

# SYNTHETIC DIFFERENTIAL JET BUNDLES ARE REDUCED

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ABSTRACT. We have previously observed that the theory of solutions of partial differential equations, regarded as *diffieties* inside jet bundles, acquires a powerful comonadic formulation after passage from the category of Fréchet smooth manifolds to the *Cahiers topos of formal smooth sets* (a well-adapted model for Synthetic Differential Geometry). However, the tacit assumption that this passage preserves the projective limits that define infinite jet bundles had remained unproven. Here we provide a detailed proof.

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## 1. INTRODUCTION

The tenet of *synthetic differential geometry* (SDG, cf. [Koc79; Lav96; Koc10]) is that concepts and results in differential geometry often find a more natural and powerful formulation after faithfully embedding the category of smooth manifolds into a suitable *topos* (cf. [MM92; Bor94]) where *infinitesimal spaces* exist as actual objects.

Such a “well-adapted” SDG topos is the *Cahiers topos* due to E. Dubuc ([Dub79], cf. [GS25b, §2]), which we may think of as the category of *formal smooth sets* (recalled in § A.1 — here “formal” is common jargon for “infinitesimally thickened” or “haloed”). This is a differential-geometric analog of working with *formal schemes* in algebraic geometry.

One striking example is from the geometry of partial differential equations. On top of describing classical solutions, SDG can treat formal (power series) solutions of partial differential equations as primary objects. Classically, these formal solutions appeared in the guise of *diffieties* inside infinite jet bundles (a very fruitful perspective due to [Vin81], brief background is recalled in § A.3). Their appearance

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in SDG turns out to become largely structural (namely *comonadic*) when viewed in the synthetic context of formal (infinitesimally thickened) smooth sets. This is the result of [KS26]. However, a key technical step in this identification had remained unproven, recalled as Thm. 2.1 below:

Namely, with *infinite jet bundles* (A.9) being projective limits of finite-dimensional manifolds, there are *a priori* two different ways of regarding them among formal smooth sets: Either by first taking the limit (in Fréchet manifolds) and then passing to formal smooth sets, or the other way around! We prove here that that these two constructions are naturally isomorphic.

This has crucial implications for applications in variational calculus and in Lagrangian field theory, cf. Cor. 2.2 below.

The result may be “intuitively expected” to be true, but proving it is not entirely trivial. Here we provide a detailed and largely constructive proof. Besides filling this particular gap in the previous literature, our proof may be interesting for the technique it uses, which seems to be new, and may be adapted to other situations, cf. Rem. 2.7 below.

## 2. STATEMENT AND PROOF

We write  $i_!$  (Def. A.7) for the inclusion of smooth sets — such as Fréchet manifolds (Lem. A.9) — among formal smooth sets.

**Theorem 2.1.** *For  $E \rightarrow \Sigma$  a finite-dimensional surjective submersion of smooth manifolds, passage to formal smooth sets preserves the projective limit involved in forming its infinite jet bundle  $J_\Sigma^\infty E$  (cf. § A.3), in that there is a natural isomorphism:*

$$(2.1) \quad i_! \left( \varprojlim_{k \in \mathbb{N}} J_\Sigma^k E \right) \simeq \varprojlim_{k \in \mathbb{N}} \left( i_! J_\Sigma^k E \right).$$

By the characterization [KS26, Prop. 2.29; GS25a, Cor. 3.6] of smooth maps out of Fréchet projective limit manifolds (“locally pro-manifolds”), Thm. 2.1 implies:

**Corollary 2.2.** *The maps of formal smooth sets from a synthetic jet bundle  $J_\Sigma^\infty E$  to a finite-dimensional manifold  $F$  is in canonical bijection with point-set smooth maps which are locally of finite jet order (as considered in [Tak79]).*

We prove Thm. 2.1 below after establishing a few lemmas, based on the following recognition principle for reduced objects among formal smooth sets:

**Lemma 2.3.** *An object  $X \in \text{FrmSmthSet}$  is reduced (A.2) iff the following two conditions hold:*

- (i) *All germs of maps out of a  $U \times \mathbb{D} \in \text{FrmCrtSp}$  into  $X$  factor through some  $V \in \text{CrtSp}$ :*

$$(2.2) \quad \forall U \times \mathbb{D} \begin{array}{c} \xrightarrow{\quad \vee \quad} \\ \dashrightarrow \exists \iota \quad \dashrightarrow V \quad \dashrightarrow \exists f \quad \dashrightarrow X \end{array}$$

