The image of the Burnside ring in the Representation ring for binary Platonic groups

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Abstract

We describe an efficient algorithm that computes, for any finite group G, the linear span of its virtual permutation representations inside all its linear representations, hence the image of the canonical morphism $A(G) \xrightarrow{\beta} R_k(G)$ from the Burnside ring to the representation ring. We use this to determine the image and cokernel of β for binary Platonic groups, hence for finite subgroups of $\mathrm{SU}(2)$, over $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. We find explicitly that for the three exceptional subgroups and for the first seven binary dihedral subgroups, β surjects onto the sub-lattice of the real representation ring spanned by the integer-valued characters. We conjecture that, generally, this holds true for all the binary dihedral groups.

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1 Introduction

Finite group actions are ubiquitous in various areas of mathematics as well as in applications to physics. For G a finite group one may consider linear as well as purely combinatorial *actions* of G on some set (see e.g. [Dr71][tDi79][Ke99][LP12]). We will be concerned with the relation between these two types of actions (see e.g. [tDi79][Be91][Bo10]).

Basics. Traditionally, for k any field the linear actions receive more attention as they are the k-linear *representations* of G, namely the group homomorphisms $G \to \mathrm{Aut}_k(V)$ from G to the k-linear invertible maps from a given k-vector space V to itself. More elementary than this concept is that of plain *G-sets*, which are instead group homomorphism of the form $G \to \mathrm{Aut}(S)$, from G to all the invertible functions from some set S to itself.

To emphasize that these two concepts, while clearly different, are conceptually related, one may appeal to the lore of the "field with one element" $\mathbb{F}_1$ [Ti56][KS][So04][CCM09][Ma08] (see [Th16][Lo18] for recent surveys) and regard plain sets as vector spaces over $\mathbb{F}_1$, and plain set-theoretic permutations as being the $\mathbb{F}_1$-linear maps $\mathrm{Aut}(S) = \mathrm{Aut}_{\mathbb{F}_1}(S)$.

In any case, for every finite group G and field k, the isomorphism classes of finite nonlinear actions and of finite-dimensional linear actions of G form two rings-without-negatives,

$$\Big(G\mathrm{Set}^{\mathrm{fin}}/_{\sim},\ \sqcup, \times\Big) = \Big(G\mathrm{Rep}^{\mathrm{fin}}_{\mathbb{F}_1}/_{\sim},\ \oplus_{\mathbb{F}_1}, \otimes_{\mathbb{F}_1}\Big) \qquad \text{and} \qquad \Big(G\mathrm{Rep}^{\mathrm{fin}}_{k}/_{\sim},\ \oplus_{k}, \otimes_{k}\Big)\,, \tag{1}$$

where addition is given by disjoint union of G-sets and by direct sum of G-representations, respectively, while the product operation is given by Cartesian product of G-sets and by tensor product of G-representations, respectively.

We will be interested in a canonical comparison map from G-sets to G-representation: By forming k-linear combinations of elements of a finite set, every G-set $G \circlearrowright S$ in $A(G)$ spans a k-linear representation

$$\overset{\overset{G}{\curvearrowright}}{k[S]} \qquad g(v) = g\Big(\sum_{s \in S} \underbrace{v_s}_{\in k} \cdot \underbrace{s}_{\in S}\Big) \;:=\; \sum_{h \in G} v_i \cdot g(s)\;. \tag{2}$$

These are called the *permutation representations*. The archetypical example is the *regular representation* $k[G]$, which is the linearization of G acting on its own underlying set by group multiplication from the left. A simple but important example is the trivial 1-dimensional representation $\mathbf{1}$ of G, which is the linearization of the point, regarded as a G-set:

$$\mathbf{1} \simeq k[*]\;. \tag{3}$$

The construction (2) of permutation representations from G-sets is clearly linear and multiplicative, in that it extends to a homomorphism between the above rings-without-negatives (1)

$$\big(G\mathrm{Set}^{\mathrm{fin}}/_{\sim}, \sqcup, \times\big) \xrightarrow{\;\;k[-]\;\;} \big(G\mathrm{Rep}^{\mathrm{fin}}/_{\sim}, \oplus, \otimes\big)\;. \tag{4}$$

While this establishes the canonical comparison map between nonlinear and linear G-actions, there is nothing much of interest to be said about it.

Passage to K-theory. This situation changes drastically as soon as we consider not just plain G-sets and G-representations, but also their "anti-G-sets" and "anti-G-representations", namely as we group-complete the rings-without-negatives in (1) to actual rings, by adjoining additive inverses for all elements. Concretely, a *virtual* G-representation is represented by a pair (V^+, V^-) of two G-representations, thought of as a plain G-representation V^+ and an *anti*-G-representation V^-; and the K-group completion is obtained by quotienting out from the evident group (ring) that these virtual representation/anti-representation pairs form the equivalence relation

$$(V, V) \sim 0 \qquad \text{for all } V \in G\mathrm{Rep}_k^{\mathrm{fin}}/_\sim\,.$$

This can viewed as exhibiting pair-creation/annihilation of bound states of a representation with its own anti-representation. The resulting ring is called the *representation ring* of G, denoted

$$R_k(G) \;:=\; K\left(G\mathrm{Rep}_k^{\mathrm{fin}}/_\sim, \oplus, \otimes\right)\,. \tag{5}$$

An analogous construction applies to virtual G-sets represented by pairs consisting of a G-set and an anti-G-set, subject to the relation of pair-creation/annihilation of bound states of a G-set S with its anti-G-set:

$$(S, S) \sim 0 \qquad \text{for all } S \in G\mathrm{Set}^{\mathrm{fin}}.$$

Now the resulting ring is known as the *Burnside ring* of G (see e.g. [So67][Dr71][tDi79][Ke99]), denoted

$$A(G) \;:=\; K\left(G\mathrm{Set}^{\mathrm{fin}}/_\sim, \sqcup, \times\right)\,. \tag{6}$$

While plain permutation representations (2) generally form just a very small subset of all isomorphism classes of k-linear representations, the passage to *virtual* permutation representations drastically changes the picture: Since every plain permutation representation decays into a direct sum of irreducible linear representations, the *formal difference* of two permutation representations in a virtual permutation representation may partially cancel out to become equal, in the representation ring, to a representation that is not itself a plain permutation representation.

For example, in the simplest non-trivial case, where $G = C_2$ is the finite group of order 2, the 1-dimensional alternating representation $\mathbf{1}_{\mathrm{alt}}$ is clearly not a permutation representation itself. But it is a direct summand in the regular representation $k[C_2] = \mathbf{1} + \mathbf{1}_{\mathrm{alt}}$. Since the other summand is a permutation representation, $\mathbf{1} = k[*]$, the alternating representation may then be isolated as the formal difference $\mathbf{1}_{\mathrm{alt}} = k[C_2] - k[*]$, thus as a virtual permutation representation.

The comparison morphism. Therefore, while the step from plain to *virtual* actions and representations is small, it has drastic consequences, as it potentially reduces much of linear representation theory to pure combinatorics. In order to quantify this effect, one observes that the construction (4) of permutation representations evidently extends linearly to virtual G-sets and virtual G-representations, to a homomorphism

$$\boxed{A(G) \xrightarrow{\beta := k[-]} R_k(G)} \tag{7}$$

from the Burnside ring (6) to the representation ring (5), taking virtual G-sets to virtual G-representations. We may associate to the representation theory of G over k the following interpretation.

Space	**Meaning**
Cokernel of β	Linear algebra invisible to pure combinatorics
Kernel of β	Pure combinatorics invisible to linear algebras

Intuition might suggest that generally the cokernel of β is large, while the kernel of β is generally small. This is indeed the case for the restriction (4) of β to actual (as opposed to virtual) G-sets and G-representations. However, the inclusion of anti-G-sets and anti-representations and passage to the K-groups of virtual G-sets and virtual G-representations completely changes the picture. It turns out that the kernel of β almost never vanishes: in characteristic zero its rank is the difference of the number of non-cyclic by cyclic subgroups of G (Prop. 2.7 below). At the same time, classical results give that the cokernel of β often vanishes: for instance, over $k = \mathbb{Q}$ it vanishes for all cyclic groups (Prop. 2.6 below), as well as for all p-groups ([Seg72], recalled as Prop. 2.8 below), while for $G = S_n$ a symmetric group, the cokernel of β vanishes even for all fields k of characteristic zero (Prop. 2.9 below).

Goal. This is a remarkable state of affairs, which deserves further investigation. Here our modest goal is to give explicit descriptions of the cokernel of β, over the rational, real and complex numbers, for further concrete examples of finite groups G. We are particularly interested in the case of the *binary Platonic groups*, namely the finite subgroups of SU(2) (recalled in Section A.1).

Outline. This paper is organized as follows. In Section 2 we recall what is known about the image of β. In Section 3 we describe an algorithm for the image of β, Theorem 3.28 below. In Section 4 we apply this algorithm and compute the cokernel of β in various concrete examples, Theorem 4.1. In Section A we collect some background material.

Results. The table in Theorem 4.1 shows that in all Examples computed here, notably for the three exceptional finite subgroups of SU(2) as well as the seven first cases of binary dihedral groups, the image of β in the real representation ring consists precisely of the sub-lattice of integer characters, hence (by Prop. 2.3) of non-irrational characters. The same holds true for larger classes of Examples which we have computed, but are not showing here. We conjecture that it holds true for all binary dihedral groups.

An application in string theory. Finally we mention that these results are motivated by and have interesting implications in string theory, which we will discuss in detail elsewhere [SS19]. Briefly: It is a long-standing conjecture [Wit98, 5.1] that the charge lattice of fractional D-branes [DGM97] stuck at G-orientifold singularities is the G-equivariant KO-theory of the singular point, hence the real representation ring $R_{\mathbb{R}}(G)$. However, it was argued already in [BDHKMMS02, 4.5.2] that not all elements of the representation ring can correspond to viable D-brane charges, and a rationale was sought for identifying a smaller sub-lattice. Independently, in [BDS00, (2.8)], [Tay00, Zho01, Rajan02] it was argued that irrational D-brane RR-charge is unphysical. Since for fractional D-branes the RR-charge is rationally proportional to the character of the corresponding representation (by [DGM97, (3.8)], [BCR00, (4.65)][ReSc13, (4.102)]) this means that representations with irrational characters should reflect physically spurious fractional D-brane charges. Precisely these are singled out, by our result, Theorem 4.1, as not being in the image of β, for orbifold singularities indicative of M-theory lifts [Sen97]. This suggests that fractional D-brane charge should be given not by the full representation ring, but by the image of β inside the representation ring, which by [Seg70] means that it should be in the image of equivariant stable Cohomotopy inside equivariant K-theory. This provides further indication of the role of Cohomotopy in M-theory [Sa18, 2.5] [HSS18] [BSS18] [FSS19a] [FSS19b] [FSS19c], as we discuss in [SS19].

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2 The image of β – General facts

In order to set the scene for the considerations below, we record some well-known general facts about the image of the comparison morphism β (7).

Definition 2.1 (Sub-lattice of integer-valued characters)**.** For G a finite group and k a field, write

$$R_k^{\text{int}}(G) \hookrightarrow R_k(G) \tag{8}$$

for the sub-lattice of the representation ring given by those representations $V \in R_k(G)$ whose characters χ_V (Def. 3.19) are integer-valued

$$\chi_V \;:\; g \longmapsto \chi_V(g) \in \mathbb{Z} \subset k\,.$$

Proposition 2.2 (β takes values in integer-valued characters)**.** *The comparison morphism* β (7) *has its image inside the integer-valued characters (Def. 2.1), hence it factors as*

$$\beta \ : \ A(G) \xrightarrow{\beta_k^{\mathrm{int}}} R_k^{\mathrm{int}}(G) \hookrightarrow R_k(G)\ . \tag{9}$$

Proof. By Example 3.20. ☐

The following is elementary, but important:

Proposition 2.3 (e.g. [Naik-CharCycl])**.** *If the ground field k has characteristic zero, then a character (Def. 3.19) that takes values in the rational numbers $\mathbb{Q} \subset k$ in fact already takes values in the integers $\mathbb{Z} \subset \mathbb{Q} \subset k$. Hence if $R_k^{\mathrm{rat}}(G) \subset R_k(G)$ denotes the sublattice of rational-valued characters, in analogy to the sub-lattice of integer-valued characters in Def. 2.1, then these sub-lattices are in fact equal:*

$$R_k^{\mathrm{int}}(G) \xrightarrow{=} R_k^{\mathrm{rat}}(G) \subset R_k(G) \qquad \text{for } \mathbb{Q} \subset k\,.$$

In particular, if the ground field $k = \mathbb{Q}$ is itself the rational numbers, then all characters are integer-valued characters (Def. 2.1), hence in this case the canonical inclusion (8) *is an isomorphism:*

$$R_{\mathbb{Q}}^{\mathrm{int}}(G) \xrightarrow{=} R_{\mathbb{Q}}(G)\ .$$

Proof. In general, characters are cyclotomic integers. Over the rationals the only cyclotomic integers are the actual integers. ☐

Remark 2.4 (Factorizations of the comparison map β)**.** In summary, Prop. 2.2 and Prop. 2.3 say that we have the following commuting diagram of factorizations of the morphism β (7) that sends G-sets to their linear permutation representation:

$$\begin{tikzcd}
A(G) \arrow[dr, "\beta_{\mathbb{Q}}^{\mathrm{int}}"'] \arrow[drrr, "\beta_{\mathbb{R}}^{\mathrm{int}}"] \arrow[drrrrr, "\beta_{\mathbb{C}}^{\mathrm{int}}"] \arrow[dddrrrrrrr, bend left, "\beta_{\mathbb{C}}"] \arrow[dddrrr, bend right, "\beta_{\mathbb{Q}}"'] & & & & & & & \\
& R_{\mathbb{Q}}^{\mathrm{int}}(G) \arrow[rr, hook] \arrow[dr, equal] & & R_{\mathbb{R}}^{\mathrm{int}}(G) \arrow[rr, hook] \arrow[dr, equal] & & R_{\mathbb{C}}^{\mathrm{int}}(G) \arrow[dr, equal] & & \\
& & R_{\mathbb{Q}}^{\mathrm{rat}}(G) \arrow[rr, hook] \arrow[dr, equal] & & R_{\mathbb{R}}^{\mathrm{rat}}(G) \arrow[rr, hook] \arrow[dr, hook] & & R_{\mathbb{C}}^{\mathrm{rat}}(G) \arrow[dr, hook] & \\
& & & R_{\mathbb{Q}}(G) \arrow[rr, hook] & & R_{\mathbb{R}}(G) \arrow[rr, hook] & & R_{\mathbb{C}}(G)
\end{tikzcd}$$

