Introduction to Hypothesis H

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Abstract

The key open question of contemporary mathematical physics is elucidation of the currently elusive fundamental laws of strongly-interacting "non-perturbative" quantum states, including bound states as mundane as nucleons but more generally of quarks confined inside hadrons, as well as strongly-correlated ground states of topologically ordered quantum materials.

The seminal strategy of regarding such systems as located on branes inside a higher dimensional string-theoretic spacetime (the "holographic principle") shows all signs of promise but has been suffering from the ironic shortcoming that also string theory has only really been defined perturbatively. However, string theory exhibits a web of *hints* towards the nature of its non-perturbative completion, famous under the working title "M-Theory" but still elusive. Thus, mathematically constructing M-theory should imply a mathematical understanding of quantum brane worldvolumes which should solve non-perturbative quantum physics: the M-strategy for attacking the Millennium Problem.

After a time of stagnation in research towards M-theory, we have recently formulated and extensively tested a hypothesis on the precise mathematical nature of at least a core part of the theory: We call this *Hypothesis H* since it postulates that M-branes are classified by Co-*H*omotopy-theory in much the same way that D-branes are expected to be classified by *K*-theory (a widely held but just as conjectural belief which might analogously be called *Hypothesis K*). In fact, stabilized coHomotopy is equivalently the algebraic K-Theory over "$\mathbb{F}_1$", the "absolute base field with one element". Last not least, coHomotopy is equivalent to framed Cobordism cohomology.

In these lecture notes we try to give an introduction to (1.) the motivation and (2.) some consequences of Hypothesis H, assuming an audience with a little background in electromagnetism, differential geometry and algebraic topology.

Contents

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The latest version of this document is available at: ncatlab.org/schreiber/show/Introduction+to+Hypothesis+H.

These are notes under development,
prepared for a series of talks and lectures;
parts are still not more than a slide show.

The first half aims to be elementary explanation of
Hypothesis H as a good question to ask about physics:
whether it is right or wrong, it deserves checking.

The second half explains evidence
that Hypothesis H is in fact correct
and some insights gained from it.

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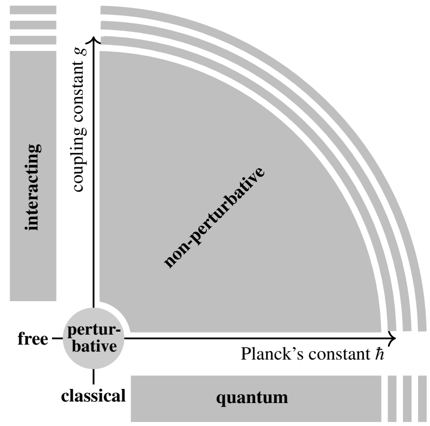
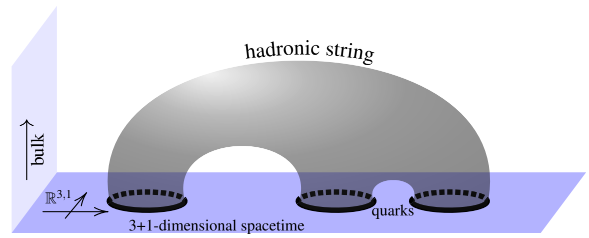
The key open question of fundamental quantum physics is not primarily the lack of coherent quantum gravity theory as such, as often portrayed, but the general lack of *non-perturbative* quantum theory of almost any sort, due to which exotic quantum states of matter – such as topologically ordered solid states thought to be needed for topological quantum computation – but even mundane phenomena – such as room-temperature matter, namely "confined" quarks in hadron bound states, reflected (just as are topological phases!) in a "mass gap" – remain theoretically ill-understood, to the extent that one speaks of an open *Millennium Problem*[1].

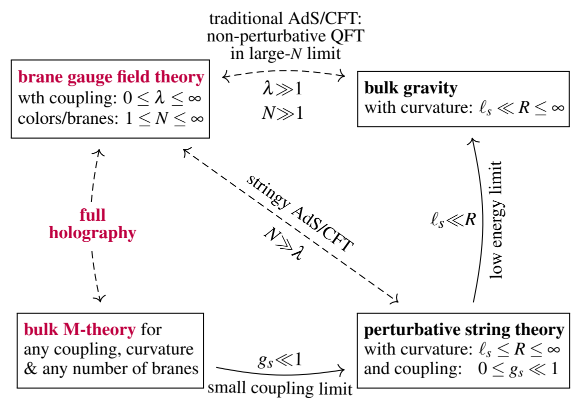


The role of string theory. String theory originates as a model for these elusive hadron bound states, specifically for the string-like "flux tubes" between pairs of quarks, conceptually explaining both their confinement and their scattering behaviour. The unexpected discovery that subtle quantum effects make these *hadronic* strings propagate in an effectively higher dimensional space – with only their endpoint quarks attached to observed 3+1 dimensional spacetime (now: the "brane") or else carrying gravitons into an otherwise unobserved higher dimensional "bulk" – came to be appreciated as a "holographic" description of non-perturbative quantum physics.[2]



The role of M-theory. Ironically, string dynamics is itself primarily understood only perturbatively, which makes holography require the unrealistic assumption of a large (in fact: humongous) number N of coincident branes, to be tractable. But understanding branes as physical objects yields a web of hints as to what non-perturbative string theory should be like, enough so that it famously has a working title (since 1995): "M-theory".

To highlight, in conclusion: One strategy for addressing the "Millennium Problem" of formulating non-perturbative QFT is to mathematically formulate M-theory:[3] With this it ought to be possible to define and investigate, with precision, individual quantum branes whose intersections should exhibit non-perturbative quantum dynamics such as anyonic topological order (which we discuss in §7) and eventually confined hadrodynamics.



bare gauge coupling	g_{YM}
number of colors/branes	N
actual (t'Hooft-)coupling	$\lambda = g_s N = g_{\mathrm{YM}}^2 N$
string coupling	$g_s = g_{\mathrm{YM}}^2 = \lambda / N$
spacetime curvature	$1/R$
string length	$\ell_s = R / \sqrt[4]{\lambda}$

The role of Algebraic Topology. After initial excitement, progress on actually formulating M-theory had stagnated and efforts had been largely abandoned[4], arguably due to a lack of appropriate mathematical tools: Where famous examples of physical theories were formulated within a fairly well-understood framework of mathematical principles (e.g. general relativity in differential geometry or quantum physics in functional analysis), the real problem with formulating M-theory is (or was) that even its underlying mathematical principles remained unclear. It was the vision of [Sa10] (review in [FSS19]) that M-theory ought to find its formulation in *algebraic topology*; initiating a program of looking for algebro-topological patterns in the available information on M-brane physics, deducing clues as to their fundamental mathematical meaning.

The role of Hypothesis H. This analysis eventually culminated in a formulation of a hypothesis – *Hypothesis H* – of what M-theory really is about [FSS20], namely about the generalized non-abelian cohomology theory called *CoHomotopy Theory*. This we explain below in §3.

It is noteworthy here that algebraic topology is not a field of mathematics as any other, but has recently been understood to serve, in its guise of *homotopy theory*, as an alternative *foundation* for mathematics itself (`HoTT`[5]). Moreover, within algebraic topology, cohomotopy is not a (multiplicative) cohomology theory as any other, but is *initial* among all of them. This may be more than a coincidence given that M-theory is meant to be not just a theory of physics as any other, but the initial foundation of all of them.

	Physics	**Mathematics**
Principles	Gauge Principle[6]	Homotopy Th.
Foundations	M-Theory	Alg. Topology
Basic Notions	Fluxes/Charges	Cohomology
Initial Notions	M-Brane Charge	Cohomotopy
Derived Notions	D-brane charge	K-Theory

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[1] The pronounciation of the problem of confinement of quarks, hence of the "mass gap in Yang-Mills theory", as a "Millennium Problem" is due to the Clay Mathematics Institute [CMI00][JaWi00][Atiyah00] and referenced as such by nuclear physicists in [KT10][Shn05, §9.1][Ha12][BaGo19, p. 2][RS20, p. 2][Ro21, ftn. 2][Ro22, p. 2].

[2] The string theory mainstream had actually abandoned the identification of strings with hadronic flux tubes in the 1970s (cf. [Ve12, pp. 30]), but came back full circle to re-appreciate this perspective in the 2010s, now under the name "holographic QCD" (cf. [Ha12, §6.4][Er15][RZ16, §4][DBLM21]), after both AdS/CFT duality and intersecting D-brane models had been intensively studied for a decade, for assessment see [Witten23, 40:13-]. But all along this holographic perspective on hadronic strings had been promoted by Alexander Polyakov [Pol80][Pol97][GKP98][Pol99][Pol02] under the name *gauge-string correspondence*, referring to the brane/bulk-perspective as the *wall and the cave* in reference to Plato, see the historical reminiscences in [Pol12].

[3] The possibility that the Yang-Mills Mass Gap Problem might be solved by developments in M-theory was mentioned already, albeit briefly, in Atiyah's original presentation of the Clay Millennium Problems, see [Atiyah00], starting at 41:47.

[4] Cf. G. Moore: *Keep true to the dreams of thy youth: M-theory*, §12 of: *Physical Math and the Future*, talk at Strings2014 [www.physics.rutgers.edu/ gmoore/PhysicalMathematicsAndFuture.pdf]

[5] A review, in our context, of Homotopy Type Theory (`HoTT`) as a foundations of mathematics may be found in [TQC2], complete with a comprehensive list of further references.

[6] Here by the *gauge principle* we mean the principle which says that: In general it makes no sense to ask whether a pair of things (field configurations) are equal or not, but that one has to ask for a *gauge transformation* between the two. And so forth: Given a pair of such gauge transformations, it makes in general no sense to ask whethether they are equal or not, but one has to ask for a *gauge-of-gauge* transformation between them, a *higher gauge transformation*. Mathematically this is the notion of *homotopy* and *higher homotopy* in homotopy theory.

1 Branes imprinted on flux

The concept of *branes* in string theory (see [IU12, §6][Fr13, §7][HSS19, §2]) is the key ingredient of the historical re-thinking of string theory that came to be known as the "second superstring revolution" [Schw96], in that it is the key for the non-perturbative completion of the theory [Du00] (the "M" in "M-theory" originates [HW96, p. 2] as a "non-committal" abbreviation for *membrane*). But conversely this means that the precise meaning of "brane" has been almost as elusive as that of "M-theory" itself. Or rather: There is a range of specialized meanings of the term, some versions of which do have precise definitions, but it has remained unclear how exactly any and all of these notions are aspects of a unified concept of "branes".

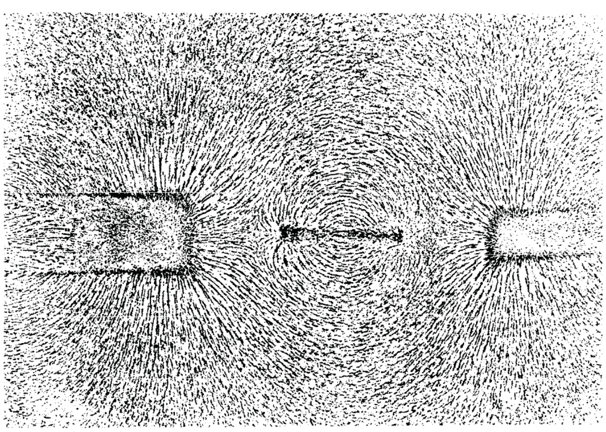
Our strategy in formalizing the concept of "branes" is conceptually straightforward:
- We recall here the notion of branes as higher-dimensional generalizations of *poles* and hence as *concentrations of flux*.
- We rephrase (§2) this notion in appropriate algebro-topological terms, from which we motivate (§3) *Hypothesis H*.
- All further exploration of the notion of branes proceeds by mathematical inspection of Hypothesis H.

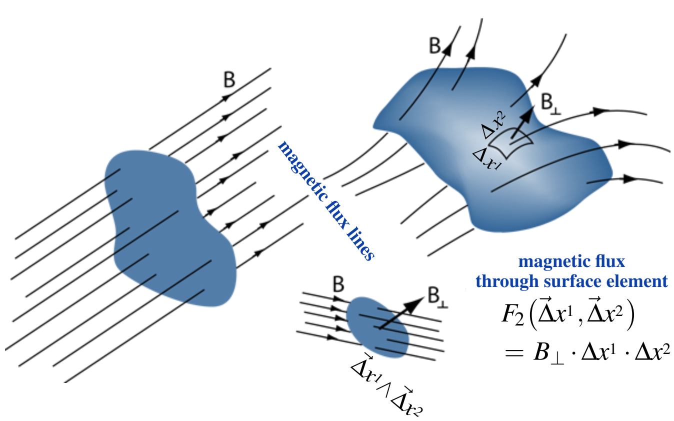
1.1 Branes as concentrations of flux

To get ground under our feet, it is expedient – our ambitious goal nonwithstanding – to start with elementary reflections on *flux lines* (*flux densities*) sourced by charged *poles* as originally conceived by Faraday, and as more generally sourced by higher dimensional charged *branes*, like the charged *membranes* already considered by [Dirac1962]. While most of these objects (famously including magnetic mono-poles) are notorious for remaining hypothetical entities not currently seen in experiment (possibly because the do not actually exist, possibly because they do exist but remain undetectable by present means), we highlight (p. 6) the example of *vortex strings* in superconductors which have been observed in detail and which – whether one likes to refer to them as "1-branes" or not – do constitute an example of the general notion of classical branes in question.[7] This may seem of little relevance to a reader used to the zoo of (hypothetical) branes considered in string theory, but since we are going to completely sidestep traditional discussion of string theory and instead bootstrap M-theory out of just a close mathematical inspection of the possible nature of charged membranes in 11-dimensional supergravity, it may be noteworthy.

Field flux.	
$X \in \mathrm{Mfds}$	spacetime manifold
$\Omega^r_{\mathrm{dR}}(X) \in \mathrm{Sets}$	differential r-forms
$F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X)$	flux density form
$\star : \Omega^r_{\mathrm{dR}} \to \Omega^{D-r}_{\mathrm{dR}}$	Hodge star (§1.2)

Classical Example: Electromagnetic flux	
$X = \mathbb{R}^{3,1}$	Minkowski spacetime
$\Omega^1_{\mathrm{dR}}(\mathbb{R}^{3,1}) = \left\{ A_i\, \mathrm{d}x^i + \phi\, \mathrm{d}t \right\}$	vector potentials
$F_2 \;:\; \Omega^2_{\mathrm{dR}}(\mathbb{R}^{3,1})$	Faraday tensor
$= E_i\, \mathrm{d}x^i \wedge \mathrm{d}t$	electric field strength
$+ B_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$	magnetic flux density





Magnetic flux lines. On the left: Faraday's original iron filings in the magnetic field of two rod magnets (*Faradays diary of experimental investigation*, entry of 11th Dec 1851, reproduced by Martin 2009). On the right: schematics (adapted from hyperphysics.phy-astr.gsu.edu/hbase/magnetic/fluxmg.html)

Imprinted on the flux density may be *two kinds* of branes, called:[8]

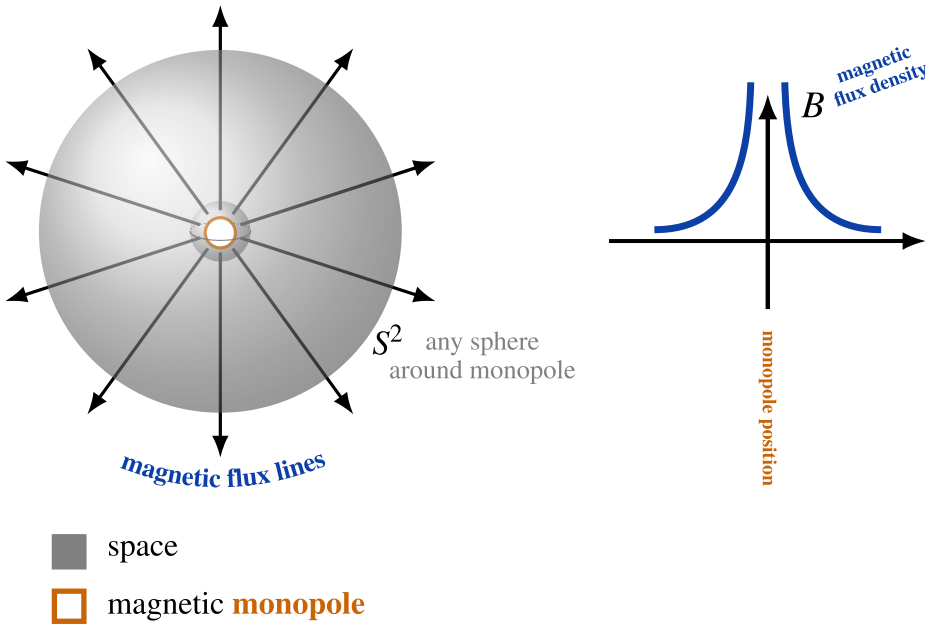
(1.) **singular branes** (*black branes*) reflected in **diverging flux density** at **singular loci** in spacetime,

(2.) **solitonic branes** reflected in *localized* but **finite flux density**, namely **vanishing at infinity**.

[7] [Pol12, p. 1] regrets not to have understood vortex lines as strings. See also the emphasis on vortex *worldsheets* in [Beekman & Zaanen 2011].

The **singular branes of 4d electromagnetism** are the (would-be) **magnetic monopoles**:

$$\overset{\text{Faraday tensor}}{F_2} \;=\; \overset{\text{magnetic flux density}}{B(r)}\,\mathrm{dvol}_{S^2}\,, \qquad \overset{\substack{\text{Gauss law}\\ \text{away from singular locus}}}{\mathrm{d}F_2 \;=\; 0}\,, \qquad \int_{S^2} F_2 \;=\; \text{total magnetic flux prop. to total charge}$$



But the **solitonic branes of 4d electromagnetism** are the **vortex strings** in type II super-conductors ("Abrikosov vortices") inside an external electromagnetic field. Here the 1-brane is the central locus (the eye of the storm) of a (non-singular) *vortex* in the electron current J, localized by the requirement that fields *vanish at infinity* (cf. eg. [Timm 2020 (6.101)]):

$$\overset{\text{Faraday tensor}}{F_2} \;=\; \overset{\text{magnetic flux density}}{B(r)}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \qquad \overset{\text{Gauss law}}{\mathrm{d}F_2 \;=\; 0}\,, \qquad \int_{\mathbb{R}_1 \times \mathbb{R}_2} F_2 \;=\; \int B\,\mathrm{d}x^1\,\mathrm{d}x^2 \;=\; \text{total magnetic flux prop. to \# vortices}$$

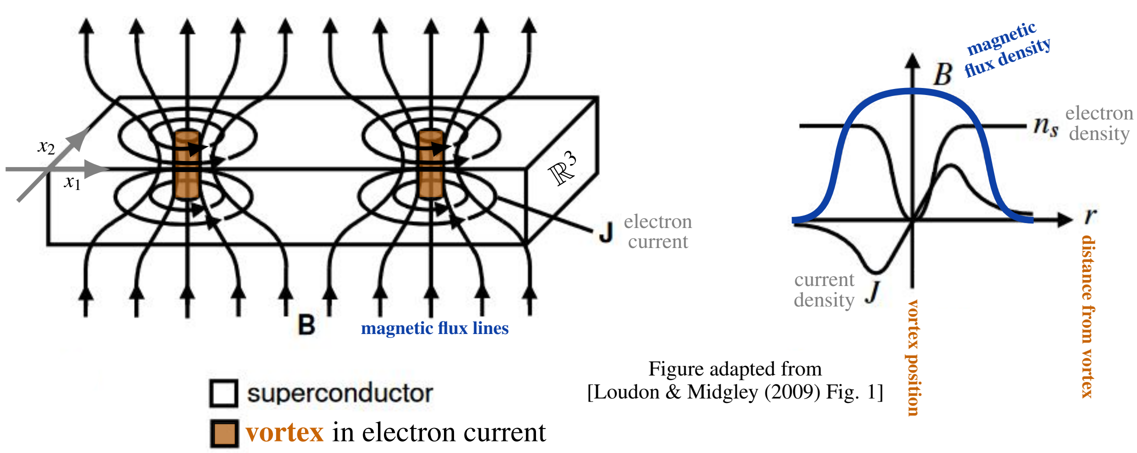


Figure adapted from
[Loudon & Midgley (2009) Fig. 1]

Abrikosov- Dirac-	*vortex strings* *magnetic monopoles*	are the	solitonic branes singular branes	associated with the EM field

[8]The terminology "solitonic brane" is wide-spread but its exact meaning differs between authors (as does the term "soliton" that it is derived from): It was introduced in [DKL92][DKL95][DL94] to mean (topologically stable) *non-singular* brane-like solutions to (supergravity/flux) equations of motion, which is how we use it here. But already [St99] uses "solitonic" to instead mean the "electromagnetic dual singular brane", eg. calling the singular NS_5 the soliton of the fundamental string, cf. (49). Somewhat in this vein, many later authors (eg. [Sm03]) use "solitonic" for any singular or non-singular brane-like supergravity solution, thus regarding it as the antonym to the fundamental sigma-model branes discussed in §4.1. This is how we ourselves use it elsewhere.

The **formalization of the difference between singular and solitonioc branes** is via choice of *domains* on which the flux densities are actually defined (following [HpH2, §2.1]).

type of brane	$\Leftrightarrow$	**domain of flux density** (19)	
singular brane		complement of brane Σ inside spacetime X, removing the singular locus from spacetime	$X \setminus \Sigma$
solitonic brane		Alexandroff-compactification of transverse space $\Sigma^{\perp}$, adjoining a "transverse point at infinity" to spacetime	$\Sigma^{\perp}_{\cup\{\infty\}}$

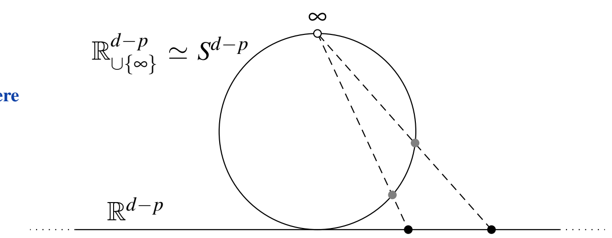
This is most transparent for the special case of "flat" branes in flat Minkowski spacetime:

- **singular branes** have spacetime singularities which are *removed from spacetime*: the field flux sourced by the singularity is that through spheres in the normal bundle around these loci and *would diverge* at the singular brane locus.

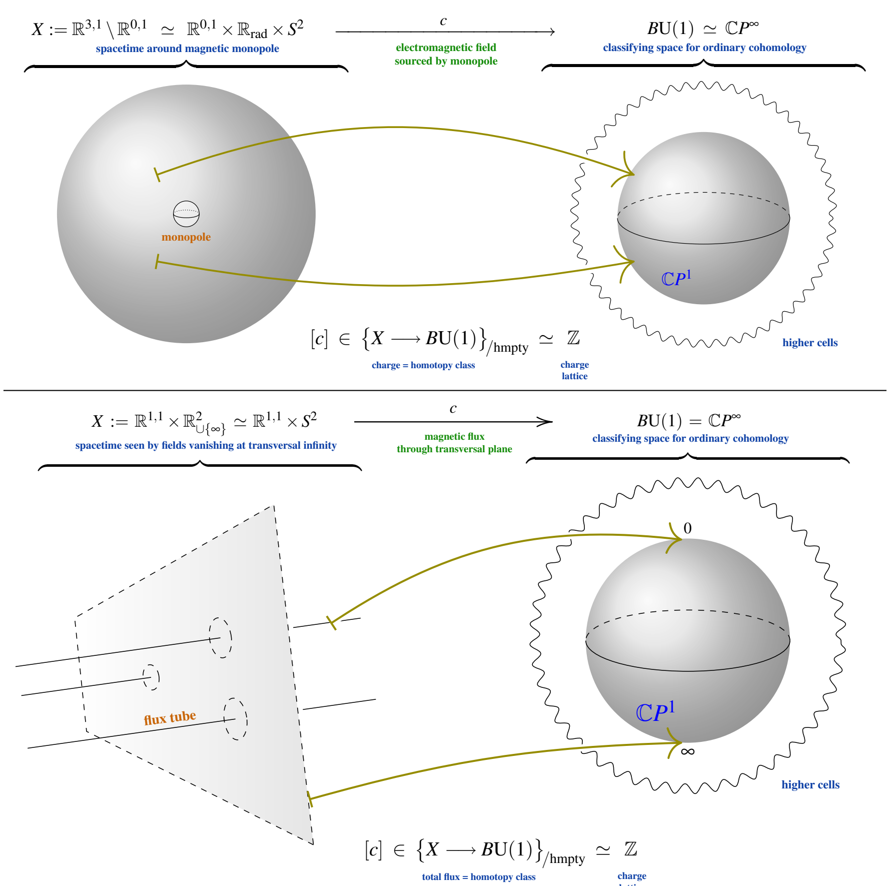
$$\overset{\text{bulk}}{\mathbb{R}^{d+1}} \setminus \overset{\substack{\text{singular}\\ \text{brane}}}{\mathbb{R}^{p+1}} \underset{\text{homeomorphism}}{\simeq} \overset{\substack{\text{punctured}\\ \text{transverse space}}}{\big(\mathbb{R}^{d-p} \setminus \{0\}\big)} \times \mathbb{R}^{p+1} \underset{\text{homotopy equivalence}}{\simeq} \overset{\text{encircling sphere}}{S^{d-p-1}} \tag{1}$$

- **solitonic branes** are witnessed by non-singular "local bumps" in the flux densities: Their flux *vanishes at infinity* which means that it is measured on the 1-point compactification of their transverse space, which is again a sphere:

$$\overset{\substack{\text{solitonic}\\ \text{brane}}}{\mathbb{R}^{p+1}_{\sqcup\{\infty\}}} \wedge \underset{\substack{\text{with point}\\ \text{at infinity}}}{\overset{\substack{\text{transv.}\\ \text{space}}}{\mathbb{R}^{d-p}_{\cup\{\infty\}}}} \simeq \mathbb{R}^{d-p}_{\cup\{\infty\}} \underset{\text{homeomorphism}}{\simeq} \overset{\text{transverse sphere}}{S^{d-p}} \tag{2}$$



Towards flux quantization. The laws of flux discussed so far are laws of "classical physics": By themselves they do not explain, for instance, why the flux carried by Abrikosov vortices (p. 6) is *quantized* to appear in integer multiples of a unit flux, or why, as argued long ago by Dirac, magnetic monopoles would be quantized to appear in integer multiples of unit charged monopoles. Apparently the electromagnetic flux density $F_2 = \Omega^2_{\mathrm{dR}}(X)$ is just one aspect of the true nature of the electromagnetic field. In modern mathematical language, the argument underlying *Dirac charge quantization* says that an electromagnetic field configuration on a spacetime X *also* involves a "charge map" $c : X \to BU(1)$ to the *classifying space* of the circle group. This may be understood as the infinite complex projective space $BU(1) \underset{\text{whe}}{\simeq} \mathbb{C}P^{\infty}$, but crucially it is a *classifying space* for ordinary integral cohomology in degree 2, meaning that homotopy classes of such maps are in natural bijection with $H^2(X;\mathbb{Z})$.



Formalizing generalized flux quantization is the topic of §2.

1.2 Laws of flux

As we now turn to the classical laws of motion for flux densities (the analogs of Maxwell's equations), the key move towards identifying possible flux quantization laws (below in §2) is to arrange these equations of motion, equivalently, as:

(1.) a purely **cohomological** system of differential equations known as higher **Bianchi identities**,

(2.) a purely **geometric** system of linear equations expressing a Hodge **self-duality**,

the point being that the first item is entirely "topological", while dependency on geometry, namely on the spacetime metric (the field of gravity) is all isolated in the second item:

$$
\begin{tikzcd}[row sep=small]
 & \boxed{\begin{array}{c}\textbf{higher Bianchi identities}\\ \text{(pre-geometric)}\end{array}} \\
\boxed{\begin{array}{c}\textbf{equations of motion}\\ \text{for higher flux forms}\\ \text{in background gravity}\end{array}} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\
 & \boxed{\begin{array}{c}\textbf{higher self-duality}\\ \text{(geometric)}\end{array}}
\end{tikzcd}
\tag{3}
$$

This move of looking at "pre-metric flux equations" first, later to be supplemented by a "constitutive" duality constraint has a curious status in the literature: On the one hand it is elementary and immediate as an equivalent re-formulation of the usual form of equations of motion, and as such has been highlighted, in the case of electromagnetism (4), a century ago [Ko1922][Cartan1924, §80][vDa1934][Whit1953, pp. 192], and re-amplified more recently under the name "pre(geo)metric electromagnetism" [HO03][Del05][HIO16][Del]; but the broader community does not seem to have taken much note of this yet. On the other hand, we may observe below in (5) and (8) that just the same "pregeometric" perspective, applied to higher degree flux forms, evidently underlies, entirely independently, what string theorists call "duality-symmetric" or (for better or worse) "democratic" formulations of supergravity fields, notably underlying the highly recognized conjecture that RR-fields are flux-quantized in K-theory ("Hypothesis K", to which we come below in §3, when we have discussed twistings).

Pregeometric electromagnetism. Concretely, Maxwell's equations for electromagnetic flux encoded in the Faraday tensor F_2 as shown on the left (cf. [Th78, v2 §1.3][Fr97, §7.2b][Na11, §2.2]) have (pre)geometric decomposition as shown on the right (cf. already [Cartan1924, §80]):

$$
\begin{tikzcd}[row sep=small]
 & \boxed{\begin{array}{rcl}\mathrm{d}F_2 &=& 0\\ \mathrm{d}G_2 &=& J_3\end{array}} \\
\boxed{\begin{array}{rcl}\mathrm{d}F_2 &=& 0\\ \mathrm{d}\star F_2 &=& J_3\end{array}} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\
 & \boxed{G_2 = \star F_2}
\end{tikzcd}
\tag{4}
$$

While trivially equivalent, some authors found deep significance to the pre-geometric decomposition on the right of (4), (cf. the careful discussion of classical electromagnetism in [HO03]) highlighting that it maximally disentangles gauge from gravitational degrees of freedom (already in [Whit1953, p. 192][9]) and thus possibly helping with understanding unification of the two (cf. [Del15]). None of what we say here refers to or relies on any of these previous discussions of premetric electromagnetism, but the inclined reader may find value in comparing to them.

It is clear that an analogous transmutation – by first "doubling" the flux degrees of freedom and then cutting them back down by half via a self-duality constaint – applies also to equations of motion for higher degree fluxes (made explicit for instance in [Fr00, Exp. 3.8]):

Pregeometric RR-fields. In evident higher-degree generalization of vacuum electromagnetism, consider a "higher gauge field" whose flux density is a tuple of differential forms F_{2p} in every second degree smaller or equal the spacetime dimension D: and satisfying the evident higher-degree generalization of Maxwell's equations, as shown on the left, where the equivalent pregeometric formulation shown on the right is rather more suggestive in its conceptual simplicity (here "$\bullet$" may be taken to range through $\mathbb{N}$ or even $\mathbb{Z}$):

$$
\begin{tikzcd}[row sep=small]
 & \boxed{\begin{array}{c}\mathrm{d}F_{2\bullet} = 0\\ {\scriptstyle 2\bullet \,\leq\, D}\end{array}} \\
\boxed{\begin{array}{rcl}\mathrm{d}F_{2\bullet} &=& 0\\ \mathrm{d}\star F_{2\bullet} &=& 0\\ & {\scriptstyle 2\bullet \,\leq\, D/2} & \end{array}} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\
 & \boxed{F_{(D-2\bullet)} = \star F_{2\bullet}}
\end{tikzcd}
\tag{5}
$$

[9] [Whit1953, p. 192]: "Since the notion of metric is a complicated one, which requires measurements with clocks and scales, generally with *rigid* bodies, which themselves are systems of great complexity, it seems undesirable to take metric as fundamental, particularly for phenomena which are simpler and actually independent of it."

