Introduction to Hypothesis H

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Abstract

The key open question of contemporary mathematical physics is the elucidation of the currently elusive fundamental laws of strongly interacting "non-perturbative" quantum states, including bound states as mundane as nucleons but more generally of quarks confined inside hadrons, as well as strongly correlated ground states of topologically ordered quantum materials.

The seminal strategy of regarding such systems as located on branes inside a higher dimensional string-theoretic spacetime (the "holographic principle") shows all signs of promise but has been suffering from the ironic shortcoming that string theory has also only really been defined perturbatively. However, string theory exhibits a web of *hints* towards the nature of its non-perturbative completion, famous under the working title "M-Theory" but still elusive. Thus, mathematically constructing M-theory should imply a mathematical understanding of quantum brane worldvolumes which should solve non-perturbative quantum physics: the M-strategy for attacking the Millennium Problem.

After a time of stagnation in research towards M-theory, we have recently formulated and tested a hypothesis on the precise mathematical nature of at least a core part of the theory: We call this *Hypothesis H* since it postulates that M-branes are classified by twisted co-*H*omotopy-theory in much the same way that D-branes are expected to be classified by twisted *K*-theory (a widely held but not uncontested similar conjectural belief which might analogously be called *Hypothesis K*). In fact, stabilized coHomotopy is equivalent to the algebraic K-Theory over the "absolute base field $\mathbb{F}_1$", as well as to framed Cobordism cohomology.

In these lecture notes, we give an introduction to (1.) the motivation and (2.) some consequences of Hypothesis H, assuming an audience with a little background in electromagnetism, differential geometry, and algebraic topology.

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These are notes under development,
prepared for a series of talks and lectures;
parts are still not more than a slide show.

The first part §1 aims to be elementary explanation of
Hypothesis H as a good question to ask about physics:
whether it is right or wrong, it deserves checking.

The second part §2 explains evidence
that Hypothesis H is in fact correct
and some insights gained from it.

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The key open question of fundamental quantum physics is not primarily the lack of coherent quantum gravity theory as such, as often portrayed, but the general lack of *non-perturbative* quantum theory of relevance[1], due to which exotic quantum states of matter – such as topologically ordered solid states thought to be needed for topological quantum computation – but even mundane phenomena – such as room-temperature matter, namely "confined" quarks in hadron bound states, reflected (just as are topological phases!) in a "mass gap" – remain theoretically ill-understood, to the extent that one speaks of an open *Millennium Problem*[2].

The role of string theory. String theory originates as a model for these elusive hadron bound states, specifically for the string-like "flux tubes" between pairs of quarks, conceptually explaining both their confinement and their scattering behavior. The unexpected discovery that subtle quantum effects make these *hadronic* strings propagate in an effectively higher dimensional space – with only their endpoint quarks attached to observed 3+1 dimensional spacetime (now: the "brane") or else carrying gravitons into an otherwise unobserved higher dimensional "bulk" – came to be appreciated as a "holographic" description of non-perturbative quantum physics.[3]

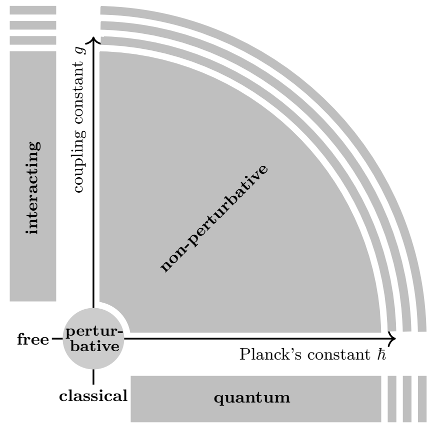
The role of M-theory. Ironically, string dynamics is itself primarily understood only perturbatively, which makes holography require the unrealistic assumption of a large (in fact: humongous) number N of coincident branes, to be tractable. But understanding branes as physical objects yields a web of hints as to what non-perturbative string theory should be like, enough so that it famously has a working title (since 1995): "M-theory".

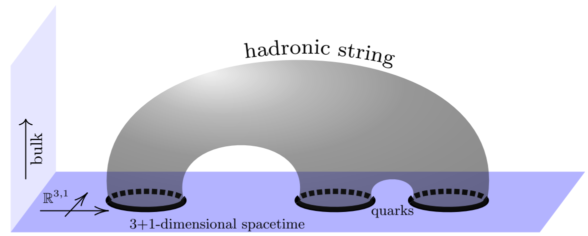
To highlight, in conclusion: One strategy for addressing the "Millennium Problem" of formulating non-perturbative QFT is to mathematically formulate M-theory:[4] With this it ought to be possible to define and investigate, with precision, individual quantum branes whose intersections should exhibit non-perturbative quantum dynamics such as anyonic topological order (which we discuss in §2.4) and eventually confined hadrodynamics.

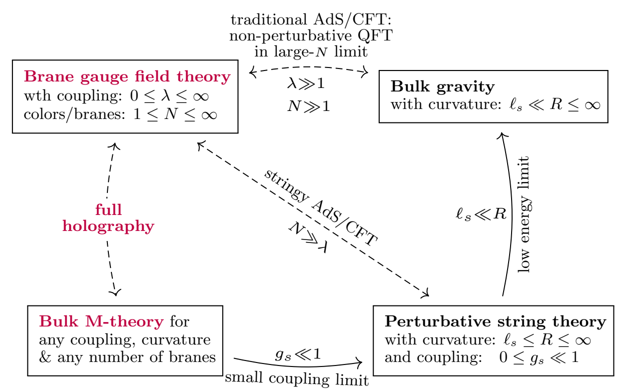
The role of Algebraic Topology. After initial excitement, progress on actually formulating M-theory had stagnated and efforts had been largely abandoned[5], arguably due to a lack of appropriate mathematical tools: Where famous examples of physical theories were formulated within a fairly well-understood framework of mathematical principles (e.g. general relativity in differential geometry or quantum physics in functional analysis), the real problem with formulating M-theory is (or was) that even its underlying mathematical principles remained unclear. It was the vision of [Sa10] (review in [FSS19]) that M-theory ought to find its formulation in *algebraic topology*; initiating a program of looking for algebro-topological patterns in the available information on M-brane physics, deducing clues as to their fundamental mathematical meaning.

The role of Hypothesis H. This analysis eventually culminated in a formulation of a hypothesis – *Hypothesis H* – of what M-theory really is about [FSS20], namely about the generalized non-abelian cohomology theory called *CoHomotopy Theory*. This we explain below in §1.3.

It is noteworthy here that algebraic topology is not a field of mathematics as any other, but has recently been understood to serve, in its guise of *homotopy theory*, as an alternative *foundation* for mathematics itself (`HoTT`[6]). Moreover, within algebraic topology, cohomotopy is not a (multiplicative) cohomology theory as any other, but is *initial* among all of them. This may be more than a coincidence given that M-theory is meant to be not just a theory of physics as any other, but the initial foundation of all of them.







Bare gauge coupling	g_{YM}
Number of colors/branes	N
Actual (t'Hooft-)coupling	$\lambda = g_s N = g_{\mathrm{YM}}^2 N$
String coupling	$g_s = g_{\mathrm{YM}}^2 = \lambda / N$
Spacetime curvature	$1/R$
String length	$\ell_s = R / \sqrt[4]{\lambda}$

	Physics	**Mathematics**
Principles	Gauge Principle	Homotopy Th.
Foundations	M-Theory	Alg. Topology
Basic Notions	Fluxes/Charges	Cohomology
Initial Notions	M-Brane Charge	Cohomotopy
Derived Notions	D-brane charge	K-Theory

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[1]For (rare) discussion of aspects of non-perturbative QFT see: [BaSh10][FS10][St13]

[2]The pronounciation of the problem of confinement of quarks, hence of the "mass gap in Yang-Mills theory", as a "Millennium Problem" is due to the Clay Mathematics Institute [CMI00][JaWi00][Atiyah00] and referenced as such by nuclear physicists in [KT10][Shn05, §9.1][Ha12][BaGo19, p. 2][RS20, p. 2][Ro21, ftn. 2][Ro22, p. 2].

[3]The string theory mainstream had actually abandoned the identification of strings with hadronic flux tubes in the 1970s (cf. [Ve12, pp. 30]), but came back full circle to re-appreciate this perspective in the 2010s, now under the name "holographic QCD" (cf. [Ha12, §6.4][Er15][RZ16, §4][DBLM21]), after both AdS/CFT duality and intersecting D-brane models had been intensively studied for a decade, for assessment see [Witten23, 40:13-]. But all along this holographic perspective on hadronic strings had been promoted by Alexander Polyakov [Pol80][Pol97][GKP98][Pol99][Pol02] under the name *gauge-string correspondence*, referring to the brane/bulk-perspective as the *wall and the cave* in reference to Plato, see the historical reminiscences in [Pol12].

[4]The possibility that the Yang-Mills Mass Gap Problem might be solved by developments in M-theory was mentioned already, albeit briefly, in Atiyah's original presentation of the Clay Millennium Problems, see [Atiyah00], starting at 41:47.

[5]Cf. G. Moore: *Keep true to the dreams of thy youth: M-theory*, §12 of: *Physical Math and the Future*, talk at Strings2014 [www.physics.rutgers.edu/ gmoore/PhysicalMathematicsAndFuture.pdf]

[6]For an exposition to Homotopy Type Theory (HoTT) in our context see [TQC2] in §2.4.

1 Flux Quantization

In higher gauge theories [Al24, §2][BF+24], *flux* of *field lines* is sourced by charged branes (§1.1), and *flux quantization* makes fluxes/charges form a discrete space, reflecting individual brane sources (§1.2). A choice of flux quantization is a hypothesis about or specification of the non-perturbative completion of the given theory (§1.3).

Tradition, originating in the ancient past, is to define any physical theory by a *stationary action principle* embodied by a Lagrangian density (e.g. [HT92]), from which one systematically extracts a perturbative phase space in the guise of a BRST-BV complex. But flux-quantization laws used to be imposed in ad-hoc fashion to "cancel anomalies" (cf. pp. 25).

In contrast, we observe [SS23-FQ] that all admissible flux-quantization laws $\mathcal{A}$ are algebro-topologically determined by the duality-symmetric form of the Bianchi identity or Gauß law (30) satisfied by the flux densities.

Hypothesizing an admissible flux quantization law $\mathcal{A}$, the non-perturbative phase space is the moduli stack of differential $\mathcal{A}$-cohomology on any Cauchy surface.
Among the admissible flux-quantization laws is typically an "evident" one. In traditional examples like electromagnetic or RR-fields this evident choice is the traditional choice, whose hypothetical nature tends to be forgotten.

The "Hypothesis H" is essentially nothing but the corresponding evident choice of flux quantization for the C-field in 11d supergravity.

The reason why this was not so "evident" earlier is that the admissible flux-quantization laws of the C-field are *non-abelian* (unstable) forms of generalized cohomology (owing to the non-linear sourcing of M2-brane flux by M5-brane flux), whose theory was fully established only in [?].

	Survey of Part 1	**Higher gauge field** of Maxwell-type	**A-field** in $D=4$	**B&RR-field** in $D=10$	**C-field** in $D=11$
§1.1	Flux densities	$\vec{F} \equiv \big(F^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(X^D)\big)_{i\in I}$	F_2 magnetic G_2 electric	H_3 NS5 H_7 F1 $F_{2\bullet}$ $D_{8-2\bullet}$	G_4 M5 G_7 M2
	Self-duality	$\star\vec{F} = \vec{\mu}(\vec{F})$	$\star F_2 = G_2$	$\star H_3 = H_7$ $\star F_{2\bullet} = F_{10-2\bullet}$	$\star G_4 = G_7$
	Bianchi identities	$\mathrm{d}\vec{F} = \vec{P}(\vec{F})$	$\mathrm{d}F_2 = 0$ $\mathrm{d}G_2 = 0$	$\mathrm{d}H_3 = 0$ $\mathrm{d}H_7 = 0$ $\mathrm{d}F_{2\bullet} = H_3 \wedge F_{2\bullet-2}$	$\mathrm{d}G_4 = 0$ $\mathrm{d}G_7 = -\tfrac{1}{2}G_4 \wedge G_4$
§1.2	CE-algebra of characteristic L_∞-algebra	$\mathrm{CE}(\mathfrak{a}) \equiv \mathbb{R}[\vec{b}\,]/(\mathrm{d}\vec{b} = \vec{P}(\vec{b}))$	$\mathrm{d}f_2 = 0$ $\mathrm{d}g_2 = 0$	$\mathrm{d}h_3 = 0$ $\mathrm{d}h_7 = 0$ $\mathrm{d}f_{2\bullet} = h_3 \wedge f_{2\bullet-2}$	$\mathrm{d}g_4 = 0$ $\mathrm{d}g_7 = -\tfrac{1}{2}g_4 \wedge g_4$
	Solution space of fluxes on $X^D = \mathbb{R}^{0,1}\times X^d$	Gauß law = $\mathfrak{a}$-closedness $\mathbf{\Omega}_{\mathrm{dR}}(X^d;\,\mathfrak{a})_{\mathrm{clsd}} \equiv \mathrm{Hom}\big(\mathrm{CE}(\mathfrak{a}),\, \Omega^\bullet_{\mathrm{dR}}(X^d)\big)$	$\Omega^2_{\mathrm{dR}}(X^d)_{\mathrm{clsd}} \times \Omega^2_{\mathrm{dR}}(X^d)_{\mathrm{clsd}}$ can. momenta	3-twisted de Rham cocycles	"4-twisted" de Rham cocycles
	Characteristic L_∞-algebra	$\mathfrak{a}$	$b\mathfrak{u}(1) \oplus b\mathfrak{u}(1)$	$[b_2, v_{2\bullet-1}] = v_{2\bullet+1}$	$[v_3,\, v_3] = v_6$ M-theory gauge algebra
	as rational Whitehead L_∞-algebra	$\mathfrak{a} \simeq \mathfrak{l}\mathcal{A}$	$\mathfrak{l}(B^2\mathbb{Z} \times B^2\mathbb{Z})$	$\mathfrak{l}\big((\mathrm{KU}_0 /\!\!/ B^2\mathbb{Z}) \times B^7\mathbb{Z}\big)$	$\mathfrak{l}(S^4)$
§1.3	Evident choice of classifying space	$\mathcal{A}$	$B^2\mathbb{Z} \times B^2\mathbb{Z}$ Dirac's hypothesis	$(\mathrm{KU}_0 /\!\!/ B^2\mathbb{Z}) \times B^7\mathbb{Z}$ Hypothesis K	S^4 Hypothesis H
	Corresponding cohomology theory	generalized cohomology	ordinary cohomology	twisted K-theory	unstable CoHomotopy
	Flux-quantized phase space	$\mathbf{\Omega}_{\mathrm{dR}}(X^d;\mathfrak{a})_{\mathrm{clsd}} \underset{L^{\mathbb{R}}\mathcal{A}(X^d)}{\times} \mathcal{A}(X^d)$	differential cohomology	differential twisted K-theory	differential CoHomotopy

1.1 Branes imprinted on flux

The concept of *branes* (see [IU12, §6][Fr13, §7][HSS19, §2]) is the core aspect of the historical re-thinking of string theory that came to be known as the "second superstring revolution" [Schw96], in that it is the key for the non-perturbative completion of the theory [Du00]: The "M" in "M-theory" originates [HW96, p. 2] as a "non-committal" abbreviation for *membrane*.

Conversely this means that the precise meaning of "brane" has been almost as elusive as that of "M-theory" itself. Or rather: There is a range of specialized meanings of the term, some versions of which do have precise definitions, but it has remained unclear how exactly any and all of these notions are aspects of a unified concept of "branes".

Our strategy for formalizing the concept of branes is by careful examination of the **nature of the flux** they source.

(1.) We recall here (§1.1) the notion of **branes as** higher-dimensional generalizations of *poles* and hence as **sources of flux**.

(2.) We **express this notion in algebro-topological terms** (§1.2), from which we motivate *Hypothesis H* (§1.3).

(3.) We observe a natural notion of **light-cone quantization** by passage to Pontrjagin homology algebras (§2.2).

(4.) All inspection of brane physics in §2 proceeds by mathematical unravelling of this algebro-topological stucture.

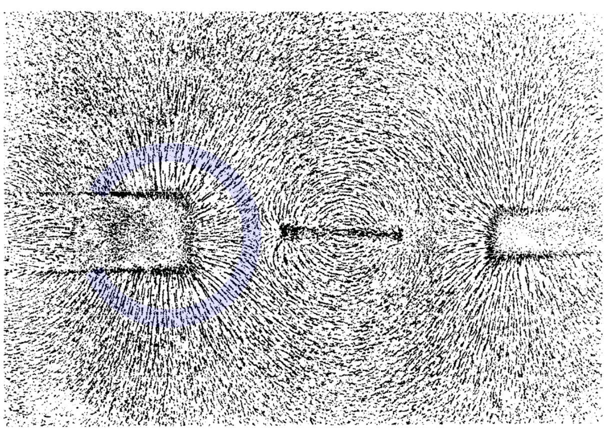
1.1.1 Branes as concentrations of flux

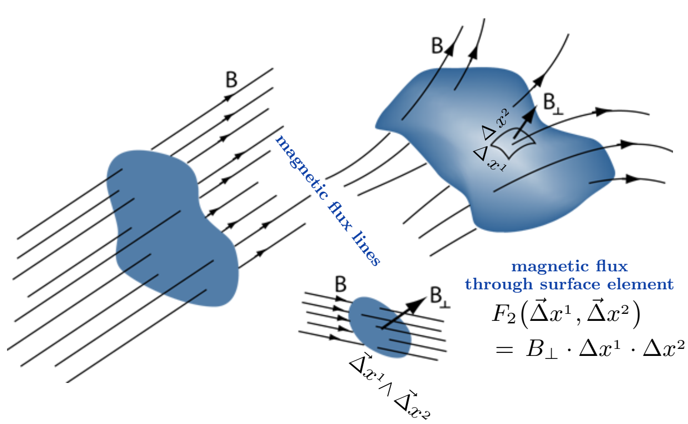
To get ground under our feet, it is expedient — our ambitious goal nonwithstanding — to start with elementary reflections on *flux lines* (*flux densities*) sourced by charged *poles* as originally conceived by Faraday in the 19th century, and as more generally sourced by higher dimensional charged *branes*, like the charged *membranes* already considered by [Dirac1962]. While most of these objects (famously including magnetic mono-poles) are notorious for remaining hypothetical entities not currently seen in experiment, we highlight (p. 7) the example of *vortex strings* in superconductors which have been observed in detail and which — whether one likes to refer to them as "1-branes" or not — do constitute an example of the general notion of classical branes in question.[a] This highlights the relevance of the distinction between singular and solitonic branes (and their difference in dimension) which may not to have received due attention before, cf. §2.3.3.

[a] A suggestion originally due to [Nielsen & Oleson 1973], cf. also [Beekman & Zaanen 2011]. [Polyakov 12, p. 1] regrets not to have understood electromagnetic vortices as strings.

General flux densities.		
X^D	$\in$ Mfds	Spacetime manifold
$\Omega^r_{\mathrm{dR}}(X^D)$	$\in$ Sets	Differential r-forms
$F^{(i)}$	$\in \Omega^{\deg_i}_{\mathrm{dR}}(X^D)$	Flux density
$\star : \Omega^r_{\mathrm{dR}}(X^D) \to \Omega^{D-r}_{\mathrm{dR}}(X^D)$		Hodge star (§1.1.2)

Electromagnetic flux density.	
X^4	Spacetime 4-fold
$F_2 \in \Omega^2_{\mathrm{dR}}(X^4)$	Faraday tensor
$= \star(E_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j)$	Electric flux density
$+ B_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$	Magnetic flux density





Magnetic flux lines. On the left: Iron filings in the magnetic field around magnetic poles (from *Faraday's diary of experimental investigation*, entry of 11th Dec 1851, reproduced by Martin 2009). The light circle around one of the poles is our addition, for emphasis. On the right: Magnetic flux(-line) density as a differential 2-form (adapted from hyperphysics.phy-astr.gsu.edu/hbase/magnetic/fluxmg.html).

Solitonic vs. Singular branes. Imprinted on the flux densities may be *two kinds* of branes, to be called:[7]

(1.) **singular branes** (*black branes*) reflected in **diverging flux density** at **singular loci** in spacetime,

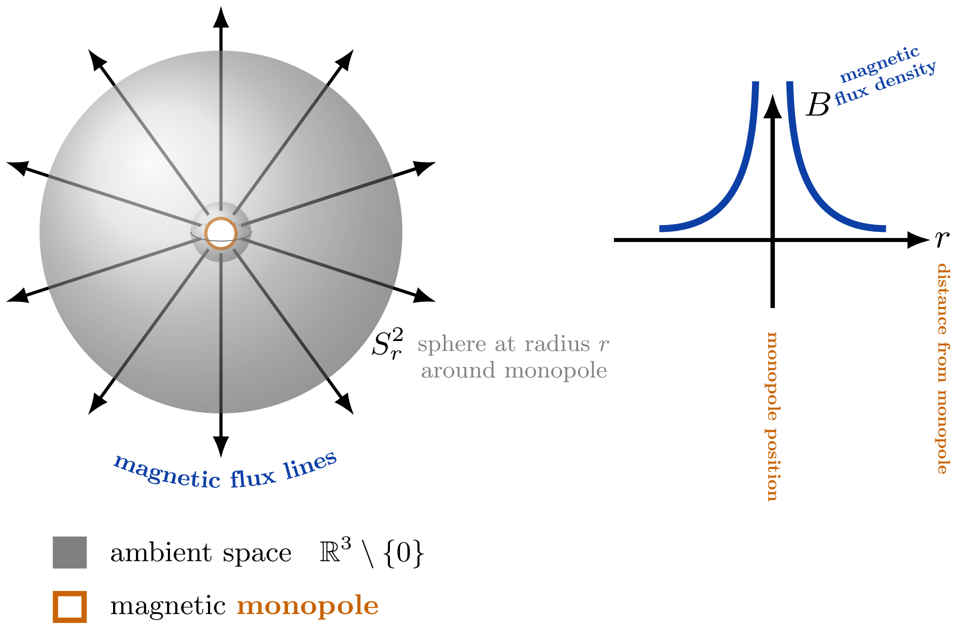
(2.) **solitonic branes** reflected in *localized* but **finite flux density**, namely **vanishing at infinity**, transversally

This distinction is often not made quite clear in the literature, but it is crucial for the analysis of brane effects (§2). Therefore we first highlight the issue in the familar case of electromagnetism:

The **singular branes of 4d electromagnetism** are the (would-be) **magnetic monopoles**:

$$F_2 \;=\; \underbrace{B(r)}_{\text{magnetic flux density}} \mathrm{dvol}_{S^2_r}\,, \qquad \overbrace{\mathrm{d}\,F_2 \;=\; 0}^{\text{Gauß law, away from singular locus}}\,, \qquad \int_{S^2_r} F_2 \;=\; \text{integrated magnetic flux through any sphere around the monopole, prop. to monopole charge} \tag{1}$$

(Annotations: F_2 — Faraday tensor; $B(r)$ — magnetic flux density.)



But the **solitonic branes of 4d electromagnetism** are the **vortex strings** in type II super-conductors ("Abrikosov vortices") inside an external electromagnetic field. Here the 1-brane is the central locus (the eye of the storm) of a (non-singular) *vortex* in the electron current J, localized by the requirement that fields *vanish at infinity* (cf. e.g. [Timm 2020 (6.101)]):

$$F_2 \;=\; B(r)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \qquad \overbrace{\mathrm{d}\,F_2 \;=\; 0}^{\text{Gauß law}}\,, \qquad \int_{\mathbb{R}_1\times\mathbb{R}_2} F_2 \;=\; \int B\,\mathrm{d}x^1\,\mathrm{d}x^2 \;=\; \text{total magnetic flux prop. to \# vortices} \tag{2}$$

(Annotations: F_2 — Faraday tensor; $B(r)$ — magnetic flux density.)

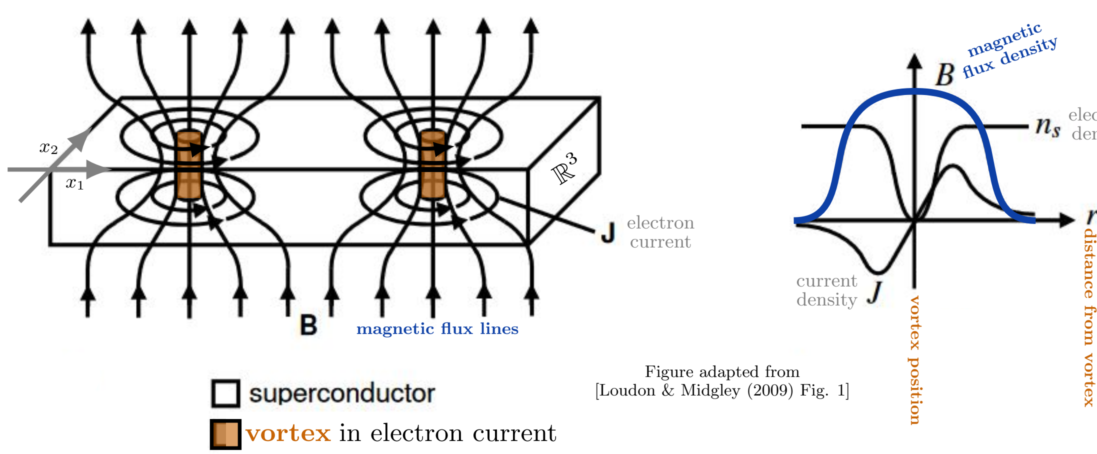


Figure adapted from [Loudon & Midgley (2009) Fig. 1]

Dirac-	*monopoles*	are the	singular branes	associated with the EM field,
Abrikosov-	*vortex strings*		solitonic branes	

[7]Beware that, while the terminology "solitonic brane" is wide-spread, its exact meaning differs between authors (as does the term "soliton" that it is derived from): It was introduced in [DKL92][DKL95][DL94] to mean (topologically stable) *non-singular* brane-like solutions to (supergravity/flux) equations of motion, which is how we use it here. But already [St99] uses "solitonic" to instead mean the "electromagnetic-dual singular brane", e.g. calling the singular NS_5 the soliton of the fundamental string, cf. (80). Somewhat in this vein, many later authors (e.g. [Sm03]) use "solitonic" for any singular or non-singular brane-like supergravity solution, thus regarding it as the antonym to the fundamental sigma-model branes discussed in §2.1.1. This is how we ourselves use it elsewhere.

The **formalization of the difference between singular and solitonioc branes** is via choices of *domains* on which the flux densities are actually defined (following [SS23-HpH2, §2.1]).

Type of brane	Spacetime domain of flux density	
Singular brane	complement of singular worldvolume locus Q^{p+1} inside spacetime X^{d+1}	$X^{d+1} \setminus Q^{p+1}$
Solitonic brane	Transverse space Y^{d-p} to worldvolume equipped with a "point at infinity"	$\big(Y^{d-p},\, \infty_Y\big)$

This is most transparent for the special case of "flat" branes in flat Minkowski spacetime:

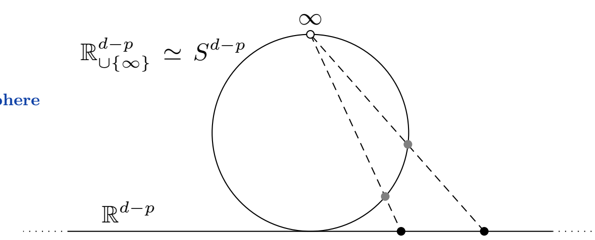
– **singular branes** have spacetime singularities which are *removed from spacetime*: the field flux sourced by the singularity is that through spheres in the normal bundle around these loci and *would diverge* at the singular brane locus (cf. (14) below):

$$\underbrace{\mathbb{R}^{d+1}}_{\text{bulk}} \setminus \underbrace{\mathbb{R}^{p+1}}_{\text{singular brane}} \underset{\text{homeomorphism}}{\simeq} \underbrace{\big(\mathbb{R}^{d-p} \setminus \{0\}\big)}_{\text{punctured transverse space}} \times \mathbb{R}^{p+1} \underset{\text{homotopy equivalence}}{\simeq} \underbrace{S^{d-p-1}}_{\text{encircling sphere}} \tag{3}$$

– **solitonic branes** are witnessed by non-singular "local bumps" in the flux densities: Their flux *vanishes at infinity*, which means that it is measured on the 1-point compactification of their transverse space, which is again a sphere:

$$\underbrace{\mathbb{R}^{p+1}_{+}}_{\text{solitonic brane}} \wedge \underbrace{\mathbb{R}^{d-p}_{\cup\{\infty\}}}_{\text{transv. space}} \underset{\text{homotopy}}{\simeq} \mathbb{R}^{d-p}_{\cup\{\infty\}} \underset{\text{homeo}}{\simeq} \underbrace{S^{d-p}}_{\text{transverse sphere}} \tag{4}$$

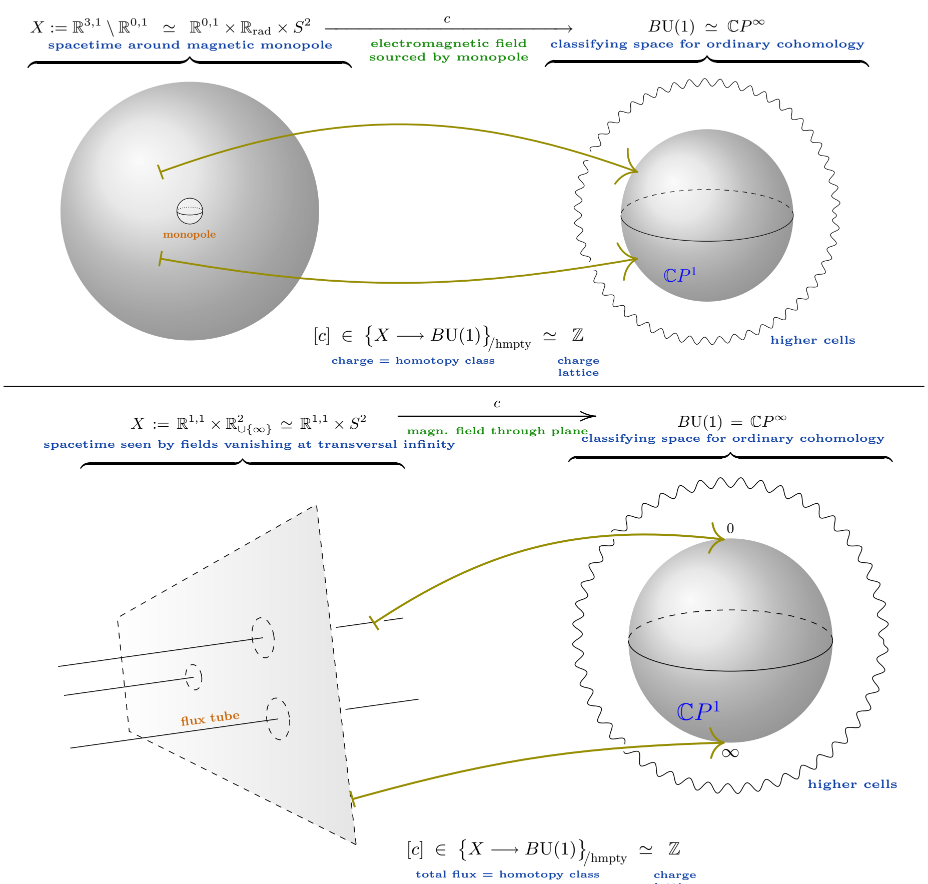
with point at infinity



Towards flux quantization. The laws of flux discussed so far are laws of "classical physics": By themselves, they do not explain, for instance, why the flux carried by Abrikosov vortices (p. 7) is *quantized* to appear in integer multiples of a unit flux, or why, as argued long ago by Dirac, magnetic monopoles would be quantized to appear in integer multiples of unit charged monopoles. Apparently the electromagnetic flux density $F_2 = \Omega^2_{\mathrm{dR}}(X)$ is just one aspect of the true nature of the electromagnetic field.

In modern mathematical language, the argument underlying *Dirac charge quantization* says that an electromagnetic field configuration on a spacetime X *also* involves a "charge map" $c : X \to B\mathrm{U}(1)$ to the *classifying space* of the circle group. This may be understood as the infinite complex projective space $B\mathrm{U}(1) \underset{\text{whe}}{\simeq} \mathbb{C}P^\infty$, but crucially it is a *classifying space* for ordinary integral cohomology in degree 2, meaning that homotopy classes of such maps are in natural bijection with $H^2(X;\mathbb{Z})$.

Formalizing generalized flux quantization is the topic of §1.2.



1.1.2 Laws of flux

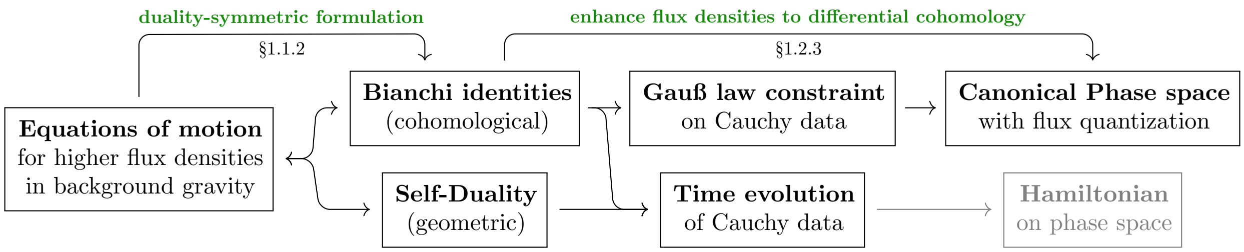
As we now turn to the equations of motion for flux densities (the analogs of Maxwell's equations), the key move towards identifying possible flux quantization laws (below in §1.2) is to arrange these equations, equivalently, as:

(i) a purely **cohomological** system of differential equations known as higher **Bianchi identities**,

(ii) a purely **geometric** system of linear equations expressing a Hodge **self-duality**,

the point being that the first item is entirely "algebro-topological" (homotopy-theoretic), while dependency on geometry, namely on the spacetime metric (the field of gravity) is all isolated in the second item.

It turns out [SS23-FQ] that from such *duality-symmetric laws of flux*, the canonical phase space of the higher gauge theory, including the flux-quantization structure may be obtained straightforwardly, *without* going through the traditional and thorny route of BRST-BV analysis based on an action principle given by a Lagrangian density.



(5)

Duality-symmetric equations of motion The move of isolating "pre-metric flux equations" supplemented by a "constitutive" duality constraint has a curious status in the literature. On the one hand, it is elementary and immediate as an equivalent re-formulation of the usual form of (higher) Maxwell-type equations of motion, and as such has been highlighted, in the case of electromagnetism (6), a century ago [Ko1922][Cartan1924, §80][vDa1934][Whit1953, pp. 192], and re-amplified more recently under the name "pre(geo)metric electromagnetism" [HO03][Del05][HIO16][Del]; but the broader community does not seem to have taken much note of this yet. On the other hand, we may observe below in (8) and (10) that just the same "pregeometric" perspective, applied to higher degree flux forms, evidently underlies what in supergravity and string thoery is called "duality-symmetric" or (for better or worse: "democratic") formulations of supergravity fields – see the following examples.

Electromagnetism. Concretely, Maxwell's equations for electromagnetic flux encoded in the Faraday tensor F_2 as shown on the left (cf. [Th78, v2 §1.3][Fr97, §7.2b][Na11, §2.2]) have (pre)geometric decomposition as shown on the right (cf. already [Cartan1924, §80]):

$$\boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\star F_2 &=& J_3 \end{array}} \;\longleftrightarrow\; \left\{ \begin{array}{l} \boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\,G_2 &=& J_3 \end{array}} \\[6pt] \boxed{G_2 \;=\; \star F_2} \end{array} \right. \tag{6}$$

While trivially equivalent, some authors found deep significance to the pre-geometric decomposition on the right of (6), (cf. the careful discussion of classical electromagnetism in [HO03]) highlighting that it maximally disentangles gauge from gravitational degrees of freedom (already in [Whit1953, p. 192][8]) and thus possibly helping with understanding unification of the two (cf. [Del15]). None of what we say here refers to or relies on any of these previous discussions of premetric electromagnetism, but the inclined reader may find value in comparing to them.

Here we are interested in the "vacuum"-case where the electric current density J_3 vanishes:

$$\boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\star F_2 &=& 0 \end{array}} \;\longleftrightarrow\; \left\{ \begin{array}{l} \boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\,G_2 &=& 0 \end{array}} \\[6pt] \boxed{G_2 \;=\; \star F_2} \end{array} \right. \tag{7}$$

[8] [Whit1953, p. 192]: "Since the notion of metric is a complicated one, which requires measurements with clocks and scales, generally with *rigid* bodies, which themselves are systems of great complexity, it seems undesirable to take metric as fundamental, particularly for phenomena which are simpler and actually independent of it."

It is clear that an analogous transmutation – by first "doubling" the flux degrees of freedom and then cutting them back down by half via a self-duality constaint – applies also to equations of motion for higher degree fluxes (made explicit for instance in [Fr00, Ex. 3.8]):

RR-fields in 10d supergravity. In evident higher-degree generalization of vacuum electromagnetism, consider a "higher gauge field" whose flux density is a tuple of differential forms F_{2p} in every second degree smaller or equal the spacetime dimension D: and satisfying the evident higher-degree generalization of Maxwell's equations, as shown on the left, where the equivalent pregeometric formulation shown on the right is rather more suggestive in its conceptual simplicity (but its physical justification remains subtle, see below):

$$\boxed{\begin{array}{rcl} \mathrm{d}\,F_{2\bullet} &=& 0 \\ \mathrm{d}\star F_{2\bullet} &=& 0 \\ \forall\, 2\bullet+\sigma &\leq& 5 \end{array}} \longleftarrow \begin{cases} \boxed{\mathrm{d}\,F_{2\bullet+\sigma} \;=\; 0 \quad \forall\, 2\bullet+\sigma} \\[6pt] \boxed{F_{(10-2\bullet-\sigma)} \;=\; \star\, F_{2\bullet+\sigma}} \end{cases} \qquad \begin{array}{l} \bullet \in \mathbb{N} \\[4pt] \sigma = \begin{cases} 0 & \text{for type IIA} \\ 1 & \text{for type IIB} \end{cases} \end{array} \tag{8}$$

More generally:

$$\boxed{\begin{array}{rclrcl} \mathrm{d}\,F_{2\bullet+\sigma} &=& H_3 \wedge F_{2\bullet+\sigma-2} & \mathrm{d}\,H_3 &=& 0 \\ \mathrm{d}\star F_{2\bullet+\sigma} &=& H_3 \wedge \star F_{D-2\bullet-\sigma+2} & \mathrm{d}\star H_3 &=& \cdots \end{array}} \longleftarrow \begin{cases} \boxed{\mathrm{d}\,F_{2\bullet+\sigma} \;=\; H_3 \wedge F_{2\bullet+\sigma-2} \quad \begin{array}{rcl} \mathrm{d}\,H_3 &=& 0 \\ \mathrm{d}\,H_7 &=& \cdots \end{array}} \\[6pt] \boxed{F_{D=2\bullet-\sigma} \;=\; \star\, F_{2\bullet+\sigma} \qquad H_7 \;=\; \star\, H_3} \end{cases}$$

But this is just the situation of the "RR-fields" in $D = 10$-dimensional massive type IIA supergravity for the special case where the background NS-fields besides the metric vanish (we discuss the more general case in §1.1.3): On the left of (8) we have the equations of motion of the RR-field fluxes in their original "geometric" form (e.g. [Po95, (3)][IU12, §4.2.5]), while on the right of (8) we have the RR-fields in their pregeometric form, now commonly called the "duality-symmetric" or "democratic" form (see [CJLP98, §3][BKORV01][MV23]), this being the form which plays into the "Hypothesis K" that D-brane charge is quantized in K-theory (56).

Self-dual higher gauge theory. In type IIB the equations (8) describe one flux density which is actually Hodge dual (not just to another flux density in the tuple but) to itself:

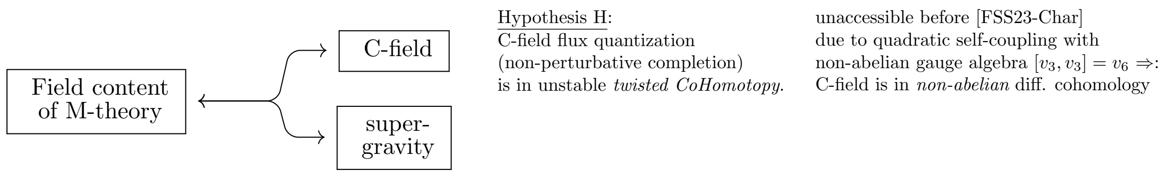
$$F_5 \;=\; \star F_5\,.$$

Such genuinely "self-dual higher gauge theories" are notoriously subtle to describe traditionally via Lagrange-densities, but the equations of motion of their flux densities, at least, is just a particular special case of general duality-symmetric equations of motion:

$$\boxed{\begin{array}{c} \text{equations of motion of} \\ \text{self-dual higher gauge field} \\ \text{in } D = 4k+2 \end{array}} \longleftarrow \begin{cases} \boxed{\mathrm{d}\,F_{D/2} \;=\; 0} \\[6pt] \boxed{F_{D/2} \;=\; \star\, F_{D/2}} \end{cases}$$

C-field in 11d supergravity. The primary example of interest here is that of "C-field flux" G_4 in 11-dimensional supergravity (aka the "3-index A-field" [CJS78], see [DF82, §p. 131][MiSc06, p. 32][vPF12, §10]), which is meant to be the low-energy approximation of M-theory (see [Du00, §1]).

It is noteworthy that the C-field is the *only* field in $D = 11$ SuGra, besides the field of (super-)gravity itself (quite in contrast to the zoo of fields that appear in lower dimensional supergravities). This may make it plausible that the proper non-perturbative completion of the C-field alone may go at least halfway towards a full definition of M-theory.



(9)

In any case, the higher Maxwell-type equations for the C-field flux are famously as follows:

$$
\boxed{\begin{aligned} \mathrm{d}\, G_4 &= 0 \\ \mathrm{d} \star G_4 &= -\tfrac{1}{2} G_4 \wedge G_4 \end{aligned}}
\;\longleftarrow\;
\left\{
\begin{aligned}
&\boxed{\begin{aligned} \mathrm{d}\, G_4 &= 0 \\ \mathrm{d}\, G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \end{aligned}} \\
&\boxed{G_7 \;=\; \star G_4}
\end{aligned}
\right.
\tag{10}
$$

On the left of (10) we have the equations of motion in their traditional geometric form ([DF82, p. 131], detailed review in [CDF91, §III.8.53][MiSc06, (3.23)]), on the right their pregeometric form, known as the *duality-symmetric* form, cf. [BBS98][CJLP98][BNS04, §2][Nu03, §3].

The double-dimensional reduction of these duality-symmetric equations of motion over a circle-bundle with non-trivial first Chern class F_2 is ([MaSa04, §4])

$$
\begin{array}{ccc}
S^1 \hookrightarrow & Y^{11} & \\
 & \big\downarrow{\scriptstyle p} & \\
 & X^{10} &
\end{array}
\qquad
\begin{aligned}
\mathrm{d}\,\theta &= p^* F_2 \\
G_4 &= p^* F_4 - \theta \wedge p^* H_3 \\
G_7 &= p^* H_7 - \theta \wedge p^* F_6
\end{aligned}
$$

$$
\boxed{\begin{array}{c} \text{dimensionally reduced} \\ \text{equations of motion} \\ \text{of C-field flux in 11d} \\ \text{to B\&RR-field flux in 10d} \\ \text{+ KK-monopole flux EoM} \end{array}}
\;\longleftarrow\;
\left\{
\begin{aligned}
&\boxed{\begin{aligned} \mathrm{d}\, F_2 &= 0 & \mathrm{d}\, H_3 &= 0 \\ \mathrm{d}\, F_4 &= H_3 \wedge F_2 & \mathrm{d}\, H_7 &= -\tfrac{1}{2} F_4 \wedge F_4 \\ \mathrm{d}\, F_6 &= H_3 \wedge F_4 & &\quad + F_2 \wedge F_6 \\ \mathrm{d}\, F_8 &= H_3 \wedge F_6 & & \end{aligned}}\;\text{11d C-field gravity} \\
&\boxed{\begin{aligned} \star F_4 &= F_6 & \star H_3 &= H_7 \\ \star F_2 &= F_8 & & \end{aligned}}\;\text{C-field gravity}
\end{aligned}
\right.
$$

Here the new gravitational flux density F_2 is understood as witnessing singular D_6-brane sources in the geometric guise of "KK-monopoles" in 11d. Notice that its Hodge dual $F_8 := \star F_2$ is purely a reflection of the gravitational field (the Hodge star encodes the 10d metric and F_2 is an aspect of the 11d fiber geometry) and hence not encoded by the equations for the C-field alone. But invoking the supergravity equations gives [CJLP98, (3.4)] that $\mathrm{d}\, F_8 = H_3 \wedge F_6$.

Notice also the presence of non-linear Bianchi identity for H_7 (cf. again [CJLP98, (3.4)])

B-field in 5d supergravity. Just to amplify that the pre-geometric decomposition of field-flux equations of motion is a generic phenomenon, we briefly mention one more example:

The flux forms in 5-dimensional supergravity (cf. [CDF91, §III.5.70]) satisfy an equation of motion analogous to the equation (10) for the C-field in 11-dimensional supergravity (cf. [GGHPR03, (2.2)]):

$$
\boxed{\begin{aligned} \mathrm{d}\, F_2 &= 0 \\ \mathrm{d} \star F_2 &= F_2 \wedge F_2 \end{aligned}}
\;\longleftarrow\;
\left\{
\begin{aligned}
&\boxed{\begin{aligned} \mathrm{d}\, F_2 &= 0 \\ \mathrm{d}\, H_3 &= F_2 \wedge F_2 \end{aligned}} \\
&\boxed{H_3 \;=\; \star F_2}
\end{aligned}
\right.
\tag{11}
$$

Summarizing this list of examples (which could be much expanded) of duality-symmetric equations of motion of flux densities:

Flux species	equations of motion (of fields in background gravity)	$\Leftrightarrow$	Bianchi identities (purely cohomological)	with	duality constraint (wrt background metric)
free A-field in 4d gravity	$\left.\begin{aligned} \mathrm{d} \star F_2 &= 0 \\ \mathrm{d}\, F_2 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}\, G_2 &= 0 \\ \mathrm{d}\, F_2 &= 0 \end{aligned}\right.$	where	$G_2 = \star F_2$
A-field & B-field in 5d supergravity	$\left.\begin{aligned} \mathrm{d} \star F_2 &= F_2 \wedge F_2 \\ \mathrm{d}\, F_2 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}\, H_3 &= F_2 \wedge F_2 \\ \mathrm{d}\, F_2 &= 0 \end{aligned}\right.$	where	$H_3 = \star F_2$
C-field in 11d supergravity	$\left.\begin{aligned} \mathrm{d} \star G_4 &= G_4 \wedge G_4 \\ \mathrm{d}\, G_4 &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{\begin{aligned} \mathrm{d}\, G_7 &= G_4 \wedge G_4 \\ \mathrm{d}\, G_4 &= 0 \end{aligned}\right.$	where	$G_7 = \star G_4$
free RR-field in 10d supergravity	$\left.\begin{aligned} \mathrm{d} \star F_{2\bullet \leq 5} &= 0 \\ \mathrm{d}\, F_{2\bullet \leq 5} &= 0 \end{aligned}\right\}$	$\Leftrightarrow$	$\left\{ \mathrm{d}\, F_{2\bullet} = 0 \right.$	where	$F_{10-2\bullet} = \star F_{2\bullet}$

(12)

The singular branes of string/M-theory. The sources for the RR-field flux (8) are, of course, the *D-branes* originally proposed in [Po95, (14)]; and for the present purpose this may be regarded as the *definition* of (classical) D-branes: concentrations of RR-field flux (cf. exposition in [Ha12]).
Similarly, we identify other (classical) brane species as sources of corresponding flux:

	Field	Flux	Singular source
$D=4$ Maxwell theory	A-field	F_2	monopole 0-branes
$D=10$ supergravity	B-field	H_3	NS5-brane
		H_7	F1-branes
	RR-field	F_{8-p}	Dp-branes
$D=11$ supergravity	C-field	G_4	M5-branes
		G_7	M2-branes

(13)

More in detail, traditionally the singular branes of string/M-theory are defined as singular solutions of the corresponding supergravity equations of motion ("black branes" in higher dimensional generalization of black holes), preserving some amount of supersymemtry ("BPS branes"), for further pointers see [HSS19, §2.2].

But the *near-horizon geometry* of $> 1/4$-BPS black p-brane spacetimes are all Cartesian products of an anti-de Sitter spacetime with (a free discrete quotient of) a sphere around the singularity, such that the result is a warped metric cone over the p-brane singularity, as shown here ([FF98, dMFF+09]).