Hence it is worthwhile to first record what is known about the image of β over $\mathbb{Q}$;

Proposition 2.5 (e.g. [tDi09, proof of Prop. 4.5.4])**.** *Over the rational numbers, $k = \mathbb{Q}$, the image of $A(G) \xrightarrow{\beta} R_{\mathbb{Q}}(G)$* (7) *is at least a sub-lattice of full rank (i.e. has the same* number *of generators as $R_{\mathbb{Q}}(G)$). This full-rank sublattice is spanned by the permutation representations* (2) *of the form $\mathbb{Q}[G/C_i]$ for $C_i \subset C_n$ ranging over the cyclic subgroups.*

Proposition 2.6 (e.g. [tDi09, Example (4.4.4)])**.** *For $G = C_n = \mathbb{Z}/n$ a cyclic group and $k = \mathbb{Q}$ the rational numbers, β is an isomorphism*

$$A(C_n) \underset{\simeq}{\xrightarrow{\beta}} R_{\mathbb{Q}}(C_n)\ .$$

Proposition 2.7. *The only finite groups G for which $A(G) \xrightarrow{\beta_k} R_k(G)$ is injective over $k = \mathbb{Q}$ (hence over $k = \mathbb{R}, \mathbb{C}$) are the cyclic groups $G = C_n$.*

Proof. We know that a linear basis for $A(G)$ is given by the cosets G/H for H ranging over conjugacy classes of all subgroups of G, while a linear basis for $R_{\mathbb{Q}}(G)$ is given by the isomorphism classes of irreducible $\mathbb{Q}$-linear representations. But the latter are in bijection to just the *cyclic* subgroups of G (e.g. [tDi09, Prop. 4.5.4]). This means that when G is not itself cyclic, then the cardinality of a linear basis for $A(G)$ is strictly larger than the cardinality of a linear basis for $R_{\mathbb{Q}}(G)$, so that no morphism $A(G) \to R_{\mathbb{Q}}(G)$ can be injective. On the other hand, when G is a cyclic group then β is an isomorphism by Prop. 2.6, and hence in particular injective. ☐

Less immediate is the following result:

Proposition 2.8 ([Seg72])**.** *If the finite group G is a p-group, hence if its number of elements is the nth power p^n of some prime number p by some natural number $n \in \mathbb{N}$*

$$|G| \;=\; p^n\,,$$

then over $k = \mathbb{Q}$ the comparison morphism $A(G) \overset{\beta}{\twoheadrightarrow} R_{\mathbb{Q}}(G)$ (7) *is surjective.*

The standard representation theory of symmetric groups in terms of Young diagrams and Specht modules yields the following statement:

Proposition 2.9 (e.g. [Dre86, Section 3])**.** *Over any ground field k of characteristic zero, and for $G = S_n$ any symmetric group of permutations of $n \in \mathbb{N}$ elements, the comparion map* $A(S_n) \twoheadrightarrow R_k(S_n)$ (7) *is surjective.*

For some other classes of finite groups, formulas for the cokernel and kernel of β are known, see e.g. [BaDo16].

3 An algorithm for the image of β

We describe here an algorithm for computing the image and cokernel of β (7). The end result is Theorem 3.28 below. Establishing the algorithm involves only elementary representation theory (see [Dr71, tDi79, Be91, Ke99, Rob06, Bo10, LP12]) and basic monoidal category theory (see [Mac65, Bo94]) but seems to be new.

Throughout, G is a finite group and k is a field.

Definition 3.1 (Category of G-sets)**.** We write $G\mathrm{Set}^{\mathrm{fin}}$ for the category of finite sets equipped with G-action, called *G-sets*, for short.

This is a symmetric monoidal category ([Bo94, vol 2, 6.1]) with respect to Cartesian product of G-sets, which is given by the plain Cartesian product of underlying sets, equipped with the diagonal G-action. For example, the underlying set of the group G becomes a G-set by the left multiplication action of G on itself. More generally, for $H \subset G$ any subgroup, the set G/H of cosets is still a G-set by the left action of G on itself. The elements of G/H are equivalence classes of elements g of G, often denoted gH, for which we will write

$$[g] := gH\,.$$

Generally, we use square brackets to indicate the equivalence classes or isomorphism classes. In particular we write $[G/H]$ for the isomorphism class of the G-set G/H as an object of $G\mathrm{Set}$.

Definition 3.2 (Category of k-linear G-representations)**.** We write $G\mathrm{Rep}_k^{\mathrm{fin}}$ for the category of finite dimensional k-linear G-representations.

This is a symmetric monoidal category ([Bo94, vol 2, 6.1]) with respect to the standard tensor product of representations, which we denote simply by "$\otimes$".

Definition 3.3 (The trivial irrep)**.** We write

$$\mathbf{1} \simeq k[*] \in G\mathrm{Rep}_k^{\mathrm{fin}}$$

for the trivial 1-dimensional G-representation (3), equivalently the permutation representation (2) of the singleton G-set. This is the *tensor unit* for the tensor monoidal structure on $G\mathrm{Rep}_k$: For $V \in G\mathrm{Rep}_k$ any representation, the hom-space out of $\mathbf{1}$ into V is the vector space V^G of *G-invariants* in V, hence of elements which are fixed by G:

$$V^G \;\simeq\; \mathrm{Hom}(\mathbf{1}, V) \;\in\; \mathrm{Vect}_k\,.$$

Definition 3.4 (The irreducible G-sets)**.** The action of G on a G-set S is called *transitive* if for all pairs of elements $s_1, s_2 \in S$ there exists a group element that takes them into each other: $gs_2 = g_2$.

Every transitive G-set S is isomorphic to a set of cosets G/H, equipped with the canonical G-action induced from the left action of G on itsef, where H is isomorphic to the stabilizer subgroup $\mathrm{Stab}(s) \subset G$ of s in S. Two such G-sets of cosets are isomorphic, $G/H_1 \simeq G/H_2$, precisely if H_1 and H_2 are conjugate to each other, as subgroups of G. If we denote isomorphism classes of G-sets by square brackets, and also denote conjugacy classes of subgroups by square brackets, then this means that

$$[G/H_1] = [G/H_2] \qquad \Longleftrightarrow \qquad [H_1] = [H_2]\,.$$

Consequently, we have the following.

Proposition 3.5 (Canonical linear basis for Burnside ring)**.** *Every finite G-set is a disjoint union of such transitive G-sets (Def. 3.4). Hence the abelian group underlying the Burnside ring is the free abelian group on elements $[G/H]$, one for each conjugacy class $[H]$ of subgroups of H:*

$$\begin{tikzcd} \underset{\substack{[H]\\ H \subset G}}{\oplus} \mathbb{Z}[H] \arrow[r, "\simeq"] & A(G) \\ {[H]} \arrow[r, mapsto] & {[G/H]}\,. \end{tikzcd}$$

We would like to get a handle on the following object:

Definition 3.6 (Multiplicities multiplication table of the Burnside ring)**.** Let $\big\{[H_i]\big\}_i$ be an indexing of the set of conjugacy classes $[H]$ of subgroups $H \subset G$.

- **(i)** The *structure constants* of the Burnside ring $A(G)$ is the set of natural numbers $\{n_{ij}^{\ell}\}$ defined by

$$k[G/H_i] \otimes k[G/H_j] \;\simeq\; \underset{\ell}{\oplus} n_{ij}^{\ell}\, k[G/H_\ell]\,. \tag{10}$$

- **(ii)** The *total multiplicities table* of the Burnside ring $A(G)$ is the quadratic matrix whose (i,j)-entry is

$$M_{ij} \;:=\; \sum_{\ell} n_{ij}^{\ell}\,. \tag{11}$$

Before discussing the crucial role of the total multiplicies (11) for our purpose, (to which we come in Prop. 3.13 below) we first record an efficient way of computing them:

Definition 3.7 (e.g [Pf97, Def. 1.1])**.** Given a finite group G, its *table of marks* is the square matrix m indexed by the conjugacy classes $[H]$ of subgroups $H \subset G$ whose $[H_i], [H_j]$-entry is the number of fixed points of the H_j-action on G/H_i

$$m_{ij} \;:=\; \left| (G/H_i)^{H_j} \right| \;\in\; \mathbb{Z}\,.$$

Proposition 3.8. *There exists a linear ordering $\leq$ of the set of conjugacy classes of subgroups of G which extends the inclusion relation of subgroups, in that*

$$\Big(H_i \subset H_j \Big) \;\Rightarrow\; \Big([H_i] \leq [H_j] \Big)\,.$$

With respect to any such linear ordering, the table of marks m (Def. 3.7) is a lower triangular matrix with positive entries on its diagonal, hence in particular an invertible matrix.

Proof. First of all, inclusion of subgroups defines a partial order and every partial order extends to a linear order. (For our finite ordered set this follows immediately by induction, splitting off a minimal element in each step; more generally see [Mar30].)

Then, observe that H_j having any fixed points on G/H_i means that it is conjugate to a subgroup of the stabilizer group of $[e] \in G/H_i$. But the latter is manifestly H_i itself. Hence

$$\Big(M_{ij} = \left| (G/H_i)^{G_j} \right| \;>\; 0 \Big) \;\Rightarrow\; \Big([H_j] \leq [H_i] \Big)\,,$$

which says that m is lower triangular.

Finally, it is clear that at least $[e] \in G/H_i$ is fixed by H_i, hence that the diaginal entries are positive. ☐

Proposition 3.9. *The Burnside multiplicities m_{ij}^{ℓ} (Def. 3.6) are given by the following algebraic expression in terms of the entries of the table of marks m (Def. 3.7) and its inverse matrix m^{-1} (from Prop. 3.8):*

$$n_{ij}^{\ell} \;=\; \sum_k m_{ik} \cdot m_{jk} \cdot (m^{-1})^{k\ell}\,. \tag{12}$$

Proof. Notice that the entry m_{ij} of the table of marks may equivalently be thought of as the cardinality of the set of homomorphism from the G-set G/H_j to the G-set G/H_i:

$$m_{ij} \;=\; \left| \, \mathrm{Hom}_{G\mathrm{Set}}\big(G/H_j, G/H_i\big) \right| \,.$$

(Because, by transitivity of the action, any such homomorphism is determined by its image of $[e] \in G/H_j$ and by G-equivariance this has to be sent to any H_j-fixed point of G/H_i.)

With this we compute as follows:

$$\begin{aligned} \sum_\ell n_{ij}^{\ell} \cdot m_{\ell k} &= \sum_\ell n_{ij}^{\ell} \cdot \left| \mathrm{Hom}_{G\mathrm{Set}}\big(G/H_k, G/H_\ell\big) \right| \\ &= \left| \mathrm{Hom}_{G\mathrm{Set}}\Big(G/H_k, \sum_\ell n_{ij}^{\ell} \cdot G/H_\ell\Big) \right| \\ &= \left| \mathrm{Hom}_{G\mathrm{Set}}\big(G/H_k,\; G/H_i \times G/H_j\big) \right| \\ &= \left| \mathrm{Hom}_{G\mathrm{Set}}\big(G/H_k,\; G/H_i\big) \right| \cdot \left| \mathrm{Hom}_{G\mathrm{Set}}\big(G/H_k,\; G/H_j\big) \right| \\ &= m_{ik} \cdot m_{jk}\,. \end{aligned}$$

Here in the third step we used the defining equation (10) of the Burnside multiplicities n_{ij}^{ℓ}, and otherwise we used evident properties of sets of homomorphisms.

Now matrix multiplication of both sides of this equation with the inverse matrix m^{-1} yields the claimed relation. ☐

In order to understand the meaning of the total multiplicities (11), we consider now some basic facts, all elementary.