But this is just the situation of the "RR-fields" in $D = 10$-dimensional massive type IIA supergravity for the special case where the background NS-fields besides the metric vanish (we discuss the more general case in §1.3): On the left of (5) we have the equations of motion of the RR-field fluxes in their original "geometric" form (e.g. [Po95, (3)][IU12, §4.2.5]), while on the right of (5) we have the RR-fields in their pregeometric form, now commonly called the "duality-symmetric" or "democratic" form (see [CJLP98, §3][BKORV01][MV23]), this being the form which plays into the "Hypothesis K" that D-brane charge is quantized in K-theory (26).

Of course, the sources for the RR-field flux are *D-branes* as originally proposed in [Po95, (14)], and for the present purpose this may be regarded as the *definition* of (classical) D-branes: concentrations of RR-field flux (cf. exposition in [Ha12]). Similarly, we identify other (classical) brane species as sources of corresponding flux densities, as shown on the right.

	field	**flux**	**sourced by**
4d Maxwell theory	A-field	F_2	monopoles
type II 10d SuGra	RR-field	F_{8-p}	Dp-branes
	B-field	H_3	NS5-brane
11d supergravity	C-field	G_4	M5-branes
		G_7	M2-branes

(6)

Pregeometric C-field in 11d supergravity. The primary example of interest here is that of "C-field flux" G_4 in 11-dimensional supergravity (the 3-index A-field [CJS78], see [DF82, §III.8][MiSc06, p. 32][vPF12, §10]), which is meant to be the low-energy approximation of M-theory (see [Du00, §1]).

It is noteworthy that the C-field is the *only* field in $D = 11$ SuGra, besides the field of (super-)gravity itself (quite in constrast to the zoo of fields that appear in lower dimensional supergravities). This may make plausible that a proper treatment of the pregeometric C-field flux alone may go at least half-way towards a full definition of M-theory.

$$
\boxed{\begin{array}{c}\text{field content}\\ \text{of M-theory}\end{array}} \longleftarrow \left\{ \begin{array}{l} \rightarrow \boxed{\text{C-field}} \\[6pt] \rightarrow \boxed{\text{gravity}} \end{array} \right. \tag{7}
$$

In any case, the higher Maxwell-type equations for the C-field flux are famously as follows:

$$
\boxed{\begin{array}{rcl} \mathrm{d}G_4 &=& 0 \\ \mathrm{d}\star G_4 &=& -\tfrac{1}{2}G_4 \wedge G_4 \end{array}} \longleftarrow \left\{ \begin{array}{l} \rightarrow \boxed{\begin{array}{rcl} \mathrm{d}G_4 &=& 0 \\ \mathrm{d}G_7 &=& -\tfrac{1}{2}G_4 \wedge G_4 \end{array}} \\[6pt] \longrightarrow \boxed{G_7 = \star G_4} \end{array} \right. \tag{8}
$$

On the left of (8) we have the equations of motion in their traditional geometric form ([DF82, p. 131], detailed review in [CDF91, §III.8.53][MiSc06, (3.23)]), on the right their pregeometric form, known as the *duality-symmetric* form, cf. [BBS98][CJLP98][BNS04, §2][Nu03, §3].

Pregeometric flux in 5d supergravity. Just to amplify that the pre-geometric decomposition of field-flux equations of motion is a generic phenomenon, we briefly mention one more example:

The flux forms in 5-dimensional supergravity (cf. [CDF91, §III.5.70]) satisfy an equation of motion analogous to the equation (8) for the C-field in 11-dimensional supergravity (cf. [GGHPR03, (2.2)]):

$$
\boxed{\begin{array}{rcl} \mathrm{d}F_2 &=& 0 \\ \mathrm{d}\star F_2 &=& F_2 \wedge F_2 \end{array}} \longleftarrow \left\{ \begin{array}{l} \rightarrow \boxed{\begin{array}{rcl} \mathrm{d}F_2 &=& 0 \\ \mathrm{d}H_3 &=& F_2 \wedge F_2 \end{array}} \\[6pt] \longrightarrow \boxed{H_3 = \star F_2} \end{array} \right. \tag{9}
$$

Summarizing this list of examples (which could be much expanded) of pre-geometric equations of motion of flux densities:

Flux species	**equations of motion** (of fields in background gravity)	$\Leftrightarrow$	**Bianchi identities** (purely cohomological)	with	**duality constraint** (wrt background metric)
free A-field in 4d gravity	$\left.\begin{aligned} \mathrm{d}\star F_2 &= 0 \\ \mathrm{d}F_2 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}G_2 &= 0 \\ \mathrm{d}F_2 &= 0 \end{aligned}\right.$	where	$G_2 = \star F_2$
A-field & B-field in 5d supergravity	$\left.\begin{aligned} \mathrm{d}\star F_2 &= F_2 \wedge F_2 \\ \mathrm{d}F_2 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}H_3 &= F_2 \wedge F_2 \\ \mathrm{d}F_2 &= 0 \end{aligned}\right.$	where	$H_3 = \star F_2$
C-field in 11d supergravity	$\left.\begin{aligned} \mathrm{d}\star G_4 &= G_4 \wedge G_4 \\ \mathrm{d}G_4 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}G_7 &= G_4 \wedge G_4 \\ \mathrm{d}G_4 &= 0 \end{aligned}\right.$	where	$G_7 = \star G_4$
free RR-field in 10d supergravity	$\left.\begin{aligned} \mathrm{d}\star F_{2\bullet\leq 5} &= 0 \\ \mathrm{d}F_{2\bullet\leq 5} &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{ \mathrm{d}F_{2\bullet} = 0 \right.$	where	$F_{10-2\bullet} = \star F_{2\bullet}$

(10)

In conclusion so far, when looking at classical flux densities sourced by branes, we are looking at systems of differential forms on a (pseudo-)Riemannian manifold satisfying polynomial exterior differential equations subject to a Hodge-duality constraint:

$$
\overbrace{
\underbrace{
\underset{\substack{\text{flux densities}\\ \text{differential forms on spacetime}}}{\Big(\overset{\text{flux densities}}{F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X)}\Big)_{1\leq a\leq a_{\max}}}
\qquad
\underset{\substack{\text{Bianchi identities}\\ \text{cohomological part of field equations}}}{\overset{\text{de Rham diff.}}{\mathrm{d}}F^{(a)}_{r_a} = \overset{\text{polynomial}}{P^{(a)}}\Big(\big\{F^{(b)}_{r_b}\big\}_{b\leq a}\Big)}
}_{\text{pre-geometric cohomological aspect}}
\qquad
\underbrace{
\underset{\substack{\text{Hodge duality constraint}\\ \text{gravity-dependent part of field equations}}}{\begin{gathered}\text{involution}\\ \sigma : \{1,\cdots,a_{\max}\} \to \{1,\cdots,a_{\max}\} \\ \sigma\circ\sigma = \mathrm{id}\,, \quad F^{(a)} = \star F^{(\sigma(a))}\end{gathered}}
}_{\text{Riemannian geometric aspect}}
}^{\text{classical higher gauge field theory in gravitational background}}
\tag{11}
$$

But for which cohomology theory, really?
We answer this question in §2, §3.

Not further discussed here.

The appearance of *quadratic* Bianchi identities in the above examples (10) — in particular for the C-field flux in 11d supergravity (8) and generally of *non-linear* polynomial Bianchi identities (11) — is a crucial effect not seen in classical electromagnetism and outside the scope of previous mathematical discussions of flux quantization. It is to handle the quantization of such *non-linear* flux that we invoke the *non-abelian* character theory developed in [Char] and surveyed below in §2, which eventually allows to identify M-theory flux quantization in non-abelian (namely: unstable) cohomotopy (in §3).

But first we here discuss the physics encoded by non-linear terms in the pregeometric equations of motion of flux:

1.3 Brane intersections imprinted in non-linear flux

Bound states of *intersecting* black branes are seen (eg. [Sm03]) as solutions of full supergravity equations of motion; but in the pre-geometric spirit of §1.2 we highlight here that the qualitative aspects of the *brane intersection laws* may largely be deduced from the pre-geometric flux Bianchi identities (11) alone.

Namely it is the *non-linear* (quadratic and higher) polynomial source terms which encode the possibility that the branes which source these fluxes may "intersect" or "end on" each other in certain ways, as we explain now. Notice that it is only "on" such brane intersections that modern string phenomenology (namely all type I/II/M/F phenomenology, excluding only the traditional HET models) expects to model quasi-realistic physics (see [Ha12, §6.1, §6.4][IU12][RZ16, §15]).

branes ending on branes (brane intersection laws)	$\longleftrightarrow$	**non-linear pregeometric flux** (polynomial Bianchi identities)	$\overset{\text{[Char]}}{\longleftrightarrow}$	**non-abelian cohomology** (polynomial characters)

The relation on the left is fairly well-known in the case of D-branes ending on NS_5-branes, to be briefly recalled now, which is a "mild" form of non-linear flux since it may still be understood as "parameterized" or "twisted" linear flux (see around (23) below) and as such can and has been discussed by conventional means of flux quantization:

D-branes ending on NS5-branes. From the perspective (6) of flux densities, *NS 5-branes* are what source 3-form flux H_3 in type II supergravity, whose pregeometric equation of motion we may take to simply be[10] $\mathrm{d}H_3 = 0$. In the presence of such flux, the pregeometric equations for RR-field fluxes (5) are modified as follows (see references around (5) and [RW86, (23)]):

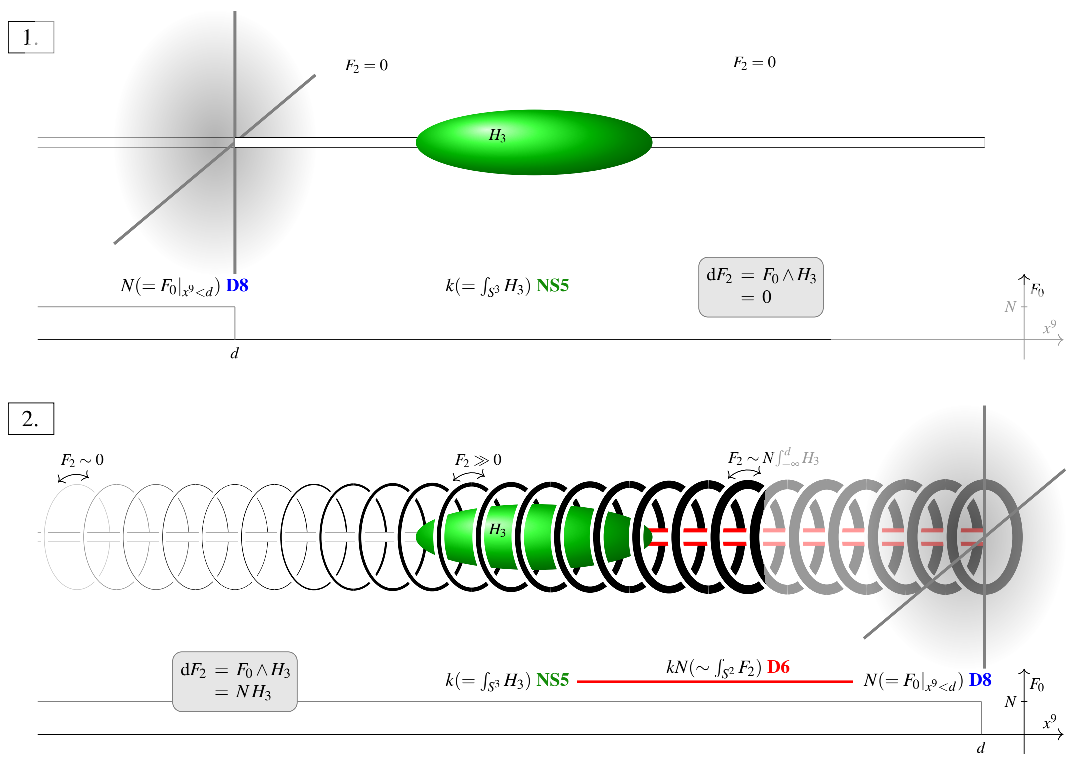
$$
\boxed{\mathrm{d}H_3 = 0}
\qquad
\boxed{\begin{array}{rcl}
\mathrm{d}F_0 &=& 0 \\
\mathrm{d}F_2 &=& H_3 \wedge F_0 \\
\mathrm{d}F_4 &=& H_3 \wedge F_2 \\
\vdots & &
\end{array}}
\qquad
\mathbf{NS5} \xrightarrow[\mathrm{D}_p]{\ \mathrm{d}F_{8-p} \,=\, H_3 \wedge F_{6-p}\ } \mathrm{D}_{p+2}
\tag{12}
$$

These differential equations, in particular the one for F_2, are not unlike the Maxwell equations (4) with a source term J_3, meaning here that NS5-branes via their H_3-flux but also D_{p+2}-branes via their F_{6-p}-flux act as a source or sink for RR-field flux F_{8-p} and hence for D_p-branes, in some way, suggesting that D_p-branes may *emanate from* or *end on* NS5-branes and D_{p+2}-branes (cf. eg. [EGKRS08] and references in [Fa17]).

The full supergravity equations of motion for such $\mathrm{NS}_5/\mathrm{D}_p/\mathrm{D}_{p+2}$-brane systems are complicated and satisfactory discussion is hard to cite, but we can readily give a full qualitative analysis of the solutions to the pre-geometric flux equations of motion (12) which already reveals the expected effects. This is going to be instructive for understanding the case of M2/M5-brane intersections that we are after further below in (13):

D6-brane creation and the Hanany-Witten effect. A popular conjecture by [HW97] states that the expected D_p-branes stretching between NS_5 and D_{p+2} are "created" as the D_{p+2}-branes are "dragged over" the NS_5, intuitively like a pole will cause a spike in a rubber sheet that is pulled over its tip. It was suggested in [Mar01, §2] that this *Hanany-Witten effect* should be understandable entirely from analysis of the flux Bianchi identities (12). At least for the case $p = 6$ of NS5/D6/D8-brane intersections [HZ98, §2.4][BLO98, p. 60] this is indeed the case, as we explain now. Here the flux F_0 of D_8-branes (the "Romans mass") is a locally constant function which vanishes in the vacuum and jumps by N units across the locus of N D_8-branes (cf. eg. [Fa17, p. 40]). But this means that:

1. When the NS_5-brane is located in the vacuum then its sourcing of F_2-flux is "switched off" by the vanishing F_0-factor in (12), hence if F_2 vanishes at infinity then the PDE demands it vanishes everywhere, reflecting the absence of D_6-branes.
2. When the NS_5-brane is located on the other side of the D_8-branes, where $F_0 = N$, then the equation (12) shows that F_2-flux/D_6-number density which vanishes far away will increase along the coordinate axis x^9 orthogonal to the D_8-branes in proportionality of the $\mathrm{d}x^9$-component of the flux H_3, and hence pronouncedly so as one crosses the NS_5-brane locus.



[10] We ignore here the dual NS-flux $H_7 = \star H_3$ in 10d supergravity: Its presence is actually a problem for the traditional *Hypothesis K* (§3.1) that NS/D-brane charge is in 3-twisted K-theory, while it plays no *direct* role in the formulation of *Hypothesis H* that we are after (§3.2). But interestingly, one proposal for incorparating H_7 into *Hypothesis K* also proceeds via cohomotopy, see [arXiv:1405.5844, §7.4] and [Char, Exp. 3.6]

D_3-branes and M_2-branes streching between 5-branes. Consider now the case of flux configurations which should reflect branes stretching between pairs of 5-branes. By the previous discussion this occurs either for D_3-branes between NS_5/D_5-branes or for M_2-branes between M_5-branes, according to the following pregeometric equations of motion:

$$
\begin{array}{llcll}
D=10: & \mathrm{d}F_5 & = & H_3 \wedge F_3 & (12) \\
D=11: & \mathrm{d}G_7 & = & \tfrac{1}{2} G_4 \wedge G_4 & (8) \\
 & & = & G_4^{(1)} \wedge G_4^{(2)} & G_4 := \underbrace{G_4^{(1)}}_{\text{homog.}} + \underbrace{G_4^{(2)}}_{\text{homog.}}
\end{array}
\tag{13}
$$

Considering a background configuration given by a pair of parallel flat 5-branes at some positive distance $2d > 0$:

$$
\begin{array}{ccc}
\mathbb{R}^{1,5}_{(i)} & \hookrightarrow & \mathbb{R}^{1,D} \\
(t, \vec{x}) & \mapsto & \big(t, \vec{x}, (-1)^i d, \vec{0}\big)
\end{array}
$$

hence reflected in flux densities of the following form (or any multiples of these, if you like)

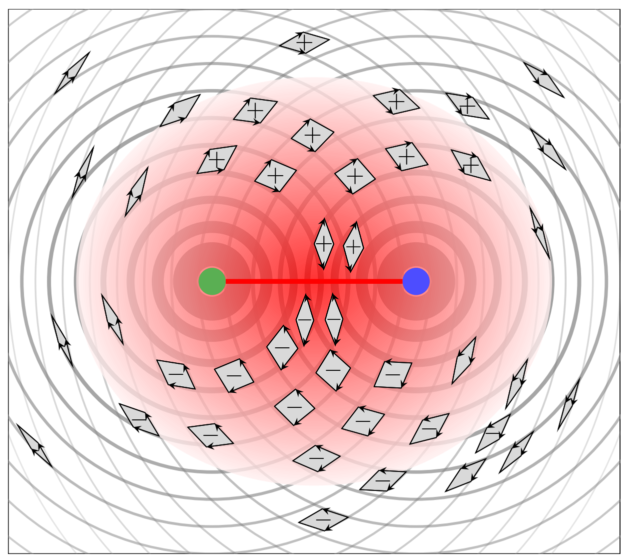
$$
\begin{array}{lcl}
H_3 & := & \mathrm{dvol}_{S^3} \in \Omega^3_{\mathrm{dR}}(S^3) \xhookrightarrow{\mathrm{pr}^*_{S^3}} \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(1)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^3\Big) \simeq \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,9} \setminus \mathbb{R}^{1,5}_{(1)}\Big) \\
F_3 & := & \mathrm{dvol}_{S^3} \in \Omega^3_{\mathrm{dR}}(S^3) \xhookrightarrow{\mathrm{pr}^*_{S^3}} \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(2)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^3\Big) \simeq \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,9} \setminus \mathbb{R}^{1,5}_{(2)}\Big) \\
G_4^{(i)} & := & \mathrm{dvol}_{S^4} \in \Omega^4_{\mathrm{dR}}(S^4) \xhookrightarrow{\mathrm{pr}^*_{S^4}} \Omega^4_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(i)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^4\Big) \simeq \Omega^4_{\mathrm{dR}}\Big(\mathbb{R}^{1,10} \setminus \mathbb{R}^{1,5}_{(i)}\Big)
\end{array}
$$

and assuming that D_3- or M_2-brane flux vanishes at infinity, then these differential equations will imply D_3- or M_2-brane flux, respectively as soon as the wedge products $H_3 \wedge F_3$ and $G_4^{(1)} \wedge G_4^{(2)}$ are *multi-poles* concentrated roughly between the given pair of branes. That and why this is indeed the case is illustrated by the following figure.

Effective dipole of quadratic brane flux. The figure on the right means to indicate the nature of the differential 2-form which is the wedge product of two copies of the pullback of dvol_{S^1} to around either of the punctures (the brane loci) in the 2-punctured plane, the 2-dimensional shadow of the analogous wedge products on the right of (13). Here:

- the strength of the circular lines indicates the absolute value of the flux density sourced by the respective 5-brane,
- the arrows indicate the orientation of the flux density of either 5-brane,
- the parallelograms indicate the orientation of their wedge product.

It is evident that the absolute value of the wedge product is concentrated near the 5-branes and particularly between them when they are close. But the point is that the orientation of the wedge product changes sign across the axis connecting the branes, as shown. This means that the flux *sourced* by this wedge product, according to (13), is, if vanishing at infinity, concentrated between the branes.



Such $M_5/M_2/M_5$-brane intersections are expected in the literature (eg. [HLV14, Fig. 3]) but rarely discussed in more detail.

M_2-Branes ending on M_1-waves and C-field tadpoles. Less widely appreciated is that M_2-branes are also argued [BPST10, §2.2.3][HSS19, Prop. 4.19] to possibly end on 1-brane-like loci known as "M-waves". In terms of pregeometric fluxes this means, by the previous arguments, that there ought to be an $9-1=8$-form flux density I_8 and a modification of the C-field flux Bianchi identity roughly of the form

$$
\mathrm{d}G_7 = \tfrac{1}{2} G_4 \wedge G_4 + c I_8 \,. \tag{14}
$$

A modified equation for M_2-brane charge of just this form was earlier argued in [DM97, (1)], based on a string perturbation-theoretic argument notorious as the "one-loop term" in the effective string action (obtained from Hypothesis H in (38)).

But from the point of view of flux densities and flux quantization, a Bianchi identity of the form (14) means that we need to understand both non-linear polynomial flux equations like that of the supergravity C-field (8) *and* their further twisting, in M-theoretic analogy of (12), hence we need to understand *twisted non-abelian cohomology* [Char] – this we turn to in §2.

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2 Brane charge quantization

Anomaly cancellation and flux quantization. Secretly, much of contemporary theory building in theoretical physics is a sophisticated process of trial, error and improvisation: The trials are Lagrangian densities ("action functionals"), the errors are "anomalies" obstructing their consistent quantization, and the improvisation is the invention of add-on rules to "cancel" the anomalies. While there is a sense of accomplishment in the community for identifying and cancelling anomalies (typically a demanding task) we should see it for what it is: *Anomaly cancellation is the patching-up of broken theories.* This can (and certainly has been) useful for exploring the space of physical theories, but it seems implausible that truly fundamental theories will come to us in broken form incrementally patched. Instead, eventually we want to understand how to construct *anomaly-free* and hence well-defined quantum theories right away.

Dirac charge quantization in integral cohomology. An early example of what we may recognize as anomaly cancellation is *Dirac's charge quantization* (p. 7) – and we are going to promote this to a rather general principle of *brane charge quantization*. In modern paraphrase, Dirac observed:

1. that the worldline theory of the electron in the background of a magnetic monopole has, in general, a quantum anomaly,
2. that this anomaly vanishes if – hence is cancelled by demanding that – electromagnetic charges are quantized in integer multiples of a unit charge, or more precisely that the *fluxes* witnessing these charges are quantized in *integral cohomology*. The latter is really the fundamental condition, whence one also speaks of *flux quantization*.

More in detail, the anomaly cancellation demand — an *ad hoc* addition to classical Maxwell theory — is that the Faraday tensor 2-form F_2, in addition to satisfying Maxwell's equations (4), also satisfies the constraint that its class in de Rham cohomology, which under de Rham's theorem is identified with $\mathbb{R}$-valued cohomology, is in fact in the image of $\mathbb{Z}$-valued cohomology:

$$\begin{array}{ccc} \underset{\text{in integral cohomology}}{\overset{\substack{\text{quantized}\\ \text{magnetic charges}}}{H^2(X;\mathbb{Z})}} & \longrightarrow & \underset{\text{in real/de Rham cohomology}}{\overset{\substack{\text{classical}\\ \text{magnetic charges}}}{H^2(X;\mathbb{R}) \simeq H^2_{\mathrm{dR}}(X^4)}} \\ [q] & \underset{\text{lift}}{\overset{\text{charge quantization}}{\longleftarrow\!\!\!\sim}} & [F_2] \end{array} \tag{15}$$

This cohomology operation of *extension-of-scalars* we are also going to refer to as the *character map* in integral cohomology (think of the differential forms in its image as "real characterizations", if you wish, of integral cohomology classes).

Notice that in general this map is not an injection (its kernel is the *torsion subgroup* of integral classes). Eventually we understand charge quantization not just as the *condition* that physical fields be in the image of this map, but we regard the actual physical fields as containing *extra structure* consisting of a choice of pre-image through this map (and yet a little more, see §2.5).

Spin-/String-structure as charge quantization in non-abelian cohomology. Dirac's argument only concerns the charge of the electron. When one also considers the spin of the electron then its worldline theory has another anomaly, which is cancelled by equipping the background spacetime with *spin-structure* (discussed this way in [Wi85, p. 65-68]). An analogous argument shows that *spinning strings* have an anomaly in their worldsheet theory which may be cancelled by equipping the background spacetime with *string-structure* (cf. [Bu11][SSS12]).

Here we are going to understand [Char, §2] phenomena such as Spin- and String-structures as examples of *non-abelian cohomology* with coefficients in a non-abelian group G ([Grothendieck55, §V][Fr1957], see also [We16, §7]) or non-abelian 2-groups etc., thus conceptually unifying them with abelian cohomology such as in (15):

$$\underset{\substack{\text{non-abelian cohomology}\\ \text{in degree 1}}}{H^1(X;G)} \;\simeq\; \underset{\substack{\text{homotopy classes of maps}\\ \text{into classifying space}}}{\pi_0\mathrm{Maps}(X;BG)} \;\simeq\; \underset{\substack{\text{isomorphism classes of}\\ \text{principal bundles}}}{G\mathrm{PrinBund}(X)_{/\sim}}$$

This way, we may understand the anomaly cancellation of the spinning electron by ambient spin-structure as of the same general cohomological form as Dirac's charge quantization (15):

$$\begin{array}{ccccc} \underset{\text{in String-cohomology}}{\overset{\substack{\text{"quantized"}\\ \text{gravitational charge}}}{H^1\big(X;\mathrm{String}(1,d)\big)}} & \longrightarrow & \underset{\text{in Spin-cohomology}}{\overset{\substack{\text{"quantized"}\\ \text{gravitational charge}}}{H^1\big(X;\mathrm{Spin}(1,d)\big)}} & \longrightarrow & \underset{\text{in nonabelian O-cohomology}}{\overset{\text{gravitational charge}}{H^1\big(X;\mathrm{O}(1,d)\big)}} \\ [\widehat{\widehat{\omega}}] & \underset{\text{lift}}{\overset{\text{string anomaly cancellation}}{\longleftarrow\!\!\!\sim}} & [\widehat{\omega}] & \underset{\text{lift}}{\overset{\text{spin anomaly cancellation}}{\longleftarrow\!\!\!\sim}} & [\omega] \end{array}$$

Cohomology rules. This shows that, at least in key examples, "anomaly cancellation" amounts to understanding that fields/fluxes which a priori seem to be given by differential forms actually need to be *flux quantized* by promoting them to cocycles in possibly non-abelian generalized cohomology theories. Together with the observation in §1.2 that pre-geometric equations of motion always form a kind of non-abelian de Rham cohomology, this paints a compelling picture that quantum fields want to be understood as pre-geometric cocycles in non-abelian generalized cohomology theories.

We now explain – in survey of [Char] – that:

- §2.1 Bianchi identities characterize flux densities as flat nilpotent L_∞-algebra-valued differential forms.
- §2.2 Flat nilpotent L_∞-algebra valued forms are cocycles in non-abelian de Rham cohomology.
- §2.3 Non-abelian de Rham cohomology is the target of the character map on higher non-abelian cohomology.
- §2.4 Higher non-abelian cohomology theories thus serve as the *flux quantization laws* for higher fluxes.

Schematically:

$$
\begin{array}{ccc}
\overset{\text{non-abelian cohomology}}{A(X)} & \xrightarrow[\mathrm{ch}_A]{\text{non-abelian character}} & \overset{\text{non-abelian de Rham cohomology}}{H_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big)} \\
\underset{\text{class of } A\text{-quantized flux}}{\big[\mathscr{F}\big]} & \longmapsto & \underset{\text{class of underlying flux densities}}{\Big[\big(F^{(a)}_{r_a}\big)_{1 \le a \le \dim[\pi_\bullet(A),\mathbb{R}]}\Big]}
\end{array}
$$

- §2.5 These flux quantization laws determine the moduli ∞-stack of the higher gauge potentials.
- §2.6 Twisted versions of these cohomology theories encode brane intersections.

This follows the seminal argument of *Dirac charge quantization* for electromagnetism [Di31] (review in [Al85][Fr97, §16.4e] [Fr00, §2]) and generalizes suggestions for charge quantization in higher gauge theories [Fr00][HS05] to the case of *non-abelian* ("unstable") fluxes, such as the C-field in 11d supergravity (8).