Near horizon spacetime	$\underbrace{\overset{\text{anti-de Sitter spacetime}}{\mathrm{AdS}_{p+2}}} \times S^{D-p-2} \;/G$
Metric in horospheric coord.	$\frac{R^2}{z^2} ds^2_{\mathbb{R}^{p,1}} + \frac{R^2}{z^2} dz^2 + \underset{\text{sphere around singularity}}{ds^2_{S^{D-p-2}}}$
Causal chart	$\overset{\text{singularity}}{\mathbb{R}^{p,1}} \times \underbrace{\overset{\text{radial direction}}{\mathbb{R}_+} \times S^{D-p-2}}_{\text{transversal space } \mathbb{R}^{D-p-2} \setminus \{0\}} \;/G$
Metric in natural coord.	$\underset{\frac{r^n}{\ell^n}\cdot \text{Minkowski}}{\frac{r^n}{\ell^n} ds^2_{\mathbb{R}^{p,1}}} + \underbrace{\frac{\ell^2}{r^2} dr^2 + \ell^2 ds^2_{S^{D-p-2}}}_{\frac{\ell^2}{r^2}\cdot \text{ metric cone } C(S^{D-p-2}) \setminus \{0\}}$

(14)

On a causal chart of AdS-spacetime this is nothing but the topological complement considered in (3). For example:

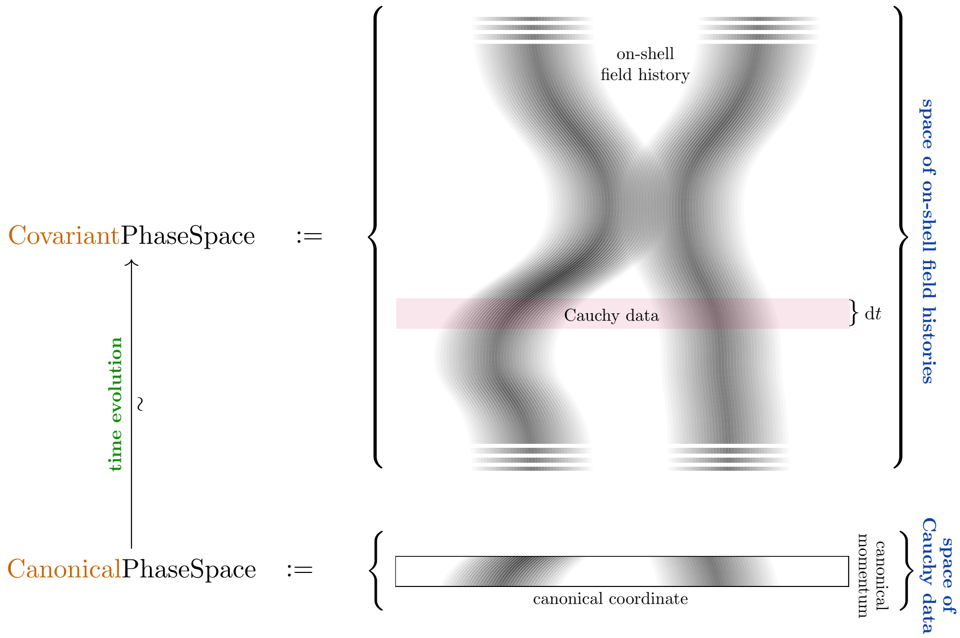
Black brane	Near horizon geometry	Causal chart
M2	$\mathrm{AdS}_4 \times S^7$	$\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7$
M5	$\mathrm{AdS}_7 \times S^4$	$\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4$

(15)

This indicates that a good deal of the nature of branes is actually independent of the metric/gravitational parameters and instead controlled purely algebro-topologically. We see why this is so by analysing the *phase space* of the flux fields sourced by the branes:

The phase space. Abstractly, the *phase space* of a field theory is nothing but the space of all those field histories that satisfy the given equations of motion (the "on-shell" field histories). Phrased this way, this is sometimes called the **covariant phase space** ([Wi86, p. 314][ČW87][HT92, §17.1]; see [Kh14][GiS23] for rigorous discussion) to emphasize that no choice of foliation of spacetime by Cauchy surfaces has been or needs to be made.

The more traditional **canonical phase space** is instead a parameterization of the covariant phase space by initial value data on a given Cauchy surface, after choosing a foliation of spacetime by spatial hypersurfaces (cf. [SS23-FQ, p. 5]). This choice breaks the "manifest covariance" of the covariant phase space. Nevertheless, if a Cauchy surface exists at all (hence on globally hyperbolic spacetimes), then both these phase spaces are equivalent, by definition, the equivalence being the map that generates from initial value data the essentially unique on-shell field history that evolves from it (possibly up to gauge transformation).



Solution space of on-shell flux densities. At this point in our discussion, we do not yet know what the full field content of our field theories really is – this will be implied by a choice of flux quantization in §1.2 – we only know the corresponding flux densities. To remember this, we shall call the space of flux densities solving their equations of motion the *solution space*, and we are after its incarnation as a *canonical solution space* of initial value data on a Cauchy surface. But the canonical phase will simply consist of all flux-quantized gauge potentials compatible with these flux solutions (cf. p. 24).

higher Maxwell-type equations of motion in duality-symmetric form

$$\boxed{\begin{array}{c}\textbf{Bianchi identities}\\ \mathrm{d}\vec{F} = \vec{P}(\vec{F}) \\ \star F = \vec{\mu}(\vec{F}) \\ \textbf{self-duality}\end{array}} \quad \boxed{\begin{array}{l} \underset{\text{flux species}}{I \in \mathrm{Set}}\,, \quad \underset{\text{flux degrees}}{\big(\deg_i \in \mathbb{N}_{\geq 1}\big)_{i\in I}}\,, \quad \underset{\text{flux densities}}{\vec{F} \equiv \Big(F^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}\big(X^D\big)\Big)_{i \in I}} \\ \underset{\text{flux self-sourcing}}{\vec{P} \text{ graded-symm. polynomial}}\,, \quad \underset{\text{vacuum permittivity}}{\vec{\mu} \text{ invertible matrix}} \end{array}} \tag{16}$$

Proposition 1.1 ([SS23-FQ]). *On a globally hyperbolic spacetime $X^D \simeq \mathbb{R}^{0,1} \times X^d$, the solution space to higher Maxwell-equations of motion brought into the duality-symmetric form (16) is isomorphic to the solution of the Bianchi identities on any Cauchy surface $\iota : X^d \hookrightarrow X^D$, then to be called the higher Gauß law:*

$$\begin{array}{rcll} \underset{\text{the equations of motion}}{\overset{\text{space of flux densities on spacetime, solving}}{\mathrm{SolSpace}}} & \equiv & \left\{ \underset{\text{electromagnetic flux densities on spacetime}}{\vec{F} \equiv \Big(F^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}\big(X^D\big)\Big)_{i\in I}} \;\middle|\; \begin{array}{c}\text{Bianchi identities}\\ \mathrm{d}\vec{F} = \vec{P}(\vec{F}) \\ \star F = \vec{\mu}(\vec{F}) \\ \text{self-duality}\end{array} \right\} & \textbf{covariant form} \\ & \underset{\iota^*}{\simeq} & \left\{ \underset{\text{magnetic flux densities on Cauchy surface}}{\vec{B} \equiv \Big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}\big(X^d\big)\Big)_{i\in I}} \;\middle|\; \begin{array}{c}\text{Gauß law}\\ \mathrm{d}\vec{B} = \vec{P}(\vec{B})\end{array} \right\} & \textbf{canonical form} \end{array} \tag{17}$$

Gravity "decouples" on canonical phase space. The inverse isomorphism (17) is given by time evolution of initial value data. Notice that the background metric (the background field of gravity) enters *only* in determining the nature of this isomorphism ι^*, but does not affect the nature of the initial value data (of the canonical phase space) as such (cf. [SS23-FQ]).

It is this "decoupling" *on the canonical phase space* of the gravity/metric effects from the phase space Gauß law constraint which allows to gain plenty of insight into brane configurations from purely cohomological analysis of fluxes on Cauchy surfaces, disregarding the full solution of the coupled (super-)gravity equations of motion, cf. the examples in §1.1.3 and §2.

In conclusion so far, when looking at classical flux densities sourced by branes, we are looking at systems of differential forms on a (pseudo-)Riemannian manifold satisfying polynomial exterior differential equations subject to a Hodge-duality constraint:

$$
\overbrace{
\underbrace{
\underset{\substack{\text{flux densities}\\ \text{differential forms on spacetime}}}{\big(F^{(i)} \in \Omega^{\mathrm{deg}_i}_{\mathrm{dR}}(X)\big)_{i\in I}}
\qquad
\underset{\substack{\text{Bianchi identities}\\ \text{cohomological part of field equations}}}{\mathrm{d}\,F^{(i)} \;=\; P^{(i)}\Big(\big\{F^{(j)}\big\}_{j\in I}\Big)}
}_{\text{pre-geometric cohomological aspect}}
\qquad
\underbrace{
\underset{\substack{\text{Hodge self-duality}\\ \text{gravity-dependent part of field equations}}}{\star\, F^{(i)} \;=\; \mu^i{}_j F^{(j)}}
}_{\text{Riemannian geometric aspect}}
}^{\text{higher gauge field fluxes in gravitational background}}
\tag{18}
$$

(Annotations: "flux densities" over $F^{(i)}$; "de Rham diff." over d; "polynomial" over $P^{(i)}$; "Hodge star" over $\star$; "permittivity" over $\mu^i{}_j$.)

But for which cohomology theory, really?
We answer this question in §1.2, §1.3.

Drops out in the canonical phase space.

The appearance of *quadratic* Bianchi identities in the above examples (12) — in particular for the C-field flux in 11d supergravity (10) and generally of *non-linear* polynomial Bianchi identities (18) — is a crucial effect not seen in ordinary electromagnetism and outside the scope of previous mathematical discussions of flux quantization. It is to handle the quantization of such *non-linear* flux that we invoke the *non-abelian* character theory developed in [?] and surveyed below in §1.2, which eventually allows identifying M-theory flux quantization in non-abelian (namely: unstable) cohomotopy (in §1.3).

But first, we here discuss the physics encoded by non-linear terms in the pregeometric equations of motion of flux.

1.1.3 Brane intersections imprinted in non-linear flux

Bound states of *intersecting* black branes are seen (e.g. [Sm03]) as solutions of full supergravity equations of motion; but in the pre-geometric spirit of §1.1.2 we highlight here that the qualitative aspects of the *brane intersection laws* may largely be deduced from the higher Gauß law on phase space (17) alone, as it ought to be by the nature of phase space.

Namely, it is the *non-linear* (quadratic and higher) polynomial source terms which encode the possibility that the branes which source these fluxes may "intersect" or "end on" each other in certain ways, as we explain now. Notice that it is only "on" such brane intersections that modern string phenomenology (namely all type I/II/M/F phenomenology, excluding only the traditional HET models) expects to model quasi-realistic physics (see [Ha12, §6.1, §6.4][IU12][RZ16, §15]).

Branes ending on branes (brane intersection laws)	⟷	**Non-linear Gauß law** (polynomial Bianchi identities)	⟷ [?]	**Non-abelian cohomology** (polynomial characters)

The relation on the left is fairly well-known in the case of D-branes ending on NS_5-branes, to be briefly recalled now, which is a "mild" form of non-linear flux since it may still be understood as "parameterized" or "twisted" linear flux (see around (52) below) and as such can and has been discussed by conventional means of flux quantization:
D-branes ending on NS5-branes. From the perspective (13) of flux densities, *NS 5-branes* are what source 3-form flux H_3 in type II supergravity, whose pregeometric equation of motion we may take to simply be[9] $\mathrm{d}\,H_3 = 0$. In the presence of such flux, the pregeometric equations for RR-field fluxes (8) are modified as follows (see references around (8)):

$$
\boxed{\mathrm{d}\,H_3 \;=\; 0}
\qquad
\boxed{\begin{array}{lcl}
\mathrm{d}\,F_0 &=& 0\\
\mathrm{d}\,F_2 &=& H_3 \wedge F_0\\
\mathrm{d}\,F_4 &=& H_3 \wedge F_2\\
\vdots
\end{array}}
\qquad
\mathrm{NS5}\; \underset{\mathrm{D}_p}{\overset{\mathrm{d}\,F_{8-p} \;=\; H_3\wedge F_{6-p}}{\rule{3cm}{1pt}}}\; \mathrm{D}_{p+2}
\tag{19}
$$

[9] We ignore here the dual NS-flux $H_7 = \star H_3$ in 10d supergravity: Its presence is actually a problem for the traditional *Hypothesis K* (§1.3.1) that NS/D-brane charge is in 3-twisted K-theory, while it plays no *direct* role in the formulation of *Hypothesis H* that we are after (§1.3.2). But interestingly, one proposal for incorporating H_7 into *Hypothesis K* also proceeds via cohomotopy, see [arXiv:1405.5844, §7.4] and [?, Ex. 3.6]

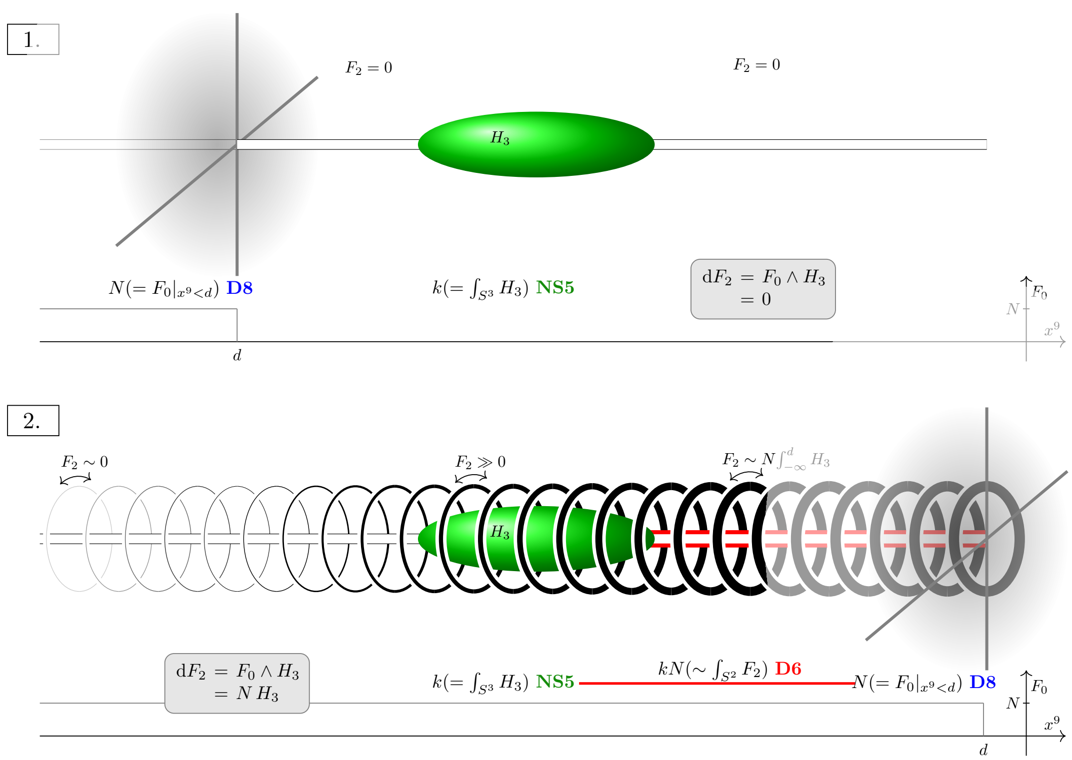
These differential equations, in particular the one for F_2, are not unlike the Maxwell equations (6) with a source term J_3, meaning here that NS5-branes via their H_3-flux but also D_{p+2}-branes via their F_{6-p}-flux act as a source or sink for RR-field flux F_{8-p} and hence for D_p-branes, in some way, suggesting that D_p-branes may *emanate from* or *end on* NS5-branes and D_{p+2}-branes (cf. e.g. [EGKRS08] and references in [Fa17]).

The full supergravity equations of motion for such $NS_5/D_p/D_{p+2}$-brane systems are complicated and satisfactory discussion is hard to cite, but we can readily give a full qualitative analysis of the solutions to the pre-metric Gauß (19) which already reveals the expected effects. This is going to be instructive for understanding the case of M2/M5-brane intersections that we are after further below in (20):

D6-brane creation and the Hanany-Witten effect. A popular conjecture by [HW97] states that the expected D_p-branes stretching between NS_5 and D_{p+2} are "created" as the D_{p+2}-branes are "dragged over" the NS_5, intuitively like a pole will cause a spike in a rubber sheet that is pulled over its tip. It was suggested in [Mar01, §2] that this *Hanany-Witten effect* should be understandable entirely from analysis of the flux Bianchi identities (19). At least for the case $p = 6$ of NS5/D6/D8-brane intersections [HZ98, §2.4][BLO98, p. 60] this is indeed the case, as we explain now. Here the flux F_0 of D_8-branes (the "Romans mass") is a locally constant function that vanishes in the vacuum and jumps by N units across the locus of N D_8-branes (cf. e.g. [Fa17, p. 40]). But this means that:

(1.) When the NS_5-brane is located in the vacuum then its sourcing of F_2-flux is "switched off" by the vanishing F_0-factor in (19), hence if F_2 vanishes at infinity then the PDE demands it vanishes everywhere, reflecting the absence of D_6-branes.

(2.) When the NS_5-brane is located on the other side of the D_8-branes, where $F_0 = N$, then the equation (19) shows that F_2-flux/D_6-number density which vanishes far away will increase along the coordinate axis x^9 orthogonal to the D_8-branes in proportionality to the dx^9-component of the flux H_3, and hence pronouncedly so as one crosses the NS_5-brane locus.



D_3-branes and M_2-branes streching between 5-branes. Consider now the case of flux configurations which should reflect branes stretching between pairs of 5-branes. By the previous discussion this occurs either for D_3-branes between NS_5/D_5-branes or for M_2-branes between M_5-branes, according to the following pregeometric equations of motion:

$$\begin{aligned} D=10: \quad \mathrm{d}\,F_5 &= H_3 \wedge F_3 && (19) \\ D=11: \quad \mathrm{d}\,G_7 &= \tfrac{1}{2} G_4 \wedge G_4 && (10) \\ &= G_4^{(1)} \wedge G_4^{(2)} \quad G_4 := \underbrace{G_4^{(1)}}_{\text{homog.}} + \underbrace{G_4^{(2)}}_{\text{homog.}} \end{aligned} \tag{20}$$

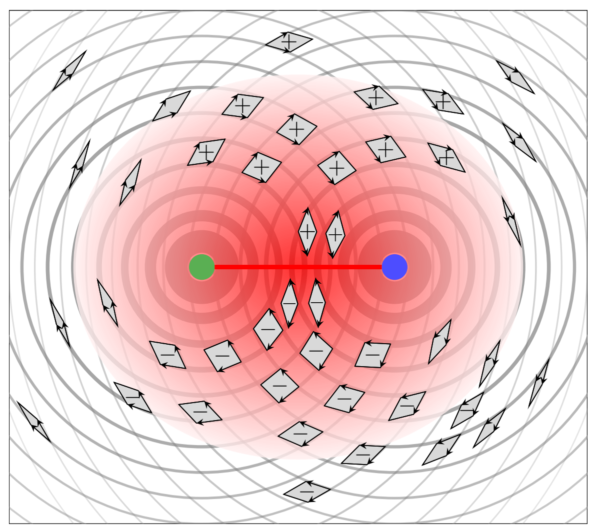
Considering a background configuration given by a pair of parallel flat 5-branes at some positive distance $2d > 0$:

$$\begin{array}{ccc} \mathbb{R}^{1,5}_{(i)} & \hookrightarrow & \mathbb{R}^{1,D} \\ (t,\vec{x}\,) & \longmapsto & \big(t,\vec{x},(-1)^i d,\vec{0}\,\big) \end{array}$$

hence reflected in flux densities of the following form (or any multiples of these, if you like)

$$\begin{aligned} H_3 &:= \mathrm{dvol}_{S^3} \in \Omega^3_{\mathrm{dR}}(S^3) \xhookrightarrow{\mathrm{pr}^*_{S^3}} \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(1)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^3\Big) \simeq \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,9} \setminus \mathbb{R}^{1,5}_{(1)}\Big) \\ F_3 &:= \mathrm{dvol}_{S^3} \in \Omega^3_{\mathrm{dR}}(S^3) \xhookrightarrow{\mathrm{pr}^*_{S^3}} \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(2)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^3\Big) \simeq \Omega^3_{\mathrm{dR}}\Big(\mathbb{R}^{1,9} \setminus \mathbb{R}^{1,5}_{(2)}\Big) \\ G_4^{(i)} &:= \mathrm{dvol}_{S^4} \in \Omega^4_{\mathrm{dR}}(S^4) \xhookrightarrow{\mathrm{pr}^*_{S^4}} \Omega^4_{\mathrm{dR}}\Big(\mathbb{R}^{1,5}_{(i)} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^4\Big) \simeq \Omega^4_{\mathrm{dR}}\Big(\mathbb{R}^{1,10} \setminus \mathbb{R}^{1,5}_{(i)}\Big) \end{aligned}$$

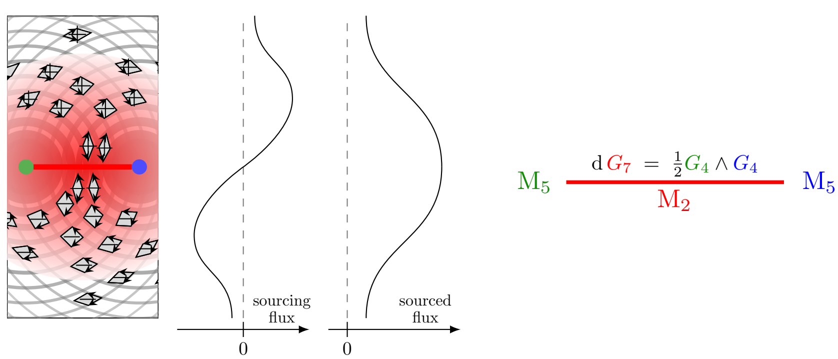
and assuming that D_3- or M_2-brane flux vanishes at infinity, then these differential equations will imply D_3- or M_2-brane flux, respectively as soon as the wedge products $H_3 \wedge F_3$ and $G_4^{(1)} \wedge G_4^{(2)}$ are *multi-poles* concentrated roughly between the given pair of branes. That and why this is indeed the case is illustrated by the following figure.



Effective dipole of quadratic brane flux. The figure on the right means to indicate the nature of the differential 2-form which is the wedge product of two copies of the pull-back of dvol_{S^1} to around either of the punctures (the brane loci) in the 2-punctured plane, the 2-dimensional shadow of the analogous wedge products on the right of (20). Here:

- the strength of the circular lines indicates the absolute value of the flux density sourced by the respective 5-brane,
- the arrows indicate the orientation of the flux density of either 5-brane,
- the parallelograms indicate the orientation of their wedge product.

Evidently the absolute value of the wedge product is concentrated near the 5-branes and particularly between them...



...but the orientation of the wedge product changes sign across the axis connecting the branes, as shown. This means that the flux *sourced* by this wedge product, according to (20), is, if vanishing at infinity, concentrated between the branes.
This sourced flux concentration (indicated in red) witnesses a brane stretching between the two 5-branes.

Such $M_5 \perp M_2 \perp M_5$-brane intersections are expected in the literature (e.g. [HLV14, Fig. 3]), but their demonstration as solutions of their flux equations of motion seems not to have been discussed before.

M_2**-Branes ending on** M_1**-waves and C-field tadpoles.** Less widely appreciated is that M_2-branes are also argued [BPST10, §2.2.3][HSS19, Prop. 4.19] to possibly end on 1-brane-like loci known as "M-waves". In terms of pregeometric fluxes this means, by the previous arguments, that there ought to be an $9 - 1 = 8$-form flux density I_8 and a modification of the C-field flux Bianchi identity roughly of the form

$$\mathrm{d}\, G_7 \;=\; \tfrac{1}{2} G_4 \wedge G_4 + c I_8 \,. \tag{21}$$

A modified equation for M_2-brane charge of just this form was earlier argued in [DM97, (1)], based on a string perturbation-theoretic argument notorious as the "one-loop term" in the effective string action (obtained from Hypothesis H in (69)).

But from the point of view of flux densities and flux quantization, a Bianchi identity of the form (21) means that we need to understand both non-linear polynomial flux equations like that of the supergravity C-field (10) *and* their further twisting, in M-theoretic analogy of (19), hence we need to understand *twisted non-abelian cohomology* [**?**] – this we turn to in §1.2.4.

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1.2 Brane charge quantization

We have seen that the densities of electromagnetic fluxes satisfy a polynomial differential equation, the *Gauß law* (17), which expresses how charged branes are the sources of this flux.

But so far, with flux densities $\vec{F}$ witnessing the presence of a source brane, also a weighted rescaling of $\vec{F}$ by real numbers will satisfy the Gauß law and hence potentially witness the presence of fractional or even irrational "numbers" of charged branes.

Hence *brane charge quantization* (in the sense of *brane charge discretization*) means to adjoin further structure to the Gauß law which enforces that integrated fluxes take quantized (discretized) values, reflecting brane charges which are essentially integral multiples of certain unit charges.

Here by integrated fluxes we are to understand the deformation classes of flux densities, reflecting the total flux but not its local density profile.

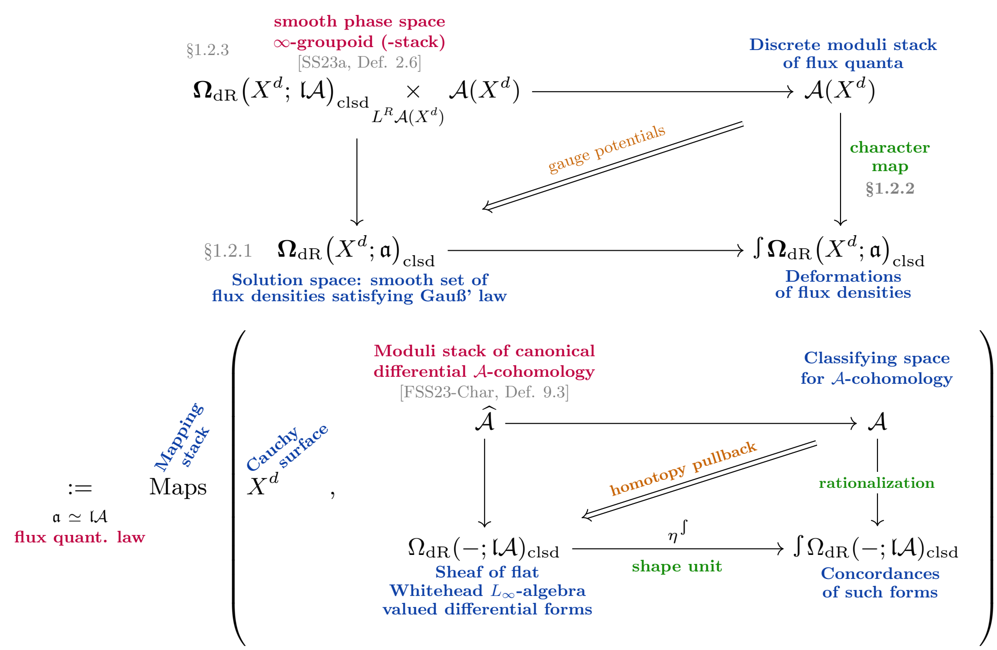
In conclusion this means to impose brane charge quantization by equipping systems of flux densities satisfying their Gauß law with deformations to systems of flux densities taken from a discrete space that are certified have admissible quantized total flux.

The key to understanding which flux quantizations are "admissible" (on purely mathematical grounds) is to understand the Gauß law constraint as the closure-condition on differential forms with coefficients in a characteristic L_∞-algebra.

Via the non-abelian generalization of the Chern-Dold character map from [FSS23-Char] this shows that the admissible laws are generalized cohomology theories (in general: non-abelian) such that the rational Whitehead L_∞-algebra of their classifying space coincides with the characteristic L_∞-algebra.

Remarkably, this construction equips the fluxes with *gauge potentials* and hence produces the full phase space of the higher gauge theory [SS23a].

Here we explain, in survey of [FSS23-Char][10], how this amounts to the following homotopy pullback of smooth ∞-groupoids:

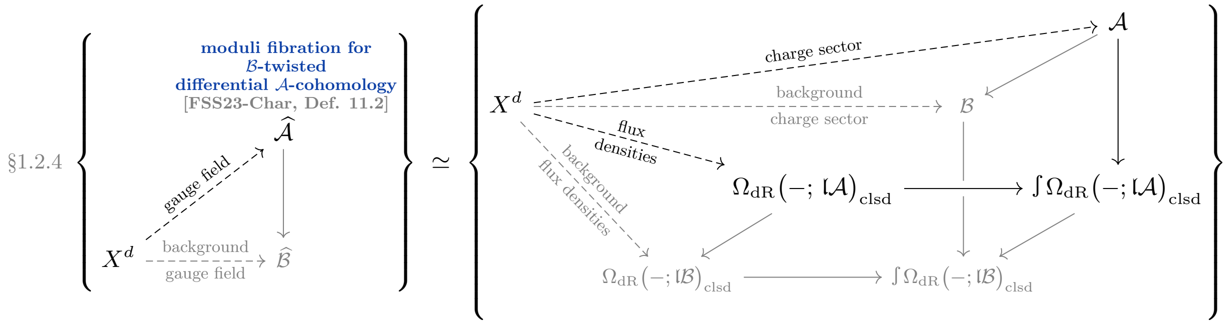


(22)

This follows the seminal argument of *Dirac charge quantization* for electromagnetism [Dirac1931] (review in [Al85][Fr97, §16.4e] [Fr00, §2]) and generalizes suggestions for charge quantization in higher gauge theories [Fr00][HS05] to the case of fluxes with non-linear self-sourcing quantized in *non-abelian* ("unstable") cohomology, such as the C-field in 11d supergravity (10).

Some of the flux densities may serve as a fixed background. In this case, the above data appears in vibrations, defining flux quantization in *twisted cohomology*:

[10] When we refer to equation-, definition-, proposition-, page-numbers in [FSS23-Char] we refer to the version published by World Scientific — see ncatlab.org/schreiber/show/The+Character+Map#PublishedVersion — which differs from the numbering in the arXiv version (otherwise the content is the same).



(23)

§1.2.1 Total flux as Nonabelian de Rham cohomology

§1.2.2 Flux quantization laws as Nonabelian cohomology

§1.2.3 Phase spaces as Differential non-abelian cohomology

§1.2.4 Background fluxes as Twisting of nonabelian cohomology

Differential-topological review of Dirac flux quantization of the electromagnetic (EM) field.

The EM flux density (Maxwell-Faraday tensor) F_2 (6)

$$F_2 \in \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}}$$

is not the full content of the EM-field.
First of all, there is also an integral class χ

$$[\chi] \in H^2(X;\mathbb{Z})$$

which coincides with the flux density in real cohomology.
This χ is the electromagnetic "instanton number":
– On the complement of a magnetic monopole worldline $X \equiv \mathbb{R}^{3,1}\backslash\mathbb{R}^{1,1}$, this χ is the integer charge of the monopole.
– On the one-point compactification of a planar type II superconductor $X \equiv \mathbb{R}^{1,1}\times\mathbb{R}^2_{\cup\{\infty\}}$, this χ is the integer number of Abrikosov vortices.

$$\begin{array}{ccc} & H^2(X;\mathbb{Z}) & [\chi] \\ & \downarrow^{\mathrm{ch}} & \updownarrow \\ \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \twoheadrightarrow & H^2_{\mathrm{dR}}(X) & \mathrm{ch}[\chi] \\ F_2 \mapsto & [F_2] & \end{array}$$

(integral charge; character; flux density; integral total flux)

But the pair (F_2,χ) is still not the full EM field content:

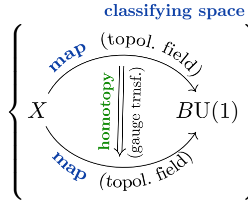
The remaining data is *how* F_2 and χ are identified.
To understand this, notice that ordinary cohomology groups have *classifying spaces*. In the case at hand, there is the space whose weak homotopy class is variously known as:

$$\underset{\text{infinite complex projective space}}{\mathbb{C}P^\infty} \underset{\mathrm{hmtp}}{\simeq} \underset{\text{projective unitary group of sep. }\infty\text{d Hilbert space}}{PU(\mathcal{H})} \underset{\mathrm{hmtp}}{\simeq} \underset{\text{classifying space of U(1)-princ. bundles}}{B\mathrm{U}(1)} \underset{\mathrm{hmtp}}{\simeq} \underset{\text{Eilenberg-MaxLane space}}{K(\mathbb{Z},2)}$$

$$H^2(X;\mathbb{Z}) \simeq \pi_0\,\mathrm{Maps}\big(X,\,\underbrace{B\mathrm{U}(1)}_{\text{classifying space}}\big)$$

(connected components of mapping space)

$$= \pi_0 \left\{ \ldots \right\}$$



and recall the notion of *higher homotopy groups* π_k of a simply connected space X, in degree $k \in \mathbb{N}_{\geq 2}$ these are abelian groups.

$$\underset{n\text{th homotopy group}}{\pi_k(X)} := \underset{\text{space of maps preserving base-point}}{\mathrm{Maps}^{*/}}\big(\underbrace{S^k}_{k\text{-sphere}},\,X\big) \in \mathrm{AbGrp}$$

Now $B\mathrm{U}(1)$ is special in that its homotopy groups are concentrated in degree 2, there being the integers.

$$\pi_k\big(B\mathrm{U}(1)\big) \simeq \begin{cases} \mathbb{Z} & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases}$$

In general, there is a (weakly homotopy-)unique connected space whose homotopy groups are concentrated in a single degree and there form an abelian group A, these are called the *Eilenberg-MacLane spaces* $K(A,n)$:

$$\pi_k\big(K(A,n)\big) \simeq \begin{cases} A & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$$

And these spaces happen to classify ordinary cohomology:

$$H^n(X;\,A) \simeq \pi_0\,\mathrm{Maps}\Big(X,\,K(A,n)\Big)$$

Hence in particular also de Rham cohomology:

$$H^n_{\mathrm{dR}}(X) \simeq \pi_0\,\mathrm{Maps}\Big(X,\,K(\mathbb{R},n)\Big)$$

Using this, we can refine the integrality condition on cohomology classes to a gauge transformation of fields:
Instead of asking the class of F_2 to equal the class of $\mathrm{ch}(\chi)$, we have a *homotopy* $\widehat{A}$ between them. This is equivalently [FSSt12, Prop. 3.2.26][FSS23-Char, Prop. 9.5] the final component of the EM-field, the *gauge potential* $\widehat{A}$.

$$\begin{array}{ccc} & \mathrm{Maps}\big(X,\,K(\mathbb{Z},2)\big) & \chi \\ & \downarrow^{\mathrm{ch}} & \updownarrow \\ \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \xrightarrow{\eta^{\int}} & \mathrm{Maps}\big(X,\,K(\mathbb{R},2)\big) & \mathrm{ch}(\chi) \\ F_2 \mapsto & \eta^{\int}(F_2) & \end{array}$$

(homotopy $\widehat{A}$ from $\eta^{\int}(F_2)$ to $\mathrm{ch}(\chi)$)

The equivalence classes of such "full EM-field" triples $(F_2,\widehat{A},\chi)$ constitute the *differential cohomology* $\widehat{H}^2(X;\mathbb{Z})$. Turns out to be equivalent to isomorphism classes of U(1)-principal bundles with Chern class χ and connection $\widehat{A}$.

$$\underset{\text{full EM-field is cocycle in ordinary differential cohomology}}{[\widehat{A}]} \in \widehat{H}^2(X;\mathbb{Z}) \xrightarrow{\ \chi\ } \underset{\text{charge sector in ordinary cohomology}}{H^2(X;\mathbb{Z})}$$

$$\widehat{H}^2(X;\mathbb{Z}) \xrightarrow{F_2} \underset{\text{flux density differential form}}{\Omega^2_{\mathrm{dR}}(X)}$$

The *reason* that this is the correct incarnation of the Maxwell field is that $(F_2,\widehat{A},\chi)$ is exactly the required data to "cancel the anomaly" (cf. p. 25) of the Lorentz force coupling term (82) in the exponentiated action functional of an electron propagating in an EM background field.

$$\begin{array}{ccc} \widehat{H}^2(X;\mathbb{Z})\times C^\infty(S^1,X) & \longrightarrow & \mathrm{U}(1) \\ (\widehat{A},\,\gamma) & \mapsto & e^{2\pi\mathrm{i}\int_{S^1}\gamma^*\widehat{A}} \end{array}$$

Up to modernized language, this is the original observation of [Dirac1931] (cf. [Al85][Fr97, §16.4e] [Fr00, §2]).

The lesson is that:

The usual differential forms entering the Lagrangian densities of (higher) gauge fields are not the full field content of the theory: Non-perturbatively, fields subsume maps to a classifying space, making the fields be cocycles in (generalized) differential cohomology thus enforcing a flux-quantization law on the differential form data.

We next explain how this works.

Flux quantization of general higher gauge theories. We explain ([FSS23-Char][SS23a]) how Dirac's flux quantization generalizes to any higher gauge theory:

Higher Maxwell-type equations have a **characteristic L_∞-algebra $\mathfrak{a}$**: The flux densities are equivalently $\mathfrak{a}$-valued differential forms, and the Gauß law (17) is equivalently the condition that these be *closed* (i.e.: flat, aka "Maurer-Cartan element"; in Italian SuGra literature: "satisfying an FDA").

$$\mathrm{SolSpace}(X^d) \simeq \left\{ \underbrace{\vec{B} \equiv \Big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}\big(X^d\big)\Big)_{i\in I}}_{\text{flux densities on Cauchy surface}} \;\middle|\; \underbrace{\mathrm{d}\,\vec{B} = \vec{P}\big(\vec{B}\big)}_{\text{satisfying Gauß's law}} \right\} \simeq \underbrace{\Omega_{\mathrm{dR}}\big(X^d;\, \mathfrak{a}\big)_{\mathrm{clsd}}}_{\text{flat differential forms valued in characteristic } L_\infty\text{-algebra}} .$$

Also every topological space $\mathcal{A}$ (under mild conditions) has a characteristic L_∞-algebra: Its $\mathbb{R}$-rational **Whitehead bracket L_∞-algebra $\mathfrak{l}\mathcal{A}$**.

$$\underset{\text{a topological space}}{\text{(homotopy type of)}}\ \mathcal{A} \underset{\mathbb{R}\text{-rationalization}}{\rightsquigarrow} \mathfrak{l}\mathcal{A}\ \ \text{Whitehead } L_\infty\text{-algebra}$$

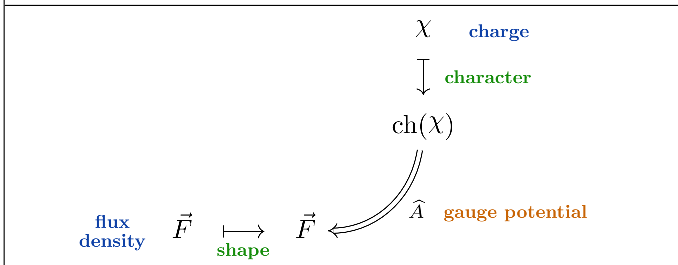
The **nonabelian** Chern-Dold **character map** turns $\mathcal{A}$-valued maps into closed $\mathfrak{l}\mathcal{A}$-valued differential forms, generalizing the Chern character for $\mathcal{A} = \mathrm{KU}_0$.

$$\text{charge}\ \big(\chi : X^d \to \mathcal{A}\big) \ \underset{\text{character map in } \mathcal{A}\text{-cohomology}}{\longmapsto}\ \mathrm{ch}(\chi) \in \Omega_{\mathrm{dR}}\big(X^d; \mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}}$$

The **possible flux quantization laws** for a given higher gauge field are those spaces $\mathcal{A}$ whose Whitehead L_∞-algebra is the characteristic one.

$$\mathrm{FluxQuantLaws} = \left\{ \underset{\text{classifying spaces}}{\mathcal{A}} \;\middle|\; \underset{\text{whose rational homotopy encodes the Gauß law}}{\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}} \right\}$$

Given a flux quantization law $\mathcal{A}$, the corresponding **higher gauge potentials** are deformations of the flux densities into characters of a $\mathcal{A}$-valued map, witnessing the flux densities as reflecting discrete charges quantized in $\mathcal{A}$-cohomology.
(It is not obvious that this reduces to the usual notion of gauge potentials, but it does.)



These non-perturbatively completed higher gauge fields form a *smooth higher groupoid*: the "canonical **differential $\mathcal{A}$-cohomology** moduli stack". Since these are now the flux-quantized on-shell fields, this is the **phase space** of the flux-quantized higher gauge theory (p. 13).

$$\underset{\substack{\text{differential}\\ \mathcal{A}\text{-cohomology}\\ \text{moduli stack}}}{\overset{\substack{\text{flux-quantized}\\ \text{phase space}\\ \text{stack is}}}{\widehat{\mathcal{A}}(X^d)}} := \left\{ \left(\begin{array}{lll} \vec{F} \in \Omega_{\mathrm{dR}}(X^d; \mathfrak{l}\mathcal{A})_{\mathrm{clsd}} & & \text{flux} \\ \chi \in \mathrm{Map}(X; \mathcal{A}) & & \text{charge} \\ \widehat{A} : \mathrm{ch}(\chi) \Rightarrow \vec{F} & & \text{gauge} \end{array} \right) \right\}$$

The topological sector of the phase space. The flux-quantized phase space hence subsumes the "solitonic" fields with non-trivial charge sectors χ, and as such is a non-perturbative completion of the traditional phase spaces (which correspond to a fixed charge sector only, typically to $\chi = 0$).

The shape (topological realization) of this phase space stack is the **space of topological fields**,

$$\int \widehat{\mathcal{A}}(X^d) \simeq \mathcal{A}(X^d) = \mathrm{Map}\big(X^d, \mathcal{A}\big).$$

which implies that the ordinary homology of the phase space stack constitutes the **topological observables** on the higher gauge theory.

$$H_\bullet\Big(\widehat{\mathcal{A}}(X^d); \mathbb{C}\Big) \simeq H_\bullet\Big(\mathcal{A}(X^d); \mathbb{C}\Big)$$

Hence if we focus only on the solitonic or *topological field*-content of the phase space, then we see plain $\mathcal{A}$-cohomology moduli of the Cauchy surface. and the full phase space stack only serves to justify this object.

$$\underset{\substack{\text{non-abelian}\\ \mathcal{A}\text{-cohomology}\\ \text{moduli space}}}{\overset{\substack{\text{flux-quantized}\\ \text{topological}\\ \text{phase space}}}{\mathcal{A}(X^d)}} := \Big\{ \chi \in \mathrm{Map}(X, \mathcal{A}) \Big\}$$

Therefore the reader need not be further concernd with higher stack theory for the present purpose.

Before we start, as further motivation, notice that the failure of imposing brane charge quantization typically shows up in unexpected behaviour of physical theories known as "anomalies":

Anomaly cancellation and flux quantization. Secretly, much of contemporary theory building in theoretical physics is a sophisticated process of trial, error and improvisation: The trials are Lagrangian densities ("action functionals"), the errors are "anomalies" obstructing their consistent quantization, and the improvisation is the invention of add-on rules to "cancel" the anomalies. While there is a sense of accomplishment in the community for identifying and cancelling anomalies we should see it for what it is:

Anomaly cancellation is the patching-up of broken theories. This can and certainly has been useful for exploring the space of physical theories, but it seems implausible that truly fundamental theories will come to us in broken form incrementally patched. Instead, eventually we want to understand how to construct *anomaly-free* and hence well-defined quantum theories right away.

Dirac charge quantization in integral cohomology (as recalled on p. 22) cancels the anomaly in the Lorentz-coupling of the worldine theory of an electron in a background magnetic fields and is given by (in particular) a lift in ordinary cohomology:

$$
\begin{array}{ccc}
\underset{\text{in integral cohomology}}{\overset{\text{quantized magnetic charges}}{H^2(X;\mathbb{Z})}} & \longrightarrow & \underset{\text{in real/de Rham cohomology}}{\overset{\text{classical magnetic charges}}{H^2(X;\mathbb{R}) \simeq H^2_{\mathrm{dR}}(X^4)}} \\
[q] & \underset{\text{lift}}{\overset{\text{charge quantization}}{\leftsquigarrow}} & [F_2]
\end{array}
\tag{24}
$$

Spin-/String-structure as charge quantization in non-abelian cohomology. Dirac's argument only concerns the charge of the electron. When one also considers the spin of the electron then its worldline theory has another anomaly, which is cancelled by equipping the background spacetime with *spin-structure* (discussed this way in [Wi85, p. 65-68]). An analogous argument shows that *spinning strings* have an anomaly in their worldsheet theory which may be cancelled by equipping the background spacetime with *string-structure* (cf. [Bu11][SSS12]). Here we are going to understand [FSS23-Char, §2] phenomena such as Spin- and String-structures as examples of *non-abelian cohomology* with coefficients in a non-abelian group G ([Grothendieck55, §V][Fr1957], see also [We16, §7]) or non-abelian 2-groups etc., thus conceptually unifying them with abelian cohomology such as in (24):

$$
\underset{\text{non-abelian cohomology in degree 1}}{H^1(X;G)} \;\simeq\; \underset{\text{homotopy classes of maps into classifying space}}{\pi_0\mathrm{Maps}(X;BG)} \;\simeq\; \underset{\text{isomorphism classes of principal bundles}}{G\mathrm{PrinBund}(X)_{/\sim}}
$$

This way, we may understand the anomaly cancellation of the spinning electron by ambient spin-structure as of the same general cohomological form as Dirac's charge quantization (24):

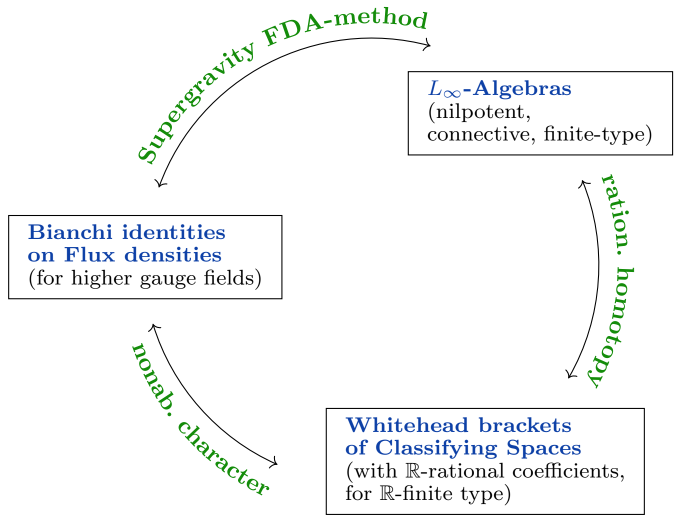
$$
\begin{array}{ccccc}
\underset{\text{in String-cohomology}}{\overset{\text{"quantized" gravitational charge}}{H^1\big(X;\mathrm{String}(1,d)\big)}} & \longrightarrow & \underset{\text{in Spin-cohomology}}{\overset{\text{"quantized" gravitational charge}}{H^1\big(X;\mathrm{Spin}(1,d)\big)}} & \longrightarrow & \underset{\text{in nonabelian O-cohomology}}{\overset{\text{gravitational charge}}{H^1\big(X;\mathrm{O}(1,d)\big)}} \\
[\widehat{\widehat{\omega}}] & \underset{\text{lift}}{\overset{\text{string anomaly cancellation}}{\leftsquigarrow}} & [\widehat{\omega}] & \underset{\text{lift}}{\overset{\text{spin anomaly cancellation}}{\leftsquigarrow}} & [\omega]
\end{array}
$$

1.2.1 Total flux as Nonabelian de Rham cohomology

We explain (30) how higher Bianchi identities (18) and their corresponding higher Gauß laws (17) are equivalently the closure (flatness) conditions on differential forms valued in a characteristic L_∞-algebra.

The notion of L_∞- or *strong homotopy Lie algebra* [LS93][LM95] is finally becoming more widely appreciated in physics, where they appear in various guises. Here we are concerned with L_∞-algebras which are (i) nilpotent, (ii) connective (iii) of finite type, in their *joint* incarnation as higher flux density coefficients *and* as higher Whitehead brackets (all to be explained in a moment), which one might refer to as **the Flux Homotopy Lie algebra triality**, indicated on the right. Classically familiar as its separate aspects are to their respective experts, the full triality may still not be widely appreciated but is key to our discussion here:

- By the **FDA method in supergravity** we refer, with some hindsight, to the observations of [vN83][DF82][CDF91], as explained in [FSS15][FSS18][HSS19], reviewed in [FSS19a].
- By **rational homotopy** we are referring here specifically to the *fundamental theorem of dg-algebraic rational homotopy theory*, mainly due to Quillen, Sullivan and Bousfield & Gugenheim, as reviewed in [FSS23-Char, §5].
- The **nonabelian character** is the generalization of the Chern-Dold character map from topological K-theory and Whitehead-generalized cohomology to higher non-abelian cohomology, constructed in [FSS23-Char].

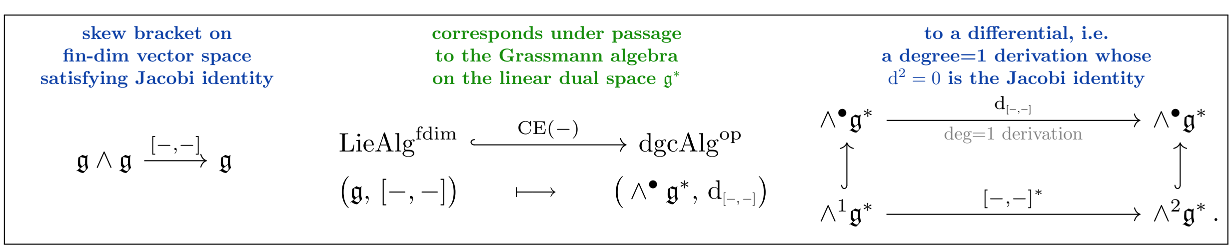


(25)

In particular, this means that L_∞-algebras as used here are *not* directly to be understood as generalizations of the gauge Lie algebras familiar from Yang-Mills theory, which are coefficients of the gauge potentials, but instead as the coefficients of their flux densities.