Proposition 3.10 (Self-duality of permutation representations)**.** *If* $\mathrm{char}(k) \neq |H|$*, then the permutation representation* $k[G/H]$ (2) *is a dualizable object in the symmetric monoidal representation category* $(G\mathrm{Rep}_k, \otimes)$ *(e.g. [Bo94, vol 2, 6.1]) and is in fact self-dual:*

$$k[G/H]^* \simeq k[G/H]\,.$$

Proof. We need to find morphisms

$$\mathbf{1} \xrightarrow{\eta} k[G/H] \otimes k[G/H] \qquad \text{and} \qquad k[G/H] \otimes k[G/H] \xrightarrow{\epsilon} \mathbf{1}$$

in $G\mathrm{Rep}_k$ that make the following triangle commutes (the "triangle identity", recalled in Section A.2):

$$\begin{tikzcd}
& k[G/H] \otimes k[G/H] \otimes k[G/H] \arrow[dr, "\mathrm{id}\otimes\epsilon"] & \\
k[G/H] \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & k[G/H]
\end{tikzcd}$$

where we are notationally suppressing the unitors and associators, as usual. With $[g] \in G/H \subset k[G/H]$ denoting both the equivalence class of an element $g \in G$ as well as the corresponding basis element of $k[G/H]$, we claim that the following choice works:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\eta"] & k[G/H] \otimes k[G/H] \\
1 \arrow[r, mapsto] & \frac{1}{|H|}\sum\limits_{g\in G} [g]\otimes[g]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
k[G/H] \otimes k[G/H] \arrow[r, "\epsilon"] & \mathbf{1} \\
{[g_1]\otimes[g_2]} \arrow[r, mapsto] & \left\{ \begin{array}{ccl} 1 & | & [g_1]=[g_2], \\ 0 & | & \text{otherwise.} \end{array} \right.
\end{tikzcd}$$

Here the fraction on the left makes sense by assumption on the characteristic of k. Also, it is immediate that these linear maps do respect the G-action and hence are morphisms in $G\mathrm{Rep}_k$. Because with this, we check for every $[\tilde{g}] \in k[G/H]$ that:

$$\begin{tikzcd}
{[\tilde{g}]} \arrow[r, mapsto, "\eta\otimes\mathrm{id}"] & \frac{1}{|H|}\sum\limits_{g\in G}[g]\otimes[g]\otimes[\tilde{g}] \arrow[r, mapsto, "\mathrm{id}\otimes\epsilon"] & \underbrace{\frac{1}{|H|}\sum\limits_{\substack{g\in G \\ [g]=[\tilde{g}]}}}_{=1}[\tilde{g}] \arrow[r, equal] & {[\tilde{g}]}\,.
\end{tikzcd}$$

□

As a direct consequence we obtain:

Proposition 3.11 (Internal hom between permutation representations)**.** *If* $\mathrm{char}(k) \neq |H_1|$*, then the* internal hom *between the permutation representation* $k[G/H_1]$ *and* $k[G/H_2]$ (2) *exists in* $(G\mathrm{Rep}_k, \otimes)$ *and is given by the tensor product of representations:*

$$\big[\, k[G/H_1]\,,\, k[G/H_2] \,\big] \;\simeq\; k[G/H_1] \otimes k[G/H_2]\,.$$

Proof. Generally, if duals exist, the internal hom is given by

$$[V_1, V_2] \;\simeq\; V_1^* \otimes V_2\,.$$

With this, the statement follows by Prop. 3.10. □

Proposition 3.12 (Trivial irrep in transitive permutation representations)**.** *The permutation representation* $k[G/H]$ *of a transitive G-set (Def. 3.4) contains precisely one direct summand of the trivial 1-dimensional representation (Def. 3.3):*

$$\dim_k \mathrm{Hom}\big(\mathbf{1}, k[G/H]\big) \;=\; \dim_k\big(k[G/H]\big)^G \;=\; 1\,.$$

Proof. By definition, every element $v \in k[G/H]$ is a formal linear combination of cosets $[g]$:

$$v \;=\; \sum_{[g] \in G/H} v_{[g]}\,[g]$$

for coefficients $v_{[g]} \in k$. If there exists $[g_1], [g_2] \in G/H$ such that $v_{[g_1]} \neq v_{[g_2]}$ then, by transitivity of the G-action, there exist $g \in G$ with $g[g_1] = [g_2]$. But since the $[g]$ constitute a basis of $k[G/H]$, this implies that $gv \neq v$, hence that v is not G-invariant. Therefore, the only G-invariant vectors $v \in k[G/H]$ are those all whose coefficients $v_{[g]}$ agree. These clearly form a 1-dimensional subspace. □

As a corollary we obtain:

Proposition 3.13 (Multiplication table via invariants and via hom-spaces). *The entries $M_{ij} := \sum\limits_{\ell} n_{ij}^{\ell}$ in the table of multiplication multiplicities in the Burnside ring (Def. 3.6) are equivalently*

- **(i)** *the dimensions of the subspaces of G-invariants in the Burnside products;*
- **(ii)** *the dimensions of the external homs of the two given basis elements;*

$$M_{ij} \;:=\; \sum_{\ell} n_{ij}^{\ell} \;=\; \dim_k \big(k[G/H_i] \otimes k[G/H_j] \big)^G \;=\; \dim_k \mathrm{Hom}\big(k[G/H_i]\,,\, k[G/H_j] \big)\,.$$

Proof. The first equality follows from Prop. 3.12 applied to the Definition 3.6 of the structure constants, which gives the following isomorphism

$$\begin{aligned} \big(k[G/H_i] \otimes k[G/H_j] \big)^G &:= \Big(\underset{\ell}{\oplus} n_{ij}^{\ell}\, k[G/H_\ell] \Big)^G \\ &\simeq \underset{\ell}{\oplus} n_{ij}^{\ell} \big(k[G/H_\ell] \big)^G \\ &\simeq \underset{\ell}{\oplus} n_{ij}^{\ell} k \;. \end{aligned}$$

The second equality comes from the following sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}\Big(k[G/H_1], k[G/H_2] \Big) &\simeq \mathrm{Hom}\Big(\mathbf{1}, \big[k[G/H_1], k[G/H_2] \big] \Big) \\ &\simeq \Big(\big[k[G/H_1], k[G/H_2] \big) \Big)^G \\ &\simeq \big(k[G/H_1] \otimes k[G/H_2] \big)^G \,. \end{aligned}$$

Here the first equivalence expresses a general relation between external and internal homs, via the tensor unit (Def. 3.3), the second is from (3.3) and the last one is Prop. 3.11. □

Now it is useful to relate this to Schur's Lemma. For this purpose it turns out to be convenient to think in terms of the following inner product.

Definition 3.14 (Inner product on Burnside ring). For $V_1, V_2 \in G\mathrm{Rep}_k$, write

$$\big\langle V_1, V_2 \big\rangle \;:=\; \dim_k \mathrm{Hom}\big(V_1, V_2 \big) \;\in\; \mathbb{N}$$

for the dimension of the vector space of representation homomorphism between them. By $\mathbb{Z}$-linearity this extends to a $\mathbb{Z}$-valued pairing on the Burnside ring:

$$R_k(G) \times R_k(G) \xrightarrow{\;\langle -,- \rangle\;} \mathbb{Z}\,.$$

In terms of this pairing, *Schur's Lemma* says the following:

Lemma 3.15 (Schur's Lemma)**.** *The pairing $\langle -,-\rangle$ from Def. 3.14 is a symmetric and $\mathbb{Z}$-bilinear inner product on the abelian group underlying the representation ring $R_k(G)$. With respect to this inner product, the set of isomorphism classes ρ_i of irreducible representations of G*

- **(i)** *is always an* orthogonal basis*, where each basis element has positive norm-square;*
- **(ii)** *is even an* orthonormal basis *if the field k is algebraically closed.*

Using this perspective, we amplify the following:

Remark 3.16. The statement of Prop. 3.13 is that, in terms of the inner product $\langle -,-\rangle$ from Def. 3.14, the multiplicities multiplication table coincides with the table of inner products of Burnside-basis elements:

$$M_{ij} \;:=\; \sum_{\ell} n_{ij}^{\ell} \;=\; \big\langle k[G/H_i]\,,\, k[G/H_j]\big\rangle\,. \tag{13}$$

Also notice that Prop. 3.13 implies that the k-linear dimension of k-linear hom-spaces between k-linear permutation representations of transitive G-sets is *independent* of the ground field k:

$$\langle k[G/H_i]\,,\, k[G/H_j]\rangle \;:=\; \dim_k \mathrm{Hom}\left(k[G/H_i]\,,\, k[G/H_j]\right) \;=\; M_{ij}$$

since the multiplication multiplicities matrix M of the Burnside ring is manifestly independent of k.

We now use this to discuss explicit matrix representations of β.

Definition 3.17 (Upper triangular form of the Burnside multiplicities matrix)**.** For G a finite group, let

$$H := U \cdot M \quad \in \mathrm{Mat}_{N\times N}(\mathbb{N}) \tag{14}$$

be an integral upper triangular form (e.g. [GiPa90, p. 3,4]) of the Burnside multiplication multiplicities matrix (Def. 3.6), hence with $U \in \mathrm{GL}(N,\mathbb{Z})$ an invertible matrix whose left-multiplication implements row reduction on M (N the number of conjugacy classes of subgroups of G). Write

$$\tilde{H} := \tilde{U} \cdot H \tag{15}$$

for the result of deleting the zero-rows from (14). Write

$$V_i \;:=\; \sum_{\ell} \tilde{U}_i^{\ell} \cdot [G/H_\ell] \;\in\; A(G) \tag{16}$$

for the corresponding permutation representations and write

$$d_i \;:=\; \big\langle V_i, V_j \big\rangle \tag{17}$$

for their norm-square with respect to the inner product (Def. 3.14).

Proposition 3.18. *Consider the upper triangular form $\tilde{H}$ of the Burnside multiplication matrix from Def. 3.17. Then the corresponding permutation representations $V_i \in R_k(G)$ (16) are orthogonal, in that their inner products (Def. 3.14) satisfy*

$$\big\langle V_i\,,\, V_j \big\rangle \;=\; \delta_{ij}\, d_i \qquad d_i \in \mathbb{N}\,; \tag{18}$$

and they linearly span the image of β:

$$\langle V_i \rangle_i \;\simeq\; \mathrm{im}(\beta) \subset R_k(G)\,.$$

Specifically, the matrix that represents β with respect to the basis of the $G/H_j \in A(G)$ and the basis of $V_i \in \mathrm{Im}(\beta) \subset R_k(G)$ is

$$\beta_{ij} \;=\; \tfrac{1}{d_i}\tilde{H}_{ij}\,.$$

(In particular this means that the ith row $\tilde{H}_{i\bullet}$ of $\tilde{H}$ is divisible by d_i.) Hence for every subgroup $H_j \subset G$, the image of β on the corresponding G-set G/H_j is the following linear combination of the representations V_i (16), from Def. 3.17:

$$\beta\big(G/H_j\big) \;=\; \sum_i \tfrac{1}{d_i} \tilde{H}_{ij}\, V_i \,. \tag{19}$$

Proof. The first statement follows from [PT91]: By (13) we have that $M = A^T \cdot A$ is a positive semi-definite matrix of inner products, where A is the matrix whose columns are the permutation representations $k[G/H]$ expanded in terms of the irreps of G. By [PT91, top of p. 5] this implies that the same row reduction which turns M into upper-triangular form takes A to a matrix whose non-vanishing columns V_i constitute an orthogonal basis of the linear span of the $k[G/H]$.

The remaining statement just spells this out by immediate computation:

$$\begin{aligned}
\tilde{H}_{ij} &= \sum_\ell \tilde{U}_i^\ell M_{\ell j} \\
&= \sum_\ell \tilde{U}_i^\ell \, \langle k[G/H_\ell], k[G/H_j] \rangle \\
&= \Big\langle \sum_\ell \tilde{U}_i^\ell k[G/H_\ell], k[G/H_j] \Big\rangle \\
&= \big\langle V_i, k[G/H_j] \big\rangle \\
&= \langle V_i, \beta(G/H_j) \rangle \\
&= d_i \beta_{ij} \,.
\end{aligned}$$

Here the first line is by Def. 3.17, and in the second step we used Prop. 3.11 in the inner product notation from Def. 3.14 (as in Remark 3.16). In the third step we used the linearity of the inner product from Prop. 3.15, in the fourth step we inserted the definition of V_i (16). In the fifth step we just identified $\beta(H_j) = k[G/H_j]$, for emphasis. Finally we used the assumption (18) to identify the coefficient β_{ij} of V_i in $\beta(H_j)$. □

With a linear basis for the image of β thus in hand, it just remains to express it in a form that may directly be compared to standard classifications available from the linear representation theory of finite groups. For completeness, recall:

Definition 3.19 (Characters)**.** For G a finite group and k a field, a function from the underlying set of G to k is called a *class function* if it is constant on conjugacy classes of G, hence if it factors as

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr] & & k \,. \\
& \mathrm{ConjCl}(G) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Hence class functions form a k-vector space of dimension the number of conjugacy classes in G:

$$k^{|\mathrm{ConjCl}(G)|}. \tag{20}$$

For $V \in R_k(G)$ a representation, the map sending any $g \in G$ to its *trace*, when regarded as a linear map $V \xrightarrow{g} V$ via this representation, is a class function (by basic properties of the trace), called the *character* χ_V of V:

$$\begin{tikzcd}
\mathrm{ConjCl}(G) \arrow[r] & k \\
{[g]} \arrow[r, mapsto] & \mathrm{tr}_V(g)
\end{tikzcd}$$

The following example is immediate but important:

Example 3.20 (Character of permutation representation)**.** For $S \in A(G)$ a finite G-set, the character (Def. 3.19) of its permutation representation $k[S]$ (2) is the function that sends g to the number of elements in S that are fixed by the given action of g:

$$\chi_{k[S]} \;:\; [g] \longmapsto |S^g| \in \mathbb{N} \subset k\,.$$

The relevance of characters is that, in characteristic zero, they already completely characterize linear representations, while being more manifestly tractable:

Proposition 3.21 (e.g. [tDi09, Theorem 2.2.5])**.** *If the field k is of characteristic zero, then the map that sends a k-linear G-representation to its character (Def. 3.19) is an injection of the k-vector space of isomorphism classes of finite-dimensional G-representations into the vector space* (20) *of class functions (Def. 3.19)*

$$\begin{tikzcd} G\mathrm{Rep}_k/_\sim \arrow[r, hook] & k^{|\mathrm{ConjCl}(G)|} \\ V \arrow[r, mapsto] & \chi_V \end{tikzcd}$$

If k is in addition a splitting field for G (notably if $k = \mathbb{C}$ is the complex numbers), then this map is even an isomorphism.

As usual, it is convenient to organize this data in *character tables.* In order to make our list of examples in Section 4 be unambiguously intelligible, we briefly dwell on the notation for character tables.

Definition 3.22 (Character table)**.** For $(W_i \in R_k(G))_{i\in\{1,\cdots,\}}$ a tuple of (possibly virtual) k-linear representations of a finite group G, their *character table* is the $n \times |\mathrm{ConjCl}(G)|$-matrix with values in k whose (i,j)-th entry is the value $\chi_{W_i}(g_j)$ of the character χ_{W_i} (Def. 3.19) on any element g_j of the jth conjugacy class $[g_j] \in \mathrm{ConjCl}(G)$.

Example 3.23 (Irreducible character table over $\mathbb{C}$)**.** By Prop. 3.21 the characters of irreducible representations over $k = \mathbb{C}$ the complex numbers form a linear basis of the representation ring $R_{\mathbb{C}}(G)$. We will denote these irreducible representations by $(\rho_i \in R_{\mathbb{C}})(G)$ and will display the corresponding character table (Def. 3.22) as follows (conjucagy classes being labeled by the order of their elements):

		conjugacy class				
		1	3	4A	4B	$\cdots$
irred. repr.	ρ_1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
	ρ_2	$\cdot$	$\chi_{\rho_2}(g_2)$	$\chi_{\rho_2}(g_3)$	$\cdot$	
	ρ_3	$\cdot$	$\chi_{\rho_3}(g_2)$	$\chi_{\rho_3}(g_3)$	$\cdot$	
	ρ_4	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
	$\vdots$					$\ddots$

The character tables of irreducible representations over the complex numbers, for many finite groups of small order, have been tabulated in the literature, for instance in [Dok-GroupNames].