2.1 Bianchi identities characterize flux densities as flat nilpotent L_∞-algebra valued forms

The structure of a system of higher Bianchi identities (11) is all encoded in the polynomials $P^{(a)}$. At the same time, the de Rham condition $\mathrm{d}^2 = 0$ imposes constraints on systems of polynomials that may arise. With these constraints, the coefficients of $P^{(a)}$ in the expansion

$$
P^{(a)}\big(\{F^{(b)}\}\big) \;=\; \sum_{n=0}^{\infty} P^{(a)}_{a_1 \cdots a_n} F^{(a_1)} \wedge \cdots \wedge F^{(a_n)}
$$

are equivalently the structure constants of an L_∞-algebra $\mathfrak{a}$, namely the unique L_∞-algebra whose Maurer-Cartan equation (the higher flatness condition) is the given Bianchi identities:

Flux densities satisfying Bianchi identities are flat L_∞-algebra-valued differential forms.

$$
\underset{\text{insert spacetime manifold here}}{\overset{\text{sheaf of flat } L_\infty\text{-algebra-valued differential forms}}{\Omega_{\mathrm{dR}}\big(-;\mathfrak{a}\big)_{\mathrm{flat}}}} \;=\; \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}(\mathfrak{a}),\, \Omega^\bullet_{\mathrm{dR}}(-)\Big) \;=\; \Big\{ \overset{\text{systems of flux densities}}{F^{(a)} \in \Omega^{r_a}_{\mathrm{dR}}(-)} \;\Big\vert\; \overset{\text{satisfying these Bianchi identities}}{\mathrm{d}\,F^{(a)} \;=\; P^{(a)}\Big(\big\{F^{(b)}\big\}_{b \le a}\Big)} \Big\}_{1 \le a \le a_{\mathrm{max}}}
$$

$$\updownarrow$$

$$
\overset{\text{Chevalley-Eilenberg algebra of } L_\infty\text{-algebra}}{\mathrm{CE}(\mathfrak{a})} \;=\; \overset{\text{free differential graded-commutative algebra on these graded generators}}{\mathbb{R}\Big[\big\{f^{(a)}\big\}_{1 \le a \le a_{\mathrm{max}}}\Big]} \Big/ \Big(\overset{\text{satisfying these differential relations}}{\mathrm{d}\,f^{(a)} \;=\; P^{(a)}\Big(\big\{f^{(b)}\big\}_{b \le a}\Big)}\Big)
$$

$$\updownarrow$$

$$
\overset{L_\infty\text{-algebra}}{\mathfrak{a}} \;=\; \overset{\text{graded vector space spanned by these graded generators}}{\mathbb{R}\Big\langle \big\{f_{(a)}\big\}_{1 \le a \le a_{\mathrm{max}}} \Big\rangle} \qquad \overset{\text{equipped with these higher Lie brackets}}{\big[f_{(a_1)}, \cdots, f_{(a_n)}\big] \;=\; P^{(a)}_{a_1 \cdots a_n}\, f_{(a)}}
$$

2.2 Flat nilpotent L_∞-algebra-valued forms are cocycles in non-abelian de Rham cohomology

Say that a pair $\big(\{F_0^{(a)}\}_a, \{F_1^{(a)}\}_a\big)$ of flat $\mathfrak{a}$-valued differential forms are *cohomologous* if they can can be *deformed into each other*, hence if they are *concordant*, in that they are boundary data of a flat $\mathfrak{a}$-form on the cylinder $X \times [0,1]$ over X:

$$\overset{\text{deformation of flux densities}}{\big\{F_0^{(a)}\big\}_{1\le a\le a_{\max}} \sim \big\{F_1^{(a)}\big\}_{1\le a\le a_{\max}}} \qquad :\Leftrightarrow \qquad \exists \;
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \\
\Omega^\bullet_{\mathrm{dR}}\big(X\times[0,1]\big) \arrow[u, "i_0{}^*"] \arrow[d, "i_1{}^*"'] & & \mathrm{CE}(\mathfrak{a}) \arrow[l, "\big\{\widehat{F}^{(a)}\big\}_{1\le a\le a_{\max}}"'] \arrow[ull, "\big\{F_0{}^{(a)}\big\}_{1\le a\le a_{\max}}"'] \arrow[dll, "\big\{F_1{}^{(a)}\big\}_{1\le a\le a_{\max}}"] \\
\Omega^\bullet_{\mathrm{dR}}(X) & &
\end{tikzcd}$$

This is an equivalence relation whose equivalence classes we call the **flat $\mathfrak{a}$-valued non-abelian de Rham cohomology** of X:

$$\overset{\substack{\text{deformation class}\\ \text{of flux densities}}}{\Big[\big\{F^{(a)}\big\}_{1\le a\le a_{\max}}\Big]} \;\in\; \overset{\substack{\mathfrak{a}\text{-valued}\\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \;:=\; \left\{
\begin{tikzcd}[column sep=large]
\Omega^\bullet_{\mathrm{dR}}(X) & \mathrm{CE}(\mathfrak{l}A) \arrow[l, bend right=50, dashed, "\underset{\text{cocycle (dga-hom)}}{\{F_0^{(a)}\}}"'{name=U}] \arrow[l, bend left=50, dashed, "\overset{\text{another cocycle}}{\{F_1^{(a)}\}}"{name=D}] \arrow[from=U, to=D, Rightarrow, "\substack{\text{coboundary}\\ \text{(concordance)}}"]
\end{tikzcd}
\right\}\Big/\sim .$$

2.3 Nonabelian de Rham cohomology is target of character map on nonabelian cohomology

Classifying spaces for cohomology. Notice that reasonable cohomology theories have *classifying spaces*:

$$
\begin{array}{rcl}
\overset{\text{ordinary cohomology}}{H^n(X;\mathbb{Z})} & \simeq & \pi_0 \,\mathrm{Maps}\Big(X,\, \overset{\text{Eilenberg-MacLane space}}{K(\mathbb{Z},n)}\Big)
\\[10pt]
\overset{\text{nonabelian cohomology}}{H^1(X;G)} & \simeq & \pi_0 \,\mathrm{Maps}\Big(X,\, \overset{\substack{\text{classifing space of}\\ \text{principal } G\text{-bundles}}}{BG}\Big)
\\[10pt]
\overset{\text{topological K-theory}}{K^0(X)} & \simeq & \pi_0 \,\mathrm{Maps}\Big(X,\, \overset{\substack{\text{space of}\\ \text{Fredholm operators}}}{\mathrm{Fred}_{\mathbb{C}}}\Big)
\\[10pt]
\overset{\substack{\text{Whitehead}\\ \text{generalized cohomology}}}{E^n(X)} & \simeq & \pi_0 \,\mathrm{Maps}\Big(X,\, \overset{\substack{\text{stage in}\\ \text{spectrum of spaces}}}{E_n}\Big)
\\[10pt]
\overset{\text{coHomotopy}}{\pi^n(X)} & \simeq & \pi_0 \,\mathrm{Maps}\Big(X,\, \overset{\text{sphere}}{S^n}\Big)
\end{array}
\tag{16}$$

$$\text{hence consider:} \qquad \overset{\substack{\text{higher}\\ \text{non-abelian}\\ \text{cohomology}}}{A(X)} \;:=\; \pi_0\,\mathrm{Maps}\Big(X,\, \overset{\text{any space}}{A}\Big) \;=\; \left\{
\begin{tikzcd}[column sep=large]
X \arrow[r, bend left=50, dashed, "\overset{\text{cocycle (map)}}{\mathscr{F}_0}"{name=U}] \arrow[r, bend right=50, dashed, "\underset{\text{another cocycle}}{\mathscr{F}_1}"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\substack{\text{coboundary}\\ \text{(homotopy)}}"]
\end{tikzcd}
\right\}\Big/\sim$$

Reduced cohomology and solitonic charges. For the charge quantization of *solitonic* branes (in §6) one needs to implement in cohomology theory their *localization* in space (cf. §1.1) which forces their fluxes to *vanish at infinity*.

We may observe that a formalization of this phenomenon is already captured by the standard notion of *reduced cohomology* on *pointed spaces* if we regard the basepoint of a domain space as its point-at-infinity and the basepoint of a coefficient space as its zero-element

$$\begin{tikzcd} \textbf{domain space} & X \arrow[rr, dashed, "\textbf{flux cocycle}", "\text{in reduced } A\text{-cohomology}"'] && A & \textbf{coefficient space} \\ \begin{array}{c}\textbf{basepoint is}\\ \textbf{point at } \infty\end{array} & \{\infty\} \arrow[u] \arrow[rr, "\text{flux vanishes at infinity}"] && \{0\} \arrow[u] & \begin{array}{c}\textbf{basepoint is}\\ 0\textbf{-element}\end{array} \end{tikzcd}$$

$$\underset{\substack{\textbf{reduced}\\ \text{higher}\\ \text{non-abelian}\\ \text{cohomology}}}{\widetilde{A}(X)} := \pi_0 \mathrm{Maps}^{*/}\big((X,\infty),\,(A,0)\big) = \left\{ \begin{tikzcd} X \arrow[rr, bend left, dashed, "\overset{\textbf{cocycle}\text{ (map)}}{\mathscr{F}_0}"] \arrow[rr, bend right, dashed, "\underset{\textbf{cocycle}}{\mathscr{F}_1}"'] \arrow[rr, phantom, "\Downarrow\,\substack{\textbf{coboundary}\\ \text{(homotopy)}}"] && A \\ \{\infty\} \arrow[u] \arrow[rr, "\text{vanishing at }\infty"'] && \{0\} \arrow[u] \end{tikzcd} \right\}_{/\sim}$$

Notice that the point at infinity may or may not be reachable by continuous paths in the space:

$$X \text{ a plain space} \quad \vdash \quad \begin{array}{lll} X_{\sqcup\{\infty\}} & \text{disjoint point adjoined} & \text{paths starting in cnctd } X \text{ never reach } \infty \\ X_{\cup\{\infty\}} & \text{one-point-compactification} & \text{paths starting in cnctd } X \text{ may reach } \infty \end{array} \tag{17}$$

Given a pointed space, we may first delete the point at infinity and then adjoint it back disjointly, making it un-reachable:

$$(X,\infty) \text{ a pointed space} \quad \vdash \quad \big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}} \quad \text{make } \infty \text{ un-reachable} \tag{18}$$

Plain cohomology (16) is subsumed in reduced cohomology as the case where the point at infinity is unreachable (17)

$$A(X) \;\simeq\; \widetilde{A}\big(X_{\sqcup\{\infty\}}\big)$$

and making ∞ unreachable (18) projects reduced into plain cohomology.

The charges that thus disappear existed only due to their localization, hence are *purely solitonic*, while those that do not vanish at ∞ are *purely singular* (cf. §1.1):

$$\begin{array}{c}(X,\infty)\\ \text{pntd space}\end{array} \;\vdash\; \begin{tikzcd} \overset{\substack{\textbf{purely}\\ \textbf{solitonic charges}}}{\ker\big(A(\varepsilon\iota_X)\big)} \arrow[r, hook] & \overset{\substack{\textbf{reduced}\\ \textbf{cohomology}}}{\widetilde{A}(X)} \arrow[r, "A(\varepsilon\iota_X)"] & \overset{\substack{\textbf{reduced}\\ \textbf{cohomology}\\ \textbf{for disjoint } \infty}}{\widetilde{A}\Big(\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}}\Big)} & \overset{\textbf{plain cohomology}}{= A\big(X\setminus\{\infty\}\big)} \arrow[r, two heads] & \overset{\substack{\textbf{purely}\\ \textbf{singular charges}}}{\mathrm{coker}\big(\widetilde{A}(\varepsilon\iota_X)\big)} \\ & X & \underset{\text{make } \infty \text{ unreachable}}{\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}}} \arrow[l, "\varepsilon\iota_X"'] \end{tikzcd} \tag{19}$$

(...)

Notice the mapping space adjunction

$$\mathrm{Maps}^{*/}\Big(X, \mathrm{Maps}^{*/}\big(Y,Z\big)\Big) \;\simeq\; \mathrm{Maps}^{*/}\big(X\wedge Y, Z\big) \;\simeq\; \mathrm{Maps}^{*/}\Big(Y, \mathrm{Maps}^{*/}\big(X,Z\big)\Big) \tag{20}$$

With generalized/non-abelian cohomology theory understood via classifying spaces this way, the *fundamental theorem of dg-algebraic rational homotopy theory* provides the generalization of the Chern character:

(1) The **rational homotopy type** of any nilpotent (classifying) space A is encoded in its *Whitehead bracket L_∞-algebra* $\mathfrak{l}A$ whose Chevalley-Eilenberg algebra is the *minimal Sullivan model* of A:

$$
\begin{array}{ccccccc}
A & \xmapsto{\text{rationalization}} & \underset{\text{Chevalley-Eilenberg algebra of its Whitehead } L_\infty\text{-algebra}}{\overset{\text{Sullivan model of } A}{\mathrm{CE}(\mathfrak{l}A)}} & \longleftrightarrow & \overset{\text{Whitehead } L_\infty\text{-algebra}}{\mathfrak{l}A} & = & \text{“infinitesimal version } \mathrm{Lie}(\Omega A) \text{ of loop group”}
\\
\overset{\text{Eilenberg-MacLane space}}{K(\mathbb{Z},n)} & \longmapsto & \mathbb{R}[c_n]/(\mathrm{d}\,c_n = 0) & & & &
\\
\overset{\text{classifying space of principal } G\text{-bundles}}{BG} & \longmapsto & \overset{\text{invariant polynomials on the Lie algebra}}{\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, \mathrm{d}=0\big)} & & & &
\\
\overset{\text{space of Fredholm operators}}{\mathrm{Fred}_{\mathbb{C}}} & \longmapsto & \mathbb{R}\Big[\{f_{2k}\,|\, k \in \mathbb{N}\}\Big]\Big/(\mathrm{d}\,f_{2k} = 0) & & & &
\\
\overset{\text{stage in spectrum of spaces}}{E_n} & \longmapsto & \Big(\mathrm{Sym}\big(\mathrm{Hom}_{\mathbb{Z}}(\pi_\bullet(E_n),\mathbb{R})\big), \mathrm{d}=0\Big) & & & &
\\
\overset{\text{even dimensional sphere}}{S^{2k}} & \longmapsto & \mathbb{R}[g_{2k},\, g_{4k-1}]\Big/\left(\begin{array}{ll}\mathrm{d}\,g_{4k-1} & = g_{2k}\wedge g_{2k} \\ \mathrm{d}\,g_{2k} & = 0\end{array}\right) & & & &
\end{array}
$$

This is an equivalence between:

weak rational homotopy types of nilpotent connected **spaces** with degreew. fin-dim rational cohomology

$\wr\wr$

quasi-isomorphism types of nilpotent connective **L_∞-algebras** on degreew. fin-dim $\mathbb{Q}$-vector spaces

here shown/used after extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$

notice the difference to L_∞-algebras of *gauge potentials*: $\mathfrak{su}(n)$, $\mathfrak{string}(n)$, ... which are *not nilpotent*

here: L_∞-algebras of gauge *fields/fluxes*

(2) The **higher non-abelian character map** ch_A universally approximates A-cohomology classes by $\mathfrak{l}A$-valued de Rham classes:

$$
\begin{array}{ccc}
\overset{\text{non-abelian cohomology}}{A(X)} & \xrightarrow[\mathrm{ch}_A]{\text{non-abelian character}} & \overset{\text{non-abelian de Rham cohomology}}{H_{\mathrm{dR}}(X;\mathfrak{l}A)}
\\
\underset{\text{class of } A\text{-quantized flux}}{[\mathscr{F}]} & \longmapsto & \underset{\text{class of underlying flux densities}}{\Big[\big(F_{r_a}^{(a)}\big)_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big]}
\end{array}
$$

$$
\begin{array}{lccc}
\text{specializing to:} & \overset{\text{ordinary cohomology}}{H^n(X;\mathbb{Z})} & \xrightarrow{\text{de Rham map}} & H^n_{\mathrm{dR}}(X)
\\
& \overset{\text{nonabelian cohomology}}{H^1(X;G)} & \xrightarrow{\text{Chern-Weil homomorphism}} & H_{\mathrm{dR}}\big(X;\mathrm{inv}(\mathfrak{g})\big)
\\
& \overset{\text{topological K-theory}}{K^0(X)} & \xrightarrow{\text{Chern character}} & \bigoplus_{k\in\mathbb{N}} H^{2k}_{\mathrm{dR}}(X)
\\
& \overset{\text{Whitehead generalized cohomology}}{E^n(X)} & \xrightarrow{\text{Chern-Dold character}} & \bigoplus_{k\in\mathbb{Z}} H^{n+k}_{\mathrm{dR}}\big(X;\pi_k(E)\otimes_{\mathbb{Z}}\mathbb{R}\big)
\\
& \overset{\text{coHomotopy in even degree}}{\pi^{2k}(X)} & \xrightarrow{\text{coHomotopical character}} & H_{\mathrm{dR}}\big(X;\mathfrak{l}S^{2k}\big)
\\
& & & \Vert
\\
& & & \left\{\begin{array}{ll} G_{4k-1} \in \Omega^{4k-1}_{\mathrm{dR}}(X) \\ G_{2k} \;\;\;\in \Omega^{2k}_{\mathrm{dR}}(X)\end{array} \,\middle|\, \begin{array}{ll}\mathrm{d}\,G_{4k-1} & = G_{2k}\wedge G_{2k} \\ \mathrm{d}\,G_{2k} & = 0\end{array}\right\}\Big/_{\text{concordance}}
\end{array}
\tag{21}
$$

2.4 Higher non-abelian cohomology theories serve as flux quantization laws

We have now seen that
the cohomological sector of classical higher gauge theory exhibits
the flux densities as classes in $\mathfrak{a}$-valued de Rham cohomology, $\mathfrak{a} \in L_\infty\mathrm{Alg}$
which is the character shadow of A-cohomology, for *any* $A \in \mathrm{Spaces}$ with $\mathfrak{l}A \;\simeq\; \mathfrak{a}$

Thus **flux quantization** in A-cohomology means to:
impose the constraint that the deformation classes of the fluxes
must be in the image of the character map (the "charge lattice").

For example, **quantization of electromagnetic flux** in integral cohomology means to require F_2 to have integral periods:

$$\begin{tikzcd}
 & & \underset{\text{flux densities}}{\overset{\text{naïve}}{}}\ \Omega^2_{\mathrm{dR}}(X) \arrow[dr] & \\
 & \text{EM-flux}\ \{F_2\} \arrow[ur] \arrow[dr] & & H^2_{\mathrm{dR}}(X)\ \begin{smallmatrix}\text{deformation classes}\\ \text{of naïve flux densities}\end{smallmatrix} \\
\underset{\text{integral cohomology}}{H^2(X;\mathbb{Z})} \arrow[urrr, "\mathrm{ch}"] \arrow[rr, two heads, "\text{forget torsion}"'] & & \underset{\text{charge lattice}}{H^2_{\mathrm{dR}}(X)_{\mathrm{int}}} \arrow[ur, hook] &
\end{tikzcd}$$

But this is not enough...

2.5 Flux quantization laws determine the moduli ∞-stack of the higher gauge potentials

Instead of just asking that *there exists* an A-cohomology class $[\mathscr{F}]$ whose character image is deformation equivalent to the $\mathfrak{a}$-valued flux densities $F^{(a)}$, the A-cocycle $\mathscr{F}$ and the gauge equivalence $\mathrm{ch}_A(\mathscr{F}) \simeq \{F^{(a)}\}$ should be part of the field content.
In order to achieve this, one needs a unified context which accommodates both

	differential forms like flux densities $F^{(a)}$
&	**homotopy types** of classifying spaces A
in	**differential homotopy theory**

In order to handle	differential structures	it is convenient to model them	on the basic charts.
In order to handle	homotopy types of spaces	it is convenient to model them	as simplicial sets.

For the present purpose we consider:

the site CartSp **of abstract smooth charts**	
an abstract *coordinate chart*	is a Cartesian space $\mathbb{R}^n$ for any $n \in \mathbb{N}$
an abstract *coordinate transformation*	is any smooth function $\mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2}$
a *covering of coordinate charts*	is an open cover $\left\{\mathbb{R}^n \simeq U_i \hookrightarrow \mathbb{R}^n\right\}_{i\in I}$ which is *differentiably good* in that finite non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are all diffeomorphic to $\mathbb{R}^n$

Given any site of Charts, serving as local model spaces:

- A **generalized space** $\mathscr{X}$ *probeable* **by such charts** is bootstrapped into existence by declaring the simplicial sets of ways of plotting out abstract coordinate charts inside $\mathscr{X}$:

$$\begin{array}{rccl}
\mathscr{X} & : & \mathrm{Charts}^{\mathrm{op}} & \longrightarrow \mathrm{sSets} \\
 & & \mathbb{R}^n & \mapsto \mathrm{Plots}(\mathbb{R}^n, \mathscr{X})
\end{array}$$

Such **simplicial presheaves** naturally form an sSet-enriched category $\mathrm{sPSh}_{\mathrm{Charts}}$; denote its simplicial hom-complexes by:

$$
\begin{array}{ccc}
\mathrm{sPSh}^{\mathrm{op}}_{\mathrm{Charts}} \times \mathrm{sPSh}_{\mathrm{Charts}} & \longrightarrow & \mathrm{sSet} \\
(\mathscr{X}, \mathscr{Y}) & \mapsto & \mathrm{Maps}(\mathscr{X}, \mathscr{Y})
\end{array}
$$

- Any $U \in \mathrm{Charts}$ becomes a generalized space by declaring its plots to be the morphisms of charts (representable presheaf):

$$
\begin{array}{ccccc}
U & : & \mathrm{Charts}^{\mathrm{op}} & \longrightarrow & \mathrm{sSets} \\
 & & V & \mapsto & \mathrm{Charts}(V, U)
\end{array}
$$

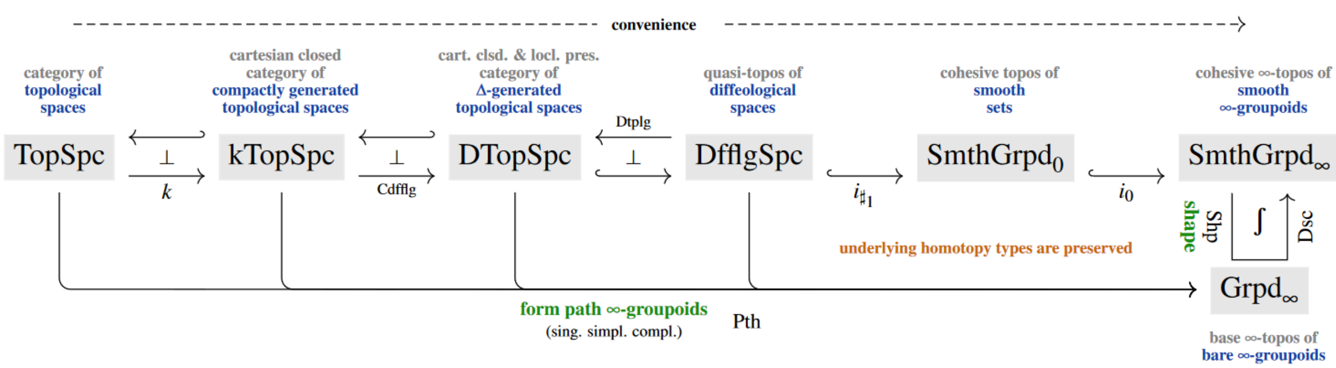
- **Consistency** of this bootstrap of generalized spaces demands:
  (1.) natural identifications between plots by and maps from charts:

$$
U \in \mathrm{Charts}, \quad \mathscr{X} \in \mathrm{sPSh}_{\mathrm{Charts}} \qquad \vdash \qquad \mathrm{Plots}(U, \mathscr{X}) \simeq \mathrm{Maps}(U, \mathscr{X})
$$

  This is the case by the (enriched) **Yoneda lemma**.
  (2.) that maps of generalized spaces are equivalences — to be denoted $(f : \mathscr{X} \to \mathscr{Y}) \in \mathrm{W}$ —
  iff *locally on all charts* i.e.: stalk-wise they are *higher gauge equivalence*, i.e.: simplicial weak homotopy equivalences
  this we *enforce* by **simplicial localization**, yielding the ∞-topos $\mathbf{H} := L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{Charts}}$

	by this principle:	**(probes of) spaces are**:
	locality principle	sheaves on charts with
&	higher gauge principle	values in simplicial sets
=	homotopy topos of simplicial sheaves on charts	

Specifically for Charts = CartSp we obtain the cohesive ∞-topos $\quad \mathrm{SmoothGrpd}_\infty \; := \; L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{CartSp}}$



In $\mathrm{SmthGrpd}_\infty$ all ingredients of higher gauge field theory find a natural home:

smooth moduli space of flat $\mathfrak{l}A$-valued forms
(genuine differential structure)

$$
\begin{array}{ccccc}
\Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{flat}} & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \mapsto & \Omega_{\mathrm{dR}}(\mathbb{R}^n; \mathfrak{l}A)_{\mathrm{flat}}
\end{array}
$$

smooth moduli stack of deformation classes of flat $\mathfrak{l}A$-valued forms
(rational homotopy type of A)

$$
\begin{array}{ccccc}
\int \Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{flat}} & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \mapsto & \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^\bullet_{\mathrm{smth}}; \mathfrak{l}A\big)_{\mathrm{flat}}
\end{array}
$$

homotopy type of A
(geometrically discrete ∞-groupoid)

$$
\begin{array}{ccccc}
A & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \mapsto & \mathrm{Sing}(A)
\end{array}
$$

Now given a choice of flux quantization law A, the corresponding moduli stack $\widehat{A}$ of *potentials* or *gauge fields* is the homotopy pullback of the sheaf of flux densities along the character map.

This classifies (higher non-abelian) *differential cohomology*, in a way that takes care of all "Dirac strings" of gauge fields and of higher gauge fields ("conts. higher form symmetries").

$$
\begin{tikzcd}
\substack{\text{moduli stack of higher gauge fields} \\ \text{with } A\text{-quantized flux densities}} \;\; \widehat{A} \arrow[r, "\text{flux densities}"] \arrow[d, "\text{charges}"'] \arrow[dr, phantom, "\text{potentials}\ (\mathrm{pb})"] & \Omega_{\mathrm{dR}}(-;\, \mathfrak{l}A)_{\mathrm{flat}} \arrow[d] \\
\text{flux quant. law} \;\; A \arrow[r, "\mathbf{ch}_A"', "\substack{\text{character map =} \\ \text{rationalization over the reals}}"] & \int \Omega_{\mathrm{dR}}(-;\, \mathfrak{l}A)_{\mathrm{flat}} \simeq A^{\mathbb{R}} .
\end{tikzcd}
\tag{22}
$$

2.6 Brane intersections and twisted cohomology

Finally, all these considerations generalize to fluxes in twisted cohomology, describing brane intersections.

Twisted RR-fields as a fibration. Notice that the twist*ed* RR-fields (12) form a fibration over the twist*ing* NS B-field whose fiber is (a torsor over) the untwisted RR-fields.

$$
\begin{tikzcd}
\substack{\text{flux of free} \\ \text{RR-fields}} \; \Big\{ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \,\Big|\, \mathrm{d}F_{2\bullet} = 0 \Big\} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \left\{ \begin{array}{l|l} H_3 \in \Omega^3_{\mathrm{dR}}(X) & \mathrm{d}H_3 = 0 \\ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) & \mathrm{d}F_{2\bullet} = H_3 \wedge F_{2\bullet-2} \end{array} \right\} \; \substack{\text{flux of} \\ \text{RR-fields coupled} \\ \text{to NS B-field}} \arrow[d, two heads] \\
\{0\} \arrow[r, hook] & \Big\{ H_3 \in \Omega^3_{\mathrm{dR}}(X) \,\Big|\, \mathrm{d}H_3 = 0 \Big\} \; \text{NS B-field}
\end{tikzcd}
\tag{23}
$$

In general, in the case of branes ending on branes, the Bianchi identities for the latter fluxes include polynomial "twists" by the former

$$
\underbrace{\overset{\text{de Rham diff.}}{\mathrm{d}}\, F^{(a)}_{r_a} \;=\; \overset{\text{polynomial}}{P_{r_a}} \Big(\big\{ F^{(b)}_{r_b} \big\}_{b \leq a},\, \overset{\text{twisting fluxes}}{\big\{ H^{(i)}_{r_i} \big\}_{1 \leq i \leq i_{\max}}} \Big)}_{\text{twisted higher "Bianchi identities"}} ,
$$

and the previous classifying spaces (16) generalize to *classifying fibrations*

$$
\begin{tikzcd}[column sep=small]
& \text{classifying spaces for...} & \\
\substack{\text{intersect}\textit{ed} \\ \text{branes}} \;\; A \arrow[r] & A /\!\!/ \mathscr{G} \arrow[d, "\substack{\text{fibrations}}"', "\substack{\text{classifying}}"] & \substack{\text{brane} \\ \text{intersections}} \\
& B\mathscr{G} & \substack{\text{intersect}\textit{ing} \\ \text{branes}}
\end{tikzcd}
$$

which classify *twisted* non-abelian cohomology theories:

$$
\underset{\substack{\text{twisted} \\ \text{non-abelian} \\ \text{cohomology}}}{A^{\tau}(X)} \;:=\; \underset{\substack{\text{vertical homotopy classes} \\ \text{of slice maps}}}{\pi_0 \mathrm{Maps}\Big((X, \tau),\, A/\!\!/\mathscr{G} \Big)_{B\mathscr{G}}} \;=\; \left\{
\begin{tikzcd}
X \arrow[rr, dashed, bend left, "\mathscr{F}_\tau", "\text{twist}\textit{ed}\text{ cocycle}"'] \arrow[rr, dashed, bend right, "\mathscr{F}'_\tau"'] \arrow[dr, "\tau"', "\text{twist}\textit{ing}\text{ cocycle}"] & & A/\!\!/\mathscr{G} \arrow[dl] \\
& B\mathscr{G} &
\end{tikzcd}
\right\}_{\big/\sim}
$$

on which the *twisted non-abelian character map*

$$\begin{array}{ccc} \overset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{cohomology}}}{A^{\tau}(X)} & \xrightarrow[\mathrm{ch}_A]{\text{twisted non-abelian character}} & \overset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big)} \\ \underset{\substack{\mathscr{H}\text{-twisted class of}\\ A\text{-quantized flux}}}{\big[\mathscr{F}_{\mathscr{H}}\big]} & \longmapsto & \underset{\substack{\text{class of}\\ \text{underlying flux densities}}}{\Big[\big(F^{(a)}_{r_a}\big)_{1 \le a \le \dim[\pi_\bullet(A),\mathbb{R}]}\Big]} \end{array} \tag{24}$$

computes the classes of underlying flux densities satisfying twisted Bianchi identities:

$$\Big[\big(F^{(a)}_{r_A}\big)_{1 \le a \le \dim[\pi_\bullet(A),\mathbb{R}]}\Big] \;\in\; \overset{\substack{\tau_{\mathrm{dR}}\text{-twisted}\\ \mathfrak{l}A\text{-valued}\\ \text{non-abelian}\\ \text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big)} \;:=\; \left\{ \begin{tikzcd} & \overset{\text{twist}\textit{ed}\text{ cocycle (dga-hom)}}{\big(F^{(a)}_{r_a}\big)} \arrow[d, equal] & \\ \Omega^\bullet_{\mathrm{dR}}(X) & \substack{\text{coboundary}\\ \text{(concordance)}} \arrow[d, Rightarrow] & \mathrm{CE}\big(\mathfrak{l}(A /\!\!/ \mathscr{G})\big) \arrow[ll, dashed, bend right] \arrow[ll, dashed, bend left] \\ & \big(F^{(a)}_{r_a}\big)' & \\ & \mathrm{CE}(\mathfrak{l}B\mathscr{G}) \arrow[uul, "\tau_{\mathrm{dR}}", "\text{twisting cocycle}"'] \arrow[uur] & \end{tikzcd} \right\}_{\Big/\sim} .$$

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3 Hypothesis H on M-theory

With a general understanding of charge quantization in hand (§2) we are in position to motivate and state *Hypothesis H* (§3.2). In order to put this hypothesis in perspective, we first review (§3.1) the widely accepted *Hypothesis K* that D-brane charges are quantized in twisted K-theory:

3.1 Hypothesis K — D/NS-brane flux quantization in K-CoHomology.

The conjecture that D-brane charges are quantized in topological K-theory (for more review and pointers see [BMSS19, §1]) originates with the observation [GHV97][MM97] that the differential RR-flux form data (5) which apparently characterizes D-brane charge has the form of the *Chern character* on topological K-theory classes (cf. [FM00, p. 8][BMRS08, §2.2][11]):

Hypothesis K for vanishing NS flux:

D-brane charges are quantized in topological K-theory, hence
RR-field fluxes are in the Chern character

$$K(X) \xrightarrow{\ \mathrm{ch}\ } H_{\mathrm{dR}}(X;\mathfrak{l}\mathrm{KU}_0) = \left\{ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \;\middle|\; \mathrm{d}F_{2\bullet} = 0 \right\}_{/\text{concordance}} \tag{25}$$

Here we have written the Chern character in the form reviewed in §2 (see [Char, Exp. 7.2]), highlighting (for comparison below in §3.2) that it may be understood as defined on homotopy-classes of maps to the classifying space KU_0 for complex topological K-theory

$$\begin{array}{c}\text{complex topological}\\ \text{K-cohomology (deg 0)}\end{array} \qquad K(X) = \left\{ X \xrightarrow{\ \text{K-cocycle}\ } \mathrm{KU}_0 \right\}_{/\text{homotopy}}$$

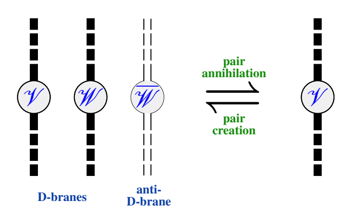
and as taking values in differential forms with coefficients in its Whitehead L_∞-algebra:

RR-field flux
away from NS_5

pre-geometric equations of motion of flux densities	$\mathrm{d}F_{2\bullet} = 0$	(5)
corresponding Sullivan model dg-algebra ("FDA")	$\mathrm{d}f_{2\bullet} = 0$	e.g. [FOT08, §1.81, 1.86]
candidate classifying space	$\mathrm{KU}_0 \simeq B\mathrm{U} \times \mathbb{Z}$	
cohomology theory classified by this space	topological K-theory $K(X) := \pi_0 \mathrm{Maps}(X, \mathrm{KU}_0)$	e.g. [Kar78, §II Thm. 1.33]

Perspective. Important to notice here is that all formulas in [GHV97][MM97] which led to the original *Hypothesis K* (25) concern differential form expressions and as such are purely "rational". It is (only) the resemblance of the differential relations satisfied by these differential forms with the image of a character map which suggests that the non-rational domain of this character map (here: K-theory) may be the true home of the brane charges: Among all cohomology theories with this form of character images, K-theory seems to be the most natural or immediate choice (for one, it is essentially the only choice with an established name and geometric interpretation, certainly when the twisting is incorporated below).

