Introduction to Hypothesis H

Hisham Sati[*†] and Urs Schreiber[*]

December 6, 2023

Abstract

The key open question of contemporary mathematical physics is elucidation of the currently elusive fundamental laws of strongly-interacting "non-perturbative" quantum states, including bound states as mundane as nucleons but more generally of quarks confined inside hadrons, as well as strongly-correlated ground states of topologically ordered quantum materials.

The seminal strategy of regarding such systems as located on branes inside a higher dimensional string-theoretic spacetime (the "holographic principle") shows all signs of promise but has been suffering from the ironic shortcoming that also string theory has only really been defined perturbatively. However, string theory exhibits a web of *hints* towards the nature of its non-perturbative completion, famous under the working title "M-Theory" but still elusive. Thus, mathematically constructing M-theory should imply a mathematical understanding of quantum brane worldvolumes which should solve non-perturbative quantum physics: the M-strategy for attacking the Millennium Problem.

After a time of stagnation in research towards M-theory, we have recently formulated and extensively tested a hypothesis on the precise mathematical nature of at least a core part of the theory: We call this *Hypothesis H* since it postulates that M-branes are classified by Co-*H*omotopy-theory in much the same way that D-branes are expected to be classified by *K*-theory (a widely held but just as conjectural belief which might analogously be called *Hypothesis K*). In fact, stabilized coHomotopy is equivalently the algebraic K-Theory over "$\mathbb{F}_1$", the "absolute base field with one element". Last not least, coHomotopy is equivalent to framed Cobordism cohomology.

In these lecture notes, we try to give an introduction to (1.) the motivation and (2.) some consequences of Hypothesis H, assuming an audience with a little background in electromagnetism, differential geometry and algebraic topology.

Contents

[*] Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

[†] The Courant Institute for Mathematical Sciences, NYU, NY

The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

The latest version of this document is available at: ncatlab.org/schreiber/show/Introduction+to+Hypothesis+H.

These are notes under development,
prepared for a series of talks and lectures;
parts are still not more than a slide show.

The first half aims to be elementary explanation of
Hypothesis H as a good question to ask about physics:
whether it is right or wrong, it deserves checking.

The second half explains evidence
that Hypothesis H is in fact correct
and some insights gained from it.

For comprehensive referencing see:
ncatlab.org/schreiber/show/Hypothesis+H

1 Brane charge quantization

Anomaly cancellation and flux quantization. Secretly, much of contemporary theory building in theoretical physics is a sophisticated process of trial, error and improvisation: The trials are Lagrangian densities ("action functionals"), the errors are "anomalies" obstructing their consistent quantization, and the improvisation is the invention of add-on rules to "cancel" the anomalies. While there is a sense of accomplishment in the community for identifying and cancelling anomalies (typically a demanding task) we should see it for what it is: *Anomaly cancellation is the patching-up of broken theories.* This can (and certainly has been) useful for exploring the space of physical theories, but it seems implausible that truly fundamental theories will come to us in broken form incrementally patched. Instead, eventually we want to understand how to construct *anomaly-free* and hence well-defined quantum theories right away.

Dirac charge quantization in integral cohomology. An early example of what we may recognize as anomaly cancellation is *Dirac's charge quantization* (p. **??**) – and we are going to promote this to a rather general principle of *brane charge quantization.* In modern paraphrase, Dirac observed:

1. that the worldline theory of the electron in the background of a magnetic monopole has, in general, a quantum anomaly,
2. that this anomaly vanishes if – hence is cancelled by demanding that – electromagnetic charges are quantized in integer multiples of a unit charge, or more precisely that the *fluxes* witnessing these charges are quantized in *integral cohomology.* The latter is really the fundamental condition, whence one also speaks of *flux quantization.*

More in detail, the anomaly cancellation demand — an *ad hoc* addition to classical Maxwell theory — is that the Faraday tensor 2-form F_2, in addition to satisfying Maxwell's equations (**??**), also satisfies the constraint that its class in de Rham cohomology — which under de Rham's theorem is identified with $\mathbb{R}$-valued cohomology — is in fact in the image of $\mathbb{Z}$-valued cohomology:

$$
\begin{array}{ccc}
\underset{\text{in integral cohomology}}{\overset{\substack{\text{quantized}\\ \text{magnetic charges}}}{H^2(X;\mathbb{Z})}} & \longrightarrow & \underset{\text{in real/de Rham cohomology}}{\overset{\substack{\text{classical}\\ \text{magnetic charges}}}{H^2(X;\mathbb{R}) \;\simeq\; H^2_{\mathrm{dR}}(X^4)}}
\\[10pt]
[q] & \underset{\text{lift}}{\overset{\text{charge quantization}}{\leftsquigarrow}} & [F_2]
\end{array}
\tag{1}
$$

This cohomology operation of *extension-of-scalars* we are also going to refer to as the *character map* in integral cohomology (think of the differential forms in its image as "real characterizations", if you wish, of integral cohomology classes).

Notice that in general this map is not an injection (its kernel is the *torsion subgroup* of integral classes). Eventually we understand charge quantization not just as the *condition* that physical fields be in the image of this map, but we regard the actual physical fields as containing *extra structure* consisting of a choice of pre-image through this map (and yet a little more, see §1.4).

Spin-/String-structure as charge quantization in non-abelian cohomology. Dirac's argument only concerns the charge of the electron. When one also considers the spin of the electron then its worldline theory has another anomaly, which is cancelled by equipping the background spacetime with *spin-structure* (discussed this way in [Wi85, p. 65-68]). An analogous argument shows that *spinning strings* have an anomaly in their worldsheet theory which may be cancelled by equipping the background spacetime with *string-structure* (cf. [Bu11][SSS12]).

Here we are going to understand [Char, §2] phenomena such as Spin- and String-structures as examples of *non-abelian cohomology* with coefficients in a non-abelian group G ([Grothendieck55, §V][Fr1957], see also [We16, §7]) or non-abelian 2-groups etc., thus conceptually unifying them with abelian cohomology such as in (1):

$$
\underset{\substack{\text{non-abelian cohomology}\\ \text{in degree 1}}}{H^1(X;G)} \;\simeq\; \underset{\substack{\text{homotopy classes of maps}\\ \text{into classifying space}}}{\pi_0 \mathrm{Maps}(X;BG)} \;\simeq\; \underset{\substack{\text{isomorphism classes of}\\ \text{principal bundles}}}{G\mathrm{PrinBund}(X)_{/\sim}}
$$

This way, we may understand the anomaly cancellation of the spinning electron by ambient spin-structure as of the same general cohomological form as Dirac's charge quantization (1):

$$
\begin{array}{ccccc}
\underset{\text{in String-cohomology}}{\overset{\substack{\text{“quantized”}\\ \text{gravitational charge}}}{H^1\big(X;\mathrm{String}(1,d)\big)}} & \longrightarrow & \underset{\text{in Spin-cohomology}}{\overset{\substack{\text{“quantized”}\\ \text{gravitational charge}}}{H^1\big(X;\mathrm{Spin}(1,d)\big)}} & \longrightarrow & \underset{\text{in nonabelian O-cohomology}}{\overset{\text{gravitational charge}}{H^1\big(X;\mathrm{O}(1,d)\big)}}
\\[10pt]
[\widehat{\widehat{\omega}}] & \underset{\text{lift}}{\overset{\text{string anomaly cancellation}}{\leftsquigarrow}} & [\widehat{\omega}] & \underset{\text{lift}}{\overset{\text{spin anomaly cancellation}}{\leftsquigarrow}} & [\omega]
\end{array}
$$

Cohomology rules. This shows that, at least in key examples, "anomaly cancellation" amounts to understanding that fields/fluxes which a priori seem to be given by differential forms actually need to be *flux quantized* by promoting them to cocycles in possibly non-abelian generalized cohomology theories. Together with the observation in **??** that pre-geometric equations of motion always form a kind of non-abelian de Rham cohomology, this paints a compelling picture that quantum fields want to be understood as pre-geometric cocycles in non-abelian generalized cohomology theories.

We now explain – in survey of [Char][1] – that:

§1.1 Bianchi identities characterize flux densities as closed differential forms valued in nilpotent L_∞-algebras.

§1.2 Non-abelian de Rham cohomology is the target of the character map on higher non-abelian cohomology.

§1.3 Higher non-abelian cohomology theories thus serve as the *flux quantization laws* for higher fluxes.

Schematically:

$$
\begin{array}{ccc}
\underset{\text{cohomology}}{\overset{\text{non-abelian}}{}} & & \underset{\text{de Rham cohomology}}{\overset{\text{non-abelian}}{}} \\
A(X) & \xrightarrow[\mathrm{ch}_A]{\text{non-abelian character}} & H_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big) \\
\big[\mathcal{F}\big] & \longmapsto & \Big[\big(F^{(a)}_{r_a}\big)_{1 \,\leq\, a \,\leq\, \dim[\pi_\bullet(A),\mathbb{R}]}\Big] \\
\underset{A\text{-quantized flux}}{\overset{\text{class of}}{}} & & \underset{\text{underlying flux densities}}{\overset{\text{class of}}{}}
\end{array}
$$

§1.4 These flux quantization laws determine the moduli ∞-stack of on-shell higher gauge potentials (phase space).

§1.5 Twisted versions of these cohomology theories encode brane intersections.

This follows the seminal argument of *Dirac charge quantization* for electromagnetism [Di31] (review in [Al85][Fr97, §16.4e] [Fr00, §2]) and generalizes suggestions for charge quantization in higher gauge theories [Fr00][HS05] to the case of *non-abelian* ("unstable") fluxes, such as the C-field in 11d supergravity (**??**).

[1] When we refer to equation-, definition-, proposition-, page-numbers in [Char] we refer to the version published by World Scientific — see ncatlab.org/schreiber/show/The+Character+Map#PublishedVersion — which differs from the numbering in the arXiv version (otherwise the content is the same).

1.1 Biachi identities characterize flux densities as closed L_∞-valued differential forms

We explain (7) how higher Bianchi identities (**??**) correspond to nilpotent finite-type L_∞-algebras.

The notion of *strong homotopy Lie algebra* or *L_∞-algebras* [LS93][LM95] is finally becoming more widely appreciated in physics, where they appear in various guises. Here we are mainly concerned with L_∞-algebras which are (i) nilpotent, (ii) connective (iii) of finite type, in their *joint* incarnation as higher Whitehead brackets *and* as higher flux density coefficients (all to be explained in a moment):

$$
\begin{tikzcd}
 & \boxed{\begin{array}{l}\text{Higher Whitehead brackets}\\ \text{of classifying spaces in}\\ \text{rational homotopy theory}\end{array}} \arrow[dd, leftrightarrow] \\
\boxed{\begin{array}{l}\text{Strong homotopy Lie algebras}\\ \text{aka } L_\infty\text{-algebras}\\ \text{(connective, nilpotent, finite-type)}\end{array}} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\
 & \boxed{\begin{array}{l}\text{Coefficients of higher flux densities}\\ \text{satisfying generalized Bianchi identities}\end{array}}
\end{tikzcd}
\tag{2}
$$

The full triality expressed above, classically familiar as its separate aspects are to the respective experts, may still not be widely appreciated, but is key to our discussion here:

- the top correspondence refers to what we may call the *fundamental theorem of dg-algebraic rational homotopy theory* as reviewed in [Char, §5],
- the right correspondence may, with some hindsight, be recognized as the "FDA"-method in supergravity [vN83][DF82] [CDF91], as explained in [FSS15][FSS18][HSS19], review in [FSS19][Char].

In particular, this means that L_∞-algebras as used here are *not* directly to be understood as generalizations of the gauge Lie algebras familiar from Yang-Mills theory (which are coefficients of the potential fields, instead of their flux densities), though they are intricately related. We expand on this point in Remark ... below.

