z}_{/C}\,. \tag{8}$$

These are the effective *layers*, cyclically permuted upon translation T_x by one lattice cell (cf. Fig. 2):

$$T_x\big|W_n(k_y)\big\rangle \;=\; \big|W_{n+1}(k_y)\big\rangle, \qquad n \mapsto n+1 \mod C \quad \text{in } \mathbb{Z}_{/C}. \tag{9}$$

Thus continuation around the transverse momentum cycle connects orbitals within each layer, while lattice translation cycles between the layers [BQ12, Sec. II].

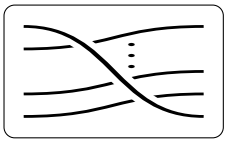


Figure 2: The eigenstates of projected x-position operators in an FCI at Chern number $C \neq 0$ exhibit an orbital representation consisting of $|C|$ Quantum Hall layers that are cyclically permuted under lattice translation in the x-direction (9).

(iii) With this, one finds [Qi11, below (5)] that these Wannier states in a layer

$$n = i + Cq, \qquad i \in \{0, \ldots, |C| - 1\}, \qquad q \in \mathbb{Z},$$

correspond to Quantum Hall Landau orbitals at transverse momentum

$$K := k_y + q\tfrac{2\pi}{a_y}\,. \tag{10}$$

Multilayer FQH wavefunctions can then be transferred to the Chern band by replacing their Landau-level orbitals with these Wannier orbitals while retaining their occupation-number coefficients.

This layered organization is encoded in the spectral flow of projected position; the metric bound in (4) controls the physical widths of the orbitals that realize it. (We will see this structure reappear from novel topological considerations in Thm. 3.5 below.)

2.2 FCI Hopfion Anyons

But this raises a key question that may not have been addressed before:

> If the bounded quantum geometry of FCIs is effectively that of cyclic FQH layers, how may the corresponding layers of FQH anyons arise?

The answer we consider here crucially draws on the recent observation [SS26b; SS25b] that 2-band FCIs in principle admit momentum-space localized anyons in the guise of Hopfion solitons in the Bloch Hamiltonian. We briefly recall how this comes about.

Topological Order from Parameter Monodromy. Anyonic statistics may be understood as a special case of the more general notion of *topological adiabatic holonomy/monodromy* ([ZR99; PZR99; PZ01], cf. [MSS24, §3; SS25b, §II.B]). Consider a family of gapped many-body ground-state spaces $\mathcal{H}_p$, depending on parameters $p \in \mathcal{P}$. When the topological part of adiabatic transport depends only on the homotopy class of the parameter path, these spaces form a *local system*, equivalently a flat unitary vector bundle over $\mathcal{P}$. Its holonomy on each connected component is a representation of the *fundamental group* of continuous deformation paths of the system's parameters in topological phase $C \in \pi_0(\mathcal{P})$:

$$\widehat{(-)} \;:\; \pi_1(\mathcal{P}, C) \longrightarrow \mathrm{U}(\mathcal{H}_C)\,. \tag{11}$$

In the familiar case where the parameters are configurations of N quasiparticles on a surface Σ,

$$\mathcal{P}_{\mathrm{trd}} \equiv \left\{(x_1, \cdots, x_N) \in \Sigma^N \,\middle|\, \forall_{i\neq j} : x_i \neq x_j\right\}, \quad \Rightarrow \quad \pi_1(\mathcal{P}_{\mathrm{trd}}) \simeq \mathrm{PBr}(\Sigma)\,, \tag{12}$$

the fundamental group (11) is the surface's (pure) *braid group*, whose unitary representations are the familiar anyon representations. But parameter monodromy describes topological order more generally.

In particular, a crystalline system's external parameters are instead hopping amplitudes or crystal potentials, encoded by single-electron *Bloch Hamiltonians* H, and topologically by their *spectral flattenings* $\widehat{H}$, which are continuous maps $T^d \longrightarrow \mathrm{Gr}_v(\mathbb{C}^N)$:

(i) from the crystal's *Brillouin torus* T^d of crystal quasi-momenta,

(ii) to the *Grassmannian space* $\mathrm{Gr}_v(\mathbb{C}^N)$ of $v \in \mathbb{N}$ valence states among N available bands,

and hence form the d-fold free loop space L^d of the Grassmannian:

$$\mathcal{P}_{\mathrm{crs}} \equiv \left\{ \begin{array}{ccc} T^d & \longrightarrow & \mathrm{Gr}_v(\mathbb{C}^N) \\ \mathbf{k} & \longmapsto & \widehat{H}(\mathbf{k}) \end{array} \right\} =: \mathrm{Map}\left(T^d, \mathrm{Gr}_v\left(\mathbb{C}^N\right)\right) \simeq L^d \mathrm{Gr}_v\left(\mathbb{C}^N\right). \tag{13}$$

Hence for pure crystalline systems without further real-space defects, the available parameter monodromy is that of deformations of (spectrally flattened) Bloch Hamiltonians:

$$\pi_1\big(\mathcal{P}_{\mathrm{crs}}, [\widehat{H}]\big) \;=\; \left\{ T^d \xrightarrow{\;\widehat{H}\;} \mathrm{Gr}_v(\mathbb{C}^N) \right\}. \tag{14}$$

For example, in a two-band system,

$$\mathrm{Gr}_1(\mathbb{C}^2) \equiv \mathbb{C}P^1 \simeq S^2 \tag{15}$$

is the Bloch sphere (cf. [Ser23, p. 112]). The observation of [SS26b; SS25b] is that in this case the Bloch monodromy (14) is precisely that expected for abelian anyons on a torus (here the momentum-space Brillouin torus):

Proposition 2.1 ([SS26b, Thm. 1; KSS26, Thm. 3.5], following [Han74; LT80; Kal01])**.** *For 2-dimensional crystalline 2-band systems, with Bloch parameter space given by* (13) *and* (15)*,*

$$\mathcal{P}_{\mathrm{crs}} \;=\; LLS^2 \tag{16}$$

(i) *the topological phases are labeled by the integers* (Chern numbers, as usual)*:*

$$\pi_0(LLS^2) \;\simeq\; \mathbb{Z}, \tag{17}$$

(ii) *the topological monodromy groups are* integer Heisenberg groups*:*[3]

$$\pi_1\big(LLS^2,\, C\big) \simeq \left\langle \begin{array}{c} W_a, \\ W_b, \\ \zeta \end{array} \middle| \begin{array}{rcl} [W_a, W_b] & = & \zeta^2 \\ [W_a, \zeta] & = & 1 \\ [W_b, \zeta] & = & 1 \\ \zeta^{2C} & = & 1 \end{array} \right\rangle. \tag{18}$$