(ii) The equivalence relation on the set of such factorizations  $(\iota, f)$  which is generated by the relation

$$(2.3) \quad (\iota, f) \sim (\iota', f') \Leftrightarrow U \times \mathbb{D} \begin{array}{ccc} \xrightarrow{\iota} & V & \xrightarrow{f} \\ & \exists! \downarrow & \nearrow \\ \xrightarrow{\iota'} & V' & \xrightarrow{f'} \end{array} X$$

is that of equal composites, hence:

$$(2.4) \quad (\iota, f) \sim \dots \sim (\iota', f') \Leftrightarrow f \circ \iota = f' \circ \iota'.$$

*Proof.* This is an immediate reformulation of the following *coend* formula for the left Kan extension on presheaves (cf. [Kel82, p. 4.25; Lor21, (2.27)]):

$$(2.5) \quad (i_! X)(U \times \mathbb{D}) \simeq \int^{V \in \text{CrtSp}} \text{FrmCrtSp}(U \times \mathbb{D}, V) \times X(V),$$

which (seen under the Yoneda lemma) says that maps  $U \times \mathbb{D} \rightarrow i_! X$  in  $\text{FrmSmthSet}$  are in natural bijection to equivalence classes of pairs consisting of a map  $U \times \mathbb{D} \rightarrow V$  (in  $\text{FrmCrtSp} \hookrightarrow \text{FrmSmthSet}$ ) into a Cartesian space  $V \in \text{CrtSp} \xrightarrow{\hookrightarrow} \text{FrmCrtSp}$  and a map  $V \rightarrow X$  in  $\text{SmthSet}$ , subject to the equivalence relation which is generated by compatible maps  $\phi : V \rightarrow V'$  (in  $\text{CartSp}$ ) between the factor spaces:

$$\left\{ U \times \mathbb{D} \rightarrow i_! X \right\} \simeq \left\{ \left( U \times \mathbb{D} \xrightarrow{\iota} V, V \xrightarrow{f} X \right) \right\} / \left\{ \left( U \times \mathbb{D} \begin{array}{ccc} \xrightarrow{\iota} & V & \xrightarrow{f} \\ & \phi \downarrow & \downarrow \phi \\ \xrightarrow{\iota'} & V' & \xrightarrow{f'} \end{array} X \right) \right\},$$

where on the right we mean that a pair as on the top of the diagram is in relation to a pair as shown at the bottom if  $\phi$  exists that makes the two triangles commute.

Under passage to sheaves (A.3) this statement holds on germs of maps, as claimed.  $\square$

The archetype of Thm. 2.1 is the statement that the projective limit defining the projectively infinite-dimensional Cartesian space (cf. [KS26, Def. 2.24, Prop. 2.26]),

$$(2.6) \quad \mathbb{R}^\infty := \varprojlim \left( \cdots \twoheadrightarrow \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^1 \twoheadrightarrow \mathbb{R}^0 \right) \in \text{FrchtMfd} \xrightarrow{(A.8)} \text{SmthSet},$$

is preserved:

**Proposition 2.4.** *Passage to formal smooth sets preserves the projective limit involved in forming the infinite Cartesian space (2.6):*

$$(2.7) \quad i_! \left( \varprojlim_{k \in \mathbb{N}} \mathbb{R}^k \right) \simeq \varprojlim_{k \in \mathbb{N}} \left( i_! \mathbb{R}^k \right).$$

*Proof.* It will suffice to observe that the right-hand side of (2.7) is reduced (proved below in Lem. 2.5), whence (using (A.7)):

$$(2.8) \quad \varprojlim_{k \in \mathbb{N}} \left( i_! \mathbb{R}^k \right) = \mathfrak{R} \varprojlim_{k \in \mathbb{N}} \left( i_! \mathbb{R}^k \right) \simeq i_! i^* \varprojlim_{k \in \mathbb{N}} \left( i_! \mathbb{R}^k \right).$$

With that and by full faithfulness of  $i_!$  (Lem. A.8), the claim is equivalent to:

$$(2.9) \quad \varprojlim_{k \in \mathbb{N}} \mathbb{R}^k \simeq i^* \left( \varprojlim_{k \in \mathbb{N}} (i_! \mathbb{R}^k) \right).$$

But this holds since  $i^*$  commutes with (projective) limits (being a right adjoint) and then using (A.5).  $\square$