L_∞-algebras. Since we are assuming L_∞-algebras to be connective and of finite type (meaning that they are degreewise finite-dimensional and concentrated in non-negative degrees) we may *define* them through their CE-algebras in the following manner, which is not only convenient for dealing with the otherwise intricate sign rules for L_∞-algebras, but also essential to their alternative perspectives in (2):

Chevalley-Eilenberg algebras of Lie algebras. Namely, recall first that for $\mathfrak{g}$ a finite-dimensional Lie algebra (our ground field is the real numbers, throughout) with Lie bracket a skew-symmetric linear map $[-,-] \,:\, \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, its linear dual vector space $\mathfrak{g}^*$ is equipped with the dual bracket $[-,-]^* \,:\, \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ which extends uniquely to a degree=1 derivation on the graded Grassmann algebra $\wedge^\bullet \mathfrak{g} := \bigoplus_{n \in \mathbb{N}} \underbrace{\mathfrak{g}^* \wedge \cdots \wedge \mathfrak{g}^*}_{n \text{ factors}}$:

$$
[-,-] \,:\, \mathfrak{g}\wedge\mathfrak{g} \to \mathfrak{g}
\qquad \leftrightarrow \qquad
\begin{tikzcd}
\wedge^\bullet \mathfrak{g}^* \arrow[r, "\mathrm{d}_{[-,-]}", "\text{deg=1 derivation}"'] & \wedge^\bullet \mathfrak{g}^* \\
\wedge^1 \mathfrak{g}^* \arrow[u, hook] \arrow[r, "{[-,-]^*}"] & \wedge^2 \mathfrak{g}^* \arrow[u, hook] .
\end{tikzcd}
$$

One readily checks that this derivation squares to zero iff the bracket satisfies its Jacobi identity(!):

$$
\text{Jacobi identity for } [-,-] \qquad \Leftrightarrow \qquad \mathrm{d}_{[-,-]} \circ \mathrm{d}_{[-,-]} \;=\; 0 \;.
$$

The resulting differential graded-commutative (dgc) algebra $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ is known as the *Chevalley-Eilenberg complex* $\mathrm{CE}(\mathfrak{g})$ whose cochain cohomology computes the Lie algebra cohomology of $\mathfrak{g}$ (with trivial coefficients); but the key point at the moment is that its construction is a *fully faithful* contravariant functor embedding the category of finite-dimensional Lie algebras into the opposite of that of dgc-algebras:

$$
\begin{array}{ccc}
\mathrm{LieAlg}^{\mathrm{fdim}} & \xhookrightarrow{\;\;\mathrm{CE}(-)\;\;} & \mathrm{dgcAlg}^{\mathrm{op}} \\
\big(\mathfrak{g},\, [-,-]\big) & \longmapsto & \big(\wedge^\bullet\, \mathfrak{g}^*,\, \mathrm{d}_{[-,-]}\big)\,.
\end{array}
\tag{3}
$$

L_∞-algebras of finite type. With Lie algebras viewed as special dgc-algebras this way (3), it is *immediate* to generalize them to the case where $\mathfrak{g}$ may be a graded vector space of degreewise finite dimension ("of finite type"): Namely, with $\mathfrak{g}^\vee$ denoting the degreewise dual graded space

$$
(\mathfrak{g}^\vee)_n \;\equiv\; (\mathfrak{g}_n)^*
$$

we understand

$$\wedge^\bullet \mathfrak{g}^\vee \;\equiv\; \mathrm{Sym}(\mathfrak{g}^\vee[1])$$

as the free graded-symmetric algebra on the shift of $\mathfrak{g}^\vee$ up by one degree and otherwise use *verbatim* the same construction:

A degree=1 derivation on $\wedge^\bullet \mathfrak{g}^\vee$ is determined by its restriction to $\wedge^1 \mathfrak{g}^\vee$, where it is a sum of co-n-ary linear maps, whose linear duals we may think of as n-ary degree=(-1) brackets on $\mathfrak{g}[1]$:

$$\begin{array}{cccccc} [-] & : & \mathfrak{g}[1] & \longrightarrow & \mathfrak{g}[1] \\ [-,-] & : & \mathfrak{g}[1]\wedge\mathfrak{g}[1] & \longrightarrow & \mathfrak{g}[1] \\ [-,-] & : & \mathfrak{g}[1]\wedge\mathfrak{g}[1]\wedge\mathfrak{g}[1] & \longrightarrow & \mathfrak{g}[1] \\ \vdots \end{array} \quad\leftrightarrow\quad \begin{tikzcd} \wedge^\bullet \mathfrak{g}^\vee \arrow[rr, "\mathrm{d}_{[-,\cdots,-]}"] && \wedge^\bullet \mathfrak{g}^\vee \arrow[d, equal] \\ \wedge^1 \mathfrak{g}^\vee \arrow[u, hook] \arrow[rr, "{[-]^* + [-,-]^* + [-,-,-]^* + \cdots}"'] && \bigoplus_{n\in\mathbb{N}} \wedge^n \mathfrak{g}^\vee \end{tikzcd} \tag{4}$$

Now the simple condition that $\mathrm{d}_{[-\cdots,-]}$ be a differential implies a tower of conditions on these brackets, which generalize the Jacobi identity on an ordinary Lie algebra:

$$\begin{array}{c}\text{higher Jacobi identity for} \\ [-],\,[-,-],\,[-,-,-],\,\cdots\end{array} \quad\Leftrightarrow\quad \mathrm{d}_{[-,\cdots,-]}\circ \mathrm{d}_{[-,\cdots,-]} \;=\; 0\,.$$

These are exactly the conditions that make $\big(\mathfrak{g},[-],[-,-],[-,-,-],\cdots\big)$ an L_∞-algebra.

In other words, we may identify L_∞-algebras of finite type as the formal dual to dgc-algebras of the form (4) (cf. [SSS09, Def. 13][FSSt12, §4.1][Char, (4.27)]):

$$\begin{array}{ccc} L_\infty\mathrm{Alg}^{\mathrm{ftype}} & \xhookrightarrow{\;\;\mathrm{CE}(-)\;\;} & \mathrm{dgcAlg}^{\mathrm{op}} \\ \big(\mathfrak{g},\,[-],\,[-,-],\,[-,-,-],\,\cdots\big) & \longmapsto & \big(\wedge^\bullet\,\mathfrak{g},\,\mathrm{d}_{[-,\cdots,-]}\big) \end{array} \tag{5}$$

Closed L_∞-algebra valued differential forms. The perspective (5) makes it immediate to define closed L_∞-algebra valued differential forms; these are the dg-algebra homomorphism from their CE-algebras into de Rham algebras (cf. [SSS09, (261)][Def. 6.1][Char], aka "Maurer-Cartan elements" in de Rham algebras):

$$\left.\begin{array}{l}\mathfrak{a}\,\in\,L_\infty\mathrm{Alg}^{\mathrm{ftype}}, \\ X\,\in\,\mathrm{SmthMfd}\end{array}\right\} \quad\vdash\quad \Omega_{\mathrm{dR}}(X;\,\mathfrak{a})_{\mathrm{clsd}} \;\equiv\; \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{g}),\,\Omega_{\mathrm{dR}}(X)\big)\,. \tag{6}$$

Bianchi identities and L_∞-algebras. The structure of a system of higher Bianchi identities (**??**) is all encoded in the polynomials $P^{(a)}$. At the same time, the de Rham condition $\mathrm{d}^2 = 0$ imposes constraints on systems of polynomials that may arise. With these constraints, the coefficients of $P^{(a)}$ in the expansion

$$P^{(a)}\big(\{F^{(b)}\}\big) \;=\; \sum_{n=0}^{\infty} P^{(a)}_{a_1\cdots a_n} F^{(a_1)}\wedge\cdots\wedge F^{(a_n)}$$

are equivalently the structure constants of an L_∞-algebra $\mathfrak{a}$ (5), namely the unique L_∞-algebra whose Maurer-Cartan equation (the higher flatness condition) is the given Bianchi identities:

Flux densities satisfying Bianchi identities are closed L_∞-algebra-valued differential forms.

$$\underbrace{\Omega_{\mathrm{dR}}\big(\underset{\text{insert spacetime manifold here}}{-};\,\mathfrak{a}\big)_{\mathrm{clsd}} \;=\; \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}(\mathfrak{a}),\,\Omega^\bullet_{\mathrm{dR}}(-)\Big)}_{\text{sheaf of closed } L_\infty\text{-algebra-valued differential forms}} \;=\; \Big\{\underbrace{F^{(a)}\,\in\,\Omega^{r_a}_{\mathrm{dR}}(-)}_{\text{systems of flux densities}} \;\Big\vert\; \underbrace{\mathrm{d}\,F^{(a)} \;=\; P^{(a)}\Big(\big\{F^{(b)}\big\}_{b\leq a}\Big)}_{\text{satisfying these Bianchi identities}}\Big\}_{1\leq a\leq a_{\mathrm{max}}}$$

$$\updownarrow$$

$$\underbrace{\mathrm{CE}(\,\mathfrak{a}\,)}_{\text{Chevalley-Eilenberg algebra of } L_\infty\text{-algebra}} \;=\; \underbrace{\mathbb{R}\Big[\big\{f^{(a)}\big\}_{1\leq a\leq a_{\mathrm{max}}}\Big]}_{\text{free differential graded-commutative algebra on these graded generators}}\Big/\underbrace{\Big(\mathrm{d}\,f^{(a)} \;=\; P^{(a)}\Big(\big\{f^{(b)}\big\}_{b\leq a}\Big)\Big)}_{\text{satisfying these differential relations}} \tag{7}$$

$$\updownarrow$$

$$\underbrace{\mathfrak{a}}_{L_\infty\text{-algebra}} \;=\; \underbrace{\mathbb{R}\Big\langle\big\{f_{(a)}\big\}_{1\leq a\leq a_{\mathrm{max}}}\Big\rangle}_{\text{graded vector space spanned by these graded generators}} \quad \underbrace{\big[f_{(a_1)},\cdots,f_{(a_n)}\big] \;=\; P^{(a)}_{a_1\cdots a_n}\,f_{(a)}}_{\text{equipped with these higher Lie brackets}}$$

Non-abelian de Rham cohomology. Say that a pair $\{F_0^{(a)}\}_a,\ \{F_1^{(a)}\}_a \in \Omega_{\mathrm{dR}}\big(X;\, \mathfrak{a}\big)_{\mathrm{clsd}}$ of closed $\mathfrak{a}$-valued differential forms (6) are *cohomologous* if they can can be *deformed into each other*, hence if they are *concordant*, in that they are boundary data of a closed $\mathfrak{a}$-form on the cylinder $X \times [0,1]$ over X:

$$
\overset{\text{deformation of flux densities}}{\big\{F_0^{(a)}\big\}_{1\le a\le a_{\max}} \sim \big\{F_1^{(a)}\big\}_{1\le a\le a_{\max}}}
\quad :\Leftrightarrow \quad
\exists
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \\
\Omega^\bullet_{\mathrm{dR}}\big(X \times [0,1]\big) \arrow[u, "i_0{}^*"] \arrow[d, "i_1{}^*"'] & & \mathrm{CE}(\mathfrak{a}) \arrow[ull, "\{F_0{}^{(a)}\}_{1\le a\le a_{\max}}"'] \arrow[ll, "\{\widehat{F}^{(a)}\}_{1\le a\le a_{\max}}"'] \arrow[dll, "\{F_1{}^{(a)}\}_{1\le a\le a_{\max}}"] \\
\Omega^\bullet_{\mathrm{dR}}(X) & &
\end{tikzcd}
$$

This is an equivalence relation whose equivalence classes we call
the **closed $\mathfrak{a}$-valued non-abelian de Rham cohomology** of X [Char, Def. 6.1]:

$$
\underset{}{\overset{\substack{\text{deformation class}\\ \text{of flux densities}}}{\Big[\big\{F^{(a)}\big\}_{1\,\le\, a\,\le\, a_{\max}}\Big]}}
\ \in\
\overset{\substack{\mathfrak{a}\text{-valued}\\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}\big(X;\, lA\big)}
\ :=\
\left\{
\begin{array}{c}
\text{cocycle (dga-hom)} \\
\begin{tikzcd}
& \big\{F_0^{(a)}\big\} \arrow[dl, dashed] \arrow[d, Rightarrow, dashed] & \\
\Omega^\bullet_{\mathrm{dR}}(X) & \substack{\text{coboundary}\\ \text{(concordance)}} \arrow[d, Rightarrow, dashed] & \mathrm{CE}(lA) \arrow[ul, dash, dashed] \arrow[dl, dash, dashed] \\
& \big\{F_1^{(a)}\big\} \arrow[ul, dashed] &
\end{tikzcd} \\
\text{another cocycle}
\end{array}
\right\}_{\Big/\sim}
.
$$

1.2 Nonabelian de Rham cohomology is target of character map on nonabelian cohomology

Classifying spaces for cohomology. Notice that reasonable cohomology theories have *classifying spaces* (cf. [Char, §2]):

$$
\begin{array}{ccl}
\overset{\text{ordinary cohomology}}{H^n(X;\mathbb{Z})} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{Eilenberg-MacLane space}}{K(\mathbb{Z},n)}\big) \\[1.5em]
\overset{\text{nonabelian cohomology}}{H^1(X;G)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{classifing space of principal } G\text{-bundles}}{BG}\big) \\[1.5em]
\overset{\text{topological K-theory}}{K^0(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{space of Fredholm operators}}{\mathrm{Fred}_{\mathbb{C}}}\big) \\[1.5em]
\overset{\substack{\text{Whitehead} \\ \text{generalized cohomology}}}{E^n(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{stage in spectrum of spaces}}{E_n}\big) \\[1.5em]
\overset{\text{coHomotopy}}{\pi^n(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{sphere}}{S^n}\big)
\end{array}
\tag{8}
$$

$$
\text{hence consider:}\qquad \overset{\substack{\text{higher} \\ \text{non-abelian} \\ \text{cohomology}}}{A(X)} \;:=\; \pi_0\,\mathrm{Maps}\big(X,\; \overset{\text{any space}}{A}\big) \;=\; \left\{ \begin{tikzcd} X \arrow[rr, bend left, dashed, "\substack{\text{cocycle (map)} \\ \mathcal{F}_0}"] \arrow[rr, bend right, dashed, "\substack{\mathcal{F}_1 \\ \text{another cocycle}}"'] \arrow[rr, phantom, "\substack{\text{coboundary} \\ \text{(homotopy)}}"] & & A \end{tikzcd} \right\}\Big/\sim
$$

Reduced cohomology and solitonic charges. For the charge quantization of *solitonic* branes (in **??**) one needs to implement in cohomology theory their *localization* in space (cf. **??**) which forces their fluxes to *vanish at infinity*.