Here:

(i) W_a, W_b describe transport around a basis of torus 1-cycles, cf. Fig. 3;

(ii) ζ generates the phase picked up by an elementary braiding process taking place locally, in a chart $\mathbb{R}^2 \hookrightarrow T^2$ of the torus, mathematically reflected [SS26a; SS25c] as a Bloch *Hopfion* [WZ83]:

$$\pi_1 \mathrm{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \simeq \pi_3(S^2) \simeq \mathbb{Z}\langle\zeta\rangle. \tag{19}$$

Hence in a superselection sector, being an irreducible unitary representation $\widehat{(-)}$ (11) of the monodromy group (18) on the system's Hilbert space of gapped ground states, we have

$$\begin{aligned} \widehat{\zeta} &= e^{i\theta}\mathbf{1} \\ \widehat{W}_a \widehat{W}_b &= e^{2i\theta} \widehat{W}_b \widehat{W}_a, \end{aligned} \tag{20}$$

[3] For a pair of group elements $g_1, g_2 \in G$, their *group commutator* is $[g_1, g_2] := g_1\, g_2\, g_1^{-1}\, g_2^{-1} \in G$.

for $e^{i\theta}$ a root of unity (assuming a finite-dimensional space of gapped ground states, cf. [SS25a, Lem. 3.32]). Remarkably, this is exactly the torus Wilson-loop algebra expected for abelian anyons in FQH systems ([WN90, (4.9); Ien92, (4.14)], reviewed in [Fra13, (4.21); Ton16, (5.28)], cf. Fig. 3) but here appearing over the Brillouin torus of crystal momenta.

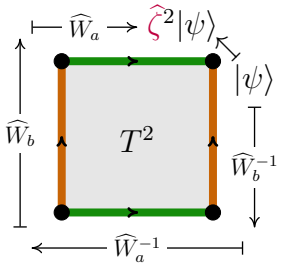


Figure 3: The relations (18) and (20) say that an adiabatic parameter loop of Bloch Hopfions around a basis of Brillouin torus cycles (cf. Fig. 1) induces transformation of the topological states by multiplication with the square of a complex phase ζ, a root of unity, to be identified [SS25a, Prop. 3.21] with an anyon braiding phase.

General Covariance and Modular Data. In more detail, topological states, subject to the laws of topological field theory, are *generally covariant* [Wit88], in that they must see diffeomorphisms of space as gauge equivalences. Here this means [SS25b, §II.C; SS25a, §2.2] that the actual parameter space for crystalline topological phases is not quite the space $\mathcal{P}_{\mathrm{crs}}$ of Bloch Hamiltonians (13) itself, but its homotopy quotient $(-)/\!\!/(-)$ (Borel construction) by the precomposition action with the (orientation-preserving) diffeomorphism group $\mathrm{Diff}^+{}_{(-)}$:

$$\mathcal{P}^{\mathrm{cov}}_{\mathrm{crs}} \;:=\; \mathrm{Map}\big(T^d, \mathrm{Gr}_v(\mathbb{C}^N)\big)/\!\!/\mathrm{Diff}^+(T^d)\,. \tag{21}$$

Recall that the connected components of the diffeomorphism group of the 2-torus

$$\pi_0\,\mathrm{Diff}^+(T^2) \;\simeq\; \mathrm{SL}(2,\mathbb{Z}) \;\simeq\; \langle S, T\rangle \tag{22}$$

form the classical *modular group*, generated by the familiar matrices (cf. [SS25a, Ex. 2.19])

$$S \;=\; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T \;=\; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{23}$$

Proposition 2.2 ([SS25a, Prop. 3.20])**.** *For even $C \in 2\mathbb{Z}$, the isomorphism in (18) may be chosen so that the diffeomorphism action (21) on $LLS^2 = \mathrm{Map}(T^2, S^2)$ gives the semidirect product*

$$\pi_1\big(\mathcal{P}^{\mathrm{cov}}_{\mathrm{crs}}, C\big) \;\equiv\; \pi_1\big((LLS^2)/\!\!/\mathrm{Diff}^+(T^2), C\big) \;\simeq\; \left\langle W_a, W_b, \zeta \;\middle|\; \begin{aligned} [W_a, W_b] &= \zeta^2 \\ [W_a, \zeta] &= 1 \\ [W_b, \zeta] &= 1 \\ \zeta^{2C} &= 1 \\ W_a^C &= 1 \\ \zeta^{C^2} W_b^C &= 1 \end{aligned} \right\rangle \rtimes \mathrm{SL}(2,\mathbb{Z})\,, \tag{24}$$

with the modular group element $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ (22) *acting on these generators as*

$$\begin{array}{ccc} \pi_1(LLS^2, C) & \longrightarrow & \pi_1(LLS^2, C) \\ W_a\zeta^C & \longmapsto & (W_a\zeta^C)^p (W_b)^q \zeta^{-pq} \\ W_b & \longmapsto & (W_a\zeta^C)^r (W_b)^s \zeta^{-rs} \\ \zeta & \longmapsto & \zeta\,. \end{array} \tag{25}$$

These results Props. 2.1 and 2.2 predict the possibility of abelian anyons localized in FCI momentum spaces, in the form of Hopf solitons in the Bloch Hamiltonian. In fact, they already predict the possibility of nonabelian topological order, reflecting modular transformations $S, T \in \mathrm{SL}(2,\mathbb{Z})$ (24), but the anyon braiding phase ζ itself remains central, hence abelian (by Prop. 2.2 above).

However, this derivation is subject to a mathematical idealization, in that the above classifies topological monodromy (14) under arbitrary excursions of system deformations in the mathematical space of Bloch Hamiltonians. Realistic systems will instead explore only a bounded subspace of all theoretically available deformations.

Our key result in § 3 is that considering a natural quantification of such a bound actually refines and strengthens the prediction of Bloch Hopfion anyons.