Hence the key is to see that:

**Lemma 2.5.** *The limiting object  $\varprojlim_k (i_! \mathbb{R}^k)$  is reduced (A.2).*

*Proof.* In view of Lem. 2.3, we first show that every top map of the following form factors through some  $V \in \text{CrtSp}$  as shown at the bottom:

$$\begin{array}{ccc} & \xrightarrow{\quad} & \\ U \times \mathbb{D} & \dashrightarrow & V \dashrightarrow \varprojlim_{k \in \mathbb{N}} (i_! \mathbb{R}^k). \end{array}$$

Since we are dealing with maps into a limit, we need to equivalently see that every commuting diagram of maps of the form

$$\begin{array}{ccccccc} & & U \times \mathbb{D} & & & & \\ & \swarrow \cdots & \downarrow & \searrow & & & \\ \cdots & \dashrightarrow & i_! \mathbb{R}^3 & \dashrightarrow & i_! \mathbb{R}^2 & \dashrightarrow & i_! \mathbb{R}^1 & \dashrightarrow & i_! \mathbb{R}^0 \end{array}$$

factors in this form:

$$\begin{array}{ccccccc} & & U \times \mathbb{D} & & & & \\ & \swarrow \cdots & \downarrow & \searrow & & & \\ \cdots & \dashrightarrow & V & \dashrightarrow & i_! \mathbb{R}^2 & \dashrightarrow & i_! \mathbb{R}^1 & \dashrightarrow & i_! \mathbb{R}^0. \end{array}$$

Now, a map  $U \times \mathbb{D} \rightarrow i_! \mathbb{R}^k$  in  $\text{FrmSmthSet}$  is just a morphism in  $\text{FrmCrtSp}$  of the form  $U \times \mathbb{D} \rightarrow \mathbb{R}^k$  (because  $i_!$  preserves representables, cf. Lem. A.8), hence is a  $k$ -tuple  $([f_i])_{i=1}^k$  of elements of  $C^\infty(U \times \mathbb{D})$ . By Def. A.6, every such element  $[f_i]$  is an equivalence class of an element  $f_i \in C^\infty(U \times \mathbb{R}^d)$ , for some  $d \in \mathbb{N}$  depending only on  $\mathbb{D}$ . Hence, if we take  $V := U \times \mathbb{R}^d$  then we may choose these  $f_i$  for each given  $[f_i]$ :

$$\begin{array}{ccc} & U \times \mathbb{D} & \\ & \downarrow & \\ ([f_i])_{i=1}^{k+1} & U \times \mathbb{R}^d & ([f_i])_{i=1}^k \\ & \swarrow \quad \searrow & \\ & (f_i)_{i=1}^{k+1} \quad (f_i)_{i=1}^k & \\ \mathbb{R}^{k+1} & \dashrightarrow & \mathbb{R}^k \end{array}$$

to achieve the desired factorization.

In order to conclude with Lem. 2.3, we still need to show that any two such factorizations of the same map are related by a compatible zig-zag (2.4) of maps between the intermediate  $V$ s. This follows from: (i) the evident observation that every factorization is related to one where  $\iota$  is a formal proper embedding of arbitrarily high

codimension (Lem. A.11), combined with (ii) the more laborious verification that any two of these whose composites are equal are in zig-zag relation to each other. This is finally shown as Lem. 2.8 below.  $\square$

With this we conclude:

*Proof of Thm. 2.1.* The key is to observe that the right hand side of (2.1) is reduced, in that

$$\varprojlim_{k \in \mathbb{N}} (i_! J_\Sigma^k E) \simeq i_! i^* \varprojlim_{k \in \mathbb{N}} (i_! J_\Sigma^k E),$$

from which the claim will follow formally, just as in the proof of Prop. 2.4: Since  $i_!$  is fully faithful, the desired condition (2.1) thereby becomes equivalent to:

$$\varprojlim_{k \in \mathbb{N}} (J_\Sigma^k E) \simeq i^* \varprojlim_{k \in \mathbb{N}} (i_! J_\Sigma^k E),$$

which holds, since  $i^*$  commutes with projective limits (being a right adjoint) and then using that  $i^* \circ i_! \simeq \text{id}$  (A.5).