We may observe that a formalization of this phenomenon is already captured by the standard notion of *reduced cohomology* on *pointed spaces* if we regard the basepoint of a domain space as its point-at-infinity and the basepoint of a coefficient space as its zero-element

$$
\begin{tikzcd}
\text{domain space} & X \arrow[rr, dashed, "\text{flux cocycle}", "\text{in reduced } A\text{-cohomology}"'] & & A & \text{coefficient space} \\
\substack{\text{basepoint is} \\ \text{point at } \infty} & \{\infty\} \arrow[u, hook] \arrow[rr, "\text{flux vanishes at infinity}"] & & \{0\} \arrow[u, hook] & \substack{\text{basepoint is} \\ 0\text{-element}}
\end{tikzcd}
$$

$$
\overset{\substack{\text{reduced} \\ \text{higher} \\ \text{non-abelian} \\ \text{cohomology}}}{\widetilde{A}(X)} \;:=\; \pi_0\,\mathrm{Maps}^{*/}\big((X,\infty),\,(A,0)\big) \;=\; \left\{ \begin{tikzcd} X \arrow[rr, bend left, dashed, "\substack{\text{cocycle (map)} \\ \mathcal{F}_0}"] \arrow[rr, bend right, dashed, "\substack{\mathcal{F}_1 \\ \text{cocycle}}"'] \arrow[rr, phantom, "\substack{\text{coboundary} \\ \text{(homotopy)}}"] & & A \\ \{\infty\} \arrow[u, hook] \arrow[rr, "\text{vanishing at } \infty"'] & & \{0\} \arrow[u, hook] \end{tikzcd} \right\}\Big/\sim
$$

Notice that the point at infinity may or may not be reachable by continuous paths in the space:

$$
X \text{ a plain space} \qquad \vdash \qquad
\begin{array}{lll}
X_{\sqcup\{\infty\}} & \text{disjoint point adjoined} & \text{paths starting in cnctd } X \text{ never reach } \infty \\
X_{\cup\{\infty\}} & \text{one-point-compactification} & \text{paths starting in cnctd } X \text{ may reach } \infty
\end{array}
\tag{9}
$$

Given a pointed space, we may first delete the point at infinity and then adjoint it back disjointly, making it un-reachable:

$$
(X,\infty) \text{ a pointed space} \qquad \vdash \qquad \big(X \setminus \{\infty\}\big)_{\sqcup\{\infty\}} \quad \text{make } \infty \text{ un-reachable}
\tag{10}
$$

Plain cohomology (8) is subsumed in reduced cohomology as the case where the point at infinity is unreachable (9)

$$A(X) \;\simeq\; \widetilde{A}\big(X_{\sqcup\{\infty\}}\big)$$

and making ∞ unreachable (10) projects reduced into plain cohomology.

The charges that thus disappear existed only due to their localization, hence are *purely solitonic*,
while those that do not vanish at ∞ are *purely singular* (cf. **??**):

$$\begin{array}{ccc}
\underset{\text{pntd space}}{(X,\infty)} & \vdash & \underset{\text{purely solitonic charges}}{\ker\big(A(\epsilon\iota_X)\big)} \hookrightarrow \underset{\text{reduced cohomology}}{\widetilde{A}(X)} \xrightarrow{A(\epsilon\iota_X)} \underset{\text{reduced cohomology for disjoint }\infty}{\widetilde{A}\Big(\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}}\Big)} = \underset{\text{plain cohomology}}{A(X\setminus\{\infty\})} \twoheadrightarrow \underset{\text{purely singular charges}}{\mathrm{coker}\big(\widetilde{A}(\epsilon\iota_X)\big)} \\
& & X \xleftarrow{\;\epsilon\iota_X\;} \underset{\text{make }\infty\text{ unreachable}}{\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}}}
\end{array} \tag{11}$$

(...)

Notice the mapping space adjunction

$$\mathrm{Maps}^{*/}\Big(X,\,\mathrm{Maps}^{*/}(Y,\,Z)\Big) \quad\simeq\quad \mathrm{Maps}^{*/}\big(X\wedge Y,\,Z\big) \quad\simeq\quad \mathrm{Maps}^{*/}\Big(Y,\,\mathrm{Maps}^{*/}\big(X,\,Z\big)\Big) \tag{12}$$

With generalized/non-abelian cohomology theory understood via classifying spaces this way, the *fundamental theorem of dg-algebraic rational homotopy theory* provides the generalization of the Chern character:

(1) The rational homotopy type of A is encoded in the Whitehead L_∞-algeba $\mathfrak{l}A$ of its loop ∞-group ΩA,

(2) The non-abelian character map ch_A approximates A-cohomology classes by $\mathfrak{l}A$-valued de Rham classes.

We now say this in more detail:

(1) The **rational homotopy type** of any simply-connected (classifying) space A (of finite rational type) is encoded in its *Whitehead bracket L_∞-algebra* $\mathfrak{l}A$.

Namely, first of all, every connected homotopy type A is equivalently the classifying space of its loop ∞-group ΩA (cf. [Char, Prop. 2.2]):

$$A \;\simeq\; B(\Omega A)\,. \tag{13}$$

It is in this way that the generalized cohomology theory classified any such A is always a non-abelian higher gauge theory of sorts, with underlying homotopy type of the gauge ∞-group being that of ΩA.

Moreover, while ΩA here is a "geometrically discrete" ∞-group (no smooth structure, at this point), if A is nilpotent (eg. simply connected) and of finite rational type, then homotopy theory nevertheless allows to assign a homotopy Lie algebra (L_∞-algebra) to it, namely its *Whitehead bracket L_∞-algebra* (cf. [Char, Prop. 5.11]) whose underlying graded vector space is that of the rationalized homotopy groups of the loop group ΩA

$$(\mathfrak{l}A)_\bullet \;\simeq\; \pi_\bullet(\Omega A)\otimes_{\mathbb{Z}}\mathbb{R} \tag{14}$$

whose binary brackets are the classical Whitehead brackets, and whose Chevalley-Eilenberg algebra is the *minimal Sullivan model* of A.

Examples:

homotopy type (topological space) A	**Sullivan model** ("FDA") $\mathrm{CE}(\mathfrak{l}A)$	**Quillen model** (Whitehead L_∞-algebra) $\mathfrak{l}A$	compare
S^4	$\mathbb{R}\begin{bmatrix}\omega_7,\\ \omega_4\end{bmatrix}\Big/\left(\begin{array}{ll}\mathrm{d}\,\omega_7 & = -\tfrac{1}{2}\omega_4\wedge\omega_4\\ \mathrm{d}\,\omega_4 & = 0\end{array}\right)$	$\mathbb{R}\left\langle\begin{array}{c}v_6,\\ v_3\end{array}\right\rangle,\;\;[v_3,v_3]=v_6$	[SV22b, §2.2]

(15)

This Whitehead L_∞-algebra actually fully determines the rational homotopy type of A. For example:

$$A \xmapsto{\text{rationalization}} \underset{\text{Chevalley-Eilenberg algebra of its Whitehead } L_\infty\text{-algebra}}{\overset{\text{Sullivan model of } A}{\mathrm{CE}\big(\mathfrak{l}A\big)}} \longleftrightarrow \overset{\text{Whitehead } L_\infty\text{-algebra}}{\mathfrak{l}A} = \text{“infinitesimal version } \mathrm{Lie}(\Omega A) \text{ of loop group”}$$

Eilenberg-MacLane space $K(\mathbb{Z}, n)$	$\longmapsto$	$\mathbb{R}\big[c_n\big]\big/\big(\mathrm{d}\, c_n = 0\big)$
classifying space of principal G**-bundles** BG	$\longmapsto$	**invariant polynomials on the Lie algebra** $\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, \mathrm{d} = 0\big)$
space of Fredholm operators $\mathrm{Fred}_{\mathbb{C}}$	$\longmapsto$	$\mathbb{R}\Big[\big\{ f_{2k} \,\vert\, k \in \mathbb{N}\big\}\Big]\Big/\big(\mathrm{d}\, f_{2k} \;=\; 0\big)$
stage in spectrum of spaces E_n	$\longmapsto$	$\Big(\mathrm{Sym}\big(\mathrm{Hom}_{\mathbb{Z}}(\pi_\bullet(E_n),\, \mathbb{R})\big), \mathrm{d} = 0\Big)$
even dimensional sphere S^{2k}	$\longmapsto$	$\mathbb{R}\big[g_{2k},\, g_{4k-1}\big]\Big/\left(\begin{array}{lcl} \mathrm{d}\, g_{4k-1} &=& g_{2k} \wedge g_{2k} \\ \mathrm{d}\, g_{2k} &=& 0 \end{array}\right)$

This is an equivalence between:

weak rational homotopy types of nilpotent connected **spaces** with degreew. fin-dim rational cohomology

$\wr\wr$

quasi-isomorphism types of nilpotent connective L_∞**-algebras** on degreew. fin-dim $\mathbb{Q}$-vector spaces

here shown/used after extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$

notice the difference to L_∞-algebras of *gauge potentials*: $\mathfrak{su}(n)$, $\mathfrak{string}(n)$, ... which are *not nilpotent*

here: L_∞-algebras of gauge *fields/fluxes*

Remark – The gauge-theoretic nature of Whitehead L_∞-alebras. The second example above makes manifest how the homotopical Whitehead L_∞-algebra differs from the actual gauge Lie algebra (that will appear later, as we pass to differential refinements of A-cohomology theory): If G is a Lie group like $SU(n)$, then the Whitehead bracket L_∞-algebra $\mathfrak{l}BG$ is not the actual Lie algebra $\mathfrak{g}$ but may be thought of as consisting dually of all its abelian Lie algebra cocycles, starting with the ordinary Chern-Simons cocycle $\langle -, [-,-]\rangle$ and so on. In other words, $\mathfrak{l}BG$ are the coefficients for all the higher curvature invariants of a $\mathfrak{g}$-values connection (the Chern classes in the case of SU(2)).

Since in this classical case of ordinary gauge field the curvature coefficients are all just shifted copies of $\mathfrak{u}(1)$ they have traditionally received little to no attention as such.

But for higher gauge fields with non-linear self-coupling – such as the supergravity C-field – the curvature invariants will have coefficients in non-trivial L_∞-algebras.

This is the general pattern: The nilpotent L_∞-algebras $\mathfrak{l}A$ here are to be thought of not as the fundamental gauge Lie algebras, but as their homotopical shadow in their higher curvature invariants. Concretely, in the discussion of differential refinement below, $\mathfrak{l}A$ is the L_∞-algebra in which the higher *curvature forms* take values, while the more familiar Lie algebra in which the actual gauge potentials take values is more implicitly encoded in the homotopy that makes the differential cohomology square (see below...).