An argument meaning to justify the choice of K-theory beyond its rational approximation was then given in [Wi98, §3], where it is observed that the expected brane/anti-brane annihiliation (by tachyon condensation in the open super-strings stretching between them) broadly resembles the Grothendieck equivalence relation which famously expresses (eg. [Kar78, §II 1]) the K-cohomology group $K(X)$ for a compact space X as the equivalence classes of pairs of vector bundles and "anti-bundles" (virtual bundles) subject to a relation expressing that equal but opposite vector bundles cancel.



[11] Starting with [MM97], many authors insist on multiplying the Chern character with a differential form representative of the square root $\sqrt{\hat{A}}$ of the A-roof genus of the tangent bundle of spacetimes before referring to it as D-brane charge. But since $\sqrt{\hat{A}}$ is multiplicatively invertible (being a unit plus a sum of inhomogeneous differential forms which are nilpotent under wedge product) this is not intrinsic to the notion of D-brane charge and may be disregarded for the purpose of charge quantization (cf. [FM00, ftn. 12]) — its role is rather in making the Chern character natural under push-forward (cf. [BMRS08, §2]).

Hypothesis K in the presence of NS-flux. In view of our above discussion, the more general conjecture [Wi98, §5.3][BM01] that D-brane charges in the presence of NS 5-brane charges are classified by 3-twisted K-theory (see [GS22] for more) is now fairly immediate from the observation (cf. [Char, Rem. 10.1]) that the differential relations satisfied by the twisted Chern character are just the pregeometric equations of motion (12):

RR-field flux
in presence of NS-flux

pre-geometric equations of motion of flux densities	$\begin{aligned} \mathrm{d}F_{2\bullet} &= H_3 \wedge F_{2\bullet-2} \\ \mathrm{d}H_3 &= 0 \end{aligned}$	(12)	(26)
corresponding relative Sullivan model dg-algebra ("FDA")	$\begin{aligned} \mathrm{d}f_{2\bullet} &= h_3 \wedge f_{2\bullet} \\ \mathrm{d}h_3 &= 0 \end{aligned}$	[FHT07, p. 6] [BMSS19, Lem. 2.31]	
candidate classifying fibration	$\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \to B^2\mathrm{U}(1)$		
cohomology theory classified by this fibration	twisted K-theory $K^\tau(X) := \pi_0 \Gamma_X\big(\tau^*(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))\big)$	[FHT07, (2.6)] [AS04, Def. 3.3]	

And so the general conjecture for D-branes, widely (though not universally) expected, is this:

Hypothesis K:

D-brane charges are quantized in twisted topological K-theory, hence
RR-field fluxes are in the twisted Chern character

$$K^\tau(X) \xrightarrow{\;\mathrm{ch}\;} H^\tau_{\mathrm{dR}}\big(X; \mathfrak{l}_{B^2\mathrm{U}(1)}\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) = \Big\{ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \;\Big\vert\; \mathrm{d}F_{2\bullet} = H_3 \wedge F_{2\bullet-2} \Big\}_{\big/\text{concordance}} \qquad (27)$$

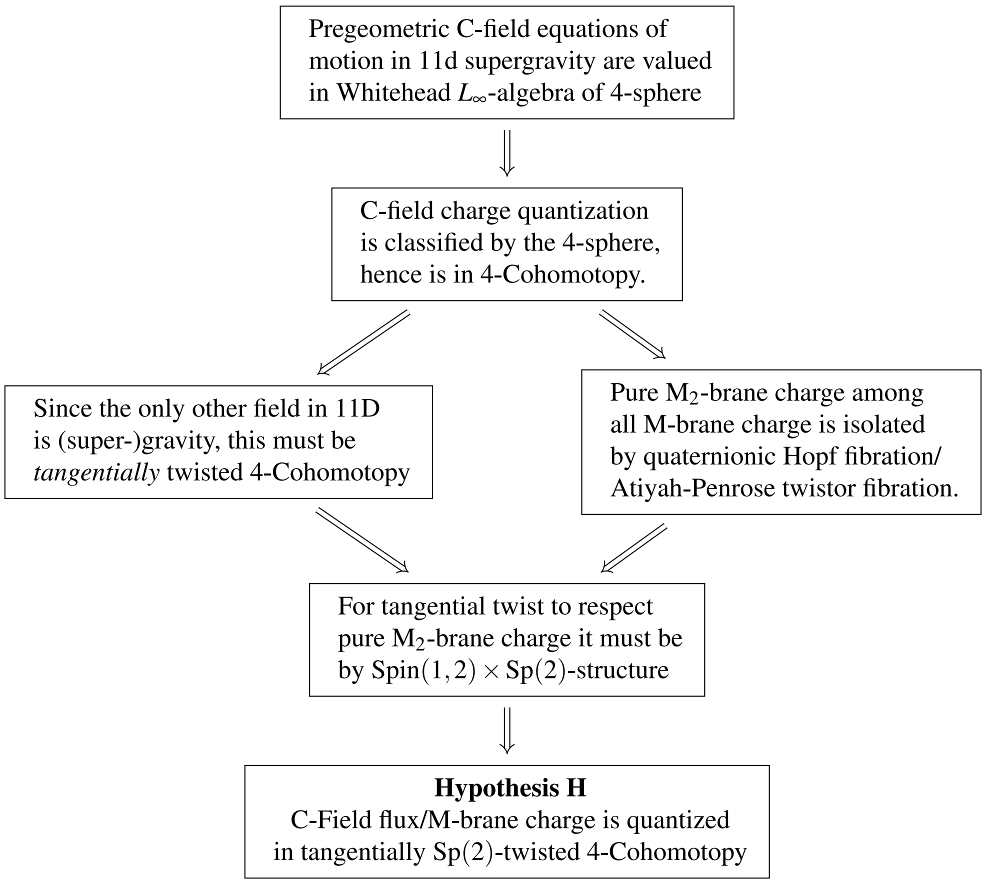
Here we have written the twisted Chern character in the form reviewed in §2 (see [Char, Exp. 6.6, Prop. 10.1]), highlighting (for comparison below in §3.2) that it is defined on homotopy classes of sections of pullbacks along the twisting map of the universal KU_0-bundle ([FHT07, (2.6)][Char, Exp. 3.4]) and takes values in differential forms with coefficients in its relative Whitehead L_∞-algebra:

$$\underset{\text{topological K-theory}}{\text{twisted}} \quad K^\tau(X) = \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, "\tau", "\text{twist}"'] && \mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \arrow[dl] \\ & B^2\mathrm{U}(1) & \end{tikzcd} \right\}_{\big/\text{rel homotopy}}$$

In the next section §3.2 we develop *Hypothesis H* in close analogy to this now classical argument for *Hypothesis K*, which is possible due to the understanding [Char] of the (twisted) Chern character on K-theory as just a special case of a general notion of (twisted) characters on non-abelian generalized cohomology theories whose images capture also non-linear Bianchi identities such as those of the C-field in 11d supergravity.

3.2 Hypothesis H — M-brane flux quantization in CoHomotopy.

We are finally ready to motivate, state and explain *Hypothesis H*. To make it transparent, we start with its formulation on flat spacetimes and then incrementally bring in the coupling to gravitational background charges in the form of appropriate tangential twisting of the charge cohomology theory. The logic [Sa13, §2.5][HpH1][GS1] is summarized by the following schematic diagram:



Alongside the development of the hypothesis we highlight here its foremost **implications** on M-brane charge quantization:
- (35) the shifted flux quantization of the C_3-field and hence of M_5-brane charge,
- (46) the shifted flux quantization of the C_6-field and hence of M_2-brane charge ("Page charge").

Further implications are discussed in following sections, notably the resulting topological M5-brane model in §4.

C-Field flux on flat spacetimes. By the discussion in §2, the admissible flux quantizations of the pre-geometric C-field flux (8) in 11-dimensional supergravity on flat spacetimes are classified by spaces whose minimal Sullivan dg-algebra satisfies the analogous equations. But in rational homotopy theory just these equations are well-known as the Sullivan model for the 4-sphere! [Sa13, §2.5], see also [Sphe, §2][ℚStruc, p. 14]. This means, by the discussion in §2, that the cohomology theory classified by the 4-sphere is an admissible quantization law for the C-field flux in 11-dimensional supergravity: This cohomology theory is *4-Cohomotopy*:

C-field flux
on flat spacetimes

pre-geometric equations of motion of flux densities	$\begin{aligned} \mathrm{d}\,G_4 &= 0 \\ \mathrm{d}\,G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \end{aligned}$	(8)
corresponding Sullivan model dg-algebra ("FDA")	$\begin{aligned} \mathrm{d}\,g_4 &= 0 \\ \mathrm{d}\,g_7 &= -\tfrac{1}{2} g_4 \wedge g_4 \end{aligned}$	e.g. [De76, Exp. 3.5 (a)] [FHT00, p. 142] [Me15, §1.2]
candidate classifying space	4-sphere: S^4	
Whitehead bracket L_∞-algebra	$\pi_3(\Omega S^4) \otimes \mathbb{R} = \mathbb{R}\langle \gamma_3 \rangle$ $\pi_6(\Omega S^4) \otimes \mathbb{R} = \mathbb{R}\langle \gamma_6 \rangle$ $[\gamma_3, \gamma_3] = \gamma_7$	cf. [CJLP98, (2.6)][LLPS99, (3.4)] [KS03, (75)] [BNS04, (86)] [Sa10, (4.9)]
cohomology theory classified by this space	4-Cohomotopy: $\pi^4(X) := \pi_0 \mathrm{Maps}(X, S^4)$	[Pontrjagin1938] [Spanier1949] [Peterson1956]

$$\text{plain 4-coHomotopy} \quad \pi^4(X) := \left\{ \underset{\text{spacetime}}{X} \xrightarrow[\text{cocycle}]{c_3} \underset{\text{4-sphere}}{S^4} \right\}_{/\text{homotopy}}$$

Moreover, the 4-sphere is the minimal such choice of flux-quantization law for 11-dimensional supergravity, in that it is the smallest CW-complex with this property. In this sense the universal choice of C-field flux quantization is by 4-cohomotopy. The hypothesis that this universal choice is the correct choice of flux-quantization for M-theory is:

Hypothesis H over flat spacetimes ([Sa13, §2.5][HpH2]):

M-brane charges are quantized in 4-cohomotopy, hence
C-field fluxes are in the 4-cohomotopical character

$$\pi^4(X) \xrightarrow{\mathrm{ch}_{\pi^4}} H_{\mathrm{dR}}(X; \mathfrak{l}S^4) = \left\{ \begin{array}{l} G_4 \in \Omega^4_{\mathrm{dR}}(X) \\ G_7 \in \Omega^7_{\mathrm{dR}}(X) \end{array} \middle| \begin{array}{l} \mathrm{d}\,G_4 = 0, \\ \mathrm{d}\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4 \end{array} \right\}_{/\text{concordance}} \tag{28}$$

Perspective. To re-iterate how this hypothesis comes about: The general theory of flux quantization (§2) says that any cohomology theory flux-quantizing the C-field fluxes (8) has a classifying space whose Sullivan model has as generators the pre-geometric field species subject to differential relations of the same form as the pre-geometric Bianchi identities of the C-field; and rational homotopy theory shows that these are precisely the spaces of the rational homotopy type of the 4-sphere. Among all of these, the 4-sphere itself (and hence the coHomotopy cohomology theory that it classifies) is in some sense the canonical/universal choice — therefore it is natural to hypothesize that this is the choice needed for M-theory.

Should Hypothesis H be false (not quite correspond to M-theory), it would mean that we have to add cells to the 4-sphere (without changing its rational homotopy type) in order to find the correct classifying space for flux quantization in M-theory. Since there are infinitely many choices involved in doing so, it will help to know *how* Hypothesis H fails, if it does, as this will indicate how the canonical choice of classifying space S^4 needs to be adjusted. In this sense the analysis of the predictions of Hypothesis H is essentially an inevitable step towards understanding charge-quantization in M-theory, either way.

Cohomotopical M-brane charges and homotopy groups of spheres. The character map in (28) is given by the abstract rationalization construction described in §2, but in degree 4 we may readily describe it explicitly: Given a cocycle $c_3 : X \longrightarrow S^4$ in 4-cohomotopy, the corresponding 4-flux density G_4 is the pullback along c^3 of the *volume form* dvol_{S^4}, hence its real cohomology class may be identified with the pullback of the fractional Euler class $\frac{1}{2}\chi_4$ on the 4-sphere:

$$
\begin{array}{lccc}
\textbf{cocycle in 4-cohomotopy} & X & \overset{c_3}{\dashrightarrow} & S^4 \\[1.5em]
\textbf{induced 4-flux} & \Omega^\bullet_{\mathrm{dR}}(X) & \xleftarrow{\;\;c_3^*\;\;} & \Omega^\bullet_{\mathrm{dR}}(S^4) \\
 & G_4 & \longleftarrow\!\!\shortmid & \mathrm{dvol}_{S^4} \\[1.5em]
\begin{array}{c}\textbf{induced } \mathrm{M}_5\textbf{-charge} \\ \textbf{in integral cohomology}\end{array} & H^\bullet\big(X;\mathbb{Z}\big) & \xleftarrow{\;\;c_3^*\;\;} & H^\bullet\big(S^4;\mathbb{Z}\big) \simeq \mathbb{Z}\big[\tfrac{1}{2}\chi_4\big] \\
 & [G_4] & \longleftarrow\!\!\shortmid & \tfrac{1}{2}\chi_4
\end{array}
\tag{29}
$$

Hence Hypothesis H implies, first of all, that singular flat M_5-branes $\mathbb{R}^{1,5} \hookrightarrow \mathbb{R}^{1,10}$ carry integral charge, as expected.

Generally, Hypothesis H implies that brane charges on flat spacetimes are given by the *homotopy groups of the 4-sphere* (cf. [HpH2]):

$$
\begin{array}{lccccl}
\begin{array}{c}\textbf{singular} \\ p\textbf{-brane charge}\end{array} & \pi^4\Big(\mathbb{R}^{1,10}\setminus\mathbb{R}^{\overbrace{1,9-n}^{p}}\Big) & & & & \\
 & \| & & & & \\
\begin{array}{c}\textbf{4th co-homotopy group} \\ \textbf{of } n\textbf{-sphere}\end{array} & \pi^4(S^n) & = & \Big\{ S^n \overset{c_3}{\dashrightarrow} S^4 \Big\}_{/\mathrm{homotopy}} & = \pi_n(S^4) & \begin{array}{c}n\textbf{-th homotopy group} \\ \textbf{of 4-sphere}\end{array} \\
 & \| & & & & \\
\begin{array}{c}\textbf{solitonic} \\ p\textbf{-brane charge}\end{array} & \pi^4\Big(\mathbb{R}^{\overbrace{1,10-n}^{p}} \times \mathbb{R}^n_{\cup\{\infty\}}\Big) & & & &
\end{array}
\tag{30}
$$

$n =$	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\pi_n(S^4)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\cdots$
	exotic branes (§6.2)			M_5			M_2				

All these groups are finite (hence are "torsion effects" predicted by charge-quantization not seen on differential form data) except in exactly two dimensions, corresponding to the existence of integer charged singular M_5-branes and M_2-branes, respectively:

$$
\begin{array}{ll}
\begin{array}{c}\textbf{cohomotopy charge of flat} \\ \textbf{singular } \mathrm{M}_5\textbf{-branes}\end{array} & \pi^4\big(\mathbb{R}^{1,10}\setminus\mathbb{R}^{5,1}\big) \simeq \pi^4\big(\mathbb{R}^{5,1}\times\mathbb{R}_{\sqcup\{\infty\}}\times S^4\big) \simeq \pi^4\big(S^4\big) \simeq \pi_4\big(S^4\big) \simeq \mathbb{Z} \\[1em]
\begin{array}{c}\textbf{cohomotopy charge of flat} \\ \textbf{singular } \mathrm{M}_2\textbf{-branes}\end{array} & \pi^4\big(\mathbb{R}^{1,10}\setminus\mathbb{R}^{1,2}\big) \simeq \pi^4\big(\mathbb{R}^{1,2}\times\mathbb{R}_{\sqcup\{\infty\}}\times S^7\big) \simeq \pi^4\big(S^7\big) \simeq \pi_7\big(S^4\big) \simeq \mathbb{Z}\oplus\text{torsion}
\end{array}
$$

To amplify this point: Any classifying space for charge quantization in 11d which implies integer-charged singular p-branes exactly for the expected values $p=2$ and $p=5$ will need to have non-torsion homotopy groups precisely in degree $9-2=7$ and $9-5=4$. The 4-sphere is the minimal cell complex with this property.

The reason for this is the existence of the *quaternionic Hopf fibration*:

Cohomotopical M_2-brane charge and the quaternion Hopf fibration. The generator of the integer summand $\mathbb{Z}\in\pi^7(S^4)$ (30) is the homotopy class of a S^3-fibration called the *quaternionic Hopf fibration*:

$$
\begin{array}{ccc}
 & \begin{array}{c}\textbf{quaternionic} \\ \textbf{Hopf fibration}\end{array} & \\
S^3 & \hookrightarrow & S^7 \\
 & & \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\
 & & S^4
\end{array}
\qquad
\big[S^7 \xrightarrow{h_{\mathbb{H}}} S^4\big] = 1 \in \mathbb{Z} \hookrightarrow \pi_7(S^4)\,.
$$

But this means that we may regard S^7 as the classifying space of integral M_2-brane charges and the quaternionic Hopf fibration as classifying the cohomology operation which injects pure M_2-brane charge into the full set of M-brane charges:

$$
\begin{tikzcd}
 & S^7 \arrow[d, "h_{\mathbb{H}}"] & \in \pi^7(X) \arrow[d, "(h_{\mathbb{H}})_*"] & \text{pure } \mathrm{M}_2\text{-brane charges} \\
X \arrow[ur, dashed, "c_6"] \arrow[r, dashed, "c_3 = (h_{\mathbb{H}})_* c_6"'] & S^4 & \in \pi^4(X) & \text{full M-brane charges}
\end{tikzcd}
\qquad \text{on flat spacetimes } X
\tag{31}
$$

Remark: Flat solitonic M-branes. With the prediction of flat singular 5-branes, Hypothesis H necessarily also predicts (30) integer-charged *solitonic 6-branes* (cf. p. 5). We discuss in §6 (following [Qnt1]) how these may be identified with the non-singular (and thus "solitonic") 6-brane-like solutions of 11d-supergavity known as the *KK-monopole*, the M-theoretic incarnation of D_6-branes.

Cohomotopy as K-theory over $\mathbb{F}_1$. The argument for *Hypothesis H* to this point is the fairly complete M-theoretic analog of the original argument for D-brane charge quantization in K-theory from inspection of the nature of the flux forms, as reviewed above in §3.1. There we highlighted that, beyond this "rational" evidence, the particular choice of topological K-theory as the charge quantization law for D-branes is further justified by the observation that the Grothendieck-equivalence relation on virtual vector bundles which defines topological K-theory reflects, at least informally, the expected cancellation of Chan-Paton gauge bundles under expected brane/anti-brane creation/annihilation processes.

But the characterization in terms of such Grothendieck equivalence relations is not specific to "topological" K-theory: It applies (in degree 0) also to "algebraic" K-theory over any ring.

If we forget the non-linear effects of non-abelian Cohomotopy by passing to its shadow in stable Cohomotopy (80), then the same plausibility check regarding brane/anti-brane annihiliation holds, in that stable CoHomotopy is also a form of K-theory, namely[12] the algebraic K-theory of the absolute base "field" $\mathbb{F}_1$.

[12] For pointers see ncatlab.org/nlab/show/stable+cohomotopy#AsAlgebraicKTheoryOverTheFieldWithOneElement

Coupling to gravitational charges and tangential twisting. We motivate the generalization of Hypothesis H to spacetimes which are not necessarily flat:

As remarked in (7), 11-dimensional supergravity stands out in its C-field being the *only* field besides that of gravity. This means that possible twistings of the C-field flux-quantization can only be by the gravitational field, namely by the Spin-frame-bundle of spacetime X (the principal bundle underlying its tangent bundle). By the general rules of twisted cohomology (§2.6) and assuming Hypothesis H on flat spacetimes (28) this means that possible twistings are given by ∞-actions of (subgroups of) the Spin-group on the 4-sphere. The canonical[13] such action is that of $\mathrm{Spin}(5)$ via the defining action of $\mathrm{SO}(5)$ on $S^4 = S(\mathbb{R}^5)$ regarded as the unit sphere in $\mathbb{R}^5$.

This leads to *tangentially twisted 4-cohomotopy theory* [HpH1, §2.1], consisting of homotopy classes of sections of the 4-sphere bundle associated with a $\mathrm{Spin}(5)$-structure τ on spacetime:

$$
\underset{\text{4-coHomotopy}}{\text{tangentially twisted}}\quad \pi^{4+\tau}(X) \;:=\; \left\{
\begin{tikzcd}
\underset{X}{\text{spacetime}} \arrow[rr, dashed, "\text{cocycle}", "c_3"'] \arrow[d, "\mathrm{Fr}(X)"'] \arrow[dr, "\tau"', "\text{twist}"] & & \underset{S^4 /\!\!/ \mathrm{Spin}(5)}{\text{universal orthogonal 4-sphere bundle}} \arrow[dl] \\
B\mathrm{Spin}(D) & B\mathrm{Spin}(5) \arrow[l] &
\end{tikzcd}
\right\}_{\big/\text{homotopy}}
\tag{32}
$$

(Such nonabelian/unstable twisted cohomotopy had previously been considered in [Cr03, Lem. 5.2], for more see [HpH1, §2.1].)

Remark: The role of G-structure [HpH1, §2.2]. Using the tangentially twisted 4-cohomotopy (32) for flux quantization means that a choice of $\mathrm{Spin}(5)$-structure on spacetime is part of the flux-quantized C-field datum (or rather of isomorphic but subtly different $\mathrm{Sp}(2)$-structure, which we come to in a moment.) Lest this seems overly restrictive, notice that the structure group of the tangent bundle may still be all of $\mathrm{Spin}(1,5) \times \mathrm{Spin}(5)$ in order that τ exists. On the other hand, the existence and choice of a cocycle in $\pi^{4+\tau}(X)$ then equivalently means that and how the $\mathrm{Spin}(5)$-structure factor is further reduced to $\mathrm{Spin}(4)$, due to

$$
\begin{tikzcd}
S^4 \arrow[r] \arrow[d, equal] & \underset{S^4 /\!\!/ \mathrm{Spin}(5)}{\text{universal orthogonal 4-sphere bundle}} \arrow[d, "\wr"] \\
\mathrm{Spin}(5)/\mathrm{Spin}(4) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & B\mathrm{Spin}(4) \arrow[d] \\
* \arrow[r] & B\mathrm{Spin}(5)
\end{tikzcd}
\tag{33}
$$

Shifted C-field flux quantization. Hence the generalization of Hypothesis H (28) away from the special case of flat spacetimes should say that C-field flux is quantized not in plain 4-cohomotopy π^4, but in tangentially twisted 4-cohomotopy $\pi^{4+\tau}$ (32). In a moment we will refine this statement a little further, but first to record the following:

The first non-trivial check of the tangential twisting is its implication of the notorious *shifted integral flux quantization* of the 4-flux density, an unusual-looking condition which however is a widely expected hallmark of M-theory (originally proposed in [Wi97a][Wi97b], see also [Wi00, §2][GS02, p. 21][CS12a][CS12b]) – it says that not the de Rham cohomology class of G_4 but its shift by one *fourth* of the Pontrjagin 4-form $p_1(\nabla)$ on spacetime (for any connection ∇ on the tangent bundle) is the real image of an integral cohomology class:

$$
\underset{\text{in ordinary cohomology}}{\text{M}_5\text{-brane charge image}}\quad \big[\widetilde{G}_4\big] \;:=\; \underset{\text{shifted 4-flux density}}{\overset{\text{class of}}{\big[G_4 + \tfrac{1}{4}p_1\big]}} \;\in\; \overset{\text{integral cohomology}}{H^4(X;\mathbb{Z})} \to H^4(X;\mathbb{R})\,.
\tag{34}
$$

Notice that a deeper cohomological understanding of this condition was the motivation for the seminal development of abelian (i.e. stable) generalized differential cohomology in [HS05]. But in our context of non-abelian cohomology the condition falls out naturally:

Namely ([HpH1, §3.4]), the integral cohomology of $S^4 /\!\!/ \mathrm{Spin}(5) \simeq B\mathrm{Spin}(4)$ (33) is generated from $\frac{1}{2}p_1$ and the combination $\frac{1}{2}\chi_4 + \frac{1}{4}p_1$ [CV98a, Lem 2.1]. But since the pullback of half the Euler class, $\frac{1}{2}\chi_4$, being the volume form on the S^4-fibers [BC98, §2], is interpreted (29) as the G_4-flux under Hypothesis H, and since the pullback of the universal $\frac{1}{4}p_1$ is the

[13] There is an isomorphic but subtly different action of $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$ on S^4, which we come to further below.

actual such class on spacetime X by the nature of tangential twisting, this means that $G_4 + \frac{1}{4}p_1$ is the image of the pullback of an integral form, and hence itself integral ([HpH1, §3.4]):

$$
\begin{array}{lccc}
\begin{array}{c}\textbf{cocycle in}\\ \textbf{tangentially twisted}\\ \textbf{4-cohomotopy}\end{array} &
\begin{tikzcd}
X \arrow[rr, dashed, "c_3"] \arrow[dr, "\vdash Fr(X)"'] & & S^4 /\!\!/ \mathrm{Spin}(5) \simeq B\mathrm{Spin}(4) \arrow[dl] \\
& B\mathrm{Spin}(d) &
\end{tikzcd}
\\[10pt]
\begin{array}{c}\textbf{induced charge in}\\ \textbf{real cohomology}\end{array} &
H^\bullet\big(X;\mathbb{R}\big) & \xleftarrow{\;\;c_3^*\;\;} & H^\bullet\big(B\mathrm{Spin}(4);\mathbb{R}\big) = \mathbb{R}\big[p_1,\,\chi_4\big]
\\
& \big[p_1(\nabla)\big] & \longleftarrow\!\shortmid & p_1 \;\; \textbf{first Pontrjagin class}
\\
& \big[G_4\big] & \longleftarrow\!\shortmid & \tfrac{1}{2}\chi_4 \;\; \textbf{fractional Euler class}
\\[10pt]
\begin{array}{c}\textbf{induced charge in}\\ \textbf{integral cohomology}\end{array} &
H^\bullet\big(X;\mathbb{Z}\big) & \xleftarrow{\;\;c_3^*\;\;} & H^\bullet\big(B\mathrm{Spin}(4);\mathbb{Z}\big) = \mathbb{Z}\Big[\tfrac{1}{2}p_1,\,\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1\Big]
\\
\begin{array}{c}\textbf{integral class of}\\ \textbf{shifted C-field flux}\end{array} &
\underbrace{\big[G_4 + \tfrac{1}{4}p_1(\nabla)\big]}_{[\widetilde{G}]} & \longleftarrow\!\shortmid & \tfrac{1}{2}\chi + \tfrac{1}{4}p_1 \;\; \begin{array}{c}\textbf{universal integral}\\ \textbf{characteristic class}\end{array}
\end{array}
\tag{35}
$$

Isolating M_2**-brane charge on curved spacetimes** [HpH1, §2.3]. We saw in (31) that the quaternionic Hopf fibration $S^4 \xrightarrow{h_{\mathbb{H}}} S^4$ serves to identify pure M_2-brane charge inside all M-brane charges, under Hypothesis H on flat spacetimes. In order to retain such an identification as we generalize M-brane charges to curved spacetimes via tangentially twisted cohomotopy (32), we need to find a Spin-group which acts on both S^4 and S^7 in a compatible way, namely such that the Hopf fibration is equivariant under this action.

Remarkably, the quaternionic Hopf fibration is indeed $\mathrm{Spin}(5)$-equivariant — or rather it is equivariant under the isomorphic quaternionic unitary group $\mathrm{Sp}(n) \simeq \mathrm{U}(n,\mathbb{H}) \subset \mathrm{GL}(n,\mathbb{H})$ (cf. [M5b, §A]) in quaternionic dimension 2, via its canonical action on $S^7 = S(\mathbb{H}^2)$, due to the following coset-space realization of the quaternionic Hopf fibration [HT09, Tab. 1][GWZ, Prop. 4.1]:

$$
\begin{tikzcd}
S^3 \arrow[rr, "\mathrm{fib}(h_{\mathbb{H}})"] \arrow[d, no head, "\wr"] & & S^7 \arrow[out=120, in=60, loop, "\mathrm{Sp}(2)"] \arrow[rr, "h_{\mathbb{H}}"] \arrow[d, no head, "\wr"] & & S^4 \arrow[out=120, in=60, loop, "\mathrm{Sp}(2)"] \arrow[d, no head, "\wr"] \\
\frac{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)} \arrow[rr, "\frac{\iota}{\mathrm{id}}"] \arrow[d, no head, "\wr"] & & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \arrow[rr, "\frac{\mathrm{id}}{q \mapsto (q,1)}"] \arrow[d, no head, "\wr"] & & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \arrow[d, no head, "\wr"] \\
\frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} & & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(3)} & & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(4)}
\end{tikzcd}
\tag{36}
$$

An important subtlety here is that $\mathrm{Spin}(5)$ and $\mathrm{Sp}(2)$, while isomorphic as abstract Lie groups, are *not isomorphic* as subgroups of $\mathrm{Spin}(8)$, but as such they are exchanged under the *triality automorphism* $\mathrm{tri} : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$. This subtlety is ultimately responsible for the appearance of the "one-loop term" I_8 (14) from Hypothesis H (see below), in that [HpH1, (97)]:

$$
\begin{array}{c}
\begin{tikzcd}
B\mathrm{Sp}(2) \arrow[r, "\sim"] \arrow[d, hook] & B\mathrm{Spin}(5) \arrow[d, hook] \\
B\mathrm{Spin}(8) \arrow[r, "B\mathrm{tri}", "\sim"'] & B\mathrm{Spin}(8)
\end{tikzcd}
\\[10pt]
\begin{array}{ccc}
H^\bullet\big(B\mathrm{Sp}(2);\mathbb{R}\big) & \xleftarrow{\;(B\mathrm{tri})^*\;} & H^\bullet\big(B\mathrm{Spin}(5);\mathbb{R}\big) \\
\tfrac{1}{2}p_1 & \longleftarrow\!\shortmid & \tfrac{1}{2}p_1 \\
(\tfrac{1}{4}p_1)^2 - 24\cdot I_8 & \longleftarrow\!\shortmid & \tfrac{1}{4}p_2
\end{array}
\end{array}
\tag{37}
$$

Namely, by [CV97, 2.2, 4.1, 4.2] and [CV98b, 8.1, 8.2] we have, respectively:

$$
\begin{array}{lccccc}
 & H^8\big(B\mathrm{Spin}(8)\big) & \xrightarrow{(B\mathrm{tri})^*} & H^8\big(B\mathrm{Spin}(8)\big) & \longrightarrow & H^8\big(B\mathrm{Sp}(2)\big) \\
 & \tfrac{1}{2}p_1 & \leftrightarrow & \tfrac{1}{2}p_1 & \mapsto & \tfrac{1}{2}p_1 \\
 & \tfrac{1}{4}\big(p_2 - (\tfrac{1}{2}p_1)^2\big) - \tfrac{1}{2}\chi_8 & \leftrightarrow & -\chi_8 & \mapsto & -\tfrac{1}{2}\big(p_2 - \tfrac{1}{4}(p_1)^2\big) \\
\text{and hence} & \tfrac{1}{4}p_2 & \leftrightarrow & \begin{array}{c} -\chi_8 + \tfrac{1}{4}(\tfrac{1}{2}p_1)^2 \\ -\tfrac{1}{2}\Big(\tfrac{1}{4}\big(p_2 - (\tfrac{1}{2}p_1)^2\big) - \tfrac{1}{2}\chi_8\Big) \end{array} & \mapsto & (\tfrac{1}{4}p_1)^2 - \tfrac{1}{2}\underbrace{\big(p_2 - \tfrac{1}{4}(p_1)^2\big)}_{=:48\cdot I_8}
\end{array}
\tag{38}
$$

This way we arrive at the general form of (28):

Hypothesis H ([HpH1]):

M-brane charges are quantized in tangentially $\mathrm{Sp}(2)$*-twisted 4-cohomotopy*, hence
C-field fluxes are in the twisted 4-cohomotopical character

$$
\pi^{4+\tau}(X) \xrightarrow{\mathrm{ch}^\tau} H^\tau\big(X; \mathfrak{l}S^4\big) \;=\; \left\{ \begin{array}{l|l} G_4 \in \Omega^4_{\mathrm{dR}}(X) & \mathrm{d}\,G_7 \;=\; -\tfrac{1}{2}\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{2}p_1) - 12 \cdot I_8 \\ G_7 \in \Omega^7_{\mathrm{dR}}(X)) & \mathrm{d}\,G_4 \;=\; 0 \end{array} \right\}_{\big/\mathrm{concordance}}
\tag{39}
$$

$$
\begin{array}{c}\textbf{tangentially}\\ \textbf{Sp(2)-twisted}\\ \textbf{4-coHomotopy}\end{array}
\quad
\pi^{4+\tau}(X) \;:=\; \left\{
\begin{tikzcd}
\overset{\text{spacetime}}{X} \arrow[rr, dashed, "c_3", "\text{cocycle}"'] \arrow[d, "\vdash \mathrm{Fr}(X)", "\text{tangent structure}"'] \arrow[drr, "\tau", "\text{twist}"'] & & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{universal orthogonal 4-sphere bundle}}{S^4 /\!\!/ \mathrm{Spin}(5)} \arrow[d] \\
\begin{array}{c} B\mathrm{Spin}(1,2) \\ \times B\mathrm{Spin}(8) \end{array} & & B\mathrm{Sp}(2) \arrow[r, "\sim"'] \arrow[d] & B\mathrm{Spin}(5) \arrow[d] \\
 & & B\mathrm{Spin}(8) \arrow[llu] \arrow[r, "B\mathrm{tri}", "\sim"'] & B\mathrm{Spin}(8)
\end{tikzcd}
\right\}_{\big/\text{rel. homotopy}}
\tag{40}
$$

That the twisted cohomotopical character is of this form (39) follows [HpH1, Prop. 3.8] essentially by the formula for the Sullivan model of $\mathrm{Spin}(5)$-associated S^4-fibrations, which in itself gives [HpH1, Prop. 2.5] $\mathrm{d}\,2G_7 \;=\; -G_4 \wedge G_4 + \tfrac{1}{4}p_2(\nabla^{\mathrm{Spin}(5)})$ and then plugging in the expression for $\tfrac{1}{4}p_2$ from (37) to account for the fact that the twist is actually by $\mathrm{Sp}(2)$-structure. Finally we have cleaned up the formula by completing the resulting square in terms of the shifted flux density $\widetilde{G}_4$ (34):

$$
-G_4 \wedge G_4 + \tfrac{1}{4}p_1 \wedge \tfrac{1}{4}p_1 \;=\; -(G_4 + \tfrac{1}{4}p_1) \wedge (G_4 - \tfrac{1}{4}p_1) \;=\; -\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{2}p_1)
$$

Notice that the factor $\mathrm{Spin}(1,2)$ may be included in the spacetime tangent structure in (40) without changing this conclusion nor that of the shifted flux quantization (35), since it contributes neither to p_1 nor to p_2.