To see that the right-hand side of (2.1) is indeed reduced, consider a dashed horizontal map of the form

$$\begin{array}{ccc} U \times \mathbb{D} & \dashrightarrow & \varprojlim_k i_! J_\Sigma^k E \\ & \searrow & \downarrow \\ & & i_! \Sigma, \end{array}$$

where at the bottom we are indicating the induced map to  $\Sigma$ , over which everything is fibered. By Lem. 2.3 we need to show that a germ of the horizontal dashed map factors through a Cartesian space. By the fact that  $E \rightarrow \Sigma$  is assumed to be a surjective submersion, and hence similarly all of  $J_\Sigma^k E \rightarrow J_\Sigma^{k-1} E \rightarrow \dots \rightarrow \Sigma$ , there is a representative of the germ of each  $U \times \mathbb{D} \rightarrow i_! J_\Sigma^k E$  which factors through a trivial fibration  $U^k \times \mathbb{D} \hookrightarrow i_! J_{\mathbb{R}^n}^k \mathbb{R}^{n+m}$ . But upon further restriction to common overlaps, these yield a representative for the germ of the original map  $U \times \mathbb{D} \rightarrow \varprojlim_k i_! J_\Sigma^k E$  (with domain to be denoted  $U' \hookrightarrow U$ ) which factors through the corresponding jet bundle of a trivial fibration:

$$\begin{array}{ccccc} U' \times \mathbb{D} & \dashrightarrow & \varprojlim_k i_! J_{\mathbb{R}^n}^k \mathbb{R}^{n+m} & \hookrightarrow & \varprojlim_k i_! J_\Sigma^k E \\ & \searrow & \downarrow & & \downarrow \\ & & i_! \mathbb{R}^{n+m} & \hookrightarrow & i_! E \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & i_! \mathbb{R}^n & \hookrightarrow & i_! \Sigma. \end{array}$$

Thereby we have reduced the situation to essentially that of Prop. 2.4 and the proof follows analogously.  $\square$

It just remains to prove the very last statement in the proof of Lem. 2.5. This is the step that requires most of the work, we prove it as Lem. 2.8 below.

**Notation 2.6.** For ease of notation we shall now abbreviate the ‘‘synthetic differential version’’ of  $\mathbb{R}^\infty$  (2.6) as

$$(2.10) \quad \mathbf{R}^\infty := \varprojlim_k i_! \mathbb{R}^k \in \text{FrmSmthSet}$$

(though once we are done with the proof of Lem. 2.8, Prop. 2.4 asserts that  $\mathbf{R}^\infty$  is isomorphic to  $i_! \mathbb{R}^\infty$ ).

*Remark 2.7.* Since  $i_! \mathbb{R}^k = \mathbb{R}^k \in \text{FrmCrtSp}$  (A.4), the plots of  $\mathbf{R}^\infty$  (2.10) are nothing but  $\mathbb{N}$ -tuples of smooth functions:

$$\begin{aligned} \text{Hom}(U \times \mathbb{D}, \mathbf{R}^\infty) &\simeq \varinjlim_{k \in \mathbb{N}} \text{Hom}(U \times \mathbb{D}, \mathbb{R}^k) \simeq \prod_{k \in \mathbb{N}} \text{Hom}(U \times \mathbb{D}, \mathbb{R}^1) \\ &\simeq \prod_{k \in \mathbb{N}} C^\infty(U \times \mathbb{D}). \end{aligned}$$

This means in particular that we may add and subtract among these plots (used in (2.14) and (2.17) below) and apply Hadamard's lemma to them (used in (2.13) and (2.16) below).

In fact, in the following proof we only use that elements of  $C^\infty(U, \mathbf{R}^\infty)$  (for  $U \in \text{CrtSp}$ ) are additive and subject to Hadamard's lemma, and hence the analogous argument applies in case  $\mathbf{R}^\infty$  is replaced by any other object for which this is the case.

**Lemma 2.8.** *Given an outer commuting diagram as follows:*

$$\begin{array}{ccccc} & & V & & \\ & \nearrow \iota & \downarrow \alpha & \searrow f & \\ U \times \mathbb{D} & \dashrightarrow W & \dashrightarrow \phi & \dashrightarrow \mathbf{R}^\infty, & \\ & \searrow \iota' & \uparrow \alpha' & \nearrow f' & \\ & & V' & & \end{array}$$

where  $\iota$  and  $\iota'$  are formal proper embeddings (Def. A.10) of sufficiently high codimension, there exists  $W \in \text{CrtSp}$  such that the diagram may be filled by dashed maps as shown, making all triangles commute.