(2) The **higher non-abelian character map** ch_A universally approximates A-cohomology classes by $\mathfrak{l}A$-valued de Rham classes:

$$
\begin{array}{ccc}
\underset{\textbf{cohomology}}{\textbf{non-abelian}} & & \underset{\textbf{de Rham cohomology}}{\textbf{non-abelian}} \\
A(X) & \xrightarrow[\mathrm{ch}_A]{\textbf{non-abelian character}} & H_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big) \\
\big[\mathcal{F}\big] & \longmapsto & \Big[\big(F^{(a)}_{r_a}\big)_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big] \\
\underset{A\textbf{-quantized flux}}{\textbf{class of}} & & \underset{\textbf{underlying flux densities}}{\textbf{class of}}
\end{array}
$$

$$
\begin{array}{lccc}
\text{specializing to:} & \overset{\textbf{ordinary cohomology}}{H^n(X;\,\mathbb{Z})} & \xrightarrow{\textbf{de Rham map}} & H^n_{\mathrm{dR}}(X) \\
 & \overset{\textbf{nonabelian cohomology}}{H^1(X;\,G)} & \xrightarrow{\textbf{Chern-Weil homomorphism}} & H_{\mathrm{dR}}\big(X;\, \mathrm{inv}(\mathfrak{g})\big) \\
 & \overset{\textbf{topological K-theory}}{K^0(X)} & \xrightarrow{\textbf{Chern character}} & \underset{k \in \mathbb{N}}{\bigoplus} H^{2k}_{\mathrm{dR}}(X) \\
 & \overset{\textbf{Whitehead generalized cohomology}}{E^n(X)} & \xrightarrow{\textbf{Chern-Dold character}} & \underset{k \in \mathbb{Z}}{\bigoplus} H^{n+k}_{\mathrm{dR}}\big(X;\, \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}\big) \\
 & \overset{\textbf{coHomotopy in even degree}}{\pi^{2k}(X)} & \xrightarrow{\textbf{coHomotopical character}} & H_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^{2k}\big) \\
 & & & \| \\
 & & & \left\{ \begin{array}{ll|ll} G_{4k-1} & \in \Omega^{4k-1}_{\mathrm{dR}}(X) & \mathrm{d}\, G_{4k-1} & = G_{2k} \wedge G_{2k} \\ G_{2k} & \in \Omega^{2k}_{\mathrm{dR}}(X) & \mathrm{d}\, G_{2k} & = 0 \end{array} \right\}_{\big/ \mathrm{concordance}}
\end{array}
\tag{16}
$$

1.3 Higher non-abelian cohomology theories serve as flux quantization laws

We have now seen that
the cohomological sector of classical higher gauge theory exhibits
the flux densities as classes in $\mathfrak{a}$-valued de Rham cohomology, $\mathfrak{a} \in L_\infty\mathrm{Alg}$
which is the character shadow of A-cohomology, for *any* $A \in \mathrm{Spaces}$ with $\mathfrak{l}A \;\simeq\; \mathfrak{a}$

Thus **flux quantization** in A-cohomology means to:
impose the constraint that the deformation classes of the fluxes
must be in the image of the character map (the "charge lattice").

For example, **quantization of electromagnetic flux** in integral cohomology means to require F_2 to have integral periods:

$$
\begin{tikzcd}
 & & \overset{\substack{\textbf{naïve}\\ \textbf{flux densities}}}{\Omega^2_{\mathrm{dR}}(X)} \arrow[dr] & \\
 & \textbf{EM-flux}\ \{F_2\} \arrow[ur] \arrow[dr] & & H^2_{\mathrm{dR}}(X)\ \ \substack{\textbf{deformation classes}\\ \textbf{of naïve flux densities}} \\
\underset{\textbf{integral cohomology}}{H^2(X;\mathbb{Z})} \arrow[urrr, "\mathrm{ch}"] \arrow[rr, two heads, "\textbf{forget torsion}"'] & & \underset{\textbf{charge lattice}}{H^2_{\mathrm{dR}}(X)_{\mathrm{int}}} \arrow[ur, hook] &
\end{tikzcd}
$$

But this is not enough...

1.4 Flux quantization laws determine the moduli ∞-stack of the higher gauge potentials

Instead of just asking that *there exists* an A-cohomology class $[\mathcal{F}]$ whose character image is deformation equivalent to the $\mathfrak{a}$-valued flux densities $F^{(a)}$, the A-cocycle $\mathcal{F}$ and the gauge equivalence $\mathrm{ch}_A(\mathcal{F}) \simeq \{F^{(a)}\}$ should be part of the field content.

In order to achieve this, one needs a unified context which accommodates both

	differential forms like flux densities $F^{(a)}$
&	**homotopy types** of classifying spaces A
in	**differential homotopy theory**

This unified context is naturally provided by *higher topos theory* (for exposition in our context see [1]).

In order to handle	differential structures	it is convenient to model them	on the basic charts.
In order to handle	homotopy types of spaces	it is convenient to model them	as simplicial sets.

For the present purpose we consider:

the site CartSp **of abstract smooth charts**	
an abstract *coordinate chart*	is a Cartesian space $\mathbb{R}^n$ for any $n \in \mathbb{N}$
an abstract *coordinate transformation*	is any smooth function $\mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2}$
a *covering of coordinate charts*	is an open cover $\left\{\mathbb{R}^n \simeq U_i \hookrightarrow \mathbb{R}^n\right\}_{i \in I}$ which is *differentiably good* in that finite non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are all diffeomorphic to $\mathbb{R}^n$

Given any site of Charts, serving as local model spaces:

- A **generalized space** $\mathcal{X}$ *probeable* **by such charts** is bootstrapped into existence
by declaring the simplicial sets of ways of plotting out abstract coordinate charts inside $\mathcal{X}$:

$$
\begin{array}{ccc}
\mathcal{X} \;:\; \mathrm{Charts}^{\mathrm{op}} & \longrightarrow & \mathrm{sSets} \\
\mathbb{R}^n & \mapsto & \mathrm{Plots}(\mathbb{R}^n, \mathcal{X})
\end{array}
$$

Such **simplicial presheaves** naturally form an sSet-enriched category $\mathrm{sPSh}_{\mathrm{Charts}}$; denote its simplicial hom-complexes by:

$$
\begin{array}{ccc}
\mathrm{sPSh}^{\mathrm{op}}_{\mathrm{Charts}} \times \mathrm{sPSh}_{\mathrm{Charts}} & \longrightarrow & \mathrm{sSet} \\
(\mathcal{X},\, \mathcal{Y}) & \mapsto & \mathrm{Maps}(\mathcal{X},\, \mathcal{Y})
\end{array}
$$

- Any $U \in$ Charts becomes a generalized space by declaring its plots to be the morphisms of charts (representable presheaf):

$$
\begin{array}{ccccc}
U & : & \mathrm{Charts}^{\mathrm{op}} & \longrightarrow & \mathrm{sSets} \\
 & & V & \mapsto & \mathrm{Charts}(V, U)
\end{array}
$$

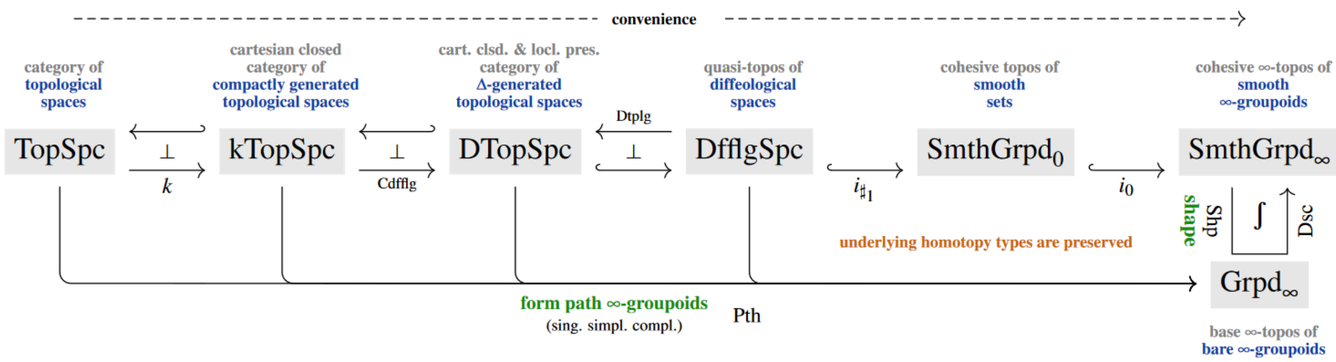
- **Consistency** of this bootstrap of generalized spaces demands:
  (1.) natural identifications between plots by and maps from charts:

$$
U \in \mathrm{Charts}, \quad \mathcal{X} \in \mathrm{sPSh}_{\mathrm{Charts}} \qquad \vdash \qquad \mathrm{Plots}(U, \mathcal{X}) \simeq \mathrm{Maps}(U, \mathcal{X})
$$

  This is the case by the (enriched) **Yoneda lemma**.
  (2.) that maps of generalized spaces are equivalences — to be denoted $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{W}$ —
  iff *locally on all charts* i.e.: stalk-wise they are *higher gauge equivalence*, i.e.: simplicial weak homotopy equivalences
  this we *enforce* by **simplicial localization**, yielding the ∞-topos $\mathbf{H} := L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{Charts}}$

	by this principle:	**(probes of) spaces are:**
	locality principle	sheaves on charts with
&	higher gauge principle	values in simplicial sets
=	homotopy topos of simplicial sheaves on charts	

Specifically for Charts = CartSp we obtain the cohesive ∞-topos $\mathrm{SmoothGrpd}_\infty := L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{CartSp}}$



In $\mathrm{SmthGrpd}_\infty$ all ingredients of higher gauge field theory find a natural home:

smooth moduli space of closed $\mathfrak{l}A$-valued forms (genuine differential structure)

$$
\begin{array}{ccccc}
\Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{clsd}} & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \longmapsto & \Omega_{\mathrm{dR}}(\mathbb{R}^n; \mathfrak{l}A)_{\mathrm{clsd}}
\end{array}
$$

smooth moduli stack of deformation classes of closed $\mathfrak{l}A$-valued forms (rational homotopy type of A)

$$
\begin{array}{ccccc}
\int \Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{clsd}} & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \longmapsto & \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^\bullet_{\mathrm{smth}}; \mathfrak{l}A\big)_{\mathrm{clsd}}
\end{array}
$$

homotopy type of A (geometrically discrete ∞-groupoid)

$$
\begin{array}{ccccc}
A & : & \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\
 & & \mathbb{R}^n & \longmapsto & \mathrm{Sing}(A)
\end{array}
$$

Now given a choice of flux quantization law A, the corresponding moduli stack $\widehat{A}$ of *potentials* or *gauge fields* is the homotopy pullback of the sheaf of flux densities along the character map.
This classifies (higher non-abelian) *differential cohomology*, in a way that takes care of all "Dirac strings" of gauge fields and of higher gauge fields ("conts. higher form symmetries").

moduli stack of higher gauge fields with A-quantized flux densities

$$
\begin{tikzcd}
\widehat{A} \arrow[r, "\text{flux densities}"] \arrow[d, "\text{charges}"'] \arrow[dr, phantom, "\text{(pb)}"] & \Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{clsd}} \arrow[d] \arrow[dl, Rightarrow, "\text{potentials}"'] \\
\text{flux quant. law}\ A \arrow[r, "\mathbf{ch}_A"', "\substack{\text{character map} = \\ \text{rationalization over the reals}}"'] & \int \Omega_{\mathrm{dR}}(-; \mathfrak{l}A)_{\mathrm{clsd}} \simeq A^{\mathbb{R}} .
\end{tikzcd}
\tag{17}
$$

Finally, given a (pseudo-)Riemannian manifold (X, g), we may consider the subset of suitably self-dual forms

$$\mathrm{Map}\big(X,\,\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{clsd}}\big)_{\mathrm{selfdual}} \hookrightarrow \mathrm{Map}\big(X,\,\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{clsd}}\big)$$

and obtain the genuine on-shell field space as the further pullback

$$\begin{tikzcd}
\mathrm{PhaseSpace}(X) \arrow[r] \arrow[d] & \mathrm{Map}\big(X,\,\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{clsd}}\big)_{\mathrm{selfdual}} \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
\mathrm{Map}\big(X,\,\widehat{A}\big) \arrow[r] & \mathrm{Map}\big(X,\,\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{clsd}}\big)
\end{tikzcd}$$

1.5 Brane intersections and twisted cohomology

Finally, all these considerations generalize to fluxes in twisted cohomology, describing brane intersections.