3 Results

For a spectrally flattened Bloch Hamiltonian (13) of a 2-band system (15),

$$\widehat{H}(\mathbf{k}) \;=\; \mathbf{n}(\mathbf{k}) \cdot \boldsymbol{\sigma} \quad \text{with} \quad \mathbf{n}(\mathbf{k}) \in S^2 \subset \mathbb{R}^3\,, \tag{26}$$

the quantum metric (2) is (cf. [Pié+16, (8)])

$$g_{ij} \;=\; \tfrac{1}{4}\, \partial_{k_i}\mathbf{n} \cdot \partial_{k_j}\mathbf{n}\,,$$

hence its x-average (4) at a given k_y is

$$\overline{g}_{xx}(k_y) \;=\; \tfrac{a_x^2}{16\pi^2} \int_0^1 \left|\partial_s \mathbf{n}\big(\tfrac{2\pi s}{a_x}, k_y\big)\right|^2 \, \mathrm{d}s\ . \tag{27}$$

For an averaged metric bound

$$M \;\in\; \mathbb{R}_{>0} \tag{28}$$

write

$$L_{\leq M} S^2 \;:=\; \left\{ \gamma \,\middle|\, \tfrac{a_x^2}{16\pi^2} \textstyle\int_0^1 |\partial_s \gamma|^2 \, \mathrm{d}s \;\;\leq\;\; M \right\} \subset W^{1,2}(S^1, S^2) \subset L S^2 \tag{29}$$

for the subspace of the free loop space on the absolutely continuous loops for which the Dirichlet functional (27) is well-defined and bounded by M.

Proposition 3.1. *The first critical value of M (28) at which the homotopy type of $L_{\leq M} S^2$ (29) changes is*

$$M_1 \;:=\; \big(\tfrac{a_x}{2}\big)^2\,. \tag{30}$$

Proof. This follows from classical results of *Morse–Bott theory*, recalled in [GSS26, §3.2]. □

Remark 3.2. This bound (30) says that the root-mean-square width of the corresponding hybrid Wannier state (4) in the x-direction is, by (5), at most half the crystal cell's x-width. Hence with this bound, the Wannier states are localized just on the scale of a lattice cell, in the x-direction.

Definition 3.3 (The configuration space of x-bounded Bloch Hamiltonians)**.** We abbreviate the bounded loop space (29) at the critical bound (30) of the averaged quantum metric by

$$L_1 S^2 \;:=\; L_{\leq M_1} S^2 \tag{31}$$

and call this the *small loop space* of S^2 (following [GSS26, §3.2]). This means that the corresponding subspace

$$L L_1 S^2 \;\subset\; L L S^2 \;\simeq\; \mathrm{Map}(T^2, S^2) \tag{32}$$

of the space of 2D 2-band Bloch Hamiltonians (16) is that of those Hamiltonians whose x-averaged quantum metric is bounded by (30).

Dependence of topological quantum states on this bounded parameter space may support different topological order, and this is what we will classify now.

First, we have the following simplified description of (31), which the impatient reader may regard as a more explicit definition:

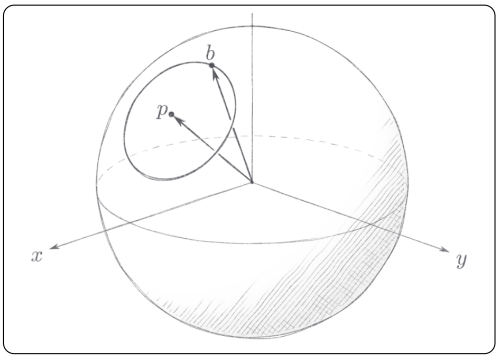


Figure 4: The space $L_1S^2 \subset LS^2$ (31) of *small loops* in the (Bloch) sphere is homotopy equivalent (by Lem. 3.4) to its further subspace of loops that trace out round circles on S^2, once at a constant rate.
These in turn are equivalently determined by their base point $b \in S^2$ and by the oriented rotation axis $p \in S^2 \subset \mathbb{R}^3$ that they are encircling clockwise.
Here constant loops are included, regarded as circles of vanishing radius that as such may be regarded as oriented either by the axis $p = b$ or $p = -b$.

Lemma 3.4. *The space L_1S^2 (31) of "small loops" in S^2:*

(i) *is homotopy equivalent more specifically to the space of loops that traverse (once, at a constant rate) round circles embedded in S^2, including degenerate circles corresponding to constant loops,*

(ii) *as such may be coordinatized by pairs of points $(b, p) \in S^2 \times S^2$ indicating the loop's basepoint b and the oriented axis p encircled clockwise by their round-circle trajectory, subject only to the relation $\forall_{b\in S^2} : (b, b) \sim (b, -b)$, reflecting that a degenerate circle constant on a point b is oriented either way* (cf. Fig. 4)*:*

$$\begin{array}{ccc} \frac{S^2 \times S^2}{\forall_b:(b,b)\sim(b,-b)} & \xrightarrow{\quad\sim\quad} & L_1S^2 \\ [b, p] & \longmapsto & \begin{array}{c}\text{loop } \gamma : S^1 \to S^2 \text{ starting at } \gamma(0) = b \\ \text{going once clockwise at a constant rate} \\ \text{through the round circle around } p.\end{array} \end{array} \tag{33}$$

Proof. This follows by combining classical results in Morse-Bott theory given in [Kli78, §2.3–2.5] (using Thm. 2.4.10, Cor. 2.4.11 & Prop. 2.5.3 there), in direct analogy with the description of L_1S^4 that appears as [GSS26, Lem. 3.4] (cf. the proof of Prop. 3.12 there). □

3.1 The FQH Layers

Our central observation is the following variant of Prop. 2.1:

Theorem 3.5 (Fundamental groups of LL_1S^2)**.** *The parameter space (32) of x-bounded Bloch Hamiltonians has:*

(i) *connected components still indexed by the Chern number $C \in \mathbb{Z}$:*

$$\pi_0\big(LL_1S^2\big) \simeq \mathbb{Z}\,, \tag{34}$$

(ii) *fundamental groups*

(a) *for* $C = 0$*:*

$$\pi_1\big(LL_1S^2, 0\big) \cong \left\langle \begin{matrix} W_a, \\ W_b, \\ \{\zeta_i\}_{i\in\mathbb{Z}} \end{matrix} \middle| \begin{matrix} [W_a, W_b] &=& \zeta_0^2, \\ [W_a, \zeta_i] &=& \zeta_{i+1}\zeta_i^{-1}, \\ [W_b, \zeta_i] &=& 1, \\ [\zeta_i, \zeta_j] &=& 1 \end{matrix} \text{ for } i,j\in\mathbb{Z} \right\rangle, \tag{35}$$

(b) *for* $C \neq 0$*:*

$$\pi_1\big(LL_1S^2, C\big) \cong \left\langle \begin{matrix} W_a, \\ W_b, \\ \zeta_0\,, \\ \vdots\ , \\ \zeta_{|C|-1} \end{matrix} \middle| \begin{matrix} [W_a, W_b] &=& \zeta_0^2, \\ [W_a, \zeta_i] &=& \zeta_{i+1}\zeta_i^{-1}, \\ [W_b, \zeta_i] &=& 1, \\ [\zeta_i, \zeta_j] &=& 1, \\ \zeta_0^2\zeta_1^2\cdots\zeta_{|C|-1}^2 &=& 1 \end{matrix} \text{ for } i,j\in\mathbb{Z}_{/C} \right\rangle. \tag{36}$$