In order to make the proof of Lem. 2.8 transparent, we present it in stages of increasing generality. We begin with the simple case where  $U \equiv *$  and  $\mathbb{D} \equiv \mathbb{D}^1(1) \equiv \text{Spec}(C^\infty(\mathbb{R}^1)/(x^2))$ .

**Lemma 2.9.** *Given a commuting diagram of smooth functions of the form of the outer solid arrows in the following diagram:*

$$\begin{array}{ccccc} & & V & & \\ & \nearrow \iota & \downarrow \alpha & \searrow f & \\ \mathbb{D}^1(1) & \dashrightarrow W & \dashrightarrow \phi & \dashrightarrow \mathbf{R}^\infty & \leftrightarrow & * & \dashrightarrow TW & \dashrightarrow T\phi & \dashrightarrow T\mathbf{R}^\infty, \\ & \searrow \iota' & \uparrow \alpha' & \nearrow f' & & \tilde{\iota} & \nearrow TV & \searrow Tf & \\ & & V' & & & \tilde{\iota}' & \searrow TV' & \nearrow Tf' & \end{array}$$

where  $\iota$  and  $\iota'$  are monomorphisms, then there exists  $W \in \text{CrtSp}$  and smooth maps corresponding to the dashed arrows that make the whole diagram commute.

*Proof.* We may assume without restriction of generality that the base points picked by  $\iota$  and  $\iota'$  are the origins of  $0 \in V$  and  $0 \in V'$ , respectively (otherwise reparametrize), hence that

$$\begin{aligned} \tilde{\iota}(\bullet) &=: v_0 \in T_0V, \\ \tilde{\iota}'(\bullet) &=: v'_0 \in T_0V'. \end{aligned}$$

The assumption that  $\iota$  and  $\iota'$  are monomorphic means that  $v_0 \neq 0$  and  $v'_0 \neq 0$ . Therefore (writing  $V_\perp \subset V$  for the orthogonal subspace to  $v_0$  and  $V'_\perp \subset V'$  for the orthogonal subspace to  $v'_0$ ), we may coordinatize  $V$  and  $V'$  by:

$$\begin{aligned} \mathbb{R} \times V_\perp &\xrightarrow{\sim} V & \mathbb{R} \times V'_\perp &\xrightarrow{\sim} V' \\ (t, \vec{x}) &\mapsto (tv_0, \vec{x}), & (t', \vec{x}') &\mapsto (t'v'_0, \vec{x}'). \end{aligned}$$

Now set

$$W := V \times V' \times \mathbb{R}^1$$

and

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ (t, \vec{x}) & \mapsto & ((t, \vec{x}), (t, 0), 0), \end{array} \quad \begin{array}{ccc} V' & \xrightarrow{\alpha'} & W \\ (t', \vec{x}') & \mapsto & ((t', 0), (t', \vec{x}'), t'^2). \end{array}$$

Since

$$\begin{aligned} \alpha(0, 0) &= ((0, 0), (0, 0), 0), & \frac{\partial}{\partial t} \alpha(0, 0) &= ((1, 0), (1, 0), 0), \\ \alpha'(0, 0) &= ((0, 0), (0, 0), 0), & \frac{\partial}{\partial t'} \alpha'(0, 0) &= ((1, 0), (1, 0), 0), \end{aligned}$$

this makes the left part of our diagram commute.

Finally, to define  $\phi$ , first consider

$$(2.11) \quad \delta : \mathbb{R} \rightarrow \mathbf{R}^\infty$$

$$(2.12) \quad t \mapsto f'(t, 0) - f(t, 0).$$

By the commutativity of the outer diagram, we have  $\delta(0) = 0$  and  $\frac{d}{dt} \delta(0) = 0$ . Hence by Hadamard's lemma there is a smooth function

$$(2.13) \quad \mu : \mathbb{R} \rightarrow \mathbf{R}^\infty, \quad \text{such that } \delta(t) = t^2 \cdot \mu(t).$$

With that function in hand, finally define

$$(2.14) \quad \phi : \left( (t, \vec{x}), (t', \vec{x}'), j \right) \mapsto f(t, \vec{x}) - f'(t, 0) + f'(t, \vec{x}') + j \cdot \mu(t).$$