Twisted RR-fields as a fibration. Notice that the twist*ed* RR-fields (**??**) form a fibration over the twist*ing* NS B-field whose fiber is (a torsor over) the untwisted RR-fields.

$$\begin{tikzcd}
\underset{\text{RR-fields}}{\text{flux of free}}\ \Big\{F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \,\Big\vert\, \mathrm{d}F_{2\bullet} = 0\Big\} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \left\{ \begin{array}{l|l} H_3 \in \Omega^3_{\mathrm{dR}}(X) & \mathrm{d}H_3 = 0 \\ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) & \mathrm{d}F_{2\bullet} = H_3 \wedge F_{2\bullet-2} \end{array} \right\}\ \text{flux of RR-fields coupled to NS B-field} \arrow[d, two heads] \\
\{0\} \arrow[r, hook] & \Big\{H_3 \in \Omega^3_{\mathrm{dR}}(X) \,\Big\vert\, \mathrm{d}H_3 = 0\Big\}\ \text{NS B-field}
\end{tikzcd} \tag{18}$$

In general, in the case of branes ending on branes, the Bianchi identities for the latter fluxes include polynomial "twists" by the former

$$\underset{\text{twisted higher “Bianchi identities”}}{\overset{\text{de Rham diff.}}{\mathrm{d}}\, F^{(a)}_{r_a} \;=\; \overset{\text{polynomial}}{P_{r_a}}\Big(\big\{F^{(b)}_{r_b}\big\}_{b \leq a},\, \overset{\text{twisting fluxes}}{\big\{H^{(i)}_{r_i}\big\}_{1 \leq i \leq i_{\max}}}\Big)},$$

and the previous classifying spaces (8) generalize to *classifying fibrations*

$$\overset{\text{classifying spaces for...}}{\begin{tikzcd}
\underset{\text{branes}}{\text{intersected}}\ A \arrow[r] & A /\!\!/ \mathcal{G}\ \underset{\text{intersections}}{\text{brane}} \arrow[d, "\text{classifying}", "\text{fibrations}"'] \\
& B\mathcal{G}\ \underset{\text{branes}}{\text{intersecting}}
\end{tikzcd}}$$

which classify *twisted* non-abelian cohomology theories:

$$\underset{A^\tau(X)}{\overset{\text{twisted non-abelian cohomology}}{}} \;:=\; \underset{\pi_0\,\mathrm{Maps}\big((X,\tau),\, A/\!\!/\mathcal{G}\big)_{B\mathcal{G}}}{\overset{\text{vertical homotopy classes of slice maps}}{}} \;=\; \left\{ \begin{tikzcd}
X \arrow[rr, dashed, bend left, "\mathcal{F}_\tau", "\text{twisted cocycle}"'{above=12pt}] \arrow[rr, dashed, bend right, "\mathcal{F}'_\tau"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "\tau"', "\text{twisting cocycle}"{below left}] & & A/\!\!/\mathcal{G} \arrow[dl] \\
& B\mathcal{G} &
\end{tikzcd} \right\}\Big/\sim$$

on which the *twisted non-abelian character map*

$$\begin{array}{ccc}
\overset{\text{twisted non-abelian cohomology}}{A^\tau(X)} & \xrightarrow[\mathrm{ch}_A]{\text{twisted non-abelian character}} & \overset{\text{twisted non-abelian de Rham cohomology}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)} \\
\underset{\mathcal{H}\text{-twisted class of } A\text{-quantized flux}}{[\mathcal{F}_{\mathcal{H}}]} & \longmapsto & \underset{\text{class of underlying flux densities}}{\Big[\big(F^{(a)}_{r_a}\big)_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big]}
\end{array} \tag{19}$$

computes the classes of underlying flux densities satisfying twisted Bianchi identities:

$$
\Big[\big(F^{(a)}_{r_A}\big)_{1\leq a\leq \dim[\pi_\bullet(A),\mathbb{R}]}\Big]
\;\in\;
\underset{\substack{\tau_{\mathrm{dR}}\text{-twisted} \\ \mathfrak{l}A\text{-valued} \\ \text{non-abelian} \\ \text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\,\mathfrak{l}A\big)}
\;:=\;
\left\{
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathrm{CE}\big(\mathfrak{l}(A/\!\!/\mathcal{G})\big) \arrow[ll, dashed, bend right, "{\text{twisted cocycle (dga-hom)}\;\big(F^{(a)}_{r_a}\big)}"'] \arrow[ll, dashed, bend left, "\big(F^{(a)}_{r_a}\big)'"] \arrow[ll, phantom, "{\Downarrow\ \text{coboundary (concordance)}}"] \\
& \mathrm{CE}(\mathfrak{l}B\mathcal{G}) \arrow[ul, "{\tau_{\mathrm{dR}}}", "\text{twisting cocycle}"'] \arrow[ur] &
\end{tikzcd}
\right\}\Big/\sim .
$$

References

[Al85] O. Alvarez, *Topological quantization and cohomology*, Comm. Math. Phys. **100** (1985), 279-309, [euclid:1103943448].

[Bu11] U. Bunke, *String structures and trivialisations of a Pfaffian line bundle*, Commun. Math. Phys. **307** (2011) : 675-712 [arXiv:0909.0846] [doi:10.1007/s00220-011-1348-0]

[CDF91] L. Castellani, R. D'Auria and P. Fré, *Supergravity and Superstrings – A Geometric Perspective*, World Scientific, (1991) [doi:10.1142/0224]

[DF82] R. D'Auria and P. Fré, *Geometric Supergravity in $D = 11$ and its hidden supergroup*, Nuclear Physics B **201** (1982) 101-140 [doi:10.1016/0550-3213(82)90376-5]

[Di31] P.A.M. Dirac, *Quantized Singularities in the Electromagnetic Field*, Proc. Royal Soc. **A133** (1931), 60-72 [doi:10.1098/rspa.1931.0130].

[FSS15] D. Fiorenza, H. Sati and U. Schreiber, *Super Lie n-algebra extensions, higher WZW models and super p-branes with tensor multiplet fields*, Int. J. Geom. Meth. Mod. Phys. **12** 02 (2015) 1550018 [arXiv:1308.5264] [doi:10.1142/S0219887815500188]

[FSS17] D. Fiorenza, H. Sati and U. Schreiber, *Rational sphere valued supercocycles in M-theory and type IIA string theory*, J. Geometry and Physics, **114** (2017) 91-108 [arXiv:1606.03206] [doi:10.1016/j.geomphys.2016.11.024]

[FSS18] D. Fiorenza, H. Sati and U. Schreiber, *T-Duality from super Lie n-algebra cocycles for super p-branes*, ATMP **22** 5 (2018) [arXiv:1611.06536] [doi:10.4310/ATMP.2018.v22.n5.a3]

[FSS19] D. Fiorenza, H. Sati and U. Schreiber, *The rational higher structure of M-theory*, Fortschritte der Physik **67** 8-9 (2019) [arXiv:1903.02834] [doi:10.1002/prop.201910017]

[Char] D. Fiorenza, H. Sati, and U. Schreiber:
The Character map in Nonabelian Cohomology — Twisted, Differential and Generalized,
World Scientific (2023) [doi:10.1142/13422] [arXiv:2009.11909] [ncatlab.org/schreiber/show/The+Character+Map]

[FSSt12] D. Fiorenza, U. Schreiber and J. Stasheff, *Čech cocycles for differential characteristic classes*, Adv. Theor. Math. Physics, **16** 1 (2012) 149-250 [arXiv:1011.4735] [doi:10.1007/BF02104916]

[Fr97] T. Frankel, *The Geometry of Physics – An Introduction*, Cambridge University Press (1997, 2004, 2012) [doi:10.1017/CBO9781139061377]

[Fr00] D. Freed, *Dirac charge quantization and generalized differential cohomology*, Surveys in Differential Geometry **7**, Int. Press (2000) 129-194 [arXiv:hep-th/0011220] [doi:10.4310/SDG.2002.v7.n1.a6]

[Fr1957] J. Frenkel, *Cohomologie non abélienne et espaces fibrés*, Bulletin de la Société Mathématique de France, **85** (1957) 135-220 [numdam:BSMF_1957__85__135_0]

[Grothendieck55] A. Grothendieck, *A General Theory of Fibre Spaces With Structure Sheaf*, University of Kansas, Report No. 4 (1955, 1958) [ncatlab.org/nlab/files/Grothendieck-FibreSpaces.pdf]

[HS05] M. Hopkins and I. Singer, *Quadratic Functions in Geometry, Topology, and M-Theory*, J. Differential Geom., **70** 3 (2005) 329-452 [arXiv:math/0211216] [doi:10.4310/jdg/1143642908]

[HSS19] J. Huerta, H. Sati and U. Schreiber, *Real ADE-equivariant (co)homotopy and Super M-branes* Comm. Math. Phys. **371** (2019) 425 [arXiv:1805.05987] [doi:10.1007/s00220-019-03442-3]

[LM95] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Alg. **23** 6 (1995) [arXiv:hep-th/9406095] [doi:10.1080/00927879508825335]

[LS93] T. Lada and J. Stasheff, *Introduction to sh Lie algebras for physicists*, Int. J. Theo. Phys. **32** (1993) 1087–1103 [arXiv:hep-th/9209099] [doi:10.1080/00927879508825335]

[Sa13] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018) 062304 [arXiv:1310.1060] [doi:10.1063/1.5007185]

[SSS09] H. Sati, U. Schreiber and J. Stasheff, *L_∞-algebra connections and applications to String- and Chern-Simons n-transport*, in *Quantum Field Theory*, Birkhäuser (2009) 303-424 [arXiv:0801.3480] [doi:10.1007/978-3-7643-8736-5_17]

[SSS12] H. Sati, U. Schreiber and J. Stasheff, *Twisted Differential String and Fivebrane Structures*, Comm. Math. Phys. **315** 1 (2012) 169-213 [arXiv:0910.4001, doi:10.1007/s00220-012-1510-3]

[SV22b] H. Sati and A. Voronov, *Mysterious Triality and M-Theory* [arXiv:2212.13968]

[1] U. Schreiber, *Higher Topos Theory in Physics*, Encyclopedia of Mathematical Physics 2nd ed., Elsevier (2024) [arXiv:2311.11026]

[vN83] P. van Nieuwenhuizen, *Free Graded Differential Superalgebras*, in: *Group Theoretical Methods in Physics*, Lecture Notes in Physics **180**, Springer (1983) 228–247 [doi:10.1007/3-540-12291-5_29]

[We16] T. Wedhorn, *Manifolds, Sheaves, and Cohomology*, Springer (2016) [doi:10.1007/978-3-658-10633-110.1007/978-3-658-10633-1]

[Wi85] E. Witten, *Global anomalies in string theory*, in: W. Bardeen and A. White (eds.) *Symposium on Anomalies, Geometry, Topology*, World Scientific (1985) 61-99 [inspire:214913]

2 Brane lightcone quantization

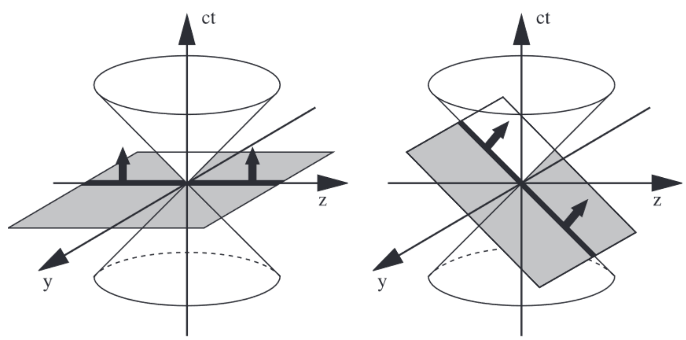
We explain how every charge-quantization of branes (§1) induces a notion of lightcone quantum system of pregeometric brane charges (**??**) on spacetimes with a circle factor (21), due to the fact that in this case the pregeometric phase space (25) is a loop space, so that homological brane observables become a star-algebra under the Pontrjagin product (with star-involution given by lightcone time inversion), making them quantum observables in the sense of algebraic quantum theory.