Example 3.24 (Irreducible character table over $\mathbb{R}$)**.** For $k \subset \mathbb{C}$ a subfield, under the ring homomorphism of "extension of scalars"

$$R_k(G) \xrightarrow{(-)_k\mathbb{C}} R_{\mathbb{C}}(G)\;,$$

the values of characters to not change. Hence if k is in characteristic zero, $\mathbb{Q} \subset k \subset \mathbb{C}$, then, by Prop. 3.21, we may equivalently express the character of any linear representation $W \in R_k(G)$ over k after tensoring it with $\mathbb{C}$. This is a linear combination of the complex irreducible characters χ_{ρ_i} from above

$$W \otimes_k \mathbb{C} \;=\; \sum_i \underbrace{w_i}_{\in\mathbb{N}} \cdot \rho_i\;, \qquad\qquad \chi_{(W\otimes_k\mathbb{C})} \;=\; \sum_i w_i \cdot \chi_{\rho_i}\,.$$

Therefore, when $\mathbb{Q} \subset k \subset \mathbb{C}$ and for $(\cdots, W \in R_k(G), \cdots)$ a tuple of k-linear representations, we may and will express the corresponding character table as a table of linear combinations of the irreducible complex characters:

		conjugacy class				
		1	3	4A	4B	$\cdots$
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
irred. repr.	$W \otimes_k \mathbb{C} = \sum_i w_i \cdot \rho_i$	$\cdot$	$\sum_i w_i \cdot \chi_{\rho_i}(g_2)$	$\sum_i w_i \cdot \chi_{\rho_i}(g_3)$	$\cdot$	
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
	$\vdots$					$\ddots$

In this fashion we will in particular state the irreducible character tables over $k = \mathbb{R}$ the real numbers, which may again be found in the literature for many finite groups of small order.

Specifically, the complex character tables available in the literature (e.g. [Dok-GroupNames]) list the *type* of the corresponding complex representation, from which the character table of irreducible representations over the real numbers may be extracted (or conversely, as in [Mon-Representations]), via the following basic fact:

Proposition 3.25 (e.g. [Rob06, p. 4])**.** *Let G be a finite group, and consider the complexification map $R_{\mathbb{R}}(G) \xrightarrow{(-)\otimes_{\mathbb{R}}\mathbb{C}} R_{\mathbb{C}}(G)$. Then every irreducible complex representation $V \in R_{\mathbb{C}}(G)$ is of exactly one of the following three* types*, depending on how it arises as a direct summand of an irreducible real representation $W \in R_{\mathbb{R}}(G)$:*

$$W \otimes_{\mathbb{R}} \mathbb{C} \;\simeq\; \left\{ \begin{array}{lccc} V & | & \textit{real type} & / \quad \textit{orthogonal} \\ V \oplus V^* & | & \textit{complex type} & \\ V \oplus V & | & \textit{quaternionic type} & / \quad \textit{symplectic} \end{array} \right.$$

In this fashion we may now identify the image of β via the character table of its basis elements:

Proposition 3.26 (Character table of linear basis of image of β)**.** *The character (Def. 3.19) of a basis element V_i* (16) *of the image of β (Prop. 19) is the class function given by*

$$\chi_{V_i} \;:\; [g] \mapsto \sum_{\ell} \tilde{U}_i^{\ell} \cdot \left|(G/H_\ell)^g\right| , \tag{21}$$

where on the right we have the sum over conjugacy classes H_ℓ of subgroups of G of the product of the entries of the base change matrix from (15) *with the number of elements in G/H_ℓ that are fixed by the action of g.*

Proof. By example 3.20. ☐

Hence in conclusion we have the following.

Proposition 3.27 (Recognizing surjectivity of β)**.** *For k of characteristic zero, let $(\rho_i \in R_k(G))$ be the irreducible k-linear representations, spanning the representation ring $R_k(G)$. Then the comparison morphism* (7) *is surjective precisely if the corresponding tuple of characters χ_{ρ_i} (Def. 3.19) is related to the set of characters χ_{V_i} in Def. 3.26 by an invertible integer matrix*

$$\beta \textit{ is surjective over } k \quad \Leftrightarrow \quad \chi_{\rho_i} \;=\; \sum_j T_i^j \cdot \chi_{V_j} \quad T \in \mathrm{GL}(N, \mathbb{Z})\,.$$

Proof. Since, by construction, the V_j (16) are linearly independent and span the image of β (Prop. 3.18) and since the ρ_i span $R_k(G)$, the number of the V_j is smaller or equal to the number of ρ_i, hence the number must be equal if β is surjective. This means that for surjectivity there must be an invertible integer matrix relating the (V_j) to the (ρ_i). But by Prop. 3.21 this is the case precisely if there is such a matrix relating the characters of these representations. ☐

This concludes our algorithmic description of the image of β. To summarize, the algorithm proceeds as follows:

Theorem 3.28 (**Algorithm for the cokernel of** β)**.** *Let G be a finite group and k a field in characteristic zero.*

(1) *Extract from standard literature:*

- *the* character table of irreducible k-linear representations *(Examples 3.23, 3.24)*

$$\big(\chi_{\rho_i}([g])\big) \quad \in \mathrm{Mat}_{n,N}(k)\;.$$

(2) *Compute:*

(a) *the* multiplication table $\left(n^{\ell}_{ij}\right)$ (10) *of the Burnside ring, efficiently so via* (12)*;*

(b) *the resulting* table of total multiplication multiplicities $(M_{ij}) :=\ \big(\sum_{\ell} n^{\ell}_{ij}\big)$ (11)*;*

(c) *its upper triangular form $H := U \cdot M$* (14)*;*

(d) *the result $\tilde{H} = \tilde{U} \cdot M$* (15) *of deleting its zero-rows;*

(e) *the* character table of the resulting linear basis for the image of β (21)

$$\Big(\chi_{V_i}([g]) \;=\; \sum_{\ell} \tilde{U}^{\ell}_{i} \cdot \left|(G/H_\ell)^g\right|\Big) \;\in\; \mathrm{Mat}_{r,N}(\mathbb{N}) \subset \mathrm{Mat}_{r,N}(k)\,.$$

(3) *Read off the quotient of the lattice spanned by the vectors χ_{ρ_i} by that spanned by the vectors χ_{V_j}:*

$$\mathrm{coker}(\beta_k) \;:=\; \frac{\mathbb{Z}[\chi_{\rho_i}]_{i=1}^{n}}{\mathbb{Z}[\chi_{V_j}]_{j=1}^{N}}\,.$$

Here

- $N := |\mathrm{ConjCl}(G)|$ is the number of conjugacy classes of G;
- $r := \mathrm{rank}(\mathrm{image}(\beta))$ is the rank of the image of β (the number of V_j);
- n is the number of isomorphism classes of irreducible representations ρ_i.

Remark 3.29 (Simplification in examples)**.** In all examples that we compute in Section 4, upper triangular form of the multiplicities matrix in the third step of the algorithm 3.28 is achieved by the most straightforward row reduction, where, incrementally in the row number, a suitable integer multiple of each row is subtracted from all those beneath it. This is remarkable, since, in general, row reduction over the integers needs more and more intricate steps than this; see e.g. [GiPa90, p. 3–4]. That this happens is due to the fact that M is a positive semidefinite matrix, as explained in [PT91].

But moreover, in each case the resulting rows V_i happen to be actual representations, as opposed to virtual representations. This makes our algorithm very efficient, and makes it easy to read off the image of β in each example. It seems clear that this particularly nice behavior of row reduction on the Burnside multiplicities matrix is due to a very special property of the latter. It would be interesting to understand this phenomenon theoretically.

4 The image of β – Examples

Given a finite group G and its irreducible characters over a given field k of characteristic zero, Theorem 3.28 provides an effective algorithm for identifying the image of $A(G) \xrightarrow{\beta} R_k(G)$ (7) and checking whether β is surjective. We have implemented this algorithm in `Python`, available as an ancillary file to this arxiv submission. Here we spell out various example computations. In summary, we obtain the result shown in Theorem 4.1. In particular, the shaded entries show that over the real numbers β has vanishing cokernel/is surjective onto the ring of integer characters (i.e. onto non-irrational characters, by Prop. 2.3).

Theorem 4.1 (Cokernel of β for D- and E-series binary Platonic groups)**.** *The following table lists the cokernels*

$$\mathrm{coker}(\beta_k) := \frac{R_k(G)}{\mathrm{image}(\beta_k)}, \qquad \mathrm{coker}(\beta_k^{\mathrm{int}}) := \frac{R_k^{\mathrm{int}}(G)}{\mathrm{image}(\beta_k)}$$

of the permutation-representation morphism $A(G) \xrightarrow{\beta_k} R_k(G)$ (7) and its corestriction to the integer-valued character ring (Prop. 2.2, Remark 2.4), for finite subgroups of $SU(2)$ in the D- and E-series (via Prop. A.1) and some relatives, over ground fields $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$:

	coker	$A(G) \xrightarrow{\beta_{\mathbb{F}}} R_{\mathbb{F}}(G)$			$A(G) \xrightarrow{\beta_{\mathbb{F}}^{\mathrm{int}}} R_{\mathbb{F}}^{\mathrm{int}}(G)$			
Dynkin label	group	ground field $\mathbb{F}$			ground field $\mathbb{F}$			
	G	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	**Proof** via Thm. 3.28:
"$\mathbb{D}_1$"	C_2	0	0	0	0	0	0	*Sec. 4.1.1*
$\mathbb{D}_3$	C_4	0	$\frac{\mathbb{Z}[\rho_2,\rho_4]}{\mathbb{Z}[\rho_2+\rho_4]}$	0	0	0	$\frac{\mathbb{Z}[\rho_2,\rho_4]}{\mathbb{Z}[\rho_2+\rho_4]}$	*Sec. 4.1.2*
$\mathbb{D}_4$	$2D_4$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$	*Sec. 4.2.1*
$\mathbb{D}_5$	$2D_6$	0	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_6]}{\mathbb{Z}[\rho_3+\rho_4,2\rho_6]}$	0	0	$\frac{\mathbb{Z}[\rho_6]}{\mathbb{Z}[2\rho_6]}$	*Sec. 4.2.2*
$\mathbb{D}_6$	$2D_8$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	0	0	$\frac{\mathbb{Z}[\rho_6+\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	*Sec. 4.2.3*
$\mathbb{D}_7$	$2D_{10}$	0	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	$\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,2\rho_7,2\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_7+\rho_8]}{\mathbb{Z}[2\rho_7+2\rho_8]}$	*Sec. 4.2.4*
$\mathbb{D}_8$	$2D_{12}$	0	$\frac{\mathbb{Z}[\rho_7,\rho_8,\rho_9]}{\mathbb{Z}[2\rho_7,2\rho_8+2\rho_9]}$	$\frac{\mathbb{Z}[2\rho_8,2\rho_9]}{\mathbb{Z}[2\rho_8+2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_7]}{\mathbb{Z}[2\rho_7]}$	*Sec. 4.2.5*
$\mathbb{E}_6$	$2T$	0	0	$\frac{\mathbb{Z}[\rho_2,\rho_2^*,\rho_4,\rho_4^*,\rho_5]}{\mathbb{Z}[\rho_2+\rho_2^*,\rho_4+\rho_4^*,2\rho_5]}$	0	0	$\frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]}$	*Sec. 4.3.1*
$\mathbb{E}_7$	$2O$	0	$\frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]}$	$\frac{\mathbb{Z}[\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[2\rho_6+2\rho_7,2\rho_8]}$	0	0	$\frac{\mathbb{Z}[\rho_8]}{\mathbb{Z}[2\rho_8]}$	*Sec. 4.3.2*
$\mathbb{E}_8$	$2I$	0	$\frac{\mathbb{Z}[2\rho_2,2\rho_3,\rho_4,\rho_5]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5]}$	$\frac{\mathbb{Z}[\rho_2,\rho_3,\rho_4,\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]}$	0	0	$\frac{\mathbb{Z}[\rho_2+\rho_3,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,2\rho_7,2\rho_9]}$	*Sec. 4.3.3*
	$\mathrm{GL}(2,\mathbb{F}_3)$	0	0	$\frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[\rho_6+\rho_7]}$	0	0	0	*Sec. 4.3.4*

For emphasis we highlight by example how to read the tabel in Theorem 4.1:

- an entry “0” means that β is surjective;
- an entry “$\frac{\mathbb{Z}[\rho]}{\mathbb{Z}[2\rho]}$” means that the image of β consists of all those virtual representations whose ρ-component has even multiplicity;
- an entry “$\frac{\mathbb{Z}[\rho_1,\rho_2]}{\mathbb{Z}[\rho_1+\rho_2]}$” means that the image of β consists of all those virtual representations whose ρ_1-component has the same weight as their ρ_2-component.

Here ρ_i refers to the irreducible representations as tabulated in the respective subsection below. From the character tables given there one also reads off whether the character of ρ_i is integer-valued or else (by Prop. 2.3) irrational. The cokernel for β_k^{int} is obtained from that of β_k by removing those generators from the numerator that have irrational-valued characters.

4.1 Cyclic groups: C_n

For completeness, we include discussion of two cyclic groups, namely those in the degenerate range of the D-series of the ADE-classification of finite subgroups of SU(2) (Sec. A.1):

Dynkin label	**Name of group**
“$\mathbb{D}_1''$	$C_2 \times C_2$
$\mathbb{D}_3 = \mathbb{A}_3$	C_4

In these simple cases the computation is fairly trivial, and may serve to introduce and illustrate our notation for recording application of the algorithm.