L_∞-algebras. Since we are assuming L_∞-algebras to be connective and of finite type (meaning that they are degreewise finite-dimensional and concentrated in non-negative degrees) we may *define* them through their Chevalley-Eilenberg (CE) algebras in the following manner, which is not only convenient for dealing with the otherwise intricate sign rules, but also essential to their alternative perspectives in the above triality:

Chevalley-Eilenberg algebras of Lie algebras. Namely, for $\mathfrak{g}$ a finite-dimensional Lie algebra (our ground field is the real numbers, throughout) with Lie bracket a skew-symmetric linear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, its linear dual vector space $\mathfrak{g}^*$ is equipped with the dual bracket $[-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ which extends uniquely to a degree=1 derivation on the graded Grassmann algebra $\wedge^\bullet \mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \underbrace{\mathfrak{g}^* \wedge \cdots \wedge \mathfrak{g}^*}_{n \text{ factors}}$:



(26)

One readily checks that this derivation squares to zero iff the bracket satisfies its Jacobi identity(!):

$$\text{Jacobi identity for } [-,-] \quad \Leftrightarrow \quad \mathrm{d}_{[-,-]} \circ \mathrm{d}_{[-,-]} = 0 \, .$$

The resulting differential graded-commutative (dgc) algebra $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ is known as the *Chevalley-Eilenberg complex* $\mathrm{CE}(\mathfrak{g})$ whose cochain cohomology computes the Lie algebra cohomology of $\mathfrak{g}$ (with trivial coefficients) — but the key point at the moment is that its construction is a *fully faithful* contravariant functor embedding the category of finite-dimensional Lie algebras into the opposite of that of dgc-algebras.

L_∞-algebras of finite type. With ordinary Lie algebras viewed as special dgc-algebras this way (26), it is *immediate* to generalize them to the case where $\mathfrak{g}$ may be a graded vector space of degreewise finite dimension ("of finite type"): Namely, writing

$$(\mathfrak{g}^\vee)_n \;\equiv\; (\mathfrak{g}_n)^*, \qquad \wedge^\bullet \mathfrak{g}^\vee \;\equiv\; \mathrm{Sym}(\mathfrak{g}^\vee[1])$$

we can use *verbatim* the same construction:

A degree=1 derivation on $\wedge^\bullet \mathfrak{g}^\vee$ is determined by its restriction to $\wedge^1 \mathfrak{g}^\vee$, where it is a sum of co-n-ary linear maps, whose linear duals we may think of as n-ary degree=(-1) brackets on $\mathfrak{g}[1]$:

Higher skew brackets on graded vector space $\mathfrak{g}$ satisfying higher Jacobi identity

$$\begin{array}{ccc} \mathfrak{g}[1] & \xrightarrow{[-]} & \mathfrak{g}[1] \\ \mathfrak{g}[1] \wedge \mathfrak{g}[1] & \xrightarrow{[-,-]} & \mathfrak{g}[1] \\ \mathfrak{g}[1] \wedge \mathfrak{g}[1] \wedge \mathfrak{g}[1] & \xrightarrow{[-,-,-]} & \mathfrak{g}[1] \\ \vdots & & \vdots \end{array}$$

correspond under passage to the graded symmetric algebra on the degreewise dual space $\mathfrak{g}^\vee$

$$\begin{array}{ccc} L_\infty \mathrm{Alg}^{\mathrm{ftp}} & \xhookrightarrow{\mathrm{CE}(-)} & \mathrm{dgcAlg}^{\mathrm{op}} \\ \big(\mathfrak{g},\, [-],\, [-,-],\, [-,-,-],\, \cdots\big) & \mapsto & \big(\wedge^\bullet \mathfrak{g}^\vee,\, \mathrm{d}_{[-,\cdots,-]}\big) \end{array}$$

to a single differential, i.e. a degree=1 derivation whose $\mathrm{d}^2 = 0$ is the higher Jacobi identity

$$\begin{array}{ccc} \wedge^\bullet \mathfrak{g}^\vee & \xrightarrow{\mathrm{d}_{[-,\cdots,-]}} & \wedge^\bullet \mathfrak{g}^\vee \\ \uparrow & & \| \\ \wedge^1 \mathfrak{g}^\vee & \xrightarrow[\oplus [-,-,-]^* \oplus \cdots]{[-]^* \oplus [-,-]^*} & \bigoplus_{n \in \mathbb{N}} \wedge^n \mathfrak{g}^\vee \end{array} \tag{27}$$

Here the simple condition that $\mathrm{d}_{[-,\cdots,-]}$ be a differential implies a tower of conditions on these brackets, which generalize the Jacobi identity on an ordinary Lie algebra, known as the conditions that make $\big(\mathfrak{g}, [-], [-,-], [-,-,-], \cdots\big)$ an L_∞-algebra:

$$\begin{array}{c}\text{Higher Jacobi identity for} \\ [-],\, [-,-],\, [-,-,-],\, \cdots \end{array} \quad\Leftrightarrow\quad \mathrm{d}_{[-,\cdots,-]} \circ \mathrm{d}_{[-,\cdots,-]} \;=\; 0\,.$$

In other words, we may identify L_∞-algebras of finite type as the formal dual to dgc-algebras whose underlying graded-commutative algebra is free on a graded vector space (cf. [SSS09, Def. 13][FSSt12, §4.1][FSS23-Char, (4.27)]). Some examples:

L_∞-algebra	$\mathfrak{g}$	$\mathfrak{g}^\vee[1]$	$\mathrm{d}_{[-,\cdots,-]}$
Line Lie algebra	$\mathfrak{u}(1)$	$\mathbb{R}\langle \omega_1 \rangle$	$\mathrm{d}\,\omega_1 = 0$
Special unitary Lie algebra	$\mathfrak{su}(2)$	$\mathbb{R}\langle \omega_1^{(1)},\, \omega_1^{(2)},\, \omega_1^{(3)} \rangle$	$\mathrm{d}\,\omega_1^{(i)} = \epsilon_{ijk}\,\omega_1^{(j)} \wedge \omega_1^{(k)}$
Line Lie 2-algebra	$b\,\mathfrak{u}(1)$	$\mathbb{R}\langle \omega_2 \rangle$	$\mathrm{d}\,\omega_2 = 0$
String Lie 2-algebra cf. [FSS14, §A]	$\mathfrak{string}(3)$	$\mathbb{R}\langle \omega_1^{(1)},\, \omega_1^{(2)},\, \omega_1^{(3)},\, \omega_2 \rangle$	$\mathrm{d}\,\omega_1^{(i)} = -\tfrac{1}{2}\epsilon_{ijk}\,\omega_1^{(j)} \wedge \omega_1^{(k)}$ $\mathrm{d}\,\omega_2 = \epsilon_{ijk}\,\omega_1^{(i)} \wedge \omega_1^{(j)} \wedge \omega_1^{(k)}$
Line Lie 3-algebra	$b^2\,\mathfrak{u}(1)$	$\mathbb{R}\langle \omega_3 \rangle$	$\mathrm{d}\,\omega_3 = 0$
T-duality Lie 3-algebra [FSS18, §7]	$b\mathcal{T}_1$	$\mathbb{R}\langle \omega_2^{(i)}, \omega_2^{(B)}, h_3 \rangle$	$\mathrm{d}\,\omega_2^{(i)} = 0$ $\mathrm{d}\,\omega_2^{(B)} = 0$ $\mathrm{d}\,h_3 = \omega_2^{(i)} \wedge \omega_2^{(B)}$
Line Lie 4-algebra	$b^3\,\mathfrak{u}(1)$	$\mathbb{R}\langle \omega_4 \rangle$	$\mathrm{d}\,\omega_4 = 0$
M-theory gauge Lie 7-algebra cf. [SV22b, §2.2]	$\mathfrak{l}\,S^4$	$\mathbb{R}\langle \omega_4, \omega_7 \rangle$	$\mathrm{d}\,\omega_4 = 0$ $\mathrm{d}\,\omega_7 = -\omega_4 \wedge \omega_4$
Cyclified M-theory gauge Lie 7-algebra [FSS17, Ex. 3.3] [BMSS19, Ex. 2.47]	$\mathfrak{l}\big(\mathcal{L}S^4 /\!\!/ S^1\big)$	$\mathbb{R}\left\langle \begin{array}{l} \omega_2, \omega_4, \omega_6 \\ h_3, h_7 \end{array} \right\rangle$	$\mathrm{d}\,h_3 = 0$ $\mathrm{d}\,\omega_2 = 0$ $\mathrm{d}\,\omega_4 = h_3 \wedge \omega_2$ $\mathrm{d}\,\omega_6 = h_3 \wedge \omega_4$

(28)

Flat L_∞-algebra valued differential forms now have an immediate definition from the perspective (27): They are the dg-algebra homomorphism from their CE-algebras into de Rham algebras (cf. [SSS09, (261)][Def. 6.1][FSS23-Char], aka "MC elements"):

$$\left.\begin{array}{l} \mathfrak{a} \,\in\, L_\infty\mathrm{Alg}^{\mathrm{ftp}}, \\ X \,\in\, \mathrm{SmthMfd} \end{array}\right\} \vdash \begin{array}{l} \Omega^1_{\mathrm{dR}}(X;\, \mathfrak{a})_{\mathrm{clsd}} \;\;\equiv \\ \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{g}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)\,. \end{array} \tag{29}$$

$$\begin{array}{ccc}
\mathrm{CE}\big(b\mathfrak{u}(1)\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\
\omega_2 & \longmapsto & F \\
\big\downarrow \mathrm{d}_{b\mathfrak{u}(1)} & & \big\downarrow \mathrm{d}_{\mathrm{dR}} \\
0 & \longmapsto & \mathrm{d}_{\mathrm{dR}} F = 0
\end{array}$$

$$\begin{array}{ccc}
\mathrm{CE}\big(\mathfrak{string}(3)\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\
\omega_1^{(i)} & \longmapsto & A^i \\
\big\downarrow \mathrm{d}_{\mathfrak{string}(3)} & & \big\downarrow \mathrm{d}_{\mathrm{dR}} \\
-\tfrac{1}{2}\epsilon_{ijk}\,\omega_1^{(j)} \wedge \omega_1^{(k)} & \longmapsto & \mathrm{d}_{\mathrm{dR}} A^i = -\tfrac{1}{2} A^j \wedge A^j \\
\omega_2 & \longmapsto & B \\
\big\downarrow \mathrm{d}_{\mathfrak{string}(3)} & & \big\downarrow \mathrm{d}_{\mathrm{dR}} \\
\epsilon_{ijk}\,\omega_1^{(i)} \wedge \omega_1^{(j)} \wedge \omega_1^{(k)} & \longmapsto & \mathrm{d}_{\mathrm{dR}} B = \epsilon_{ijk}\, A^i \wedge A^j \wedge A^k
\end{array}$$

$$\begin{array}{ccc}
\mathrm{CE}\big(\mathfrak{l}\,S^4\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\
\omega_4 & \longmapsto & G_4 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}S^4} & & \\
0 & \longmapsto & \mathrm{d}_{\mathrm{dR}} G_4 = 0 \\
\omega_7 & \longmapsto & 2G_7 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}S^4} & & \big\downarrow \mathrm{d}_{\mathrm{dR}} \\
-\omega_4 \wedge \omega_4 & \longmapsto & 2\mathrm{d}_{\mathrm{dR}} G_7 = -G_4 \wedge G_4
\end{array}$$

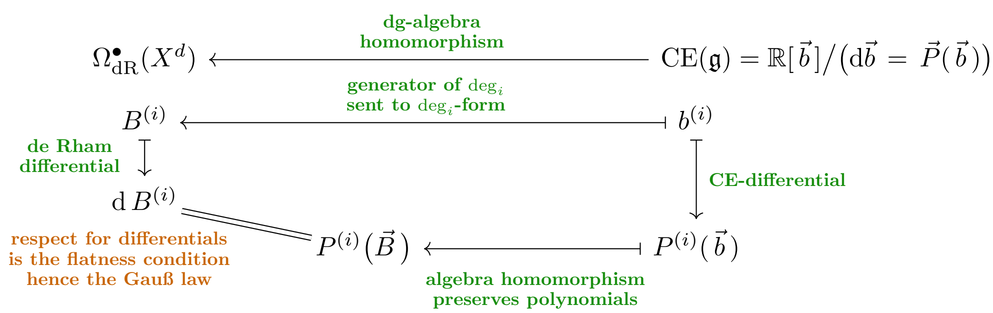
$$\begin{array}{ccc}
\mathrm{CE}\big(\mathfrak{l}(\mathcal{L}S^4 /\!\!/ S^1)\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\
h_3 & \longmapsto & H_3 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}(\mathcal{L}S^4/\!\!/S^1)} & & \big\downarrow \\
0 & \longmapsto & \mathrm{d}_{\mathrm{dR}} H_3 = 0 \\
\omega_2 & \longmapsto & F_2 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}(\mathcal{L}S^4/\!\!/S^1)} & & \big\downarrow \\
0 & \longmapsto & \mathrm{d}_{\mathrm{dR}} F_2 = 0 \\
\omega_4 & \longmapsto & F_4 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}(\mathcal{L}S^4/\!\!/S^1)} & & \big\downarrow \\
h_3 \wedge \omega_2 & \longmapsto & \mathrm{d}_{\mathrm{dR}} F_4 = H_3 \wedge F_2 \\
\omega_6 & \longmapsto & F_6 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}(\mathcal{L}S^4/\!\!/S^1)} & & \big\downarrow \\
h_3 \wedge \omega_4 & \longmapsto & \mathrm{d}_{\mathrm{dR}} F_6 = H_3 \wedge F_4 \\
h_7 & \longmapsto & H_7 \\
\big\downarrow \mathrm{d}_{\mathfrak{l}(\mathcal{L}S^4/\!\!/S^1)} & & \big\downarrow \\
\begin{array}{l} -\tfrac{1}{2}\omega_4 \wedge \omega_4 \\ +\omega_2 \wedge \omega_6 \end{array} & \longmapsto & \mathrm{d}_{\mathrm{dR}} H_7 = \begin{array}{l} -\tfrac{1}{2} F_4 \wedge F_4 \\ +F_2 \wedge F_6 \end{array}
\end{array}$$

Flux densities satisfying Bianchi/Gauß laws are flat L_∞-algebra-valued differential forms. Remarkably, it follows that polynomials $\vec{P}$ defining Bianchi identities (18) and Gauß laws (17) are equivalently structure constants of L_∞-algebras $\mathfrak{a}$ (27), such that the Bianchi/Gauß law is the flatness condition on $\mathfrak{a}$-valued forms:

$$
\begin{array}{rcl}
\underbrace{\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{a}\big)_{\mathrm{clsd}}}_{\text{sheaf of closed } L_\infty\text{-algebra-valued differential forms}} & = & \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}(\mathfrak{a}),\, \Omega^\bullet_{\mathrm{dR}}(-)\Big) \;=\; \Big\{ \underbrace{\vec{B} \equiv \big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(-)\big)}_{\text{systems of flux densities}} \;\Big\vert\; \underbrace{\mathrm{d}\,\vec{B} \,=\, \vec{P}(\vec{B})}_{\text{satisfying this Gauß law}} \Big\} \\
& & \updownarrow \\
\underbrace{\mathrm{CE}(\,\mathfrak{a}\,)}_{\text{Chevalley-Eilenberg algebra of } L_\infty\text{-algebra}} & = & \underbrace{\mathbb{R}\Big[\big\{b^{(i)}_{\deg_i}\big\}_{i\in I}\Big]}_{\text{free differential graded-commutative algebra on these graded generators}} \Big/ \underbrace{\Big(\mathrm{d}\,\vec{b} \,=\, \vec{P}(\vec{b})\Big)}_{\text{satisfying these differential relations}} \\
& & \updownarrow \\
\underbrace{\mathfrak{a}}_{L_\infty\text{-algebra}} & = & \underbrace{\mathbb{R}\Big\langle \big\{v^{(i)}_{\deg_i - 1}\big\}_{i\in I}\Big\rangle}_{\text{graded vector space spanned by these graded generators}} \quad \underbrace{\big[v^{(i_1)}, \cdots, v^{(i_n)}\big] \;=\; \textstyle\sum_{i\in I} P^{(i)}_{i_1\cdots i_n}\, v^{(i)}}_{\text{equipped with these higher Lie brackets}}
\end{array}
\tag{30}
$$

insert spacetime manifold here

Again in more detail: Homomorphism from a CE-algebra to differential forms assign, as graded algebra homomorphisms, flux densities $B^{(i)}$ to each CE-generator $b^{(i)}$, and the respect for the differentials enforces on them the Gauß law:



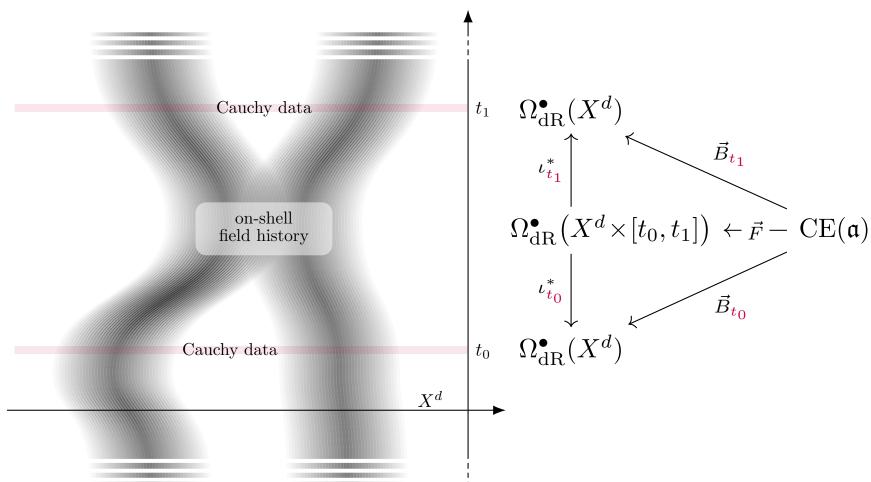
In summary so far this means that the flux solution space (17) of higher gauge theories is further identified with space of closed (flat) differential forms (on the Cauchy surface) valued in a characteristic L_∞-algebra $\mathfrak{a}$ which defines and is defined by the given Bianchi identities.

$$
\begin{array}{rcl}
\underbrace{\mathrm{SolSpace}(X^D)}_{\text{space of flux densities on spacetime, solving the equations of motion}} & \equiv & \left\{ \underbrace{\vec{F} \,\equiv\, \Big(F^{(i)} \,\in\, \Omega^{\deg_i}_{\mathrm{dR}}\big(X^D\big)\Big)_{i\in I}}_{\text{electromagnetic flux densities on spacetime}} \;\middle\vert\; \begin{array}{l} \overset{\text{Bianchi identities}}{\mathrm{d}\,\vec{F} \,=\, \vec{P}(\vec{F})} \\ \underset{\text{self-duality}}{\star\,\vec{F} \,=\, \vec{\mu}(\vec{F})} \end{array} \right\} \quad \text{covariant form} \\
& \underset{\iota^*}{\simeq} & \left\{ \underbrace{\vec{B} \,\equiv\, \Big(B^{(i)} \,\in\, \Omega^{\deg_i}_{\mathrm{dR}}\big(X^d\big)\Big)_{i\in I}}_{\text{magnetic flux densities on Cauchy surface}} \;\middle\vert\; \overset{\text{Gauß law}}{\mathrm{d}\,\vec{B} \,=\, \vec{P}(\vec{B})} \right\} \quad \text{canonical form} \\
& \simeq & \underbrace{\Omega^1_{\mathrm{dR}}\big(X^d;\,\mathfrak{a}\big)_{\mathrm{clsd}}}_{\text{space of closed (flat) } \mathfrak{a}\text{-valued differential forms}}
\end{array}
\tag{31}
$$

Non-abelian de Rham cohomology. From the flux *densitites* we are to extract a measure for the *total* flux, thought of as the *integration* of the flux densities. To make sense of general integrals of $\mathfrak{a}$-valued flux densities for non-abelian L_∞-algebras $\mathfrak{a}$, notice that in the abelian case the integrals of differential forms over cycles are exactly what is captured by their de Rham cohomology class, which is equivalently their *deformation class* [FSS23-Char, Prop. 6.4] in the sense of the following definition, which makes immediate sense also with L_∞-algebraic coefficients:

We say [FSS23-Char, §6] that a pair $\vec{B}_0, \vec{B}_1 \in \Omega_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big)_{\mathrm{clsd}}$ of closed $\mathfrak{a}$-valued differential forms (29) are *cohomologous* if they can can be *deformed into each other*, hence if they are *concordant*, in that they are boundary data of a closed $\mathfrak{a}$-valued form on the cylinder $X^d \times [0,1]$ over X^d:

$$\underbrace{\vec{B}_0 \sim \vec{B}_1}_{\text{deformation of flux densities}} \quad :\Leftrightarrow \quad \exists\ \vec{F} \in \Omega^1_{\mathrm{dR}}\big(X^d \times [0,1]\big)_{\mathrm{clsd}} \ \text{with} \ \begin{cases} \iota_0^* \vec{F} = \vec{B}_0 \\ \iota_1^* \vec{F} = \vec{B}_1 \end{cases} \tag{32}$$



This is an equivalence relation whose equivalence classes we call
the **$\mathfrak{a}$-valued non-abelian de Rham cohomology** of X [FSS23-Char, Def. 6.3]:

$$\underbrace{\big[\vec{B}\big]}_{\substack{\text{deformation class}\\ \text{of flux densities}}} \in \underbrace{H^1_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big)}_{\substack{\mathfrak{a}\text{-valued}\\ \text{de Rham cohomology}}} := \pi_0 \left\{ \Omega^\bullet_{\mathrm{dR}}(X^d) \ \underset{\text{another cocycle}\ \vec{B}'}{\overset{\text{cocycle (dga-hom)}\ \vec{B}}{\Big\Downarrow\ \text{coboundary (concordance)}}}\ \mathrm{CE}(\mathfrak{a}) \right\}. \tag{33}$$

$$\begin{array}{ccc} \underset{\text{differential forms}}{\text{closed } \mathfrak{a}\text{-valued}} & & \underset{\text{de Rham cohomology}}{\text{nonabelian } \mathfrak{a}\text{-valued}} \\ \Omega^1_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big)_{\mathrm{clsd}} & \twoheadrightarrow & H^1_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big) \\ \underset{\text{flux densities}}{\vec{B}} & \longmapsto & \underset{\text{total flux}}{\big[\vec{B}\big]} \end{array}$$

Eg. for $\mathfrak{a} \equiv b^n\mathfrak{u}(1)$ being the abelian line Lie n-algebra (28), the above definition reduces to ordinary de Rham cohomology [FSS23-Char, Prop. 6.4]:

$$H^1_{\mathrm{dR}}\big(X; b^n\mathbb{R}\big) \simeq H^{n+1}_{\mathrm{dR}}(X)\,.$$

1.2.2 Flux quantization laws as Nonabelian cohomology

Classifying spaces for cohomology. Reasonable cohomology theories have *classifying spaces* (cf. [FSS23-Char, §2]):

$$
\begin{array}{rcl}
\overset{\text{ordinary cohomology}}{H^n(X;\mathbb{Z})} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{Eilenberg-MacLane space}}{K(\mathbb{Z},n)}\big)
\\[10pt]
\overset{\text{topological K-theory}}{K^0(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{space of Fredholm operators}}{\mathrm{Fred}_{\mathbb{C}}}\big)
\\[10pt]
\overset{\text{Whitehead-generalized cohomology}}{E^n(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{stage in spectrum of spaces}}{E_n}\big)
\\[10pt]
\overset{\text{nonabelian cohomology}}{H^1(X;G)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{classifing space of principal } G\text{-bundles}}{BG}\big)
\\[10pt]
\overset{\text{coHomotopy}}{\pi^n(X)} & \simeq & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{sphere}}{S^n}\big)
\\[10pt]
\overset{\text{generalized nonabelian cohomology}}{H^1\big(X,\,\Omega\mathcal{A}\big)} & := & \pi_0\,\mathrm{Maps}\big(X,\ \overset{\text{any space}}{\mathcal{A}}\big)
\\[10pt]
& = & \pi_0 \left\{ X \underset{\mathcal{F}' \ \text{another cocycle}}{\overset{\mathcal{F}\ \text{cocycle (map)}}{\underset{\Downarrow\ \text{coboundary (homotopy)}}{\rightrightarrows}}} \mathcal{A} \right\}
\end{array}
\tag{34}
$$

Moreover, over smooth manifolds, reasonable cohomology theories have differential form representatives of their non-torsion content,

via *character maps*. This is classical for generalized abelian and ordinary non-abelian cohomology (cf. [FSS23-Char, §7, §8]):

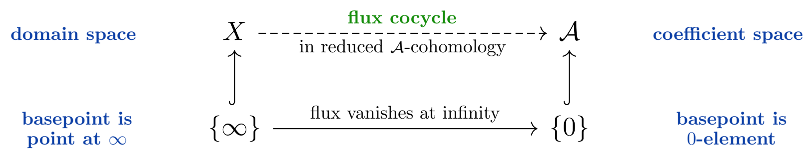
Ordinary integral cohomology	$H^n(X;\mathbb{Z})$	$\xrightarrow{\text{de Rham map}}$	$H^n_{\mathrm{dR}}(X) \simeq \mathrm{Hom}_{\mathbb{R}}\big(\mathbb{R}\langle\omega_n\rangle,\, H^\bullet_{\mathrm{dR}}(X)\big)$	**differential forms in degree n**
Traditional nonabelian cohomology	$H^1(X;G)$	$\xrightarrow{\text{Chern-Weil homomorphism}}$	$\mathrm{Hom}_{\mathbb{R}}\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, H^\bullet_{\mathrm{dR}}(X)\big)$	**differential forms for $\mathfrak{g}$-invariant polynomials**
Topological K-theory	$K^0(X)$	$\xrightarrow{\text{Chern character}}$	$\mathrm{Hom}_{\mathbb{R}}\big(\mathbb{R}\langle\omega_0,\omega_2,\omega_4,\cdots\rangle,\, H^\bullet_{\mathrm{dR}}(X)\big)$	**differential forms in every even degree**
abelian Whitehead-generalized cohomology	$E^n(X)$	$\xrightarrow{\text{Chern-Dold character}}$	$\mathrm{Hom}_{\mathbb{R}}\big((\pi_\bullet(E)\otimes_{\mathbb{Z}}\mathbb{R})^\vee,\, H^{\bullet+n}_{\mathrm{dR}}(X)\big)$	**differential forms for rational homotopy groups of the classifying space**
Generalized non-abelian cohomology	$H^1\big(X;\Omega\mathcal{A}\big)$	$\xrightarrow{\text{nonabelian character}}$	$H^1_{\mathrm{dR}}(X;\mathfrak{l}A) := \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)/_\sim$	**differential forms with coefficients in Whitehead L_∞-algebra**

Such character maps were not known in the generality of generalized non-abelian cohomology, such as CoHomotopy, but they do exist: the general *non-abelian character map* is constructed in [FSS23-Char, Part IV]:

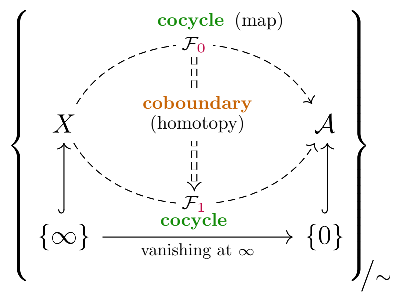
This subsumes all the other cases. We now indicate how this works.

Aside: **Reduced cohomology and solitonic charges.** For the charge quantization of *solitonic* branes (in §2.3) one needs to implement in cohomology theory their *localization* in space (cf. §1.1.1) which forces their fluxes to *vanish at infinity*.

We may observe that a formalization of this phenomenon is already captured by the standard notion of *reduced cohomology* on *pointed spaces* if we regard the basepoint of a domain space as its point-at-infinity and the basepoint of a coefficient space as its zero-element



$$\underset{\text{reduced higher non-abelian cohomology}}{\widetilde{H}^1\big(X;\,\Omega\mathcal{A}\big)} \;:=\; \pi_0\,\mathrm{Maps}^{*/}\big((X,\infty),\,(\mathcal{A},0)\big) \;=\; \tag{35}$$



Notice that the point at infinity may or may not be reachable by continuous paths in the space:

$$X \text{ a plain space} \quad \vdash \quad \begin{array}{lll} X_{\sqcup\{\infty\}} & \text{disjoint point adjoined} & \text{paths starting in cnctd } X \text{ never reach } \infty \\ X_{\cup\{\infty\}} & \text{one-point-compactification} & \text{paths starting in cnctd } X \text{ may reach } \infty \end{array} \tag{36}$$

Given a pointed space, we may first delete the point at infinity and then adjoint it back disjointly, making it un-reachable:

$$(X,\infty) \text{ a pointed space} \quad \vdash \quad \big(X \setminus \{\infty\}\big)_{\sqcup\{\infty\}} \quad \text{make } \infty \text{ un-reachable} \tag{37}$$

Plain cohomology (34) is subsumed in reduced cohomology as the case where the point at infinity is unreachable (36)

$$H^1\big(X;\,\Omega\mathcal{A}\big) \;\simeq\; \widetilde{H}^1\big(X_{\sqcup\{\infty\}};\,\Omega\mathcal{A}\big)$$

and making ∞ unreachable (37) projects reduced into plain cohomology.

The charges that thus disappear existed only due to their localization, hence are *purely solitonic*, while those that do not vanish at ∞ are *purely singular* (cf. §1.1.1):

$$\underset{\text{purely solitonic charges}}{\ker\big(\epsilon\iota_X^*\big)} \hookrightarrow \underset{\text{reduced cohomology}}{\widetilde{H}^1\big(X;\,\Omega\mathcal{A}\big)} \xrightarrow{\;\epsilon\iota_X^*\;} \underset{\text{reduced cohomology for disjoint }\infty}{\widetilde{H}^1\Big(\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}};\,\Omega\mathcal{A}\Big)} = \underset{\text{plain cohomology}}{H^1\big(X\setminus\{\infty\};\,\Omega\mathcal{A}\big)} \twoheadrightarrow \underset{\text{purely singular charges}}{\mathrm{coker}\big(\epsilon\iota_X^*\big)} \tag{38}$$

$$X \xleftarrow{\quad \epsilon\iota_X \quad} \underset{\text{make }\infty\text{ unreachable}}{\big(X\setminus\{\infty\}\big)_{\sqcup\{\infty\}}}$$

(...)

Notice the mapping space adjunction

$$\mathrm{Maps}^{*/}\Big(X,\,\mathrm{Maps}^{*/}(Y,\,Z)\Big) \;\simeq\; \mathrm{Maps}^{*/}\big(X\wedge Y,\,Z\big) \;\simeq\; \mathrm{Maps}^{*/}\Big(Y,\,\mathrm{Maps}^{*/}\big(X,\,Z\big)\Big) \tag{39}$$

The **Whitehead bracket L_∞-algebra** $\mathfrak{l}\mathcal{A}$ of topological spaces $\mathcal{A}$.

Quillen-Sullivan theorem. [FSS23-Char, Prop. 4.23, 5.6 & 5.13]
For a topological space $\mathcal{A}$ which is
(1.) simply connected: $\pi_0(i) = *$, $\pi_1(i) = 1$,
(2.) of finite rational type: $\dim_{\mathbb{Q}}\big(H^n(\mathcal{A};\mathbb{Q})\big) < \infty$.
there is a unique-up-to-iso polynomial dgc-algebra over $\mathbb{R}$, whose:
(1.) generators are the $\mathbb{R}$-rational homotopy groups of $\mathcal{A}$,

$$\mathrm{CE}(\mathfrak{l}\mathcal{A}) \;=\; \Big(\wedge^\bullet\, \big(\pi_\bullet(\Omega\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee,\; \mathrm{d}_{\mathrm{CE}(\mathfrak{l}\mathcal{A})}\Big)$$

(2.) cochain cohomology is the $\mathbb{R}$-cohomology of $\mathcal{A}$.

$$H^\bullet\big(\mathrm{CE}(\mathfrak{l}\mathcal{A})\big) = H^\bullet(\mathcal{A};\,\mathbb{R})$$

This is the CE-algebra of the Whitehead L_∞-algebra of $\mathcal{A}$, without unary bracket; the binary bracket is Whitehead's.

Notice. Every connected homotopy type $\mathcal{A}$ is equivalently the classifying space of its loop ∞-group $\Omega\mathcal{A}$ (cf. [FSS23-Char, Prop. 2.2]):

$$\mathcal{A} \;\simeq\; B(\Omega\mathcal{A})\,. \tag{40}$$

In this sense all $\mathcal{A}$-cohomology is higher gauge theory for a gauge ∞-group $G \simeq \Omega\mathcal{A}$.
The *Whitehead bracket L_∞-algebra* may be thought of as a homotopical version of the would-be Lie algebra of this ∞-group:

$$(\mathfrak{l}\mathcal{A})_\bullet \;\simeq\; \pi_\bullet\big(\Omega\mathcal{A}\big)\otimes_{\mathbb{Z}}\mathbb{R} \tag{41}$$

The "**Sullivan model**" of $\mathcal{A}$ is the CE-algebra of this L_∞-algebra.

Examples:

Circle: $\mathcal{A} \equiv S^1 \simeq B\mathbb{Z}$.
$\big(\pi_\bullet(S^1)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle\omega_1\rangle$, $\quad H^\bullet(S^1;\mathbb{R}) \simeq \mathbb{R}[\omega_1]$
Since $\mathbb{R}[\omega_1]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{S^1} = 0$ and hence

$$\mathrm{CE}\big(\mathfrak{l}S^1\big) \;\simeq\; \mathbb{R}[\omega_1]/\big(\mathrm{d}\,\omega_1 = 0\big)$$

While the circle is not simply connected, it is a "nilpotent space", and Sullivan's theorem actually applies in this generality.
Nilpotent spaces have nilpotent fundamental group (eg.: abelian) such that all higher homotopy groups are nilpotent modules (eg.: trivial modules).

2-Sphere: $\mathcal{A} \equiv S^2$.
$\big(\pi_\bullet(S^2)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \;\simeq\; \mathbb{R}\langle\omega_2,\,\omega_3\rangle$, $\;H^\bullet(S^2;\mathbb{R}) \simeq \mathbb{R}[\omega_2]/\big(\omega_2^2\big)$
The differential on $\mathbb{R}[\omega_2,\omega_3]$ needs to remove ω_2^2 and ω_3 from cohomology, hence it must be that:

$$\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}\begin{bmatrix}\omega_3,\\ \omega_2\end{bmatrix}\Big/\left(\begin{aligned}&\mathrm{d}\,\omega_3 = -\tfrac{1}{2}\omega_2\wedge\omega_2\\ &\mathrm{d}\,\omega_2 = 0\end{aligned}\right)$$

The homotopy group coresponding to the generator ω_3 is that represented by the *complex Hopf fibration*

$$S^3 \xrightarrow{h_{\mathbb{C}}} S^2\,.$$

3-Sphere: $\mathcal{A} \equiv S^3$.
$\big(\pi_\bullet(S^3)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle\omega_3\rangle$, $\quad H^\bullet(S^3;\mathbb{R}) \simeq \mathbb{R}[\omega_3]$
Since $\mathbb{R}[\omega_3]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{S^3} = 0$ and hence

$$\mathrm{CE}\big(\mathfrak{l}S^3\big) \;\simeq\; \mathbb{R}[\omega_3]/\big(\mathrm{d}\,\omega_3 = 0\big)$$

While $S^3 \simeq \mathrm{SU}(2)$, we see that $\mathfrak{l}\,\mathrm{SU}(2)$ is different from $\mathfrak{su}(2)$. But the former captures the cocycles of the latter:

$$\begin{array}{ccc}\mathfrak{su}(2) & \longrightarrow & \mathfrak{l}\,\mathrm{SU}(2)\\ \mathrm{CE}\big(\mathfrak{su}(2)\big) & \longleftarrow & \mathrm{CE}\big(\mathfrak{l}\,\mathrm{SU}(2)\big)\\ \mathrm{tr}\big(-,\,[-,-]\big) & \longleftarrow\!\shortmid & \omega_3\end{array}$$

4-Sphere: $\mathcal{A} \equiv S^4$.
$\big(\pi_\bullet(S^4)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \;\simeq\; \mathbb{R}\langle\omega_4,\,\omega_7\rangle$, $\;H^\bullet(S^4;\mathbb{R}) \simeq \mathbb{R}[\omega_4]/\big(\omega_4^2\big)$
The differential on $\mathbb{R}[\omega_4,\omega_7]$ needs to remove ω_4^2 and ω_7 from cohomology, hence it must be that:

$$\mathrm{CE}\big(\mathfrak{l}S^4\big) \;\simeq\; \mathbb{R}\begin{bmatrix}\omega_7,\\ \omega_4\end{bmatrix}\Big/\left(\begin{aligned}&\mathrm{d}\,\omega_7 = -\tfrac{1}{2}\omega_4\wedge\omega_4\\ &\mathrm{d}\,\omega_4 = 0\end{aligned}\right)$$

The homotopy group corresponding to the generator ω_7 is that represented by the *quaternionic Hopf fibration*

$$S^7 \xrightarrow{h_{\mathbb{H}}} S^4$$

Complex Projective space: $\mathcal{A} \equiv \mathbb{C}P^n$. $\big(\pi_\bullet(\mathbb{C}P^n)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_2,\, \omega_{2n+1}\rangle$, $H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\omega_2]/\big(\omega_2^{n+1}\big)$ The differential on $\mathbb{R}[\omega_2, \omega_{2n+1}]$ needs to remove ω_2^{n+1} from cohomology, hence it must be that: $$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{C}P^n\big) \;\simeq\; \mathbb{R}\begin{bmatrix}\omega_{2n+1},\\ \omega_2\end{bmatrix}\Big/\left(\begin{aligned}\mathrm{d}\,\omega_{2n+1} &= -\omega_2^{n+1}\\ \mathrm{d}\,\omega_2 &= 0\end{aligned}\right)$$	This is related to the above sequence of examples by the fact that $\mathbb{C}P^n$ is an S^1-quotient of S^{2n+1}: $$\begin{array}{ccc} S^1 & \hookrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{C}P^n \end{array}$$
Infinite Projective space: $\mathcal{A} \equiv \mathbb{C}P^\infty \simeq B\mathrm{U}(1) \simeq B^2\mathbb{Z}$. $\big(\pi_\bullet(\mathbb{C}P^\infty)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_2\rangle$, $H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\omega_2]$ Since $\mathbb{R}[\omega_2]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{\mathbb{C}P^\infty} = 0$: $$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{C}P^\infty\big) \;\simeq\; \mathbb{R}\,[\omega_2]\,/\,\big(\mathrm{d}\,\omega_2 = 0\big)$$	This is the Lie 2-algebra of the shifted circle group: $$\mathfrak{l}B\mathrm{U}(1) \;\simeq\; b\,\mathfrak{u}(1)$$
Eilenberg-MacLane space: $\mathcal{A} \equiv B^n\mathrm{U}(1) \simeq B^{n+1}\mathbb{Z}$. $\big(\pi_\bullet(B^{n+1}\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle\omega_{n+1}\rangle$, $H^\bullet(B^{n+1}\mathbb{Z}) \simeq \mathbb{R}[\omega_{n+1}]$ Since $\mathbb{R}[\omega_{n+1}]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{B^{n+1}\mathbb{Z}} = 0$: $$\mathrm{CE}\big(\mathfrak{l}\,B^{n+1}\mathbb{Z}\big) \;\simeq\; \mathbb{R}\,[\omega_{n+1}]\,/\,\big(\mathrm{d}\,\omega_{n+1} = 0\big)$$	This is the Lie $(n+1)$-algebra of the circle $(n+1)$-group: $$\mathfrak{l}B^n\mathrm{U}(1) \;\simeq\; b^n\,\mathfrak{u}(1)$$
Classifying space: $\mathcal{A} \equiv BG$ of cpt. 1-conn. Lie group. $H^\bullet(BG;\mathbb{R}) \simeq \mathrm{inv}^\bullet(\mathfrak{g})$ the invar. polynomials on Lie alg. (Chern-Weil theory) Since $H^\bullet(BG;\mathbb{R})$ is already a free graded-symmetric ring it must be that $\mathrm{d}_{BG} = 0$ (cf. [FSS23-Char, Lem. 8.2]): $$\mathrm{CE}\big(\mathfrak{l}\,BG\big) \;\simeq\; \mathrm{inv}^\bullet(\mathfrak{g})/\,(\mathrm{d}_{BG} = 0)$$	$\mathfrak{l}BG$ captures all the curvature invariants hence all the invariant flux densities of $\mathfrak{g}$-connections $A \in \Omega^1_{\mathrm{dR}}(X)\otimes\mathfrak{g}$, e.g. $$\begin{array}{ccc}\mathrm{CE}\big(\mathfrak{l}B\mathrm{SU}(2)\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\ \mathrm{tr}(-,-) & \mapsto & \delta_{ij}F_A^{(i)}\wedge F_A^{(j)}\end{array}$$

The M-Theory gauge algebra models is the Whitehead bracket of the 4-sphere. Of particular interest to us, for the formulation of *Hypothesis H* in §1.3, is the example of the Whitehead L_∞-algebra of the 4-sphere, because this happens to coincide [Sa10, §4][SV22b, §2.2] with the *M-theory gauge algebra* ([CJLP98, (2.5)]):

Homotopy type (topological space) $\mathcal{A}$	**Sullivan model** ("FDA") $\mathrm{CE}(\mathfrak{l}\mathcal{A})$	**Quillen model** (Whitehead L_∞-algebra) $\mathfrak{l}\mathcal{A}$
S^4 **4-sphere**	$\mathbb{R}\begin{bmatrix}\omega_7,\\ \omega_4\end{bmatrix}\Big/\left(\begin{aligned}\mathrm{d}\,\omega_7 &= -\tfrac{1}{2}\omega_4\wedge\omega_4\\ \mathrm{d}\,\omega_4 &= 0\end{aligned}\right)$ **abstract Bianchi identity of duality-symmetric C-field fluxes**	$\mathbb{R}\left\langle\begin{matrix}v_6,\\ v_3\end{matrix}\right\rangle$, $[v_3, v_3] = v_6$ **C-field gauge algebra**

(42)

On the right we are including a factor $\frac{1}{2}$ to match physics normalization conditions. The subtle deeper origin of this factor, which goes beyond rational homotopy theory, is discussed in detail in [FSS19b], in the context of the M5-brane model (see §2.1).

Rational homotopy theory: Discarding torsion in non-abelian cohomology. From the perspective that any topological space $\mathcal{A}$ serves as the *classifying space* of a generalized non-abelian cohomology theory (34), the idea of Rational Homotopy Theory (survey in [He07][FSS23-Char, §4]) becomes that of extracting the non-torsion content of such a cohomology theory, which we will see is that shadow of it which, over smooth manifolds, can be reflected in the non-abelian de Rham cohomology of $\mathfrak{l}\mathcal{A}$-valued differential forms.

regard spaces as classifying spaces ↓

Homotopy theory	Rational	Sullivan model
Nonabelian cohomology	Non-torsion	de Rham cohomology

Now, in a sense, the signature of any $\mathcal{A}$-cohomology theory is its reduced cohomology groups (35) on spheres, equal to the homotopy groups of the classifying space:

$$\underset{\text{of the } n\text{-sphere}}{\overset{\text{reduced } \mathcal{A}\text{-cohomology}}{}} \quad \widetilde{H}^1\big(S^n;\, \Omega\mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}^{*/}\big(S^n,\, \mathcal{A}\big) \;\equiv\; \pi_n(\mathcal{A}) \quad \underset{\text{of classifying space}}{\overset{n\text{th homotopy group}}{}}$$

Assuming throughout (for ease of exposition) that $\mathcal{A}$ is simply-connected, the remaining non-trivial homotopy groups are abelian $\pi_{n\geq 2}(i) \in \mathrm{AbGrp}$. Discarding torsion elements (nilpotent group elements) from these groups is achieved by tensoring with the abelian group of rational numbers:

$$\begin{array}{ccc}
\underset{\text{of the } n\text{-sphere}}{\overset{\text{reduced } \mathcal{A}\text{-cohomology}}{}}\;\widetilde{H}^1(S^n;\Omega\mathcal{A}) \simeq \pi_n(i) & \xrightarrow{\text{rationalization}} & \pi_n(i)\otimes_{\mathbb{Z}}\mathbb{Q} \;\; \underset{\text{of the } n\text{-sphere}}{\overset{\text{rationalized reduced } \mathcal{A}\text{-cohomology}}{}} \\
[c] \text{ with } k\cdot[c] = 0 & \longmapsto & [c]\otimes 1 \;=\; [c]\otimes k\cdot\tfrac{1}{k} \;=\; k\cdot[c]\otimes\tfrac{1}{k} \;=\; 0
\end{array}$$

This is a "projection operation" (jargon: "localization"), in that doing it twice has no further effect:

$$\begin{array}{ccc}
\text{double rationalizaton}\;\; \pi_n(i)\otimes_{\mathbb{Z}}\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} & \underset{\sim}{\overset{\text{isomorphic}}{\longleftrightarrow}} & \pi_n(i)\otimes_{\mathbb{Z}}\mathbb{Q}\;\; \text{single rationalizaton} \\
[c]\otimes\frac{p_1}{q_1}\otimes\frac{p_2}{q_2} = [c]\otimes\frac{p_1}{q_1}\otimes q_1\frac{p_2}{q_1q_2} & \longleftrightarrow & [c]\otimes\frac{p_1p_2}{q_1q_2}
\end{array}$$

Hence to have a classifying space for the non-torsion part of $\mathcal{A}$-cohomology means to ask for:

The rationalization of $\mathcal{A}$:

A topological space	$L^{\mathbb{Q}}\mathcal{A}$
all whose homotopy groups have the structure of $\mathbb{Q}$-vector spaces	$\pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big) \in \mathrm{Mod}_{\mathbb{Q}}$
equipped with a map from $\mathcal{A}$	$\mathcal{A} \xrightarrow{\eta^{\mathbb{Q}}_{\mathcal{A}}} L^{\mathbb{Q}}\mathcal{A}$
which induces isomorphisms on rationalized homotopy groups	$\pi_n(i)\otimes_{\mathbb{Z}}\mathbb{Q} \underset{\sim}{\xrightarrow{\eta^{\mathbb{Q}}_{\mathcal{A}}\otimes_{\mathbb{Z}}\mathbb{Q}}} \pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big)\otimes_{\mathbb{Z}}\mathbb{Q}$
and is universal with this property	

For example, the rationalization of $B^n\mathbb{Z}$ classifies ordinary rational cohomology, mapping further to ordinary de Rham cohomology:

$$\begin{array}{ccccccc}
\underset{B^n\mathbb{Z}}{\overset{\text{integral EM-space}}{}} & \underset{\eta^{\mathbb{Q}}_{B^n\mathbb{Z}}}{\xrightarrow{\text{rationalization}}} & \underset{L^{\mathbb{Q}}B^n\mathbb{Z}\simeq B^n\mathbb{Q}}{\overset{\text{rational EM-space}}{}} & \underset{B^n\left((-)\otimes_{\mathbb{Q}}\mathbb{R}\right)}{\xrightarrow{\text{extension of scalars}}} & \underset{B^n\mathbb{R}}{\overset{\text{real EM space}}{}} & & \\
\pi_0\mathrm{Map}\big(X,\, B^n\mathbb{Z}\big) & \xrightarrow[\pi_0\mathrm{Map}\left(X,\,\eta^{\mathbb{Q}}_{B^n\mathbb{Z}}\right)]{} & \pi_0\mathrm{Map}\big(X,\, B^n\mathbb{Q}\big) & \xrightarrow[\pi_0\mathrm{Map}\left(X,B^n((-)\otimes_{\mathbb{Q}}\mathbb{R})\right)]{} & \pi_0\mathrm{Map}\big(X,\, B^n\mathbb{R}\big) & & \\
\wr\| & & \wr\| & & \wr\| & & \\
H^n(X;\,\mathbb{Z}) & \xrightarrow{\text{cohomology operation}} & H^n(X;\,\mathbb{Q}) & \xrightarrow{\text{cohomology operation}} & H^n(X;\,\mathbb{R}) & \underset{\text{isomorphism}}{\xrightarrow{\text{de Rham}}} & H^n_{\mathrm{dR}}(X) \\
\text{integral ordinary cohomology} & & \text{rational ordinary cohomology} & & \text{real ordinary cohomology} & & \text{de Rham cohomology}
\end{array} \tag{43}$$

(ordinary character map: from $H^n(X;\mathbb{Z})$ to $H^n_{\mathrm{dR}}(X)$)

We may regard this as the archetype of a *character map* and ask for its generalization to any $\mathcal{A}$-cohomology theory. The pivotal observation of [FSS23-Char] is that for this purpose one may invoke the *fundamental theorem of dg-algebraic rational homotopy theory*:

The Fundamental Theorem of dg-Algebraic Rational Homotopy Theory (reviewed as [FSS23-Char, Prop. 5.6]) says that the homotopy theory of rational spaces (simply-connected with fin-dim rational cohomology) is all encoded by their Whitehead L_∞-algebra (41) over the rational numbers.