Remark: Normalization of the one-loop term in the Bianchi identity. The factor of "12" in (39) may seem unexpected, since an old argument [SVW96, p. 2][DM97, (1)] (which, incidentally, neglects the shifting (34)) might lead one to expect a factor of "1" here, instead — but this depends in turn on the prefactor which translates between the integrated flux density $\int_{S^7} G_7$ and the actual number of M_2-branes. In [HpH1, p. 12-13] we argue that proper counting of 2-brane charge in Cohomotopy does resolve this apparent discrepancy.

On the other hand, we discuss next that in order for the "M_2 Page-charge" to be integral and the M_5-brane sigma-model in the background of M2-brane flux to be well-defined, this characteristic polynomial has to vanish (an M-theoretic form of anomaly cancellation by "Fivebrane structure"), in which case this issue disappears anyway, see (47) below.

M_2-charge quantization and the Hopf-Wess-Zumino coupling in the M_5. Hypothesis H in the form (39) implies (by design, recalling (31) and (36)) that a notion of *pure* M_2*-brane charge* is retained after M-brane charge quantization in twisted Cohomotopy, namely given by those twisted 4-Cohomotopy cocycles which factor through the $B\mathrm{Sp}(2)$-parameterized quaternionic Hopf fibration $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ up to homotopy, or more specifically, by the *choice* (c_6, b_2) of such a homotopy-factorization:

$$
\begin{tikzcd}
 & & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] & \in & \pi^{3+c_3}(X) \arrow[r] & \pi^{7+\tau}(X) \arrow[d, "(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_*"] & \text{pure } \mathrm{M}_2\text{-brane charges} \\
X \arrow[urr, dashed, "c_6"] \arrow[rr, "c_3 = (h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_*(c_6)"'] \arrow[dr, "\tau"'] & & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[dl] & \in & & \pi^{4+\tau}(X) & \text{full M-brane charges} \\
 & B\mathrm{Sp}(2) & & & & &
\end{tikzcd}
\tag{41}
$$

Here for fixed c_3 we may interpret the compatible M_2-brane charges (c_6, b_2) with the c_3-twisted non-abelian cohomology classified by $S^7 /\!\!/ \mathrm{Sp}(2)$. This is a twisted form of 3-Cohomotopy, because the homotopy fiber of $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ is still the 3-sphere ([GS1, Lem. 2.8]):

$$
\begin{tikzcd}
S^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] \\
* \arrow[r] & S^4 /\!\!/ \mathrm{Sp}(2)
\end{tikzcd}
\tag{42}
$$

As such we have the corresponding character differential forms for pure M_2-brane charge, which pick up a 3-form flux H_3 ([HpH1, Prop. 3.20], cf. (63) below):

implication of **Hypothesis H** on M_2-brane charge:

pure M_2-*brane charges in given background* $\mathrm{M}_2/\mathrm{M}_5$-*charge* c_3 *are quantized in* c_3-*twisted 3-cohomotopy*, hence (C_6, B_2)-field fluxes are in the twisted 3-cohomotopical character

$$
\pi^{3+c_3}(X) \xrightarrow{\mathrm{ch}} H^{c_3}_{\mathrm{dR}}\Big(X; \mathfrak{l}\big(S^7 /\!\!/ \mathrm{Sp}(2)\big)\Big) = \left\{ \begin{array}{l|l} G_4 \in \Omega^4_{\mathrm{dR}}(X) & \mathrm{d}\, G_4 = 0, \\ G_7 \in \Omega^7_{\mathrm{dR}}(X) & \mathrm{d}\, G_7 = -\tfrac{1}{2}\widetilde{G}_4 \wedge \widetilde{G}_4 - \tfrac{1}{2}p_1 - 12 \cdot I_8 \\ H_3 \in \Omega^3_{\mathrm{dR}}(X) & \mathrm{d}\, H_3 = \widetilde{G}_4 - \tfrac{1}{2}p_1 \end{array} \right\}_{\big/ \text{concordance}}
\tag{43}
$$

The literature on M_2-brane charge expects (though throughout ignoring the shift by p_1 in $\widetilde{G}_4$ (34)) that given such an H_3-"potential" in 11d supergravity (then typically regarded as the C-field gauge potential and denoted "C_3") the following expression — known as the *Page charge* — is the M_2-brane charge [Pa83, (8)][DS91, (43)][BLMP13, p. 21]:

$$
\underset{\text{in ordinary cohomology}}{\overset{\mathrm{M}_2\text{-brane charge image}}{}} \quad [\widetilde{G}_7] := \big[G_7 + \tfrac{1}{2}H_3 \wedge \widetilde{G}_4\big] \in H^7\big(X; \mathbb{R}\big)\,.
\tag{44}
$$

The same expression gives the "Hopf-Wess-Zumino term" in the M5-brane sigma-model (we come to this in §4), hence the coupling of the fundamental five-brane to the background C-field analogous to the coupling of an electron the electromagnetic field.

What had remained open (and hardly discussed at all) is that, how and why this term is integral: Regarded as M_2-brane charge such an integrality is necessary at least to justify common discussion of M_2-brane counting, while regarded as the Hopf-WZ term for the M5-brane such an integrality is necessary for the M5-brane sigma-model to actually be well-defined (anomaly-free) — by the exact same argument of Dirac charge quantization, up to degree, we expand on this in §4.

Experience with the NS5-brane sigma-model suggests that its anomaly-cancellation requires a topological condition on spacetime that is a higher-degree analog of "String structure" (whence called "Fivebrane structure" in [SSS09]) requiring an degree-8 polynomial in the Pontrjagin forms of spacetime to vanish.

The M-theoretic analog of Fivebrane structure as implied by Hypothesis H is the trivialization of the Euler class, hence of the "one loop term" $\chi_8 = 24 \cdot I_8$ (38), which we may refer to as M_5-*brane structure* [M5b, Exp. 3.2]:

$$
\begin{tikzcd}
 & & B\widehat{\mathrm{Sp}}(2) \arrow[d] \arrow[rr] \arrow[drr, phantom, "\text{(pb)}"] & & * \arrow[d] \\
X \arrow[urr, dashed, bend left, "\text{M5-brane structure}"] \arrow[r, "\vdash \mathrm{Fr}(X)"'] & B\big(\mathrm{Sp}(1,2) \times \mathrm{Sp}(2)\big) \arrow[r] & B\mathrm{Sp}(2) \arrow[r, "\sim"] & B\mathrm{Spin}(5) \arrow[r, "\chi_8"'] & B^8\mathbb{Z}
\end{tikzcd}
\tag{45}
$$

in that this is what, under Hypothesis H, implies the (half-)integrality of the M_2-brane Page charge, hence of the Hopf-WZ terms of the M_5-brane sigma-model:

Theorem 3.1 ([M5a, Thm. 4.8])**.** *Hypothesis H* (39) *implies that, on spacetimes admitting* M_5*-brane structure* (45)*, the resulting* M_2*-brane charge quantization* (43) *makes twice the Page charge/Hopf-WZ term* (44) *an integral cohomology class:*

$$\begin{array}{c}\text{M}_2\text{-brane charge image}\\ \text{in ordinary cohomology}\end{array} \quad \overset{\substack{\text{class of}\\ \text{shifted 7-flux density}}}{\big[2G_7 + H_3 \wedge \widetilde{G}_4\big]} \;\in\; \overset{\text{integral cohomology}}{H^7\big(X;\mathbb{Z}\big)} \to H^7\big(X;\mathbb{R}\big) \tag{46}$$

Discussion and interpretation of the factor of 2 here is given in [M5a, (3)][M5b, p. 3].

For example, the condition of M_5-brane structure is satisfied if the structure group reduces further along $\mathrm{Sp}(1)\times\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2)$ (since the Euler of a direct sum of vector bundles is the cup product of that of the summands, but the Euler 8-class of a single $B\mathrm{Sp}(2)$ vanishes by degree reasons). This special case subsumes the important example of M_5-branes at ADE-singularities, see [M5e, (1)].

Hence if one insists — which is reasonable — that M-brane charge quantization should imply Page charge quantization (46) and thus conistency of the M_5-brane sigma model in charged backgrounds, then one will want to include the demand of M_5-brane $\widehat{\mathrm{Sp}(2)}$-structure (45) into the hypothesis (39):

Hypothesis $\widehat{\mathrm{H}}$ ([M5a]):

M-brane charges are quantized in tangentially $\widehat{\mathrm{Sp}(2)}$*-twisted 4-cohomotopy*, hence
C-field fluxes are in the twisted 4-cohomotopical character

$$\pi^{4+\tau}(X) \xrightarrow{\;\mathrm{ch}^\tau\;} H^\tau\big(X;\,\mathfrak{l}S^4\big) \;=\; \left\{ \begin{array}{l} G_4 \in \Omega^4_{\mathrm{dR}}(X) \\ G_7 \in \Omega^7_{\mathrm{dR}}(X)) \end{array} \;\middle|\; \begin{array}{l} \mathrm{d}\,G_7 = -\tfrac{1}{2}\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{2}p_1) \\ \mathrm{d}\,G_4 = 0 \end{array} \right\}_{\big/\text{concordance}} \tag{47}$$

$$\begin{array}{c}\text{tangentially}\\ \widehat{\mathrm{Sp}(2)}\text{twisted}\\ \text{4-coHomotopy}\end{array} \quad \pi^{4+\tau}(X) := \left\{ \begin{tikzcd} \overset{\text{spacetime}}{X} \arrow[r, dashed, "\overset{\text{cocycle}}{c_3}"] \arrow[dr, "\tau"', "\text{twist}"] \arrow[dd, "\vdash \mathrm{Fr}(X)", "\text{tangent structure}"'] & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\substack{\text{universal orthogonal}\\ \text{4-sphere bundle}}}{S^4 /\!\!/ \mathrm{Spin}(5)} \arrow[d] \\ & \underset{\text{M}_5\text{-brane structure}}{B\widehat{\mathrm{Sp}(2)}} \arrow[r] & B\mathrm{Sp}(2) \arrow[r, "\sim"'] \arrow[d] & B\mathrm{Spin}(5) \arrow[d] \\ \begin{array}{c} B\mathrm{Spin}(1,2) \\ \times B\mathrm{Spin}(8) \end{array} & & B\mathrm{Spin}(8) \arrow[ll] \arrow[r, "B\mathrm{tri}", "\sim"'] & B\mathrm{Spin}(8) \end{tikzcd} \right\}_{\big/\text{rel. homotopy}} \tag{48}$$

Hypothesis H for heterotic M-theory. Finally, Hypothesis H generalizes to (and maybe comes into full bloom) in "heterotic M-theory" (Hořava-Witten theory), where Cohomotopy is enhanced to "twistorial Cohomotopy", now represented by the "twistor space" $\mathbb{C}P^3$ covering the 4-sphere through the Calabi-Penrose fibration.
This is discussed in [GS1][GS2].

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{C}}", "\substack{\text{7d complex}\\ \text{Hopf fibration}}"'] \arrow[dd, bend right, "h_{\mathbb{H}}"', "\substack{\text{quaternionic}\\ \text{Hopf fibration}}"] & \simeq & \mathrm{Sp}(2)/\,\mathrm{Sp}(1)_L \\ \mathbb{C}P^3 \arrow[d, "t_{\mathbb{H}}", "\substack{\text{Calabi-Penrose}\\ \text{twistor fibration}}"'] & \simeq & \mathrm{Sp}(2)/\big(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R\big) \\ S^4 & \simeq & \mathrm{Sp}(2)/\big(\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R\big) \end{tikzcd}$$

$$\underset{\substack{\text{manifold}\\ \text{with}\\ \text{tangential Sp(2)-structure } \tau}}{\overset{\substack{\text{Twistorial}\\ \text{Cohomotopy}}}{\mathcal{T}^\tau(X)}} \xrightarrow[\;\mathrm{ch}\;]{\substack{\text{Twisted Non-abelian}\\ \text{character map}}} \left\{ \begin{array}{l} F_2, \\ G_4, \in \Omega^\bullet(X) \\ G_7 \end{array} \;\middle|\; \begin{array}{l} \overset{\substack{\text{1st Chern form of}\\ \text{heterotic line bundle}}}{d\;\; F_2 = 0\,,} \quad \overset{\substack{\text{2nd Chern class of corresponding}\\ S(\mathrm{U}(1)^2)\text{-gauge field } \widehat{A}}}{-[F_2 \wedge F_2] \;\in\; H^4\big(X;\mathbb{Z}\big)} \\ \overset{\text{C-field 4-flux}}{d\;\; G_4 = 0\,,} \;\; \overset{\text{Hořava-Witten's Green-Schwarz mechanism (3)}}{[G_4] - [\tfrac{1}{4}p_1(\omega)] \;=\; [F_2 \wedge F_2] \;\in\; H^4\big(X;\mathbb{Z}\big)} \\ d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) \\ \overset{\substack{\text{dual}\\ \text{7-flux}}}{} \qquad -\tfrac{1}{2}\big(p_2(\omega) - \tfrac{1}{4}\big(p_1(\omega)\big)^2\big) \end{array} \right\}$$

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4 Resulting M5-brane model

After a brief recollection of the meaning of *fundamental sigma-model branes* in §4.1, we survey in §4.2 how *Hypothesis H* implies global consistency of the (Hopf-)Wess-Zumino gauge coupling of the fundamental 5-brane sigma model by implying flux quantization of the worldvolume B-field in twisted 3-Cohomotopy underlying which is a "nonabelian gerbe field" for worldvolume gauge group $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ — this result is from [HpH1, §3.7][M5a][M5b].[14]

4.1 Fundamental sigma-model branes

Besides the singular/solitonic classical branes of §1 there are supposed to be "fundamental" or "sigma-model"-branes which are not imprinted on flux, but which are *effected by* flux. Here

fundamental brane : singular brane

is like

fundamental particle : black hole

in that fundamental branes are supposed to be "massless" cousins of black branes, which have analogous attributes but instead of impacting spacetime by their backreaction on it, they trace out trajectories $\phi : \Sigma^{1+p} \to X$ in a fixed background spacetime X subject to forces exerted by spacetime fields. These forces include the force of gravity and the generalized *Lorentz force* exerted by the background gauge field. In the spirit of pre-geometric fluxes as discussed in §1.2 here we focus on these Lorentz forces.

In fact, fundamental branes include, with the *fundamental particles* that they derive their name from, the most prominent brane species: notably the *fundamental string* which gives its name to *string theory* and the *fundamental membrane* from which the term *M-theory* is derived:

	flux densities on spacetime **black branes** singular source of flux	σ-model with target spacetime **fundamental brane** subject to forces from such background flux
electromagnetism	magnetic monopole	fundamental particle (electron)
string theory	NS_5-brane	fundamental string
M-theory	M_5-brane	fundamental membrane
	M_2-brane	fundamental fivebrane

(49)

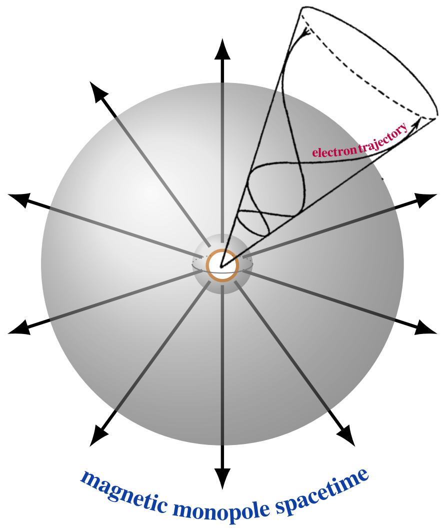
The fundamental 0-branes in electromagentism are simply the electrons – these being *fundamental particles* in the sense of particle physics, whence the general term "fundamental brane".

The following graphics shows[15] the generic trajectory of an electron in the vicinity of a magnetic monopole: The Lorentz force felt by the electrically charged electron when moving in a background magnetic field deforms the otherwise straight trajectory into a helix whose radius of curvature is the smaller the stronger the magentic flux density. In the case of the magnetic field sourced by a monopole (cf. p. 6) this makes the electron trajectories lie on a cone in space whose vertex is the

[14] Hypothesis H also implies information about the *dynamical* (i.e. geometric, non-topological) sector of the $\mathrm{Sp}(1)$-gauged M5-brane sigma-model: this is discussed in [M5d][M5e].

[15] This is discussed for instance in Ferraro (1956) *Electromagnetism Theory*, §137. The helical trajectory in (50) is adapted from Ferraro's Fig. 161.

singular locus of the monopole:



$$
\begin{tikzcd}
\textbf{worldline} & \textbf{trajectory} & \textbf{spacetime} \\
\Sigma^{1+0} \arrow[rr, "\phi"] \arrow[d, equal] & & X \arrow[d, equal] \\
\mathbb{R}^{1,0} \arrow[rr] & & \mathbb{R}^{1,3} \setminus \mathbb{R}^{1,0} \simeq \mathbb{R}^{1,0} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^2
\end{tikzcd}
\tag{50}
$$

This means in particular that there exist circular electron trajectories which lie entirely in a plane in space and are periodic in time, winding around one of the radial flux lines. Since the Faraday tensor F_2 sourced by a magnetic monopole has no temporal (electric field) component, we may consistently ignore the temporal translation of these trajectories and understand them as maps from a circular "worldline" into space:

$$
\begin{tikzcd}
\textbf{worldline} & \textbf{trajectory} & \textbf{space} & \textbf{gauge potential} & \\
S^1 \arrow[rr, "\phi"] & & \mathbb{R}_{\sqcup\{\infty\}} \times S^2 \arrow[rr, "\widehat{A}_1"] \arrow[dr, "F_2"'] & & \widehat{B^2\mathbb{Z}} \arrow[dl] \\
& & & \Omega^2_{\mathrm{cl}}(-) &
\end{tikzcd}
\qquad
\begin{array}{c}
\textbf{gauge coupling action functional} \\
\rightsquigarrow \textbf{Lorentz force} \\
\exp\Big(\underbrace{2\pi i \int_{S^1} \phi^* \widehat{A}_1}_{\text{holonomy}}\Big) \in \mathbb{R}/\mathbb{Z}
\end{array}
\tag{51}
$$

The gauge-coupling part of the "exponentiated action" (in the sense of classical Lagranian physics) of such a trajectory is the holonomy of the gauge potential 1-form $\widehat{A}_1$ (§2.5) around this closed curve, in that the variation of this functional gives the contribution to the Euler-Lagrange equations of motion of the electron which expresses the Lorentz force.[16]

Recall here that it is the definition of the differential cohomology coefficients $\widehat{B^2\mathbb{Z}}$ (§2.5) which implements the Dirac charge quantization condition and makes the action functional take values in $\mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1)$:

$$
\begin{tikzcd}
& & & \Omega^2_{\mathrm{dR}}(-)_{\mathrm{flat}} \arrow[dr] \arrow[dd, Rightarrow, "\text{(pb)}"'] & \\
\underset{\Sigma^{1+0}}{\textbf{electron worldline}} \arrow[r, "\phi"] & X \arrow[urr, bend left, "\underset{F_2}{\textbf{background flux density}}"] \arrow[r, dashed, "\widehat{A}_1"', "\text{``vector potential'' gauge field}"'] \arrow[drr, bend right, "\textbf{background magnetic charge}"'] & \widehat{B^2\mathbb{Z}} \arrow[ur] \arrow[dr] & & B^2\mathbb{R} \\
& & & B^2\mathbb{Z} \arrow[ur, "\mathbf{ch}"'] &
\end{tikzcd}
\tag{52}
$$

Fundamental p-brane sigma-models. In straightforward generalization of the above situation for fundamental particles, one considers maps into space(-time) from $p+1$-dimensional manifolds Σ^{1+p} — which we may assume to be closed, for simplicity and following (51) — regarded as *worldvolumes* of fundamental p-branes. These may have higher Lorentz-force

[16] See for instance Misner, Thorne & Wheeler (1973) *Gravitation*, Exc 7.2 on p. 179, or Frankel (1997) *The Geometry of Physics*, §16.4b.

couplings to higher background gauge fields $\widehat{A}_{p+1}$ represented by cocycles in differential cohomology of degree $p+2$:

$$
\begin{tikzcd}
\overset{\text{p-brane worldvolume}}{\Sigma^{1+p}} \arrow[r, "\phi"] \arrow[drr, bend right, "\phi^*\widehat{A}_{p+1}"'] & \overset{\text{target spacetime}}{X} \arrow[r, "\widehat{A}_{p+1}"] & \widehat{\mathbf{B}^{p+2}\mathbb{Z}} \\
 & & \underset{\substack{\text{moduli stack of (flat)}\\ \text{differential cohomology}}}{\mathbf{B}^{p+1}\mathbb{R}/\mathbb{Z}} \arrow[u]
\end{tikzcd}
\tag{53}
$$

p-brane worldvolume **embedding field** **target spacetime** **higher gauge potential**

$$
\begin{array}{ccccc}
C^\infty\big(\Sigma^{1+p}, X\big) & \longrightarrow & H^{p+1}\big(\Sigma^{p+1};\mathbb{R}/\mathbb{Z}\big) & \overset{\text{exponentiated action functional}}{=\!=\!=\!=\!=} & \mathbb{R}/\mathbb{Z} \\
\phi & \longmapsto & \big[\phi^*\widehat{A}_{p+1}\big] & =: & \exp\Big(2\pi i \underbrace{\int_{\Sigma^{p+1}} \phi^*\widehat{A}_{p+1}}_{\substack{\text{gauge coupling/}\\ \text{Lorentz force/}\\ \text{Wess-Zumino term}}}\Big)
\end{array}
\tag{54}
$$

This may be understood as defining (the gauge-coupling topological sector of) a field theory on Σ^{1+p} whose:

- Fields are the brane trajectories, namely the smooth maps $\phi : \Sigma^{1+p} \to X$ – then often called "embedding fields", though not not actually required to constitute an embedding $\Sigma^{1+p} \hookrightarrow X$.
- Action functional is the higher holonomy functional (54).

Such field theories – whose fields are maps to a given *target space* X this way – are known as **non-abelian sigma-models**, for historical reasons. For the full geometric dynamics of fundamental branes one is to add another contribution (the "Nambu-Goto action") to the action functional, which we disregard here (in the pre-geometric spirit of §1.2), so that the "gauge coupling sector" of fundamental p-branes which we retain may be understood as a *worldvolume topological field theory*, here a *topological sigma-model* also called a *homotopical field theory*, see [MW20] for detailed discussion (at the classical non-quantum level) in the case at hand.

Fundamental membrane sigma-model. For example, the sigma-model for the fundamental membrane propagating along a trajectory $\phi : \Sigma^{1+2} \to U \hookrightarrow X$ inside a chart U of an 11d supergravity target spacetime X is meant [BST87][HS05, §4.4] to couple to the background C-field flux $G_4|_U$ via an (exponentiated) action functional that is *locally* of the form $\phi \mapsto \int_{\Sigma^{1+2}} \phi^* C_3$, where $C_3 \in \Omega^3_{\mathrm{dR}}(U)$ is a local gauge potential for the C-field, in that $\mathrm{d}C_3 = G_4|_U$.

It is rarely (if ever) discussed in the string theory literature that the global definition (54) of this coupling term requires an integral charge quantization of G_4; but the expected shifted integrality condition $[G_4 + \tfrac{1}{4}p_1(\nabla)] \in H^4(X;\mathbb{Z})$ (34) on the C-field flux — which is a consequence of Hypothesis H by (35) — serves this purpose if one enhances the local conditions to

$$
\mathrm{d}C_3 = G_4|_U + \tfrac{1}{4}p_1(\nabla|_U)\,. \tag{55}
$$

Globally such C_3 is to be the 3-form connection $\widehat{C}_3$ on a 2-gerbe with characteristic class $[G_4 + \tfrac{1}{4}p_1(\nabla)]$ and makes the fundamental membrane sigma-model be well-defined, via (54):

$$
\begin{tikzcd}
 & & & \Omega^4_{\mathrm{dR}}(-)_{\mathrm{flat}} \arrow[dr] \arrow[dd, Rightarrow, "\text{(pb)}"'] & \\
\underset{}{\overset{\substack{\text{membrane}\\ \text{worldvolume}}}{\Sigma^{1+2}}} \arrow[r, "\phi"] & X \arrow[urr, bend left, "\substack{\text{background flux density}\\ G_4+\frac{1}{4}p_1(\nabla)}"] \arrow[r, dashed, "\widehat{C}_3", "\text{C-field gauge potential}"'] \arrow[drr, bend right, "\substack{[G_4+\frac{1}{4}p_1]\\ \text{background M5-brane charge}}"'] & \widehat{B^4\mathbb{Z}} \arrow[ur] \arrow[dr] & & B^4\mathbb{R} \\
 & & & B^4\mathbb{Z} \arrow[ur, "\mathbf{ch}"'] &
\end{tikzcd}
\tag{56}
$$

The analogous situation for the fundamental fivebrane is richer and more subtle, this we turn to in §4.2.

4.2 The fundamental fivebrane

The fundamental fivebrane sigma-model in the literature. Apart from its dimensionality, the sigma-model for the fundamental fivebrane on 11d supergravity target spacetimes [HS97][HSW97][PST97][BLNPST97] is crucially different from the previous examples (§4.1) in that, besides the "embedding field" $\phi : \Sigma^{1+5} \to X$ (53), there is supposed to be a higher gauge field propagating on its worldvolume – the B-field – with a 3-form flux density $H_3 \in \Omega^3_{\mathrm{dR}}(\Sigma^{1+5})$ which:

1. is sourced by the restriction $\phi^* G_4$ of the C-field flux on X to the 5-brane worldvolume: $\mathrm{d}H_3 = \phi^* G_4$ ([HS97, (36)])
2. is subject to a notoriously subtle self-duality constraint (*not quite* $H_3 = \star H_3$, cf. [HS97, below (41)])

$$
\boxed{\begin{array}{c}\text{B-field flux on}\\ \text{fundamental 5-brane}\\ \phi : \Sigma^{1+5} \to X\end{array}} \longleftarrow \left\{ \begin{array}{l} \boxed{\mathrm{d}H_3 \;=\; \phi^* G_4} \\[10pt] \boxed{\text{subtle self-duality}} \end{array} \right.
\tag{57}
$$

3. enters the Wess-Zumino (WZ) term (54) for the gauge-coupling to the background M2-brane flux G_7, deforming it to the *Hopf-WZ term* [Ah96, p. 10][BLNPST97, (1)][PST97, (17)][In00, (2.4)] which for trajectories $\phi : \Sigma^{1+5} \to U \hookrightarrow X$ inside a chart U of X is meant to be of this form (cf. [M5a, §2]):

$$
(\phi, H_3) \;\mapsto\; \int_{\Sigma^{1+5}} \big(C_6 - \tfrac{1}{2} H_3 \wedge C_3\big)\,, \qquad \text{where} \quad \begin{array}{ll} C_3 \in \Omega^3_{\mathrm{dR}}(U), & \mathrm{d}C_3 \;=\; G_4\vert_U \\ C_6 \in \Omega^6_{\mathrm{dR}}(U), & \mathrm{d}C_6 \;=\; G_7\vert_U + \tfrac{1}{2} C_3 \wedge G_4\vert_U \end{array}\,.
\tag{58}
$$

An enormous (and ongoing) effort – motivated by arguments going back to [Wi02][Wi10] – has been devoted to understanding the self-duality constraint in (57), while the global understanding of the Bianchi identity $\mathrm{d}H_3 = \phi^* G_4$ and its role in the fivebrane's peculiar gauge coupling term (59) has received little to no attention in the community. In the spirit of the pre-geometric perspective §1.2 we proceed here contrariwise:

Since it is likely premature to discuss the geometric self-duality constraint on the H_3-flux before its flux quantization law has been identified, we discuss the latter – deriving it as a consequence of Hypothesis H, proving that it implies the necessary "level quantization" of the Hopf-WZ term (from [M5a]) and anlyzing the "non-abelian gerbe"-field on the 5-brane worldvolume which makes this happen.