Proof outline[4]. With Lem. 3.4 we are reduced to computing the fundamental groups of LQ for (33)

$$Q := (S^2\times S^2)_{/\sim}\,. \tag{37}$$

Since Q is the quotient of a connected space, it is connected:

$$\pi_0(Q) \simeq *\,. \tag{38}$$

With the generalized van Kampen theorem (cf. [Geo08, § 6.2, Case 2 and Prop. 6.2.2]), one finds

$$\pi_1(Q) \simeq \mathbb{Z}\langle w_a\rangle\,, \tag{39}$$

where we name group generators in angle brackets. A model for the universal cover of Q consists of an infinite chain of copies of $S^2\times S^2$, with successive copies glued along their diagonal and antidiagonal spheres. Mayer–Vietoris and the Hurewicz theorem then identify

$$\pi_2(Q) \cong \begin{matrix} \mathbb{Z}\langle w_b\rangle\oplus \\ \bigoplus_{i\in\mathbb{Z}}\mathbb{Z}\langle\, z_i\,\rangle, \end{matrix} \tag{40}$$

where under the canonical action $\rho_{(-)}(-) : \pi_1(Q)\times\pi_2(Q)\longrightarrow\pi_2(Q)$ we have

$$\rho_{w_a} : \begin{cases} w_b &\mapsto& w_b &+& 2z_0, \\ z_i &\mapsto& & & z_{i+1}\,. \end{cases} \tag{41}$$

The inclusion $LQ\subset \mathrm{Map}\big(T^2, S^2\big)$ is clearly a bijection on π_0, so that the components of LQ are labeled by winding number $C\in\mathbb{Z}$, as claimed. In component C, the homotopy exact sequence of the free-loop evaluation fibration has boundary map $\delta = 1-\rho_{w_a}^C$, yielding a split exact sequence of this form:

$$1 \longrightarrow \frac{\pi_2(Q)}{(\rho_{w_a}^C - 1)\pi_2(Q)} \xhookrightarrow{\ \iota_C\ } \pi_1\big((LQ)_C\big) \xrightarrow{\ \mathrm{ev}_*\ } \mathbb{Z}\langle w_a\rangle \longrightarrow 1\,, \tag{42}$$

$$\pi_1\big((LQ)_C\big) \hookleftarrow W_a \mapsto w_a \quad \mathbb{Z}\langle w_a\rangle$$

with

$$W_a\,\iota_C[-]\,W_a^{-1} = \iota_C\big[\rho_{w_a}(-)\big]\,. \tag{43}$$

[4] A `Lean-4` verification certificate of Thm. 3.5 is available at `ncatlab.org/schreiber/show/IIA+Anyons`.

This implies that our group is the semidirect product

$$\begin{array}{ccc} \pi_1\big((LQ)_C\big) & \xleftrightarrow{\ \sim\ } & \frac{\pi_2(Q)}{(\rho^C_{w_a}-1)\pi_2(Q)} \rtimes_\rho \mathbb{Z}\langle w_a\rangle \\ W_a & \longleftrightarrow & w_a \\ W_b & \longleftrightarrow & w_b \\ \zeta_i & \longleftrightarrow & z_i\,, \end{array} \tag{44}$$

where (41) gives

$$\begin{aligned} W_a\, W_b\, W_a^{-1} &= \zeta_0^2 W_b \\ W_a\, \zeta_i\, W_a^{-1} &= \zeta_{i+1} \end{aligned} \tag{45}$$

while (40) gives

$$W_b\, \zeta_i\, W_b^{-1} = \zeta_i. \tag{46}$$

With this, the quotient on the left of (42) enforces, for $C > 0$,

$$\begin{aligned} & W_a^C(W_b)W_a^{-C} && = W_b \\ \Leftrightarrow\ & W_a^{C-1}\big(\zeta_0^2\, W_b\big)W_a^{-C+1} && = W_b \\ \Leftrightarrow\ & W_a^{C-2}\big(\zeta_1^2\zeta_0^2\, W_b\big)W_a^{-C+2} && = W_b \\ \Leftrightarrow\ & && \vdots \\ \Leftrightarrow\ & \zeta_{C-1}^2\cdots\zeta_1^2\zeta_0^2\, W_b && = W_b \\ \Leftrightarrow\ & \zeta_{C-1}^2\cdots\zeta_1^2\zeta_0^2 && = 1\,, \end{aligned} \tag{47}$$

and

$$\begin{aligned} & W_a^C(\zeta_i)W_a^{-C} && = \zeta_i \\ \Leftrightarrow\ & W_a^{C-1}(\zeta_{i+1})W_a^{-C+1} && = \zeta_i \\ \Leftrightarrow\ & W_a^{C-2}(\zeta_{i+2})W_a^{-C+2} && = \zeta_i \\ \Leftrightarrow\ & && \vdots \\ \Leftrightarrow\ & \zeta_{i+C} && = \zeta_i\,. \end{aligned} \tag{48}$$

Equivalent relations are implied for $C < 0$, while for $C = 0$ the quotient imposes no relation.

In summary, these are equivalently the claimed generators and relations. □

Example 3.6. For Chern number $|C| = 2$, the group (36) is equivalently presented as

$$\pi_1\big(LL_1S^2, 2\big) \simeq \left\langle \begin{array}{l|lll|l} W_a, & W_a\, W_a\, W_a^{-1} = W_a\,, & W_b\, W_a\, W_b^{-1} = \zeta_0^{-2} W_a\,, & \\ W_b, & W_a\, W_b\, W_a^{-1} = \zeta_0^2\, W_b\,, & W_b\, W_b\, W_b^{-1} = W_b\,, & \\ \zeta_0, & W_a\ \zeta_0\ W_a^{-1} = \zeta_1\,, & W_b\ \zeta_0\ W_b^{-1} = \zeta_0\,, & \zeta_0\zeta_1 = \zeta_1\zeta_0 \\ \zeta_1 & W_a\ \zeta_1\ W_a^{-1} = \zeta_0\,, & W_b\ \zeta_1\ W_b^{-1} = \zeta_1\,, & \zeta_1^2 = \zeta_0^{-2} \end{array} \right\rangle. \tag{49}$$