In particular, for X a CW-complex one gets

$$\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big)_{/\mathrm{homotopy}} \;\simeq\; \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}\big(\mathfrak{l}^{\mathbb{Q}}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{PLdR}}(X)\Big)_{/\mathrm{concordance},} \tag{44}$$

where on the right we have something called the "piecewise linear de Rham complex" of the topological space X.

Notice that the right-hand side looks close to the definition of $\mathfrak{l}\mathcal{A}$-valued de Rham cohomology in (33). In order to actually connect to such smooth differential forms, we need to extend the scalars from the rational to the real numbers:

Rational homotopy theory over the Reals. [FSS23-Char, Def. 5.7, Rem. 5.2] The construction (44) also works over $\mathbb{R}$ (but is then not a "localization") to give the $\mathbb{R}$-rationalization [FSS23-Char, Def. 5.7, Prop. 5.8]:

The $\mathbb{R}$-rationalization of $\mathcal{A}$:

A topological space	$L^{\mathbb{R}}\mathcal{A}$
equipped with a map	$L^{\mathbb{Q}}\mathcal{A} \xrightarrow{\;\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}\;} L^{\mathbb{R}}\mathcal{A}$
which on homotopy groups is extension of scalars	$\pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big) \xrightarrow[\;= (-)\otimes_{\mathbb{Q}}\mathbb{R}\;]{\;\pi_n\left(\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}\right)\;} \pi_n\big(L^{\mathbb{R}}\mathcal{A}\big)$
suitably universal as such.	

With this "derived extension of scalars" [FSS23-Char, Lem 5.3] and for X a smooth manifold, the fundamental theorem (44) does relate to smooth differential forms (33) [FSS23-Char, Lem. 6.4] via a *non-abelian de Rham theorem* [FSS23-Char, Thm. 6.5]:

$$\begin{array}{ccccc}
\overset{\text{non-abelian rational cohomology}}{H^1\big(X;\, L^{\mathbb{Q}}\Omega\mathcal{A}\big)} & \xrightarrow{\;\text{derived extension of scalars}\;} & \overset{\text{non-abelian real cohomology}}{H^1\big(X;\, L^{\mathbb{R}}\Omega\mathcal{A}\big)} & & \\
\Vert & & \Vert & \nwarrow^{\text{non-abelian de Rham theorem}}_{\sim} & \\
\pi_0\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big) & \xrightarrow{\;\pi_0\mathrm{Map}\left(X,\, \eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}}\right)\;} & \pi_0\mathrm{Map}\big(X,\, L^{\mathbb{R}}\mathcal{A}\big) & & \\
{\scriptstyle\text{fundamental theorem of dg-algebraic rational homotopy}}\;\updownarrow\wr & & \updownarrow\wr & & \searrow \\
\mathrm{Hom}_{\mathrm{dgAl}}\Big(\mathrm{CE}\big(\mathfrak{l}^{\mathbb{Q}}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{PLdR}}(X)\Big)_{/\mathrm{cncd}} & \xrightarrow[\;\text{scalars}\;]{\;\text{extension of}\;} & \mathrm{Hom}_{\mathrm{dgAl}}\Big(\mathrm{CE}\big(\mathfrak{l}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{dR}}(X)\Big)_{/\mathrm{cncd}} & \equiv & \underset{\text{non-abelian de Rham cohomology}}{H^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}\mathcal{A}\big)}
\end{array} \tag{45}$$

In abelian (ie. Whitehead-generalized) cohomology theories both the rationalization step and the subsequent extension of scalars to $\mathbb{R}$ can be more easily described as forming the smash product of the coefficient spectrum with the Eilenberg-MacLane spectrum $H\mathbb{R}$ [FSS23-Char, Ex. 5.7]. This is how the Chern-Dold character map over $\mathbb{R}$ is tacitly used in all the literature on abelian Whitehead-generalized differential cohomology theory (e.g. [BN19, Def. 4.2]):

$$\overbrace{\mathrm{Spectra} \xrightarrow[\;\substack{\text{rationalization}\\ \text{localization}}\;]{\;(-)\wedge H\mathbb{Q}\;} \mathrm{Spectra} \xrightarrow[\;\substack{\text{extension}\\ \text{of scalars}}\;]{\;(-)\wedge_{H\mathbb{Q}} H\mathbb{R}\;} \mathrm{Spectra}}^{\substack{\text{rationalization over } \mathbb{R}\\ (-)\wedge H\mathbb{R}}} \tag{46}$$

The point of the non-abelian de Rham theorem (45) from [FSS23-Char] is to generalize the real-ification of spectra (46) to non-abelian cohomology, such as to Cohomotopy; and the key result that makes this work is the fundamental theorem of dg-algebraic homotopy theory (44). This, ultimately, is the 'reason' why L_∞-valued differential forms relate fluxes to their flux-quantization laws.

The general non-abelian character map is now immediate [FSS23-Char, Def. IV.2]: It is the cohomology operation induced by $\mathbb{R}$-rationalization of classifying spaces, seen under the non-abelian de Rham theorem (45):

$$\begin{array}{ccccccc}
\multicolumn{7}{c}{\overbrace{\hspace{30em}}^{\text{character map on } \mathcal{A}\text{-cohomology}}} \\
H^1(X;\Omega\mathcal{A}) & \xrightarrow{\;\text{rationalization}\;} & H^1\big(X;\, L^{\mathbb{Q}}\Omega\mathcal{A}\big) & \xrightarrow[\;\text{of scalars}\;]{\;\text{extension}\;} & H^1\big(X;\, L^{\mathbb{R}}\Omega\mathcal{A}\big) & \xrightarrow[\;\text{de Rham theorem}\;]{\;\text{nonabelian}\;} & H^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}\mathcal{A}\big) \\
\Vert & & \Vert & & \Vert & & \Vert \\
\pi_0\mathrm{Map}\big(X,\, \mathcal{A}\big) & \xrightarrow{\;(\eta^{\mathbb{Q}}_{\mathcal{A}})_*\;} & \pi_0\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big) & \xrightarrow{\;(\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}})_*\;} & \pi_0\mathrm{Map}\big(X,\, L^{\mathbb{R}}\mathcal{A}\big) & \underset{\substack{\text{fundamental theorem}\\ \text{of dg-algebraic RHT}}}{\xrightarrow{\;\sim\;}} & \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)_{/\mathrm{cncrd}}
\end{array} \tag{47}$$

Summary so far:

$\mathcal{A}$	$\longmapsto$ ($\mathbb{R}$-rationalization)	Sullivan model of $\mathcal{A}$: $\mathrm{CE}(\mathfrak{l}\mathcal{A})$ — Chevalley-Eilenberg algebra of its Whitehead L_∞-algebra	$\longleftrightarrow$	Whitehead L_∞-algebra $\mathfrak{l}\mathcal{A}$ = "infinitesimal version $\mathrm{Lie}(\Omega\mathcal{A})$ of loop group"
Eilenberg-MacLane space $K(\mathbb{Z}, n)$	$\longmapsto$	$\mathbb{R}[c_n]/(\mathrm{d}\, c_n = 0)$		
classifying space of principal G-bundles BG	$\longmapsto$	invariant polynomials on the Lie algebra $\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, \mathrm{d} = 0\big)$		
space of Fredholm operators $\mathrm{Fred}_{\mathbb{C}}$	$\longmapsto$	$\mathbb{R}\big[\{f_{2k} \,\vert\, k \in \mathbb{N}\}\big]/\big(\mathrm{d}\, f_{2k} = 0\big)$		
stage in spectrum of spaces E_n	$\longmapsto$	$\Big(\mathrm{Sym}\big(\mathrm{Hom}_{\mathbb{Z}}(\pi_\bullet(E_n),\, \mathbb{R})\big), \mathrm{d} = 0\Big)$		
even-dimensional sphere S^{2k}	$\longmapsto$	$\mathbb{R}[g_{2k},\, g_{4k-1}]\Big/\left(\begin{array}{ll}\mathrm{d}\, g_{4k-1} & = g_{2k} \wedge g_{2k} \\ \mathrm{d}\, g_{2k} & = 0\end{array}\right)$		

This is an equivalence between:

> weak rational homotopy types of nilpotent connected **spaces** with degreew. fin-dim rational cohomology
>
> $\wr\!\wr$
>
> quasi-isomorphism types of nilpotent connective **L_∞-algebras** on degreew. fin-dim $\mathbb{Q}$-vector spaces

here shown/used after extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$

notice the difference to L_∞-algebras of *gauge potentials*: $\mathfrak{su}(n)$, $\mathfrak{string}(n)$, ... which are *not nilpotent*

here: L_∞-algebras of gauge *fields/fluxes*

The general non-abelian character map $\mathrm{ch}_{\mathcal{A}}$ universally approximates $\mathcal{A}$-cohomology classes by $\mathfrak{l}\mathcal{A}$-valued de Rham classes:

$$
\begin{array}{ccc}
\text{non-abelian cohomology} & & \text{non-abelian de Rham cohomology} \\
H^1\big(X^d;\, \Omega\mathcal{A}\big) & \xrightarrow[\mathrm{ch}_{\mathcal{A}}]{\text{non-abelian character}} & H^1_{\mathrm{dR}}\big(X^d;\, \mathfrak{l}\mathcal{A}\big) \\
c & \longmapsto & [\vec{B}\,] \\
\text{total charge} & & \text{total flux}
\end{array}
$$

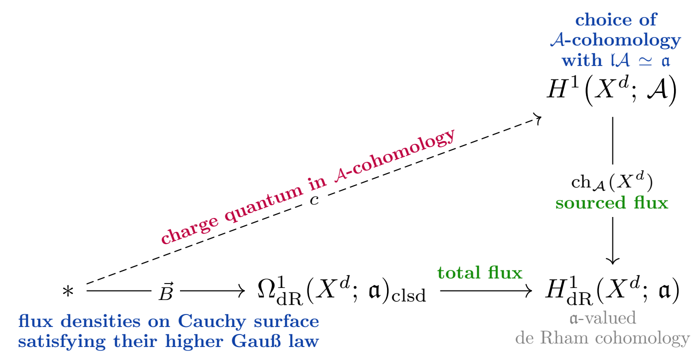
specializing to:

$$
\begin{array}{ccc}
\underset{\text{ordinary cohomology}}{H^n(X^d;\, \mathbb{Z})} & \xrightarrow{\text{de Rham map}} & H^n_{\mathrm{dR}}(X^d) \\[1em]
\underset{\text{nonabelian cohomology}}{H^1(X^d;\, G)} & \xrightarrow{\text{Chern-Weil homomorphism}} & H_{\mathrm{dR}}\big(X^d;\, \mathrm{inv}(\mathfrak{g})\big) \\[1em]
\underset{\text{topological K-theory}}{K^0(X^d)} & \xrightarrow{\text{Chern character}} & \underset{k \in \mathbb{N}}{\bigoplus} H^{2k}_{\mathrm{dR}}(X^d) \\[1em]
\underset{\text{Whitehead generalized cohomology}}{E^n(X^d)} & \xrightarrow{\text{Chern-Dold character}} & \underset{k \in \mathbb{Z}}{\bigoplus} H^{n+k}_{\mathrm{dR}}\big(X^d;\, \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}\big) \\[1em]
\underset{\text{coHomotopy in even degree}}{\pi^{2k}(X^d)} & \xrightarrow{\text{coHomotopical character}} & H^1_{\mathrm{dR}}\big(X^d;\, \mathfrak{l}S^{2k}\big)
\end{array}
\tag{48}
$$

$$
H^1_{\mathrm{dR}}\big(X^d;\, \mathfrak{l}S^{2k}\big) = \left\{ \begin{array}{ll} G_{4k-1} \in \Omega^{4k-1}_{\mathrm{dR}}(X^d) \\ G_{2k} \;\;\; \in \Omega^{2k}_{\mathrm{dR}}(X^d) \end{array} \,\middle\vert\, \begin{array}{ll} \mathrm{d}\, G_{4k-1} = G_{2k} \wedge G_{2k} \\ \mathrm{d}\, G_{2k} \;\;\; = 0 \end{array} \right\}\Big/_{\text{concordance}}
$$

With the general character map in hand, we may finally state a general definition of flux quantization:

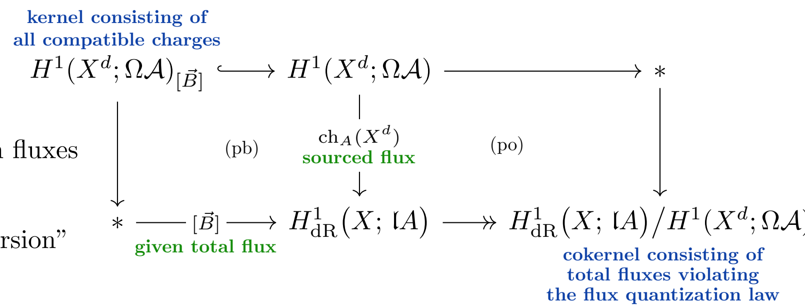
The first idea of flux quantization. Higher gauge fields on a spatial Cauchy surface satisfying their Gauß law constraint are equivalently closed L_∞-valued forms for some characteristic L_∞-algebra $\mathfrak{a}$.
These flux densities should be thought of as mere *characters* of the hidden *true field content* which encompasses also lifts of these characters to charge classes in $\mathcal{A}$-cohomology, where any choice of $\mathcal{A}$ with $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$ serves as a compatible *flux quantization law.*



Since the character map generally...

...fails to be surjective, ie. has a cokernel:
  ⇒ **flux quantization is a condition** on fluxes

...fails to be injective, ie. has a kernel:
  ⇒ **flux quantization is a choice** of "torsion"



In fact, the actual field content is larger still, involving also the *gauge potentials* – we come to this in §1.2.3 below.

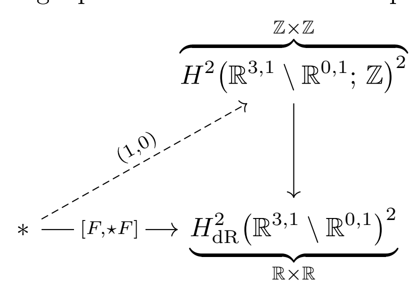
Example: Quantization of electromagnetic flux. The pregeometric fluxes of electromagnetism are (6):

$$(F, G) \;\in\; \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \times \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \;\simeq\; \Omega_{\mathrm{dR}}\big(X;\, \mathfrak{bu}(1) \oplus \mathfrak{bu}(1)\big)_{\mathrm{clsd}}$$

Admissible electromagnetic flux-quantization laws are
Spaces A whose rationalization is $B^2\mathbb{Q} \times B^2\mathbb{Q}$:

$\underbrace{B^2\mathbb{Q}}_{\text{mag}} \times \underbrace{B^2\mathbb{Q}}_{\text{el}}$	this choice imposes essentially *no* flux quantization – tacit choice since [Maxwell1873] until [Dirac1931]
$\underbrace{B^2\mathbb{Z}}_{\text{mag}} \times \underbrace{B^2\mathbb{Q}}_{\text{el}}$	this choice imposes integrality of magnetic flux but no further condition on electric flux — tacit common choice since [Dirac1931], explicit e.g. in [Fr00, Ex. 2.1.2]
$\underbrace{B^2\mathbb{Z}}_{\text{mag}} \times \underbrace{B^2\mathbb{Z}}_{\text{el}}$	this choice imposes integrality of magnetic flux and integrality of electric flux — considered in [FMS07a][FMS07b] hupf
$\underbrace{B^2\mathbb{Z}}_{\text{mag}} \rtimes \underbrace{BK \ltimes B^2\mathbb{Z}}_{\text{el}}$	for a finite group $K \to \mathbb{Z}/2 = \mathrm{Aut}(\mathbb{Z})$, this choice induces non-commutativity between el/el- and el/mag-fluxes — this was considered in [SS23b]

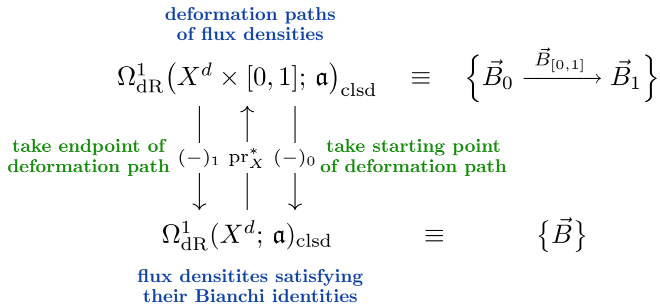
Charge quantization of Dirac monopole:



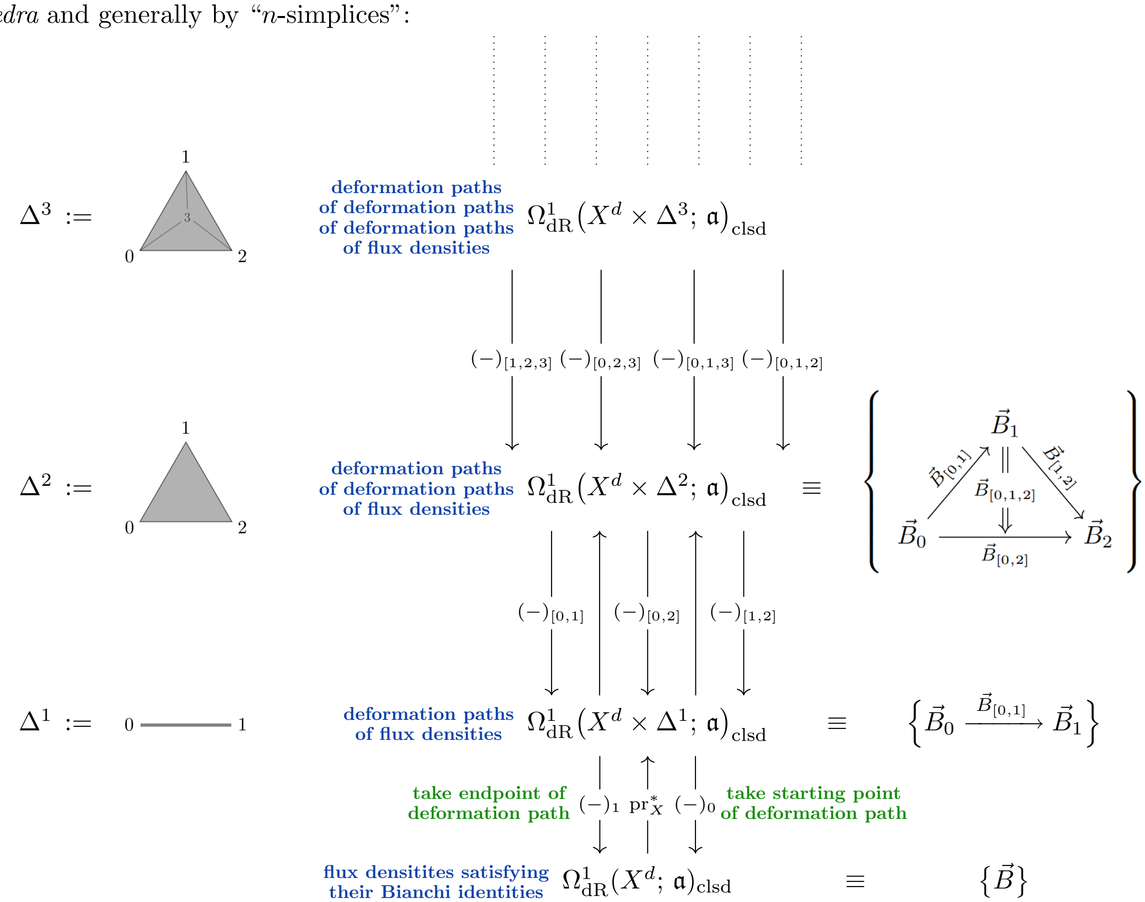
1.2.3 Phase spaces as Differential non-abelian cohomology

In the spirit of the (higher) gauge principle, flux quantization should ultimately not be imposed just as an equality of cohomology classes as in eq. (47), but as a specified coboundary between cocycles, namely as an explicit deformation of flux densities. We discuss how these coboundaries are the higher *gauge potentials* for the given field fluxes – this is the principle of *differential cohomology* in the way first understood in [HS05] and now generlized to non-abelian cohomology via [FSS23-Char, §9].

Higher deformations of flux densities. Recall from (32) that a coboundary in $\mathfrak{a}$-valued de Rham cohomology is a "concordance" of flux densities, to be thought of as a path of smooth variations of the flux densities, subject to their Bianchi identities:



But in higher gauge theories, we are also to consider deformations-of-deformations. It is intuitively plausible that these should be given by deformation paths-of-paths parameterized by *squares* $[0,1]^2$, next by cubes $[0,1]^3$, etc. It turns out to be equivalent but technically more convenient to parameterize them instead by *triangles*, then *tetrahedra* and generally by "n-simplices":



Such a system of sets indexed by higher simplices is called a *simplicial set*.
In its contravariant dependence on $X \in \mathrm{SmthMfd}$ it is called a *simplicial presheaf*
As such, we here denote it $\int \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{a}\big)$ [FSS23-Char, Def. 9.1]

Such simplicial presheaves live in *differential homotopy theory*:

Differential homotopy theory.

The geometry of higher gauge theory must unify:

	differential forms like flux densities F
&	**homotopy types** of classifying spaces A
in	**differential homotopy theory**

This unified context is naturally provided by *higher topos theory*. We briefly indicate how this works – for more exposition and pointers see [Sch24].
The impatient reader may want to skip this and just take note of important consequence (50) below.

Simplicial presheaves modelling geometric homotopy types. For the present purpose we consider:

the site CartSp **of abstract smooth charts**	
an abstract *coordinate chart*	is a Cartesian space $\mathbb{R}^n$ for any $n \in \mathbb{N}$
an abstract *coordinate transformation*	is any smooth function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$
a *covering of coordinate charts*	is an open cover $\left\{\mathbb{R}^n \simeq U_i \hookrightarrow \mathbb{R}^n\right\}_{i \in I}$ which is *differentiably good* in that finite non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are all diffeomorphic to $\mathbb{R}^n$

Given any site of Charts, serving as local model spaces:

- A **generalized space** $\mathcal{X}$ *probeable* **by such charts** is bootstrapped into existence by declaring the simplicial sets of ways of plotting out abstract coordinate charts inside $\mathcal{X}$:

$$\begin{aligned} \mathcal{X} \;:\; \mathrm{Charts}^{\mathrm{op}} &\longrightarrow \mathrm{sSets} \\ \mathbb{R}^n \quad &\mapsto \quad \mathrm{Plots}(\mathbb{R}^n, \mathcal{X}) \end{aligned}$$

  Such **simplicial presheaves** naturally form an sSet-enriched category $\mathrm{sPSh}_{\mathrm{Charts}}$; denote its simplicial hom-complexes by:

$$\begin{aligned} \mathrm{sPSh}^{\mathrm{op}}_{\mathrm{Charts}} \times \mathrm{sPSh}_{\mathrm{Charts}} &\longrightarrow \mathrm{sSet} \\ (\mathcal{X}, \mathcal{Y}) \quad &\mapsto \quad \mathrm{Maps}(\mathcal{X}, \mathcal{Y}) \end{aligned}$$

- Any $U \in$ Charts becomes a generalized space by declaring its plots to be the morphisms of charts (representable presheaf):

$$\begin{aligned} U \;:\; \mathrm{Charts}^{\mathrm{op}} &\longrightarrow \mathrm{sSets} \\ V \quad &\mapsto \quad \mathrm{Charts}(V, U) \end{aligned}$$

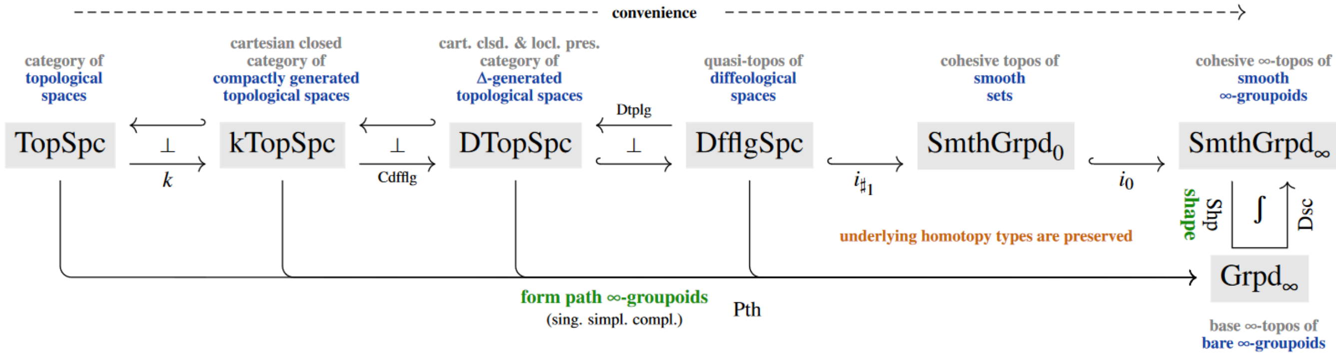
- **Consistency** of this bootstrap of generalized spaces demands:
  (1.) natural identifications between plots by and maps from charts:

$$U \in \mathrm{Charts}, \quad \mathcal{X} \in \mathrm{sPSh}_{\mathrm{Charts}} \qquad \vdash \qquad \mathrm{Plots}(U, \mathcal{X}) \simeq \mathrm{Maps}(U, \mathcal{X})$$

  This is the case by the (enriched) **Yoneda lemma**.
  (2.) that maps of generalized spaces are equivalences — to be denoted $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{W}$ —
  iff *locally on all charts* i.e.: stalk-wise they are *higher gauge equivalence*, i.e.: simplicial weak homotopy equivalences
  this we *enforce* by **simplicial localization**, yielding the ∞-topos $\mathbf{H} := L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{Charts}}$

	by this principle:	**(probes of) spaces are**:
	locality principle	sheaves on charts with
&	higher gauge principle	values in simplicial sets
=	homotopy topos of simplicial sheaves on charts	

Specifically for Charts = CartSp we obtain $\mathrm{SmoothGrpd}_\infty := L^{\mathrm{W}}\mathrm{sPSh}_{\mathrm{CartSp}}$ [FSS23-Char, Ex. 1.20][SS21]:



In $\mathrm{SmthGrpd}_\infty$ all ingredients of higher gauge field theory find a natural home:

smooth moduli space of closed $\mathfrak{l}\mathcal{A}$-valued forms (genuine differential structure)

$$\Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}} \;:\; \begin{array}{ccc} \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\ \mathbb{R}^n & \longmapsto & \Omega^1_{\mathrm{dR}}(\mathbb{R}^n;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}} \end{array}$$

smooth moduli stack of deformations of closed $\mathfrak{l}\mathcal{A}$-valued forms (rational homotopy type of $\mathcal{A}$)

$$\textstyle\int \Omega^1_{\mathrm{dR}}(-;\,\mathfrak{l}\mathcal{A})_{\mathrm{clsd}} \;:\; \begin{array}{ccc} \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\ \mathbb{R}^n & \longmapsto & \Omega^1_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^\bullet;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} \end{array}$$

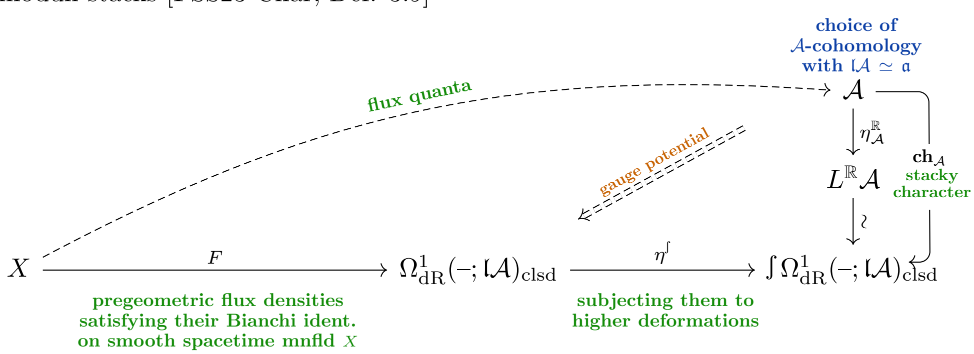
homotopy type of $\mathcal{A}$ (geometrically discrete ∞-groupoid)

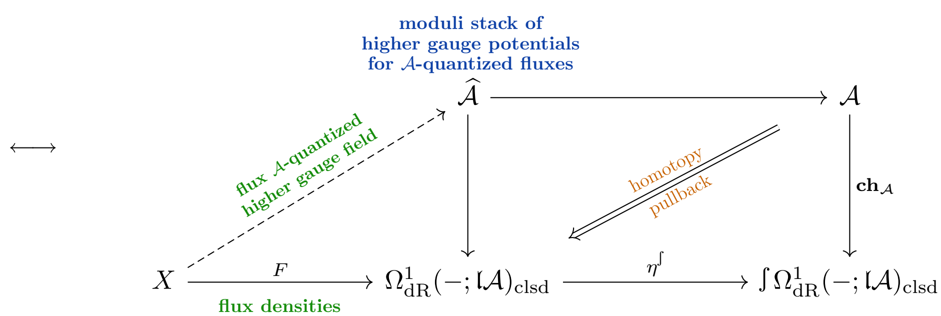
$$\mathcal{A} \;:\; \begin{array}{ccc} \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow & \mathrm{sSet} \\ \mathbb{R}^n & \longmapsto & \mathrm{Sing}(i) \end{array}$$

stacky $\mathbb{R}$-rationalization [FSS23-Char, Lem. 9.1]

$$L^{\mathbb{R}}\mathcal{A} \underset{\mathrm{lwh}}{\simeq} \textstyle\int \Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}}$$

The moduli stack of flux-quantized higher gauge fields. With this, we may enhance the flux quantization of eq. (47) to moduli stacks [FSS23-Char, Def. 3.9]





(49)

– The mapping stack $\mathrm{Map}(X,\,\widehat{\mathcal{A}})$ is the moduli stack of flux-quantized higher gauge fields on X.
– Its global points give the non-abelian *differential $\mathcal{A}$-cohomology* of X.

The canonical phase space of flux-quantized higher gauge fields is this differential cohomology stack:

Since higher Maxwell-type equations (§1.1.2) constrain exclusively the flux densities and since therefore every gauge potential with on-shell flux density is itself on-shell, the phase space must be the enhancement of the solution space (31) by all compatible flux-quantized gauge potentials. This is exactly the construction of the differential cohomology stack (1.2.3).

The physics literature is mostly familiar with the infinitesimal approximation to this higher stack, which is a higher Lie algebroid known as the **BRST complex** (e.g. [HT92]). The full higher phase space is to the BRST complex as a Lie group is to its Lie algebra, hence may be thought of as the **integrated BRST complex** (in the sense of Lie integration).

The plain moduli space of flux-quantized gauge potentials. For X a smooth manifold, the underlying homotopy type of the stack of flux-quantized gauge fields on X is the mapping space from X into $\mathcal{A}$:

$$\underset{\substack{\text{shape of moduli stack}\\ \text{of } \mathcal{A}\text{-quantized fluxes}\\ \text{on smooth mfd } X}}{\int \mathrm{Map}\big(X,\, \widehat{\mathcal{A}}\big)} \;\underset{\text{smooth Oka}}{\simeq}\; \mathrm{Map}\big(\int X,\, \int \widehat{\mathcal{A}}\big) \;\underset{\text{cohesion}}{\simeq}\; \underset{\substack{\text{mapping space}\\ \text{into } \mathcal{A}}}{\mathrm{Map}\big(\int X,\, \mathcal{A}\big)}. \tag{50}$$

Notice that this is what is seen by the ordinary homology of the gauge moduli stack, considered as the topological quantum observables on fluxes, below in §2.2:

$$H_\bullet\Big(\mathrm{Map}\big(X,\, \widehat{\mathcal{A}}\big);\, \mathbb{C}\Big) \;\simeq\; H_\bullet\Big(\int \mathrm{Map}\big(X,\, \widehat{\mathcal{A}}\big);\, \mathbb{C}\Big) \;\underset{(50)}{\simeq}\; H_\bullet\big(\mathrm{Map}(X,\, \mathcal{A});\, \mathbb{C}\big). \tag{51}$$

1.2.4 Background fluxes as Twisting of nonabelian cohomology

All these considerations generalize to fluxes in twisted cohomology [FSS23-Char, §V], describing dependency on background fluxes.

Twisted RR-fields as a fibration. Notice that the twist*ed* RR-fields (19) form a fibration over the twist*ing* NS B-field whose fiber is (a torsor over) the untwisted RR-fields.

$$
\begin{array}{ccc}
\underset{\text{RR-fields}}{\text{flux of free}}\ \left\{ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \,\middle|\, \mathrm{d}F_{2\bullet} = 0 \right\} & \hookrightarrow & \left\{ \begin{array}{l} H_3 \in \Omega^3_{\mathrm{dR}}(X) \,\big|\, \mathrm{d}H_3 = 0 \\ F_{2\bullet} \in \Omega^{2\bullet}_{\mathrm{dR}}(X) \,\big|\, \mathrm{d}F_{2\bullet} = H_3 \wedge F_{2\bullet-2} \end{array} \right\}\ \text{flux of RR-fields coupled to NS B-field} \\
\downarrow & \text{(pb)} & \downarrow \\
\{0\} & \hookrightarrow & \left\{ H_3 \in \Omega^3_{\mathrm{dR}}(X) \,\middle|\, \mathrm{d}H_3 = 0 \right\}\ \text{NS B-field}
\end{array}
\tag{52}
$$

In general, in the case of branes ending on branes, the Bianchi identities for the latter fluxes include polynomial "twists" by the former

$$
\underbrace{\overset{\text{de Rham diff.}}{\mathrm{d}}\, F^{(i)} \;=\; \overset{\text{polynomial}}{P^{(i)}}\Big(\big\{F^{(j)}\big\}_{j \in I},\ \overset{\text{twisting fluxes}}{\big\{H^{(k)}\big\}_{k \in K}} \Big)}_{\text{twisted higher "Bianchi identities"}},
$$

and the previous classifying spaces (34) generalize to *classifying fibrations*

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\text{classifying spaces for...}} \\
\underset{\text{branes}}{\text{intersect}\textit{ed}}\ A & \longrightarrow & A /\!\!/ \mathcal{G}\ \underset{\text{intersections}}{\text{brane}} \\
& & \Big\downarrow{\scriptstyle \text{classifying fibrations}} \\
& & B\mathcal{G}\ \underset{\text{branes}}{\text{intersect}\textit{ing}}
\end{array}
$$

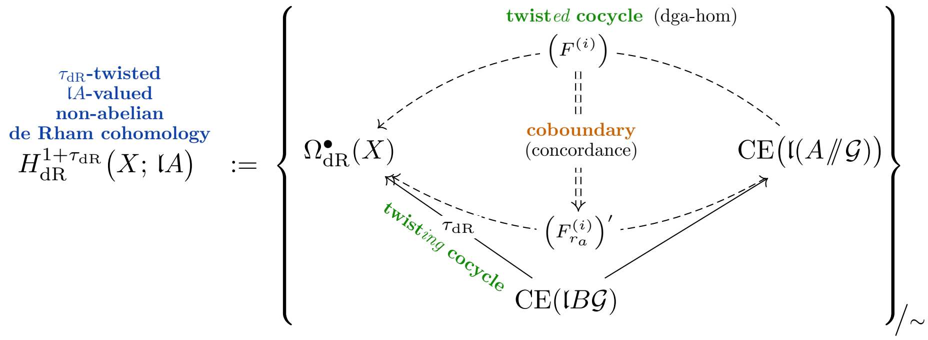
which classify *twisted* non-abelian cohomology theories:

$$
\underset{H^{1+\tau}\big(X;\, \Omega\mathcal{A}\big)}{\overset{\text{Twisted non-abelian cohomology}}{}}
\;:=\;
\underset{\pi_0\,\mathrm{Maps}\big((X,\tau),\, A/\!\!/\mathcal{G}\big)_{B\mathcal{G}}}{\overset{\text{vertical homotopy classes of slice maps}}{}}
\;=\;
\left\{
\begin{array}{ccc}
& \overset{\text{twist}\textit{ed}\text{ cocycle}}{\mathcal{F}_\tau} & \\
X & \overset{\Downarrow}{\underset{\mathcal{F}'_\tau}{\dashrightarrow}} & A/\!\!/\mathcal{G} \\
& {\scriptstyle \tau}\searrow\ {\scriptstyle \text{twist}\textit{ing}\text{ cocycle}} \quad \swarrow & \\
& B\mathcal{G} &
\end{array}
\right\}_{/\sim}
$$

on which the *twisted non-abelian character map*

$$
\begin{array}{ccc}
\overset{\text{twisted non-abelian cohomology}}{H^{1+\tau}\big(X;\, \Omega\mathcal{A}\big)} & \xrightarrow[\mathrm{ch}_A]{\text{twisted non-abelian character}} & \overset{\text{twisted non-abelian de Rham cohomology}}{H^{1+\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X;\, \mathfrak{l}A\big)} \\
\underset{\text{twisted charge}}{c} & \longmapsto & \underset{\text{total flux}}{\big[\vec{B}\,\big]}
\end{array}
\tag{53}
$$

computes the classes of underlying flux densities satisfying twisted Bianchi identities:



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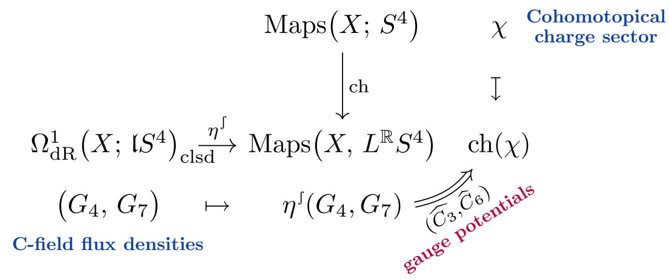
1.3 Hypothesis H on M-theory

With a general understanding of flux-quantization in hand (§1.2) we are in position to motivate and state *Hypothesis H* (§1.3.2) on M-brane charge quantization.

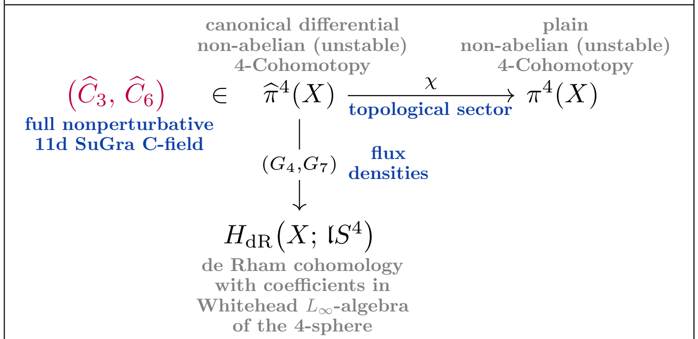
In order to put this in perspective, we first review (§1.3.1) the widely accepted *Hypothesis K* that D-brane charges are quantized in twisted K-theory.

In the special case of *flat* spacetimes X possibly with a point at infinity adjoined (36), Hypothesis H postulates the following, in direct analogy, with Dirac's EM-charge quantization (p. 22):

Hypothesis H on flat spacetimes says that the non-perturbative completion of the C-field in 11d supergravity (10) involves a map χ from spacetime to the homotopy type of the 4-sphere, so that the C-field gauge potentials $(\widehat{C}_3, \widehat{C}_6)$ exhibit the flux densities (G_4, G_7) as $\mathbb{R}$-rational representatives of χ.



In other words, on flat spacetimes Hypothesis H postulates that the non-perturbative C-field is a cocycle in canonical *differential non-abelian 4-Cohomotopy* [FSS15-M5WZW, §4][GrS20, §3.1][Char, Ex. 9.3].



As an immediate plausibility check: This implies, from the well-known homotopy groups of spheres in low degrees, cf. (60) below:

integral quantization of charges carried by singular M5-brane branes *and*

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{5,1}\big) \;=\; \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4\big) \;=\; \pi^4(S^4) \;=\; \pi_4(S^4) \;=\; \mathbb{Z}$$

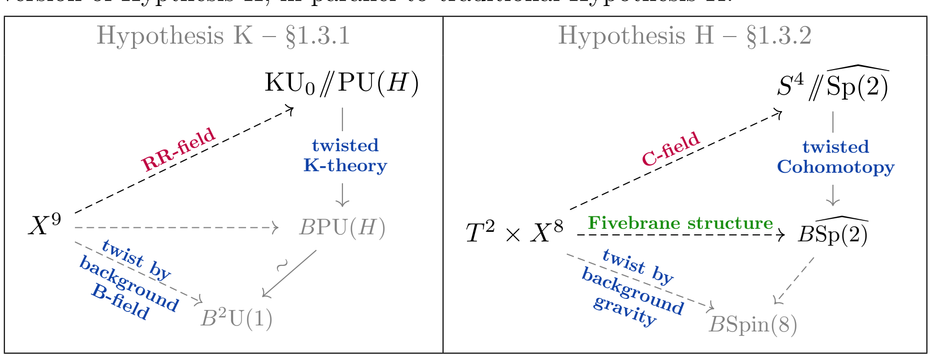
integral quantization of charges carried by singular M2-branes... plus a torsion-contribution (a first prediction of Hypothesis H).

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{2,1}\big) \;=\; \pi^4\big(\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7\big) \;=\; \pi^4(S^7) \;=\; \pi_7(S^4) \;=\; \mathbb{Z} \oplus \mathbb{Z}_{12}$$

To generalize this to non-flat spacetimes, it remains to discuss the twisting (according to §1.2.4) of Cohomotopy received by the gravitational background field.

To appreciate this it may be helpful to recall that also the B-field in 10d may be understood as part of the "generalized geometric" gravitational background flux.

Hypothesis H on gravitational backgrounds. In the following we explain this gravitationally coupled twisted version of Hypthesis H, in parallel to traditional Hypothesis K:



To distinguish M2/M5-charge, the tangential twisting needs to preserve the $\mathbb{H}$-Hopf fibration ⇒ tangential $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure [FSS20-HpH1, §2.3]. With this, integrality of M2's Page charge & anomaly-cancellation of the M5's Hopf-WZ term follows from trivialization of the Euler 8-class, which means lift to the *Fivebrane* 6-group $\widehat{\mathrm{Sp}(2)} \to \mathrm{Sp}(2)$ [FSS21-M5a, §4].

This implies [FSS20-HpH1][FSS21-M5a]:
(1.) half-integrally shifted quantization (65) of M5-brane charge in curved backgrounds, *and*
(2.) integral quantization of the "Page charge" of M2-branes (75).

$$[\widetilde{G}_4] \;:=\; \underbrace{[G_4]}_{\substack{\text{C-field}\\ \text{4-flux}}} + \tfrac{1}{2}\underbrace{\big(\tfrac{1}{2}p_1(TX^8)\big)}_{\substack{\text{integral Spin-}\\ \text{Pontrjagin class}}} \in H^4\big(X^8;\mathbb{Z}\big)$$

$$2[\widetilde{G}_7] \;:=\; 2\big([G_7] + \tfrac{1}{2}[H_3 \wedge \widetilde{G}_4]\big) \;\in\; H^7(\widehat{X}^8;\mathbb{Z})$$

Both of these quantization conditions on M-brane charge are thought to be crucial for M-theory to make any sense.

Previously the first had remained enigmatic and the second had remained wide open.

Double dimensional reduction of Hypothesis H. [FSS18-T, §3][BMSS19, §2.2][SS23-Cyc, p. 6][SV23].

Cyclicfication of classifying spaces. The free loop space of a (classifying space) carries a canonical S^1-action by rotation of loops, its homotopy quotient is the *cyclification.*

cyclified free loop space

$$\mathrm{Cyc}(\mathcal{A}) \;:=\; \mathrm{Maps}\big(S^1,\, \mathcal{A}\big) /\!\!/ S^1$$

Topological KK-Reduction. For a principal circle bundle $X^{10}_{\mathrm{M}} \to X^{9}_{\mathrm{IIA}}$, the moduli of topological $\mathcal{A}$-fields on X^{10}_{M} are equivalent to those of topological $\mathrm{Cyc}(\mathcal{A})$-fields on X^{9}_{IIA} sliced over BS^1

$$\mathrm{Maps}\big(X^{11}, \mathcal{A}\big) \underset{\text{KK-oxidation}}{\overset{\text{KK-reduction}}{\rightleftarrows}} \mathrm{Maps}_{/BS^1}\big(X^{10},\; \mathrm{Cyc}(\mathcal{A})\big) \quad (\simeq)$$

This is **double dimensional** reduction in that with the domain space dimension also the degree of the fluxes is reduced, if they "wrap" the KK-fiber.

e.g. $\mathrm{Cyc}\big(B^{n+1}\mathbb{Z}\big) \;\simeq\; \big(B^n\mathbb{Z} \times B^{n+1}\mathbb{Z}\big) /\!\!/ S^1$

(wrapped fluxes, non-wrapped fluxes, D0-WZ term)

Hence for flat X^{10}_{M} (eg. a torus bundle over a Euclidean space with a point at infinity), **Hypothesis H** implies that fluxes in type IIA string theory are quantized in $\mathrm{Cyc}(S^4)$-cohomologly.

$$\begin{array}{ccc} X^{10}_{\mathrm{M}} & \xrightarrow[c_3]{\text{cohomotopical C-field flux}} & S^4 \\ \big\downarrow{\scriptstyle \mathrm{fib}(f_2)} & \updownarrow & \\ X^{9}_{\mathrm{IIA}} & \overset{\text{and its KK-reduction}}{\underset{\widetilde{c}_3}{\dashrightarrow}} & \mathrm{Cyc}(S^4) \\ & {\scriptstyle f_2}\searrow \quad BS^1 \quad \swarrow & \end{array}$$

Rationally this cyclification is indeed like twisted K-theory, but without the "Romans mass" term F_0 sourced by singular D_8-branes (we find *solitonic* D_8-branes in hupf).
Hence in IIA, Hypothesis H predicts a non-abelian **modification of the traditional Hypothesis K**.

$$\Omega_{\mathrm{dR}}\Big(X^9_{\mathrm{IIA}}, \mathfrak{l}\mathrm{Cyc}(S^4)\Big)_{\mathrm{clsd}} \overset{\text{[FSS17-Sph, Ex. 3.3]}}{=}$$

$$\left\{ \begin{array}{lcl} \mathrm{d}F_2 &=& 0 \\ \mathrm{d}F_4 &=& H_3 \wedge F_2 \\ \mathrm{d}F_6 &=& H_3 \wedge F_4 \end{array} \quad \begin{array}{lcl} \mathrm{d}H_3 &=& 0 \\ \mathrm{d}H_7 &=& F_2 \wedge F_6 - \tfrac{1}{2} F_4 \wedge F_4 \end{array} \right\}$$

U-Duality. This process of double dimensional reduction by cyclification of the 4-sphere coefficients continues to yield, rationally, the expected U-duality symmetries of M-theory [SV23, p. 5]:

D	k	Type of E_k	Lie algebra $\mathfrak{g}$	del Pezzo	Model	Maximal Split Torus
11	0	A_{-1}	$\mathfrak{sl}_0 = \varnothing$	$\mathbb{CP}^2$	S^4	$\mathbb{G}_m$
10	1	A_0	$\mathfrak{sl}_1 = 0$	$\mathbb{B}_1$	$\mathscr{L}_c S^4$	$\mathbb{G}_m \times \mathbb{G}_m$
10	1	A_1	$\mathfrak{sl}_2$	$\mathbb{CP}^1 \times \mathbb{CP}^1$	IIB	$\mathbb{G}_m \times \mathbb{G}_m$
9	2	A_1	$\mathfrak{sl}_2$	$\mathbb{B}_2$	$\mathscr{L}_c^2 S^4$	$\mathbb{G}_m^2 \times \mathbb{G}_m$
8	3	$A_2 \times A_1$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$	$\mathbb{B}_3$	$\mathscr{L}_c^3 S^4$	$\mathbb{G}_m^3 \times \mathbb{G}_m$
7	4	A_4	$\mathfrak{sl}_5$	$\mathbb{B}_4$	$\mathscr{L}_c^4 S^4$	$\mathbb{G}_m^4 \times \mathbb{G}_m$
6	5	D_5	$\mathfrak{so}_{10}$	$\mathbb{B}_5$	$\mathscr{L}_c^5 S^4$	$\mathbb{G}_m^5 \times \mathbb{G}_m$
5	6	E_6	$\mathfrak{e}_6$	$\mathbb{B}_6$	$\mathscr{L}_c^6 S^4$	$\mathbb{G}_m^6 \times \mathbb{G}_m$
4	7	E_7	$\mathfrak{e}_7$	$\mathbb{B}_7$	$\mathscr{L}_c^7 S^4$	$\mathbb{G}_m^7 \times \mathbb{G}_m$
3	8	E_8	$\mathfrak{e}_8$	$\mathbb{B}_8$	$\mathscr{L}_c^8 S^4$	$\mathbb{G}_m^8 \times \mathbb{G}_m$

D	k	Type of E_k	Kac-Moody algebra $\mathfrak{g}$	Non-Fano Surface	Model	Maximal Split Torus
2	9	$E_9 = \widehat{E}_8$	affine $\mathfrak{e}_9 = \widehat{\mathfrak{e}}_8$	$\mathbb{B}_9$	$\mathscr{L}_c^9 S^4$	$\mathbb{G}_m^9 \times \mathbb{G}_m$
1	10	E_{10}	hyperbolic $\mathfrak{e}_{10}$	$\mathbb{B}_{10}$	$\mathscr{L}_c^{10} S^4$	$\mathbb{G}_m^{10} \times \mathbb{G}_m$
0	11	E_{11}	Lorentzian $\mathfrak{e}_{11}$	$\mathbb{B}_{11}$	$\mathscr{L}_c^{11} S^4$	$\mathbb{G}_m^{11} \times \mathbb{G}_m$

1.3.1 Hypothesis K — RR/B-flux quantization in K-CoHomology.

The original plain Hypothesis K. The conjecture/hypothesis that D-brane charges and RR-fluxes are quantized in topological K-theory (for more review and pointers see [BMSS19, §1]) originates with the observation [GHV97][MM97] that the differential RR-flux form data (8) which apparently characterizes D-brane charge has the form of the *Chern character* on topological K-theory classes (cf. [FH00, p. 8][BMRS08, §2.2]

Since [MM97], many authors insist on multiplying the Chern character with a differential form representative of the square root $\sqrt{\hat{A}}$ of the A-roof genus of the tangent bundle of spacetimes before referring to it as D-brane charge. However, since $\sqrt{\hat{A}}$ is multiplicatively invertible (being a unit plus a sum of inhomogeneous differential forms which are nilpotent under wedge product), this is not intrinsic to the notion of D-brane charge and may be disregarded for the purpose of charge quantization (cf. [FH00, ftn. 12]) — its role is rather in making the Chern character natural under push-forward (cf. [BMRS08, §2]).:

> **Hypothesis K** for vanishing NS flux:
>
> *D-brane charges are quantized in topological K-theory*, hence
> RR-field flux densities are in the image of the Chern character
>
> $$K(X) \xrightarrow{\;\mathrm{ch}\;} H_{\mathrm{dR}}(X; \mathfrak{l}\mathrm{KU}_0) \;=\; \Big\{ F_{2\bullet} \,\in\, \Omega^{2\bullet}_{\mathrm{dR}}(X) \;\big\vert\; \mathrm{d}\,F_{2\bullet} \,=\, 0 \Big\}\Big/_{\mathrm{concordance}} \qquad (54)$$

Here we have written the Chern character in the form reviewed in §1.2 (see [Char, Ex. 7.2]), highlighting (for comparison below in §1.3.2) that it may be understood as defined on homotopy classes of maps to the classifying space KU_0 for complex topological K-theory

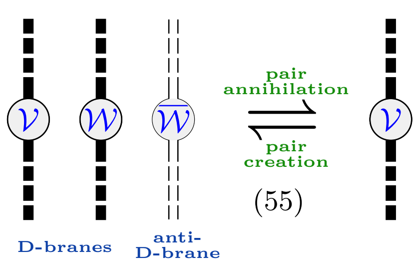
complex topological K-cohomology (deg 0)	$K(X) \;=\; \Big\{ X \xrightarrow{\text{K-cocycle}} \mathrm{KU}_0 \Big\}\Big/_{\mathrm{homotopy}}$

and as taking values in differential forms with coefficients in its Whitehead L_∞-algebra:

	RR-field flux away from NS_5	
Pre-geometric equations of motion of flux densities	$\mathrm{d}\,F_{2\bullet} \;=\; 0$	(8)
Corresponding Sullivan model dg-algebra ("FDA")	$\mathrm{d}\,f_{2\bullet} \;=\; 0$	e.g. [FOT08, §1.81, 1.86]
Candidate classifying space	$\mathrm{KU}_0 \;\simeq\; BU \times \mathbb{Z}$	
Cohomology theory classified by this space	topological K-theory $K(X) \;:=\; \pi_0 \mathrm{Maps}(X, \mathrm{KU}_0)$	e.g. [Kar78, §II Thm. 1.33]

Rational evidence for Hypothesis K. Important to notice here is that all formulas in [GHV97][MM97] which led to the original *Hypothesis K* (54) concern differential form expressions and as such are purely "rational". It is (only) the resemblance of the differential relations satisfied by these differential forms with the image of a character map which suggests that the non-rational domain of this character map (here: K-theory) may be the true home of the brane charges: Among all cohomology theories with this form of character images, K-theory seems to be the most natural or immediate choice (for one, it is essentially the only choice with an established name and geometric interpretation, certainly when the twisting is incorporated below), but it is not the only choice.