4.1.1 The cyclic group C_2

Group name: C_2 ([Dok-C_2])
Group order: 2
Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	2	1	1	✓
B	1	2	1	✓

Burnside ring product:

$\times$	A	B
A	A	B
B	B	$2B$

Table of multiplicities:

	A	B
A	1	1
B	1	2

Upper triangular form:

	A	B
V_1	1	1
V_2	.	1

Character table for image of β:

class	1	2
size	1	1
V_1	1	1
V_2	1	-1

Character table of irreps:
over $\mathbb{C}$:

		conjugacy class	
		1	2
irred. repr.	ρ_1	1	1
	ρ_2	1	-1

over $\mathbb{R}$:

		conjugacy class	
		1	2
irred. repr.	ρ_1	1	1
	ρ_2	1	-1

Hence the cokernel of β is:

$$\begin{array}{c|c} \begin{array}{c} V_1 = \rho_1 \\ V_2 = \rho_2 \end{array} & \operatorname{coker}\left(A(C_2) \xrightarrow{\beta} R_k(C_2)\right) \simeq \begin{cases} 0 & | \quad k = \mathbb{C} \\ 0 & | \quad k = \mathbb{R} \\ 0 & | \quad k = \mathbb{Q} \end{cases} \end{array}$$

4.1.2 The cyclic group C_4

Group name: C_4 ([Dok-C_4])
Group order: 4
Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	4	1	1	✓
B	2	2	1	✓
C	1	4	1	✓

Burnside ring product:

$\times$	A	B	C
A	A	B	C
B	B	$2B$	$2C$
C	C	$2C$	$4C$

Table of multiplicities:

	A	B	C
A	1	1	1
B	1	2	2
C	1	2	4

Upper triangular form:

	A	B	C
V_1	1	1	1
V_2	0	1	1
V_3	0	0	2

Character table for image of β:

class	1	2	3	4
size	1	1	1	1
V_1	1	1	1	1
V_2	1	-1	1	-1
V_3	2	0	-2	0

Character table of irreps:

over $\mathbb{C}$:

		conjugacy class			
		1	2	3	4
irred. repr.	ρ_1	1	1	1	1
	ρ_2	1	i	-1	$-i$
	ρ_3	1	-1	1	-1
	ρ_4	1	$-i$	-1	i

over $\mathbb{R}$:

		conjugacy class			
		1	2	3	4
irred. repr.	ρ_1	1	1	1	1
	ρ_3	1	-1	1	-1
	$\rho_2 + \rho_4$	2	0	-2	0

Hence the cokernel of β is:

$$\begin{array}{l|l} V_1 = \rho_1 & \\ V_2 = \rho_3 & \operatorname{coker}\Big(A(C_4) \xrightarrow{\beta} R_k(C_4)\Big) \;\simeq\; \left\{ \begin{array}{ccc} \frac{\mathbb{Z}[\rho_2,\rho_4]}{\rho_2+\rho_4} & | & k=\mathbb{C} \\ 0 & | & k=\mathbb{R} \\ 0 & | & k=\mathbb{Q} \end{array} \right. \\ V_3 = \rho_1 + \rho_4 & \end{array}$$

4.2 Binary dihedral groups: $2D_{2n} \simeq \mathrm{Dic}_n$

The binary dihedral groups have the following presentation (see e.g. [Lindh2018]):

$$2D_{2n} := \langle r, s | r^{2n} = 1, s^2 = r^n, s^{-1}rs = r^{-1} \rangle.$$

The order of $2D_{2n}$ is $4n$ and has $n+3$ conjugacy classes:

$$\begin{aligned} &\{1\}, \{s^2\}, \\ &\{r, r^{2n-1}\}, \{r^2, r^{2n-2}\}, ..., \{r^{n-1}, r^{n+1}\}, \\ &\{s, sr^2, ..., sr^{2n-2}\}, \{sr, sr^3, ..., sr^{2n-1}\}. \end{aligned}$$

The $n+3$ complex irreducible characters are given by:

$2D_{2n} = \mathrm{Dic}_n$	1	s^2	r	r^2	$\cdots$	r^{n-1}	s	sr
Triv.	1	1	1	1	$\cdots$	1	1	1
1A	1	1	1	1	$\cdots$	1	-1	-1
1B	1	$(-1)^n$	-1	1	$\cdots$	$(-1)^{n-1}$	i^n	$-i^n$
1C	1	$(-1)^n$	-1	1	$\cdots$	$(-1)^{n-1}$	$-i^n$	i^n
ρ_1	2	-2	$\zeta+\zeta^{-1}$	$\zeta^2+\zeta^{-2}$	$\cdots$	$\zeta^{n-1}+\zeta^{1-n}$	0	0
ρ_k	2	$(-1)^k 2$	$\zeta^k+\zeta^{-k}$	$\zeta^{2k}+\zeta^{-2k}$	$\cdots$	$\zeta^{k(n-1)}+\zeta^{k(1-n)}$	0	0

with $k = 2, ..., n-1$ and $\zeta = e^{2\pi i/2n}$.

The representations Triv. and 1A are always real. For n even 1B and 1C are real. For n odd 1A and 1B are complex, and 1A+1B is real irreducible. For k even ρ_k is real. For k odd ρ_k is quaternionic, and so $2\rho_k$ is real.

4.2.1 Binary dihedral group: $2D_4 \simeq \mathrm{Dic}_2 \simeq Q_8$

Group name: $2D_4 \simeq Q_8$ ([Dok-$2D_4$])
Group order: $|2D_4| = 8$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	8	1	1	
B	4	2	1	✓
C	4	2	1	✓
D	4	2	1	✓
E	2	4	1	✓
F	1	8	1	✓

Burnside ring product:

$\times$	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	$2B$	E	E	$2E$	$2F$
C	C	E	$2C$	E	$2E$	$2F$
D	D	E	E	$2D$	$2E$	$2F$
E	E	$2E$	$2E$	$2E$	$4E$	$4F$
F	F	$2F$	$2F$	$2F$	$4F$	$8F$

Table of multiplicities:

	A	B	C	D	E	F
A	1	1	1	1	1	1
B	1	2	1	1	2	2
C	1	1	2	1	2	2
D	1	1	1	2	2	2
E	1	2	2	2	4	4
F	1	2	2	2	4	8

Upper triangular form:

	A	B	C	D	E	F
V_1	1	1	1	1	1	1
V_2	.	1	.	.	1	1
V_3	.	.	1	.	1	1
V_4	.	.	.	1	1	1
V_5	.	.	.	.	.	4

Character table for image of β:

class	1	2	$4A$	$4B$	$4C$
size	1	1	2	2	2
V_1	1	1	1	1	1
V_2	1	1	-1	1	-1
V_3	1	1	-1	-1	1
V_4	1	1	1	-1	-1
V_5	4	-4	0	0	0

Character table of irreps [Dok-$2D_4$, Mon-Q_8]

over $\mathbb{C}$:

		conjugacy class				
		1	2	4A	4B	4C
irred. repr.	ρ_1	1	1	1	1	1
	ρ_2	1	1	-1	1	-1
	ρ_3	1	1	1	-1	-1
	ρ_4	1	1	-1	-1	1
	ρ_5	2	-2	0	0	0

over $\mathbb{R}$:

		conjugacy class				
		1	2	4A	4B	4C
irred. repr.	ρ_1	1	1	1	1	1
	ρ_2	1	1	-1	1	-1
	ρ_3	1	1	1	-1	-1
	ρ_4	1	1	-1	-1	1
	$2\rho_5$	4	-4	0	0	0

Hence the cokernel of β is:

$$\begin{array}{r|l} V_1 = \rho_1 & \\ V_2 = \rho_2 & \\ V_3 = \rho_4 & \operatorname{coker}\Big(A(2D_4) \overset{\beta}{\to} R_k(2D_4)\Big) \simeq \left\{ \begin{array}{ccc} \frac{\mathbb{Z}[\rho_5]}{\mathbb{Z}[2\rho_5]} & | & k = \mathbb{C} \\ 0 & | & k = \mathbb{R} \\ 0 & | & k = \mathbb{Q} \end{array} \right. \\ V_4 = \rho_3 & \\ V_5 = 2\rho_5 & \end{array}$$

4.2.2 Binary dihedral group: $2D_6 \simeq \mathrm{Dic}_3$

Group name: $2D_6$ ([Dok-$2D_6$])
Group order: $|2D_6| = 12$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	12	1	1	
B	6	2	1	✓
C	4	3	3	✓
D	3	4	1	✓
E	2	6	1	✓
F	1	12	1	✓

Burnside ring product:

×	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	$2B$	E	$2D$	$2E$	$2F$
C	C	E	$C+E$	F	$3E$	$3F$
D	D	$2D$	F	$4D$	$2F$	$4F$
E	E	$2E$	$3E$	$2F$	$6E$	$6F$
F	F	$2F$	$3F$	$4F$	$6F$	$12F$

Table of multiplicities:

	A	B	C	D	E	F
A	1	1	1	1	1	1
B	1	2	1	2	2	2
C	1	1	2	1	3	3
D	1	2	1	4	2	4
E	1	2	3	2	6	6
F	1	2	3	4	6	12

Upper triangular form:

	A	B	C	D	E	F
V_1	1	1	1	1	1	1
V_2	.	1	.	1	1	1
V_3	.	.	1	.	2	2
V_4	.	.	.	2	.	2
V_5	.	.	.	.	.	4

Character table for image of β:

class	1	2	3	4A	4B	6
size	1	1	2	3	3	2
V_1	1	1	1	1	1	1
V_2	1	1	1	−1	−1	1
V_3	2	2	−1	0	0	−1
V_4	2	−2	2	0	0	−2
V_5	4	−4	−2	0	0	2

Character table of irreps [Dok-$2D_6$]

over $\mathbb{C}$:

		conjugacy class					
		1	2	3	4A	4B	6
irred. repr.	ρ_1	1	1	1	1	1	1
	ρ_2	1	1	1	-1	-1	1
	ρ_3	1	-1	1	i	$-i$	-1
	ρ_4	1	-1	1	$-i$	i	-1
	ρ_5	2	2	-1	0	0	-1
	ρ_6	2	-2	-1	0	0	1

over $\mathbb{R}$:

		conjugacy class					
		1	2	3	4A	4B	6
irred. repr.	ρ_1	1	1	1	1	1	1
	ρ_2	1	1	1	-1	-1	1
	$\rho_3+\rho_4$	2	-2	2	0	0	-2
	ρ_5	2	2	-1	0	0	-1
	$2\rho_6$	4	-4	-2	0	0	2

Hence the cokernel of β is

$$\begin{array}{l|l} V_1 = \rho_1 & \\ V_2 = \rho_2 & \\ V_3 = \rho_5 & \operatorname{coker}(\beta_k) \ = \begin{cases} \frac{\mathbb{Z}[\rho_3,\rho_4,\rho_6]}{\mathbb{Z}[\rho_3+\rho_4,2\rho_6]} & \mid \quad k=\mathbb{C} \\ 0 & \mid \quad k=\mathbb{R} \\ 0 & \mid \quad k=\mathbb{Q} \end{cases} \\ V_4 = \rho_3+\rho_4 & \\ V_5 = 2\rho_6 & \end{array}$$

4.2.3 Binary dihedral group: $2D_8 \simeq \mathrm{Dic}_4 \simeq Q_{16}$

Group name: $2D_8$ ([Dok-$2D_8$])
Group order: $|2D_8| = 16$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	16	1	1	
B	8	2	1	✓
C	8	2	1	
D	8	2	1	
E	4	4	2	✓
F	4	4	1	✓
G	4	4	2	✓
H	2	8	1	✓
I	1	16	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I
A	1	1	1	1	1	1	1	1	1
B	1	2	1	1	1	2	1	2	2
C	1	1	2	1	2	2	1	2	2
D	1	1	1	2	1	2	2	2	2
E	1	1	2	1	3	2	2	4	4
F	1	2	2	2	2	4	2	4	4
G	1	1	1	2	2	2	3	4	4
H	1	2	2	2	4	4	4	8	8
I	1	2	2	2	4	4	4	8	16

Upper triangular form:

	A	B	C	D	E	F	G	H	I
V_1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	.	.	1	.	1	1
V_3	.	.	1	.	1	1	.	1	1
V_4	.	.	.	1	.	1	1	1	1
V_5	.	.	.	.	1	.	1	2	2
V_6	.	.	.	.	.	.	.	.	8

Character table for image of β:

class	1	2	$4A$	$4B$	$4C$	$8A$	$8B$
size	1	1	2	4	4	2	2
V_1	1	1	1	1	1	1	1
V_2	1	1	1	-1	1	-1	-1
V_3	1	1	1	-1	-1	1	1
V_4	1	1	1	1	-1	-1	-1
V_5	2	2	-2	0	0	0	0
V_6	8	-8	0	0	0	0	0

Character table of irreps [Dok-2D_8]:

over $\mathbb{C}$

		conjugacy class						
		1	2	4A	4B	4C	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1
	ρ_2	1	1	1	1	-1	-1	-1
	ρ_3	1	1	1	-1	1	-1	-1
	ρ_4	1	1	1	-1	-1	1	1
	ρ_5	2	2	-2	0	0	0	0
	ρ_6	2	-2	0	0	0	$\sqrt{2}$	$-\sqrt{2}$
	ρ_7	2	-2	0	0	0	$-\sqrt{2}$	$\sqrt{2}$

over $\mathbb{R}$

		conjugacy class						
		1	2	4A	4B	4C	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1
	ρ_2	1	1	1	1	-1	-1	-1
	ρ_3	1	1	1	-1	1	-1	-1
	ρ_4	1	1	1	-1	-1	1	1
	ρ_5	2	2	-2	0	0	0	0
	$2\rho_6$	4	-4	0	0	0	$2\sqrt{2}$	$-2\sqrt{2}$
	$2\rho_7$	4	-4	0	0	0	$-2\sqrt{2}$	$2\sqrt{2}$

Hence the cokernel of β is:

$V_1 = \rho_1$
$V_2 = \rho_3$
$V_3 = \rho_4$
$V_4 = \rho_2$
$V_5 = \rho_5$
$V_6 = 2\rho_6 + 2\rho_7$

$$\operatorname{coker}\Big(A(2D_8) \xrightarrow{\beta} R_k(2D_8)\Big) \simeq \begin{cases} \frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]} & \mid \quad k=\mathbb{C} \\ \frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]} & \mid \quad k=\mathbb{R} \\ 0 & \mid \quad k=\mathbb{Q} \end{cases}$$