This approach to brane quantum systems is due to [Qnt1][Qnt2] where it is applied to the case of M_5-brane intersections under Hypothesis H (**??**), to which we come in **??**.

Non-perturbative light-cone quantization. The solution to the problem of non-perturbative quantization of a relativistic Lagrangian field theory appears in principle straightforward: Choose a foliation of spacetime by non-timelike hypersurfaces and then consider the Hamiltonian dynamics of evolution along the leaves.

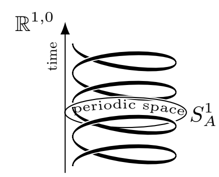
It is for technical and computational problems encountered with carrying this out for the naïve choice of spacelike foliations that the Hamiltonian approach to relativistic QFT was largely abandoned, long ago, in favor of Schwinger-Tomonoga-Feynman-Dyson perturbation theory, which is now often but erroneously regarded as synonymous with "quantum field theory".

However, one may also consider foliation by *lightlike* hypersurfaces (light wave fronts, [Dirac1949, §5]), and the resulting *light-cone quantization* turns out to be mathematically natural and more tractable, especially in application to hadronic bound states in strongly coupled QCD (e.g. [BMPP93][Zh94][BPP98][Ba$^+$13]).



spacelike foliation lightlike foliation

The greatest practical progress with non-perturbative computations in QCD has been made by additionally assuming that spacetime is periodic along one light-like direction so that the light-cone momentum values are discrete, whence one speaks of **discretized light-cone quantization** [MY76][Ca76][Th77][Th78][PB85][Pa99].
This may be understood [Sei97] as the physics seen by a lightlike observer travelling along a periodic spatial dimension.



(20)

But in itself, while computationally succesful, the fact that a spacelike periodicity is required and singled out here is puzzling from the point of view of physics in 1+3-dimensional.

M-Theory as $\mathbb{R}^{1,0} \times \mathrm{S}^1_A$ light-cone quantum mechanics. However, exactly such a circle-factor S^1_A in spacetime is meant to appear in strongly coupled type IIA string theory in the guise of M-theory (cf. p. **??**), where the radius of S^1_A scales with the string coupling seen in 10d [DHIS87][To95][Wi95, §2.3] (review in [Du96, §2(ii)][OP99, §2.1]).

Indeed, one early proposal for making sense of M-theory is (see [NH98, §10]) to regard it as the *lightcone quantum mechanics* of the fundamental membrane (**??**) propagating on a spacetime of the form (20)

$$X^{1,d} \;=\; \mathbb{R}^{1,0} \,\times\, S^1_A \,\times\, X^{d-1} \tag{21}$$

with lightcone momentum along the circle S^1_A, which in the small radius limit is thought to reduce to the D_0-brane dynamics described by the *BFSS matrix model* [BFSS97][Su97][Sei97] or rather the *BMN matrix model* [BMN02, §5] (review in [Yd18]).

A key consistency check of these M-theory matrix models have been computations recovering 11d supergravity in the form of graviton scattering amplitudes [BBPT97][HPSW99] — but brane charge quantization such as in K-theory (**??**) is not reflected in these models (cf. [AST02]). Contrariwise, we now explain a lightcone quantization of pre-geometric brane charges in general cohomology theories.

In order to appreciate the concept of *higher quantum observables* that we are about to consider, it may be useful to recall the following standard conceptualization of algebraic (quantum) mechanics, schematically:

The fundamental concepts of quantum physics in "algebraic" form are the following (good exposition in [Gl09][Gl11], for more see [La17]):

- The **covariant phase space** of a physical system is really the space of solutions to the classical equations of motion [Wi86, §5][CW87][HT92, §17.1], hence the space of physically possible ("on shell") *field histories* of the system.

$$\mathrm{PhsSpc} \;=\; \left\{ \begin{array}{c} \text{solutions to equations of motion} \\ \text{hence: possible field histories} \end{array} \right\}.$$

- The **classical observables** on a physical system are encoded in *compactly supported*[2] complex-valued functions on field histories, understood as assigning to a field history the value that the observable takes there.
  In the simplistic but relevant special case where the phase space is just a discrete set (cf. [La17, §1.2]), this means that the space of observables is the linear span of formal linear combinations of field histories:

$$\mathrm{PhsSpc} \in \mathrm{Sets} \qquad \Rightarrow \qquad \mathrm{Obsrvbls} \;=\; \mathbb{C}[\mathrm{PhsSpc}] \tag{22}$$

- The **quantum observables** are a *choice* of the structure of a (non-commutative) complex star-algebra[3] on the Obsrvbls:[4]

$$\mathrm{QObsrvbls} \;=\; \Big(\mathrm{Obsrvbls},\, (\text{-})\cdot(\text{-}),\, (\text{-})^*\Big), \qquad \big(\mathcal{O}_1 \cdot \mathcal{O}_2\big)^* \;=\; \mathcal{O}_2^* \cdot \mathcal{O}_1^*, \qquad \big((a+ib)\cdot\mathcal{O}\big)^* \;=\; (a - ib)\cdot \mathcal{O}^* \tag{23}$$

  Observable algebra and temporal order. Physically, the (dependency of the non-commutative product on the) order of quantum observables reflects **temporal order** (originally observed by [Fey42, p. 35][Fey48, p. 381] as reviewed in [Ong]), which implies that the star-operation $(-)^*$ expresses time-reversal.

  (Non-)Lagrangean origin of quantum observables. In the case of *Lagrangean field theories* there is an elaborate prescription, subject to a multitude of ad-hoc choices, occupying most of the large and still growing literature on the subject (e.g. [HT92]), for how to choose the quantum observables as a deformation controlled by Poisson structure on the phase space, at least perturbatively. But we cannot expect M-theory to be the quantization of a Lagrangean field theory (already the sector of coincident fivebranes inside M-theory is expected not to be Lagrangean) and will instead discover a natural star-algebra of quantum observables right away (33), without detour through a classical field theory.
- The **quantum states** for given quantum observables are the linear maps on the quantum observables (understood as assigning to an observable the value that it takes in the given state) which are "positive" (semidefinite), in that on elements of the form A^*A they take non-negative real values.

$$\mathrm{QStates} \;=\; \Big\{ \rho \,:\, \mathrm{QObsrvbls} \xrightarrow{\text{linear}} \mathbb{C} \;\Big\vert\; \underset{\mathcal{O}}{\forall}\;\; \rho\big(\mathcal{O}^* \cdot \mathcal{O}\big) \;\in\; \mathbb{R}_{\sqcup\{\infty\}} \subset \mathbb{C} \Big\} \tag{24}$$

But in the presence of higher gauge fields, these traditional structures of quantum physics are to be promoted to *higher structures*:

The higher phase space is a higher groupoid/stack whose
- objects are the field histories,
- higher morphisms are their higher gauge transformations.

The physics literature is mostly familiar with the infinitesimal approximation to this higher stack, which is a higher Lie algebroid known as the **BRST complex** (e.g. [HT92]). The full higher phase space is to the BRST complex as a Lie group is to its Lie algebra, hence may be thought of as the **integrated BRST complex** (in the sense of Lie integration).

Concretely, if $\widehat{A}$ is the moduli stack (17) of a (nonabelian, generalized) cohomology theory expressing the flux quantization law (§1) of a higher gauge theory, and if we consider only the *pre-geometric* field equations (**??**), then the **pregeometric higher phase space**[5] is the *mapping stack* from spacetime into $\widehat{A}$:

$$\underset{\substack{\textbf{pregeometric}\\ \textbf{higher phase space/}\\ \textbf{integrated BRST complex}}}{\mathrm{Maps}\big(X,\,\widehat{A}\big)} \quad = \quad \left\{ \begin{tikzcd} X \arrow[rr, bend left, dashed, "\overset{\textbf{gauge field}\ (\text{map})}{\widehat{\mathcal{F}}}"] \arrow[rr, bend right, dashed, "\underset{\textbf{gauge field}}{\widehat{\mathcal{F}}'}"'] \arrow[rr, phantom, "\Downarrow\ \substack{\textbf{gauge transfo.}\\ (\text{homotopy})}"] && \widehat{A} \end{tikzcd} \right\}. \tag{25}$$

---

[2]In C^*-algebraic formulations of mechanics the algebra of classical observables on a phase space is often taken to be the C^*-algebra $C_0(P)$ of continuous functions vanishing at infinity (e.g. [La17, §3]). But this may be understood as the C^*-completion of the "actual" observable algebra of compactly-supported functions $C_c(P) \subset C_0(P)$, see e.g. [La98, p. 55, 116][La17, p. 528].

[3]This means to require structure like that of a C^*-algebra but disregarding the completeness condition for a Banach algebra. In our application to "topological" charge sectors below the space of (higher) observables is (graded and) degreewise *finite*-dimensional, so that this (graded) Banach-algebra structure is automatic.

[4]In (23) we tacitly assume that the underlying space of quantum observables coincides with that of the "classical" observables. This turns out to be the case of relevance here (33). In traditional discussion the space of quantum observables can also be larger (such as in formal deformation quantization) or smaller (such as in geometric quantization) than the space of classical observables.

[5]The genuine higher phase space is obtained from the pregeometric higher phase space by adjoining at least a background field of gravity and then imposing the remaining self-duality conditions (**??**).

Underlying this higher moduli stack — after forgetting the gauge potentials and flux densities only remembering the corresponding charges — is the plain homotopy type of charges, the *cocycle space* of X in A-cohomology:

$$\mathrm{PrePhsSpc} \;=\; \underset{\substack{\textbf{pregeometric}\\ \textbf{higher phase space}\\ \textbf{of brane charges}}}{\mathrm{Maps}\big(X,\,A\big)} \;=\; \left\{ \begin{tikzcd} X \arrow[rr, bend left, dashed, "{\overset{\textbf{charges}\ \text{(map)}}{c}}"{name=U}] \arrow[rr, bend right, dashed, "{\underset{\textbf{charges}}{c'}}"'{name=D}] & & A \arrow[from=U, to=D, Rightarrow, shorten=4pt, "{\substack{\textbf{gauge transfo.}\\ \text{(homotopy)}}}"] \end{tikzcd} \right\}. \tag{26}$$

We will focus on this pre-phase space now, which may be regarded as reflecting the purely "topological" (non-geometric) sector of the higher phase space.

Higher observables. The notion of **higher observables** on a higher phase (25) is not widely discussed, but from the Dao of homotopy theory it is clear that the coefficient ring is to be promoted to a higher ring, namely a *ring spectrum* R. Then the higher analog of observables (22) on the topological sector (26) of a higher pregeometric phase space, is the *R-homology*:

$$\mathrm{PreObsrvbls} \;=\; R_\bullet\big[\mathrm{Maps}(X,\,A)\big]\,. \tag{27}$$

If $R = H\mathbb{C}$ is the Eilenberg-MacLane spectrum of the complex numbers, then this is ordinary homology:

$$\mathrm{PreObsrvbls} \;=\; H\mathbb{C}_\bullet\big[\mathrm{Maps}(X,\,A)\big] \;=\; H_\bullet\big(\mathrm{Maps}(X,\,A);\,\mathbb{C}\big)\,. \tag{28}$$

Higher quantum observables. In general there is no *canonical* (star-)algebra structure on pregeometric higher observables (27) — but if spacetime has a circle factor (21) then the pregeometric higher phase space is the loop space of the *transverse phase space*:

$$\begin{tikzcd} & & \overset{\textbf{pregeometric phase space}}{\mathrm{Maps}\big(S^1 \times X^{d-1},\,A\big)} \arrow[d, equal] & \\ \text{mixed states} \arrow[r, squiggly, leftarrow] & \overset{\substack{\textbf{based loop space of}\\ \textbf{transverse phase space}}}{\Omega_c \mathrm{Maps}\big(X^{d-1},\,A\big)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Maps}\big(S^1,\,\mathrm{Maps}(X^{d-1},\,A)\big) \arrow[d] & \\ & \{c\} \arrow[r] & \underset{\textbf{transverse phase space}}{\mathrm{Maps}\big(X^{d-1},\,A\big)} \arrow[r, squiggly] & \text{pure states} \end{tikzcd} \tag{29}$$

This means that the following basic fact of algebraic topology provides us with a canonical discrete light-cone quantization of charges:

The Pontrjagin-Hopf algebra structure on the homology of loop spaces. [BoSa53][Br61, p. 36][Ha02, §3.C]
The homology of a based loop space

$$\Omega Y \;:=\; \Big\{ \gamma : [0,1] \xrightarrow{\text{cntns}} Y \,\big|\, \gamma(0) \;=\; \gamma(1) \Big\}$$

with coefficients in a field becomes

- *a graded algebra*[6] *under concatenation of loops,*

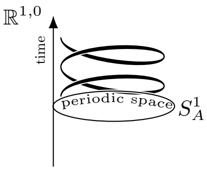
$$\begin{tikzcd}[column sep=large] H_\bullet\big(\Omega Y\big) \otimes H_\bullet\big(\Omega Y\big) \arrow[r, "\sim", "\text{Künneth}"'] \arrow[rr, bend left, "\text{Pontrjagin product}"] & H_\bullet\big(\Omega Y \times \Omega Y\big) \arrow[r, "(\text{-})\cdot(\text{-}) \,:=\, H_\bullet(\mu;\mathbb{C})", "\text{pushforward in homology}"'] & H_\bullet\big(\Omega Y\big) \\ & \Omega Y \times \Omega Y \arrow[r, "\mu"] & \Omega Y \\ & (\gamma_1,\,\gamma_2) \arrow[r, mapsto] & \left(t \mapsto \begin{cases} \gamma(t/2) & \text{for } 0 \leq t \leq 1/2 \\ \gamma(t/2 - 1/2) & \text{for } 1/2 \leq t \leq 1 \end{cases} \right) \end{tikzcd} \tag{30}$$

- *a graded star-algebra under reversal of loops*

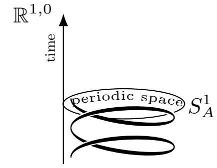
$$\begin{tikzcd}[column sep=huge] H_\bullet\big(\Omega Y\big) \arrow[r, "H_\bullet(\text{inv})"] \arrow[r, bend left=40, "\text{Pontrjagin antipode}"] & H_\bullet\big(\Omega Y\big) \\ \Omega Y \arrow[r, "\text{inv}"] & \Omega Y \\ \gamma \arrow[r, mapsto] & \gamma\big(1 - (\text{-})\big) \end{tikzcd} \tag{31}$$

[6] In fact, together with the canonical coproduct in homology the Pontrjagin product (30) becomes a Hopf algebra structure with the star-involution (31) being a Hopf antipode.

Notice that this is not quite a *complex* star-algebra in the sense of (23) yet, since the Pontrjagin-antipode[7] (31) acts trivially on the coefficient field – but we do get a complex star-algebra by composing the Pontrjagin antipode with complex conjugation on the coefficients



$$
\begin{tikzcd}[column sep=large]
 & H_\bullet(\Omega Y;\mathbb{C}) \arrow[dr, "{\substack{\text{complex conjugation}\\ H_\bullet(\Omega Y;\,(\text{-})^*)}}", "\text{time inversion}"'] & \\
H_\bullet(\Omega Y;\mathbb{C}) \arrow[ur, "{\substack{\text{Pontrjagin-antipode}\\ H_\bullet(\mathrm{inv};\,\mathbb{C})}}", "S^1_A\text{-inversion}"'] \arrow[rr, "(\text{-})^*", "\text{light-cone parameter inversion}"'] \arrow[dr, "H_\bullet(\Omega Y;\,(\text{-})^*)"'] & & H_\bullet(\Omega Y;\mathbb{C}) \\
 & H_\bullet(\Omega Y;\mathbb{C}) \arrow[ur, "H_\bullet(\mathrm{inv};\,(\text{-})^*)"'] &
\end{tikzcd}
$$



(32)

This way we have obtained **higher quantum observables** on the light-cone for pregeometric brane charges in spacetimes of the form (21):

$$
\mathrm{QObsrvbls}_c \;=\; \Big(H_\bullet\big(\Omega_c \mathrm{Maps}(X^{d-1},\, A);\, \mathbb{C}\big),\;\; (\text{-})\cdot(\text{-}),\;\; (\text{-})^* \Big)\,. \tag{33}
$$

whose star-involution is *light-cone parameter inversion.*

Notice here that

$$
\begin{aligned}
\Omega_0 \mathrm{Map}\big(X,\, A\big) \;&\simeq\; \mathrm{Map}^{*/}\big(S^1,\, \mathrm{Map}^{*\cdot}(X_{\sqcup\{\infty\}},\, A)\big) \\
&\simeq\; \mathrm{Map}^{*/}\big(X_{\sqcup\{\infty\}},\, \mathrm{Map}^{*\cdot}(S^1,\, A)\big) \\
&\simeq\; \mathrm{Map}^{*/}\big(X_{\sqcup\{\infty\}},\, \Omega_0 A\big) \\
&\simeq\; \mathrm{Map}\big(X,\, \Omega_0 A\big)\,.
\end{aligned} \tag{34}
$$

The corresponding **light-cone quantum states** are hence (24) those cohomology classes which are (semi-)positive-definite:

$$
\mathrm{QStates}_c \;=\; \Big\{ \rho \,\in\, H^\bullet\Big(\Omega_c \mathrm{Maps}\big(X^{d-1},\, A\big);\, \mathbb{C}\Big) \;\Big\vert\; \underset{\mathcal{O}}{\forall}\;\; \rho(\mathcal{O}^*\mathcal{O}) \geq 0 \Big\}\,. \tag{35}
$$

In **??** we discuss examples of 11d spacetime domains whose light-cone quantum states (35) of pregeometric (intersecting) brane charges include, under Hypothesis H:

?? quantum states of Hanany-Witten NS/D-brane configurations,

?? quantum states of transverse M_5-branes,

?? quantum states of $\mathrm{M}_5 \perp \mathrm{M}_5$ intersections,

?? quantum states of topological strings,

with provable properties of the kind expected in the string theory literature.

Pontrjagin rings as deformation quantization. For context, we indicate how Pontrjagin rings (33) are related to more familiar notions of quantization.

First recall from §1 that every connected coefficient space A is equivalently the delooping of its based loop group (13), the latter understood as an ∞-group. In this sense A describes (the topological sectors of) higher gauge theory whose higher gauge group has the homotopy type of the ∞-group $\mathcal{G} := \Omega A$. Moreover, the Whitehead L_∞-algebra $\mathfrak{l}A$ of A may be understood as the L_∞-algebra of this ∞-group, in that its underlying graded vector space is that of the rationalized homotopy groups of ΩA (14).

$$
A \;\simeq\; B\big(\Omega A\big) \;\simeq\; * /\!\!/ \Omega A\,, \qquad\qquad (\mathfrak{l}A)_n \;\simeq\; \pi_n\big(\Omega A\big) \otimes_{\mathbb{Z}} \mathbb{R}\,.
$$

In the case that X^{d-1} is contractible, this is already our pre-phase space (33): $\mathrm{Maps}(X^{d-1}, A) \;\simeq\; A \;\simeq\; * /\!\!/ \Omega A$.

Phase spaces whose homotopy type is of this form are those arising from Lie-Poisson structures, whose symplectic groupoid is the coadjoint action groupoid $\mathfrak{g}^* /\!\!/ G$ [We91, Ex. 3.2][BC05, Ex. 4.3][Nui13, pp. 111, cf. Prop. 5.2.12]. Since the underlying space of $\mathfrak{g}^*$ is contractible, the underlying homotopy type (shape) of this action groupoid is $B\textstyle\int G \simeq * /\!\!/ \textstyle\int G$, where $\textstyle\int G$ is the underlying homotopy type of G (e.g. [Orb, Prop. 3.4]):

$$
\begin{array}{ccc}
\mathrm{SmthGrpd}_\infty & \xrightarrow{\;\;\int\;\;} & \mathrm{Grpd}_\infty \\
\mathfrak{g}^* /\!\!/ G & \longmapsto & * /\!\!/ \textstyle\int G
\end{array}
$$

Now, the deformation quantization of Lie-Poisson manifolds $\mathfrak{g}^*$ is well-known to be the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ [Gu83, (4.2)][Gu11, §2.2]. Notice also that for connected compact Lie groups G, $U(\mathfrak{g})$ plays the role of the convolution algebra of G ([Ho81, §XVI][Tj92, pp. 9]), while the latter of course exists also for discrete groups.

[7] The term "Pontrjagin antipode" is not standard, but it is the natural name for the antipode of the Pontrjagin-Hopf algebra structure.

The Pontrjagin ring of ΩA unifies these two perspectives: It gives the group algebra on $\pi_0(\Omega A)$

$$\pi_{\geq 1}\big(\Omega A\big) \;\simeq\; * \qquad\quad \Rightarrow \qquad\quad H_\bullet\big(\Omega A;\, \mathbb{C}\big) \quad\simeq\quad \mathbb{C}\big[\pi_0(\Omega A)\big]$$

and the universal enveloping algebra of the binary super Lie bracket of the connected components of ΩA [MiMo65, Apd.][FHT00, Thm. 16.13]:

$$\pi_0\big(\Omega A\big) \simeq * \qquad \Rightarrow \qquad \overset{\substack{\text{Pontrjagin algebra} \\ \text{of loop } \infty\text{-group}}}{H_\bullet\big(\Omega A;\, \mathbb{R}\big)} \quad\simeq\quad \underset{\substack{\textbf{deformation quantization} \\ \textbf{of } \mathrm{Bin}(\mathfrak{l}A)^*}}{\overset{\substack{\text{universal envelope} \\ \text{of binary} \\ \text{Whitehead bracket}}}{U\big(\mathrm{Bin}(\mathfrak{l}A)\big)}}$$

Example – M-Theoretic Quantum Cohomology. The Pointrjagin ring of the loop space of the 4-sphere is, by the above and using (15),

$$H_\bullet\big(\Omega S^4;\, \mathbb{C}\big) \;\simeq\; U\big([v_3, v_3] = v_6\big) \;=\; \mathbb{C}[v_3, v_6]/\big(v_3^2 - v_6\big)\,. \tag{36}$$

Notice that this is a deformation of the cohomology ring of S^3

$$H^\bullet(S^3;\mathbb{C}) \;\simeq\; \mathbb{C}[v_3]/\big(v_3^2\big)$$

in the same way (up to degree shifts) that the *quantum cohomology* of $S^2 = \mathbb{C}P^1$:

$$\mathrm{QH}^\bullet\big(\mathbb{C}P^1\big) \;\simeq\; \mathbb{C}[v_2, v_4]/\big(v_2^2 - v_4\big)$$

is a deformation of the ordinary cohomology ring

$$H^\bullet\big(\mathbb{C}P^1\big) \;\simeq\; \mathbb{C}[v_2]/\big(v_2^2\big)\,.$$

This quantum cohomology reflects the interaction of topological strings propagating on $\mathbb{C}P^1$. But since S^3 is an S^1-bundle over $S^2 = \mathbb{C}P^1$, it stands to reason that (36) is an M-theoretic lift of this situation, possibly characterizing the interaction of topological membranes propagating on S^3.

Example: Fluxes in ordinary vacuum gauge theory. For the case of plain electromagnetism (**??**) we may choose the charge quantization law for the pre-geometric fields (**??**) to be by the space $B\mathrm{U}(1) \times B\mathrm{U}(1)$, which we will denote $B\mathrm{U}(1)_B \times B\mathrm{U}(1)_E$, just for emphasis.