Hypothesis K beyond rational: Brane/Antibrane annihilation. The (single) argument meaning to justify the choice of K-theory beyond its rational approximation was then given in [Wi98, §3], who observed that the expected brane/anti-brane annihilation (by tachyon condensation in the open super-strings stretching between them) broadly resembles the Grothendieck equivalence relation which famously expresses (eg. [Kar78, §II 1]) the K-cohomology group $K(X)$ for a compact space X as the equivalence classes of pairs of vector bundles and "anti-bundles" (virtual bundles) subject to a relation expressing that equal but opposite vector bundles cancel.



(55)

Th'is argument is rather hand-wavy (tachyon condensation in superstrings remains poorly understood) but quite suggestive of the actual physics captured by *Hypothesis K*: D-brane charge should be (the) invariant of D-brane pair-annihilation/pair-creation processes.

Various further consistency checks for *Hypothesis K* have been claimed, but unresolved issues remain, pointing to a need for a more refined description.

Hypothesis K in the presence of NS-flux. In view of our above discussion, the more general conjecture [Wi98, §5.3][BM01] that D-brane charges in the presence of NS 5-brane charges are classified by 3-twisted K-theory (see [GS22] for more) is now fairly immediate from the observation (cf. [Char, Rem. 10.1]) that the differential relations satisfied by the twisted Chern character are just the pregeometric equations of motion (19):

RR-field flux
in presence of NS-flux

Pre-geometric equations of motion of flux densities	$\begin{aligned} \mathrm{d}\, F_{2\bullet} &= H_3 \wedge F_{2\bullet-2} \\ \mathrm{d}\, H_3 &= 0 \end{aligned}$	(19)	
Corresponding relative Sullivan model dg-algebra ("FDA")	$\begin{aligned} \mathrm{d}\, f_{2\bullet} &= h_3 \wedge f_{2\bullet} \\ \mathrm{d}\, h_3 &= 0 \end{aligned}$	[FHT07, p. 6] [BMSS19, Lem. 2.31]	(56)
Candidate classifying fibration	$\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \to B^2\mathrm{U}(1)$		
Cohomology theory classified by this fibration	twisted K-theory $K^\tau(X) \;:=\; \pi_0 \Gamma_X\big(\tau^*(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))\big)$	[FHT07, (2.6)] [AS04, Def. 3.3]	

And so the general conjecture for D-branes, widely (though not universally) expected, is this:

Hypothesis K:

D-brane charges are quantized in twisted topological K-theory, hence
RR-field fluxes are in the twisted Chern character

$$K^\tau(X) \xrightarrow{\;\;\mathrm{ch}\;\;} H^\tau_{\mathrm{dR}}\big(X; \mathfrak{l}_{B^2\mathrm{U}(1)}\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) \;=\; \Big\{ F_{2\bullet} \,\in\, \Omega^{2\bullet}_{\mathrm{dR}}(X) \;\Big\vert\; \mathrm{d}\, F_{2\bullet} \,=\, H_3 \wedge F_{2\bullet-2} \Big\}\Big/_{\mathrm{concordance}} \tag{57}$$

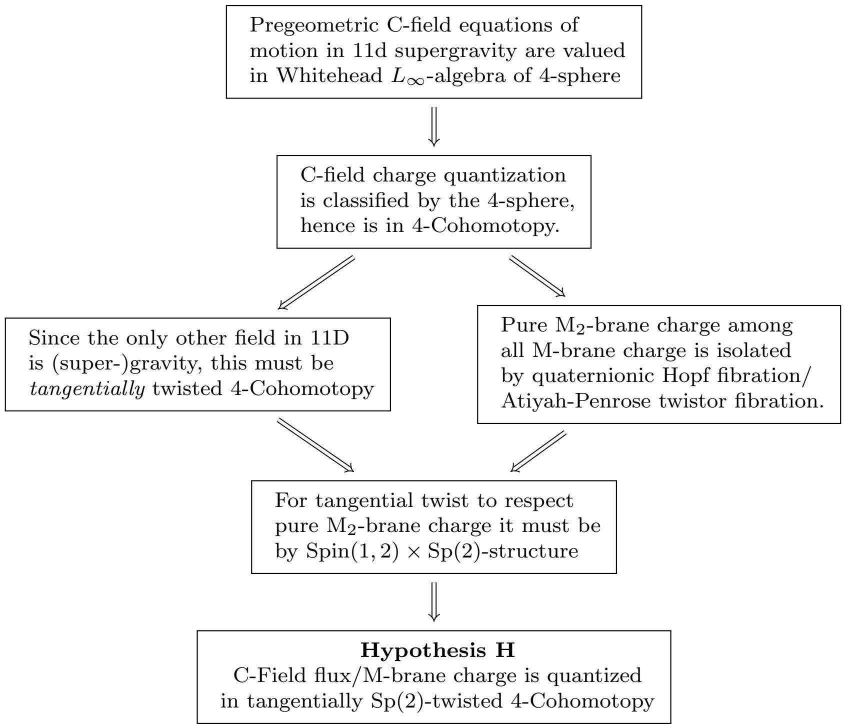
Here we have written the twisted Chern character in the form reviewed in §1.2 (see [Char, Ex. 6.6, Prop. 10.1]), highlighting (for comparison below in §1.3.2) that it is defined on homotopy classes of sections of pullbacks along the twisting map of the universal KU_0-bundle ([FHT07, (2.6)][Char, Ex. 3.4]) and takes values in differential forms with coefficients in its relative Whitehead L_∞-algebra:

$$\underset{\text{topological K-theory}}{\text{twisted}} \quad K^\tau(X) \;=\; \left\{ \begin{array}{ccc} X & \overset{\text{cocycle}}{\dashrightarrow} & \mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \\ & {\scriptstyle \text{twist}}\searrow^{\tau} \quad \swarrow & \\ & B^2\mathrm{U}(1) & \end{array} \right\}\Big/_{\text{rel homotopy}}$$

In the next section §1.3.2 we develop *Hypothesis H* in close analogy to this now classical argument for *Hypothesis K*, which is possible due to the understanding [Char] of the (twisted) Chern character on K-theory as just a special case of a general notion of (twisted) characters on non-abelian generalized cohomology theories whose images capture also non-linear Bianchi identities such as those of the C-field in 11d supergravity.

1.3.2 Hypothesis H — M-brane flux quantization in CoHomotopy.

We are finally ready to motivate, state and explain *Hypothesis H*. To make it transparent, we start with its formulation on flat spacetimes and then incrementally bring in the coupling to gravitational background charges in the form of appropriate tangential twisting of the charge cohomology theory. The logic [Sa13, §2.5][FSS20-HpH1][GS1] is summarized by the following schematic diagram:



Alongside the development of the hypothesis we highlight here its foremost **implications** on M-brane charge quantization:

- (66) the shifted flux quantization of the C_3-field and hence of M_5-brane charge,
- (77) the shifted flux quantization of the C_6-field and hence of M_2-brane charge ("Page charge").

Further implications are discussed in following sections, notably the resulting topological M5-brane model in §2.1.

Remark: Plain reduced and un-reduced Cohomotopy coincide. For the following discussion notice:

For plain Cohomotopy (as opposed to its differential refinement considered below) the reduced and unreduced theories on a given pointed space actually coincide. This follows from the long exact sequence of homotopy groups applied to the fibration which exhibits the reduced moduli as the homotopy fiber of the base-evaluation map on unreduced moduli.

$$
\begin{array}{ccc}
\mathrm{Map}^{*/}\big(X,\, S^4\big) & \longrightarrow & \mathrm{Map}\big(X,\, S^4\big) \\
\big\downarrow & \text{(pb)} & \big\downarrow \mathrm{ev} \\
\ast & \longrightarrow & \underbrace{\mathrm{Map}\big(\ast,\, S^4\big)}_{S^4}
\end{array}
$$

$$
\begin{array}{ccccccccc}
\cdots\to & \pi_1(S^4) & \longrightarrow & \pi_0\Big(\mathrm{Map}^{*/}\big(X, S^4\big)\Big) & \longrightarrow & \pi_0\Big(\mathrm{Map}\big(X, S^4\big)\Big) & \longrightarrow & \pi_0(S^4) \\
& \| & & \| & & \| & & \| \\
\cdots\to & 1 & \longrightarrow & \underset{\text{reduced Cohomotopy}}{\widetilde{\pi}^4(X)} & \xrightarrow{\;\sim\;} & \underset{\text{un-reduced Cohomotopy}}{\pi^4(X)} & \longrightarrow & \ast
\end{array}
$$

C-Field flux on flat spacetimes. By the discussion in §1.2, the admissible flux quantizations of the pre-geometric C-field flux (10) in 11-dimensional supergravity on flat spacetimes are classified by spaces whose minimal Sullivan dg-algebra satisfies the analogous equations. But in rational homotopy theory just these equations are well-known as the Sullivan model for the 4-sphere! [Sa13, §2.5], see also [FSS17-Sph, §2][ℚStruc, p. 14]. This means, by the discussion in §1.2, that the cohomology theory classified by the 4-sphere is an admissible quantization law for the C-field flux in 11-dimensional supergravity: This cohomology theory is *4-Cohomotopy*:

C-field flux
on flat spacetimes

Bianchi identities/ higher Gauß law on flux densities	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4$	(10)
Corresponding Sullivan model dg-algebra ("FDA")	$\mathrm{d}\,g_4 = 0$ $\mathrm{d}\,g_7 = -\tfrac{1}{2} g_4 \wedge g_4$	e.g. [De76, Ex. 3.5 (a)] [FHT00, p. 142] [Me15, §1.2]
Candidate classifying space	4-sphere: S^4	
Whitehead bracket L_∞-algebra	$\pi_3(\Omega S^4) \otimes \mathbb{R} = \mathbb{R}\langle \gamma_3 \rangle$ $\pi_6(\Omega S^4) \otimes \mathbb{R} = \mathbb{R}\langle \gamma_6 \rangle$ $[\gamma_3, \gamma_3] = \gamma_6$	cf. [CJLP98, (2.6)][LLPS99, (3.4)] [KS03, (75)] [BNS04, (86)] [Sa10, (4.9)][SV22, (13)]
Cohomology theory classified by this space	4-Cohomotopy: $\pi^4(X) := \pi_0 \mathrm{Maps}(X, S^4)$	[Pontrjagin1938] [Spanier1949] [Peterson1956]

$$\textbf{plain 4-coHomotopy} \quad \pi^4(X) := \left\{ \overset{\text{spacetime}}{X} \xrightarrow[\text{cocycle}]{c_3} \overset{\text{4-sphere}}{S^4} \right\}_{/\text{homotopy}}$$

Moreover, the 4-sphere is the minimal such choice of flux-quantization law for 11-dimensional supergravity, in that it is the smallest CW-complex with this property. In this sense, the universal choice of C-field flux quantization is by 4-cohomotopy. The hypothesis that this universal choice is the correct choice of flux quantization for M-theory is:

> **Hypothesis H** over flat spacetimes ([Sa13, §2.5][SS23-HpH2]):
>
> *M-brane charges are quantized in 4-cohomotopy*, hence
> C-field fluxes are in the 4-cohomotopical character
>
> $$\pi^4(X) \xrightarrow{\mathrm{ch}_{\pi^4}} H_{\mathrm{dR}}(X; \mathfrak{l}S^4) = \left\{ \begin{array}{l} G_4 \in \Omega^4_{\mathrm{dR}}(X) \\ G_7 \in \Omega^7_{\mathrm{dR}}(X) \end{array} \middle| \begin{array}{l} \mathrm{d}\,G_4 = 0, \\ \mathrm{d}\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4 \end{array} \right\}_{/\text{concordance}} \qquad (58)$$

Perspective. To re-iterate how Hypothesis H comes about:

The general theory of flux quantization (§1.2) says that any cohomology theory flux-quantizing the C-field fluxes (10) has a classifying space whose Sullivan model has as generators the pre-geometric field species subject to differential relations of the same form as the pre-geometric Bianchi identities of the C-field; and rational homotopy theory shows that these are precisely the spaces of the rational homotopy type of the 4-sphere. Among all of these, the 4-sphere itself (and hence the coHomotopy cohomology theory that it classifies) is in some sense the canonical/universal choice — therefore it is natural to hypothesize that this is the choice needed for M-theory.

Should Hypothesis H be false (not quite correspond to M-theory), it would mean that we have to add cells to the 4-sphere (without changing its rational homotopy type) in order to find the correct classifying space for flux quantization in M-theory. Since there are infinitely many choices involved in doing so, it will help to know *how* Hypothesis H fails, if it does, as this will indicate how the canonical choice of classifying space S^4 needs to be adjusted. In this sense, the analysis of the predictions of Hypothesis H is essentially an inevitable step towards understanding flux- & charge-quantization in M-theory, either way.

Cohomotopical M-brane charges and homotopy groups of spheres. The character map in (58) is given by the abstract rationalization construction described in §1.2, but in degree 4 we may readily describe it explicitly: Given a cocycle $c_3 : X \longrightarrow S^4$ in 4-cohomotopy, the corresponding 4-flux density G_4 is the pullback along c^3 of the *volume form* dvol_{S^4}, hence its real cohomology class may be identified with the pullback of the fractional Euler class $\frac{1}{2}\chi_4$ on the 4-sphere:

$$
\begin{array}{lccc}
\textbf{cocycle in 4-cohomotopy} & X & \xrightarrow{\;c_3\;} & S^4 \\
\textbf{induced 4-flux} & \Omega^\bullet_{\mathrm{dR}}(X) & \xleftarrow{\;c_3^*\;} & \Omega^\bullet_{\mathrm{dR}}(S^4) \\
 & G_4 & \longleftarrow & \mathrm{dvol}_{S^4} \\
\textbf{induced } \mathrm{M}_5\textbf{-charge in integral cohomology} & H^\bullet(X;\mathbb{Z}) & \xleftarrow{\;c_3^*\;} & H^\bullet(S^4;\mathbb{Z}) \simeq \mathbb{Z}\big[\tfrac{1}{2}\chi_4\big] \\
 & [G_4] & \longleftarrow & \tfrac{1}{2}\chi_4
\end{array}
\tag{59}
$$

Hence Hypothesis H implies, first of all, that singular flat M_5-branes $\mathbb{R}^{1,5} \hookrightarrow \mathbb{R}^{1,10}$ carry integral charge, as expected.

Generally, Hypothesis H implies that brane charges on flat spacetimes are given by the *homotopy groups of the 4-sphere* (cf. [SS23-HpH2]):

$$
\begin{array}{ll}
\textbf{singular } p\textbf{-brane charge} & \widetilde{\pi}^4\Big(\mathbb{R}^{1,10}_{\sqcup\{\infty\}} \setminus \mathbb{R}^{1,\overbrace{9-n}^{p}}_{\sqcup\{\infty\}}\Big) \\
 & \quad\| \\
\textbf{4th co-homotopy group of } n\textbf{-sphere} & \widetilde{\pi}^4(S^n) \;=\; \big\{ S^n \xrightarrow{c_3} S^4 \big\}_{/\text{homotopy}} \;=\; \pi_n(S^4) \quad \textbf{n-th homotopy group of 4-sphere} \\
 & \quad\| \\
\textbf{solitonic } p\textbf{-brane charge} & \widetilde{\pi}^4\Big(\mathbb{R}^{1,\overbrace{10-n}^{p}}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{n}_{\cup\{\infty\}}\Big)
\end{array}
\tag{60}
$$

$n =$	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\pi_n(S^4)$	$*$	1	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\cdots$

singularity: M_9 (p. 87), exotic branes (§2.3.2), M_5 (15), M_2 (15)

All these groups are finite (hence are "torsion effects" predicted by charge-quantization not seen on differential form data) except in exactly two dimensions, corresponding to the existence of integer charged singular M_5-branes and M_2-branes, respectively:

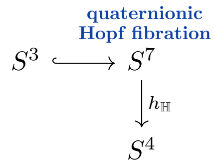
$$
\textbf{cohomotopy charge of flat singular } \mathrm{M}_5\textbf{-branes} \quad \pi^4\big(\mathbb{R}^{1,10} \setminus \mathbb{R}^{5,1}\big) \simeq \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^4\big) \simeq \pi^4\big(S^4\big) \simeq \pi_4\big(S^4\big) \simeq \mathbb{Z}
$$

$$
\textbf{cohomotopy charge of flat singular } \mathrm{M}_2\textbf{-branes} \quad \pi^4\big(\mathbb{R}^{1,10} \setminus \mathbb{R}^{1,2}\big) \simeq \pi^4\big(\mathbb{R}^{1,2} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^7\big) \simeq \pi^4\big(S^7\big) \simeq \pi_7\big(S^4\big) \simeq \mathbb{Z} \oplus \text{torsion}
$$

To amplify this point: Any classifying space for charge quantization in 11d which implies integer-charged singular p-branes exactly for the expected values $p = 2$ and $p = 5$ will need to have non-torsion homotopy groups precisely in degree $9 - 2 = 7$ and $9 - 5 = 4$. The 4-sphere is the minimal cell complex with this property.

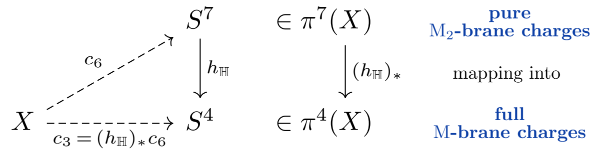
The reason for this is the existence of the *quaternionic Hopf fibration*:

Cohomotopical M_2-brane charge and the quaternion Hopf fibration. The generator of the integer summand $\mathbb{Z} \subset \pi^7(S^4)$ in (60) is the homotopy class of a S^3-fibration called the *quaternionic Hopf fibration* (cf. [FSS20-HpH1, pp. 4]):



$$\left[S^7 \xrightarrow{h_{\mathbb{H}}} S^4\right] \;=\; 1 \in \mathbb{Z} \hookrightarrow \pi_7(S^4)\,.$$

But this means that we may regard S^7 as the classifying space of integral M_2-brane charges and the quaternionic Hopf fibration as classifying the cohomology operation which injects pure M_2-brane charge into the full set of M-brane charges:

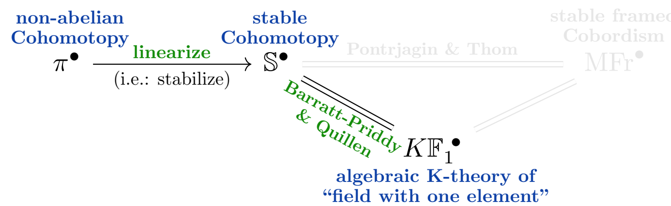


(61)

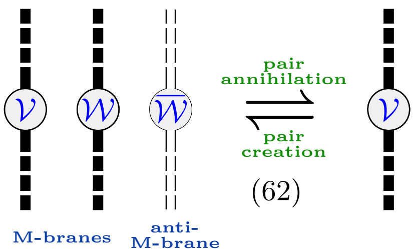
Remark: Flat solitonic M-branes. With the prediction of flat singular 5-branes, Hypothesis H necessarily also predicts (60) integer-charged *solitonic 6-branes* (cf. p. 6). We discuss in §2.3 (following [SS22-Qnt1]) how these may be identified with the non-singular (and thus "solitonic") 6-brane-like solutions of 11d-supergavity known as the *KK-monopole*, the M-theoretic incarnation of D_6-branes.

Remark: Cohomotopy is the absolute K-theory.

The abelianized shadow (stabilization) of Cohomotopy (cf. p. 83), happens to be a form of K-theory, namely the algebraic K-theory of the "absolute base field" $\mathbb{F}_1$ (cf. [CLS12, Thm. 5.9])



This being so, Witten's argument (55) that the eponymous *equivalence classes* defining K-theory ("K" is for German *Klasse*) correctly reflects brane/anti-brane annihilation applies equally to stable Cohomotopy theory, only that now the role of Chan-Paton vector bundles is played by "normal framings" (see p. 83).



(62)

Coupling to gravitational charges and tangential twisting. We motivate the generalization of Hypothesis H to spacetimes which are not necessarily flat:

As indicated in (9), 11-dimensional supergravity stands out in its C-field being the *only* field besides that of gravity. This means that possible twistings of the C-field flux-quantization can only be by the gravitational field, namely by the Spin-frame-bundle of spacetime X (the principal bundle underlying its tangent bundle). By the general rules of twisted cohomology (§1.2.4) and assuming Hypothesis H on flat spacetimes (58) this means that possible twistings are given by ∞-actions of (subgroups of) the Spin-group on the 4-sphere. The canonical[a] such action is that of Spin(5) via the defining action of SO(5) on $S^4 = S(\mathbb{R}^5)$ regarded as the unit sphere in $\mathbb{R}^5$.

[a] There is an isomorphic but subtly different action of $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$ on S^4, which we come to further below.

This leads to *tangentially twisted 4-cohomotopy theory* [FSS20-HpH1, §2.1], consisting of homotopy classes of sections of the 4-sphere bundle associated with a Spin(5)-structure τ on spacetime:

$$\underset{\text{4-coHomotopy}}{\overset{\text{tangentially twisted}}{}}\;\pi^{4+\tau}(X) \;:=\; \left\{ \begin{array}{ccccc} \overset{\text{spacetime}}{X} & & \overset{\text{cocycle}}{\dashrightarrow_{c_3}} & & \overset{\text{universal orthogonal 4-sphere bundle}}{S^4 /\!\!/ \mathrm{Spin}(5)} \\ {\scriptstyle \vdash \mathrm{Fr}(X)}\downarrow & \searrow^{\tau}_{\text{twist}} & & \swarrow & \\ B\mathrm{Spin}(D) & \longleftarrow & B\mathrm{Spin}(5) & & \end{array} \right\}_{\big/ \text{homotopy}} \tag{63}$$

(Such nonabelian/unstable twisted cohomotopy had previously been considered in [Cr03, Lem. 5.2]; for more see [FSS20-HpH1, §2.1].)

Remark: The role of G-structure [FSS20-HpH1, §2.2]. Using the tangentially twisted 4-cohomotopy (63) for flux quantization means that a choice of Spin(5)-structure on spacetime is part of the flux-quantized C-field datum (or rather of isomorphic but subtly different Sp(2)-structure, which we come to in a moment.) Lest this seems overly restrictive, notice that the structure group of the tangent bundle may still be all of $\mathrm{Spin}(1,5) \times \mathrm{Spin}(5)$ in order that τ exists. On the other hand, the existence and choice of a cocycle in $\pi^{4+\tau}(X)$ then equivalently means that and how the Spin(5)-structure factor is further reduced to Spin(4), due to

$$\begin{array}{ccc} S^4 & \longrightarrow & \overset{\text{universal orthogonal 4-sphere bundle}}{S^4 /\!\!/ \mathrm{Spin}(5)} \\ \| & & \wr\,| \\ \mathrm{Spin}(5)/\mathrm{Spin}(4) & \longrightarrow & B\mathrm{Spin}(4) \\ \downarrow & \text{(pb)} & \downarrow \\ * & \longrightarrow & B\mathrm{Spin}(5) \end{array} \tag{64}$$

Shifted C-field flux quantization. Hence the generalization of Hypothesis H (58) away from the special case of flat spacetimes should say that C-field flux is quantized not in plain 4-cohomotopy π^4, but in tangentially twisted 4-cohomotopy $\pi^{4+\tau}$ (63). In a moment we will refine this statement a little further, but first to record the following:

The first non-trivial check of the tangential twisting is its implication of the notorious *shifted integral flux quantization* of the 4-flux density, an unusual-looking condition which however is a widely expected hallmark of M-theory (originally proposed in [Wi97a][Wi97b], see also [Wi00, §2][GS02, p. 21][CS12a][CS12b]) – it says that not the de Rham cohomology class of G_4 but its shift by one *fourth* of the Pontrjagin 4-form $p_1(\nabla)$ on spacetime (for any connection ∇ on the tangent bundle) is the real image of an integral cohomology class:

$$\underset{\text{in ordinary cohomology}}{\overset{\text{M}_5\text{-brane charge image}}{}}\;\overset{\text{class of shifted 4-flux density}}{\big[\widetilde{G}_4\big] \;:=\; \big[G_4 + \tfrac{1}{4}p_1\big]} \;\in\; \overset{\text{integral cohomology}}{H^4(X;\mathbb{Z}) \to H^4(X;\mathbb{R})}\,. \tag{65}$$

Notice that the desire for a deeper cohomological understanding of this condition was previously the motivation for the seminal development of abelian (i.e. stable) generalized differential cohomology in [HS05]. But in our context of non-abelian cohomology the condition falls out naturally:

Namely ([FSS20-HpH1, §3.4]), the integral cohomology of $S^4 /\!\!/ \mathrm{Spin}(5) \simeq B\mathrm{Spin}(4)$ (64) is generated from $\frac{1}{2}p_1$ and the combination $\frac{1}{2}\chi_4 + \frac{1}{4}p_1$ [CV98a, Lem 2.1]. But since the pullback of half the Euler class, $\frac{1}{2}\chi_4$, being the volume form on the S^4-fibers [BC98, §2], is interpreted (59) as the G_4-flux under Hypothesis H, and since the pullback of the universal $\frac{1}{4}p_1$ is the actual such class on spacetime X by the nature of tangential twisting, this means that $G_4 + \frac{1}{4}p_1$ is the image of the pullback of an integral form, and hence itself integral ([FSS20-HpH1, §3.4]):

cocycle in tangentially twisted 4-cohomotopy

$$
\begin{array}{ccc}
X & \overset{c_3}{\dashrightarrow} & S^4 /\!\!/ \mathrm{Spin}(5) \;\simeq\; B\mathrm{Spin}(4) \\
& {\scriptstyle \vdash \mathrm{Fr}(X)} \searrow \quad B\mathrm{Spin}(d) \quad \swarrow &
\end{array}
$$

induced charge in real cohomology

$$
\begin{array}{ccc}
H^\bullet\big(X;\,\mathbb{R}\big) & \xleftarrow{\;c_3^*\;} & H^\bullet\big(B\mathrm{Spin}(4);\,\mathbb{R}\big) \;=\; \mathbb{R}\big[p_1,\,\chi_4\big] \\
\big[p_1(\nabla)\big] & \longleftarrow\!\shortmid & p_1 \;\; \text{first Pontrjagin class} \\
\big[G_4\big] & \longleftarrow\!\shortmid & \tfrac{1}{2}\chi_4 \;\; \text{fractional Euler class}
\end{array}
\tag{66}
$$

induced charge in integral cohomology

integral class of shifted C-field flux

$$
\begin{array}{ccc}
H^\bullet(X;\,\mathbb{Z}) & \xleftarrow{\;c_3^*\;} & H^\bullet\big(B\mathrm{Spin}(4);\,\mathbb{Z}\big) \;=\; \mathbb{Z}\big[\tfrac{1}{2}p_1,\,\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1\big] \\
\underbrace{\big[G_4 + \tfrac{1}{4}p_1(\nabla)\big]}_{[\widetilde{G}]} & \longleftarrow\!\shortmid & \tfrac{1}{2}\chi + \tfrac{1}{4}p_1 \;\; \text{universal integral characteristic class}
\end{array}
$$

Isolating M_2-brane charge on curved spacetimes [FSS20-HpH1, §2.3]. We saw in (61) that the quaternionic Hopf fibration $S^4 \xrightarrow{h_{\mathbb{H}}} S^4$ serves to identify pure M_2-brane charge inside all M-brane charges, under Hypothesis H on flat spacetimes. To retain such an identification as we generalize M-brane charges to curved spacetimes via tangentially twisted cohomotopy (63), we need to find a Spin-group which acts on both S^4 and S^7 in a compatible way, namely such that the Hopf fibration is equivariant under this action.

Remarkably, the quaternionic Hopf fibration is indeed Spin(5)-equivariant — or rather it is equivariant under the isomorphic quaternionic unitary group $\mathrm{Sp}(n) \simeq \mathrm{U}(n,\mathbb{H}) \subset \mathrm{GL}(n,\mathbb{H})$ (cf. [M5b, §A]) in quaternionic dimension 2, via its canonical action on $S^7 = S(\mathbb{H}^2)$, due to the following coset-space realization of the quaternionic Hopf fibration [HT09, Tab. 1][GWZ, Prop. 4.1]:

$$
\begin{array}{ccccc}
& & \overset{\mathrm{Sp}(2)}{\circlearrowright} & & \overset{\mathrm{Sp}(2)}{\circlearrowright} \\
S^3 & \xrightarrow{\;\mathrm{fib}(h_{\mathbb{H}})\;} & S^7 & \xrightarrow{\;h_{\mathbb{H}}\;} & S^4 \\
\wr\vert & & \wr\vert & & \wr\vert \\
\frac{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)} & \xrightarrow[\mathrm{id}]{\;\iota\;} & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xrightarrow[q \mapsto (q,1)]{\;\mathrm{id}\;} & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \\
\wr\vert & & \wr\vert & & \wr\vert \\
\frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} & & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(3)} & & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(4)}
\end{array}
\tag{67}
$$

An important subtlety here is that Spin(5) and Sp(2), while isomorphic as abstract Lie groups, are *not isomorphic* as subgroups of Spin(8), but as such they are exchanged under the *triality automorphism* $\mathrm{tri} : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$. This subtlety is ultimately responsible for the appearance of the "one-loop term" I_8 (21) from Hypothesis H (see below), in that [FSS20-HpH1, (97)]:

$$
\begin{array}{ccc}
B\mathrm{Sp}(2) & \xrightarrow{\;\sim\;} & B\mathrm{Spin}(5) \\
\big\downarrow & & \big\downarrow \\
B\mathrm{Spin}(8) & \underset{\sim}{\xrightarrow{\;B\mathrm{tri}\;}} & B\mathrm{Spin}(8) \\
\\
H^\bullet\big(B\mathrm{Sp}(2);\,\mathbb{R}\big) & \xleftarrow{\;(B\mathrm{tri})^*\;} & H^\bullet\big(B\mathrm{Spin}(5);\,\mathbb{R}\big) \\
\tfrac{1}{2}p_1 & \longleftarrow\!\shortmid & \tfrac{1}{2}p_1 \\
(\tfrac{1}{4}p_1)^2 - 24\cdot I_8 & \longleftarrow\!\shortmid & \tfrac{1}{4}p_2
\end{array}
\tag{68}
$$

Namely, by [CV97, 2.2, 4.1, 4.2] and [CV98b, 8.1, 8.2] we have, respectively:

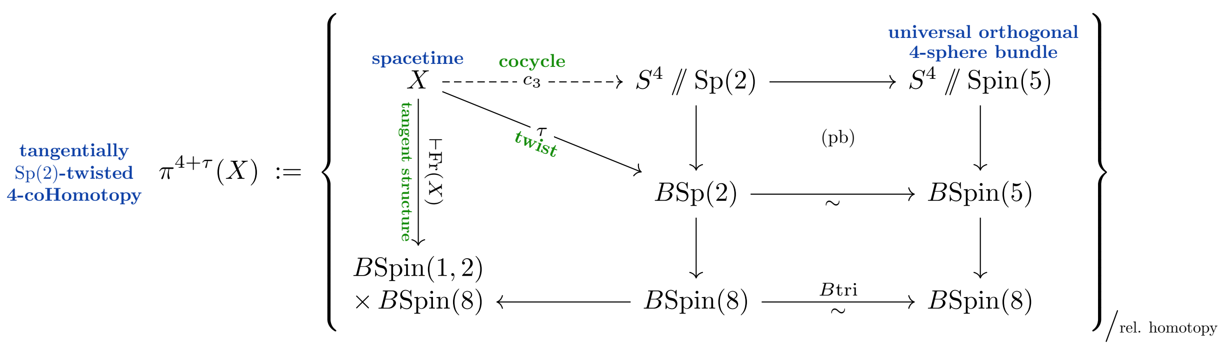
$$
\begin{array}{ccccc}
H^8\big(B\mathrm{Spin}(8)\big) & \xrightarrow{(B\mathrm{tri})^*} & H^8\big(B\mathrm{Spin}(8)\big) & \longrightarrow & H^8\big(B\mathrm{Sp}(2)\big) \\
\tfrac{1}{2}p_1 & \leftrightarrow & \tfrac{1}{2}p_1 & \mapsto & \tfrac{1}{2}p_1 \\
\tfrac{1}{4}\big(p_2 - (\tfrac{1}{2}p_1)^2\big) - \tfrac{1}{2}\chi_8 & \leftrightarrow & -\chi_8 & \mapsto & -\tfrac{1}{2}\big(p_2 - \tfrac{1}{4}(p_1)^2\big) \\
\text{and hence} \quad \tfrac{1}{4}p_2 & \leftrightarrow & \begin{array}{c} -\chi_8 + \frac{1}{4}(\frac{1}{2}p_1)^2 \\ -\frac{1}{2}\Big(\frac{1}{4}\big(p_2 - (\frac{1}{2}p_1)^2\big) - \frac{1}{2}\chi_8\Big) \end{array} & \mapsto & (\tfrac{1}{4}p_1)^2 - \tfrac{1}{2}\underbrace{\big(p_2 - \tfrac{1}{4}(p_1)^2\big)}_{=:\, 48\cdot I_8}
\end{array}
\tag{69}
$$

This way we arrive at the general form of (58):

Hypothesis H ([FSS20-HpH1]):

M-brane charges are quantized in tangentially Sp(2)*-twisted 4-cohomotopy*, hence
C-field fluxes are in the twisted 4-cohomotopical character

$$
\pi^\tau(X) \xrightarrow{\mathrm{ch}^\tau} H^\tau\big(X;\, \mathfrak{l}S^4\big) = \left\{ \begin{array}{l} G_4 \,\in\, \Omega^4_{\mathrm{dR}}(X) \\ G_7 \,\in\, \Omega^7_{\mathrm{dR}}(X)\big) \end{array} \middle| \begin{array}{l} \mathrm{d}\,G_7 \,=\, -\tfrac{1}{2}\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{2}p_1) - 12\cdot I_8 \\ \mathrm{d}\,G_4 \,=\, 0 \end{array} \right\}\Big/_{\mathrm{concordance}}
\tag{70}
$$



(71)

That the twisted cohomotopical character is of this form (70) follows [FSS20-HpH1, Prop. 3.8] essentially by the formula for the Sullivan model of Spin(5)-associated S^4-fibrations, which in itself gives [FSS20-HpH1, Prop. 2.5] $\mathrm{d}\,2G_7 = -G_4 \wedge G_4 + \frac{1}{4}p_2(\nabla^{\mathrm{Spin}(5)})$ and then plugging in the expression for $\frac{1}{4}p_2$ from (68) to account for the fact that the twist is actually by Sp(2)-structure. Finally, we have cleaned up the formula by completing the resulting square in terms of the shifted flux density $\widetilde{G}_4$ (65):

$$
-G_4 \wedge G_4 \,+\, \tfrac{1}{4}p_1 \wedge \tfrac{1}{4}p_1 \;=\; -(G_4 + \tfrac{1}{4}p_1) \wedge (G_4 - \tfrac{1}{4}p_1) \;=\; -\widetilde{G}_4 \wedge (\widetilde{G}_4 - \tfrac{1}{2}p_1)
$$

Notice that the factor $\mathrm{Spin}(1,2)$ may be included in the spacetime tangent structure in (71) without changing this conclusion nor that of the shifted flux quantization (66), since it contributes neither to p_1 nor to p_2.

Remark: Normalization of the one-loop term in the Bianchi identity. The factor of "12" in (70) may seem unexpected, since an old argument [SVW96, p. 2][DM97, (1)] (which, incidentally, neglects the shifting (65)) might lead one to expect a factor of "1" here, instead — but this depends in turn on the prefactor which translates between the integrated flux density $\int_{S^7} G_7$ and the actual number of M_2-branes. In [FSS20-HpH1, p. 12-13] we argue that proper counting of 2-brane charge in Cohomotopy does resolve this apparent discrepancy.

On the other hand, we discuss next that in order for the "M_2 Page-charge" to be integral and the M_5-brane sigma-model in the background of M2-brane flux to be well-defined, this characteristic polynomial has to vanish (an M-theoretic form of anomaly cancellation by "Fivebrane structure"), in which case this issue disappears anyway, see (78) below.

M_2-charge quantization and the Hopf-Wess-Zumino coupling in the M_5. Hypothesis H in the form (70) implies (by design, recalling (61) and (67)) that a notion of *pure M_2-brane charge* is retained after M-brane charge quantization in twisted Cohomotopy, namely given by those twisted 4-Cohomotopy cocycles which factor through the $B\mathrm{Sp}(2)$-parameterized quaternionic Hopf fibration $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ up to homotopy, or more specifically, by the *choice* (c_6, b_2) of such a homotopy-factorization:

$$
\begin{array}{ccccc}
 & & S^7 /\!\!/ \mathrm{Sp}(2) & \in & \pi^{3+c_3}(X) \longrightarrow \pi^{7+\tau}(X) \quad \text{pure M}_2\text{-brane charges} \\
 & \overset{c_6}{\nearrow} \; \overset{b_2}{\Leftarrow} & \Big\downarrow{\scriptstyle h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)} & & \Big\downarrow{\scriptstyle (h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2))_*} \\
X & \xrightarrow{\;c_3 = (h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2))_*(c_6)\;} & S^4 /\!\!/ \mathrm{Sp}(2) & \in & \pi^{4+\tau}(X) \quad \text{full M-brane charges} \\
 & {\scriptstyle \tau}\searrow & \swarrow & & \\
 & & B\mathrm{Sp}(2) & &
\end{array}
\tag{72}
$$

Here for fixed c_3 we may interpret the compatible M_2-brane charges (c_6, b_2) with the c_3-twisted non-abelian cohomology classified by $S^7 /\!\!/ \mathrm{Sp}(2)$. This is a twisted form of 3-Cohomotopy, because the homotopy fiber of $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ is still the 3-sphere ([GS1, Lem. 2.8]):

$$
\begin{array}{ccc}
S^3 & \longrightarrow & S^7 /\!\!/ \mathrm{Sp}(2) \\
\Big\downarrow & {\scriptstyle (\mathrm{pb})} & \Big\downarrow{\scriptstyle h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)} \\
* & \longrightarrow & S^4 /\!\!/ \mathrm{Sp}(2)
\end{array}
\tag{73}
$$

As such we have the corresponding character differential forms for pure M_2-brane charge, which pick up a 3-form flux H_3 ([FSS20-HpH1, Prop. 3.20], cf. (94) below):

Implication of **Hypothesis H** on M_2-brane charge:

pure M_2*-brane charges in given background* $\mathrm{M}_2/\mathrm{M}_5$*-charge* c_3 *are quantized in* c_3*-twisted 3-cohomotopy*, hence (C_6, B_2)-field fluxes are in the twisted 3-cohomotopical character

$$
\pi^{c_3}(X) \xrightarrow{\mathrm{ch}} H^{c_3}_{\mathrm{dR}}\big(X; \mathfrak{l}(S^7 /\!\!/ \mathrm{Sp}(2))\big) = \left\{ \begin{array}{l|l} G_4 \in \Omega^4_{\mathrm{dR}}(X) & \mathrm{d}\, G_4 = 0, \\ G_7 \in \Omega^7_{\mathrm{dR}}(X) & \mathrm{d}\, G_7 = -\tfrac{1}{2}\widetilde{G}_4 \wedge \big(\widetilde{G}_4 - \tfrac{1}{2}p_1\big) - 12 \cdot I_8 \\ H_3 \in \Omega^3_{\mathrm{dR}}(X) & \mathrm{d}\, H_3 = \widetilde{G}_4 - \tfrac{1}{2}p_1 \end{array} \right\}_{\Big/ \text{concordance}}
\tag{74}
$$

The literature on M_2-brane charge expects (though throughout ignoring the shift by p_1 in $\widetilde{G}_4$ (65)) that given such an H_3-"potential" in 11d supergravity (then typically regarded as the C-field gauge potential and denoted "C_3") the following expression — known as the *Page charge* — is the M_2-brane charge [Pa83, (8)][DS91, (43)][BLMP13, p. 21]:

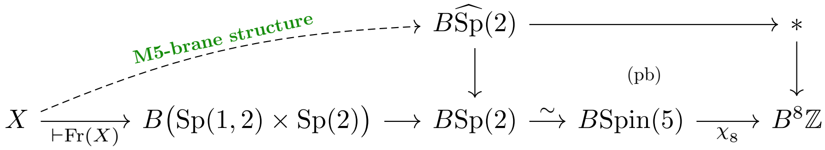
$$
\underset{\text{in ordinary cohomology}}{\text{M}_2\text{-brane charge image}} \quad [\widetilde{G}_7] := \big[G_7 + \tfrac{1}{2}H_3 \wedge \widetilde{G}_4\big] \in H^7\big(X; \mathbb{R}\big) .
\tag{75}
$$

The same expression gives the "Hopf-Wess-Zumino term" in the action functional of the M5-brane sigma-model (we come to this in §2.1), hence the coupling of the fundamental five-brane to the background C-field analogous to the coupling of an electron the electromagnetic field.

What had remained open (and hardly discussed at all) is that, how and why this term is integral: Regarded as M_2-brane charge such an integrality is necessary at least to justify common discussion of M_2-brane counting, while regarded as the Hopf-WZ term for the M5-brane such an integrality is necessary for the M5-brane sigma-model to actually be well-defined (anomaly-free) — by the exact same argument of Dirac charge quantization, up to degree, we expand on this in §2.1.

Experience with the NS5-brane sigma-model suggests that its anomaly-cancellation requires a topological condition on spacetime that is a higher-degree analog of "String structure" (whence called "Fivebrane structure" in [SSS09]) requiring an degree-8 polynomial in the Pontrjagin forms of spacetime to vanish.

The M-theoretic analog of Fivebrane structure as implied by Hypothesis H is the trivialization of the Euler class, hence of (12 times) the "one loop term" $\chi_8 = 24 \cdot I_8$ (69), which we may refer to as M_5*-brane structure* [M5b, Ex. 3.2]:



(76)

in that this is what, under Hypothesis H, implies the (half-)integrality of the M_2-brane Page charge, hence of the Hopf-WZ terms of the M_5-brane sigma-model:

Theorem 1.2 ([FSS21-M5a, Thm. 4.8])**.** *Hypothesis H* (70) *implies that, on spacetimes admitting* M_5*-brane structure* (76)*, the resulting* M_2*-brane charge quantization* (74) *makes twice the Page charge/Hopf-WZ term* (75) *an integral cohomology class:*

$$\underset{\text{in ordinary cohomology}}{\overset{\mathrm{M}_2\text{-brane charge image}}{}} \quad \overset{\substack{\text{class of} \\ \text{shifted 7-flux density}}}{2\big[G_7 + \tfrac{1}{2} H_3 \wedge \widetilde{G}_4\big]} \;\in\; \overset{\text{integral cohomology}}{H^7\big(X;\mathbb{Z}\big)} \to H^7\big(X;\mathbb{R}\big) \tag{77}$$

Discussion and interpretation of the factor of 2 here is given in [FSS21-M5a, (3)][M5b, p. 3].

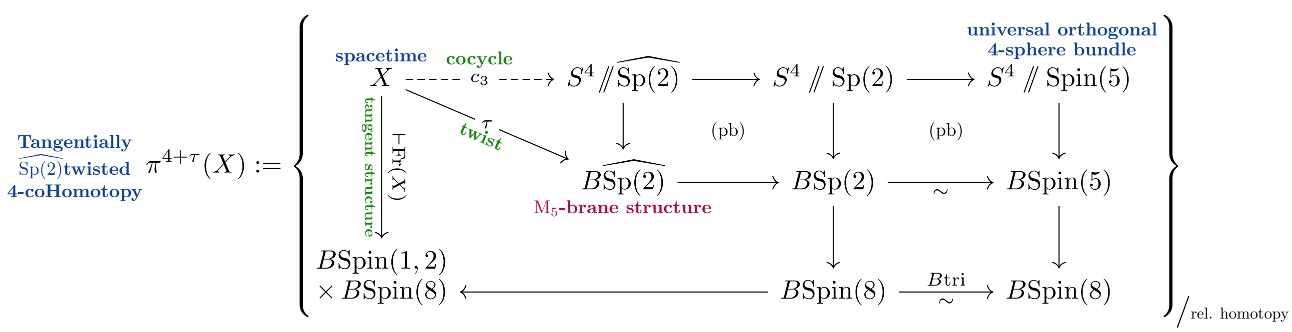
For example, the condition of M_5-brane structure is satisfied if the structure group reduces further along $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2)$ (since the Euler of a direct sum of vector bundles is the cup product of that of the summands, but the Euler 8-class of a single $B\mathrm{Sp}(2)$ vanishes by degree reasons). This special case subsumes the important example of M_5-branes at ADE-singularities, see [M5e, (1)].