Proposition 3.7 (Comparison of monodromy as the quantum metric bound is lifted)**.**

(i) *The defining inclusion* (32)

$$LL_1S^2 \xhookrightarrow{\ \iota\ } LLS^2 \tag{50}$$

induces, for each degree $C \in \mathbb{Z}$, a surjective homomorphism of fundamental groups,

which in terms of the generators of Prop. 2.1 item **(ii)** *and* Thm. 3.5 item **(ii)** *is:*

$$\begin{array}{ccc} \pi_1\big(LL_1S^2, C\big) & \xrightarrow{\iota_*} & \pi_1\big(LLS^2, C\big) \\ W_a & \longmapsto & W_a, \\ W_b & \longmapsto & W_b, \\ \zeta_i & \longmapsto & \zeta\,. \end{array} \tag{51}$$

Equivalently, this is quotienting by the relation

$$\forall_{i,j} \,:\, \zeta_i \sim \zeta_j\,. \tag{52}$$

(ii) *The kernels of the homomorphisms* (51) *are free abelian:*

$$\ker(\iota_*) \cong \begin{cases} \bigoplus_{\mathbb{Z}-\{0\}} \mathbb{Z} & \text{if } C = 0, \\ \bigoplus_{|C|-1} \mathbb{Z} & \text{if } C \neq 0, \end{cases} \tag{53}$$

with a free abelian basis given by

$$\begin{cases} \left\{\zeta_i\zeta_0^{-1}\right\}_{i\in\mathbb{Z}-\{0\}} & \text{if } C = 0, \\ \left\{\zeta_i\zeta_0^{-1}\right\}_{1\leq i\leq |C|-1} & \text{if } C \neq 0. \end{cases} \tag{54}$$

(iii) *In particular, ι_* is an isomorphism precisely for $|C| = 1$.*

Proposition 3.8. *Under the restriction to bounded quantum geometry* (50)*, the modular group* $\mathrm{SL}(2,\mathbb{Z})$*-action* (25) *breaks to the* shear subgroup *which preserves the "oriented small direction":*

$$\mathbb{Z} \simeq \langle T\rangle \simeq \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z})\,, \tag{55}$$

whose generator $T \in \mathrm{SL}(2,\mathbb{Z})$ (23) *acts on the generators from Thm. 3.5 as follows:*

$$\begin{array}{ccc} \pi_1\big(LL_1S^2, C\big) & \longrightarrow & \pi_1\big(LL_1S^2, C\big) \\ W_a & \longmapsto & W_aW_b\zeta_{-1}^{-1} \\ W_b & \longmapsto & W_b \\ \zeta_i & \longmapsto & \zeta_i\,. \end{array} \tag{56}$$

3.2 The Topological Gates

For a given parameter monodromy, the possible superselection sectors of topological orders are the corresponding irreducible unitary representations, whose operators are the predicted possible topologically protected quantum gates. We discuss the general classification and some interesting examples in our case (§ 3.1), cf. the comments in Rem. 4.1 at the end.

Classification for bounded quantum metric FCIs.

Proposition 3.9 (Irrep classification)**.** *For $C \neq 0$, the finite-dimensional irreducible unitary representations of $\pi_1\big(LL_1S^2, C\big)$* (36) *are, up to unitary isomorphism, labeled by:*

(i) *orbits $[\chi]$ under the automorphism ρ_{w_a}* (41) *of unitary characters*

$$\begin{array}{ccccc} \left\langle W_b, \{\zeta_i\}_{i=1}^{|C|}\right\rangle & \xrightarrow{\chi} & \mathrm{U}(1) & & \\ W_b & \longmapsto & \phi_b & & \\ \zeta_i & \longmapsto & \phi_i, & \big(\phi_1\cdots\phi_{|C|}\big)^2 = 1\,, & \end{array} \tag{57}$$

(ii) *further phases* $\lambda \in \mathrm{U}(1)$,

and are given on the d-dimensional Hilbert space $\mathcal{H} := \bigoplus_{j=0}^{d-1} \mathbb{C}|j\rangle$, *for*

$$\begin{aligned} d &:= \min\Big\{ r \geq 1 \,\Big|\, \chi \circ \rho_{w_a}^r = \chi \Big\} \\ &= \min\Big\{ r \geq 1 \,\Big|\, \forall_i : \phi_{i+r} = \phi_i\,,\ \Big(\textstyle\prod_{s=0}^{r-1} \phi_s\Big)^2 = 1 \Big\} \end{aligned} \tag{58}$$

the length of the orbit (a divisor of $|C|$), *by:*

$$\begin{aligned} \pi_1\big(LL_1S^2, C\big) &\xrightarrow{V_{[\chi],\lambda}} \mathrm{U}(\mathcal{H}) \\ W_a &\longmapsto \left(\widehat{W}_a \,:\, |j\rangle \mapsto \begin{cases} |j+1\rangle & \text{if } 0 \leq j < d-1 \\ \lambda|0\rangle & \text{if } j = d-1 \end{cases} \right) \\ W_b &\longmapsto \Big(\widehat{W}_b \,:\, |j\rangle \mapsto \phi_b\big(\textstyle\prod_{r=1}^{j} \phi_{-r}^{-2}\big)|j\rangle \Big) \\ \zeta_i &\longmapsto \Big(\widehat{\zeta}_i \,:\, |j\rangle \mapsto \phi_{i-j}\,|j\rangle \Big)\,. \end{aligned} \tag{59}$$

Proof. By (44) our group is a semidirect product of the abelian group on the left of (57). Therefore, Mackey theory implies that its irreps are characterized by orbits of its characters, as claimed here (cf. [CR62, Prop. 11.8; Ser77, §8.2]). □

Remark 3.10 (Cyclic Monodromy and Effective Quantum Hall Layers). The irrep classification Prop. 3.9 now allows us to return to the effective quantum Hall layers of Qi and Barkeshli–Qi [Qi11; BQ12] surveyed in § 2.1: Recall that the Wannier spectral flow (7) organizes the hybrid Wannier orbitals into $|C|$ families: Continuation around the transverse momentum cycle connects orbitals within each family, while translation by one lattice spacing in the x-direction cyclically permutes the families (9).

Our monodromy representations exhibit just this structure. The covariance relation

$$\widehat{W}_a\, \widehat{\zeta}_i\, \widehat{W}_a^{-1} = \widehat{\zeta}_{i+1}, \qquad i \in \mathbb{Z}_{|C|}, \tag{60}$$

is realized in (59) by cyclic transport between character sectors, with

$$\widehat{\zeta}_i\,|j\rangle = \phi_{i-j}|j\rangle, \qquad j = 0, \ldots, d-1. \tag{61}$$

Thus the braiding phase depends on the relative cyclic position of the observable and the character sector.