Then for spacetime of the form an orientable closed connected surface Σ times a lightlike compactification of $\mathbb{R}^{1,1}$, we find with (34):

$$\begin{aligned} \mathrm{Map}\big(\Sigma,\, B\mathrm{U}(1)_B \times B\mathrm{U}(1)_E\big) \;&\simeq\; \mathrm{Map}\big(\Sigma,\, B\mathrm{U}(1)_B\big) \times \mathrm{Map}\big(\Sigma,\, B\mathrm{U}(1)_E\big) \\ &\simeq\; \big(H^2(\Sigma;\mathbb{Z}) \times BH^1(\Sigma;\mathbb{Z}) \times B^2 H^0\big(\Sigma;\mathbb{Z}\big)\big)^2 \\ &\simeq\; \mathbb{Z}_B \times \mathbb{Z}_E \times B\big(H^1(\Sigma;\mathbb{Z})_B \times H^1(\Sigma;\mathbb{Z})_E\big) \times B^2\big(H^0(\Sigma;\,\mathbb{Z}_B) \times H^0(\Sigma;\,\mathbb{Z}_E)\big) \end{aligned}$$

Here the first two factors are the usual grading (cf. [FMS07a, p. 9] and the following comment) of the phase space by total electric fluxes $\Phi_E^\Sigma \in \mathbb{Z}_E$ and magnetic fluxes $\Phi_B^\Sigma \in \mathbb{Z}_E$ through Σ, while the second factor is picked up by looping:

$$\Omega_{\Phi_E^\Sigma, \Phi_B^\Sigma}\mathrm{Map}\big(\Sigma,\, B\mathrm{U}(1) \times B\mathrm{U}(1)\big) \;\simeq\; H^1(\Sigma;\mathbb{Z})_E \times H^1(\Sigma;\mathbb{Z})_B \times H^0(\Sigma;\mathbb{Z})_E \times H^0(\Sigma;\mathbb{Z})_B\,.$$

The Pontrjagin algebra in degree 0 is hence the group algebra

$$H_0\Big(\Omega_{\Phi_E^\Sigma, \Phi_B^\Sigma}\mathrm{Map}\big(\Sigma,\, B\mathrm{U}(1) \times B\mathrm{U}(1)\big);\, \mathbb{C}\Big) \;\simeq\; \mathbb{C}\big[H^1(\Sigma;\mathbb{Z})_E \times H^1(\Sigma;\mathbb{Z})_B\big]\,.$$

Since $H^1(-;\mathbb{Z})$ is an abelian group, this is an abelian algebra, which means in particular that all the flux observables commute with each other. This is indeed as expected, at least for non-torsion fluxes (e.g. [FM07b, (3.6)]); and in our case the light-cone Ansatz ensures that all fluxes are non-torsion, since H^1 of an orientable surface has no torsion.

(...)

References

[AST02] T. Asakawa, S. Sugimoto and S. Terashima, *D-branes, Matrix Theory and K-homology*, JHEP 0203 (2002) 034 [arXiv:hep-th/0108085] [doi:10.1088/1126-6708/2002/03/034]

[BFSS97] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M Theory As A Matrix Model: A Conjecture*, Phys. Rev. D **55** (1997) [arXiv:hep-th/9610043] [doi:10.1103/PhysRevD.55.5112]

[Ba$^+$13] B. Bakker et al., *Light-Front Quantum Chromodynamics: A framework for the analysis of hadron physics*, White Paper of International Light Cone Advisory Committee, Nuclear Physics B – Proceedings Supplements **251**- **252** (2014) 165-174 [arXiv:1309.6333] [doi:10.1016/j.nuclphysbps.2014.05.004]

[BBPT97] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, *Higher Order Graviton Scattering in M(atrix) Theory*, Phys. Rev. D **56** (1997) 3174-3178 [arXiv:hep-th/9706072] [doi:10.1103/PhysRevD.56.R3174]

[BMN02] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp waves from $\mathcal{N}=4$ Super Yang Mills*, JHEP 0204 (2002) 013 [arXiv:hep-th/0202021, doi:10.1088/1126-6708/2002/04/013]

[BoSa53] R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Commentarii Mathematici Helvetici **27** (1953) 320–337 [doi:10.1007/BF02564566]

[BPP98] S. Brodsky, H.-C. Pauli and S. S. Pinsky, *Quantum Chromodynamics and Other Field Theories on the Light Cone*, Phys. Rept. **301** (1998) 299-486 [arXiv:hep-ph/9705477] [doi:10.1016/S0370-1573(97)00089-6]

[BMPP93] S. J. Brodsky, G. McCartor, H.-C. Pauli and S. S. Pinsky, *The challenge of light-cone quantization of gauge field theory*, Particle World **3** 3 (1993) 109-124 [cds:240388]

[Br61] W. Browder, *Torsion in H-Spaces*, Annals of Mathematics, Second Series **74** 1 (1961) 24-51 [jstor:1970305]

[BC05] H. Bursztyn and M. Crainic, *Dirac structures, momentum maps and quasi-Poisson manifolds*, in: *The Breadth of Symplectic and Poisson Geometry*, Prog. Math. **232** Birkhäuser (2005) 1-40 [doi:10.1007/0-8176-4419-9_1]

[Ca76] A. Casher, *Gauge fields on the null plane*, Phys. Rev. D **14** (1976) 452 [doi:10.1103/PhysRevD.14.452]

[CP17] A. S. Cattaneo, A. Perez, *A note on the Poisson bracket of 2d smeared fluxes*, Class. Quant. Grav. 34 (2017) 107001 [arXiv:1611.08394] [doi:10.1088/1361-6382/aa69b4]

[CW87] Č. Crnković and E. Witten, *Covariant Description of Canonical Formalism in Geometrical Theories*, chapter 16 in: *Three Hundred Years of Gravitation*, Cambridge University Press (1987) 676-684 [ISBN:9780521379762] [www.ias.edu/sites/default/files/sns/files/CovariantPaper-1987.pdf]

[Dirac1949] P. A. M. Dirac, *Forms of Relativistic Dynamics*, Rev. Mod. Phys. **21** (1949) 392-399 [doi:10.1103/RevModPhys.21.392]

[Du96] M. Duff, *M-Theory (the Theory Formerly Known as Strings)*, Int. J. Mod. Phys. A **11** (1996) 5623-5642 [arXiv:hep-th/9608117] [doi:10.1142/S0217751X96002583]

[DHIS87] M. Duff, P. Howe, T. Inami and K. Stelle, *Superstrings in $D=10$ from Supermembranes in $D=11$*, Phys. Lett. B **191** (1987) 70 [doi:10.1016/0370-2693(87)91323-2]

[FHT00] Y. Félix and S. Halperin and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics **205** Springer (2000) [doi:10.1007/978-1-4613-0105-9]

[Fey42] R. P. Feynman, *The Principles of Least Action in Quantum Mechanics*, PhD thesis (1942), reprinted in L. Brown (ed.) *Feynman's Thesis – A New Approach to Quantum Theory*, World Scientific (2005) [doi:10.1142/9789812567635_0001]

[Fey48] R. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys. **20** (1948) 367 [doi:10.1103/RevModPhys.20.367]

[FMS07a] D. Freed, G. Moore and G. Segal, *The Uncertainty of Fluxes*, Commun. Math. Phys. **271** (2007) 247-274 [arXiv:hep-th/0605198] [doi:10.1007/s00220-006-0181-3]

[FM07b] D. Freed, G. Moore and G. Segal, *Heisenberg Groups and Noncommutative Fluxes*, Annals Phys. **322** (2007) 236-285 [arXiv:hep-th/0605200] [doi:10.1016/j.aop.2006.07.014]

[Gl09] J. Gleason, *The C^*-algebraic formalism of quantum mechanics* (2009) [ncatlab.org/nlab/files/GleasonAlgebraic.pdf]

[Gl11] J. Gleason, *From Classical to Quantum: The F^*-Algebraic Approach*, contribution to VIGRE REU 2011, Chicago (2011) [ncatlab.org/nlab/files/GleasonFAlgebraic.pdf]

[Gu83] S. Gutt, *An explicit $*$-product on the cotangent bundle of a Lie group*, Lett. Math. Phys. **7** (1983) 249–258 [doi:10.1007/BF00400441]

[Gu11] S. Gutt, *Deformation quantization of Poisson manifolds*, Geometry and Topology Monographs **17** (2011) 171-220 [gtm:17]

[Ha02] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002) [ISBN:9780521795401] [https://pi.math.cornell.edu/~hatcher/AT/ATpage.html]

[HPSW99] R. Helling, J. Plefka, M. Serone and A. Waldron, *Three-graviton scattering in M-theory*, Nuclear Physics B **559** 1–2 (1999) 184-204 [arXiv:hep-th/9905183] [doi:10.1016/S0550-3213(99)00451-4]

[HT92] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* Princeton University Press (1992) [jstor:j.ctv10crg0r]

[Ho81] G. P. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Graduate Texts in Mathematics **75**, Springer (1981) [doi:10.1007/978-1-4613-8114-3_16]

[La98] K. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer (1998) [doi:10.1007/978-1-4612-1680-3]

[La17] K. Landsman, *Foundations of quantum theory – From classical concepts to Operator algebras*, Springer Open (2017) [doi:10.1007/978-3-319-51777-3]

[MY76] T. Maskawa, K. Yamakawi, *The Problem of $P^+ = 0$ Mode in the Null-Plane Field Theory and Dirac's Method of Quantization*, Progress of Theoretical Physics **56** 1 (1976) 270–283 [doi:10.1143/PTP.56.270]

[MiMo65] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Annals of Math. **81** (1965) 211-264 [doi:10.2307/1970615]

[NH98] H. Nicolai and R. C. Helling, *Supermembranes and M(atrix) Theory*, In: *Nonperturbative aspects of strings, branes and supersymmetry*, Trieste (1998) 29-74 [arXiv:hep-th/9809103] [inspire:476366]

[Nui13] J. Nuiten, *Cohomological quantization of local prequantum boundary field theory*, MSc thesis, Utrecht (2013) [dspace:1874/282756]

[OP99] N.A. Obers, B. Pioline, *U-duality and M-Theory*, Phys. Rept. **318** (1999) 113-225 [arXiv:hep-th/9809039] [doi:10.1016/S0370-1573(99)00004-6]

[Ong] Y. C. Ong, *Where is the Commutation Relation Hiding in the Path Integral Formulation?* [physicstravelguide.com/_media/quantum_theory/path-integral.pdf]

[Pa99] H.-C. Pauli, *Discretized light-cone quantization and the effective interaction in hadrons*, AIP Conf. Proc. **494** (1999) 80-139 [arXiv:hep-ph/9910203] [doi:10.1063/1.1301662]

[PB85] H.-C. Pauli and S. J. Brodsky, *Discretized light-cone quantization: Solution to a field theory in one space and one time dimension*, Phys. Rev. D **32** 2001 (1985) [doi:10.1103/PhysRevD.32.2001]

[Qnt1] H. Sati and U. Schreiber, *Differential Cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams*, Adv. Theor. Math. Phys. **26** 4 (2022) [arXiv:1912.10425].

[Orb] H. Sati and U. Schreiber, *Proper Orbifold Cohomology* [arXiv:2008.01101]

[Qnt2] D. Corfield, H. Sati, and U. Schreiber, *Fundamental weight systems are quantum states*, Lett. Math. Phys. (2023) [arXiv:2105.02871].

[Sei97] N. Seiberg, *Why is the Matrix Model Correct?*, Phys. Rev. Lett. **79** (1997) 3577-3580 [arXiv:hep-th/9710009] [doi:10.1103/PhysRevLett.79.3577]

[Su97] L. Susskind, *Another Conjecture about M(atrix) Theory* [arXiv:hep-th/9704080]

[Th77] C. B. Thorn, *On the derivation of dual models from field theory*, Physics Letters B **70** 1 (1977) 85-87 [doi:10.1016/0370-2693(77)90351-3]

[Th78] C. B. Thorn, *Derivation of dual models from field theory. II*, Phys. Rev. D **17** (1978) 1073 [doi:10.1103/PhysRevD.17.1073]

[To95] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. B **350** (1995) 184-187 [arXiv:hep-th/9501068] [doi:10.1016/0370-2693(95)00397-4]

[Tj92] T. Tjin, *An introduction to quantized Lie groups and algebras*, Int. J. Mod. Phys. A **7** (1992) 6175-6213 [arXiv:hep-th/9111043] [doi:10.1142/S0217751X92002805]

[We91] A. Weinstein, *Noncommutative geometry and geometric quantization*, in *Symplectic geometry and Mathematical physics*, Progr. Math. **99** Birkhäuser (1991) 446-461

[Wi86] E. Witten, *Interacting field theory of open superstrings*, Nuclear Physics B **276** 2 (1986) 291-324 [doi:10.1016/0550-3213(86)90298-1]

[Wi95] E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl. Phys. B **443** (1995) 85-126 [arXiv:hep-th/9503124] [doi:10.1016/0550-3213(95)00158-O]

[Yd18] B. Ydri, *Matrix Models of String Theory*, IOP (2018) [ISBN:978-0-7503-1726-9] [arXiv:1708.00734]

[Zh94] W.-M. Zhang, *Light-Front Dynamics and Light-Front QCD*, Chin. J. Phys. **32** (1994) 717-808 [arXiv:hep-ph/9412244]