Hence if one insists — which is reasonable — that M-brane charge quantization should imply Page charge quantization (77) and thus consistency of the M_5-brane sigma model in charged backgrounds, then one will want to include the demand of M_5-brane $\widehat{\mathrm{Sp}}(2)$-structure (76) into the hypothesis (70):

Hypothesis $\widehat{\mathrm{H}}$ ([FSS21-M5a]):

M-brane charges are quantized in tangentially $\widehat{\mathrm{Sp}(2)}$*-twisted 4-cohomotopy*, hence C-field fluxes are in the twisted 4-cohomotopical character

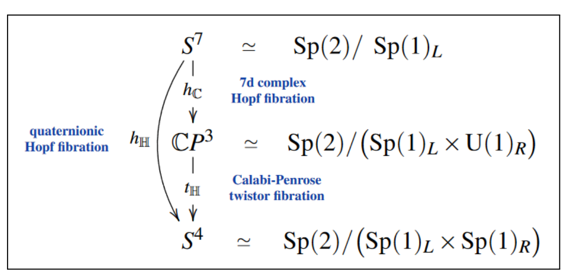
$$\pi^{4+\tau}(X) \xrightarrow{\mathrm{ch}^\tau} H^\tau\big(X;\, \mathfrak{l}S^4\big) \;=\; \left\{ \begin{array}{l} G_4 \,\in\, \Omega^4_{\mathrm{dR}}(X) \\ G_7 \,\in\, \Omega^7_{\mathrm{dR}}(X)) \end{array} \;\middle|\; \begin{array}{l} \mathrm{d}\, G_7 \;=\; -\tfrac{1}{2}\widetilde{G}_4 \wedge \big(\widetilde{G}_4 - \tfrac{1}{2}p_1\big) \\ \mathrm{d}\, G_4 \;=\; 0 \end{array} \right\} \Big/ \text{concordance} \tag{78}$$

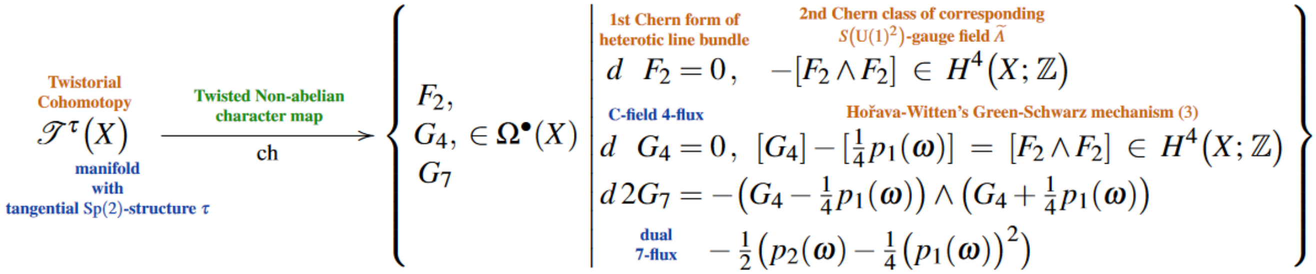


(79)

Hypothesis H for heterotic M-theory. Finally, Hypothesis H generalizes to (and maybe comes into full bloom) in "heterotic M-theory" (Hořava-Witten theory), where Cohomotopy is enhanced to "twistorial Cohomotopy", now represented by the "twistor space" $\mathbb{C}P^3$ covering the 4-sphere through the Calabi-Penrose fibration.

This is discussed in [FSS20-GS][SS20-GS].





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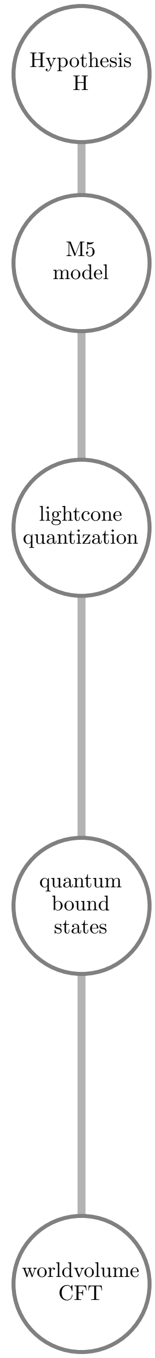
2 Quantized Flux



In this second part, we explore consequences of Hypothesis H (§1.3) and match them to expected phenomena in the M-theory folklore.

§2.1 **Resulting M5-brane model.**
First is the observation that Hypothesis H for the bulk C-field implies a good deal of subtle structure for the fundamental M5-brane probing this bulk. In particular, we find that the enigmatic 3-form flux expected on M5-brane worldvolumes appears and is implied to, in turn, be flux-quantized in a twisted form of 3-Cohomotopy. This means that with Hypothesis H, we may investigate the nature of branes (defects) *inside* the M5-worldvolume.

§2.2 **Brane lightcone quantization.**
For investigating such individual quantum branes beyond the traditional large-N limit, we need to obtain genuine quantum observables on flux-quantized fields. In previous lack of a general prescription for such non-perturbative quantization of non-Lagrangian fields, we make the following observation ([SS23-Qnt], in itself independent of Hypothesis H, but to be combined with it):
For flux-quantized fields on spacetimes with a circle factor (such as in M/IIA duality), the homology Pontrjagin algebra of the looping of the flux-quantized phase space stack (from **??**) is a good algebra of topological quantum observables (on the discretized light cone, as familiar from the BFSS matrix model).

§2.3 **Resulting quantum branes.**
We work out the phase spaces, under Hypothesis H, of various *intersecting* solitonic branes, by intersecting their respective Cohomotopical phase spaces (§1.2.3); then we determine the algebras of quantum observables according to §2.2:
– For $D_8 \perp D_6$ intersections on NS_5s we find this way much of the structure of Hanany-Witten brane systems and, M/IIA-dually, quantum observables of M2/M5 bound states as expected from the BMN matrix model ([SS22-Cnf][CSS23]).
– For $M_5 \perp M_5$ intersections on codimension=2 defects inside M_5 we find anyonic quantum observables whose quantum states are braid group representations.

§2.4 **Resulting worldvolume CFT.**
Finally, we describe in more detail the quantum states, implied this way by Hypothesis H, on these $M_5 \perp M_5$-intersections, and find them [SS23-Dfc1][SS23-Dfc2] to be MacLaughlin wavefunctions of su(2)-anyons (at any admissible level), namely conformal blocks of the $\mathfrak{su}(2)$-WZW 2d conformal field theory on the M_5 transverse to the intersecting M_5, at any admissible (possibly fractional) level.
Notice that the CFT expected in codimension=2 on M5-branes has previously been argued, informally, to be the $\mathfrak{su}(2)$-WZW model ([Wi10, p. 22]).

Conclusion and outlook. In summary, we seem to have substantial evidence that Hypothesis H gets close to the expected non-perturbative completion of 11d supergravity, aka M-theory. Better yet, it seems to provide a mathematical context for nonperturbative quantum physics (such as anyonic topological order) which can be analyzed rigorously and independently of the string/M-theory folklore.

In order to become a complete such theory, what is clearly missing from Hypothesis H, so far, is the incorporation of dynamical (super-)gravity. But we may observe that the Whitehead L_∞-algebra $\mathfrak{l}S^4$ also controls the exceptional super-Minkowski spacetime of which 11d supergravity should be the higher Cartan geometry. This points to a gravitational enhancement of Hypothesis H, but details remain to be worked out.

2.1 Resulting M5-brane model

After a brief recollection of the meaning of *fundamental sigma-model branes* in §2.1.1, we survey in §2.1.2 how *Hypothesis H* implies global consistency of the (Hopf-)Wess-Zumino gauge coupling of the fundamental 5-brane sigma model by implying flux quantization of the worldvolume B-field in twisted 3-Cohomotopy underlying which is a "nonabelian gerbe field" for worldvolume gauge group $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ — this result is from [FSS20-HpH1, §3.7][FSS21-M5a][FSS21-M5b].[11]

2.1.1 Fundamental sigma-model branes

Besides the singular/solitonic classical branes of §1.1 there are supposed to be "fundamental" or "sigma-model"-branes which are not imprinted on flux, but which are *effected by* flux. Here

fundamental brane : singular brane

is like

fundamental particle : black hole

in that fundamental branes are supposed to be "massless" cousins of black branes, which have analogous attributes, but instead of impacting spacetime by their backreaction on it they trace out trajectories $\phi : \Sigma^{1+p} \to X$ in a fixed background spacetime X subject to forces exerted by spacetime fields. These forces include the force of gravity and the generalized *Lorentz force* exerted by the background gauge field. In the spirit of pre-geometric fluxes as discussed in §1.1.2 here we focus on these Lorentz forces.

In fact, fundamental branes include, with the *fundamental particles* that they derive their name from, the most prominent brane species: notably the *fundamental string* which gives its name to *string theory* and the *fundamental membrane* from which the term *M-theory* is derived:

	Flux densities on spacetime **black branes** singular source of flux	σ-model with target spacetime **fundamental brane** subject to forces from such background flux
Electromagnetism	magnetic monopole	fundamental particle (electron)
String theory	NS_5-brane	fundamental string
M-theory	M_5-brane	fundamental membrane
	M_2-brane	fundamental fivebrane

(80)

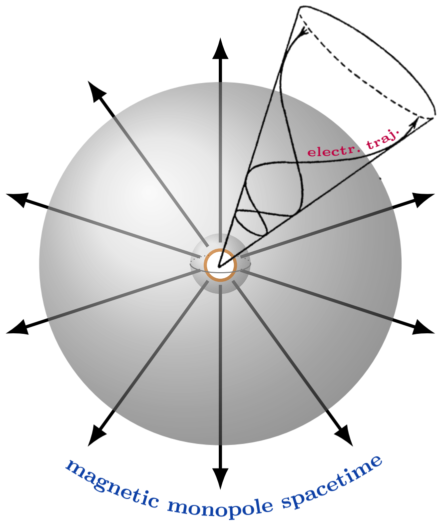
The fundamental 0-branes in electromagentism are simply the electrons – these being *fundamental particles* in the sense of particle physics, whence the general term "fundamental brane".

The following graphics shows[12] the generic trajectory of an electron in the vicinity of a magnetic monopole: The Lorentz force felt by the electrically charged electron when moving in a background magnetic field deforms the otherwise straight trajectory into a helix whose radius of curvature is the smaller the stronger the magentic flux density. In the case of the magnetic field sourced by a monopole (cf. p. 7) this makes the electron trajectories lie

[11] Hypothesis H also implies information about the *dynamical* (i.e. geometric, non-topological) sector of the Sp(1)-gauged M5-brane sigma-model: this is discussed in [FSS20-M5d][FSS21-M5e].

[12] This is discussed for instance in Ferraro (1956) *Electromagnetism Theory*, §137. The helical trajectory in (81) is adapted from Ferraro's Fig. 161.

on a cone in space whose vertex is the singular locus of the monopole:



$$
\begin{array}{ccc}
\text{worldline} & \text{trajectory} & \text{spacetime} \\
\Sigma^{1+0} & \xrightarrow{\ \phi\ } & X \\
\| & & \| \\
\mathbb{R}^{1,0} & \longrightarrow & \mathbb{R}^{1,3} \setminus \mathbb{R}^{1,0} \simeq \mathbb{R}^{1,0} \times \mathbb{R}_{\sqcup\{\infty\}} \times S^2
\end{array}
\tag{81}
$$

This means in particular that there exist circular electron trajectories which lie entirely in a plane in space and are periodic in time, winding around one of the radial flux lines. Since the Faraday tensor F_2 sourced by a magnetic monopole has no temporal (electric field) component, we may consistently ignore the temporal translation of these trajectories and understand them as maps from a circular "worldline" into space:

$$
\begin{array}{ccccc}
\text{worldline} & \text{trajectory} & \text{space} & \text{gauge potential} & \\
S^1 & \xrightarrow{\ \phi\ } & \mathbb{R}_{\sqcup\{\infty\}} \times S^2 & \xrightarrow{\ \widehat{A}_1\ } & \widehat{B^2\mathbb{Z}} \\
 & & \searrow^{F_2} & & \swarrow \\
 & & & \Omega^2_{\mathrm{cl}}(-) &
\end{array}
\qquad
\begin{array}{c}
\text{gauge coupling action functional} \\
\rightsquigarrow \text{Lorentz force} \\
\exp\Big(\underbrace{2\pi\mathrm{i}\int_{S^1}\phi^*\widehat{A}_1}_{\text{holonomy}}\Big) \in \mathbb{R}/\mathbb{Z}
\end{array}
\tag{82}
$$

The gauge-coupling part of the "exponentiated action" (in the sense of classical Lagranian physics) of such a trajectory is the holonomy of the gauge potential 1-form $\widehat{A}_1$ (§1.2.3) around this closed curve, in that the variation of this functional gives the contribution to the Euler-Lagrange equations of motion of the electron which expresses the Lorentz force.[13]

Recall here that it is the definition of the differential cohomology coefficients $\widehat{B^2\mathbb{Z}}$ (§1.2.3) which implements the Dirac charge quantization condition and makes the action functional take values in $\mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1)$:

$$
\begin{array}{ccccccc}
 & & & \overset{\text{background flux density}}{\xrightarrow{\ F_2\ }} & \Omega^2_{\mathrm{dR}}(-)_{\mathrm{flat}} & & \\
 & & \nearrow & & \nearrow & \overset{(\mathrm{pb})}{\Downarrow} & \searrow \\
\underset{\Sigma^{1+0}}{\text{electron worldline}} & \xrightarrow{\ \phi\ } & X & \underset{\text{"vector potential" gauge field}}{\overset{\widehat{A}_1}{\dashrightarrow}} & \widehat{B^2\mathbb{Z}} & & B^2\mathbb{R} \\
 & & \searrow & & \searrow & & \nearrow_{\mathbf{ch}} \\
 & & & \underset{\text{background magnetic charge}}{\longrightarrow} & B^2\mathbb{Z} & &
\end{array}
\tag{83}
$$

Fundamental p-brane sigma-models. In straightforward generalization of the above situation for fundamental particles, one considers maps into space(-time) from $p+1$-dimensional manifolds Σ^{1+p} — which we may assume to be closed, for simplicity and following (82) — regarded as *worldvolumes* of fundamental p-branes. These may

[13]See for instance Misner, Thorne & Wheeler (1973) *Gravitation*, Exc 7.2 on p. 179, or Frankel (1997) *The Geometry of Physics*, §16.4b.

have higher Lorentz-force couplings to higher background gauge fields $\widehat{A}_{p+1}$ represented by cocycles in differential cohomology of degree $p+2$:

p-brane worldvolume **embedding field** **target spacetime** **higher gauge potential**

$$\begin{array}{ccccc} \Sigma^{1+p} & \xrightarrow{\ \phi\ } & X & \xrightarrow{\ \widehat{A}_{p+1}\ } & \widehat{\mathbf{B}^{p+2}\mathbb{Z}} \\ & & & & \uparrow \\ \Sigma^{1+p} & & \xrightarrow{\ \phi^\ast \widehat{A}_{p+1}\ } & & \mathbf{B}^{p+1}\mathbb{R}/\mathbb{Z} \end{array} \tag{84}$$

moduli stack of (flat) differential cohomology

exponentiated action functional

$$\begin{array}{ccccc} C^\infty\big(\Sigma^{1+p},\, X\big) & \longrightarrow & H^{p+1}\big(\Sigma^{p+1};\mathbb{R}/\mathbb{Z}\big) & = & \mathbb{R}/\mathbb{Z} \\ \phi & \longmapsto & \big[\phi^\ast \widehat{\mathrm{A}}_{p+1}\big] & =: & \exp\Big(2\pi\mathrm{i}\int_{\Sigma^{p+1}} \phi^\ast \widehat{\mathrm{A}}_{p+1}\Big) \end{array} \tag{85}$$

gauge coupling/ Lorentz force/ Wess-Zumino term

This may be understood as defining (the gauge-coupling topological sector of) a field theory on Σ^{1+p} whose:

- Fields are the brane trajectories, namely the smooth maps $\phi : \Sigma^{1+p} \to X$ – then often called "embedding fields", though not not actually required to constitute an embedding $\Sigma^{1+p} \hookrightarrow X$.
- Action functional is the higher holonomy functional (85).

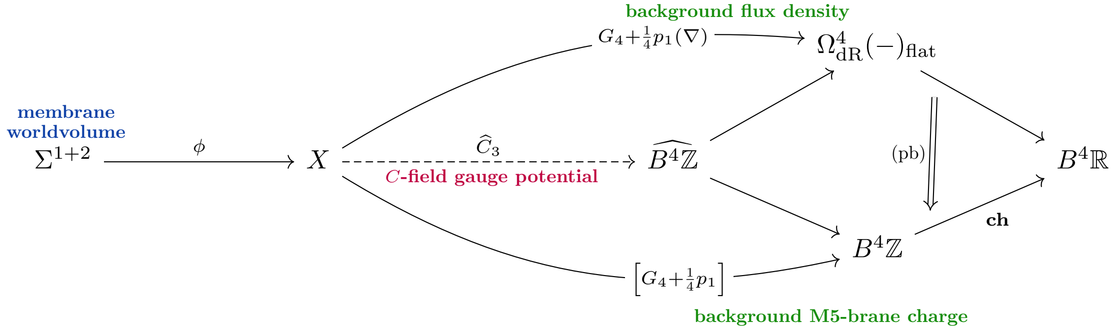
Such field theories – whose fields are maps to a given *target space* X this way – are known as **non-abelian sigma-models**, for historical reasons. For the full geometric dynamics of fundamental branes one is to add another contribution (the "Nambu-Goto action") to the action functional, which we disregard here (in the pre-geometric spirit of §1.1.2), so that the "gauge coupling sector" of fundamental p-branes which we retain may be understood as a *worldvolume topological field theory*, here a *topological sigma-model* also called a *homotopical field theory*, see [MW20] for detailed discussion (at the classical non-quantum level) in the case at hand.

Fundamental membrane sigma-model. For example, the sigma-model for the fundamental membrane propagating along a trajectory $\phi : \Sigma^{1+2} \to U \hookrightarrow X$ inside a chart U of an 11d supergravity target spacetime X is meant [BST87][HS05, §4.4] to couple to the background C-field flux $G_4|_U$ via an (exponentiated) action functional that is *locally* of the form $\phi \mapsto \int_{\Sigma^{1+2}} \phi^\ast C_3$, where $C_3 \in \Omega^3_{\mathrm{dR}}(U)$ is a local gauge potential for the C-field, in that $\mathrm{d}C_3 = G_4|_U$.

It is rarely (if ever) discussed in the string theory literature that the global definition (85) of this coupling term requires an integral charge quantization of G_4; but the expected shifted integrality condition $[G_4 + \frac{1}{4}p_1(\nabla)] \in H^4(X;\mathbb{Z})$ (65) on the C-field flux — which is a consequence of Hypothesis H by (66) — serves this purpose if one enhances the local conditions to

$$\mathrm{d}\, C_3 \;=\; G_4|_U + \tfrac{1}{4}p_1(\nabla|_U)\,. \tag{86}$$

Globally such C_3 is to be the 3-form connection $\widehat{C}_3$ on a 2-gerbe with characteristic class $[G_4 + \frac{1}{4}p_1(\nabla)]$ and makes the fundamental membrane sigma-model be well-defined, via (85):

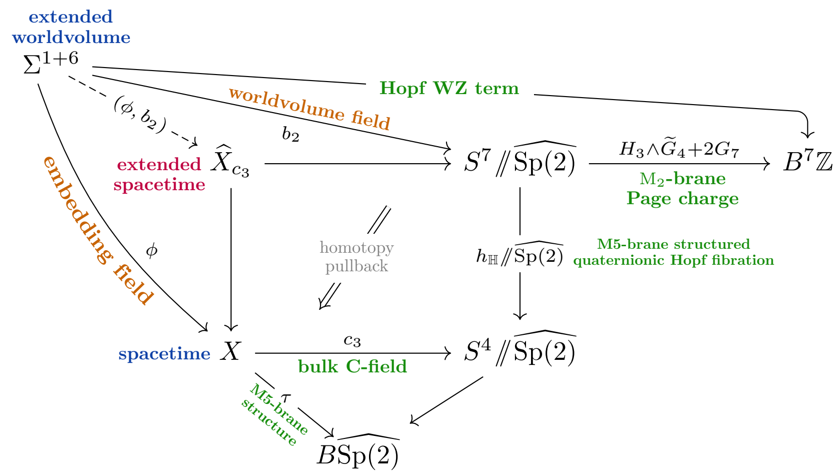


(87)

The analogous situation for the fundamental fivebrane is richer and more subtle, this we turn to in §2.1.2.

2.1.2 The fundamental Fivebrane sigma-model under Hypothesis H

The Fivebrane sigma-model is that whose topological ("Hopf"-)WZ term is the background M2-brane's Page charge, which by Hypothesis H is classified on the tangentially twisted quaternionic Hopf fibration over the 4-sphere coefficient of the background M5-brane charge. [FSS21-M5a].



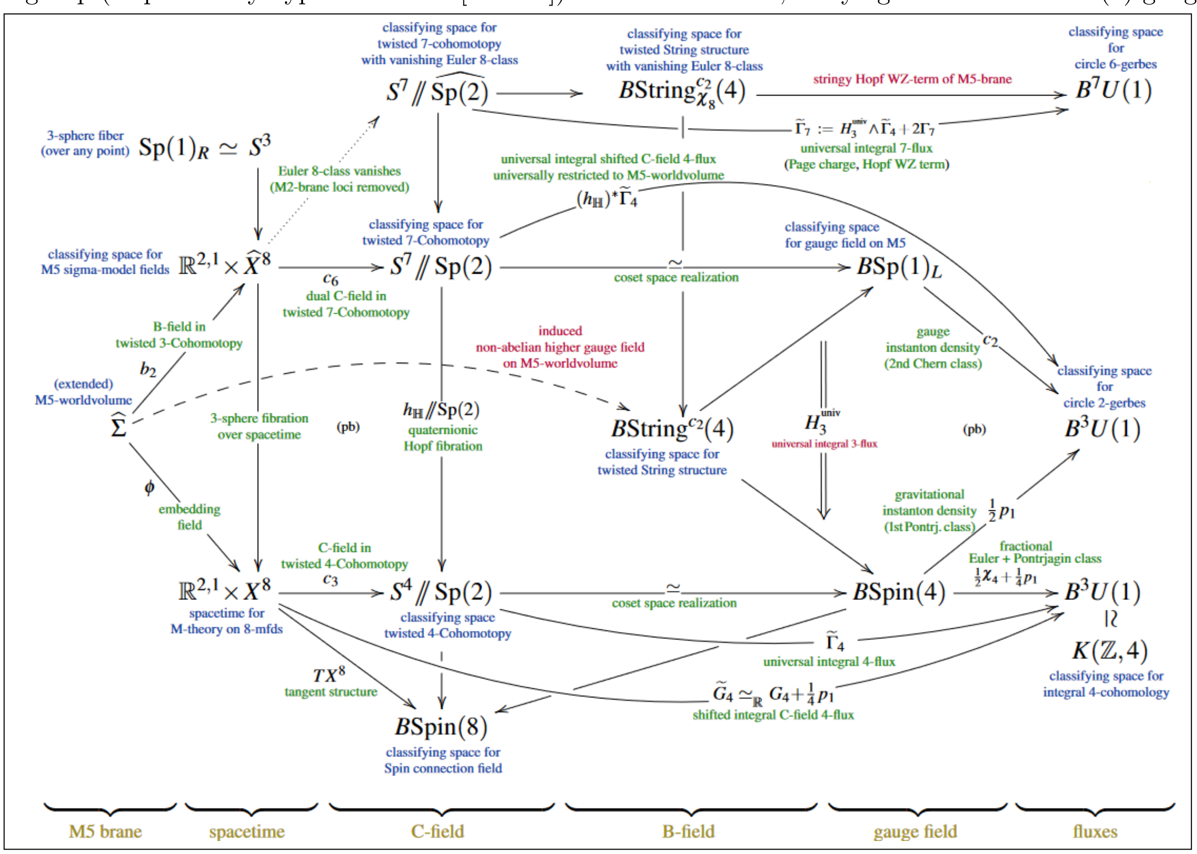
H_3 cohomotopical flux-quantization. Since the fiber of the quaternionic Hopf fibration is the 3-sphere, it follows from Hypothesis H that the worldvolume H_3-flux on the Fivebrane is quantized in 3-Cohomotopy twisted by the pullback of the bulk C-field along the Fivebrane's embedding field [FSS20-HpH1, §3.7] [FSS21-M5a, §4][SS20-GS].

In particular, when the backround C-field vanishes, then the worldvolume B-field is flux-quantized in the plain 3-Cohomotopy of the Fivebrane's extended worldvolume.

$$
c_3 = 0 \quad \overset{\text{with Hypoth. H}}{\Rightarrow} \quad
\left(
\begin{array}{ccl}
H_3 & \in & \Omega_{\mathrm{dr}}\big(\Sigma^{1+6};\, \mathfrak{l}S^3\big) \\
b_2 & \in & \mathrm{Maps}\big(\Sigma^{1+6},\, S^3\big) \\
\widehat{B}_2 & : & H_3 \Rightarrow \mathrm{ch}(b_2)
\end{array}
\right)
$$

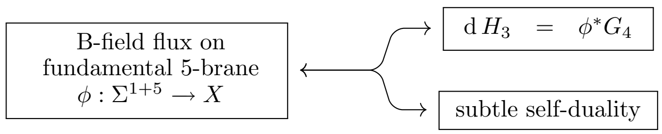
Non-abelian gerbe structure. It turns out [FSS21-M5b] that this 3-Cohomotopical flux quantization makes the Fivebrane's worldvolume B-field behave like a "non-abelian gerbe" field (as originally suggest by [Wi04, p. 16 & 15]) with structure a twisted String-2-group (as previously hypothesized in [SäSc18]).

On two coincident M5s. Indeed, careful analysis [FSS21-M5a, Thm. 4.8] shows that it is only *twice* the traditional Hopf WZ term which is generally integral, and (hence) that flux quantization in Cohomotopy sees the Fivebrane sigma model as that of *two coincident* M5-branes, carrying a non-abelian SU(2)-gauge field.



The fundamental fivebrane sigma-model in the literature. Apart from its dimensionality, the sigma-model for the fundamental fivebrane on 11d supergravity target spacetimes [HS97][HSW97][PST97][BLNPST97] is crucially different from the previous examples (§2.1.1) in that, besides the "embedding field" $\phi : \Sigma^{1+5} \longrightarrow X$ (84), there is supposed to be a higher gauge field propagating on its worldvolume – the B-field – with a 3-form flux density $H_3 \in \Omega^3_{\mathrm{dR}}(\Sigma^{1+5})$ which:

1. is sourced by the restriction $\phi^* G_4$ of the C-field flux on X to the 5-brane worldvolume: $\mathrm{d}H_3 = \phi^* G_4$ ([HS97, (36)][So00, (5.75)])
2. is subject to a notoriously subtle self-duality constraint (*not quite* $H_3 = \star H_3$, cf. [HS97, below (41)])



(88)

3. enters the Wess-Zumino (WZ) term (85) for the gauge-coupling to the background M2-brane flux G_7, deforming it to the *Hopf-WZ term* [Ah96, p. 10][BLNPST97, (1)][PST97, (17)][In00, (2.4)] which for trajectories $\phi : \Sigma^{1+5} \to U \hookrightarrow X$ inside a chart U of X is meant to be of this form (cf. [FSS21-M5a, §2]):

$$(\phi, H_3) \;\mapsto\; \int_{\Sigma^{1+5}} \big(C_6 - \tfrac{1}{2} H_3 \wedge C_3\big)\,, \qquad \text{where} \quad \begin{array}{l} C_3 \,\in\, \Omega^3_{\mathrm{dR}}(U), \quad \mathrm{d}C_3 \,=\, G_4\vert_U \\ C_6 \,\in\, \Omega^6_{\mathrm{dR}}(U), \quad \mathrm{d}C_6 \,=\, G_7\vert_U + \tfrac{1}{2} C_3 \wedge G_4\vert_U \end{array}\,. \tag{89}$$

An enormous (and ongoing) effort – motivated by arguments going back to [Wi02][Wi10] – has been devoted to understanding the self-duality constraint in (88), while the global understanding of the Bianchi identity $\mathrm{d}H_3 = \phi^* G_4$ and its role in the fivebrane's peculiar gauge coupling term (90) has received little to no attention in the community. In the spirit of the pre-geometric perspective §1.1.2 we proceed here contrariwise:

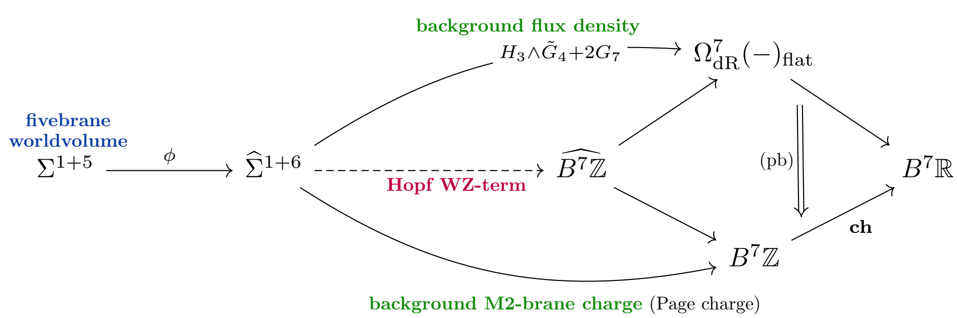
Since it is likely premature to discuss the geometric self-duality constraint on the H_3-flux before its flux quantization law has been identified, we discuss the latter – deriving it as a consequence of Hypothesis H, proving that it implies the necessary "level quantization" of the Hopf-WZ term (from [FSS21-M5a]) and anlyzing the "non-abelian gerbe"-field on the 5-brane worldvolume which makes this happen.

Level-quantization of the 5brane's Hopf-WZ term. Assuming for a moment that the H_3-flux is defined not just on Σ^{1+5} but on all of X as in (74) or at least on a 7-dimensional "extended worldvolume" [FSS21-M5a, (5)] $\widehat{\Sigma}^{1+6} \to X$ we may observe that on flat spacetimes the Hopf-WZ term (89) is a local potential for the M_2-brane Page charge (75)

$$\mathrm{d}\big(C_6 - \tfrac{1}{2} H_3 \wedge C_3\big) \;=\; \big(G_7 + \tfrac{1}{2} H_3 \wedge G_4\big)\vert_U\,,$$

and on curved spacetimes we may adapt it to include the necessary shifting (which the literature ignores) by $p_1(\nabla)$ (65) along the lines of (86), see [FSS21-M5a, (11,16)] for details.

Therefore Hypothesis $\widehat{\mathrm{H}}$ (78) implies, via Thm. 1.2, that (twice) the Hopf-WZ term for *pure* M_2-brane background charge (72) on $\widehat{\Sigma}^{1+6}$ is properly level-quatized and hence indeed a globally consistent gauge coupling — this is the main result of [FSS21-M5a]:

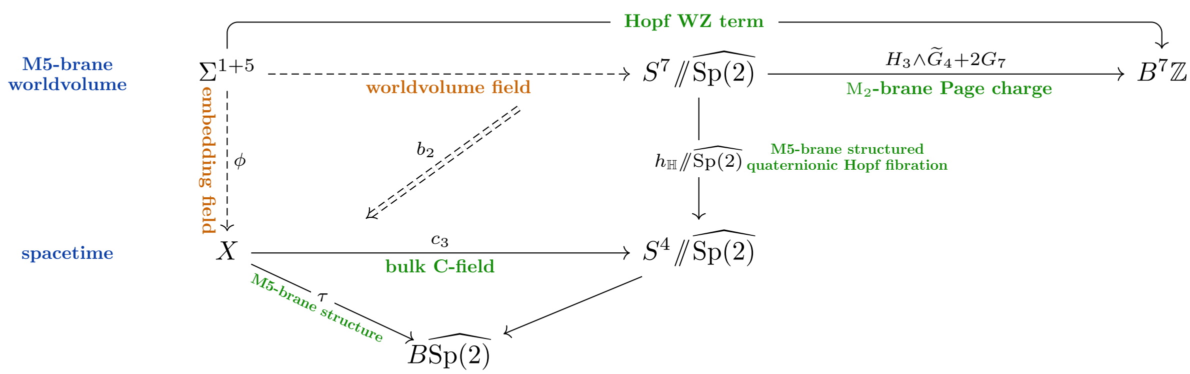


(90)

Charge quantization and non-abelian gerbe field on fivebrane worldvolume. We analyze in more detail what it is that makes the fivebrane worldvolume "anomaly cancellation" (90) work, in terms of peculiar worldvolume field content that is implied by Hypothesis H – this is the main result from [FSS21-M5b].

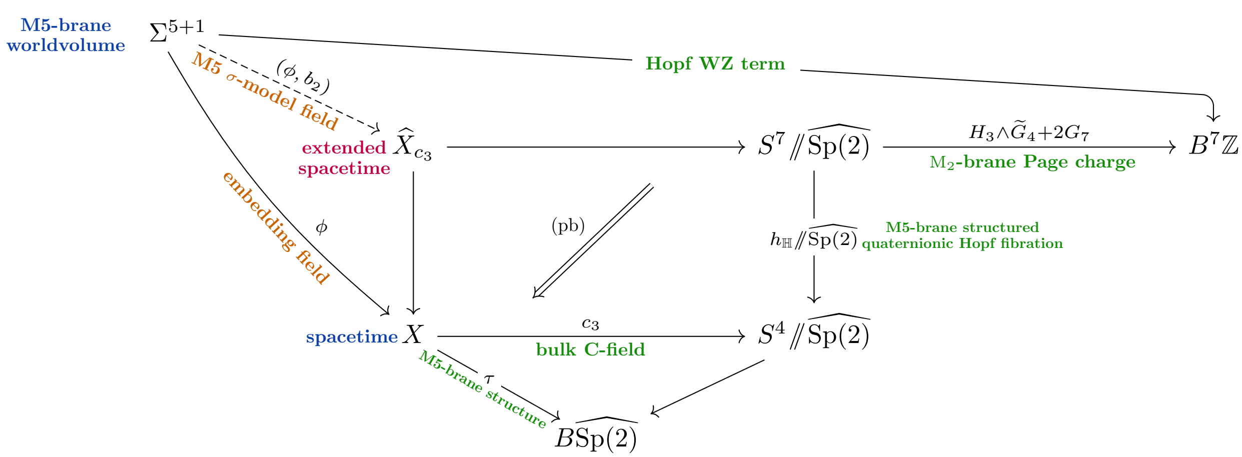
We left off in §1.3.2 with observing that "pure" M_2-brane charge – "Page charge" (75) – is reflected, via Hypothesis H, in factorizations (72) of the full cohomotopical M-brane charge through the (M5-brane structured) quaternionic Hopf fibration $h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}$. That such factorizations imply the existence of a 3-flux H_3 (74) which trivializes the background M_5-charge (relative to the pertinent shift of the vacuum by $\frac{1}{2}p_1$) is part of what it means for the M_2-brane charge to be "pure" (no M_5-brane charge admixtures) but it also means that the existence of such lifts on all of spacetime are strongly constrained.

However, to make sense of the M_5-brane sigma-model coupled to the Page charge, we only need such lifts to exists *on the worldvolume* Σ^{1+5} of the M_5, hence after pulling back the C-field along an embedding field $\phi : \Sigma^{1+5} \to X$ — and there the side-effect of trivializing the (shifted) G_4-flux by a 3-flux now makes perfect sense and identifies the 3-flux H_3 with the worldvolume 3-flux H_3 expected on the M5, which is sourced by the 1-branes inside M_5-worldvolumes known as "self-dual strings" or "M-strings":



(91)

Such "homotopy cones" (as indicated by the dashed arrows) are equivalently maps into the corresponding *homotopy pullback* (of the M5-structured quaternionic Hopf fibration along the cohomotopy cocycle c_3 for the C-field), which we denote $\widehat{X}_{c_3}$ and think of as the **C-field extended spacetime** [FSS20-HpH1, Def. 3.16] [FSS20-M5d, Rem. 3.9][FSS21-M5b, p. 7] :



(92)

This means that for given background C-field c_3 on a spacetime X, the extended spacetime $\widehat{X}_{c_3}$ is the correct "target space" for (the topological sector of) the M5-brane sigma-model, unifying the actual target spacetime X with a classifying space for the worldvolume B-field on the M5-brane.

B-field flux quantization of M_5-worldvolumes In fact, from (73) and pasting law, it follows that the extended spacetime $\widehat{X}_{c_3}$ is a 3-sphere fibration over spacetime:

$$
\begin{array}{ccccccc}
\underset{\text{fiber}}{\overset{\text{3-sphere}}{}} & & \underset{\text{spacetime}}{\overset{\text{extended}}{}} & & & & \\
S^3_x & \longrightarrow & \widehat{X}_{c_3} & \longrightarrow & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} & \longrightarrow & S^7 /\!\!/ \mathrm{Sp}(2) \\
\downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\
\underset{\text{any point}}{\{x\}} & \hookrightarrow & \underset{\text{spacetime}}{X} & \xrightarrow{\;c_3\;} & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} & \longrightarrow & S^4 /\!\!/ \mathrm{Sp}(2) \\
 & & & & \downarrow & (\mathrm{pb}) & \downarrow \\
 & & & & B\widehat{\mathrm{Sp}(2)} & \longrightarrow & B\mathrm{Sp}(2)
\end{array}
\tag{93}
$$

Since it is this 3-sphere fiber which, locally, classifies the H_3-flux, we find, in mild that variation of (74) implies that the B-field on fivebrane worldvolumes is flux-quantized in a form of twisted 3-cohomotopy:

> Implication of **Hypothesis** $\widehat{\mathrm{H}}$ on M5-worldvolumes ([FSS21-M5b, p. 7]):
>
> *M-string charges in* M_5 *worldvolumes are quantized in* $\phi^*(c_3)$*-twisted 3-cohomotopy*, hence B-field fluxes on M_5 worldvolumes are in a twisted 3-cohomotopical character

(94)

$$
\underset{\pi^{3+\phi^*(c_3)}\big(\Sigma^{1+5}\big)}{\overset{\textbf{Bckgr. C-field-twisted}}{\textbf{3-Cohomotopy}}}
\;:=\;
\left\{
\begin{array}{ccccccc}
\Sigma^{1+5} & \overset{\substack{\textbf{worldvolume}\\ \textbf{B-field}}}{\dashrightarrow_{b_2}} & \widehat{X}_{c_3} & \longrightarrow & S^7 /\!\!/ \widehat{\mathrm{Sp}(2)} & \longrightarrow & S^7 /\!\!/ \mathrm{Sp}(2) \\
 & \searrow^{\phi}_{\text{embedding field}} & \downarrow & (\mathrm{pb}) & \downarrow{\scriptstyle h_{\mathbb{H}} /\!\!/ \widehat{\mathrm{Sp}(2)}} & (\mathrm{pb}) & \downarrow{\scriptstyle h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)} \\
 & & \underset{\textbf{background C-field}}{X} & \xrightarrow{\;c_3\;} & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} & \longrightarrow & S^4 /\!\!/ \mathrm{Sp}(2)
\end{array}
\right\}_{\Big/ \text{rel. homotopy}}
\tag{95}
$$

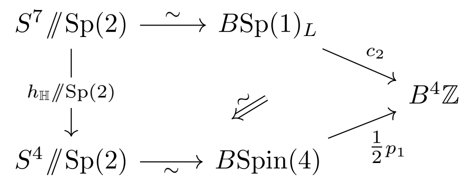
We re-iterate that this twisted 3-cohomotopical flux-quantization of the worldvolume B-field implies that the fivebrane's Hopf-WZ gauge coupling term is globally well defined – a key requirement for consistency of the M_5-brane sigma model whose solution had been a wide-open problem. All the more is the following consequence noteworthy:

Emergence of a non-abelian higher gauge field on the M_5-worldvolume. Remarkably, this particular form of twisted 3-cohomotopy (95) also has an equivalent *gauge-theoretic* interpretation, due to the coset space realization (67) of the quaternionic Hopf fibration – which implies that its tangentially twisted version is equivalently a map of classifying spaces of $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$-gauge fields:

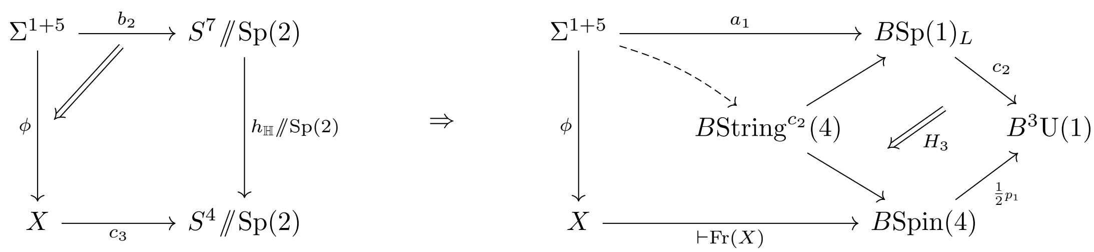
$$
\begin{array}{ccccccc}
S^7 /\!\!/ \mathrm{Sp}(2) & \xleftrightarrow{\;\sim\;} & B\mathrm{Sp}(1)_L & \times & * & \xleftrightarrow{\;\sim\;} & B\mathrm{Sp}(1)_L \\
\downarrow{\scriptstyle h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)} & & \| & & \downarrow & & \downarrow \\
S^4 /\!\!/ \mathrm{Sp}(2) & \xleftrightarrow{\;\sim\;} & B\mathrm{Sp}(1)_L & \times & B\mathrm{Sp}(1)_R & \xleftrightarrow{\;\sim\;} & B\mathrm{Spin}(4) \\
 & & H^\bullet(B\mathrm{Sp}(1);\mathbb{Z}) & \oplus & H^\bullet(B\mathrm{Sp}(1);\mathbb{Z}) & \xleftrightarrow{\;\sim\;} & H^\bullet\big(B\mathrm{Spin}(4);\,\mathbb{Z}\big) \\
 & & c_2^L & + & c_2^R & = & \tfrac{1}{2}p_1 \quad \textbf{first fractional Pontrjagin class/ C-field background charge} \\
 & & c_2^L & & & = & \tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1 = \widetilde{G}_4 \quad \textbf{shifted integral C-field charge} \\
 & & & & -c_2^R & = & \widetilde{G}_4 - \tfrac{1}{2}p_1 \quad \textbf{C-field charge relative to background charge}
\end{array}
$$

The decomposition of the cohomology generators as shown in the last line (using [CV98a, Lem 2.1], see [FSS20-HpH1, Lem. 3.9]) shows that the pullback of the fractional Pontrjagin class along the parameterized quaternionic Hopf fibration equals the Chern class on the "gauge factor" $B\mathrm{Sp}(1)_L$:

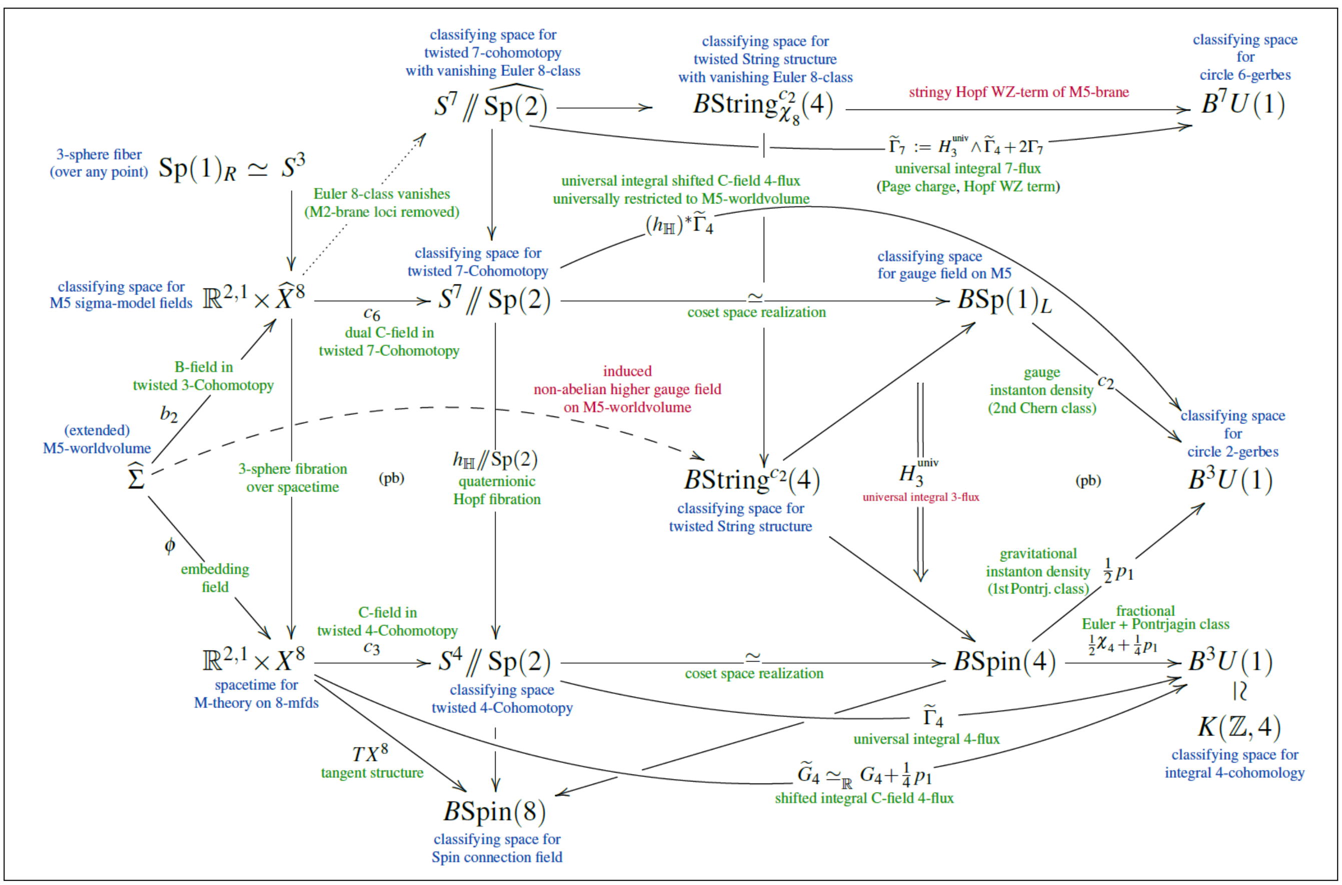
$$\big(h_{\mathbb{H}} \,/\!\!/\, \mathrm{Sp}(2)\big)^* \tfrac{1}{2} p_2 \;=\; c_2 \qquad \Leftrightarrow$$



But this means that the worldvolume B-field on the fundamental 5-brane according to (95) may be regarded as having an underlying Sp(1)-gauge field a_1 equipped with a "Green-Schwarz term" H_3 that identifies the gauge-fields Chern class (instanton density) with the first fractional Potrjagin class (pulled back to the worldvolume), hence as having an underlying $\mathrm{String}^{c_2}(4)$-valued higher gauge field:



Speculation that such a "non-abelian gerbe field" might emerge on M_5-branes originates with [Wi02, p. 6, 15] and the particular possibility of String(G)-fields for $G = \mathrm{SU}(2)$ was explored in [SäSc18] but had remained guesswork. Here the expected kind of structure drops out as a consequence of Hypothesis H, complete with its subtle charge-quantization law. For more discussion, including more pointers to related literature, see [FSS21-M5b][SS20-GS].



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2.2 Brane lightcone quantization

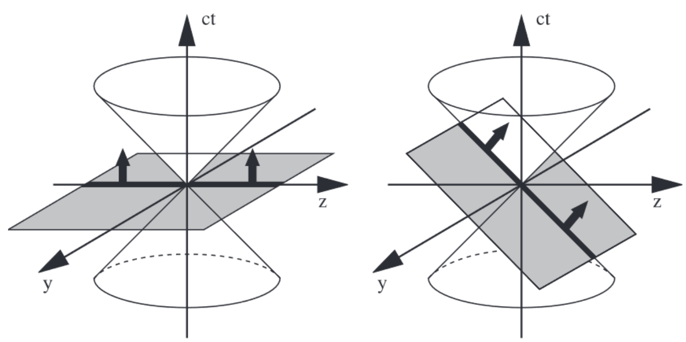
As we now turn to the actual non-perturbative quantization of phase space stacks of flux-quantized higher gauge fields (§1.2.3) and hence of the corresponding branes (§1.1) there is little traditional theory to go by, since: (1.) we have no Lagrangian density and (2.) our fields are not sections of an ordinary field bundle, the usual definition of pre-symplectic structure on (covariant) phase space (p. 13) fails on several accounts).

But we may observe that the ordinary homology of the phase space stack looks just like topological observables on higher gauge fields (cf. p. 24). Moreover, on spacetimes with a circle factor (such as for M/IIA duality) the homology algebra inherits a non-commutative product (Pontrjagin product) which makes it a star-algebra as suitable for quantum observables in "discretized light cone gauge" [SS23-Qnt][SS19-Qnt][CSS23-Qnt].

Non-perturbative light-cone quantization. The solution to the problem of non-perturbative quantization of a relativistic Lagrangian field theory appears in principle straightforward: Choose a foliation of spacetime by non-timelike hypersurfaces and then consider the Hamiltonian dynamics of evolution along the leaves.

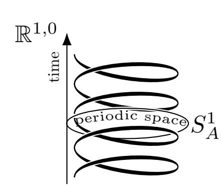
It is for technical and computational problems encountered with carrying this out for the naïve choice of spacelike foliations that the Hamiltonian approach to relativistic QFT was largely abandoned, long ago, in favor of Schwinger-Tomonoga-Feynman-Dyson perturbation theory, which is now often but erroneously regarded as synonymous with "quantum field theory".