Level-quantization of the 5brane's Hopf-WZ term. Assuming for a moment that the H_3-flux is defined not just on Σ^{1+5} but on all of X as in (43) or at least on a 7-dimensional "extended worldvolume" [M5a, (5)] $\widehat{\Sigma}^{1+6} \to X$ we may observe that on flat spacetimes the Hopf-WZ term (58) is a local potential for the M_2-brane Page charge (44)

$$
\mathrm{d}\big(C_6 - \tfrac{1}{2} H_3 \wedge C_3\big) \;=\; \big(G_7 + \tfrac{1}{2} H_3 \wedge G_4\big)\vert_U\,,
$$

and on curved spacetimes we may adapt it to include the necessary shifting (which the literature ignores) by $p_1(\nabla)$ (34) along the lines of (55), see [M5a, (11,16)] for details.

Therefore Hypothesis $\widehat{\mathrm{H}}$ (47) implies, via Thm. 3.1, that (twice) the Hopf-WZ term for *pure* M_2-brane background charge (41) on $\widehat{\Sigma}^{1+6}$ is properly level-quatized and hence indeed a globally consistent gauge coupling — this is the main result of [M5a]:

$$
\begin{tikzcd}
 & & & \Omega^7_{\mathrm{dR}}(-)_{\mathrm{flat}} \arrow[dr] \arrow[dd, Rightarrow, "\text{(pb)}"'] & \\
\underset{\Sigma^{1+5}}{\overset{\text{fivebrane worldvolume}}{}} \arrow[r, "\phi"] & \widehat{\Sigma}^{1+6} \arrow[urr, bend left, "\text{background flux density}\; H_3 \wedge \tilde{G}_4 + 2G_7"] \arrow[r, dashed, "\text{Hopf WZ-term}"'] \arrow[drr, bend right, "\text{background M2-brane charge (Page charge)}"'] & \widehat{B^7\mathbb{Z}} \arrow[ur] \arrow[dr] & & B^7\mathbb{R} \\
 & & & B^7\mathbb{Z} \arrow[ur, "\mathbf{ch}"'] &
\end{tikzcd}
\tag{59}
$$

Charge quantization and non-abelian gerbe field on fivebrane worldvolume. We analyze in more detail what it is that makes the fivebrane worldvolume "anomaly cancellation" (59) work, in terms of peculiar worldvolume field content that is implied by Hypothesis H – this is the main result from [M5b].

We left off in §3.2 with observing that "pure" M_2-brane charge – "Page charge" (44) – is reflected, via Hypothesis H, in factorizations (41) of the full cohomotopical M-brane charge through the (M5-brane structured) quaternionic Hopf fibration $h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}$. That such factorizations imply the existence of a 3-flux H_3 (43) which trivializes the background M_5-charge (relative to the pertinent shift of the vacuum by $\frac{1}{2}p_1$) is part of what it means for the M_2-brane charge to be "pure" (no M_5-brane charge admixtures) but it also means that the existence of such lifts on all of spacetime are strongly constrained.

However, to make sense of the M_5-brane sigma-model coupled to the Page charge, we only need such lifts to exists *on the worldvolume* Σ^{1+5} of the M_5, hence after pulling back the C-field along an embedding field $\phi : \Sigma^{1+5} \to X$ — and there the side-effect of trivializing the (shifted) G_4-flux by a 3-flux now makes perfect sense and identifies the 3-flux H_3 with the worldvolume 3-flux H_3 expected on the M5, which is source by the 1-branes inside M_5-worldvolumes known as "self-dual strings" or "M-strings":

$$
\begin{tikzcd}
\substack{\textbf{M5-brane}\\ \textbf{worldvolume}} & \Sigma^{1+5} \arrow[rr, dashed, "\textbf{worldvolume field}"'] \arrow[d, dashed, "\phi", "\textbf{embedding field}"'] \arrow[rrr, bend left, "\textbf{Hopf WZ term}"] \arrow[drr, phantom, "\overset{b_2}{\Swarrow}"] & & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r, "H_3 \wedge \widetilde{G}_4 + 2G_7", "\mathrm{M}_2\textbf{-brane Page charge}"'] \arrow[d, "h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}"', "\substack{\textbf{M5-brane structured}\\ \textbf{quaternionic Hopf fibration}}"] & B^7\mathbb{Z} \\
\textbf{spacetime} & X \arrow[rr, "c_3", "\textbf{bulk C-field}"'] \arrow[dr, "\tau", "\textbf{M5-brane structure}"'] & & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[dl] & \\
& & B\widehat{\mathrm{Sp}(2)} & &
\end{tikzcd}
\tag{60}
$$

Such "homotopy cones" (as indicated by the dashed arrows) are equivalently maps into the corresponding *homotopy pullback* (of the M5-structured quaternionic Hopf fibration along the cohomotopy cocycle c_3 for the C-field), which we denote $\widehat{X}_{c_3}$ and think of as the **C-field extended spacetime** [HpH1, Def. 3.16] [M5d, Rem. 3.9][M5b, p. 7] :

$$
\begin{tikzcd}
\substack{\textbf{M5-brane}\\ \textbf{worldvolume}}\ \Sigma^{5+1} \arrow[dr, dashed, "{(\phi, b_2)}", "\textbf{M5 }\sigma\textbf{-model field}"'] \arrow[ddr, bend right, "\phi", "\textbf{embedding field}"'] \arrow[drrr, bend left, "\textbf{Hopf WZ term}"] & & & \\
& \substack{\textbf{extended}\\ \textbf{spacetime}}\ \widehat{X}_{c_3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{(\mathrm{pb})}{\Swarrow}"] & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r, "H_3 \wedge \widetilde{G}_4 + 2G_7", "\mathrm{M}_2\textbf{-brane Page charge}"'] \arrow[d, "h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}"', "\substack{\textbf{M5-brane structured}\\ \textbf{quaternionic Hopf fibration}}"] & B^7\mathbb{Z} \\
& \textbf{spacetime}\ X \arrow[r, "c_3", "\textbf{bulk C-field}"'] \arrow[dr, "\tau", "\textbf{M5-brane structure}"'] & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[d] & \\
& & B\widehat{\mathrm{Sp}(2)} &
\end{tikzcd}
\tag{61}
$$

This means that for given background C-field c_3 on a spacetime X, the extended spacetime $\widehat{X}_{c_3}$ is the correct "target space" for (the topological sector of) the M5-brane sigma-model, unifying the actual target spacetime X with a classifying space for the worldvolume B-field on the M5-brane.

B-field flux quantization of M_5-worldvolumes In fact, from (42) and pasting law, it follows that the extended spacetime $\widehat{X}_{c_3}$ is a 3-sphere fibration over spacetime:

$$
\begin{tikzcd}
\overset{\substack{\text{3-sphere}\\ \text{fiber}}}{S^3_x} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \overset{\substack{\text{extended}\\ \text{spacetime}}}{\widehat{X}_{c_3}} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d] \\
\underset{\text{any point}}{\{x\}} \arrow[r, hook] & \underset{\text{spacetime}}{X} \arrow[r, dashed, "c_3"] & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[d] \\
& & B\widehat{\mathrm{Sp}(2)} \arrow[r] & B\mathrm{Sp}(2)
\end{tikzcd}
\tag{62}
$$

Since it is this 3-sphere fiber which, locally, classifies the H_3-flux, we find, in mild that variation of (43) implies that the B-field on fivebrane worldvolumes is flux-quantized in a form of twisted 3-cohomotopy, :

implication of **Hypothesis** $\widehat{\mathrm{H}}$ on M5-worldvolumes ([M5b, p. 7]): *M-string charges in* M_5 *worldvolumes are quantized in* $\phi^*(c_3)$*-twisted 3-cohomotopy*, hence B-field fluxes on M_5 worldvolumes are in a twisted 3-cohomotopical character

(63)

$$\underset{\text{3-Cohomotopy}}{\text{bckgr. C-field-twisted}} \quad \pi^{3+\phi^*(c_3)}\big(\Sigma^{1+5}\big) \;:=\; \left\{ \begin{tikzcd} \Sigma^{1+5} \arrow[r, dashed, "b_2", "\text{worldvolume B-field}"'] \arrow[dr, "\phi"', "\text{embedding field}"] & \widehat{X}_{c_3} \arrow[r] \arrow[d] & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] \arrow[d, "h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}"] \arrow[dl, phantom, "(\mathrm{pb})"] & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] \arrow[dl, phantom, "(\mathrm{pb})"] \\ & X \arrow[r, dashed, "c_3", "\text{background C-field}"'] & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[r] & S^4 /\!\!/ \mathrm{Sp}(2) \end{tikzcd} \right\}_{/\text{rel. homotopy}} \tag{64}$$

We re-iterate that this twisted 3-cohomotopical flux-quantization of the worldvolume B-field implies that the fivebrane's Hopf-WZ gauge coupling term is globally well defined – a key requirement for consistency of the M_5-brane sigam model whose solution had been a wide open problem. All the more is the following consequence remarkable:

Emergence of a non-abelian higher gauge field on the M_5-worldvolume. Remarkably, this particular form of twisted 3-cohomotopy (64) also has an equivalent *gauge-theoretic* interpretation, due to the coset space realization (36) of the quaternionic Hopf fibration – which implies that its tangentially twisted version is equivalently a map of classifying spaces of $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$-gauge fields:

$$\begin{tikzcd} S^7 /\!\!/ \mathrm{Sp}(2) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"'] & B\mathrm{Sp}(1)_L \times \ast \arrow[r, leftrightarrow, "\sim"] \arrow[d, shift right=6, equal] \arrow[d, shift left=6] & B\mathrm{Sp}(1)_L \arrow[d] \\ S^4 /\!\!/ \mathrm{Sp}(2) \arrow[r, leftrightarrow, "\sim"] & B\mathrm{Sp}(1)_L \times B\mathrm{Sp}(1)_R \arrow[r, leftrightarrow, "\sim"] & B\mathrm{Spin}(4) \end{tikzcd}$$

$$\begin{array}{ccccl} H^\bullet\big(B\mathrm{Sp}(1);\mathbb{Z}\big) \oplus H^\bullet\big(B\mathrm{Sp}(1);\mathbb{Z}\big) & \overset{\sim}{\longleftrightarrow} & H^\bullet\big(B\mathrm{Spin}(4);\mathbb{Z}\big) & & \\ c_2^L \;+\; c_2^R & = & \tfrac{1}{2}p_1 & & \textbf{first fractional Pontrjagin class/ C-field background charge} \\ c_2^L & = & \tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1 = \widetilde{G}_4 & & \textbf{shifted integral C-field charge} \\ -c_2^R & = & \widetilde{G}_4 - \tfrac{1}{2}p_1 & & \textbf{C-field charge relative to background charge} \end{array}$$

The decomposition of the cohomology generators as shown in the last line (using [CV98a, Lem 2.1], see [HpH1, Lem. 3.9]) shows that the pullback of the fractional Pontrjagin class along the parameterized quaternionic Hopf fibration equals the Chern class on the "gauge factor" $B\mathrm{Sp}(1)_L$:

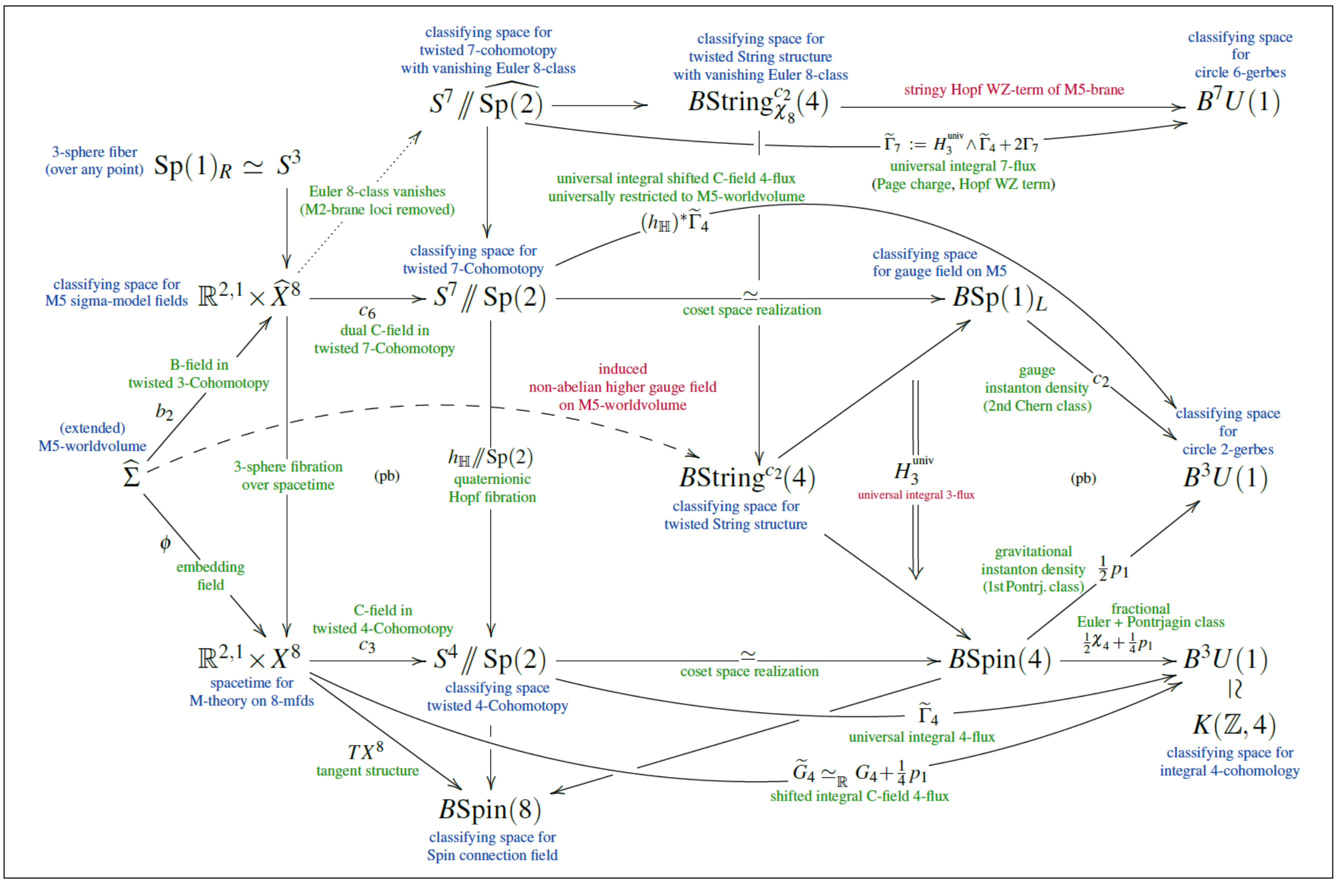
$$\big(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big)^* \tfrac{1}{2}p_2 \;=\; c_2 \qquad\Leftrightarrow\qquad \begin{tikzcd} S^7 /\!\!/ \mathrm{Sp}(2) \arrow[r, "\sim"] \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"'] & B\mathrm{Sp}(1)_L \arrow[dr, "c_2"] \arrow[d, phantom, "\Swarrow\sim"] & \\ S^4 /\!\!/ \mathrm{Sp}(2) \arrow[r, "\sim"'] & B\mathrm{Spin}(4) \arrow[r, "\frac{1}{2}p_1"'] & B^4\mathbb{Z} \end{tikzcd}$$

But this means that the worldvolume B-field on the fundamenal 5-brane according to (64) may be regarded as having an underlying $\mathrm{Sp}(1)$-gauge field a_1 equipped with a "Green-Schwarz term" H_3 that idenfies the gauge-fields Chern class (instanton density) with the first fractional Potrjagin class (pulled back to the worldvolume), hence as having an underlying

$\mathrm{String}^{c_2}(4)$-valued higher gauge field:

$$
\begin{tikzcd}
\Sigma^{1+5} \arrow[r, "b_2"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swArrow"] & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] \\
X \arrow[r, "c_3"'] & S^4 /\!\!/ \mathrm{Sp}(2)
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\Sigma^{1+5} \arrow[rr, "a_1"] \arrow[dr, dashed] \arrow[dd, "\phi"'] & & B\mathrm{Sp}(1)_L \arrow[dr, "c_2"] & \\
& B\mathrm{String}^{c_2}(4) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\swArrow H_3"] & & B^3\mathrm{U}(1) \\
X \arrow[rr, "\vdash \mathrm{Fr}(X)"'] & & B\mathrm{Spin}(4) \arrow[ur, "\frac{1}{2}p_1"'] &
\end{tikzcd}
$$

Speculation that such a "non-abelian gerbe field" might emerge on M_5-branes originates with [Wi02, p. 6, 15] and the particular possibility of $\mathrm{String}(G)$-fields for $G = \mathrm{SU}(2)$ was explored in [SäSc18] but had remained guesswork. Here the expected kind of structure drops out as a consequence of Hypothesis H, complete with its subtle charge-quantization law. For more discussion, including more pointers to related literature, see [M5b][GS2].



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5 Brane lightcone quantization

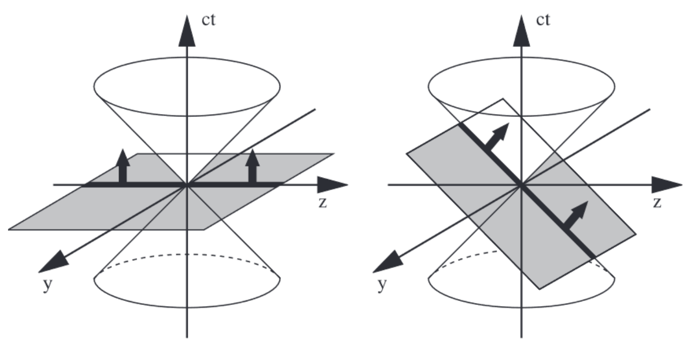
We explain here how every charge-quantization of branes (§2) induces a notion of lightcone quantum system of pregeometric brane charges (§1.2) on spacetimes with a circle factor (66), due to the fact that in this case the pregeometric phase space (70) is a loop space, so that homological brane observables become a star-algebra under the Pontrjagin product (with star-involution given by lightcone time inversion), making them quantum observables in the sense of algebraic quantum theory.

This approach to brane quantum systems is due to [Qnt1][Qnt2] where it is applied to the case of M_5-brane intersections under Hypothesis H (§3), to which we come in §6.

Non-perturbative light-cone quantization. The solution to the problem of non-perturbative quantization of a relativistic Lagrangian field theory appears in principle straightforward: Choose a foliation of spacetime by non-timelike hypersurfaces and then consider the Hamiltonian dynamics of evolution along the leaves.

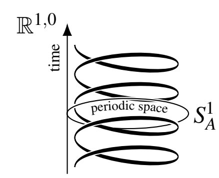
It is for technical and computational problems encountered with carrying this out for the naïve choice of spacelike foliations that the Hamiltonian approach to relativistic QFT was largely abandoned, long ago, in favor of Schwinger-Tomonoga-Feynman-Dyson perturbation theory, which is now often but erroneously regarded as synonymous with "quantum field theory".

However, one may also consider foliation by *lightlike* hypersurfaces (light wave fronts, [Dirac1949, §5]), and the resulting *light-cone quantization* turns out to be mathematically natural and more tractable, especially in application to hadronic bound states in strongly coupled QCD (eg. [BMPP93][Zh94][BPP98][Ba$^+$13]).



spacelike foliation lightlike foliation

The greatest practical progress with non-perturbative computations in QCD has been made by additionally assuming that spacetime is periodic along one light-like direction so that the light-cone energy is quantized, whence one speaks of **discretized light-cone quantization** [PB85][Pa99].
This may be understood [Sei97] as the physics seen by a lightlike observer travelling along a periodic spatial dimension.



(65)

But in itself, while computationally succesful, the fact that a spacelike periodicity is required and singled out here is puzzling from the point of view of physics in 1+3-dimensional.

M-Theory as $\mathbb{R}^{1,0}\times S^1_A$ light-cone quantum mechanics. However, exactly such a circle-factor S^1_A in spacetime is meant to appear in strongly coupled type IIA string theory in the guise of M-theory (cf. p. 3), where the radius of S^1_A scales with the string coupling seen in 10d [DHIS87][To95][Wi95, §2.3] (review in [Du96, §2(ii)][OP99, §2.1]).

Indeed, one early proposal for making sense of M-theory is (see [NH98, §10]) to regard it as the *lightcone quantum mechanics* of the fundamental membrane (49) propagating on a spacetime of the form (65)

$$X^{1,d} \;=\; \mathbb{R}^{1,0}\times S^1_A \times X^{d-1} \tag{66}$$

with lightcone momentum along the circle S^1_A, which in the small radius limit is thought to reduce to the D_0-brane dynamics described by the *BFSS matrix model* [BFSS97][Su97][Sei97] or rather the *BMN matrix model* [BMN02, §5] (review in [Yd18]).

A key consistency check of these M-theory matrix models have been computations recovering 11d supergravity in the form of graviton scattering amplitudes [BBPT97][HPSW99] — but brane charge quantization such as in K-theory (§3.1) is not reflected in these models (cf. [AST02]). Contrariwise, we now explain a lightcone quantization of pre-geometric brane charges in general cohomology theories.

In order to appreciate the concept of *higher quantum observabes* that we are about to consider, it may be useful to recall the following standard conceptualization of algebraic (quantum) mechanics, schematically:

The fundamental concepts of quantum physics in "algebraic" form are the following (good exposition in [Gl09][Gl11], for more see [La17]):

- The **covariant phase space** of a physical system is really the space of solutions to the classical equations of motion [Wi86, §5][CW87][HT92, §17.1], hence the space of physically possible ("on shell") *field histories* of the system.

$$\mathrm{PhsSpc} \;=\; \left\{ \begin{array}{c} \text{solutions to equations of motion} \\ \text{hence: possible field histories} \end{array} \right\}.$$

- The **classical observables** on a physical system are encoded in *compactly supported*[17] complex-valued functions on field histories, understood as assigning to a field history the value that the observable takes there.
  In the simplistic but relevant special case where the phase space is just a discrete set (cf. [La17, §1.2]), this means that the space of observables is the linear span of formal linear combinations of field histories:

$$\mathrm{PhsSpc} \in \mathrm{Sets} \qquad \Rightarrow \qquad \mathrm{Obsrvbls} \;=\; \mathbb{C}[\mathrm{PhsSpc}] \tag{67}$$

- The **quantum observables** are a *choice* of the structure of a (non-commutative) complex star-algebra[18] on the Obsrvbls:[19]

$$\mathrm{QObsrvbls} \;=\; \Big(\mathrm{Obsrvbls},\, (\text{-})\cdot(\text{-}),\, (\text{-})^*\Big), \qquad \big(\mathscr{O}_1 \cdot \mathscr{O}_2\big)^* \;=\; \mathscr{O}_2^* \cdot \mathscr{O}_1^*, \quad \big((a + \mathrm{i}b)\cdot \mathscr{O}\big)^* \;=\; (a - \mathrm{i}b)\cdot \mathscr{O}^* \tag{68}$$

  In the case of *Lagrangean field theories* there is an elaborate prescription, occupying most of the large and still growing literature on the subject (e.g. [HT92]), for how to choose the quantum observables as a deformation controlled by Poisson structure on the phase space, at least perturbatively. But we cannot expect M-theory to be the quantization of a Lagrangean field theory (already the sector of coincident fivebranes inside M-theory is expected not to be Lagrangean) and will instead discover a natural star-algebra of quantum observables right away (78), without detour through a classical field theory.
- The **quantum states** for given quantum observables are the linear maps on the quantum observables (understood as assigning to an observable the value that it takes in the given state) which are "positive" (semidefinite), in that on elements of the form A^*A they take non-negative real values.

$$\mathrm{QStates} \;=\; \Big\{\rho \;:\; \mathrm{QObsrvbls} \xrightarrow{\text{linear}} \mathbb{C} \,\Big|\, \underset{\mathscr{O}}{\forall}\;\; \rho\big(\mathscr{O}^* \cdot \mathscr{O}\big) \;\in\; \mathbb{R}_{\sqcup\{\infty\}} \subset \mathbb{C}\Big\} \tag{69}$$

But in the presence of higher gauge fields, these traditional structures of quantum physics are to be promoted to *higher structures*:

The higher phase space is a higher groupoid/stack whose

- objects are the field histories,
- higher morphisms are their higher gauge transformations.

The physics literature is mostly familiar with the infinitesimal approximation to this higher stack, which is a higher Lie algebroid known as the **BRST complex** (eg. [HT92]). The full higher phase space is to the BRST complex as a Lie group is to its Lie algebra, hence may be thought of as the **integrated BRST complex** (in the sense of Lie integration).

Concretely, if $\widehat{A}$ is the moduli stack (22) of a (nonabelian, generalized) cohomology theory expressing the flux quantization law (§2) of a higher gauge theory, and if we consider only the *pre-geometric* field equations (§1.2), then the **pregeometric**

[17] In C^*-algebraic formulations of mechanics the algebra of classical observables on a phase space is often taken to be the C^*-algebra $C_0(P)$ of continuous functions vanishing at infinity (eg. [La17, §3]). But this may be understood as the C^*-completion of the "actual" observable algebra of compactly-supported functions $C_c(P) \subset C_0(P)$, see eg. [La98, p. 55, 116][La17, p. 528].

[18] This means to require structure like that of a C^*-algebra but disregarding the completeness condition for a Banach algebra. In our application to "topological" charge sectors below the space of (higher) observables is (graded and) degreewise *finite*-dimensional, so that this (graded) Banach-algebra structure is automatic.

[19] In (68) we tacitly assume that the underlying space of quantum observables coincides with that of the "classical" observables. This turns out to be the case of relevance here (78). In traditional discussion the space of quantum observables can also be larger (such as in formal deformation quantization) or smaller (such as in geometric quantization) than the space of classical observables.

higher phase space[20] is the *mapping stack* from spacetime into $\widehat{A}$:

$$\underset{\substack{\text{pregeometric}\\ \text{higher phase space/}\\ \text{integrated BRST complex}}}{\mathrm{Maps}\big(X, \widehat{A}\big)} \;=\; \left\{ \begin{tikzcd}[column sep=huge] X \arrow[r, dashed, bend left=50, "\overset{\text{gauge field (map)}}{\widehat{\mathscr{F}}}"{name=U}] \arrow[r, dashed, bend right=50, "\underset{\text{gauge field}}{\widehat{\mathscr{F}}'}"'{name=D}] & \widehat{A} \arrow[from=U, to=D, phantom, "\substack{\text{gauge transfo.}\\ \text{(homotopy)}}\Downarrow"] \end{tikzcd} \right\}. \tag{70}$$

Underlying this higher moduli stack — after forgetting the gauge potentials and flux densities only remembering the corresponding charges — is the plain homotopy type of charges, the *cocycle space* of X in A-cohomology:

$$\mathrm{PrePhsSpc} \;=\; \underset{\substack{\text{pregeometric}\\ \text{higher phase space}\\ \text{of brane charges}}}{\mathrm{Maps}\big(X, A\big)} \;=\; \left\{ \begin{tikzcd}[column sep=huge] X \arrow[r, dashed, bend left=50, "\overset{\text{charges (map)}}{c}"{name=U}] \arrow[r, dashed, bend right=50, "\underset{\text{charges}}{c'}"'{name=D}] & A \arrow[from=U, to=D, phantom, "\substack{\text{gauge transfo.}\\ \text{(homotopy)}}\Downarrow"] \end{tikzcd} \right\}. \tag{71}$$

We will focus on this pre-phase space now, which may be regarded as reflecting the purely "topological" (non-geometric) sector of the higher phase space.

Higher observables. The notion of **higher observables** on a higher phase (70) is not widely discussed, but from the Dao of homotopy theory it is clear that the coefficient ring is to be promoted to a higher ring, namely a *ring spectrum* R. Then the higher analog of observables (67) on the topological sector (71) of a higher pregeometric phase space, is the R*-homology*:

$$\mathrm{PreObsrvbls} \;=\; R_\bullet\Big[\mathrm{Maps}\big(X, A\big)\Big]. \tag{72}$$

If $R = H\mathbb{C}$ is the Eilenberg-MacLane spectrum of the complex numbers, then this is ordinary homology:

$$\mathrm{PreObsrvbls} \;=\; H\mathbb{C}_\bullet\big[\mathrm{Maps}(X, A)\big] \;=\; H_\bullet\Big(\mathrm{Maps}\big(X, A\big); \mathbb{C}\Big). \tag{73}$$

Higher quantum observables. In general there is no *canonical* (star-)algebra structure on pregeometric higher observables (72) — but if spacetime has a circle factor (66) then the pregeometric higher phase space is the loop space of the *transverse phase space*:

$$\begin{tikzcd} & & \overset{\text{pregeometric phase space}}{\mathrm{Maps}\Big(S^1 \times X^{d-1}, A\Big)} \arrow[d, equal] \\ \text{mixed states} \arrow[r, leftsquigarrow] & \overset{\substack{\text{based loop space of}\\ \text{transverse phase space}}}{\Omega_c \mathrm{Maps}\big(X^{d-1}, A\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Maps}\Big(S^1, \mathrm{Maps}\big(X^{d-1}, A\big)\Big) \arrow[d] \\ & \{c\} \arrow[r] & \underset{\text{transverse phase space}}{\mathrm{Maps}\big(X^{d-1}, A\big)} \arrow[r, rightsquigarrow] & \text{pure states} \end{tikzcd} \tag{74}$$

This means that the following basic fact of algebraic topology provides us with a canonical discrete light-cone quantization of charges:

The Pontrjagin-Hopf algebra structure on the homology of loop spaces. [BoSa53][Br61, p. 36][Ha02, §3.C] *The homology of a based loop space*

$$\Omega Y \;:=\; \Big\{\gamma : [0,1] \xrightarrow{\text{cntns}} Y \,\Big|\, \gamma(0) \,=\, \gamma(1)\Big\}$$

with coefficients in a field becomes

[20] The genuine higher phase space is obtained from the pregeometric higher phase space by adjoining at least a background field of gravity and then imposing the remaining self-duality conditions (3).

- *a graded algebra*[21] *under concatenation of loops,*

$$
\begin{tikzcd}
H_\bullet(\Omega Y)\otimes H_\bullet(\Omega Y) \arrow[r, "\sim", "\text{Künneth}"'] \arrow[rr, bend left, "\text{Pontrjagin product}"] & H_\bullet(\Omega Y\times\Omega Y) \arrow[r, "(-)\cdot(-) := H_\bullet(\mu;\mathbb{C})", "\text{pushforward in homology}"'] & H_\bullet(\Omega Y)
\end{tikzcd}
$$

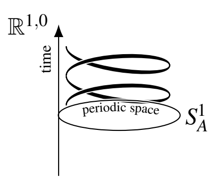
$$
\begin{array}{ccc}
\Omega Y\times\Omega Y & \xrightarrow{\quad\quad\mu\quad\quad} & \Omega Y \\
(\gamma_1,\gamma_2) & \mapsto & \left(t\mapsto \begin{cases}\gamma(t/2) & \text{for } 0\le t\le 1/2\\ \gamma(t/2-1/2) & \text{for } 1/2\le t\le 1\end{cases}\right)
\end{array}
\tag{75}
$$

- *a graded star-algebra under reversal of loops*

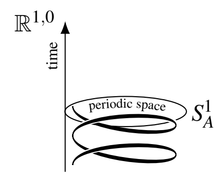
$$
\begin{tikzcd}
H_\bullet(\Omega Y) \arrow[r, "H_\bullet(\mathrm{inv})"] \arrow[r, bend left=60, "\text{Pontrjagin antipode}"] & H_\bullet(\Omega Y)
\end{tikzcd}
$$

$$
\begin{array}{ccc}
\Omega Y & \xrightarrow{\quad\mathrm{inv}\quad} & \Omega Y \\
\gamma & \longmapsto & \gamma(1-(\text{-}))
\end{array}
\tag{76}
$$

Notice that this is not quite a *complex* star-algebra in the sense of (68) yet, since the Pontrjagin-antipode[22] (76) acts trivially on the coefficient field – but we do get a complex star-algebra by composing the Pontrjagin antipode with complex conjugation on the coefficients



$$
\begin{tikzcd}
 & H_\bullet(\Omega Y;\mathbb{C}) \arrow[dr, "\substack{\text{complex conjugation}\\ H_\bullet(\Omega Y;(\text{-})^*)}", "\text{time inversion}"'] & \\
H_\bullet(\Omega Y;\mathbb{C}) \arrow[ur, "\substack{\text{Pontrjagin-antipode}\\ H_\bullet(\mathrm{inv};\mathbb{C})}", "S^1_A\text{-inversion}"'] \arrow[rr, "(\text{-})^*", "\text{light-cone parameter inversion}"'] \arrow[dr, "H_\bullet(\Omega Y;(\text{-})^*)"'] & & H_\bullet(\Omega Y;\mathbb{C}) \\
 & H_\bullet(\Omega Y;\mathbb{C}) \arrow[ur, "H_\bullet(\mathrm{inv};(\text{-})^*)"'] &
\end{tikzcd}
\tag{77}
$$



This way we have obtained **higher quantum observables** on the light-cone for pregeometric brane charges in spacetimes of the form (66):

$$
\mathrm{QObsrvbls}_c \;=\; \Big(H_\bullet\big(\Omega_c\mathrm{Maps}(X^{d-1},A);\mathbb{C}\big),\; (\text{-})\cdot(\text{-}),\; (\text{-})^*\Big).
\tag{78}
$$

whose star-involution is *light-cone parameter inversion*. Notice that the Pontrjagin product Hopf-algebra (78) is (by [MiMo65, App.]) the universal enveloping algebra of the Whitehead bracket Lie algebra $\mathfrak{l}\mathrm{Maps}(X^{d-1},A)$ (2.3) of the charge moduli space, and that universal envelopes are a standard deformation quantization of Poisson-Lie structures (eg. [Gu11, §2.2]).