Proposition 3.11 (Shear-Equivariant Extension of Irreps)**.**

(i) *An irrep* $V_{[\chi],\lambda}$ (59), from Prop. 3.9, *extends to a* (necessarily also irreducible) *unitary representation* $V_{[\chi],\lambda,\widehat{T}}$ *of the semidirect product with the shear group* (Prop. 3.8)

$$\begin{array}{ccc} \pi_1\big(LL_1S^2, C\big) & \xrightarrow{\quad V_{[\chi],\lambda} \quad} & \mathrm{U}(\mathcal{H}) \\ \big\downarrow & \overset{V_{[\chi],\lambda,\widehat{T}}}{\nearrow} & \\ \pi_1\big(LL_1S^2, C\big) \rtimes \langle T \rangle & & \end{array} \tag{62}$$

if and only if

$$\det\big(\widehat{W}_b\big) = \det\big(\widehat{\zeta}_0\big)\,, \tag{63}$$

which, in terms of the character parameters (57), *is equivalent to*

$$\phi_b^d = \textstyle\prod_{r=1}^{d} \phi_{-r}^{2(d-r)+1}, \tag{64}$$

where the subscripts on ϕ *are understood modulo* C.

(ii) *Whenever the conditions* (63) *and* (64) *hold, the unitary equivalence classes of extensions form a* U(1)*-torsor: given one extension* $V_{[\chi],\lambda,\widehat{T}}$*, any other is uniquely obtained by multiplying the operator* $\widehat{T}$ *representing* T *by an element in* U(1)*.*

Proof.

(i) By (56), an extension amounts to a choice of a unitary operator $\widehat{T}$ such that

$$\begin{aligned} \widehat{T}\,\widehat{W}_a\,\widehat{T}^{-1} &= \widehat{W}_a\,\widehat{W}_b\,\widehat{\zeta}_{-1}^{-1}, \\ \widehat{T}\,\widehat{W}_b\,\widehat{T}^{-1} &= \widehat{W}_b, \\ \widehat{T}\widehat{\zeta}_i\widehat{T}^{-1} &= \widehat{\zeta}_i \qquad \text{for all } i. \end{aligned} \tag{65}$$

Taking determinants in the first relation gives

$$\det\left(\widehat{W}_b\right) = \det(\widehat{\zeta}_{-1})\,.$$

Since $\widehat{\zeta}_{-1}$ and $\widehat{\zeta}_0$ are conjugate by $\widehat{W}_a$, this proves necessity of (63).

Conversely, suppose that (63) holds. In the character basis of (59), write

$$\widehat{W}_b|j\rangle = b_j|j\rangle, \qquad b_j := \phi_b\textstyle\prod_{r=1}^{j}\phi_{-r}^{-2}, \qquad 0 \le j < d.$$

Now define unit complex numbers u_j recursively by

$$u_0 = 1, \qquad u_{j+1} = u_j\, b_j\, \phi_{-1-j}^{-1}, \qquad 0 \le j < d. \tag{66}$$

Their product around the cyclic basis satisfies

$$\frac{u_d}{u_0} = \prod_{j=0}^{d-1} b_j\phi_{-1-j}^{-1} = \frac{\det\left(\widehat{W}_b\right)}{\det\left(\widehat{\zeta}_{-1}\right)} = 1.$$

Thus $u_d = u_0$, and the recursion closes consistently. Set

$$\begin{aligned} \mathcal{H} &\xrightarrow{\ \widehat{T}\ } \mathcal{H} \\ |j\rangle &\longmapsto u_j|j\rangle. \end{aligned} \tag{67}$$

This operator is unitary and commutes with $\widehat{W}_b$ and every $\widehat{\zeta}_i$, since these operators are diagonal in the same basis. The recursion (66) gives the first relation in (65). This also holds at the wrap from $|d-1\rangle$ to $|0\rangle$: the factor λ in $\widehat{W}_a$ occurs on both sides and cancels. Hence λ imposes no additional condition.

Since the shear subgroup $\langle T\rangle \simeq \mathbb{Z}$ is infinite cyclic, there is no further relation to impose on $\widehat{T}$. The extension is therefore given by

$$V_{[\chi],\lambda,\widehat{T}}(g, T^n) := V_{[\chi],\lambda}(g)\widehat{T}^n\,.$$

Expanding the product in the recursion (66) gives

$$u_d = \phi_b^d\textstyle\prod_{r=1}^{d}\phi_{-r}^{-(2(d-r)+1)}.$$

Consequently $u_d = 1$ is precisely (64).

(ii) If $\widehat{T}'$ is another representation of T, then $\widehat{T}^{-1}\widehat{T}'$ commutes with every $V_{[\chi],\lambda}(g)$. By the assumption that $V_{[\chi],\lambda}$ is irreducible, Schur's lemma implies that $\widehat{T}^{-1}\widehat{T}'$ operates by multiplication with a complex number, and unitarity then implies that this complex number is in U(1). □

Examples of Realizable Quantum Gates. We highlight some familiar quantum gates (cf. [Bar+95; NC00, §4.5]) that appear in the irreps classified by Prop. 3.9, insofar as their topological protection is noteworthy (cf. Rem. 4.1). All of the following examples are chosen such that they are shear-equivariant according to Prop. 3.11.

I. Diagonal gates. The key effect on possible quantum gates of passing from the unbounded parameter monodromy (18) to its bounded counterpart (36) is that the Cth power ζ^C of a single braiding phase—whose constraint $(\zeta^C)^2 = 1$ quantizes that braiding phase to a root of unity—is relaxed to a product $\zeta_1 \cdots \zeta_{|C|}$ of a tuple of phases that are individually unconstrained apart from their global condition $(\zeta_1 \cdots \zeta_{|C|})^2 = 1$ (57).

As a result, all gates represented by diagonal matrices in the computational basis may be realized via (59), possibly with an ancillary space in order to satisfy that global constraint in (57). We highlight some noteworthy examples and record the Chern numbers C needed to realize them.