We check that this makes the right part of our diagram commute, since:

$$\begin{aligned} \phi \circ \alpha(t, \vec{x}) &= \phi \left( (t, \vec{x}), (t, 0), 0 \right) \\ &= f(t, \vec{x}) + 0 + 0, \end{aligned}$$

and

$$\begin{aligned} \phi \circ \alpha'(t', \vec{x}') &= \phi \left( (t', 0), (t', \vec{x}'), t'^2 \right) \\ &= \underbrace{f(t', 0) - f'(t', 0)}_{-\delta(t')} + \underbrace{f'(t', \vec{x}') + t'^2 \cdot \mu(t')}_{\delta(t')} \\ &= f'(t', \vec{x}'). \end{aligned}$$

This completes the proof.  $\square$

Next, we generalize this proof to the case where (still  $U \equiv *$  but)  $\mathbb{D}$  is arbitrary. To that end, choose an inclusion  $\mathbb{D} \hookrightarrow \mathbb{R}^d$  with minimal  $d$  and choose a set of generators

$$(2.15) \quad \vec{h} := \left( h_i \in C^\infty(\mathbb{R}^d) \right)_{i=1}^n \quad (\text{for some } n \in \mathbb{N})$$

for the kernel of the dual algebra homomorphism  $C^\infty(\mathbb{R}^d) \twoheadrightarrow C^\infty(\mathbb{D})$ .

**Lemma 2.10.** *Given a commuting diagram of smooth functions of the form of the outer solid arrows in the following diagram:*

$$\begin{array}{ccccc}
 & & V & & \\
 & \nearrow \iota & \downarrow \alpha & \searrow f & \\
 \mathbb{D} & \dashrightarrow & W & \dashrightarrow \phi & \mathbf{R}^\infty, \\
 & \searrow \iota' & \uparrow \alpha' & \nearrow f' & \\
 & & V' & & 
 \end{array}$$

where  $\iota$  and  $\iota'$  are monomorphisms, then there exists  $W \in \text{CrtSp}$  and dashed maps also making the inner diagram commute.

*Proof.* Without restriction of generality we may assume that  $\iota(0) = 0 \in V$  and  $\iota'(0) = 0 \in V'$  (otherwise reparametrize). Since  $\iota$  and  $\iota'$  are monomorphic, we may decompose  $V$ ,  $V'$  respectively into the span of the image of  $\iota$ ,  $\iota'$  and its orthogonal complement, respectively:

$$\begin{aligned}
 \mathbb{R}^d \times V_\perp &\xrightarrow{\sim} V, \\
 \mathbb{R}^d \times V'_\perp &\xrightarrow{\sim} V'.
 \end{aligned}$$

Now take

$$W := V \times V' \times \mathbb{R}^n$$

and

$$\begin{aligned}
 \alpha(\vec{t}, \vec{x}) &:= \left( (\vec{t}, \vec{x}), (\vec{t}, 0), 0 \right) \\
 \alpha'(\vec{t}', \vec{x}') &:= \left( (\vec{t}', 0), (\vec{t}', \vec{x}'), \vec{h}(\vec{t}') \right),
 \end{aligned}$$

where the last component are the values of the generators (2.15) of the nil ideal.

Then we have the following composites (where in the lower lines we show the pullback of the canonical coordinate functions along the top map, whose inspection is sufficient for identifying these maps, cf. [GS25b, Prop. 4.6])

$$\begin{array}{ccccccc}
 \mathbb{D} & \xrightarrow{\iota} & \mathbb{R}^d \times V_\perp & \xrightarrow{\alpha} & \mathbb{R}^d \times V_\perp \times \mathbb{R}^d \times V'_\perp \times \mathbb{R}^n & & \\
 [\vec{t}] & \longleftarrow & \vec{t} & \longleftarrow & \vec{t} & & \\
 0 & \longleftarrow & \vec{x} & \longleftarrow & \vec{x} & & \\
 [\vec{t}'] & \longleftarrow & \vec{t}' & \longleftarrow & \vec{t}' & & \\
 0 & \longleftarrow & 0 & \longleftarrow & \vec{x}' & & \\
 0 & \longleftarrow & 0 & \longleftarrow & \vec{j} & & 
 \end{array}$$

and

$$\begin{array}{ccccccc}
 \mathbb{D} & \xrightarrow{\iota'} & \mathbb{R}^d \times V'_\perp & \xrightarrow{\alpha'} & \mathbb{R}^d \times V_\perp \times \mathbb{R}^d \times V'_\perp \times \mathbb{R}^n & & \\
 [\vec{t}] & \longleftarrow & \vec{t} & \longleftarrow & \vec{t} & & \\
 0 & \longleftarrow & 0 & \longleftarrow & \vec{x} & & \\
 [\vec{t}'] & \longleftarrow & \vec{t}' & \longleftarrow & \vec{t}' & & \\
 0 & \longleftarrow & \vec{x}' & \longleftarrow & \vec{x}' & & \\
 0 & \longleftarrow & \vec{h}(\vec{t}') & \longleftarrow & \vec{j} & & 
 \end{array}$$

where in the last line we use that the functions  $\vec{h}$  (2.15) vanish on  $\mathbb{D}$ , by assumption. Since these two composite maps hence agree, the left part of our diagram commutes.