4.2.4 Binary dihedral group: $2D_{10} \simeq \mathrm{Dic}_5$

Group name: $2D_{10}$ ([Dok-$2D_{10}$])
Group order: $|2D_{10}| = 20$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	20	1	1	
B	10	2	1	✓
C	5	4	1	✓
D	4	5	5	✓
E	2	10	1	✓
F	1	20	1	✓

Table of multiplicities:

	A	B	C	D	E	F
A	1	1	1	1	1	1
B	1	2	2	1	2	2
C	1	2	4	1	2	4
D	1	1	1	3	5	5
E	1	2	2	5	10	10
F	1	2	4	5	10	20

Upper triangular form:

	A	B	C	D	E	F
V_1	1	1	1	1	1	1
V_2	.	1	1	.	1	1
V_3	.	.	2	.	.	2
V_4	.	.	.	2	4	4
V_5	.	.	.	.	.	8

Character table for image of β:

class	1	2	$4A$	$4B$	5	5	10	10
size	1	1	5	5	2	2	2	2
V_1	1	1	1	1	1	1	1	1
V_2	1	1	-1	-1	1	1	1	1
V_3	2	-2	0	0	2	2	-2	-2
V_4	4	4	0	0	-1	-1	-1	-1
V_5	8	-8	0	0	-2	-2	2	2

Character table of irreps [Dok-2D_{10}]

over $\mathbb{C}$

class	1	2	$4A$	$4B$	$5A$	$5B$	$10A$	$10B$
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	1	-1	-1	1	1	1	1
ρ_3	1	-1	$-i$	i	1	1	-1	-1
ρ_4	1	-1	i	$-i$	1	1	-1	-1
ρ_5	2	2	0	0	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$
ρ_6	2	2	0	0	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$
ρ_7	2	-2	0	0	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$	$-\zeta_5^2-\zeta_5^3$	$-\zeta_5-\zeta_5^4$
ρ_8	2	-2	0	0	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$	$-\zeta_5-\zeta_5^4$	$-\zeta_5^2-\zeta_5^3$

over $\mathbb{R}$

class	1	2	$4A$	$4B$	$5A$	$5B$	$10A$	$10B$
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	1	-1	-1	1	1	1	1
$\rho_3+\rho_4$	2	-2	0	0	2	2	-2	-2
ρ_5	2	2	0	0	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$
ρ_6	2	2	0	0	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$	$\zeta_5+\zeta_5^4$	$\zeta_5^2+\zeta_5^3$
$2\rho_7$	4	-4	0	0	$2\left(\zeta_5^2+\zeta_5^3\right)$	$2\left(\zeta_5+\zeta_5^4\right)$	$-2\left(\zeta_5^2-\zeta_5^3\right)$	$-2\left(\zeta_5-\zeta_5^4\right)$
$2\rho_8$	4	-4	0	0	$2\left(\zeta_5+\zeta_5^4\right)$	$2\left(\zeta_5^2+\zeta_5^3\right)$	$-2\left(\zeta_5-\zeta_5^4\right)$	$-2\left(\zeta_5^2-\zeta_5^3\right)$

Hence the cokernel of β is:

$$
\begin{array}{l|l}
V_1=\rho_1 & \\
V_2=\rho_2 & \\
V_3=\rho_3+\rho_4 & \operatorname{coker}\left(A(2D_{10})\xrightarrow{\beta}R_k(2D_{10})\right)=\begin{cases}\frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]} & \mid\quad k=\mathbb{C}\\ \frac{\mathbb{Z}[\rho_3,\rho_4,\rho_5,\rho_6,2\rho_7,2\rho_8]}{\mathbb{Z}[\rho_3+\rho_4,\rho_5+\rho_6,2\rho_7+2\rho_8]} & \mid\quad k=\mathbb{R}\\ 0 & \mid\quad k=\mathbb{Q}\end{cases}\\
V_4=\rho_5+\rho_6 & \\
V_5=2\rho_7+2\rho_8 &
\end{array}
$$

4.2.5 Binary dihedral group: $2D_{12}\simeq\mathrm{Dic}_6$

Group name: $2D_{12}$ ([Dok-$2D_{12}$])
Group order: $|2D_{12}|=24$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	24	1	1	
B	12	2	1	✓
C	12	2	1	
D	12	2	1	
E	8	3	3	
F	6	4	1	✓
G	4	6	3	✓
H	4	6	1	✓
I	4	6	3	✓
J	3	8	1	✓
K	2	12	1	✓
L	1	24	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I	J	K	L
A	1	1	1	1	1	1	1	1	1	1	1	1
B	1	2	1	1	1	2	1	2	1	2	2	2
C	1	1	2	1	1	2	2	1	1	2	2	2
D	1	1	1	2	1	2	1	1	2	2	2	2
E	1	1	1	1	2	1	2	3	2	1	3	3
F	1	2	2	2	1	4	2	2	2	4	4	4
G	1	1	2	1	2	2	4	3	3	2	6	6
H	1	2	1	1	3	2	3	6	3	2	6	6
I	1	1	1	2	2	2	3	3	4	2	6	6
J	1	2	2	2	1	4	2	2	2	8	4	8
K	1	2	2	2	3	4	6	6	6	4	12	12
L	1	2	2	2	3	4	6	6	6	8	12	24

Upper triangular form:

	A	B	C	D	E	F	G	H	I	J	K	L
V_1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	.	.	1	.	1	.	1	1	1
V_3	.	.	1	.	.	1	1	.	.	1	1	1
V_4	.	.	.	1	.	1	.	.	1	1	1	1
V_5	.	.	.	.	1	.	1	2	1	.	2	2
V_6	.	.	.	.	.	.	1	.	1	.	2	2
V_7	.	.	.	.	.	.	.	.	.	4	.	4
V_8	.	.	.	.	.	.	.	.	.	.	.	8

Character table for image of β:

class	1	2	3	4	4	4	6	12	12
size	1	1	2	6	6	2	2	2	2
V_1	1	1	1	1	1	1	1	1	1
V_2	1	1	1	-1	-1	1	1	1	1
V_3	1	1	1	-1	1	-1	1	-1	-1
V_4	1	1	1	1	-1	-1	1	-1	-1
V_5	2	2	-1	0	0	2	-1	-1	-1
V_6	2	2	-1	0	0	-2	-1	1	1
V_7	4	-4	4	0	0	0	-4	0	0
V_8	8	-8	-4	0	0	0	4	0	0

Character table of irreps

over $\mathbb{C}$

class	1	2	3	$4A$	$4B$	$4C$	6	$12A$	$12B$
ρ_1	1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	-1	-1	1	1	-1	-1
ρ_3	1	1	1	1	-1	-1	1	1	1
ρ_4	1	1	1	-1	1	-1	1	-1	-1
ρ_5	2	2	-1	2	0	0	-1	-1	-1
ρ_6	2	2	-1	-2	0	0	-1	1	1
ρ_7	2	-2	2	0	0	0	-2	0	0
ρ_8	2	-2	-1	0	0	0	1	$\sqrt{3}$	$-\sqrt{3}$
ρ_9	2	-2	-1	0	0	0	1	$-\sqrt{3}$	$\sqrt{3}$

over $\mathbb{R}$

class	1	2	3	$4A$	$4B$	$4C$	6	$12A$	$12B$
ρ_1	1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	-1	-1	1	1	-1	-1
ρ_3	1	1	1	1	-1	-1	1	1	1
ρ_4	1	1	1	-1	1	-1	1	-1	-1
ρ_5	2	2	-1	2	0	0	-1	-1	-1
ρ_6	2	2	-1	-2	0	0	-1	1	1
$2\rho_7$	4	-4	4	0	0	0	-4	0	0
$2\rho_8$	4	-4	-2	0	0	0	2	$2\sqrt{3}$	$-2\sqrt{3}$
$2\rho_9$	4	-4	-2	0	0	0	2	$-2\sqrt{3}$	$2\sqrt{3}$

Hence the cokernel of β is

$$
\begin{array}{l|l}
V_1 = \rho_1 & \\
V_2 = \rho_3 & \\
V_3 = \rho_2 & \\
V_4 = \rho_4 & \\
V_5 = \rho_5 & \operatorname{coker}\left(A(2D_{12}) \overset{\beta}{\to} R_k(2D_{12})\right) = \begin{cases} \frac{\mathbb{Z}[\rho_7,\rho_8,\rho_9]}{\mathbb{Z}[2\rho_7,2\rho_8+2\rho_9]} & | \quad k = \mathbb{C} \\ \frac{\mathbb{Z}[2\rho_8,2\rho_9]}{\mathbb{Z}[2\rho_8+2\rho_9]} & | \quad k = \mathbb{R} \\ 0 & | \quad k = \mathbb{Q} \end{cases} \\
V_6 = \rho_6 & \\
V_7 = 2\rho_7 & \\
V_8 = 2\rho_8 + 2\rho_9 &
\end{array}
$$

4.2.6 Binary dihedral group: $2D_{14} \simeq \mathrm{Dic}_7$

Group name: $2D_{14}$
Group order: $|2D_{14}| = 28$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	28	1	1	
B	14	2	1	✓
C	7	4	1	✓
D	4	7	7	✓
E	2	14	1	✓
F	1	28	1	✓

Table of multiplicities:

	A	B	C	D	E	F
A	1	1	1	1	1	1
B	1	2	2	1	2	2
C	1	2	4	1	2	4
D	1	1	1	4	7	7
E	1	2	2	7	14	14
F	1	2	4	7	14	28

Upper triangular form:

	A	B	C	D	E	F
V_1	1	1	1	1	1	1
V_2	.	1	1	.	1	1
V_3	.	.	2	.	.	2
V_4	.	.	.	3	6	6
V_5	.	.	.	.	.	12

Character table for image of β:

class	1	2	4	4	7	7	7	14	14	14
size	1	1	7	7	2	2	2	2	2	2
V_1	1	1	1	1	1	1	1	1	1	1
V_2	1	1	-1	-1	1	1	1	1	1	1
V_3	2	-2	0	0	2	2	2	-2	-2	-2
V_4	6	6	0	0	-1	-1	-1	-1	-1	-1
V_5	12	-12	0	0	-2	-2	-2	2	2	2

class	1	2	$4A$	$4B$	$7A$	$7B$	$7C$	$14A$	$14B$	$14C$
ρ_1	1	1	1	1	1	1	1	1	1	1
ρ_2	1	1	-1	-1	1	1	1	1	1	1
ρ_3	1	-1	$-i$	i	1	1	1	-1	-1	-1
ρ_4	1	-1	i	$-i$	1	1	1	-1	-1	-1
ρ_5	2	2	0	0	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$
ρ_6	2	2	0	0	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$
ρ_7	2	2	0	0	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$
ρ_8	2	-2	0	0	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$-\zeta_7^3-\zeta_7^4$	$-\zeta_7-\zeta_7^6$	$-\zeta_7^2-\zeta_7^5$
ρ_9	2	-2	0	0	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$\zeta_7+\zeta_7^6$	$-\zeta_7^2-\zeta_7^5$	$-\zeta_7^3-\zeta_7^4$	$-\zeta_7-\zeta_7^6$
ρ_{10}	2	-2	0	0	$\zeta_7+\zeta_7^6$	$\zeta_7^2+\zeta_7^5$	$\zeta_7^3+\zeta_7^4$	$-\zeta_7-\zeta_7^6$	$-\zeta_7^2-\zeta_7^5$	$-\zeta_7^3-\zeta_7^4$

4.2.7 **Binary dihedral group:** $2D_{16} \simeq \mathrm{Dic}_8 \simeq Q_{32}$

Group name: $2D_{16}$
Group order: $|2D_{16}| = 32$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	32	1	1	
B	16	2	1	
C	16	2	1	✓
D	16	2	1	
E	8	4	1	✓
F	8	4	2	
G	8	4	2	
H	4	8	4	✓
I	4	8	4	✓
J	4	8	1	✓
K	2	16	1	✓
L	1	32	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I	J	K	L
A	1	1	1	1	1	1	1	1	1	1	1	1
B	1	2	1	1	2	1	2	2	1	2	2	2
C	1	1	2	1	2	1	1	1	1	2	2	2
D	1	1	1	2	2	2	1	1	2	2	2	2
E	1	2	2	2	4	2	2	2	2	4	4	4
F	1	1	1	2	2	3	2	2	3	4	4	4
G	1	2	1	1	2	2	3	3	2	4	4	4
H	1	2	1	1	2	2	3	5	4	4	8	8
I	1	1	1	2	2	3	2	4	5	4	8	8
J	1	2	2	2	4	4	4	4	4	8	8	8
K	1	2	2	2	4	4	4	8	8	8	16	16
L	1	2	2	2	4	4	4	8	8	8	16	32

Upper triangular form:

	A	B	C	D	E	F	G	H	I	J	K	L
V_1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	.	1	.	1	1	.	1	1	1
V_3	.	.	1	.	1	.	.	.	.	1	1	1
V_4	.	.	.	1	1	1	.	.	1	1	1	1
V_5	.	.	.	.	.	1	1	1	1	2	2	2
V_6	.	.	.	.	.	.	.	2	2	.	4	4
V_7	.	.	.	.	.	.	.	.	.	.	.	16

Character table for image of β:

class	1	2	4	4	4	8	8	16	16	16	16
size	1	1	8	8	2	2	2	2	2	2	2
V_1	1	1	1	1	1	1	1	1	1	1	1
V_2	1	1	1	−1	1	1	1	−1	−1	−1	−1
V_3	1	1	−1	−1	1	1	1	1	1	1	1
V_4	1	1	−1	1	1	1	1	−1	−1	−1	−1
V_5	2	2	0	0	2	−2	−2	0	0	0	0
V_6	4	4	0	0	−4	0	0	0	0	0	0
V_7	16	−16	0	0	0	0	0	0	0	0	0

class	1	2	$4A$	$4B$	$4C$	$8A$	$8B$	$16A$	$16B$	$16C$	$16D$
ρ_1	1	1	1	1	1	1	1	1	1	1	1
ρ_2	1	1	1	1	−1	1	1	−1	−1	−1	−1
ρ_3	1	1	1	−1	1	1	1	−1	−1	−1	−1
ρ_4	1	1	1	−1	−1	1	1	1	1	1	1
ρ_5	2	2	2	0	0	−2	−2	0	0	0	0
ρ_6	2	2	−2	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
ρ_7	2	2	−2	0	0	0	0	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
ρ_8	2	−2	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$\zeta_{16} - \zeta_{16}^7$	$-\zeta_{16} + \zeta_{16}^7$	$-\zeta_{16}^3 + \zeta_{16}^5$	$\zeta_{16}^3 - \zeta_{16}^5$
ρ_9	2	−2	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\zeta_{16} + \zeta_{16}^7$	$\zeta_{16} - \zeta_{16}^7$	$\zeta_{16}^3 - \zeta_{16}^5$	$-\zeta_{16}^3 + \zeta_{16}^5$
ρ_{10}	2	−2	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-\zeta_{16}^3 + \zeta_{16}^5$	$\zeta_{16}^3 - \zeta_{16}^5$	$-\zeta_{16} + \zeta_{16}^7$	$\zeta_{16} - \zeta_{16}^7$
ρ_{11}	2	−2	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\zeta_{16}^3 - \zeta_{16}^5$	$-\zeta_{16}^3 + \zeta_{16}^5$	$\zeta_{16} - \zeta_{16}^7$	$-\zeta_{16} + \zeta_{16}^7$

4.3 Binary exceptional groups: $2T$, $2I$, $2O$

We discuss the three exceptional cases in the E-series of the finite subgroups of SU(2) (from Prop. A.1).