Example 3.12 (T-Gate Realization)**.** Consider $C = 2$ as in Ex. 3.6. If we take the parameter phases (57) to be:

$$\phi_b = \phi_1 = e^{\mathrm{i}\pi/8}, \qquad \phi_2 = e^{-\mathrm{i}\pi/8}, \qquad \lambda = 1\,, \tag{68}$$

then Prop. 3.9 demonstrates an irrep (topological order) of dimension $d = 2$, given by:

$$\begin{array}{ccc} \pi_1\big(LL_1S^2, 2\big) & \xrightarrow{V_{\chi,1}} & \mathrm{U}(\mathbb{C}^2) \\ W_a & \longmapsto & \widehat{W}_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ W_b,\ \zeta_1 & \longmapsto & \widehat{W}_b = \widehat{\zeta}_1 = \begin{pmatrix} e^{\mathrm{i}\pi/8} & 0 \\ 0 & e^{-\mathrm{i}\pi/8} \end{pmatrix} \\ \zeta_2 = \zeta_0 & \longmapsto & \widehat{\zeta}_2 = \widehat{\zeta}_0 = \widehat{\zeta}_1^{-1}\,. \end{array} \tag{69}$$

In terms of quantum gates we see that:

(i) $\widehat{W}_a = X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the *Pauli X-gate*,

(ii) $\widehat{\zeta}_0 = e^{-\mathrm{i}\pi/8}\, T := e^{-\mathrm{i}\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\pi/4} \end{pmatrix}$ is (up to a global phase) the *T-gate* (cf. [NC00, p. xxx]).

Example 3.13 (Controlled Z-Gate Realization)**.** For $N \in \mathbb{N}_{\geq 2}$ with $C = 2^N$, consider the following character phases in (57):

$$\begin{aligned} \phi_b &= e^{\mathrm{i}\pi/C}, \\ \phi_i &= \begin{cases} -1 & \text{if } i = 1 \pmod{2^N} \\ +1 & \text{otherwise} \end{cases} \\ \lambda &= 1\,. \end{aligned} \tag{70}$$

In this case Prop. 3.9 demonstrates an irrep given by:

$$\begin{array}{ccc} \pi_1\big(LL_1S^2, C\big) & \xrightarrow{V_{\chi,1}} & \mathrm{U}\big(\mathbb{C}^{2^N}\big) \\ W_a & \longmapsto & \Big(\widehat{W}_a \,:\, |j\rangle \longmapsto |j{+}1\rangle\Big) \\ W_b & \longmapsto & \Big(\widehat{W}_b \,:\, |j\rangle \longmapsto e^{\mathrm{i}\pi/C}|j\rangle\Big) \\ \zeta_i & \longmapsto & \left(\widehat{\zeta}_i \,:\, |j\rangle \longmapsto \begin{cases} -|j\rangle & \text{if } i-j \,=\, 1 \pmod{2^N} \\ +|j\rangle & \text{otherwise} \end{cases} \right). \end{array} \tag{71}$$

Identifying the Hilbert space with that of N qubits, by binary expansion:

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes N} & \xrightarrow{\;\simeq\;} & \mathbb{C}^{2^N} \\ |b_1\rangle \otimes \cdots \otimes |b_N\rangle & \longmapsto & |j\rangle\,, \quad j \,=\, \textstyle\sum_{r=1}^N 2^{N-r} b_r, \quad b_r \,\in\, \{0,1\}\,, \end{array} \tag{72}$$

and observing that the binary expansion of $-1 \,=\, 2^N - 1 \pmod{2^N}$ has $\forall_r \,:\, b_r \,=\, 1$, we see that

$$\widehat{\zeta}_0 \,:\, |b_1, \ldots, b_N\rangle \longmapsto (-1)^{b_1 \cdots b_N} |b_1, \ldots, b_N\rangle \tag{73}$$

is the $(N-1)$-*controlled Z gate* $\mathrm{C}^{N-1}\mathrm{Z}$ (cf. [Güh+14, (1); TYC17; Bev+20, §4.1]). In particular:

(i) For $C \,=\, 2^2$, this (73) is the two-qubit *controlled-Z gate*

$$\widehat{\zeta}_0 \,=\, \mathrm{CZ} \,=\, \mathrm{diag}(1,1,1,-1). \tag{74}$$

This is an entangling gate (cf. [Bre+02]), since

$$\mathrm{CZ}\big((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\big) \,=\, |00\rangle + |01\rangle + |10\rangle - |11\rangle \tag{75}$$

is an entangled state.

(ii) For $C \,=\, 2^3$, this (73) is the three-qubit *controlled-controlled Z-gate*

$$\widehat{\zeta}_0 \,=\, \mathrm{CCZ} \,=\, \mathrm{diag}(1,1,1,1,1,1,1,-1). \tag{76}$$

This gate is in addition non-Clifford (cf. [CTV17]): conjugation sends the Pauli operator $X \otimes I \otimes I$ to

$$\mathrm{CCZ}(X \otimes I \otimes I)\mathrm{CCZ}^{-1} \,=\, X \otimes \mathrm{CZ}\,, \tag{77}$$

which is not a Pauli operator.

Example 3.14 (Metaplectic Qutrit-Gate Realization)**.** Similarly to Ex. 3.13 but for $C \,=\, 3$ we may choose

$$\phi_1 \,=\, -1\,, \quad \phi_2 \,=\, \phi_3 \,=\, \lambda \,=\, 1\,, \quad \phi_b \,=\, e^{\mathrm{i}\pi/C} \tag{78}$$

and obtain the *metaplectic qutrit gate* (cf. [Gla+22])

$$\widehat{\zeta}_0 \,=\, \mathrm{diag}(1,1,-1)\,. \tag{79}$$

Example 3.15 (Controlled Phase-Gate Realizations)**.** In a joint generalization of Exs. 3.12 and 3.13, consider $C \,=\, 4$ with

$$\phi_1 \,=\, \varphi\, e^{\mathrm{i}\pi/4}\,, \quad \phi_2 \,=\, \phi_3 \,=\, \phi_4 \,=\, \varphi\,, \quad \phi_b \,=\, \varphi\,, \quad \lambda \,=\, 1\,, \tag{80}$$

for $\varphi \,:=\, e^{-\mathrm{i}\pi/16}$. This yields the *controlled T-gate*

$$\widehat{\zeta}_0 \,=\, \varphi\, \mathrm{diag}\big(1,1,1,e^{\mathrm{i}\pi/4}\big) \tag{81}$$

and hence the *controlled S-gate* (cf. [HH18])

$$\widehat{\zeta}_0^2 \,=\, \varphi^2\, \mathrm{diag}(1,1,1,\mathrm{i}) \tag{82}$$

in addition to the controlled Z-gate $\propto \widehat{\zeta}_0^4$ from (74), up to global phases.