Now consider

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{\delta} & \mathbf{R}^\infty \\ \vec{t} & \mapsto & f'(\vec{t}, 0) - f(\vec{t}, 0). \end{array}$$

By the commutativity of the outer part of our diagram, the components of this map are in the ideal spanned by the  $h_i$  (2.15), so that

$$(2.16) \quad \delta(\vec{t}) = \sum_i h_i(\vec{t}) \mu^i(\vec{t}) \quad \text{for some} \quad (\mu^i : \mathbb{R}^d \rightarrow \mathbf{R}^\infty)_{i=1}^n.$$

With this in hand, finally define

$$(2.17) \quad \phi\left((\vec{t}, \vec{x}), (\vec{t}', \vec{x}'), \vec{j}\right) := f(\vec{t}, \vec{x}) - f'(\vec{t}, 0) + f'(\vec{t}, \vec{x}') + \sum_i j_i \mu^i(\vec{t}).$$

We check by direct computation that this indeed makes the right part of our diagram commute:

$$\begin{aligned} \phi \circ \alpha(\vec{t}, \vec{x}) &= \phi\left((\vec{t}, \vec{x}), (\vec{t}, 0), 0\right) \\ &= f(\vec{t}, \vec{x}) + 0 + 0 \end{aligned}$$

and

$$\begin{aligned} \phi \circ \alpha'(\vec{t}', \vec{x}') &= \phi\left((\vec{t}', 0), (\vec{t}', \vec{x}'), \vec{h}(\vec{t}')\right) \\ &= \underbrace{f(\vec{t}', 0) - f'(\vec{t}', 0)}_{-\delta(\vec{t}')} + f'(\vec{t}', \vec{x}') + \underbrace{\sum_i h_i(\vec{t}') \mu^i(\vec{t}')}_{\delta(\vec{t}')} \\ &= f'(\vec{t}', \vec{x}'). \end{aligned}$$

This completes the proof.  $\square$

Finally, it is not much more work to lift this proof to full generality:

*Proof of Lem. 2.8.* By Lem. A.12 we may assume, without restriction of generality, that, up to diffeomorphism on the target:

$$\begin{aligned} \iota|_{U \times \{*\}} &\simeq (\text{id}, 0) : U \hookrightarrow U \times \tilde{V} \simeq V \\ \iota'|_{U \times \{*\}} &\simeq (\text{id}, 0) : U \hookrightarrow U \times \tilde{V}' \simeq V'. \end{aligned}$$

But under this identification, we have over each point  $u \in U$  the exact situation of Lem. 2.10, and hence the solution is as given there, just with  $W$  having another factor of  $U$  and with  $\alpha$ ,  $\alpha'$  and  $\phi$  depending on a further variable  $u \in U$ .

Concretely, in generalizing Lem. 2.9 this way:

- (i) The parametrization of  $V$  is now as a bundle over  $U \times \mathbb{R}$ , with  $(u, t \cdot v(u), \vec{x}_{v(u)}) \in V$ , where  $\vec{x}_{v(u)}$  are elements perpendicular to  $v(u)$ .
- (ii) The corresponding  $\delta$  is now a map  $U \times \mathbb{R} \rightarrow \mathbf{R}^\infty$  which satisfies  $\delta(u, 0) = 0$ ,  $\frac{\partial}{\partial t} \delta(u, 0) = 0$  and so, by the *partial/parameterized version* of Hadamard's lemma Thm. A.5, may be written as  $\delta(u, t) = t^2 \cdot \mu(u, t)$  for a smooth  $\mu : U \times \mathbb{R} \rightarrow \mathbf{R}^\infty$ .  $\square$

## APPENDIX A. BACKGROUND

For reference in the main text, here we briefly record some basic definitions and facts.