4.3.1 Binary tetrahedral group: $2T = \mathrm{SL}(2,3)$.

Group name: $2T$ ([Dok-$2T$])
Group order: $|2T| = 24$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	24	1	1	
B	8	3	1	
C	6	4	4	✓
D	4	6	3	✓
E	3	8	4	✓
F	2	12	1	✓
G	1	24	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G
A	1	1	1	1	1	1	1
B	1	3	1	3	1	3	3
C	1	1	2	2	2	4	4
D	1	3	2	4	2	6	6
E	1	1	2	2	4	4	8
F	1	3	4	6	4	12	12
G	1	3	4	6	8	12	24

Upper triangular form:

	A	B	C	D	E	F	G
V_1	1	1	1	1	1	1	1
V_2	.	2	.	2	.	2	2
V_3	.	.	1	1	1	3	3
V_4	.	.	.	.	2	.	4
V_5	.	.	.	.	.	.	4

Character table for image of β:

class	1	2	$3A$	$3B$	4	$6A$	$6B$
size	1	1	4	4	6	4	4
V_1	1	1	1	1	1	1	1
V_2	2	2	−1	−1	2	−1	−1
V_3	3	3	0	0	−1	0	0
V_4	4	−4	1	1	0	−1	−1
V_5	4	−4	−2	−2	0	2	2

Character table of irreps [Dok-2T, Mon-2T]:

over $\mathbb{C}$:

		conjugacy class						
		1	2	3A	3B	4	6A	6B
irred. repr.	ρ_1	1	1	1	1	1	1	1
	ρ_2	1	1	ζ_3^2	ζ_3	1	ζ_3^2	ζ_3
	ρ_2^*	1	1	ζ_3	ζ_3^2	1	ζ_3	ζ_3^2
	ρ_3	3	3	0	0	−1	0	0
	ρ_4	2	−2	$-\zeta_3^2$	$-\zeta_3$	0	ζ_3^2	ζ_3
	ρ_4^*	2	−2	$-\zeta_3$	$-\zeta_3^2$	0	ζ_3	ζ_3^2
	ρ_5	2	−2	−1	−1	0	1	1

over $\mathbb{R}$

		conjugacy class						
		1	2	3A	3B	4	6A	6B
irred. repr.	ρ_1	1	1	1	1	1	1	1
	$\rho_2 + \rho_2^*$	2	2	-1	-1	2	-1	-1
	ρ_3	3	3	0	0	-1	0	0
	$\rho_4 + \rho_4^*$	4	-4	1	1	0	-1	-1
	$2\rho_5$	4	-4	-2	-2	0	2	2

where $\zeta_3 := \frac{1}{2}\left(-1 + \sqrt{3}i\right)$

Hence the cokernel of β is:

$$\operatorname{coker}\Big(A(2T) \overset{\beta}{\to} R_k(2T)\Big) \simeq \begin{cases} \frac{\mathbb{Z}[\rho_2,\rho_2^*,\rho_4,\rho_4^*,\rho_8]}{\mathbb{Z}[\rho_2+\rho_2^*,\rho_4+\rho_4^*,2\rho_8]} & \mid \quad k = \mathbb{C} \\ 0 & \mid \quad k = \mathbb{R} \\ 0 & \mid \quad k = \mathbb{Q} \end{cases}$$

4.3.2 Binary octahedral group: $2O \simeq \mathrm{CSU}(2, \mathbb{F}_3)$.

Group name: $2O$ ([Dok-$2O$])
Group order: $|2O| = 48$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	48	1	1	
B	24	2	1	
C	16	3	3	
D	12	4	4	
E	8	6	3	
F	8	6	1	
G	8	6	3	✓
H	6	8	4	✓
I	4	12	6	✓
J	4	12	3	✓
K	3	16	4	✓
L	2	24	1	✓
M	1	48	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I	J	K	L	M
A	1	1	1	1	1	1	1	1	1	1	1	1	1
B	1	2	1	1	1	2	1	2	1	2	2	2	2
C	1	1	2	1	2	3	2	1	2	3	1	3	3
D	1	1	1	2	2	1	1	2	3	2	2	4	4
E	1	1	2	2	3	3	2	2	4	4	2	6	6
F	1	2	3	1	3	6	3	2	3	6	2	6	6
G	1	1	2	1	2	3	3	2	3	4	2	6	6
H	1	2	1	2	2	2	2	4	4	4	4	8	8
I	1	1	2	3	4	3	3	4	7	6	4	12	12
J	1	2	3	2	4	6	4	4	6	8	4	12	12
K	1	2	1	2	2	2	2	4	4	4	8	8	16
L	1	2	3	4	6	6	6	8	12	12	8	24	24
M	1	2	3	4	6	6	6	8	12	12	16	24	48

Upper triangular form:

	A	B	C	D	E	F	G	H	I	J	K	L	M
V_1	1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	.	.	1	.	1	.	1	1	1	1
V_3	.	.	1	.	1	2	1	.	1	2	.	2	2
V_4	.	.	.	1	1	.	.	1	2	1	1	3	3
V_5	.	.	.	.	.	.	1	1	1	1	1	3	3
V_6	.	.	.	.	.	.	.	.	.	.	4	.	8
V_7	.	.	.	.	.	.	.	.	.	.	.	.	8

Character table for image of β:

class	1	2	3	$4A$	$4B$	6	$8A$	$8B$
size	1	1	8	6	12	8	6	6
V_1	1	1	1	1	1	1	1	1
V_2	1	1	1	1	−1	1	−1	−1
V_3	2	2	−1	2	0	−1	0	0
V_4	3	3	0	−1	1	0	−1	−1
V_5	3	3	0	−1	−1	0	1	1
V_6	8	−8	2	0	0	−2	0	0
V_7	8	−8	−4	0	0	4	0	0

Character table of irreps [Dok-2O, Mon-2O]
over $\mathbb{C}$

		conjugacy class							
		1	2	3	4A	4B	6	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1
	ρ_2	1	1	1	1	−1	1	−1	−1
	ρ_3	2	2	−1	2	0	−1	0	0
	ρ_4	3	3	0	−1	−1	0	1	1
	ρ_5	3	3	0	−1	1	0	−1	−1
	ρ_6	2	−2	−1	0	0	1	$\sqrt{2}$	$-\sqrt{2}$
	ρ_7	2	−2	−1	0	0	1	$-\sqrt{2}$	$\sqrt{2}$
	ρ_8	4	−4	1	0	0	−1	0	0

over $\mathbb{R}$

		conjugacy class							
		1	2	3	4A	4B	6	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1
	ρ_2	1	1	1	1	-1	1	-1	-1
	ρ_3	2	2	-1	2	0	-1	0	0
	ρ_4	3	3	0	-1	-1	0	1	1
	ρ_5	3	3	0	-1	1	0	-1	-1
	$2\rho_6$	4	-4	-2	0	0	2	$2\sqrt{2}$	$-2\sqrt{2}$
	$2\rho_7$	4	-4	-2	0	0	2	$-2\sqrt{2}$	$2\sqrt{2}$
	$2\rho_8$	8	-8	2	0	0	-2	0	0

Hence the cokernel of β is:

$$\operatorname{coker}\left(A(2O) \xrightarrow{\beta} R_k(2O)\right) \simeq \begin{cases} \frac{\mathbb{Z}[\rho_6,\rho_7,\rho_8]}{\mathbb{Z}[2\rho_6+2\rho_7,2\rho_8]} & \mid \quad k = \mathbb{C}; \\ \frac{\mathbb{Z}[2\rho_6,2\rho_7]}{\mathbb{Z}[2\rho_6+2\rho_7]} & \mid \quad k = \mathbb{R}; \\ 0 & \mid \quad k = \mathbb{Q} \end{cases}$$

4.3.3 Binary icosahedral group: $2I \simeq \mathrm{SL}(2,5)$.

Group name: $2I$ ([Dok-$2I$])
Group order: $|2I| = 120$

Subgroups:

subgroup	order	cosets	conjugates	cyclic
A	120	1	1	
B	24	5	5	
C	20	6	6	
D	12	10	10	
E	10	12	6	✓
F	8	15	5	
G	6	20	10	✓
H	5	24	6	✓
I	4	30	15	✓
J	3	40	10	✓
K	2	60	1	✓
L	1	120	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I	J	K	L
A	1	1	1	1	1	1	1	1	1	1	1	1
B	1	2	1	2	1	2	3	1	3	3	5	5
C	1	1	2	2	2	3	2	2	4	2	6	6
D	1	2	2	3	2	4	4	2	6	4	10	10
E	1	1	2	2	4	3	4	4	6	4	12	12
F	1	2	3	4	3	6	5	3	9	5	15	15
G	1	3	2	4	4	5	8	4	10	8	20	20
H	1	1	2	2	4	3	4	8	6	8	12	24
I	1	3	4	6	6	9	10	6	16	10	30	30
J	1	3	2	4	4	5	8	8	10	16	20	40
K	1	5	6	10	12	15	20	12	30	20	60	60
L	1	5	6	10	12	15	20	24	30	40	60	120

Upper triangular form:

	A	B	C	D	E	F	G	H	I	J	K	L
V_1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	1	.	1	2	.	2	2	4	4
V_3	.	.	1	1	1	2	1	1	3	1	5	5
V_4	.	.	.	.	2	.	2	2	2	2	6	6
V_5	.	.	.	.	.	.	.	4	.	4	.	12
V_6	.	.	.	.	.	.	.	.	.	4	.	8
V_7	.	.	.	.	.	.	.	.	.	.	.	8

Character table for image of β:

class	1	2	3	4	$5A$	$5B$	6	$10A$	$10B$
size	1	1	20	30	12	12	20	12	12
V_1	1	1	1	1	1	1	1	1	1
V_2	4	4	1	0	-1	-1	1	-1	-1
V_3	5	5	-1	1	0	0	-1	0	0
V_4	6	6	0	-2	1	1	0	1	1
V_5	12	-12	0	0	2	2	0	-2	-2
V_6	8	-8	2	0	-2	-2	-2	2	2
V_7	8	-8	-4	0	-2	-2	4	2	2

Character table of irreps [Dok-2I]
over $\mathbb{C}$

		conjugacy class								
		1	2	3	4	5A	5B	6	10A	10B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1	1
	ρ_2	2	-2	-1	0	$\phi - 1$	$-\phi$	1	ϕ	$1 - \phi$
	ρ_3	2	-2	-1	0	$-\phi$	$\phi - 1$	1	$1 - \phi$	ϕ
	ρ_4	3	3	0	-1	$1 - \phi$	ϕ	0	ϕ	$1 - \phi$
	ρ_5	3	3	0	-1	ϕ	$1 - \phi$	0	$1 - \phi$	ϕ
	ρ_6	4	4	1	0	-1	-1	1	-1	-1
	ρ_7	4	-4	1	0	-1	-1	-1	1	1
	ρ_8	5	-5	-1	1	0	0	-1	0	0
	ρ_9	6	-6	0	0	1	1	0	-1	-1

over $\mathbb{R}$

		conjugacy class								
		1	2	3	4	5A	5B	6	10A	10B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1	1
	$2\rho_2$	4	-4	-2	0	$2(\phi-1)$	-2ϕ	2	2ϕ	$2(1-\phi)$
	$2\rho_3$	4	-4	-2	0	-2ϕ	$2(\phi-1)$	2	$2(1-\phi)$	2ϕ
	ρ_4	3	3	0	-1	$1-\phi$	ϕ	0	ϕ	$1-\phi$
	ρ_5	3	3	0	-1	ϕ	$1-\phi$	0	$1-\phi$	ϕ
	ρ_6	4	4	1	0	-1	-1	1	-1	-1
	$2\rho_7$	8	-8	2	0	-2	-2	-2	2	2
	ρ_8	5	5	-1	1	0	0	-1	0	0
	$2\rho_9$	12	-12	0	0	2	2	0	-2	-2

where $\phi := \frac{1}{2}\left(1+\sqrt{5}\right)$ is the golden ratio.
Hence the cokernel of β is

$$
\begin{array}{l|l}
V_1 = \rho_1 & \\
V_2 = \rho_6 & \\
V_3 = \rho_8 & \\
V_4 = \rho_4 + \rho_5 & \operatorname{coker}(\beta_k) \;=\; \begin{cases} \frac{\mathbb{Z}[\rho_2,\rho_3,\rho_4,\rho_5,\rho_7,\rho_9]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5,2\rho_7,2\rho_9]} & \mid \;\; k=\mathbb{C} \\ \frac{\mathbb{Z}[2\rho_2,2\rho_3,\rho_4,\rho_5]}{\mathbb{Z}[2\rho_2+2\rho_3,\rho_4+\rho_5]} & \mid \;\; k=\mathbb{R} \\ 0 & \mid \;\; k=\mathbb{Q} \end{cases} \\
V_5 = 2\rho_9 & \\
V_6 = 2\rho_7 & \\
V_7 = 2\rho_2 + 2\rho_3 &
\end{array}
$$