The corresponding **light-cone quantum states** are hence (69) those cohomology classes which are (semi-)positive-definite:

$$
\mathrm{QStates}_c \;=\; \Big\{\rho\in H^\bullet\big(\Omega_c\mathrm{Maps}(X^{d-1},A);\mathbb{C}\big)\;\Big\vert\; \underset{\mathscr{O}}{\forall}\; \rho(\mathscr{O}^*\mathscr{O})\ge 0\Big\}.
\tag{79}
$$

In §6 we discuss examples of 11d spacetime domains whose light-cone quantum states (79) of pregeometric (intersecting) brane charges include, under Hypothesis H:

§6.3.1 quantum states of Hanany-Witten NS/D-brane configurations,

§6.3.2 quantum states of transverse M_5-branes,

§6.3.3 quantum states of $\mathrm{M}_5\perp\mathrm{M}_5$ intersections,

§6.3.4 quantum states of topological strings,

with provable properties of the kind expected in the string theory literature.

---

[21] In fact, together with the canonical coproduct in homology the Pontrjagin product (75) becomes a Hopf algebra structure with the star-involution (76) being a Hopf antipode.

[22] The term "Pontrjagin antipode" is not standard, but it is the natural name for the antipode of the Pontrjagin-Hopf algebra structure.

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6 Resulting quantum branes

We discuss the nature of solitonic M-branes (§1), according to Hypothesis H (§3), and their lightcone quantum mechanics (§5).

Worldvolumes of solitonic branes. Much of existing literature insists of thinking of branes as *worldvolume* submanifolds in spacetime. But in general, solitonic branes are whatever is seen by the reduced charge-cohomology theory $\widetilde{A}$ (19), hence solitonic branes under Hypothesis H are whatever is seen by reduced 4-Cohomotopy $\widetilde{\pi}$. In general there need not and will not be an identifiable *worldvolume* manifold sourcing such cohomology classes — but classical theorems about Cohomotopy assert that in this case such geometric worldvolumes often do have meaning:

§6.1 – the *Pontrjagin theorem*.

§6.2 – the *May-Segal theorem*.

6.1 Solitonic branes and their Cobordism classes

Solitonic branes seen in Cohomotopy are equivalently cobordism classes of normally framed submanifolds [HpH2, §2.2] [Orb1, §2.1]:

Brane charge and Pontrjagin-Thom collapse. The above motivation of Cobordism cohomology as the natural home for p-brane charge, summarized in Table I, is in itself only a plausibility argument, just as is the traditional motivation ([Wi98, §3]) of K-theory as the natural home for D-brane charge. However, this physically plausible conclusion is rigorously *implied by Hypothesis H*, and hence is supported by and adds to the other evidence for that Hypothesis: This implication is the statement of *Pontrjagin's isomorphism*, which says that the operation of assigning to a normally framed closed submanifold its *asymptotic directed distance* function (traditionally known as the *Pontrjagin-Thom collapse construction*)

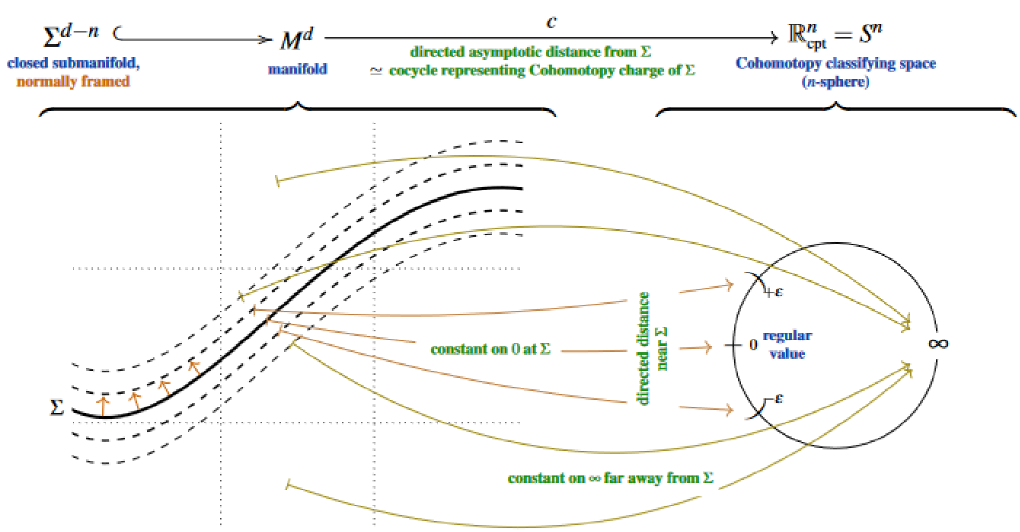


Figure D – The Pontrjagin construction. The *charge in Cohomotopy* of a manifold M^d, sourced by a normally framed closed submanifold Σ^{d-n}, is the homotopy class of the function that assigns directed asymptotic distance from Σ, measured along its normal framing.

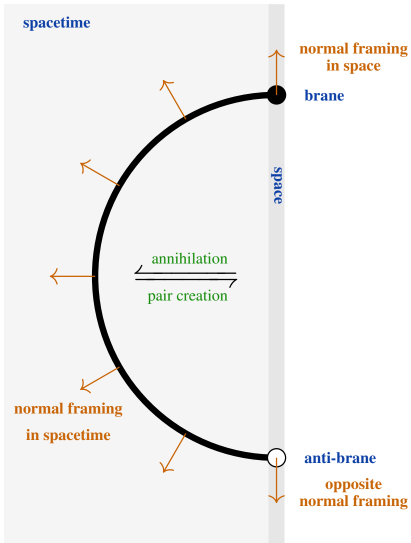
identifies framed Cobordism with Cohomotopy, as non-abelian cohomology theories, over *closed manifolds* M^d:

$$\underset{\substack{\text{framed unstable}\\ n\text{-Cobordism of } M^d}}{\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)} := \mathrm{NFramedSubmflds}_{d-n}\big(M^d\big)_{/\mathrm{brdsm}} \underset{\text{pre-image of regular value 0}}{\overset{\text{directed asymptotic distance}}{\rightleftarrows}} \mathrm{Maps}\big(M^d, S^n\big)_{/\mathrm{hmtpy}} =: \underset{\substack{\text{closed}\\ \text{manifold}}}{\overset{\substack{\text{unstable}\\ n\text{-Cohomotopy of } M^d}}{\pi^n\big(M^d\big)}}.$$

$$\text{assign Cohomotopy charge: } \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big) \longrightarrow \pi^n\big(M^d\big), \qquad \text{reconstruct submanifold from its charge: } \pi^n\big(M^d\big) \longrightarrow \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)$$

For example, let

This equivalence of CoHomotopy with unstable framed Cobordism reflects exactly the expected brane/anti-brane reactions:



In fact, in linear approximation to the Bianchi identities, the resulting *stable CoHomotopy* is equivalent to *framed Cobordism Cohomology*

$$
\begin{tikzcd}
\underset{\text{Cohomotopy}}{\overset{\text{non-abelian}}{\pi^\bullet}} \arrow[r, "\text{linearize}", "\text{(i.e.: stabilize)}"'] & \underset{\text{Cohomotopy}}{\overset{\text{abelian}}{\mathbb{S}^\bullet}} \arrow[r, equal, "\text{Pontrjagin-Thom}"] \arrow[d, equal, "\text{Barratt-Priddy-Quillen}"] & \overset{\text{framed Cobordism}}{\mathrm{Mfr}^\bullet} \\
& \underset{\text{algebraic K-theory of ``field with one element''}}{K\mathbb{F}_1^\bullet} &
\end{tikzcd}
\tag{80}
$$

Linearized Hypothesis H:

$\boxed{\textit{M-brane flux is quantized in (tangentially twisted) framed Cobordism.}}$

(Possibly this relates *Hypothesis H* to Vafa's *cobordism conjecture* cf. [HpH2, §4]).

In conclusion: the Pontrjagin theorem and its variants give, under Hypothesis H, a detailed description of *worldvolumes* of M-branes as (cobordism classes of normally framed) sub-manifolds of spacetime.

E.g. this allows to study exotic defect branes in low codimension $\longrightarrow$

6.2 Exotic branes and their configuration spaces

Exotic branes. In the string theory literature, by a *non-standard brane* [BR12] or *exotic brane* [dBS13] one means, foremost, a p-brane species of low codimension $D-(1+p) \leq 2$ — which in M-theory generally corresponds to codimensions $D-(1+p) \leq 3$. (We observe that this are exactly the solitonic branes which have vanishing classical charge under Hypothesis H, according to (30).)

The most familiar examples of exotic branes in 10d string theory are the comparatively well-understood D_7 branes (of codimension 2, hence "defect branes" [BOR12]), the D_8-branes (of codimension 1, hence "domain walls") and the D_9 (codimension 0). Beyond these there is expected a plethora of further exotic branes (cf. [dBS13, Fig 1]) whose existence is argued indirectly by assumption of the famous but largely hypothetical "U-duality"-symmetry of string/M-theory, but which are not known to arise as supergravity solutions.

Already the low-codimension D-branes push the boundaries of common string theory lore: For instance the $\mathrm{SL}(2,\mathbb{Z})$-charges crucially meant to be carried by D_7-branes had no reflection in *Hypothesis K* (§3.1, a shortcoming which we argue is resolved by Hypothesis H, see §6.3) and the lift of D_8-branes to M-theory had remained at least subtle, even by informal arguments:

The problem with the D_8**-brane** is part of the general problem of integrating the relevant "massive" variant of type IIA string theory into the non-pertubative picture of M-theory, which fails in its most naive form since 11d supergravity provably does not admit the analogous "massive" deformation. While it has been argued that massive type IIA supergravity does arise from plain 11d supergravity, after all, by twisting the usual KK-compactification by U-duality transformations (which remain fairly conjectural themselves), the nature of the resulting lift of the D_8-brane, commonly called the M_9-brane, remained so elusive that more recent authors argued it should rather be called the M_8*-brane* to be understood only somewhat tautologically as 'an object that exists only as a lift of the D8-brane".[23]

Notice that the traditional *Hypothesis K* (§3.1) inherits all these conceptual problems since the "Romans mass"-term F_0 (the flux sourced by the D_8-brane) which is at the root of the problem is key part of its pre-geometric derivation (12). But in §6.3 we will argue in detail that and how the D_8-brane in its M-theoretic incarnation as an $\mathrm{M}_9/\mathrm{M}_8$-brane is implied by *Hypothesis H*.

Higher observables of exotic branes. Indeed, Hypothesis H implies right away, via (30), that there is *no* charge sourced by flat solitonic exotic branes (and most other admissible flux-quantization laws for the C-field would imply the same) but that ((73)) there are non-trivial higher observables of exotic branes, encoded in the higher homotopy type of the Cohomotopy cocycle space

$$X \text{ a flat space(time) with a (base)point (at) } \infty \qquad \vdash \qquad \overbrace{\widetilde{\boldsymbol{\pi}}^4(X)}^{\text{4-Cohomotopy flux moduli space}} \;:=\; \underbrace{\mathrm{Maps}^{*/}\big(X, S^4\big)}_{\text{pointed mapping space into 4-sphere}} \tag{81}$$

in that

$$\overbrace{\pi^4\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{p\leq 3}_{\cup\{\infty\}}\Big) \;\simeq\; \pi_{p\leq 3}(S^4) \;=\; 0}^{\text{exotic solitonic M-branes carry no clasical charge}} \qquad \text{but} \qquad \overbrace{\widetilde{\boldsymbol{\pi}}^4\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{p\leq 3}_{\cup\{\infty\}}\Big) \;\neq\; *}^{\text{have non-trivial higher observables}}.$$

We will argue in §6.3 for the example of M_9-brane intersections that these higher observables on Cohomotopy moduli indeed reflect a wealth of expected patterns in D_8-brane intersections. This way, Hypothesis H seems to nicely capture the ethereal nature of exotic branes.

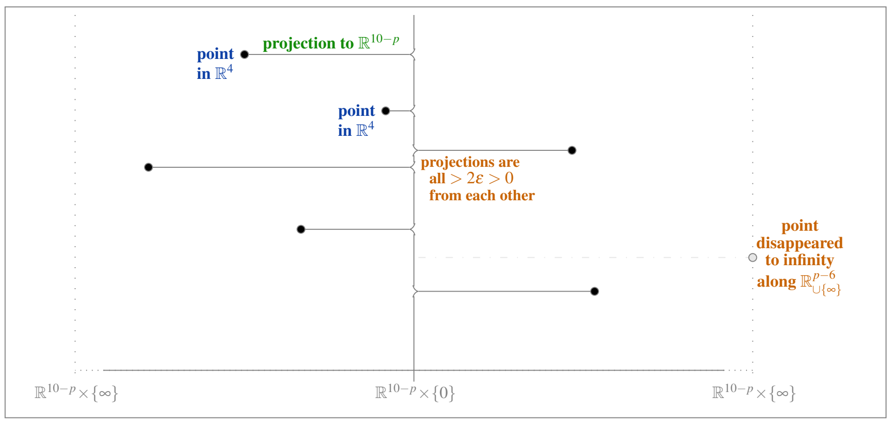
[23] [BMO18, p. 65]: "However, as remarked in [OP99, p. 109], [the M9 brane] should more properly be called an M8-brane or perhaps KK8 following its mass formula designation $8^{(1,0)}$. It is, perhaps, to be understood as an object that exists only as a lift of the D8-brane of Type IIA."

Configuration spaces of exotic branes. We observe [Qnt1] that the analog of the Pontrjagin theorem (§6.1) in the case of exotic branes, under Hypothesis H, is the *May-Segal theorem* [May72, Thm. 2.7][Segal73, Thm. 3] which, with Hypothesis H, equivalently says [Qnt1, Prop. 2.5] that the moduli space of flat solitonic $p \geq 7$-branes in 11d is (homotopy equivalent to) a *configuration space of points* in their transverse space:

cohomotopical moduli space of flat solitonic $p \geq 7$-branes with $\lesssim 3$-dim. transverse space

$$
\begin{array}{l}
\widetilde{\boldsymbol{\pi}}^4\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{10-p}_{\cup\{\infty\}}\Big) \;:=\; \mathrm{Maps}^{*/}\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{10-p}_{\cup\{\infty\}},\, S^4\Big) \\
\quad\wr| \\
\mathrm{Maps}^{*/}\Big(\mathbb{R}^{10-p}_{\cup\{\infty\}},\, S^4\Big) \\
\quad \Big\uparrow \wr \;\; \text{May-Segal theorem} \;\; \textbf{cohomotopy charge map} \\
\mathrm{Conf}\Big(\mathbb{R}^{10-p}_{\sqcup\{\infty\}},\, \mathbb{R}^{p-6}_{\cup\{\infty\}}\Big) = \left\{ \text{(see figure)} \right\}
\end{array}
\qquad (82)
$$

configuration space of points in transverse space



$\mathrm{Conf}\big(\mathbb{R}^{10-p}_{\sqcup\{\infty\}}, \mathbb{R}^{p-6}\big)$ is the pointed space of • un-ordered tuples of points in $\mathbb{R}^4 \simeq \mathbb{R}^{10-p} \times \mathbb{R}^{p-6}$ — as such they look like flat solitonic 6-branes. • which have pairwise distinct projections to $\mathbb{R}^{10-p}$ — as such they look like flat solitonic p-branes • and may escape to (or emerge from) ∞ along $\mathbb{R}^{p-6}_{\cup\{\infty\}}$ — like partially de-localized 6-brane solitons

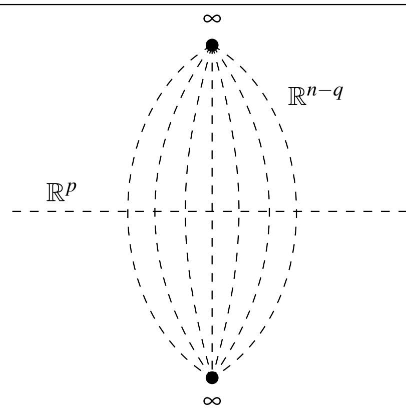
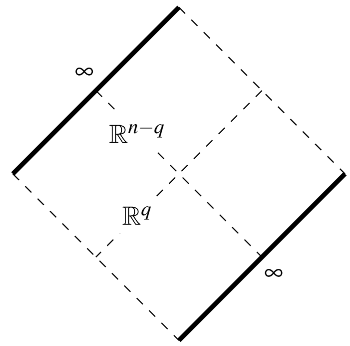
Indeed, the cohomotopy charge map (82) (aka inverse "electric field map" [Segal73, §1][McD75, §1] or "scanning map") evaluates on the configuration space by [Segal73, §3] assigning to each point in the configuration the unit $(10-p)$-cohomotopy charge of a solitonic p-brane, but regarded after inclusion into the cohomotopy charge space of solitonic 6-branes.

$$
\begin{tikzcd}
\substack{\textbf{moduli space of}\\ \textbf{solitonic } p\textbf{-branes}}\;\mathrm{Conf}\big(\mathbb{R}^{10-p}, \mathbb{R}^{p-6}_{\cup\{\infty\}}\big) \arrow[rr, bend left, "\text{4-cohomotopy charge map for solitonic } p \geq 7\text{-brane}"] & \widetilde{\boldsymbol{\pi}}^{10-p}\big(\mathbb{R}^{10-p}_{\cup\{\infty\}}\big) & \widetilde{\boldsymbol{\pi}}^{4}\big(\mathbb{R}^{10-p}_{\cup\{\infty\}}\big) \\
\substack{\textbf{single-brane}\\ \textbf{subspace}}\;\mathbb{R}^{10-p} \times \mathbb{R}^{p-6} \arrow[u, hook] \arrow[r, "\substack{\text{pure solitonic}\\ p\text{-brane charge}}"] & \mathrm{Maps}^{*/}\big(\mathbb{R}^{10-p}_{\cup\{\infty\}}, \mathbb{R}^{10-p}_{\cup\{\infty\}}\big) \arrow[u, equal] \arrow[r, "\substack{\text{regarded as solitonic}\\ \text{6-brane charge}}"] & \mathrm{Maps}^{*/}\big(\mathbb{R}^{10-p}_{\cup\{\infty\}}, \mathbb{R}^{4}_{\cup\{\infty\}}\big) \arrow[u, equal] \\
(x, y) \arrow[r, mapsto] & \left(x' \mapsto \begin{cases} \dfrac{x'-x}{\exp\left(\frac{-1}{\varepsilon - |x'-x|^2}\right)} & \text{if } |x'-x| < \varepsilon \\ \infty & \text{otherwise} \end{cases} \right) \arrow[r, mapsto] & \left(x' \mapsto \begin{cases} \dfrac{(x'-x,\, y)}{\exp\left(\frac{-1}{\varepsilon - |x'-x|^2}\right)} & \text{if } |x'-x| < \varepsilon \\ \infty & \text{otherwise} \end{cases} \right)
\end{tikzcd}
\qquad (83)
$$

Notice the dichotomy: • *If branes can not escape to* ∞ • then their *fluxes vanish at* ∞. and vice versa.

This is nicely brought out by notationally retaining a contractible wedge factor in the general statement of the May-Segal theorem, because then the passage between the configuration space of branes and the spacetime domain on which their charges are evaluated is given by swapping the subscripts $(\text{-})_{\sqcup\{\infty\}} \leftrightarrow (\text{-})_{\cup\{\infty\}}$

$$\underbrace{\mathrm{Conf}\Big(\mathbb{R}^{n-q}_{\sqcup\{\infty\}}, \mathbb{R}^{q}_{\cup\{\infty\}}\Big)}_{\substack{\text{brane}\\ \text{configuration space}}} \xrightarrow[\sim]{\substack{\text{cohomotopy}\\ \text{charge map}}} \mathrm{Maps}^{*/}\Big(\mathbb{R}^{n-q}_{\cup\{\infty\}} \wedge \mathbb{R}^{q}_{\sqcup\{\infty\}}, S^n\Big) = \underbrace{\widetilde{\boldsymbol{\pi}}^n\Big(\mathbb{R}^{n-q}_{\cup\{\infty\}} \wedge \mathbb{R}^{q}_{\sqcup\{\infty\}}\Big)}_{\substack{\text{flux}\\ \text{cocycle space}}} \qquad (\text{for } n > q > 0) \tag{84}$$

Smash product of pointed topological spaces	Visualization with point at infinity	Visualization as Penrose diagram
$\overbrace{\mathbb{R}^{n-q}_{\cup\{\infty\}}}^{\substack{\text{fluxes vanish at infinity}\\ \text{along these directions}}} \wedge \underbrace{\mathbb{R}^{q}_{\sqcup\{\infty\}}}_{\substack{\text{...but not necessarily}\\ \text{along these}}}$		

M_9**-Branes.** We obtain now from Hypothesis H a rigorous definition of the otherwise elusive M_9-brane (cf. p. 54) and can investigate it by mathematical analysis.

Namely, given that the (flat) M_9 is supposed to be:

1. solitonic
   (as there is no corresponding singular supergravity solution)
2. as such localized along a single transverse direction
   (since this is what it means to be a 9-brane in 11d)
3. but necessarily compactified on the M-theory circle S^1_A,
   (since it "exists only as a lift of the D_8-brane of type IIA", cf. ftn. 23)

the (flat) M_9-brane should, assuming Hypothesis H, be addressed as whatever it is that 4-Cohomotopy sees on the following spacetime domain:

$$\begin{array}{ll} & \mathbb{R}^{1,0}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{5}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{1}_{\cup\{\infty\}} \wedge \mathbb{R}^{3}_{\sqcup\{\infty\}} \times S^1_{\sqcup\{\infty\}} \\ \mathrm{M}_9 & \end{array} \tag{85}$$

Here the bar shows, in the tradition of brane diagrams, across which dimensions the M_9-brane is extended (which we have decomposed in anticipation of the brane intersections below in (88)) — but the light shading is to indicate that this remains somewhat ambiguous – since we are dealing with an exotic brane of low codimension so that the Pontrjgin theorem (§6.1) does not apply, while also the May-Segal theorem (82) does not quite apply, due to the presence of the $S^1_{\sqcup\{\infty\}}$-factor. Therefore one cannot *quite* translate the cohomotopical M_9-brane moduli on (85) into submanifolds in a spacetime domain. Of course, just such an ambiguity is expected for the M_9 brane (aka M_8-brane), cf. ftn. 23.

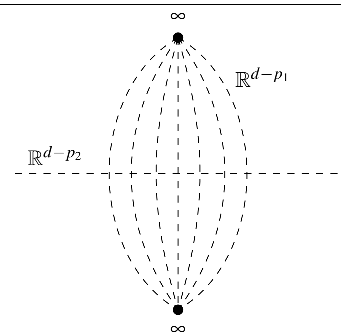
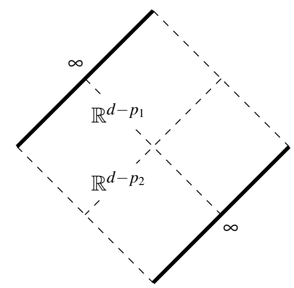
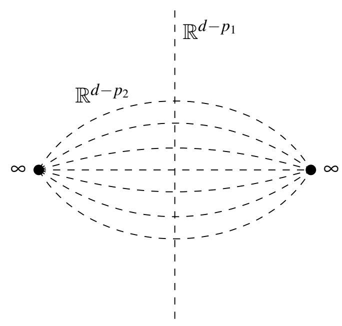
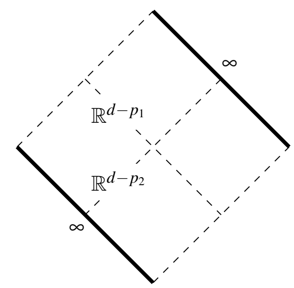
On the other hand, by the above discussion we can describe the M_9 moduli quite explicitly: They are given by loops in the configuration space of solitonic 6-branes which are delocalized in 3 transverse directions:

$$
\begin{aligned}
& \overset{\textbf{M}_9\textbf{-brane moduli}}{\widetilde{\boldsymbol{\pi}}^4\big(\mathbb{R}^{1,8}_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}} \wedge \mathbb{R}^1_{\cup\{\infty\}}\big)} \\
:=\; & \mathrm{Maps}^{*/}\Big(\mathbb{R}^{1,5}_{\sqcup\{\infty\}} \wedge \mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}},\, S^4\Big) && \text{charge moduli space (81) on M}_9\text{-domain (85)} \\
\simeq\; & \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}},\, S^4\big) && \mathbb{R}^{1,5} \text{ is contractible} \\
\simeq\; & \mathrm{Maps}^{*/}\Big(S^1_{\sqcup\{\infty\}},\, \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}},\, S^4\big)\Big) && \text{mapping space adjunction (20)} \\
\simeq\; & \mathrm{Maps}^{*/}\Big(S^1_{\sqcup\{\infty\}},\, \mathrm{Conf}\big(\mathbb{R}^1_{\sqcup\{\infty\}},\, \mathbb{R}^3_{\cup\{\infty\}}\big)\Big) && \text{May-Segal theorem (82)} \\
=\; & \underset{\substack{\textbf{loop space of configuration space} \\ \textbf{of 3-fold delocalized 6-branes}}}{\mathscr{L}\,\mathrm{Conf}\Big(\mathbb{R}^1_{\sqcup\{\infty\}},\, \mathbb{R}^3_{\cup\{\infty\}}\Big)}
\end{aligned}
\tag{86}
$$

In order to say more about the M_9-brane, we need to see it interact (intersect) with other branes. This we turn to in (88) below.

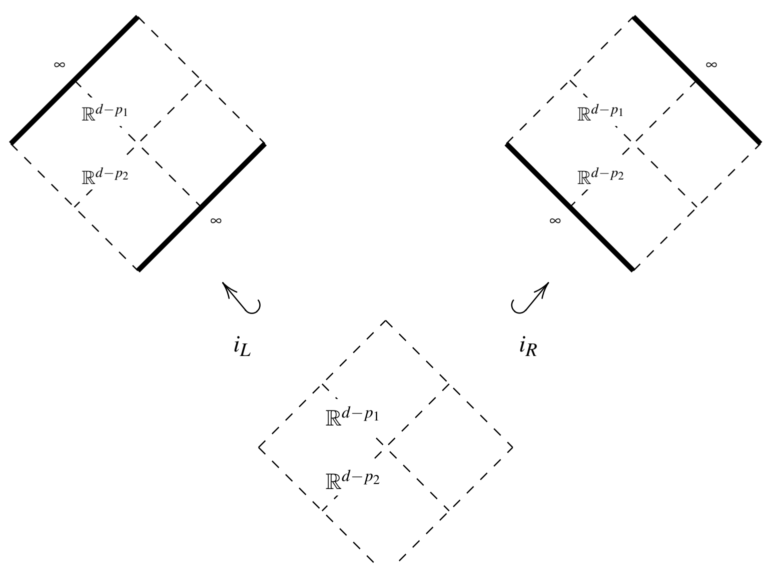
6.3 Solitonic brane intersections

The situation in (84) suggests that to measure the charges and hence the presence of *intersecting solitonic* branes we need to evaluate cohomotopy on a kind of amalgamation of their respective domains (84):

	Smash product of pointed topological spaces	**Visualization** with point at infinity	**Visualization** as Penrose diagram
transverse space of p_1-soliton	$\overbrace{\mathbb{R}^{d-p_1}_{\cup\{\infty\}}}^{\text{fluxes vanish at infinity along these directions}} \wedge \underbrace{\mathbb{R}^{d-p_2}_{\sqcup\{\infty\}}}_{\text{...but not necessarily along these}}$		
transverse space of p_2-soliton	$\underbrace{\mathbb{R}^{d-p_1}_{\sqcup\{\infty\}}}_{\text{...but not necessarily along these}} \wedge \overbrace{\mathbb{R}^{d-p_2}_{\cup\{\infty\}}}^{\text{fluxes vanish at infinity along these direction}}$		

The topos theory of intersecting solitonic brane spaces. A sensible amalgamation of these transverse spaces does not exist *as a topological space*. But we may pass to the universal mathematical context where it does exist: this is the *presheaf topos* over the category of "Penrose diagrams" of this form [Qnt1, Def. 2.2]: In this topos, the amalgamation space transverse to flat intersecting branes is the "pushout" or "cofiber coproduct" of the two separately compactified transverse spaces over the uncompactified transverse space, hence their "gluing" according to the following diagram:

$$
\left(
\begin{tikzcd}
\mathbb{R}^{d-p_1}_{\cup\{\infty\}} \wedge \mathbb{R}^{d-p_2}_{\sqcup\{\infty\}} & & \mathbb{R}^{d-p_1}_{+} \wedge \mathbb{R}^{d-p_2}_{\cup\{\infty\}} \\
& \mathbb{R}^{2d-p_1-p_2}_{\sqcup\{\infty\}} \arrow[ul, hook, "i_L"] \arrow[ur, hook', "i_R"'] &
\end{tikzcd}
\right)
$$



Intersecting brane charges in Cohomotopy. Furthermore, we may understand the May-Segal theorem (82) as providing a *differential refinement* of cohomotopy on such spaces, in that the configuration space canonically carries the structure of a manifold which represents the homotopy type of the cohomotopy space.