However, one may also consider foliation by *lightlike* hypersurfaces (light wave fronts, [Dirac1949, §5]), and the resulting *light-cone quantization* turns out to be mathematically natural and more tractable, especially in application to hadronic bound states in strongly coupled QCD (e.g. [BMPP93][Zh94][BPP98][Ba+13]).



spacelike foliation lightlike foliation

The greatest practical progress with non-perturbative computations in QCD has been made by additionally assuming that spacetime is periodic along one light-like direction so that the light-cone momentum values are discrete, whence one speaks of **discretized light-cone quantization** [MY76][Ca76][Th77][Th78][PB85][Pa99]. This may be understood [Sei97] as the physics seen by a lightlike observer travelling along a periodic spatial dimension.



(96)

But in itself, while computationally succesful, the fact that a spacelike periodicity is required and singled out here is puzzling from the point of view of physics in 1+3-dimensional.

M-Theory as $\mathbb{R}^{1,0} \times \mathrm{S}^1_A$ light-cone quantum mechanics. However, exactly such a circle-factor S^1_A in spacetime is meant to appear in strongly coupled type IIA string theory in the guise of M-theory (cf. p. 3), where the radius of S^1_A scales with the string coupling seen in 10d [DHIS87][To95][Wi95, §2.3] (review in [Du96, §2(ii)][OP99, §2.1]).

Indeed, one early proposal for making sense of M-theory is (see [NH98, §10]) to regard it as the *lightcone quantum mechanics* of the fundamental membrane (80) propagating on a spacetime of the form (96)

$$X^{1,d} \;=\; \mathbb{R}^{1,0} \times S^1_A \times X^{d-1} \tag{97}$$

with lightcone momentum along the circle S^1_A, which in the small radius limit is thought to reduce to the D_0-brane dynamics described by the *BFSS matrix model* [BFSS97][Susk97][Sei97] or rather the *BMN matrix model* [BMN02, §5] (review in [Yd18]).

A key consistency check of these M-theory matrix models have been computations recovering 11d supergravity in the form of graviton scattering amplitudes [BBPT97][HPSW99][HM23] — but brane charge quantization such as in K-theory (§1.3.1) is not reflected in these models (cf. [AST02]). Contrariwise, we now explain a lightcone quantization of pre-geometric brane charges in general cohomology theories.

2.2.1 Quantum observables on flux-quantized fields.

Recall:

Quantum observables and quantum states.

Given a *star-algebra of quantum observables* consider their	$\big(\mathrm{Obs},\ \cdot\ , (\text{-})^*\big)$
expectation values for a given *quantum state* (which is defined thereby)	$\langle - \rangle \,:\, \mathrm{Obs} \xrightarrow{\text{linear}} \mathbb{C}$
respecting involution	$A \in \mathrm{Obs} \ \vdash\ \langle A^* \rangle \,=\, \overline{\langle A \rangle}$
and being "positive" on normal observables.	$A \in \mathrm{Obs} \ \vdash\ \langle A^* A \rangle \in \mathbb{R}_{\geq 0} \subset \mathbb{C}$
From this a (Hilbert-)space of states is induced	$\mathcal{H} \,:=\, \mathrm{Obs}/\big\{A \in \mathrm{Obs} \,\big\vert\, \langle A^* \cdot A \rangle \,=\, 0\big\}$
(the *GNS construction*) with "ground" state	$\vert\psi_0\rangle := [1]$
supporting an operator-state correspondence	$A\vert\psi_0\rangle \,=\, [A]$
for the induced inner product, reproducing	$\langle A\psi_0, \vert B\psi_0 \rangle \,=\, \langle \psi_0 \vert A^* B \vert \psi_0 \rangle \,:=\, \langle A^* B \rangle$
the expectation values in this ground state.	$\langle \psi_0 \vert - \vert \psi_0 \rangle \,=\, \langle - \rangle$

Non-perturbative quantization

of a Poisson-manifold phase space	$\{-,-\} \,:\, C^\infty(P) \otimes C^\infty(P) \to C^\infty(P)$
is a bundle of C^*-algebras	$\Big\{\mathrm{Obs} \overset{(-)_\hbar}{\twoheadrightarrow} \mathrm{Obs}_\hbar \in C^*\mathrm{Alg}\Big\}_{\hbar \in \mathbb{R}}$, $\quad f \in C^0(\mathbb{R}),\ A \in \mathrm{Obs}\ \vdash\ fA \in \mathrm{Obs}\,, (fA)_\hbar = f(\hbar) A_\hbar$
which continuously	$\underset{A \in \widehat{\mathrm{Obs}}}{\forall} \Big(\hbar \mapsto \vert A_\hbar \vert\Big) \in C^0(\mathbb{R})\,, \qquad \vert A \vert = \sup_\hbar \vert A_\hbar \vert$
deforms the classical observables	$Q : C^\infty_{\mathrm{cpt}}(P) \to \mathrm{Obs}\,, \quad Q(-)_0 \,:\, C^\infty_{\mathrm{cpt}}(P) \xhookrightarrow{\text{dense}} A_0$
satisfying Dirac's quantization condition.	$\lim\limits_{\hbar \to 0} \Big\vert \big[Q(f)_\hbar,\, Q(g)_\hbar\big] - \mathrm{i}\hbar\, Q\big(\{f,g\}\big)_\hbar \Big\vert \,=\, 0\,.$

Yang-Mills flux observables.

in $\mathfrak{g}$-Yang-Mills theory on $\mathbb{R}^{0,1} \times X^3$, with an oriented closed surface $\Sigma^2 \hookrightarrow X^3$ measure the weighted integrals of the electric & magnetic flux densities over Σ	$\Phi^\omega_E \,=\, \int_\Sigma \langle \omega, E \rangle$, $\quad \Phi^\omega_B \,=\, \int_\Sigma \langle \omega, B \rangle \quad$ for $\omega \in \Omega^0_{\mathrm{dR}}(X^3; \mathfrak{g})$

Proposition [SS23-Qnt]: **Non-perturbative quantum observables on quantized YM fluxes.**

A non-perturbative quantization of the Φ_E and Φ_A	for a choice of Lie group G with Lie algebra $\mathfrak{g}$
is the Fréchet-group *convolution algebra* of semidirect product group-valued functions on Σ	$\mathrm{Obs} \,=\, C^\infty\big(\Sigma,\, G \rtimes \mathfrak{g}_0/\Lambda\big)$
Hence the *topological flux observables* form a convolution group algebra of cohomology of Σ:	$\mathrm{Obs}^{\mathrm{top}} \,=\, \mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma;\, G \ltimes B\Lambda\big)\Big] \,\simeq\, \mathbb{C}\Big[H^0\big(\Sigma; G\big) \ltimes H^1(\Sigma; \Lambda)\Big]$
For Maxwell theory on $\mathbb{R}^{1,1} \times \Sigma$ this coincides with the Pontrjagin-Hopf algebra of loops in the moduli of flux-quantized topological field sectors on Σ	$\simeq H_0\Big(\Omega\, \mathrm{Maps}\big(\Sigma,\, B\mathrm{U}(1) \times B\mathrm{U}(1)\big)\Big)$

This **remarkable re-formulation of quantization**

Non-perturbative quantum observables on topological flux observables in Yang-Mills theory on $\mathbb{R}^{1,1} \times \Sigma$.	$\longleftrightarrow$	Pontrjagin-Hopf algebra of loops in flux-quantized topological fields on Σ.

we next take as the blueprint for the quantization of topological fluxes in M-theory.

Lightcone quantization of topological flux observables
for flux-quantization in $\mathcal{A}$-theory [SS23-Qnt].

Nonprtrbtv BRST complex of topological fields. The observables on charge (super-selection) sectors are evidently the linear combinations of $\pi_0 \mathrm{Maps}\big(X^d, \mathcal{A}\big)$.

$$\mathrm{Obs}_0(X^d, \mathcal{A}) \;=\; H_0\big(\mathrm{Maps}\big(X^d, \mathcal{A}\big); \mathbb{C}\big)\,.$$

The higher homotopies $\pi_n \mathrm{Maps}\big(X^d, \mathcal{A}\big)$ are higher gauge transformations, whence higher chains are (topological) higher "BRST-ghost" field observables.

$$\mathrm{BRST}_\bullet(X^d, \mathcal{A}) \;=\; C_\bullet\big(\mathrm{Maps}\big(X^d, \mathcal{A}\big); \mathbb{C}\big)$$

Therefore the gauge invariant observables are the chain-homology of this BRST complex, hence are the complex homology of the topological phase space.

$$\mathrm{Obs}_\bullet(X^d, \mathcal{A}) \;=\; H_\bullet\big(\mathrm{Maps}\big(X^d, \mathcal{A}\big); \mathbb{C}\big)$$

Non-perturbative M-theory spacetime domain. In lifting a type IIA spacetime domain. X^9_{IIA} (a pointed space) to fully non-perturbative M-theory, the IIA-circle fiber S^1_A is meant to appear decompactified as $\mathbb{R}^1$. But assuming that fluxes vanish at infinity along this direction, the corresponding fiber domain is $\mathbb{R}^1_{\cup\{\infty\}}$ and hence the M-theory spacetime domain is $\mathbb{R}^1_{\cup\{\infty\}} \wedge X^9_{\mathrm{IIA}}$.

Makes topological phase space a loop space. This implies that the phase space of flux-quantized topological fields in M-theory is a based loop space:

$$\mathrm{Maps}^{*/}\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \;\equiv\; \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge X^9_{\mathrm{IIA}}, \mathcal{A}\big)$$
$$\simeq \Omega\, \mathrm{Maps}^{*/}\big(X^9_{\mathrm{IIA}}, \mathcal{A}\big)$$

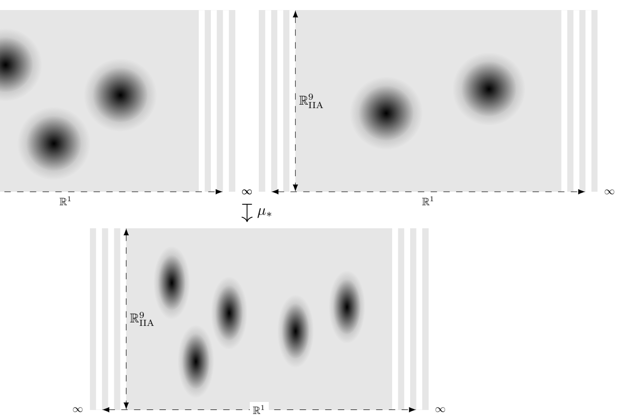
M-theory quantizes itself. From this, the topological observables inherit a non-commutative *Pontrjagin-Hopf algebra* structure, which makes them be quantum observables [CSS23-Qnt]:

$$\mathrm{QObs}_\bullet\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \;=\; H_\bullet\Big(\Omega\, \mathrm{Maps}\big(X^9_{\mathrm{IIA}}, \mathcal{A}\big)\Big)$$

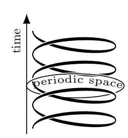
whose operator product is given by by translation followed by "fusion" of solitons in the M-theory circle-direction.

Pontrjagin product:

$$H_\bullet(\Omega Y) \otimes H_\bullet(\Omega Y) \xrightarrow[\text{Künneth}]{\sim} H_\bullet(\Omega Y \times \Omega Y) \xrightarrow[\text{pushforward in homology}]{(\text{-})\cdot(\text{-}) := \mu_*} H_\bullet(\Omega Y)$$
$$\Omega Y \times \Omega Y \xrightarrow[\text{concatenation of loops}]{\mu} \Omega Y$$



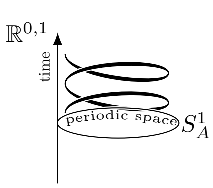
The Discrete lightcone emerges. But, generally, the operator product of quantum observables reflects *temporal* order (originally observed by [Fey42, p. 35][Fey48, p. 381], cf. [Ong]), whence we are faced with a topological version of "discretized light cone" quantization (cf. [BFSS97][Susk97]).



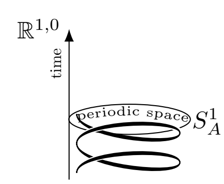
The star-involution of Light-cone time-reversal must hence be the combination of the Pontrjagin antipode (spatial inversion) with complex conjugation (plain temporal inversion), which together makes the quantum observables into a complex Hopf algebra.

Pontrjagin antipode:

$$H_\bullet(\Omega Y) \xrightarrow{\mathrm{inv}_*} H_\bullet(\Omega Y)$$
$$\Omega Y \xrightarrow[\text{reversal of loops}]{\mathrm{inv}} \Omega Y$$



$$\begin{array}{ccccc}
 & & H_\bullet(\Omega Y; \mathbb{C}) & & \\
 & \nearrow^{\text{Pontrjagin-antipode}}_{H_\bullet(\mathrm{inv};\mathbb{C}),\ S^1_A\text{-inversion}} & & \searrow^{\text{complex conjugation}}_{H_\bullet(\Omega Y;(\text{-})^*),\ \text{time inversion}} & \\
H_\bullet(\Omega Y; \mathbb{C}) & & \xrightarrow[\text{light-cone parameter inversion}]{(\text{-})^*} & & H_\bullet(\Omega Y; \mathbb{C}) \\
 & \searrow_{H_\bullet(\Omega Y;(\text{-})^*)} & & \nearrow_{H_\bullet(\mathrm{inv};(\text{-})^*)} & \\
 & & H_\bullet(\Omega Y; \mathbb{C}) & &
\end{array}$$



Quantum states of topological fields are therefore the positive linear functionals on this complex Pontrjagin-Hopf homology algebra:

$$\mathrm{QStates}\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \;=\; \Big\{ \rho \;:\; \mathrm{QObs}_\bullet\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \xrightarrow{\text{linear}} \mathbb{C} \;\Big\vert\; \underset{\mathcal{O}}{\forall}\;\; \rho\big(\mathcal{O}^* \cdot \mathcal{O}\big) \;\in\; \mathbb{R}_{\geq 0} \subset \mathbb{C} \Big\}$$

In order to appreciate the concept of *higher quantum observables* that we are about to consider, it may be useful to recall the following standard conceptualization of algebraic (quantum) mechanics, schematically:

The fundamental concepts of quantum physics in "algebraic" form are the following (good exposition in [Gl09][Gl11], for more see [La17]):

- The **covariant phase space** of a physical system is really the space of solutions to the classical equations of motion [Wi86, §5][CW87][HT92, §17.1], hence the space of physically possible ("on shell") *field histories* of the system.

$$\mathrm{PhsSpc} \;=\; \left\{ \begin{array}{c} \text{solutions to equations of motion} \\ \text{hence: possible field histories} \end{array} \right\}.$$

- The **classical observables** on a physical system are encoded in *compactly supported*[14] complex-valued functions on field histories, understood as assigning to a field history the value that the observable takes there.
  In the simplistic but relevant special case where the phase space is just a discrete set (cf. [La17, §1.2]), this means that the space of observables is the linear span of formal linear combinations of field histories:

$$\mathrm{PhsSpc} \in \mathrm{Sets} \qquad \Rightarrow \qquad \mathrm{Obsrvbls} \;=\; \mathbb{C}[\mathrm{PhsSpc}] \tag{98}$$

- The **quantum observables** are a *choice* of the structure of a (non-commutative) complex star-algebra[15] on the Obsrvbls:[16]

$$\mathrm{QObsrvbls} \;=\; \Big(\mathrm{Obsrvbls},\, (\text{-})\cdot(\text{-}),\, (\text{-})^\ast\Big)\,, \qquad \big(\mathcal{O}_1 \cdot \mathcal{O}_2\big)^\ast \;=\; \mathcal{O}_2^\ast \cdot \mathcal{O}_1^\ast\,, \qquad \big((a+\mathrm{i}b)\cdot\mathcal{O}\big)^\ast \;=\; (a-\mathrm{i}b)\cdot\mathcal{O}^\ast \tag{99}$$

  Observable algebra and temporal order. Physically, the (dependency of the non-commutative product on the) order of quantum observables reflects **temporal order** (originally observed by [Fey42, p. 35][Fey48, p. 381] as reviewed in [Ong]), which implies that the star-operation $(-)^\ast$ expresses time-reversal.

  (Non-)Lagrangean origin of quantum observables. In the case of *Lagrangean field theories* there is an elaborate prescription, subject to a multitude of ad-hoc choices, occupying most of the large and still growing literature on the subject (e.g. [HT92]), for how to choose the quantum observables as a deformation controlled by Poisson structure on the phase space, at least perturbatively. But we cannot expect M-theory to be the quantization of a Lagrangean field theory (already the sector of coincident fivebranes inside M-theory is expected not to be Lagrangean) and will instead discover a natural star-algebra of quantum observables right away (107), without detour through a classical field theory.

- The **quantum states** for given quantum observables are the linear maps on the quantum observables (understood as assigning to an observable the value that it takes in the given state) which are "positive" (semidefinite), in that on elements of the form $A^\ast A$ they take non-negative real values.

$$\mathrm{QStates} \;=\; \Big\{ \rho \,:\, \mathrm{QObsrvbls} \xrightarrow{\text{linear}} \mathbb{C} \,\Big\vert\, \underset{\mathcal{O}}{\forall}\;\; \rho\big(\mathcal{O}^\ast \cdot \mathcal{O}\big) \;\in\; \mathbb{R}_{\geq 0} \subset \mathbb{C} \Big\} \tag{100}$$

But in the presence of higher gauge fields, these traditional structures of quantum physics are to be promoted to *higher structures*:

Higher observables. The notion of **higher observables** on a higher phase is not widely discussed, but from the Dao of homotopy theory it is clear that the coefficient ring is to be promoted to a higher ring, namely a *ring spectrum R*. Then the higher analog of topological observables (98) is the *R-homology*:

$$\mathrm{TopObsrvbls} \;\;=\;\; R_\bullet\big[\mathrm{Maps}(X,\,\mathcal{A})\big]\,. \tag{101}$$

If $R = H\mathbb{C}$ is the Eilenberg-MacLane spectrum of the complex numbers, then this is ordinary homology (51):

$$\mathrm{TopObsrvbls} \;\;=\;\; H\mathbb{C}_\bullet\big[\mathrm{Maps}(X,\,\mathcal{A})\big] \;\;=\;\; H_\bullet\big(\mathrm{Maps}(X,\,\mathcal{A});\,\mathbb{C}\big)\,. \tag{102}$$

[14] In $C^\ast$-algebraic formulations of mechanics the algebra of classical observables on a phase space is often taken to be the $C^\ast$-algebra $C_0(P)$ of continuous functions vanishing at infinity (e.g. [La17, §3]). But this may be understood as the $C^\ast$-completion of the "actual" observable algebra of compactly-supported functions $C_c(P) \subset C_0(P)$, see e.g. [La98, p. 55, 116][La17, p. 528].

[15] This means to require structure like that of a $C^\ast$-algebra but disregarding the completeness condition for a Banach algebra. In our application to "topological" charge sectors below the space of (higher) observables is (graded and) degreewise *finite*-dimensional, so that this (graded) Banach-algebra structure is automatic.

[16] In (99) we tacitly assume that the underlying space of quantum observables coincides with that of the "classical" observables. This turns out to be the case of relevance here (107). In traditional discussion the space of quantum observables can also be larger (such as in formal deformation quantization) or smaller (such as in geometric quantization) than the space of classical observables.

Higher quantum observables. In general there is no *canonical* (star-)algebra structure on pregeometric higher observables (101) — but if spacetime has a circle factor (97) then the pregeometric higher phase space is the loop space of the *transverse phase space*:

$$
\begin{array}{ccccc}
 & & & & \underset{\textbf{pregeometric phase space}}{\mathrm{Maps}\big(S^1 \times X^{d-1},\, \mathcal{A}\big)} \\
 & & & & \| \\
\text{mixed states} & \leftsquigarrow & \underset{\textbf{based loop space of transverse phase space}}{\Omega_c \mathrm{Maps}\big(X^{d-1},\, \mathcal{A}\big)} & \longrightarrow & \mathrm{Maps}\big(S^1,\, \mathrm{Maps}(X^{d-1},\, \mathcal{A})\big) \\
 & & \downarrow & \text{(pb)} & \downarrow \\
 & & \{c\} & \longrightarrow & \underset{\textbf{transverse phase space}}{\mathrm{Maps}\big(X^{d-1},\, \mathcal{A}\big)} \quad \rightsquigarrow \quad \text{pure states}
\end{array}
\tag{103}
$$

This means that the following basic fact of algebraic topology provides us with a canonical discrete light-cone quantization of charges:

The Pontrjagin-Hopf algebra structure on the homology of loop spaces. [BoSa53][Br61, p. 36][Ha02, §3.C]
The homology of a based loop space

$$
\Omega Y \;:=\; \Big\{ \gamma : [0,1] \xrightarrow{\;\text{cntns}\;} Y \,\Big|\, \gamma(0) \;=\; \gamma(1) \Big\}
$$

with coefficients in a field becomes

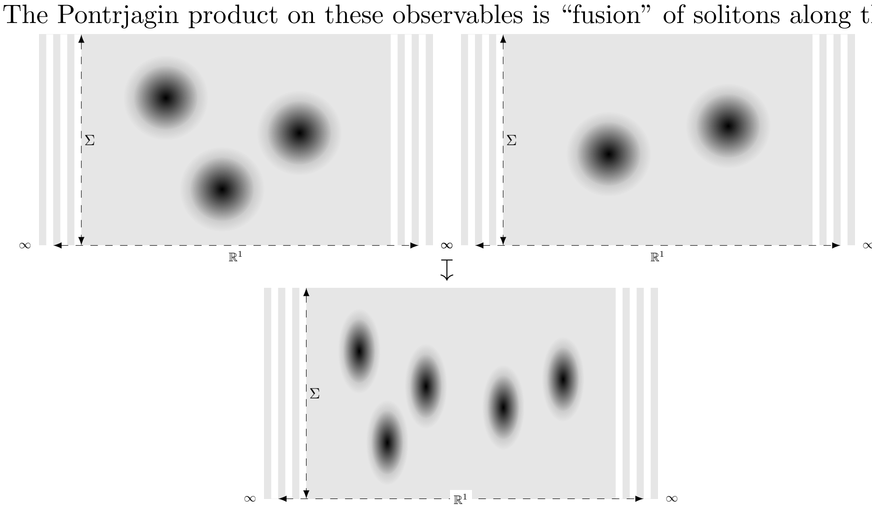
- *a graded algebra*[17] *under concatenation of loops,*

$$
\begin{array}{ccccc}
H_\bullet\big(\Omega Y\big) \otimes H_\bullet\big(\Omega Y\big) & \underset{\text{Künneth}}{\xrightarrow{\;\sim\;}} & H_\bullet\big(\Omega Y \times \Omega Y\big) & \underset{\text{pushforward in homology}}{\xrightarrow{\;(\text{-})\cdot(\text{-}) \,:=\, H_\bullet(\mu;\mathbb{C})\;}} & H_\bullet\big(\Omega Y\big) \\
\multicolumn{5}{c}{\text{(composite: Pontrjagin product)}} \\
 & & \Omega Y \,\times\, \Omega Y & \xrightarrow{\qquad \mu \qquad} & \Omega Y \\
 & & (\gamma_1,\, \gamma_2) & \longmapsto & \left(t \mapsto \begin{cases} \gamma(t/2) & \text{for } 0 \leq t \leq 1/2 \\ \gamma(t/2 - 1/2) & \text{for } 1/2 \leq t \leq 1 \end{cases} \right)
\end{array}
\tag{104}
$$

- *a graded star-algebra under reversal of loops*

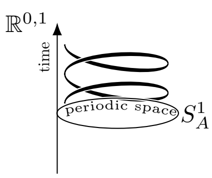
$$
\begin{array}{ccc}
H_\bullet\big(\Omega Y\big) & \xrightarrow{\;H_\bullet(\text{inv})\;} & H_\bullet\big(\Omega Y\big) \\
\multicolumn{3}{c}{\text{(composite: Pontrjagin antipode)}} \\
\Omega Y & \xrightarrow{\;\text{inv}\;} & \Omega Y \\
\gamma & \longmapsto & \gamma\big(1 - (\text{-})\big)
\end{array}
\tag{105}
$$

The Pontrjagin product on these observables is "fusion" of solitons along the lightcone direction:

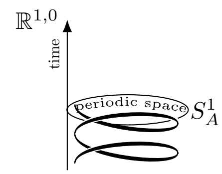


[17] In fact, together with the canonical coproduct in homology the Pontrjagin product (104) becomes a Hopf algebra structure with the star-involution (105) being a Hopf antipode.

Notice that this is not quite a *complex* star-algebra in the sense of (99) yet, since the Pontrjagin-antipode[18] (105) acts trivially on the coefficient field – but we do get a complex star-algebra by composing the Pontrjagin antipode with complex conjugation on the coefficients



$$
\begin{array}{ccccc}
 & & H_\bullet(\Omega Y;\mathbb{C}) & & \\
 & \overset{\text{Pontrjagin-antipode}}{\underset{S^1_A\text{-inversion}}{\nearrow^{H_\bullet(\mathrm{inv};\mathbb{C})}}} & & \overset{\text{complex conjugation}}{\underset{\text{time inversion}}{\searrow^{H_\bullet(\Omega Y;(\text{-})^*)}}} & \\
H_\bullet(\Omega Y;\mathbb{C}) & & \xrightarrow[\text{light-cone parameter inversion}]{(\text{-})^*} & & H_\bullet(\Omega Y;\mathbb{C}) \\
 & \searrow_{H_\bullet(\Omega Y;(\text{-})^*)} & & \nearrow_{H_\bullet(\mathrm{inv};(\text{-})^*)} & \\
 & & H_\bullet(\Omega Y;\mathbb{C}) & &
\end{array}
\tag{106}
$$



This way we have obtained **higher quantum observables** on the light-cone for pregeometric brane charges in spacetimes of the form (97):

$$
\mathrm{QObsrvbls}_c \;=\; \Big(H_\bullet\big(\Omega_c \mathrm{Maps}(X^{d-1},\, A);\, \mathbb{C}\big),\;\; (\text{-})\cdot(\text{-}),\;\; (\text{-})^* \Big)\,. \tag{107}
$$

whose star-involution is *light-cone parameter inversion*.

Notice here that

$$
\begin{aligned}
\Omega_0 \mathrm{Map}\big(X,\, A\big) \;&\simeq\; \mathrm{Map}^{*/}\big(S^1,\, \mathrm{Map}^{*\cdot}(X_{\sqcup\{\infty\}},\, A)\big) \\
&\simeq\; \mathrm{Map}^{*/}\big(X_{\sqcup\{\infty\}},\, \mathrm{Map}^{*\cdot}(S^1,\, A)\big) \\
&\simeq\; \mathrm{Map}^{*/}\big(X_{\sqcup\{\infty\}},\, \Omega_0 A\big) \\
&\simeq\; \mathrm{Map}\big(X,\, \Omega_0 A\big)\,.
\end{aligned}
\tag{108}
$$

The corresponding **light-cone quantum states** are hence (100) those cohomology classes which are (semi-)positive-definite:

$$
\mathrm{QStates}_c \;=\; \left\{ \rho \,\in\, H^\bullet\Big(\Omega_c \mathrm{Maps}\big(X^{d-1},\, A\big);\, \mathbb{C}\Big) \;\Big\vert\; \underset{\mathcal{O}}{\forall}\;\; \rho(\mathcal{O}^*\mathcal{O}) \geq 0 \right\}. \tag{109}
$$

In §2.3 we discuss examples of 11d spacetime domains whose light-cone quantum states (109) of pregeometric (intersecting) brane charges include, under Hypothesis H:

§2.3.3 quantum states of Hanany-Witten NS/D-brane configurations,

example 2.1 quantum states of transverse M_5-branes,

§2.3.4 quantum states of $\mathrm{M}_5 \perp \mathrm{M}_5$ intersections,

with provable properties of the kind expected in the string theory literature.

Pontrjagin rings as deformation quantization. For context, we indicate how Pontrjagin rings (107) are related to more familiar notions of quantization.

First recall from §1.2 that every connected coefficient space A is equivalently the delooping of its based loop group (40), the latter understood as an ∞-group. In this sense A describes (the topological sectors of) higher gauge theory whose higher gauge group has the homotopy type of the ∞-group $\mathcal{G} := \Omega A$. Moreover, the Whitehead L_∞-algebra $\mathfrak{l}A$ of A may be understood as the L_∞-algebra of this ∞-group, in that its underlying graded vector space is that of the rationalized homotopy groups of ΩA (41).

$$
A \;\simeq\; B\big(\Omega A\big) \;\simeq\; \ast /\!\!/ \Omega A\,, \qquad\quad (\mathfrak{l}A)_n \;\simeq\; \pi_n\big(\Omega A\big) \otimes_{\mathbb{Z}} \mathbb{R}\,.
$$

In the case that X^{d-1} is contractible, this is already our pre-phase space (107): $\mathrm{Maps}(X^{d-1}, A) \;\simeq\; A \;\simeq\; \ast /\!\!/ \Omega A$.

Phase spaces whose homotopy type is of this form are those arising from Lie-Poisson structures, whose symplectic groupoid is the coadjoint action groupoid $\mathfrak{g}^* /\!\!/ G$ [We91, Ex. 3.2][BC05, Ex. 4.3][Nui13, pp. 111, cf. Prop. 5.2.12]. Since the underlying space of $\mathfrak{g}^*$ is contractible, the underlying homotopy type (shape) of this action groupoid is $B\int G \;\simeq\; \ast /\!\!/ \int G$, where $\int G$ is the underlying homotopy type of G (e.g. [Orb, Prop. 3.4]):

$$
\begin{array}{ccc}
\mathrm{SmthGrpd}_\infty & \xrightarrow{\;\;\int\;\;} & \mathrm{Grpd}_\infty \\
\mathfrak{g}^* /\!\!/ G & \longmapsto & \ast /\!\!/ \int G
\end{array}
$$

[18]The term "Pontrjagin antipode" is not standard, but it is the natural name for the antipode of the Pontrjagin-Hopf algebra structure.

Now, the deformation quantization of Lie-Poisson manifolds $\mathfrak{g}^*$ is well-known to be the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ [Gu83, (4.2)][Gu11, §2.2]. Notice also that for connected compact Lie groups G, $U(\mathfrak{g})$ plays the role of the convolution algebra of G ([Ho81, §XVI][Tj92, pp. 9]), while the latter of course exists also for discrete groups.

The Pontrjagin ring of ΩA unifies these two perspectives: It gives the group algebra on $\pi_0(\Omega A)$

$$\pi_{\geq 1}(\Omega A) \simeq * \qquad \Rightarrow \qquad H_\bullet(\Omega A; \mathbb{C}) \;\; \simeq \;\; \mathbb{C}\big[\pi_0(\Omega A)\big]$$

and the universal enveloping algebra of the binary super Lie bracket of the connected components of ΩA [MiMo65, Apd.][FHT00, Thm. 16.13]:

$$\pi_0(\Omega A) \simeq * \qquad \Rightarrow \qquad \overset{\text{Pontrjagin algebra of loop } \infty\text{-group}}{H_\bullet(\Omega A; \mathbb{R})} \;\; \simeq \;\; \underset{\text{deformation quantization of } \mathrm{Bin}(\mathfrak{l}A)^*}{\overset{\text{universal envelope of binary Whitehead bracket}}{U\big(\mathrm{Bin}(\mathfrak{l}A)\big)}}$$

Example – M-Theoretic Quantum Cohomology. The Pointrjagin ring of the loop space of the 4-sphere is, by the above and using (42),

$$H_\bullet\big(\Omega S^4; \mathbb{C}\big) \;\simeq\; U\big([v_3, v_3] = v_6\big) \;=\; \mathbb{C}[v_3, v_6]/\big(v_3^2 - v_6\big)\,. \tag{110}$$

Notice that this is a deformation of the cohomology ring of S^3

$$H^\bullet(S^3; \mathbb{C}) \;\simeq\; \mathbb{C}[v_3]/\big(v_3^2\big)$$

in the same way (up to degree shifts) that the *quantum cohomology* of $S^2 = \mathbb{C}P^1$:

$$\mathrm{QH}^\bullet\big(\mathbb{C}P^1\big) \;\simeq\; \mathbb{C}[v_2, v_4]/\big(v_2^2 - v_4\big)$$

is a deformation of the ordinary cohomology ring

$$H^\bullet\big(\mathbb{C}P^1\big) \;\simeq\; \mathbb{C}[v_2]/\big(v_2^2\big)\,.$$

This quantum cohomology reflects the interaction of topological strings propagating on $\mathbb{C}P^1$. But since S^3 is an S^1-bundle over $S^2 = \mathbb{C}P^1$, it stands to reason that (110) is an M-theoretic lift of this situation, possibly characterizing the interaction of topological membranes propagating on S^3.

Example: Topological quantum observables in vacuum Maxwell theory. [SS23-Qnt] (...)

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2.3 Resulting quantum branes

We discuss the nature of solitonic M-branes (§1.1), according to Hypothesis H (§1.3), and their lightcone quantum mechanics (§2.2).

Worldvolumes of solitonic branes. Traditionally, the notion of *worldvolumes* for solitonic branes — whose presence is reflected only in localized concentrations of flux density, cf. §1.1 — is far from clear, even classically, and yet it is widely assumed to make sense.

But assuming Hypothesis H (§1.3), we may observe:

(1.) That [SS23-Mf, §2.2] [SS20-Orb, §2.1] flux in Cohomotopy is tightly related to actual submanifolds as envisioned for brane worldvolumes, due to *Pontrjagin's theorem* [Po38][Ko93, §IX]. In fact, (un-)stable Cohomotopy is equivalent to (un-)stable frame Cobordism cohomology, and under this equivalence Cohomotopy charge reflects *exactly* the (cobordism classes of) normally submanifolds in spacetime, where the "normal framing" is a form of charge carried by these submanifolds that locally reflects brane/antibrane annihilation in quite the way traditionally envisioned for Hypothesis K.

(2.) That [SS22-Cnf] the analog of the Pontrjagin theorem (§2.3.1) in the case of exotic branes, under Hypothesis H, is the *May-Segal theorem* [May72, Thm. 2.7][Segal73, Thm. 3] which, with Hypothesis H, equivalently says [SS22-Cnf, Prop. 2.5] that the moduli space of flat solitonic branes of low codimension ≤ 3 is (homotopy equivalent to) a *configuration space* of unodered points in their transverse space.

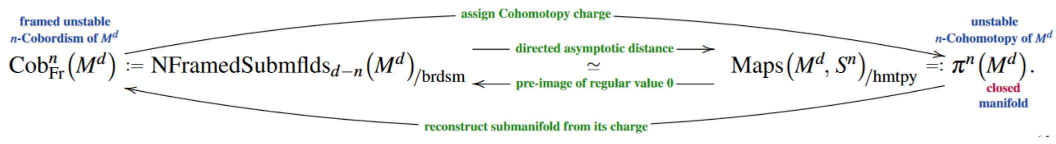
(3.) That [SS22-Cnf], therefore, the *intersections* with codimension=1 branes form an *ordered* configuration space of points, so that the light-cone quantum observables (according to §2.2) on such solitonic brane intersctions are given by the homology of the loop space of ordered configurations. This is described by the Fadell-Husseini theorem, and we observe that it reflects various structures expected in Hanany-Witten theory of such brane intersections.

Remark. The definition of phase spaces of solitonic branes that we used above and now in §2.3.1 and §2.3.2 is, while rarely made as formally explict, by and large tacitly understood in the literature. However, despite the popularity of "intersecting brane models" there has been no previous attempt to cast into a definition what their phase spaces should actually be in the case of *solitonic* branes, given that they not imprinted in localized singularities of their flux densities. Therefore, as we proceed to discuss this matter in §2.3.3, we have to introduce such a definition (from [SS22-Cnf]), which is a postulate about physics on top of Hypothesis H. However, the postulate is most natural: We declare that the phase space of intersecting solitonic branes is the fiber product of the phase spaces of the separate solitonic branes that participate in the intersection. Conversely, this *defines* what it means to speak of intersecting solitonic branes, in the first place.

2.3.1 Solitonic brane cobordism via Pontrjagin's Theorem

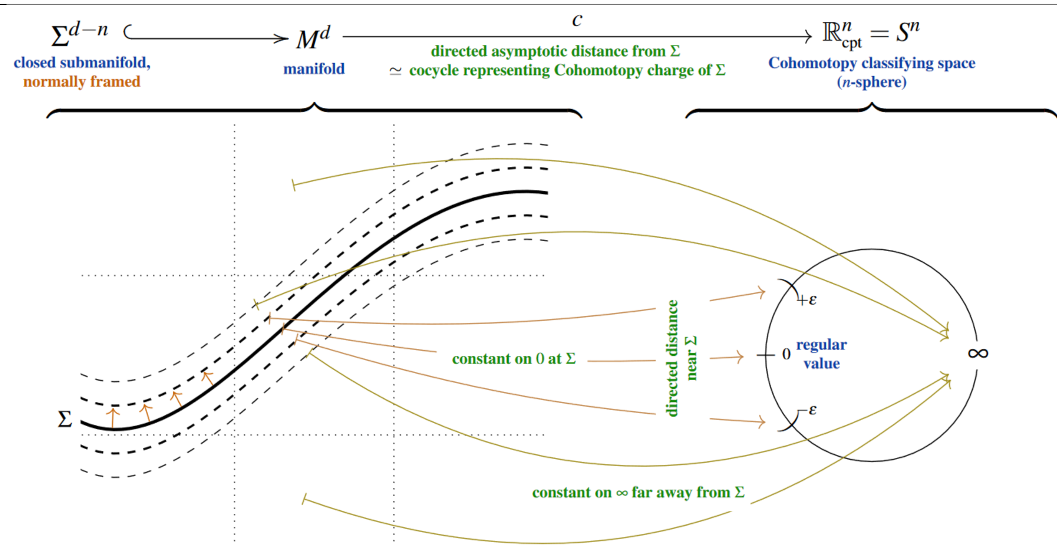
Remarkably, there is a tight relation between Cohomotopy of spacetime and cobordism classes of submanifolds that behave like branes carrying a corresponding Cohomotopy charge (cf. [SS23-Mf, §2.2] [SS20-Orb, §2.1]):

The **Pontrjagin theorem** [Po38][Ko93, §IX] identifies the unstable n-Cohomotopy of a closed manifold with the cobordism classes of its normally framed submanifolds of co-dimension n.



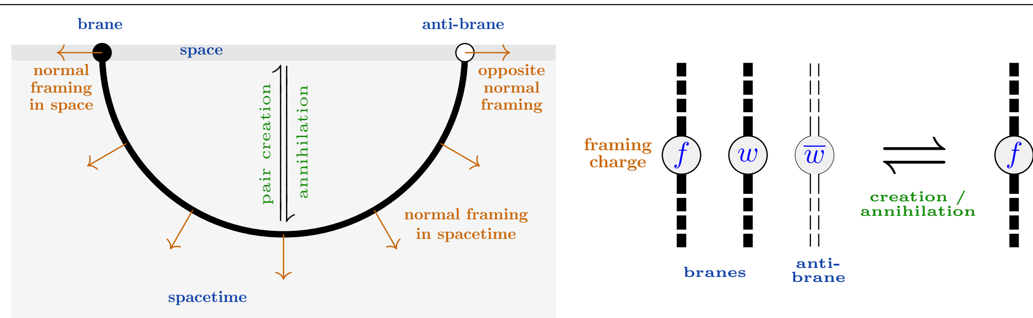
The **Cohomotopy charge** of a normally framed submanifold (aka *scanning map* or *Pontrjagin-Thom collapse*) is represented by mapping points of the ambient space to their directed distance if inside a tubular neighbourhood, else to ∞.

Conversely, every Cohomotopy class is represented by a smooth map with 0 a regular value, whose pre-image is a normally framed submanifold with that Cohomotopy charge.

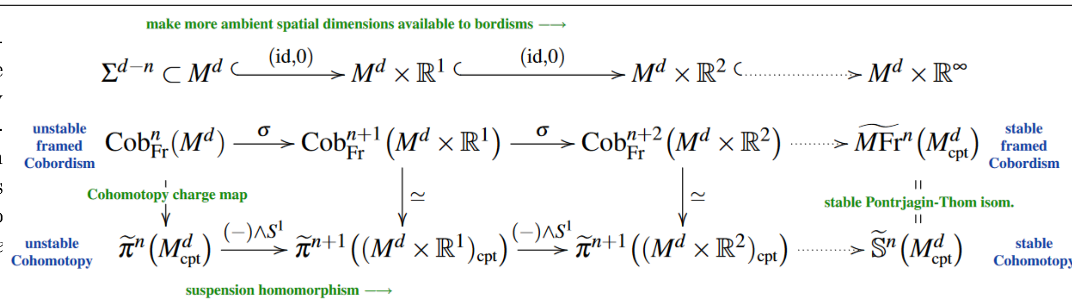


Under this relation, homotopy of charge maps corresponds to nrml. framed **cobordism** of submnflds.

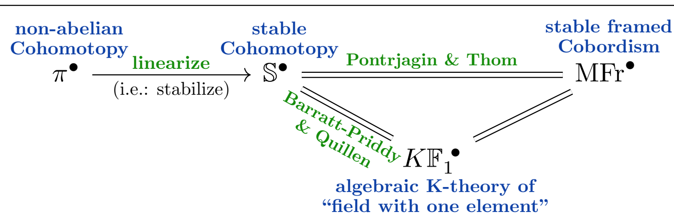
The cobordism relation exhibits a form of pair creation/annihilation of submanifolds carrying opposite Cohomotopy charges.



When making more ambient dimensions available, the cobordism classes eventually (quickly) exhibit **stabilization** on abelian cobordism cohomology groups. (This might relate *Hypothesis H* to Vafa's *cobordism conjecture* cf. [SS23-Mf, §4]).

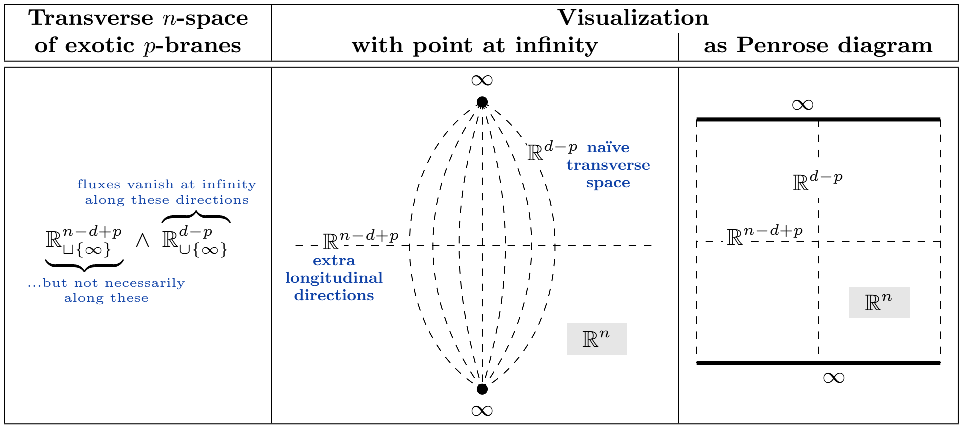


This "linearized" Cohomotopy/Cobordism is a **form of K-theory**: algebraic K-theory over the "absolute base field $\mathbb{F}_1$" (cf. [CLS12, Thm. 5.9]).



2.3.2 Exotic brane configurations via May-Segal's theorem

Cohomotopy charge in low codimension. The Pontrjagin theorem (§2.3.1) suggests that every solitonic brane seen by n-Cohomotopy "wants to be" a $d-n$-brane. Indeed, if the available transverse space is $< n$-dimensional as for exotic branes (e.g. the M9), then the *May-Segal theorem* [May72, Thm. 2.7][Segal73, Thm. 3] may be understood as saying that n-Cohomotopy still sees $(d-n)-$*branes*, but "delocalized" to look like exotic branes.



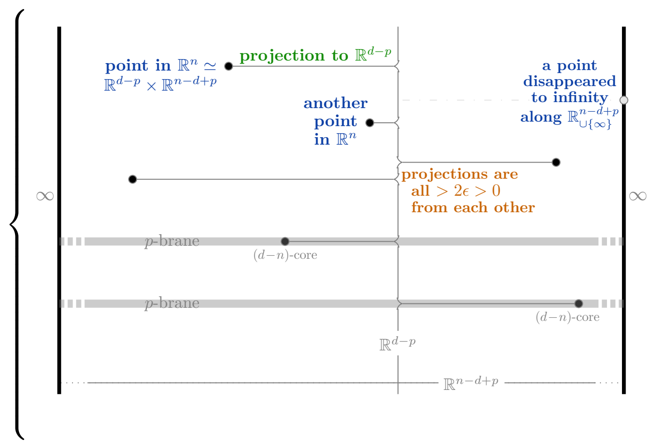
May-Segal's theorem indeed identifies the n-Cohomotopy moduli of $< n$-dimensional transverse spaces with the configurations of (unordered) points in $\mathbb{R}^n$ which are distinct as points in $\mathbb{R}^{d-p}$ (and as such look like transverse positions of flat p-branes) while their "core" may escape to infinity in the tangential direction, reflecting the fact that the n-Cohomotopy flux is not constrained to vanish at infinity in these directions.

$$
\mathrm{Maps}^{*/}\Big(\mathbb{R}^{n-d+p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{d-p}_{\cup\{\infty\}},\, S^n\Big) \;=:\; \widetilde{\boldsymbol{\pi}}^n\Big(\mathbb{R}^{1,d-n}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{n-d+p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{d-p}_{\cup\{\infty\}}\Big)
$$

(n-cohomotopical moduli space of flat solitonic p-brane worldvolumes with $< n$-dim. transverse space)

$$
\begin{array}{c}
\wr\!\!\mid \\
\mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}},\, S^n\Big) \\
\uparrow \wr \;\; \text{cohomotopy charge map (May-Segal theorem)} \\
\mathrm{Conf}\Big(\mathbb{R}^{d-p}_{\sqcup\{\infty\}},\, \mathbb{R}^{p-d+n}_{\cup\{\infty\}}\Big) =
\end{array}
$$

configuration space of points in transverse space $\mathbb{R}^{d-p}$ with "labels" in $\mathbb{R}^{n-d+p}_{\cup\{\infty\}}$



Notice the dichotomy: If branes *can not escape to* ∞ then their *fluxes vanish at* ∞ and vice versa. This is why in passing from the Cohomotopy charge to the corresponding brane configurations the subscripts swap as $(\text{-})_{\sqcup\{\infty\}} \leftrightarrow (\text{-})_{\cup\{\infty\}}$.

configuration space of points

$\mathrm{Conf}\big(\mathbb{R}^{d-p}_{\sqcup\{\infty\}}, \mathbb{R}^{n-d+p}_{\cup\{\infty\}}\big)$ is the pointed space of

- un-ordered tuples of points in $\mathbb{R}^n \simeq \mathbb{R}^{d-p} \times \mathbb{R}^{n-d+p}$ — as such they look like flat solitonic $d-n$-branes.
- which have pairwise distinct projections to $\mathbb{R}^{d-p}$ — as such they look like flat solitonic p-branes
- and may escape to or emerge from ∞ along $\mathbb{R}^{n-d+p}_{\cup\{\infty\}}$ — like partially de-localized $d-n$-brane solitons

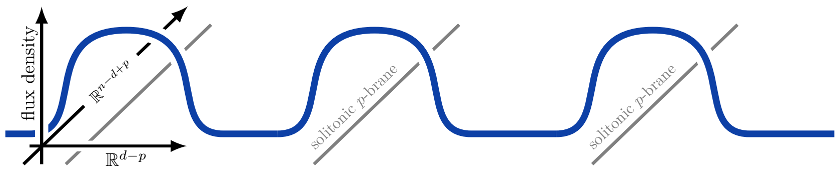
(NB: These configuration spaces are connected: The moduli are all in higher homotopy, invisible to traditional treatment.)

Pontrjagin's Cohomotopy charge map still exhibits the equivalence of the May-Segal theorem, now known as the inverse "electric field map" [Segal73, §1][McD75, §1] or "scanning map" and evaluated on the configuration space by [Segal73, §3], assigning to each point in the configuration the unit $(d-p)$-cohomotopy charge of a solitonic p-brane, but regarded after inclusion into the cohomotopy charge space of solitonic $(d-n)$-branes:

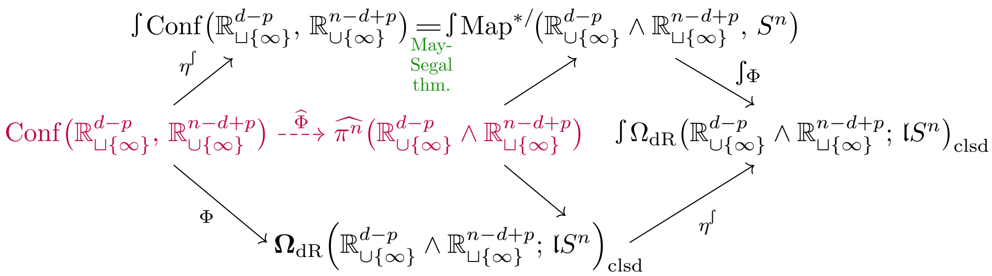
n-cohomotopy charge map for solitonic $p > d-n$-branes

$$
\begin{array}{ccccc}
\underset{\text{Moduli space of solitonic } p\text{-branes}}{\mathrm{Conf}\Big(\mathbb{R}^{d-p},\, \mathbb{R}^{n-d+p}_{\cup\{\infty\}}\Big)} & & \widetilde{\boldsymbol{\pi}}^{d-p}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) & & \widetilde{\boldsymbol{\pi}}^{n}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) \\
\uparrow & & \| & & \| \\
\underset{\text{single-brane subspace}}{\mathbb{R}^{d-p} \times \mathbb{R}^{n-d+p}} & \xrightarrow{\text{pure solitonic } p\text{-brane charge}} & \mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}},\, \mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) & \xrightarrow{\text{regarded as solitonic } d-n\text{-brane charge}} & \mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}},\, \mathbb{R}^{n}_{\cup\{\infty\}}\Big) \\
(x,\, y) & \longmapsto & \left(x' \mapsto \left\{ \begin{array}{ll} \frac{x'-x}{\exp\left(\frac{1}{(|x'-x|-\epsilon)^2}\right)} & \text{if } |x'-x|^2 < \epsilon \\ \infty & \text{otherwise} \end{array}\right.\right) & \longmapsto & \left(x' \mapsto \left\{ \begin{array}{ll} \frac{(x'-x,\, y)}{\exp\left(\frac{1}{(|x'-x|-\epsilon)^2}\right)} & \text{if } |x'-x| < \epsilon \\ \infty & \text{otherwise} \end{array}\right.\right)
\end{array}
\qquad (111)
$$

A cohomotopical ADHM construction. Pullback of the volume form on S^n along the Cohomotopy charge map (111) assigns to solitonic codim $< n$ branes (p. 84) their flux density, cf. eq. (2).