4.3.4 The general linear group: $\mathrm{GL}(2,\mathbb{F}_3)$

Group name: $\mathrm{GL}(2,\mathbb{F}_3)$ ([Dok-GL(2, 3)][1])
Group order: $|\mathrm{GL}(2,\mathbb{F}_3)| = 48$

Subgroups:

---

[1] The representation theory of this group $\mathrm{GL}(2,\mathbb{F}_3)$ is deceptively similar to that of the binary octahedral group $2O \simeq \mathrm{CSU}(2,\mathbb{F}_3)$ (discussed Sect. 4.3.2): Both have the same character table over $\mathbb{C}$, the only difference being in the Schur indices, hence in the real character table. In fact, several online databases of character tables had misidentified the two groups, which became apparent when our computation of the image of β revealed real representations of $2O$ that contradicted available character tables. We are indebted to James Montaldi for patiently double-checking computations with us and to Tim Dokchitser for swiftly recognizing and fixing the issue with the databases.

subgroup	order	cosets	conjugates	cyclic
A	48	1	1	
B	24	2	1	
C	16	3	3	
D	12	4	4	
E	8	6	3	✓
F	8	6	1	
G	8	6	3	
H	6	8	4	
I	6	8	4	
J	6	8	4	✓
K	4	12	6	
L	4	12	3	✓
M	3	16	4	✓
N	2	24	1	✓
P	2	24	12	✓
Q	1	48	1	✓

Table of multiplicities:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q
A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B	1	2	1	1	1	2	1	2	1	1	1	2	2	1	2	2
C	1	1	2	1	2	3	2	1	1	1	2	3	1	2	3	3
D	1	1	1	2	1	1	2	2	2	2	3	2	2	3	4	4
E	1	1	2	1	3	3	2	2	1	1	3	4	2	3	6	6
F	1	2	3	1	3	6	3	2	1	1	3	6	2	3	6	6
G	1	1	2	2	2	3	3	2	2	2	4	4	2	4	6	6
H	1	2	1	2	2	2	2	4	2	2	4	4	4	4	8	8
I	1	1	1	2	1	1	2	2	3	3	3	2	4	5	4	8
J	1	1	1	2	1	1	2	2	3	3	3	2	4	5	4	8
K	1	1	2	3	3	3	4	4	3	3	7	6	4	7	12	12
L	1	2	3	2	4	6	4	4	2	2	6	8	4	6	12	12
M	1	2	1	2	2	2	2	4	4	4	4	4	8	8	8	16
N	1	1	2	3	3	3	4	4	5	5	7	6	8	13	12	24
P	1	2	3	4	6	6	6	8	4	4	12	12	8	12	24	24
Q	1	2	3	4	6	6	6	8	8	8	12	12	16	24	24	48

Upper triangular form:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q
V_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	.	1	.	.	.	1	.	1	.	.	.	1	1	.	1	1
V_3	.	.	1	.	1	2	1	.	.	.	1	2	.	1	2	2
V_4	.	.	.	1	.	.	1	1	1	1	2	1	1	2	3	3
V_5	.	.	.	.	1	.	.	1	.	.	1	1	1	1	3	3
V_6	.	.	.	.	.	.	.	.	1	1	.	.	2	2	.	4
V_7	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	4

Character table for image of β:

class	1	$2A$	$2B$	3	4	6	$8A$	$8B$
size	1	1	12	8	6	8	6	6
V_1	1	1	1	1	1	1	1	1
V_2	1	1	-1	1	1	1	-1	-1
V_3	2	2	0	-1	2	-1	0	0
V_4	3	3	1	0	-1	0	-1	-1
V_5	3	3	-1	0	-1	0	1	1
V_6	4	-4	0	1	0	-1	0	0
V_7	4	-4	0	-2	0	2	0	0

Character table of irreps [Dok-GL(2, 3)]:

over $\mathbb{C}$

		conjugacy class							
		1	2A	2B	3	4	6	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1
	ρ_2	1	1	-1	1	1	1	-1	-1
	ρ_3	2	2	0	-1	2	-1	0	0
	ρ_4	3	3	-1	0	-1	0	1	1
	ρ_5	3	3	1	0	-1	0	-1	-1
	ρ_6	2	-2	0	-1	0	1	$-\sqrt{2}$	$\sqrt{2}$
	ρ_7	2	-2	0	-1	0	1	$\sqrt{2}$	$-\sqrt{2}$
	ρ_8	4	-4	0	1	0	-1	0	0

over $\mathbb{R}$

		conjugacy class							
		1	2A	2B	3	4	6	8A	8B
irred. repr.	ρ_1	1	1	1	1	1	1	1	1
	ρ_2	1	1	-1	1	1	1	-1	-1
	ρ_3	2	2	0	-1	2	-1	0	0
	ρ_4	3	3	-1	0	-1	0	1	1
	ρ_5	3	3	1	0	-1	0	-1	-1
	$\rho_6 + \rho_7$	4	-4	0	-2	0	2	0	0
	ρ_8	4	-4	0	1	0	-1	0	0

Hence the cokernel of β is

$$\mathrm{coker}\Big(A\big(\mathrm{GL}(2,\mathbb{F}_3)\big) \overset{\beta_k}{\to} R_k\big(\mathrm{GL}(2,\mathbb{F}_3)\big)\Big) \simeq \left\{ \begin{array}{ccc} \frac{\mathbb{Z}[\rho_6,\rho_7]}{\mathbb{Z}[\rho_6+\rho_7]} & \mid & k = \mathbb{C} \\ 0 & \mid & k = \mathbb{R} \\ 0 & \mid & k = \mathbb{Q} \end{array} \right.$$

A Background

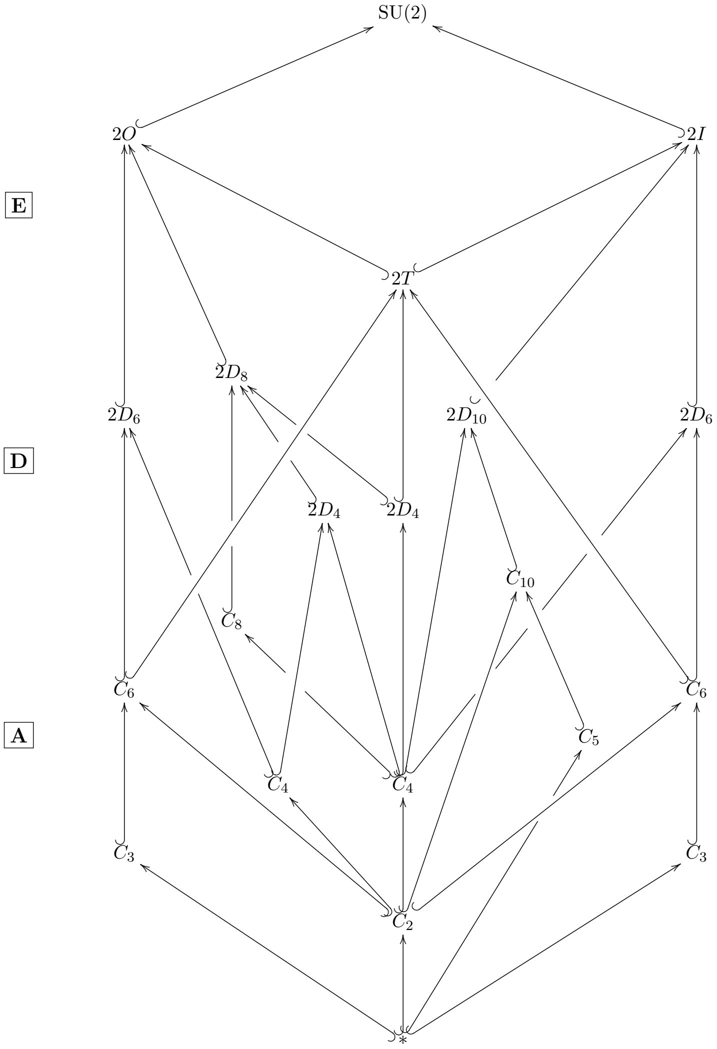
To be reasonably self-contained, we briefly collect some background material.

A.1 The Platonic groups

Proposition A.1 (ADE-classification of the finite rotation groups [Klein1884])**.** *The finite subgroups of* SU(2) *are given, up to conjugacy, by the following classification (where $n \in \mathbb{N}$):*

Dynkin label	**Finite subgroup of** $\mathrm{SO}(3)$	**Name of group**	**Finite subgroup of** $\mathrm{SU}(2)$	**Name of group**
$\mathbb{A}_{n\geq 1}$	C_{n+1}	*Cyclic*	C_{n+1}	*Cyclic*
$\mathbb{D}_{n\geq 4}$	$\mathrm{D}_{2(n-2)}$	*Dihedral*	$2\mathrm{D}_{2(n-2)}$	*Binary dihedral*
$\mathbb{E}_6$	T	*Tetrahedral*	2T	*Binary tetrahedral*
$\mathbb{E}_7$	O	*Octahedral*	2O	*Binary octahedral*
$\mathbb{E}_8$	I	*Icosahedral*	2I	*Binary icosahedral*

Full proof for finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ is in [MBD1916], recalled in [Ser14, Sec. 2]. Full proof for $\mathrm{SO}(3)$ is spelled out in [Ree05, Theorem 11]; from this the case of $\mathrm{SU}(2)$ is given in [Kee03, Theorem 4]. See also [Lindh2018, Chapter 1,2] for an elementary treatment.



The full subgroup lattice of $SU(2)$ **under the three exceptional finite subgroups** from Prop. A.1 (using the subgroup lattices from [Dok-$2T$, Dok-$2O$, Dok-$2I$]).

A.2 Categorical algebra

For ease of reference, we briefly recall the concept of internal homs in compact closed categories [Bo94, vol 2, 6.1], also [Mac65].

The categories $G\mathrm{Set}^{\mathrm{fin}}$ (Def. 3.1) and $G\mathrm{Rep}_k^{\mathrm{fin}}$ (Def. 3.2) enjoy various properties that are directly analogous to familiar properties of the category $\mathrm{Vect}_k^{\mathrm{fin}}$ of finite-dimensional k-vector spaces. The language of categorical algebra allows to make these analogies explicit, and such that one may reason uniformly in all three cases.

For $V, W \in \mathrm{Vect}_k^{\mathrm{fin}}$ two vector spaces, the set $\mathrm{Hom}(V,W)$ of linear maps ("homomorphisms") $V \to W$ between them becomes itself canonically a vector space, by pointwise multiplication with k and pointwise addition of values of functions. When we want to emphasize that we regard the set $\mathrm{Hom}(V,W)$ as equipped with this vector space structure, we write $[V,W]$ for it.

One way to make this vector space of linear functions $[V,W]$ more explicit is to consider the *dual* vector space V^*: With that in hand we have a canonical linear isomorphism

$$\begin{tikzcd}
W \otimes V^* \arrow[r, "\simeq"] & {[V,W]} \\
(|w\rangle \otimes \langle v|) \arrow[r] & \Big(|q\rangle \mapsto |w\rangle \cdot \underbrace{\langle v, q\rangle}_{\in k}\Big)
\end{tikzcd}$$

which identifies the tensor product space of V^* with W as the vector space of linear maps from V to W. Here

$$\langle -, - \rangle \;:\; V^* \otimes V \longrightarrow k = \mathbf{1}$$

denotes the pairing map that defines the dual vector space, and we denote elements of V by $|q\rangle \in V$ and those of the dual vector space by $\langle v| \in V^*$, just so as to bring out the pattern better. Note that this pairing map is itself k-linear. Hence if we regard the ground field k as the canonical 1-dimensional k-vector $\mathbf{1}$, as indicated, then this is actually a morphism in $\mathrm{Vect}_k^{\mathrm{fin}}$. There is also a closely related linear going the other way around:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\eta"] & {[V,V]} & \simeq \; V \otimes V^* \\
1 \arrow[r, mapsto] & \mathrm{id}_V &
\end{tikzcd}$$

which, under the above identification, sends any element $c \in k$ to the linear map from V to itself that is given by multiplication with c. One readily checks that these two functions make the following triangles commute

$$\begin{tikzcd}
& V \otimes V^* \otimes V \arrow[dr, "\mathrm{id}\otimes\langle -,-\rangle"] & \\
V \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, "\mathrm{id}"] & & V
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^* \otimes V \otimes V^* \arrow[dr, "\langle -,-\rangle\otimes\mathrm{id}"] & \\
V^* \arrow[ur, "\mathrm{id}\otimes\eta"] \arrow[rr, "\mathrm{id}"] & & V^*
\end{tikzcd}$$

whence called the *triangle identities*.

The quickest way to convince oneself that this indeed holds is to choose linear identifications $V \simeq \mathbb{R}^n$ and $W \simeq \mathbb{R}^m$, which means to choose *linear bases*. This in turn induces a canonical identification $V^* \simeq (\mathbb{R}^n)^* \simeq \mathbb{R}^n$ (the *dual linear basis*), hence a linear identification

$$[V,W] \simeq V^* \otimes W \;\simeq\; \mathbb{R}^n \otimes \mathbb{R}^m \simeq \mathbb{R}^{n\times m} \simeq \mathrm{Mat}_{n\times m}(k)$$

of the vector space of linear maps $V \to W$ with the vector space of $n \times m$ matrices.

The description of dual vector spaces in terms of pairing and co-pairing maps satisfying triangle identities, as above, turns out to be *equivalent* to the traditional definition. It may seem more involved than the direct definition, but it has the great advantage that it makes sense without any actual reference to the nature of vector spaces: all that is needed to speak of *dual objects* is the analogue of the tensor product $\otimes$.

Categorical algebra shows that the *triangle identities* guarantee that $V^* \otimes W$ behaves like an "internalized" version of the Hom-set. The same applies to the tensor product of representations used in Section 3.

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