Example 3.16 ($\sqrt{\mathrm{T}}$-Gate Realization)**.** In the same $C = 4$ irrep of Ex. 3.15, the square root of the T-gate (cf. [Bev+20, §A.2]),

$$\sqrt{\mathrm{T}} := \mathrm{diag}\big(1, e^{\mathrm{i}\pi/8}\big)\,, \tag{83}$$

may also be realized, with an ancillary space and up to a global phase, as

$$\widehat{W}_b\,\widehat{\zeta}_0\,\widehat{\zeta}_3^{-1} \;=\; e^{-\mathrm{i}\pi/16}\mathrm{id}_2 \otimes \sqrt{\mathrm{T}}\,. \tag{84}$$

II. Non-diagonal gates. The principal non-diagonal gate realized by the irreps of Prop. 3.9 is the *quantum incrementer*, represented by $\widehat{W}_a$ in (36) whenever $\lambda = 1$. Since the split braiding phases ζ_i do not commute with W_a, we obtain interesting operator conjugations.

Example 3.17 (Realizing Magic State Stabilizers)**.** In the situation of Ex. 3.13 item **(ii)**, observe that in the binary qubit encoding (72) we have $\widehat{W}_a^4|b_1, b_2, b_3\rangle = |b_1 + 1, b_2, b_3\rangle$ and hence

$$\widehat{W}_a^4 \;=\; \mathrm{X}_1 \;:=\; \mathrm{X}\otimes\mathrm{id}\otimes\mathrm{id} \;\in\; \mathrm{U}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)\,. \tag{85}$$

Therefore, with (76), we may realize the operator

$$\widehat{\zeta}_0\,\widehat{W}_a^4\,\widehat{\zeta}_0^{-1} \;=\; \mathrm{CCZ}\,\mathrm{X}_1\,\mathrm{CCZ}^{-1} \;=\; \mathrm{X}\otimes\mathrm{CZ}\,. \tag{86}$$

This quantum gate and its cousins are used for the verification of CCZ magic states (cf. [HH18, §5.2]).

4 Conclusion

We set out to look for new theoretical pathways towards possibly realizing topological quantum gates on crystalline topological insulators, based on the previously possibly underappreciated observation of *holonomic quantum computing*: The theoretically achievable [5] topological orders are the *parameter monodromies* of gapped ground-state Hilbert spaces classified in their superselection sectors by (finite-dimensional unitary) irreps of the fundamental groups of the gapped quantum system's external parameter space. Here we highlighted that, for crystalline insulators, the external parameters are the crystal's potential/binding parameters, which are encoded in the single-electron Bloch Hamiltonians. Hence:

> Possible topological quantum orders in crystalline insulators are represented by unitary representations of the fundamental groups of the space of maps from their Brillouin torus to the Grassmannian space of valence bands among the accessible bands.

We specialized this general situation to the archetypical case of 2D 2-band insulators, where it was recently observed that these fundamental groups, when otherwise unconstrained, are integer Heisenberg groups, just those groups whose representations are the expected quantum states of abelian anyons on a torus, traditionally discussed as such for fractional quantum Hall (FQH) systems with their anyonic quasi-particles localized in real

[5] Throughout, we are concerned here with the theoretically achievable quantum orders, leaving the question of their lattice or experimental realization to be addressed elsewhere.

space. Here the torus is instead the Brillouin torus of crystal momenta, whence the anyons predicted are localized in momentum space, with their braids being Hopf solitons in paths of Bloch Hamiltonian deformations.

Possible topological quantum orders of 2D crystalline insulators with 2 accessible bands and no further constraints are just the momentum space duals of abelian FQH anyons on a torus.

The new aspect studied here is the case where the deformation classes accessible to the topological insulator system are further subject to a bound on the averaged quantum metric along one basis a-cycle (in the k_x-direction) of the Brillouin torus. We observed that this bound equivalently localizes the projected x-position eigenstates of electrons in fractional Chern insulators to the scale of the crystal lattice. Qi & Barkeshli had previously observed that such projected position eigenstate bases exhibit the FCI as a cyclic layer of $|C|$ quantum Hall systems. This raises the question:

How does the cyclic layer of quantum Hall states exhibited by projected position eigenbases in FCIs support an analogously layered anyonic topological order?

Due to the generality of the holonomic parameter monodromy classification, we could address this question by rigorous topological analysis: Theory predicts that this expected layered topological order must, if indeed it exists, be reflected in the irreps of the fundamental groups of the space of 2D 2-band Bloch Hamiltonians subject to the bound on their averaged quantum metric in the x-direction. Our main observation is therefore a mathematical theorem, and it turns out to state that

When subjected to the bound on the x-averaged quantum metric, the single braiding phase $\widehat{\zeta}$ of the momentum-space localized Hopfionic Bloch anyons, subject to the root-of-unity constraint $\widehat{\zeta}^C = \pm I$, splits into a cyclic layer of $|C| \geq 1$ operators $\widehat{\zeta}_i$ subject jointly to the decomposed constraint $\widehat{\zeta}_1 \cdots \widehat{\zeta}_{|C|} = \pm I$, and transforming under lattice translation as expected from Qi-Barkeshli layers.

The key points here are:

(i) The emerging QH-layer structure is suggestive of how this abstractly classified topological order may be identified within the known orbital structure of FCIs,

(ii) the class of topological quantum gates in principle realizable on the quantum metric-bounded topological order is considerably richer than that of ordinary abelian FQH anyons seen without the metric bound, due to:

- **(a)** the appearance of tuples of largely freely tunable braiding phases ζ_i,
- **(b)** their mild non-abelianness, being cyclically permuted by conjugation with x-lattice translation operators W_a.

Remark 4.1.

(i) These two facts imply that the quantum metric-bounded topological order in principle allows the realization of protected quantum gates beyond the notorious restrictions of surface-code based approaches ([CTV17]), such as T-gates.

(ii) As the downside of its higher generality, the theory developed here does not select specific braiding phases among a continuum of possible solutions to the single constraint $\zeta_1 \cdots \zeta_{|C|} = \pm 1$. The question of how specific choices may be stably realized in lattice models or experiment is not addressed here.

(iii) And while mildly non-abelian braiding effects do appear here (cf. Ex. 3.17), the predicted set of topologically protected quantum gates remains far from universal under our assumptions.

In conclusion, we have shown, by revisiting theoretical foundations of topological quantum order in general and of crystalline insulators specifically, that:

> There are interesting previously unrecognized theoretical possibilities for topological quantum gates potentially realizable on 2D 2-band crystalline insulators, compatible with the known cyclically layered quantum Hall orbital structure of fractional Chern insulators.

While their actual realization in lattice models and experiment is beyond the scope of this letter, these theoretical considerations may help guide searches for such realizations.

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