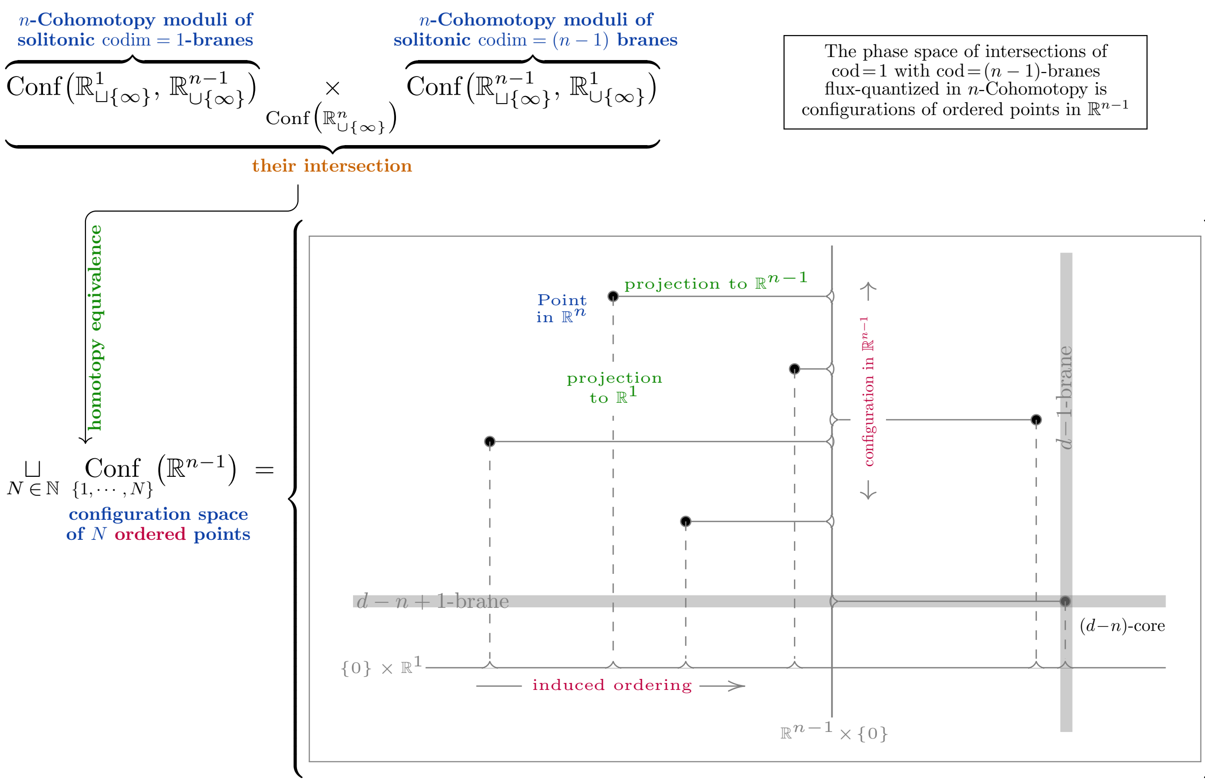


[SS24-Cnf]: This map Φ represents the cohomotopical character, and thus induces a shape-equivalence $\widehat{\Phi}$ to differential Cohomotopy, showing that the **configuration space is a gauge-fixed phase space of multi-core solitons** representing every solitonic Cohomotopy charge sector.



Intersecting solitonic brane charges in Cohomotopy. Noticing that the n-flux density arising this way has vanishing cup-square (simply by degree reasons in low codimension) hence behaves linearly, the gauge-fixed phase space of *intersecting* flat branes of low codimension must be the fiber product of these configuration spaces [SS22-Cnf, Ex. 2.3].

Gauge enhancement on domain wall intersections. In the special case that one of the intersecting brane species is of codimension=1 something remarkable happens [SS22-Cnf, Prop. 2.4. 2.11]: The fiber product of the "labelled" configuration spaces (p. 84) is homotopy-equivalent to a configuration space of *ordered* points in the remaining $n-1$ transverse dimensions that may no longer escape to ∞:



(112)

Now, the homotopy type of such configuration spaces where points are no longer allowed to escape to ∞ is quite rich (see eg. [Kn18]) considerably richer than that of the "labeled" configuration spaces on p. 84. With Hypothesis H this provides a substantiation of the expection of rich physics appearing on intersecting branes. We next check this by computing the lightcone quantum observables of these configurations.

Exotic branes. In the string theory literature, by a *non-standard brane* [BR12] or *exotic brane* [dBS13] one means, foremost, a p-brane species of low codimension $D-(1+p) \leq 2$ — which in M-theory generally corresponds to codimensions $D-(1+p) \leq 3$. (We observe that this are exactly the solitonic branes which have vanishing classical charge under Hypothesis H, according to (60).)

The most familiar examples of exotic branes in 10d string theory are the comparatively well-understood D_7 branes (of codimension 2, hence "defect branes" [BOR12]), the D_8-branes (of codimension 1, hence "domain walls") and the D_9 (codimension 0). Beyond these there is expected a plethora of further exotic branes (cf. [dBS13, Fig 1]) whose existence is argued indirectly by assumption of the famous but largely hypothetical "U-duality"-symmetry of string/M-theory, but which are not known to arise as supergravity solutions.

Already the low-codimension D-branes push the boundaries of common string theory lore: For instance the $\mathrm{SL}(2,\mathbb{Z})$-charges crucially meant to be carried by D_7-branes had no reflection in *Hypothesis K* (§1.3.1, a shortcoming which we argue is resolved by Hypothesis H, see §2.3.3) and the lift of D_8-branes to M-theory had remained at least subtle, even by informal arguments:

The problem with the D_8-brane is part of the general problem of integrating the relevant "massive" variant of type IIA string theory into the non-pertubative picture of M-theory, which fails in its most naive form since 11d supergravity provably does not admit the analogous "massive" deformation. While it has been argued that massive type IIA supergravity does arise from plain 11d supergravity, after all, by twisting the usual KK-compactification by U-duality transformations (which remain fairly conjectural themselves), the nature of the resulting lift of the D_8-brane, commonly called the M_9-brane, remained so elusive that more recent authors argued it should rather be called the M_8-*brane* to be understood only somewhat tautologically as 'an object that exists only as a lift of the D8-brane".[19]

Notice that the traditional *Hypothesis K* (§1.3.1) inherits all these conceptual problems since the "Romans mass"-term F_0 (the flux sourced by the D_8-brane) which is at the root of the problem is key part of its pre-geometric derivation (19). But in §2.3.3 we will argue in detail that and how the D_8-brane in its M-theoretic incarnation as an M_9/M_8-brane is implied by *Hypothesis H.*

Higher observables of exotic branes. Indeed, Hypothesis H implies right away, via (60), that there is *no* charge sourced by flat solitonic exotic branes (and most other admissible flux-quantization laws for the C-field would imply the same) but that (102) there are non-trival higher observables of exotic branes, encoded in the higher homotopy type of the Cohomotopy cocycle space

$$X \text{ a flat space(time) with a (base)point (at) } \infty \qquad \vdash \qquad \underbrace{\overset{\text{4-Cohomotopy flux moduli space}}{\widetilde{\boldsymbol{\pi}}^4(X)} \;:=\; \mathrm{Maps}^{*/}\big(X,\, S^4\big)}_{\text{pointed mapping space into 4-sphere}} \tag{113}$$

in that

$$\underbrace{\widetilde{\pi}^4\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{p\leq 3}_{\cup\{\infty\}}\Big) \;\simeq\; \pi_{p\leq 3}(S^4) \;=\; 0}_{\text{exotic solitonic M-branes carry no clasical charge}} \qquad \text{but} \qquad \underbrace{\widetilde{\boldsymbol{\pi}}^4\Big(\mathbb{R}^{1,p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{p\leq 3}_{\cup\{\infty\}}\Big) \;\neq\; *}_{\text{have non-trivial higher observables}}.$$

We will argue in §2.3.3 for the example of M_9-brane intersections that these higher observables on Cohomotopy moduli indeed reflect a wealth of expected patterns in D_8-brane intersections. This way, Hypothesis H seems to nicely capture the ethereal nature of exotic branes.

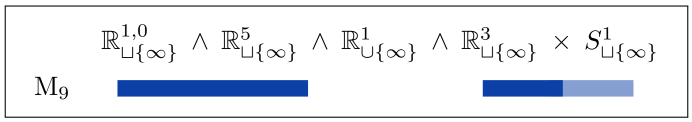
[19] [BMO18, p. 65]: "*However, as remarked in [OP99, p. 109], [the M9 brane] should more properly be called an M8-brane or perhaps KK8 following its mass formula designation* $8^{(1,0)}$. *It is, perhaps, to be understood as an object that exists only as a lift of the D8-brane of Type IIA.*"

Solitonic M_9-Branes. We obtain now from Hypothesis H a rigorous definition of the otherwise elusive M_9-brane (cf. p. 86) and can investigate it by mathematical analysis.

Namely, given that the (flat) M_9 is supposed to be:

1. solitonic
   (as there is no corresponding singular supergravity solution)
2. as such localized along a single transverse direction
   (since this is what it means to be a 9-brane in 11d)
3. but necessarily compactified on the M-theory circle S^1_A,
   (since it "exists only as a lift of the D_8-brane of type IIA", cf. ftn. 19)

the (flat) M_9-brane should, assuming Hypothesis H, be addressed as whatever it is that 4-Cohomotopy sees on the following spacetime domain:



(114)

Here the bar shows, in the tradition of brane diagrams, across which dimensions the M_9-brane is extended (which we have decomposed in anticipation of the brane intersections below in §2.3.3) — but the light shading is to indicate that this remains somewhat ambiguous – since we are dealing with an exotic brane of low codimension so that the Pontrjgin theorem (§2.3.1) does not apply, while also the May-Segal theorem (p. 84) does not quite apply, due to the presence of the $S^1_{\sqcup\{\infty\}}$-factor. Therefore one cannot *quite* translate the cohomotopical M_9-brane moduli on (114) into submanifolds in a spacetime domain. Of course, just such an ambiguity is expected for the M_9 brane (aka M_8-brane), cf. ftn. 19.

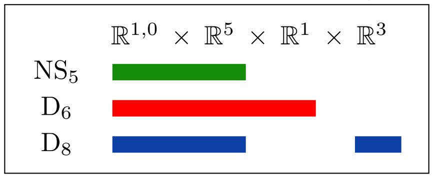
On the other hand, by the above discussion we can describe the M_9 moduli quite explicitly: They are given by loops in the configuration space of solitonic 6-branes which are delocalized in 3 transverse directions:

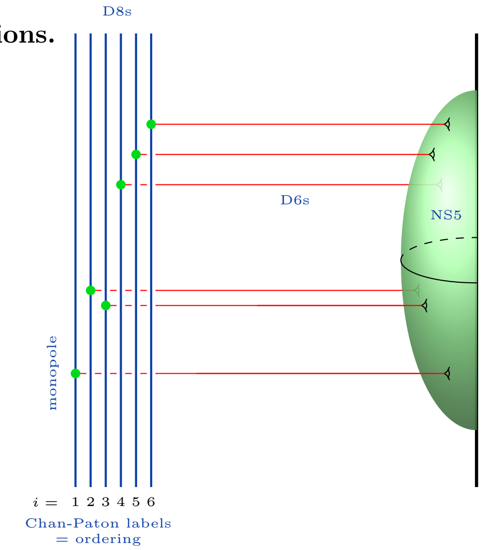
$$
\begin{aligned}
& \underbrace{\widetilde{\boldsymbol{\pi}}^4\big(\mathbb{R}^{1,8}_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}} \wedge \mathbb{R}^1_{\cup\{\infty\}}\big)}_{\text{M}_9\text{-brane moduli}} \\
& := \mathrm{Maps}^{*/}\Big(\mathbb{R}^{1,5}_{\sqcup\{\infty\}} \wedge \mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}},\, S^4\Big) && \text{charge moduli space (113) on } \mathrm{M}_9\text{-domain (114)} \\
& \simeq \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}} \wedge S^1_{\sqcup\{\infty\}},\, S^4\big) && \mathbb{R}^{1,5} \text{ is contractible} \\
& \simeq \mathrm{Maps}^{*/}\Big(S^1_{\sqcup\{\infty\}},\, \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge \mathbb{R}^3_{\sqcup\{\infty\}},\, S^4\big)\Big) && \text{mapping space adjunction (39)} \\
& \simeq \mathrm{Maps}^{*/}\Big(S^1_{\sqcup\{\infty\}},\, \mathrm{Conf}\big(\mathbb{R}^1_{\sqcup\{\infty\}},\, \mathbb{R}^3_{\cup\{\infty\}}\big)\Big) && \text{May-Segal theorem (p. 84)} \\
& = \underbrace{\mathcal{L}\,\mathrm{Conf}\Big(\mathbb{R}^1_{\sqcup\{\infty\}},\, \mathbb{R}^3_{\cup\{\infty\}}\Big)}_{\substack{\text{loop space of configuration space} \\ \text{of 3-fold delocalized 6-branes}}}
\end{aligned}
\tag{115}
$$

In order to say more about the M_9-brane, we need to see it interact (intersect) with other branes. This we turn to in §2.3.3 below.

2.3.3 Quantum $D_6 \perp D_8$-branes via Fadell-Husseini's theorem

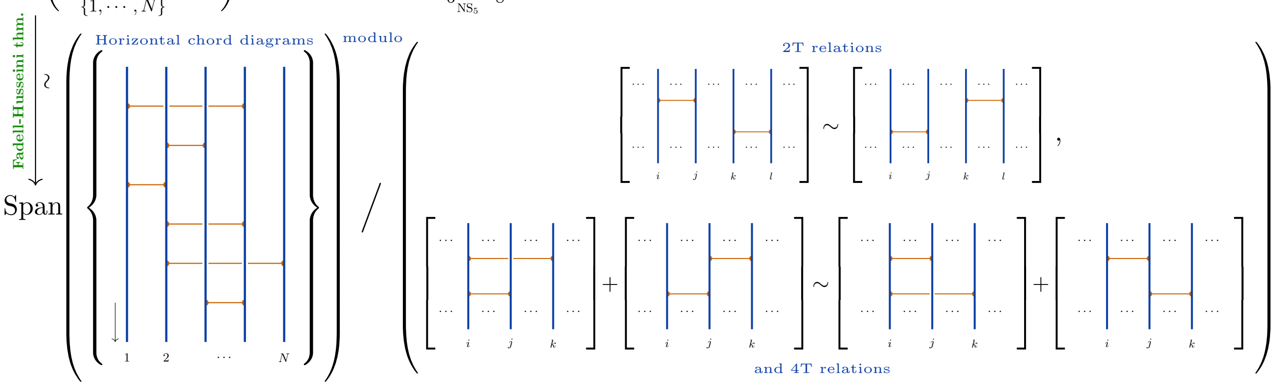
Cohomotopy Moduli of Hanany-Witten brane configurations. Consider the situation (112) for M-theory on S^1 with $D_6 \perp D_8$-intersections on NS_5-cores: hence for $n = 4,\ d = 9,\ p = 6.$ [SS22-Cnf, Rem. 2.14]



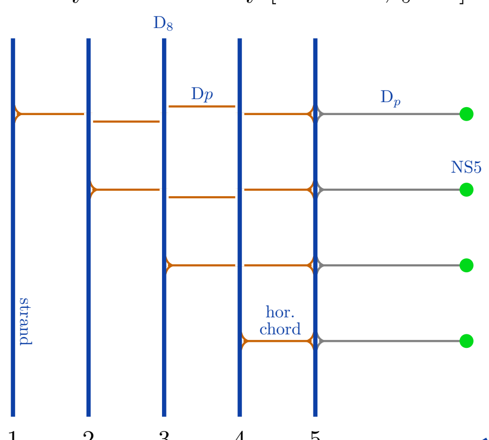


Quantum observables on Hanany-Witten configurations. It follows with the discussion in §2.2 that the light-cone quantum observables on these brane configurations form, for each number N of D_8-branes, the homology Pontrjagin-algebra of the based loop space of the ordered configuration space (112). Remarkably, by the Fadell-Husseini theorem [FH01, Thm. 2.2] this is isomorphic to the algebra of *horizontal chord diagrams* on N-strands modulo the "2T- and 4T-relations" [SS22-Cnf, Prop. 2.18]:

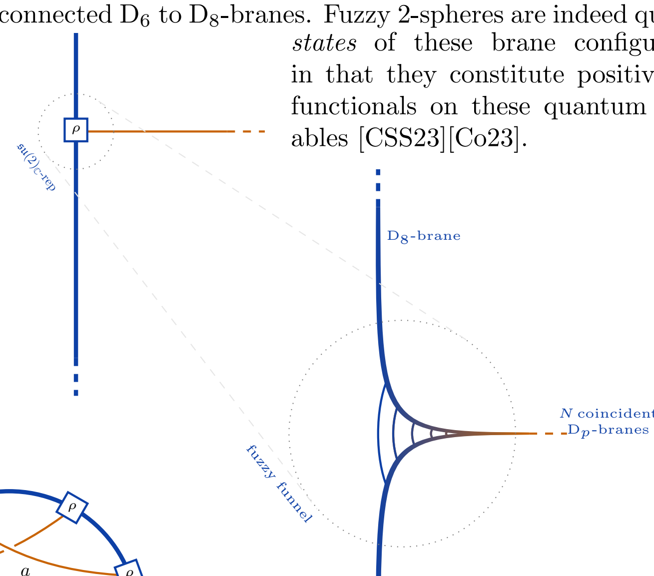
$$H_\bullet\Big(\Omega \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big) \overset{(112)}{\simeq} \mathrm{QObsrvbs}_{D_6 \underset{NS_5}{\perp} D_8}$$



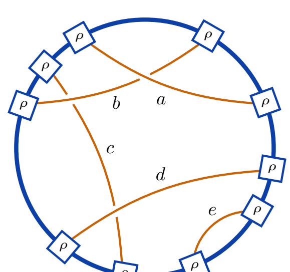
In their classical limit (graded-symmetric chord algebra) these relations match the **brane intersection rules** expected in Hanany-Witten theory [SS22-Cnf, §4.10]:



Generally, chord diagrams serve as observables on the geometry of fuzzy 2-spheres [SS22-Cnf, §4.2] as expected for **fuzzy funnels** connected D_6 to D_8-branes. Fuzzy 2-spheres are indeed quantum *states* of these brane configurations, in that they constitute positive linear functionals on these quantum observables [CSS23][Co23].



Typical value of a quantum state on chord diagrams (a "weight system") in Penrose notation.



$$= \mathrm{Tr}\big(\rho_a \cdot \rho_b \cdot \rho_c \cdot \rho^b \cdot \rho_d \cdot \rho^c \cdot \rho_a \cdot \rho^e \cdot \rho^d\big)$$

M2/M5-Brane states in their matrix model. From the 11d perspective, these $D_6 \perp D_8$ configurations are – as anything classified by 4-Cohomotopy in 11d – certain M_2/M_5-brane states, as also suggested by the expected string theory dualities (cf. [BLMP13, p. 37]).

$$\overbrace{\Big(\bigoplus_i N_i^{(M2)} \cdot \underbrace{\mathbf{N}_i^{(M5)}}_{\dim = N^{(M5)}\text{ irrep}}\Big)}^{\text{susy ground state of } M_2/M_5 \text{ according to BMN matrix model}} \in \mathfrak{su}(2)_{\mathbb{C}}\mathrm{Mod}_{/\sim}$$

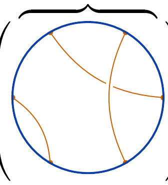
$$N := \sum_i N_i^{(M2)} N_i^{(M5)} \in \mathbb{N}$$

	M2-branes	*M5-branes*
If for all i:	$N_i^{(M5)} \to \infty$	$N_i^{(M2)} \to \infty$
with fixed	$N_i^{(M2)}$	$N_i^{(M5)}$
and fixed	$N_i^{(M5)}/N$	$N_i^{(M2)}/N$

Traditionally, the *BMN matrix model* — which is meant to be the lightcone quantization of Membranes on (Penrose limits of) singular M_2/M_5-brane backgrounds — suggests [MSJVR03][AIST17][AIST18] that the supersymmetric quantum ground states of transverse M_2/M_5-brane bound states are fuzzy 2-spheres, namely $\mathfrak{su}(2)$-modules.

With quantum Hypothesis H we find these $\mathfrak{su}(2)$-modules as quantum states of branes such that these limits make sense: Namely as weight systems on chord diagrams [SS22-Cnf, §4.9]

$$\overbrace{\frac{4\pi\, 4^n}{\Big(\big(N^{(M5)}\big)^2 - 1\Big)^{1/2+n}} w_{\mathbf{N}^{(M5)}}}^{\substack{\text{M2-brane state in BMN model} \\ \text{(multiple of } \mathfrak{su}(2)_{\mathbb{C}}\text{-weight system)}}} \left(\overbrace{\text{[round chord diagram]}}^{\substack{\text{Single-trace observable} \\ \text{(round chord diagram)}}} \right) = \frac{4\pi}{\sqrt{\big(N^{(M5)}\big)^2 - 1}} \mathrm{Tr}_{\mathbf{N}^{(M5)}} \big(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c \big)$$



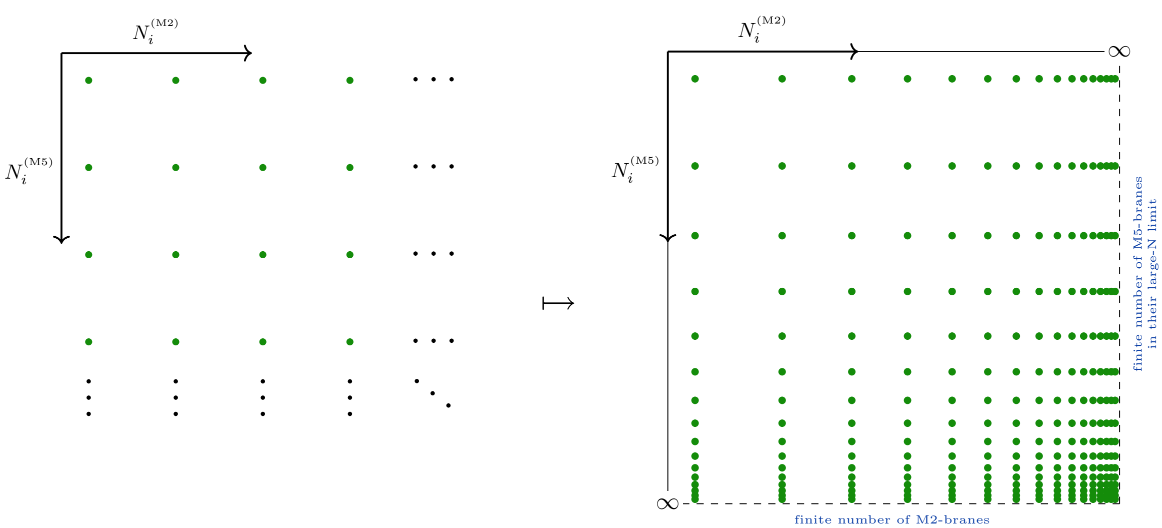
$$= \int_{S^2_{N^{(M5)}}} (R^2)^3 \,\text{[chord diagram]} \qquad \substack{\text{M2-brane quantum fluctuation amplitude} \\ \text{(fuzzy 2-sphere shape coefficient)}}$$

$$\underbrace{\mathfrak{su}(2)_{\mathbb{C}}\mathrm{MetMod}_{/\sim}}_{} \xrightarrow[\substack{\text{M2-M5-brane bound states} \\ \text{(normalized Lie algebra weights)}}]{\Psi_{(-)}} \prod_{n \in \mathbb{N}} \mathcal{W}^n$$

(Fuzzy 2-sphere geometries (metric representations of $\mathfrak{su}(2)_{\mathbb{C}}$); Supersymmetric states of BMN matrix model (weight systems on Sullivan chord diagrams))

$$\wr\!\!\downarrow \qquad\qquad \uparrow$$

$$\left\{ \bigoplus_{i\in\mathbb{N}} \Big(\overbrace{\underbrace{N_i^{(M2)}}_{\text{multiplicity}} \cdot \underbrace{\mathbf{N}_i^{(M5)}}_{\substack{\text{irrep of} \\ \dim_{\mathbb{C}} = N_i^{(M5)}}}}^{\substack{\text{Charges carried by} \\ i\text{th stack of branes}}} \Big) \;\middle|\; \underset{\in \bigoplus_{i\in\mathbb{N}} (\mathbb{N}\times\mathbb{N})}{\big\{ (N_i^{(M2)}, N_i^{(M5)}) \big\}_{i\in\mathbb{N}}} \right\} \longrightarrow \left\{ \overbrace{\frac{1}{\sum_{i\in\mathbb{N}} N_i^{(M2)}} \sum_{i\in\mathbb{N}}}^{\text{Mixture}} \overbrace{\frac{N_i^{(M2)}\, 4\pi\, 2^{2n}}{\Big(\big(N_i^{(M5)}\big)^2 - 1\Big)^{1/2+n}}}^{\text{Normalized radii}} \overbrace{w_{\mathbf{N}_i^{(M5)}}}^{\text{Lie weights}} \;\middle|\; \underset{\in \bigoplus_{i\in\mathbb{N}} (\mathbb{N}\times\mathbb{N}_{\geq 1})}{\big\{ (N_i^{(M2)}, N_i^{(M5)}) \big\}_{i\in\mathbb{N}}} \right\} \Big/ \sim$$



(116)

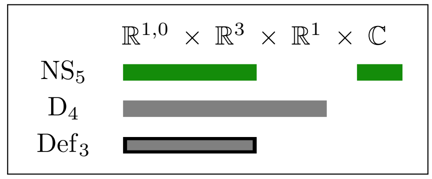
Example 2.1 (M_5/M2-Brane states)**. Light-cone quantum states of** $\mathrm{M}_5/\mathrm{M}_2$**-Brane bound states.** In the vein that, for $n = 4$, (112) gives the moduli of (nearly) coincident solitonic quantum M_5-branes in M(embrane)-theory, we should find the light-cone quantum observables from p. 88 also to match those in the membrane matrix model (p. 73) describing M_5-branes.

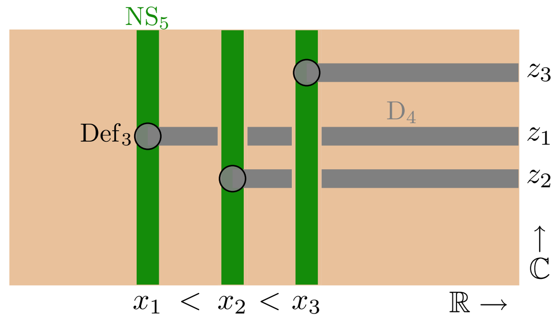
This is indeed the case [SS22-Cnf, §4.9]: The light-cone quantum ground states of the BMN matrix model are superpositions of fuzzy 2-spheres, and in suitable arrangements these encode either pure M_5-brane states or generally $\mathrm{M}_2/\mathrm{M}_5$-brane bound states [BMN02, (5.5)][MSvR03][AIST17][AIST18]. In particular, the quantum state given by multiples of the $N^{(M5)}$-dimensional irrep of $\mathfrak{su}(2)_{\mathbb{C}}$ should correspond to $N^{(M5)}$ 5-branes in the ground state.

No-ghost theorem for M_5**-branes.** Therefore, with the light-cone quantum mechanics established in §2.2, we may ask which of these light-cone quantum states are proper quantum states in that they are positive (non-ghost) states. In [SS22-Cnf, Ex. 3.11][CSS23] we showed that, beyond the trivial case of $N^{(\mathrm{M}_5)} = 1$ this is the case for fundamental representation with $N^{(M5)} = 2$ [SS22-Cnf, Ex. 3.5][CSS23]. In [Co23] it is claimed that the same conclusion still holds for all symmetric and exterior powers of the fundamental representation, which would establish the positivity (no-ghost theorem) for the corresponding mixed $\mathrm{M}_2/\mathrm{M}_5$-brane bound states.

2.3.4 Quantum $\mathrm{M}_5 \perp \mathrm{M}_5$-branes via Gelfand-Raikov theorem.

Now consider moduli of the 3-cohomotopical H_3 flux, which in §2.1 we saw appears on the ambient space[20] of 5-branes. For the moduli space of codim=2 $\mathrm{D}_4 \perp \mathrm{NS}_5 \overset{\mathrm{IIA/M}}{\rightsquigarrow} \mathrm{M}_5 \perp \mathrm{M}_5$ defects inside 5-branes [CHKS21, Fig. 1 & 3][SS23-Dfc1, pp. 28], we are to consider the situation (112) for $n = 3,\ d = 7,\ p = 4$, which yields configurations of ordered points in the transverse $\mathbb{C}$-plane.

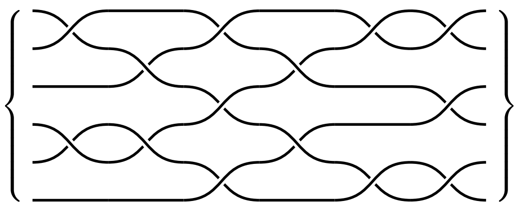




To understand the light-cone quantization (§2.2) of these brane moduli, observe that the homotopy type of this configuration space is the classifying space of the pure[21] **braid group** PBr (cf. [MySS23, pp. 12]), being the *group of motions* of the codim = 2 defects (Def_3) around each other in the transverse M_5-worldvolume $\mathbb{C}$.

$$\pi_1\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})\Big) \;\simeq\; \mathrm{PBr}(n)$$

$$=$$

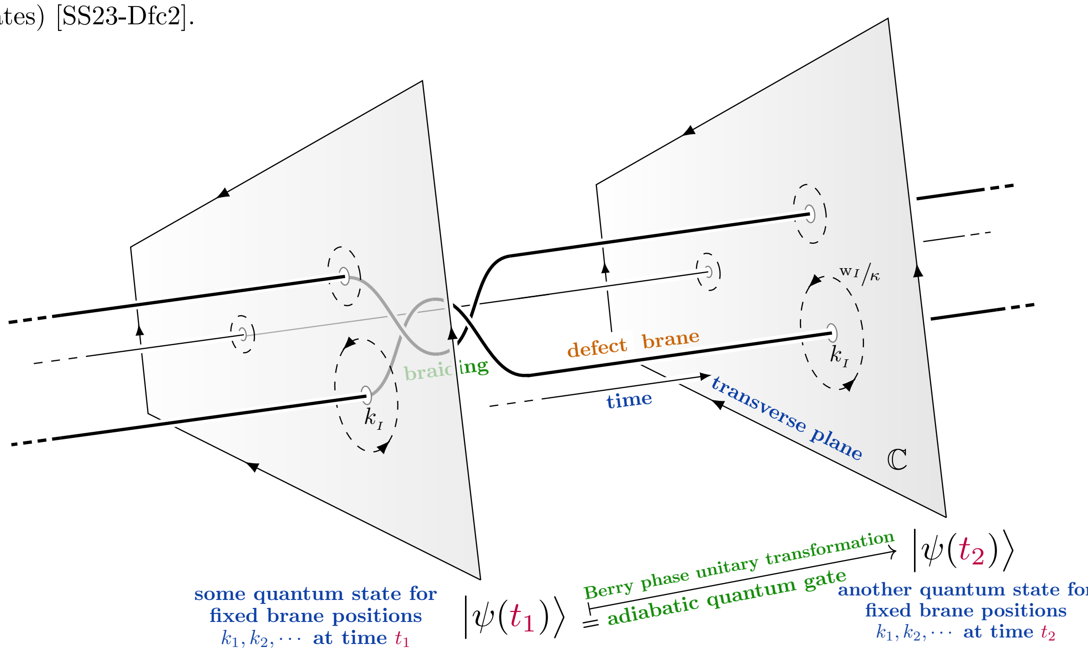


This implies that the light-cone quantum observable algebra (107) is the pure braid group algebra.

$$\begin{aligned} &\mathrm{QObsrvbls}_{N\mathrm{M}_5\perp\mathrm{M}_5} \\ &\equiv\; H_\bullet\big(\Omega \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C})\big) \;\simeq\; H_\bullet\big(\Omega B\mathrm{PBr}(N)\big) \\ &\simeq\; H_\bullet\big(\mathrm{PBr}(N)\big) \;\simeq\; \mathbb{C}\big[\mathrm{PBr}(N)\big] \quad \textbf{group algebra} \end{aligned}$$

The Quantum states. Thus with the *Gelfand-Raikov theorem* [Di77, Thm. 13.4.5.(ii)] it follows that the light-cone quantum states are given by unitary pure braid representations, hence are **anyonic states** ("topologically ordered" quantum states) [SS23-Dfc2].

$$\mathrm{QStates}_{N\mathrm{M}5\perp\mathrm{M}5} \;\simeq\; \left\{ \rho : \mathbb{C}\big[\mathrm{PBr}(N)\big] \to \mathbb{C} \;\middle|\; \begin{array}{l} g \mapsto \rho(g) = \langle\psi|U(g)|\psi\rangle \\ \text{for } U \in \mathrm{PBr}(N) \circlearrowright \mathcal{H},\ |\psi\rangle \in \mathcal{H} \end{array} \right\}.$$



[20]Hence with the M-theory circle included, the ambient space of the 5-brane on which we consider the H_3-flux is 8-dimensional.
[21]Our figures show im-pure braids, just for ease of illustration.

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2.4 Resulting worldvolume CFT

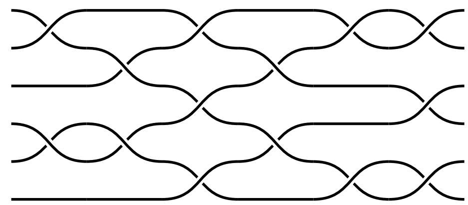
We have seen in example 2.1 that general light-cone quantum states of M_5-brane insertions transverse to a complex plane in an ambient M_5-brane are elements of unitary representations of the pure braid group. Here we discuss the specific such representation states that are singled out by the cohomology of the transverse phase space (103), which turn out to be given by the conformal blocks of su_2-affine conformal field theory.

This result is from [SS23-Dfc1], implications are developed in [SS23-Dfc2][TQC1][TQC2].

2.4.1 Conformal blocks of M5-observables

The spectral prequantum line bundle. These configuration spaces are non-simply connected: their fundamental group is the pure[22] *braid group* — being the *group of motions* of the M5 ⊥ M5-intersections around each other in the ambient M5-worldvolume:

$$\pi_1\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C}^2)\Big) \;\simeq\; \mathrm{PBr}(n) \;=\; \Big\{ \quad \Big\} \tag{117}$$



Hence the corresponding twisted ordinary cohomology (aka: "local system cohomology") is that whose cocycles are sections of "Eilenberg-MacLane-spectrum line bundles" pulled back from the classifying space $B\mathbb{Z}/\kappa$ of a cyclic group:

$$\underset{\substack{\text{twisted cohomology}\\ \text{of configuration space}\\ \text{quantum state space}}}{H^{[\omega_1]}\Big(\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C}^2)\Big)} \;=\; \left\{ \begin{array}{ccc} & & \overset{\text{local coefficient bundle}}{B^k\mathbb{C}/\!\!/\mathbb{Z}/\kappa} \\ & \overset{\text{twisted cocyle}}{\underset{\text{quntm state}}{\nearrow}} & \downarrow \\ \underset{\text{phase space}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})} & \xrightarrow{\;\omega_1\;} & \underset{\text{prequantum line bundle}}{B\mathbb{Z}/\kappa} \end{array} \right\}_{/\mathrm{hmtp}} \tag{118}$$

(Closer analysis reveals [SS23-Dfc1, §3] that κ equals the order of the $\mathbb{A}_{\kappa-1}$-singularity at which dual D7/D3-branes are placed.)

In order to analyze these quantum states, we may decompose the problem by:

(1.) holding fixed N of the branes,
(2) letting n mobile branes move around them.

$$\begin{array}{ccc} \overset{\substack{n\text{-configuration space}\\ \text{of } N\text{-punctured plane}}}{\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big)} & \longrightarrow & \overset{\substack{\text{fibration of}\\ \text{configuration spaces}}}{\underset{\{1,\cdots,N+n\}}{\mathrm{Conf}}(\mathbb{C}^2)} \\ \downarrow & (\mathrm{pb}) & \downarrow{\scriptstyle \text{forget } n \text{ points}} \\ * & \xrightarrow[\text{pick } N\text{-configuration}]{(z_1,\cdots,z_N)} & \underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\mathbb{C}^2) \end{array} \tag{119}$$

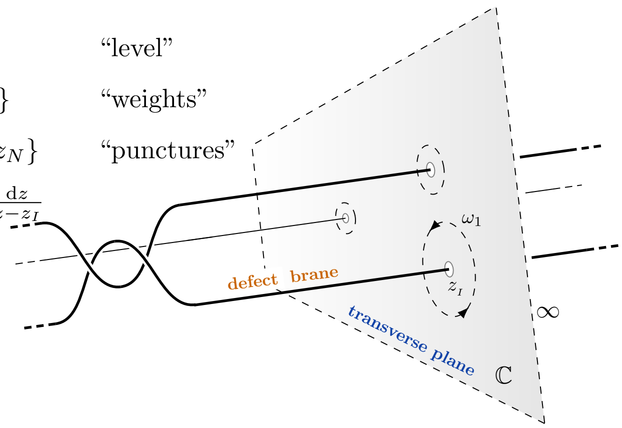
In the simple case of a single mobile brane moving – along a dashed line in (121) – among N fixed branes, we have

$$\underset{\{1,\cdots,1\}}{\mathrm{Conf}}\big(\mathbb{C}\setminus\{z_1,\cdots,z_N\}\big) \;=\; \mathbb{C}\setminus\{z_1,\cdots,z_N\} \tag{120}$$

and the twist ω_1 (118) is fixed by:

[22] Our figures show im-pure braids just for ease of illustration.

$$
\begin{array}{llll}
\kappa & := & k+2 & \text{“level”} \\
\mathrm{w}_I & \in & \{0, \cdots, k\} & \text{“weights”} \\
z_I & \in & \{z_1, \cdots, z_N\} & \text{“punctures”} \\
\text{as } \omega_1 & := & \sum_I -\frac{\mathrm{w}_I}{\kappa} \frac{\mathrm{d}z}{z - z_I} &
\end{array}
\tag{121}
$$



Brane states identified with worldvolume correlators. Curiously, such sets of labels coincide with those of “conformal blocks” – namely chiral correlation functions – in the $\widehat{\mathfrak{su}_2}^k$-*conformal quantum field theory* on the punctured Riemann sphere

$$
\mathbb{C}P^2 \setminus \big\{ z_1, \cdots, z_N, \infty \big\} \;\; \simeq \;\; \mathbb{C}^2 \setminus \big\{ z_1, \cdots, z_N \big\}\,. \tag{122}
$$

And indeed, a well-but-not-widely known theorem called the *hypergeometric integral construction* identifies these conformal blocks of “degree=1” inside the twisted cohomology (118) of the punctured plane (122)

$$
\begin{array}{ccc}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=1}}{\mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})}} & \xhookrightarrow[\text{inclusion}]{\text{natural}} & \overset{\text{1-twisted deg=1}}{\underset{}{\overset{\text{de Rham cohomology}}{H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C} \setminus \{\vec{z}\}\big),\, \mathrm{d} + \omega_1 \wedge \Big)}}} \\
& \xhookrightarrow[\text{inclusion}]{\text{natural}} & \underset{\text{K-theory of } \mathbb{A}_{\kappa-1}\text{-singularity}}{\underset{\text{inner local system-twisted deg=1}}{\mathrm{KU}^{1+\omega_1}\Big(\big(\mathbb{C} \setminus \{\vec{z}\}\big) \times * /\!\!/ C_\kappa;\, \mathbb{C}\Big)}} \quad \text{[SS23-Dfc1, Prop. 2.16]}
\end{array}
\tag{123}
$$

and generally the conformal blocks of any degree n inside n-configuration space of points, if we set

$$
\omega_1 := \sum_{1 \leq i \leq n} \sum_{I} -\frac{\mathrm{w}_I}{\kappa} \frac{\mathrm{d}z}{z - z_I} + \sum_{1 \leq i < j \leq n} \frac{2}{\kappa} \frac{\mathrm{d}z}{z^i - z^j} \quad \text{on} \quad \underset{\{1, \cdots, n\}}{\mathrm{Conf}} \big(\mathbb{C} \setminus \{\vec{z}\}\big)\,. \tag{124}
$$

namely:

$$
\begin{array}{ccc}
\underset{\text{conformal blocks}}{\overset{\mathfrak{su}(2)\text{-affine deg=}n}{\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z})}} & \hookrightarrow & \overset{\text{1-twisted deg=}n\text{ de Rham cohomology}}{\overset{\text{of configuration space of } n \text{ points}}{H^n\bigg(\Omega^\bullet_{\mathrm{dR}}\Big(\underset{\{1, \cdots, n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{\vec{z}\}\big)\Big),\, \mathrm{d} + \omega_1 \wedge \bigg)}} \\
& \hookrightarrow & \underset{\text{of configurations in } \mathbb{A}_{\kappa-1}\text{-singularity}}{\underset{\text{inner local system-twisted deg=}n\text{ K-theory}}{\mathrm{KU}^{n+\omega_1}\bigg(\Big(\underset{\{1, \cdots, n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{\vec{z}\}\big)\Big) \times * /\!\!/ C_\kappa;\, \mathbb{C}\bigg)}} \quad \text{[SS23-Dfc1, Thm. 2.18]}
\end{array}
\tag{125}
$$

Concretely, this inclusion is given by sending the canonical basis elements of conformal blocks to “Slater-determinant”-like expressions, as follows:

$$
\begin{array}{ccc}
\mathrm{CnfBlck}^n_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) & \xhookrightarrow{\qquad\qquad} & H^n\bigg(\Omega^{\bullet,0}_{\mathrm{dR}}\Big(\underset{\{1, \cdots, n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{\vec{z}\}\big)\Big)\Big|_{\overline{\partial}=0},\, \partial + \omega_1(\vec{\mathrm{w}}, \kappa) \wedge \bigg) \\
f_{I_1} \cdots f_{I_n} | v^0_1 \cdots, v^0_N \rangle & \underset{\text{generators}}{\longmapsto} & \bigg[\det\Big(\Big(\frac{\mathrm{w}_{I_j}}{\kappa} \frac{1}{z^i - z_{I_j}}\Big)_{i,j=1}^{n}\Big)\, dz^1 \wedge \cdots \wedge dz^n\bigg] \\
\text{e.g.} \quad f_I f_J | v^0_1 \cdots, v^0_N \rangle \;=\; \big[\cdots, (f \cdot v^0_I), \cdots, (f \cdot v^0_J), \cdots\big] & \longmapsto & \Big[\frac{\mathrm{w}_I}{\kappa} \frac{dz^1}{(z^1 - z_I)} \wedge \frac{\mathrm{w}_J}{\kappa} \frac{dz^2}{(z^2 - z_J)} + \frac{\mathrm{w}_I}{\kappa} \frac{dz^2}{(z^2 - z_I)} \wedge \frac{\mathrm{w}_J}{\kappa} \frac{dz^1}{(z^1 - z_J)}\Big]\,.
\end{array}
\tag{126}
$$

In summary, we have derived, from Hypothesis H, that:

$$\left.\begin{array}{c}\text{Quantum states of}\\ \text{brane configurations}\\ \text{inside an M-theoretic bulk}\end{array}\right\} \text{ are identified with } \left\{\begin{array}{c}\text{Quantum correlators of}\\ \text{a conformal field theory}\\ \text{on their worldvolume}\end{array}\right. \tag{127}$$

This is just the form of "holographic duality" that is expected in string/M-theory, here specifically in "Theory-$\mathcal{S}$"-compactifications of M5-branes on Riemann surfaces such as (122). Our result that $\mathfrak{su}_2$-conformal blocks appear on M5-branes compactified on a Riemann surface matches the conclusion in [Wi10, p. 22].

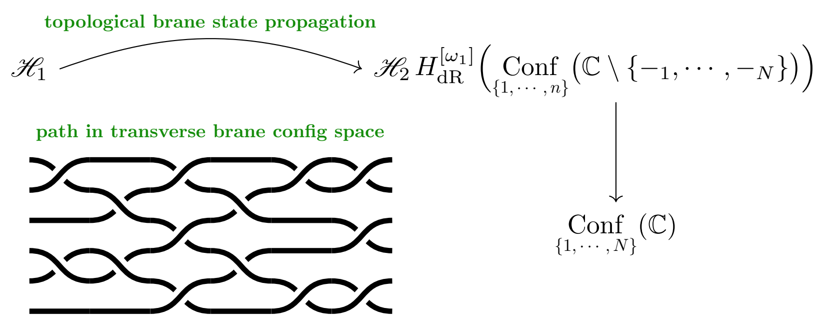
Strongly coupled holographic quantum materials. In [SS23-Dfc2] we give a detailed argument that the worldvolume CFT which we see here is that of *anyonic defects* in *topologically ordered ground states* of *crystalline quantum materials* which are in a *topological phase of matter*.

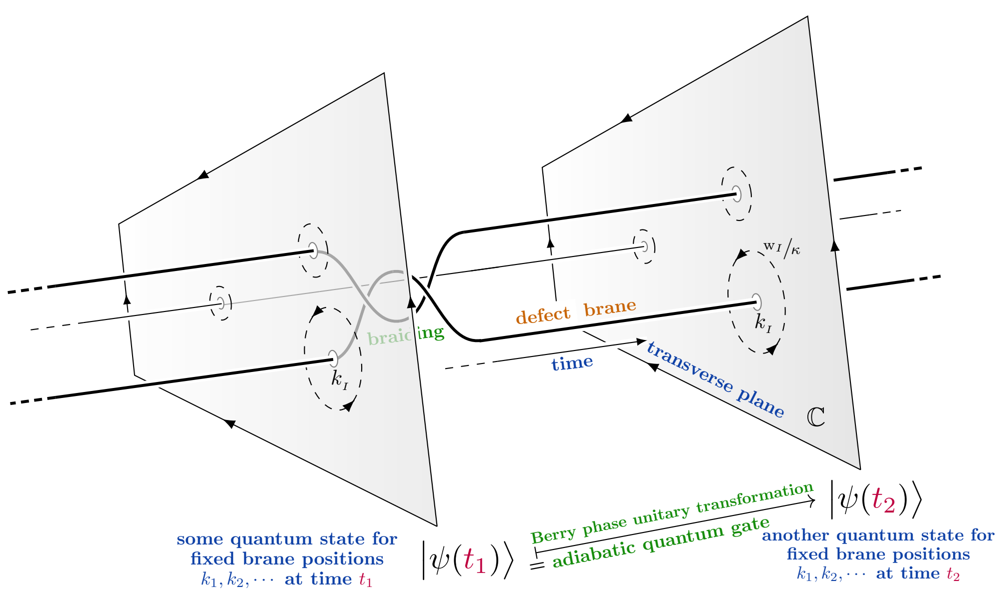
This being a strongly coupled QFT on a *small* number κ of branes, it is outside the realm of perturbative string theory and would indeed be expected to require M-theory for its holographic description (cf. p. 3).

2.4.2 Anyon braiding

We close by indicating how the "topological dynamics" of M5 $\perp$ M5 (their adiabatic movement in moduli space) acts on their quantum states just as expected for *quantum logic gates* in *topological quantum computers* based on *anyon braiding* – as they should by the duality (127). Detailed discussion may be found in [TQC1][TQC2].

Modular functor of M5 $\perp$ M5 **Hilbert spaces.** The fibration of configuration spaces (119) induces on its fiberwise twisted cohomology groups (125) a flat connection — called a *Gauss-Manin connection*, which on the spaces of conformal blocks restricts to the *Knizhnik-Zamolodchikov connection*.
The parallel transport of this connection computes the unitary transformations on the branes' quantum states induced by their adiabatic movement in moduli space:





Under the above holographic duality, such brane braiding translates to the braiding of anyonic defects in topologically ordered quantum materials, which is thought to potentially serve as quantum logic gates for topological quantum computers.

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