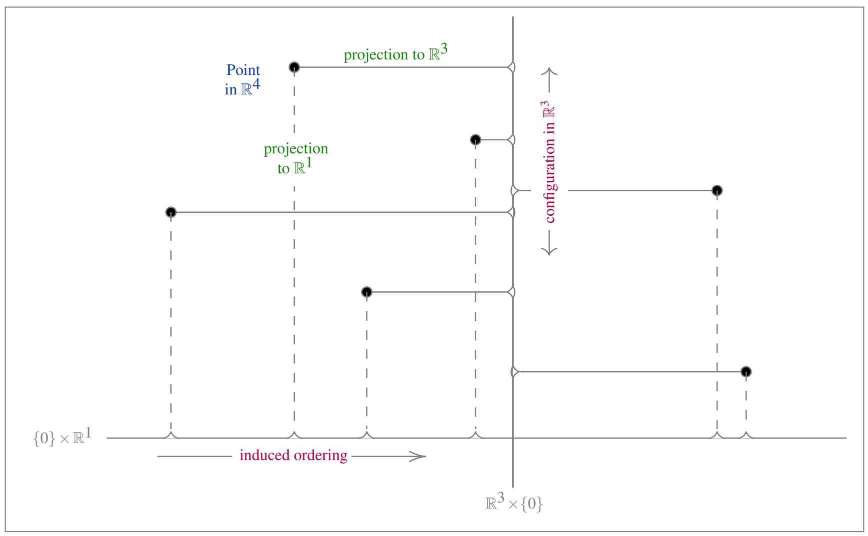
This provides a formalization of what it means to detect *intersecting* solitonic brane charges in Cohomotopy theory. Assuming this, the basic laws of topos theory imply that the (differential) Cohomotopy moduli space for intersecting solitonic branes as above is the *fiber product* of the two configuration spaces (84) for each solitonic brane separately [Qnt1, Exp. 2.3].

Gauge enhancement on domain wall intersections. In the special case that one of the intersecting brane species is of codimension=1 something remarkable happens, as then this fiber product of un-ordered configuration spaces becomes homotopy-equivalent to a configuration space of *ordered* points in the remaining $n-1$ transverse dimensions [Qnt1, Prop. 2.4. 2.11]:

$$\underbrace{\overbrace{\mathrm{Conf}\big(\mathbb{R}^1_{\sqcup\{\infty\}}, \mathbb{R}^{n-1}_{\cup\{\infty\}}\big)}^{\substack{n\text{-Cohomotopy moduli of} \\ \text{solitonic codim} = 1\text{-branes}}} \underset{\mathrm{Conf}\big(\mathbb{R}^n_{\cup\{\infty\}}\big)}{\times} \overbrace{\mathrm{Conf}\big(\mathbb{R}^{n-1}_{\sqcup\{\infty\}}, \mathbb{R}^1_{\cup\{\infty\}}\big)}^{\substack{n\text{-Cohomotopy moduli of} \\ \text{solitonic codim} = (n-1) \text{ branes}}}}_{\text{their intersection}}$$

$$\Big\downarrow \text{homotopy equivalence}$$

$$\bigsqcup_{N \in \mathbb{N}} \underbrace{\mathrm{Conf}_{\{1,\cdots,N\}}\big(\mathbb{R}^{n-1}\big)}_{\substack{\text{configuration space} \\ \text{of } N \text{ ordered points}}} = \left\{ \vphantom{\Big|} \right. \tag{87}$$

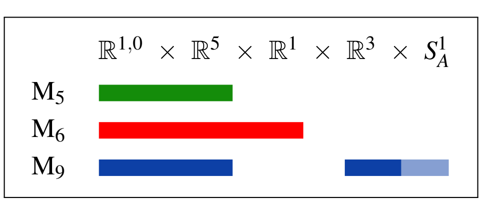


But the homotopy type of ordered configuration spaces is quite rich (see eg. [Kn18]) considerably richer than that of the un-ordered configuration spaces (82) where points may escape to ∞. With Hypothesis H this provides a substantiation of the expected phenomenon: There is *rich physics appearing on brane intersections*.

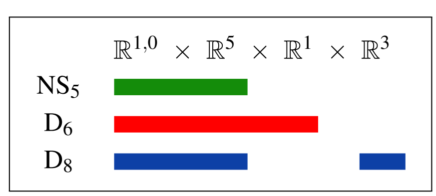
6.3.1 Hanany-Witten D_{p+2}-D_p-NS_5-brane intersections

We discuss how Hypothesis H implies a lift to M-theory of the NS_5-D_6-D_8-brane intersections (p. 11), now incarnated as solitonic (§1.1) quantum (§5) M_5-MK_6-M_9-brane intersections – this is from [Qnt1][Qnt2].

Moduli of Hanany-Witten brane configuration. Now we can analyze the intersections of M_9-branes (85) with solitonic 6-branes on 5-branes in 11d, corresponding in 10d to a solitonic incarnation of the NS_5-D_6-D_8-brane intersections whose incarnation as singular branes we discussed on p. 11, as shown by the following brane diagrams:



$\rightsquigarrow$



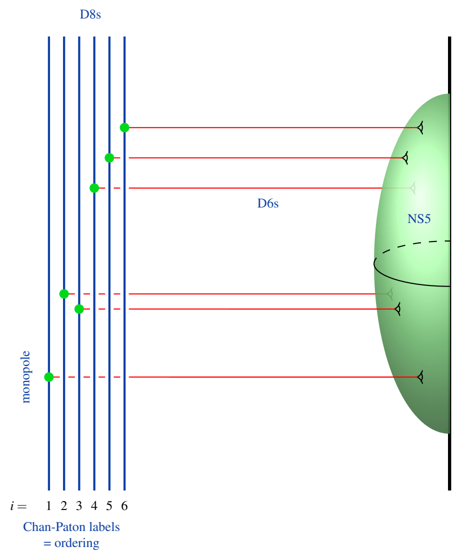
(88)

By applying the argument for the M_9-brane moduli (86) analogously to the other factors in the fiber product (87) and using that forming (free) loop spaces $\mathrm{Maps}^{*/}(S^1_{\sqcup\{\infty\}}, \text{-})$ is a right adjoint (20) and hence preserves these fiber products, we find with (87) that the moduli of these brane intersections (88) is the loop space

$$\mathscr{L} \underset{N \in \mathbb{N}}{\sqcup} \underset{\{1,\cdots,N\}}{\mathrm{Conf}} \big(\mathbb{R}^3\big) \;\simeq\; \underset{N \in \mathbb{N}}{\sqcup} \mathscr{L} \underset{\{1,\cdots,N\}}{\mathrm{Conf}} \big(\mathbb{R}^3\big)$$

of the space of ordered configurations of points in $\mathbb{R}^3$, identified here (88) as the space of configurations of flat solitonic 6-branes inside the transverse D_8-brane that they are intersecting along an NS_5-brane [Qnt1, Prop. 2.11].

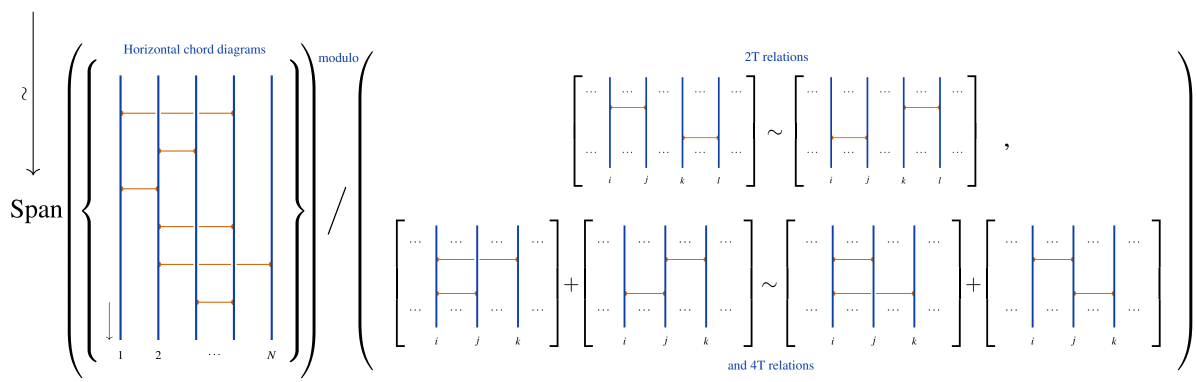
Another way of looking at all this — if we focus just on the points in the configurations (the green dots in the diagram) which the mathematical analysis gives us — is that in them we have found the solitonic incarnation of the $\mathrm{M}_5/\mathrm{NS}_5$-brane. (Notice that also *singular* $\frac{1}{2}$BPS M_5-branes necessarily sit on 6-branes once coincident, [HSS19, Exp. 2.2.6].)



(89)

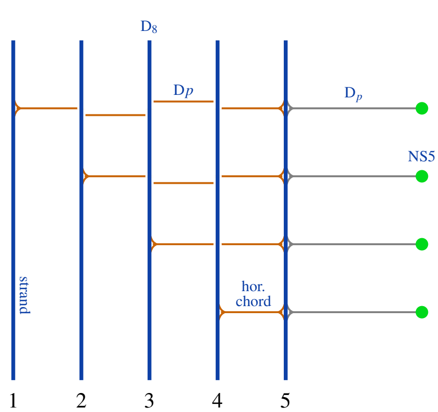
Quantum observables on Hanany-Witten configurations. It follows now with the discussion in §5 that the light-cone quantum observables on these brane configurations form, for each number N of D_8-branes, the homology Pontrjagin-algebra of the based loop space of the ordered configuration space (89). Remarkably, this is isomorphic to the algebra of *horizontal chord diagrams* on N-strands modulo certain relations [Qnt1, Prop. 2.18]

$$\mathrm{QObsrvbs}_{\mathrm{D}_8\text{-}\mathrm{D}_6\text{-}\mathrm{NS}_5}$$
$$\|$$
$$H_\bullet\Big(\Omega \underset{\{1,\cdots,N\}}{\mathrm{Conf}} \big(\mathbb{R}^3\big)\Big)$$



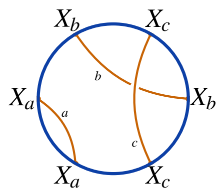
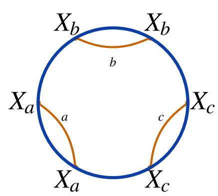
(90)

On the *skew-symmetric* sector of the algebra, these relations may be identified with the brane intersection rules expected in Hanany-Witten theory [Qnt1, §4.10]:

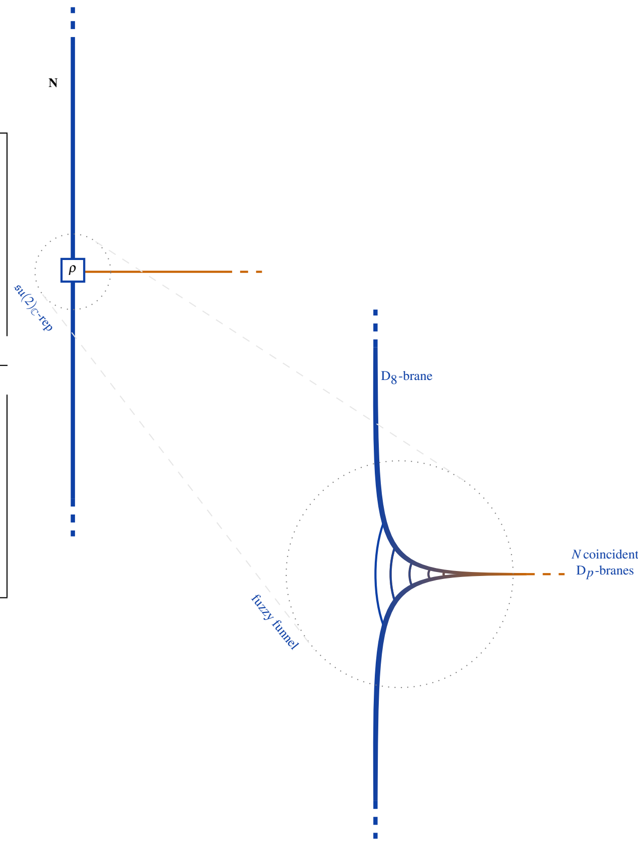


(91)

On the other hand, the full observable algebra (90) may naturally be identified with an observables on fuzzy 2-spheres [Qnt1, §4.2], which are expected to describe the quantum nature of the $D_8 \perp D_6$-brane intersections.

$\int_{S^2_N} (R^2)^3 = \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c\big)$	
$\int_{S^2_N} (R^2)^3 = \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X^b \cdot X_c \cdot X^c\big)$	

In fact, $D_{p+2} \perp D_p$-brane intersections for any p are meant to all be T-dual to each other, and all share this *fuzzy funnel*-form of their quantum intersection.
Among these the $D_4 \perp D_2$ are meant to be dual to $M_5 \perp M_2$-brane intersections.



6.3.2 M5/M-Branestates

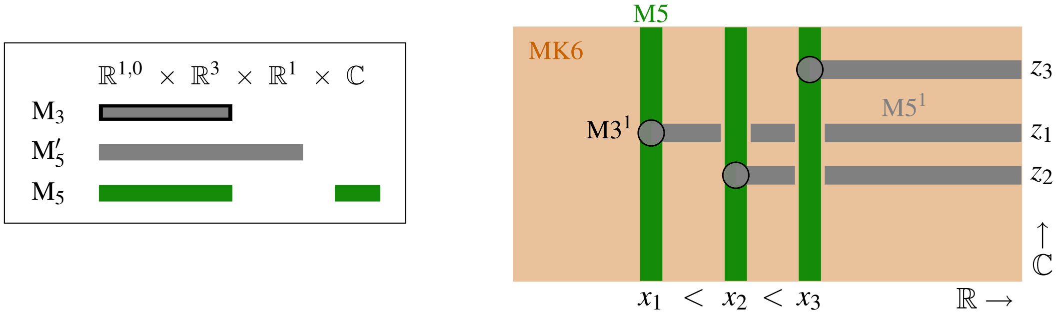
Light-cone quantum states of M_5/M_2-Brane bound states. In the vein that (89) gives the moduli of (nearly) coincident solitonic quantum M_5-branes in M(embrane)-theory, we should find the above light-cone quantum observables (90) to match those in the membrane matrix model (p. 46) describing M_5-branes.

This is indeed the case [Qnt1, §4.9]: The light-cone quantum ground states of the BMN matrix model are superpositions of fuzzy 2-spheres, and in suitable arrangements these encode either pure M_5-brane states or generally M_2/M_5-brane bound states [BMN02, (5.5)][MSvR03][AIST17][AIST18]. In particular, the quantum state given by multiples of the $N^{(M5)}$-dimensional irrep of $\mathfrak{su}(2)_{\mathbb{C}}$ should correspond to $N^{(M5)}$ 5-branes in the ground state.

No-ghost theorem for M_5-branes. Therefore, with the light-cone quantum mechanics established in §5, we may ask which of these light-cone quantum states are proper quantum states in that they are positive (non-ghost) states. In [Qnt1, Exp. 3.11][Qnt2] we showed that, beyond the trivial case of $N^{(M_5)} = 1$ this is the case for fundamental representation with $N^{(M5)} = 2$ [Qnt1, Exp. 3.5][Qnt2]. In [Co23] it is claimed that the same conclusion still holds for all symmetric and exterior powers of the fundamental representation, which would establish the positivity (no-ghost theorem) for the corresponding mixed M_2/M_5-brane bound states.

6.3.3 $M_5 \perp M_5$-Brane intersections.

Applying the analogous analysis to the 3-cohomotopy fields in 7d which in §4 we saw appear on fivebranes, we find a moduli space of 3-branes inside 5-branes, just as expected for M5 ⊥ M5-brane intersections [Dfc1, pp. 28], to be given by the configuration space of points in the transverse complex plane:



(92)

To find the light-cone quantization (§5) of these brane intersections notice that the homotopy type of this configuration space is the classifying space of the pure braid group (cf. [TQC2])

$$
\begin{array}{ccc}
\Omega \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \longrightarrow \mathscr{L}\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) & & \mathrm{PBr}(N) \longrightarrow \mathscr{L}\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) \\
\qquad\qquad\qquad\qquad \big\downarrow & \simeq & \qquad\qquad\qquad \big\downarrow \\
\qquad\qquad\qquad\qquad \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}) & & \qquad\qquad\qquad B\mathrm{PBr}(N)
\end{array}
$$

This means that the light-cone quantum observable algebra (78) in this case is the braid group algebra . Thus with the Gelfand-Raikov theorem[24] we find that the corresponding light-cone quantum states are given by unitary pure braid representations:

$$
\mathrm{QObscrbls}_{N\mathrm{M}_5 \perp \mathrm{M}_5} \;\simeq\; \mathbb{C}\big[\mathrm{PBr}(N)\big]
$$

$$
\mathrm{QStates}_{N\mathrm{M5}\perp\mathrm{M5}} \;\simeq\; \left\{ \rho : \mathbb{C}\big[\mathrm{PBr}(N)\big] \to \mathbb{C} \;\middle|\; \underset{\substack{U \in \\ \mathrm{PBr}(N) \circlearrowright \mathscr{H}, \\ |\psi\rangle \in \mathscr{H}}}{\exists} \;\; \underset{g \in \mathrm{PBr}(N)}{\forall} \; \rho(g) = \langle \psi | U(g) | \psi \rangle \right\}.
$$

But also the quantum states on the base moduli $B\mathrm{Pr}(N)$ gives rise to braid representations, this we discuss in §7.

6.3.4 Topological string states

Another interesting special case is the solitonic spacefilling brane on the flat fivebrane worldvolume. By the discussion in §4, its moduli under Hypothesis H form the loop space of the 3-sphere:

$$
\widetilde{\boldsymbol{\pi}}^3\big(\mathbb{R}^{1,5}_{\sqcup\{\infty\}}\big) \;\simeq\; \mathrm{Maps}^{*/}\big(\mathbb{R}^{1,5}_{\sqcup\{\infty\}}, S^3\big) \;\simeq\; S^3 \,.
$$

Hence its light-cone quantum observables (78) form the Pontrjagin algebra of the based loop space of the 3-sphere, which is known[25] to surject onto (become isomorphic under inversion of some elements to) the "quantum cohomology" of $\mathbb{C}P^1$, hence to the quantum algebra of 3-point functions of the topological string with target space $\mathbb{C}P^1$:

$$
\mathrm{QObsrvbls} \;\simeq\; H_\bullet\big(\Omega S^3\big) \;\simeq\; H_\bullet\big(\Omega \mathrm{SU}(2)\big) \twoheadrightarrow \mathrm{HQ}^\bullet\big(\mathbb{C}P^1\big)\,.
$$

[24] For pointers see: ncatlab.org/nlab/show/state+on+a+star-algebra#StatesOnGroupAlgebrasAreUnitaryRepresentations.

[25] For pointers see ncatlab.org/nlab/show/Pontrjagin+ring#ReferencesOnQuantumCohomologyAsPontrjaginRings.

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7 Resulting worldvolume CFT

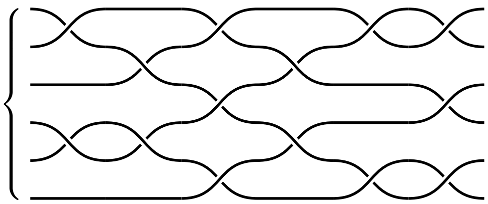
We have seen in §6.3.2 that general light-cone quantum states of M_5-brane insertions transverse to a complex plane in an ambient M_5-brane are elements of unitary representations of the pure braid group. Here we discuss the specific such representation states that are singled out by the cohomology of the transverse phase space (74), which turn out to be given by the conformal blocks of su_2-affine conformal field theory.

This result is from [Dfc1], implications are developed in [Dfc2][TQC1][TQC2].

7.1 Conformal blocks of M5-observables

The spectral prequantum line bundle. These configuration spaces are non-simply connected: their fundamental group is the pure[26] *braid group* — being the *group of motions* of the M5 $\perp$ M5-intersections around each other in the ambient M5-worldvolume:

$$\pi_1\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\,(\mathbb{C}^2)\Big) \;\simeq\; \mathrm{PBr}(n) \;=\; \Big\{ \qquad \Big\} \tag{93}$$



Hence the corresponding twisted ordinary cohomology (aka: "local system cohomology") is that whose cocycles are sections of "Eilenberg-MacLane-spectrum line bundles" pulled back from the classifying space BC_κ of a cyclic group:

$$\underset{\substack{\text{twisted cohomology}\\ \text{of configuration space}\\[2pt] \text{quantum state space}}}{H^{[\omega_1]}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\,(\mathbb{C}^2)\Big)} \;=\; \left\{ \begin{tikzcd} & \underset{\text{local coefficient bundle}}{B^k\mathbb{C}/\!\!/C_\kappa} \arrow[d] \\ \underset{\text{phase space}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\,(\mathbb{C})} \arrow[r, "\omega_1"] \arrow[ur, dashed, "\text{twisted cocyle}", "\text{quntm state}"'] & \underset{\text{prequantum line bundle}}{BC_\kappa} \end{tikzcd} \right\}_{/\mathrm{hmtp}} \tag{94}$$

(Closer analysis reveals [Dfc1, §3] that κ equals the order of the $\mathbb{A}_{\kappa-1}$-singularity at which dual D7/D3-branes are placed.)

In order to analyze these quantum states,
we may decompose the problem by:

(1.) holding fixed N of the branes,
(2) letting n mobile branes move around them.

$$\begin{tikzcd} \underset{\substack{n\text{-configuration space}\\ \text{of } N\text{-punctured plane}}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\substack{\text{fibration of}\\ \text{configuration spaces}}}{\underset{\{1,\cdots,N+n\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big)} \arrow[d, "\text{forget } n \text{ points}"] \\ * \arrow[r, "(z_1,\cdots,z_N)", "\text{pick } N\text{-configuration}"'] & \underset{\{1,\cdots,N\}}{\mathrm{Conf}}\big(\mathbb{C}^2\big) \end{tikzcd} \tag{95}$$

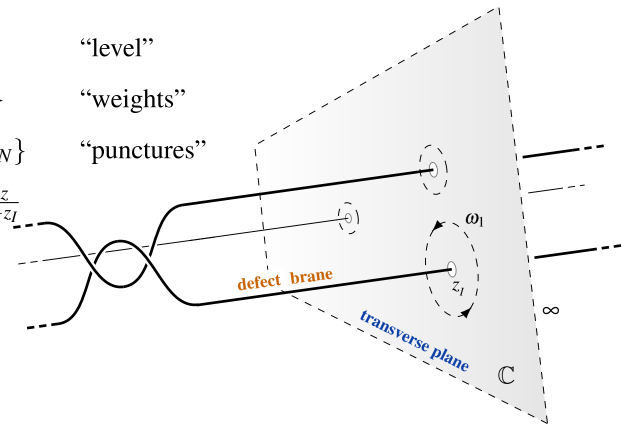
In the simple case of a single mobile brane moving – along a dashed line in (97) – among N fixed branes, we have

$$\underset{\{1,\cdots,1\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big) \;=\; \mathbb{C}\setminus\{z_1,\cdots,z_N\} \tag{96}$$

and the twist ω_1 (94) is fixed by:

[26] Our figures show im-pure braids just for ease of illustration.

$$
\begin{array}{llll}
 & \kappa & := & k+2 & \text{“level”} \\
 & \mathrm{w}_I & \in & \{0, \cdots, k\} & \text{“weights”} \\
 & z_I & \in & \{z_1, \cdots, z_N\} & \text{“punctures”} \\
\text{as} & \omega_1 & := & \sum_I -\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z}{z - z_I} &
\end{array}
\tag{97}
$$



Brane states identified with worldvolume correlators. Curiously, such sets of labels coincide with those of “conformal blocks” – namely chiral correlation functions – in the $\widehat{\mathfrak{su}_2}^k$-*conformal quantum field theory* on the punctured Riemann sphere

$$
\mathbb{C}P^2 \setminus \{z_1, \cdots, z_N, \infty\} \;\simeq\; \mathbb{C}^2 \setminus \{z_1, \cdots, z_N\}\,. \tag{98}
$$

And indeed, a well-but-not-widely known theorem called the *hypergeometric integral construction* identifies these conformal blocks of “degree=1” inside the twisted cohomology (94) of the punctured plane (98)

$$
\begin{array}{ccl}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=1}}{\mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})}} & \xhookrightarrow{\text{natural inclusion}} & \overset{\text{1-twisted deg=1 de Rham cohomology}}{H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C}\setminus\{\vec{z}\}\big), \mathrm{d} + \omega_1 \wedge\Big)} \\
 & \xhookrightarrow{\text{natural inclusion}} & \underset{\text{inner local system-twisted deg=1 K-theory of } \mathbb{A}_{\kappa-1}\text{-singularity}}{\mathrm{KU}^{1+\omega_1}\Big(\big(\mathbb{C}\setminus\{\vec{z}\}\big) \times \ast /\!\!/ C_\kappa; \mathbb{C}\Big)} \quad \text{[Dfc1, Prop. 2.16]}
\end{array}
\tag{99}
$$

and generally the conformal blocks of any degree n inside n-configuration space of points, if we set

$$
\omega_1 := \sum_{1 \leq i \leq n} \sum_I -\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z}{z - z_I} + \sum_{1 \leq i < j \leq n} \frac{2}{\kappa}\frac{\mathrm{d}z}{z^i - z^j} \quad \text{on} \quad \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\,. \tag{100}
$$

namely:

$$
\begin{array}{ccl}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=}n}{\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})}} & \hookrightarrow & \overset{\text{1-twisted deg=}n\text{ de Rham cohomology of configuration space of } n \text{ points}}{H^n\Big(\Omega^\bullet_{\mathrm{dR}}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big), \mathrm{d} + \omega_1 \wedge\Big)} \\
 & \hookrightarrow & \underset{\text{inner local system-twisted deg=}n\text{ K-theory of configurations in } \mathbb{A}_{\kappa-1}\text{-singularity}}{\mathrm{KU}^{n+\omega_1}\Big(\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big) \times \ast /\!\!/ C_\kappa; \mathbb{C}\Big)} \quad \text{[Dfc1, Thm. 2.18]}
\end{array}
\tag{101}
$$

Concretely, this inclusion is given by sending the canonical basis elements of conformal blocks to “Slater-determinant”-like expressions, as follows:

$$
\begin{array}{ccc}
\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) & \xrightarrow{\qquad\qquad} & H^n\Big(\Omega^{\bullet,0}_{\mathrm{dR}}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{\vec{z}\}\big)\Big)\big|_{\bar{\partial}=0}, \partial + \omega_1(\vec{\mathrm{w}}, \kappa) \wedge\Big) \\
f_{I_1}\cdots f_{I_n}|v^0_1 \cdots, v^0_N\rangle & \underset{\text{generators}}{\longmapsto} & \Big[\det\Big(\Big(\frac{\mathrm{w}_{I_j}}{\kappa}\frac{1}{z^i - z_{I_j}}\Big)^n_{i,j=1}\Big) dz^1 \wedge \cdots \wedge dz^n\Big] \\
\text{e.g.} \quad f_I f_J|v^0_1 \cdots, v^0_N\rangle \;=\; [\cdots, (f\cdot v^0_I), \cdots, (f \cdot v^0_J), \cdots] & \longmapsto & \Big[\frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z^1}{(z^1 - z_I)} \wedge \frac{\mathrm{w}_J}{\kappa}\frac{\mathrm{d}z^2}{(z^2 - z_J)} + \frac{\mathrm{w}_I}{\kappa}\frac{\mathrm{d}z^2}{(z^2 - z_I)} \wedge \frac{\mathrm{w}_J}{\kappa}\frac{\mathrm{d}z^1}{(z^1 - z_J)}\Big]\,.
\end{array}
\tag{102}
$$

In summary, we have derived, from Hypothesis H, that:

$$\boxed{\left.\begin{array}{c}\text{quantum states of}\\ \text{brane configurations}\\ \text{inside an M-theoretic bulk}\end{array}\right\}\ \text{are identified with}\ \left\{\begin{array}{c}\text{quantum correlators of}\\ \text{a conformal field theory}\\ \text{on their worldvolume}\end{array}\right.} \tag{103}$$

This is just the form of "holographic duality" that is expected in string/M-theory, here specifically in "Theory-$\mathscr{S}$"-compactifications of M5-branes on Riemann surfaces such as (98). Our result that $\mathfrak{su}_2$-conformal blocks appear on M5-branes compactified on a Riemann surface matches the conclusion in [Wi10, p. 22].

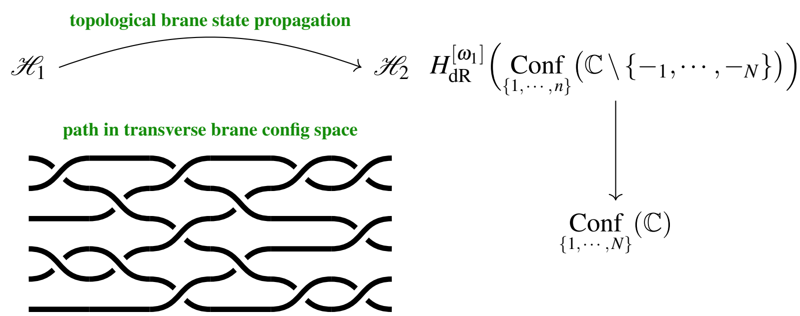
Strongly coupled holographic quantum materials. In [Dfc2] we give a detailed argument that the worldvolume CFT which we see here is that of *anyonic defects* in *topologically ordered ground states* of *crystalline quantum materials* which are in a *topological phase of matter*.

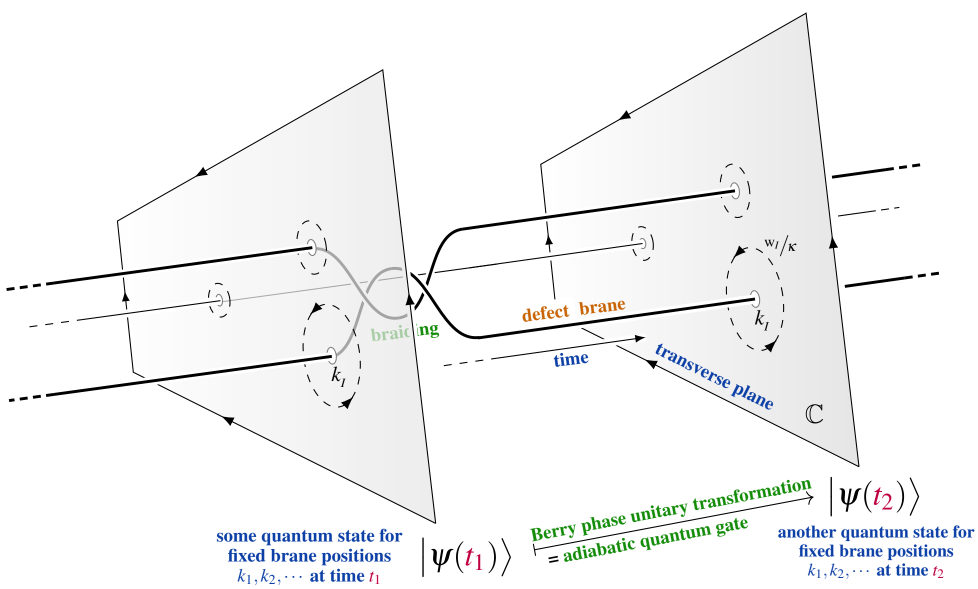
This being a strongly coupled QFT on a *small* number κ of branes, it is outside the realm of perturbative string theory and would indeed be expected to require M-theory for its holographic description (cf. p. 3).

7.2 Anyon braiding

We close by indicating how the "topological dynamics" of M5 $\perp$ M5 (their adiabatic movement in moduli space) acts on their quantum states just as expected for *quantum logic gates* in *topological quantum computers* based on *anyon braiding* – as they should by the duality (103). Detailed discussion may be found in [TQC1][TQC2].

Modular functor of M5 $\perp$ M5 **Hilbert spaces.** The fibration of configuration spaces (95) induces on its fiberwise twisted cohomology groups (101) a flat connection — called a *Gauss-Manin connection*, which on the spaces of conformal blocks restricts to the *Knizhnik-Zamolodchikov connection*.
The parallel transport of this connection computes the unitary transformations on the branes' quantum states induced by their adiabatic movement in moduli space:





Under the above holographic duality, such brane braiding translates to the braiding of anyonic defects in topologically ordered quantum materials, which is thought to potentially serve as quantum logic gates for topological quantum computers.

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