**A.1. Formal Smooth Sets.** We briefly review the topos of *formal smooth sets* ([Sch13, Def. 4.5.5; KS26, §2.1; GS25b, §2], exposition in [Sch25]), equivalent (by [GS25b, §4.1]) to Dubuc’s *Cahiers topos* [Dub79; KR87], which is a “well-adapted model” for synthetic differential geometry.

Our ground field is the real numbers. We write  $\mathbf{CAlg}$  for the category of commutative  $\mathbb{R}$ -algebras.

**Definition A.1.** (i) We write  $\mathbf{CrtSp}$  for the category whose objects are the Cartesian spaces  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , and whose morphisms are the smooth functions between these. Hence this is the full subcategory of smooth manifolds

$$\mathbf{CrtSp} \hookrightarrow \mathbf{SmthMfd}$$

on the Cartesian spaces. We will denote generic Cartesian spaces by  $U, V \in \mathbf{CrtSp}$ .

(ii) We regard this category as a site by equipping it with the coverage (Grothendieck pre-topology) of differentiably good open covers.

**Proposition A.2** (cf. [KMS93, §35]). *The functor which forms (plain)  $\mathbb{R}$ -algebras of smooth functions constitutes a full inclusion of smooth manifolds, hence in particular of Cartesian spaces (Def. A.1):*

$$\mathbf{CrtSp} \hookrightarrow \mathbf{SmthMfd} \xrightarrow{C^\infty(-)} \mathbf{CAlg}^{\mathrm{op}}$$

**Definition A.3.** The category of *infinitesimally thickened points (haloed points)* is the full subcategory of the opposite of commutative algebras

$$\begin{aligned} \mathbf{FrmPnt} &\hookrightarrow \mathbf{CAlg}^{\mathrm{op}} \\ \mathbb{D} &\longmapsto C^\infty(\mathbb{D}). \end{aligned}$$

on those what are called *Weil algebras* in the SDG literature or *Artin local algebras* in algebraic geometry, namely those of the form

$$C^\infty(\mathbb{D}) \simeq_{\mathbb{R}} \mathbb{R} \oplus V,$$

where  $V \subset C^\infty(\mathbb{D})$  is a finite-dimensional ideal which is *nilpotent* in that

$$\exists_{k \in \mathbb{N}} : V^{k+1} = 0.$$

The smallest  $k$  for which this holds is the *order of nilpotency* of  $\mathbb{D}$ .

**Example A.4.** An *infinitesimal disk* of dimension  $d \in \mathbb{N}$  and order of infinitesimality  $k \in \mathbb{N}$  is an infinitesimally thickened point  $\mathbb{D}^d(k) \in \mathbf{FrmPnt}$  (Def. A.3) of the form

$$\begin{aligned} C^\infty(\mathbb{D}^d(k)) &\simeq \mathbb{C}^\infty(\mathbb{R}^d)/(x^1, \dots, x^d)^{k+1} \\ &\simeq \mathbb{R}[\epsilon^1, \dots, \epsilon^d]/(\epsilon^1, \dots, \epsilon^d)^{k+1}. \end{aligned}$$

The second equivalence above follows by Hadamard’s lemma, which we repeatedly refer to. For reference, we state it in a version that covers both parameters and partial derivatives:

**Lemma A.5** (Hadamard's Lemma, cf. [DK04, Lem. 2.2.7], we follow [GS25b, Lem. 4.1]). *For any smooth function  $f(x, y) \in C^\infty(\mathbb{R}^k \times \mathbb{R}^m)$  and any  $l \in \mathbb{N}$ , there exist smooth functions  $\{h_{i_1 \dots i_{l+1}}\}_{i_1, \dots, i_{l+1}=1, \dots, m} \subset C^\infty(\mathbb{R}^k \times \mathbb{R}^m)$  such that*

$$\begin{aligned} f &= f(x, 0) + \sum_{i=1}^m y^i \partial_{y^i} f(x, 0) + \frac{1}{2} \sum_{i,j=1}^m y^i y^j \partial_{y^i} \partial_{y^j} f(x, 0) \\ &+ \dots + \frac{1}{l!} \sum_{i_1, \dots, i_l=1}^m y^{i_1} \dots y^{i_l} \partial_{y^{i_1}} \dots \partial_{y^{i_l}} f(x, 0) \\ &+ \sum_{i_1, \dots, i_{l+1}=1}^m y^{i_1} \dots y^{i_{l+1}} h_{i_1 \dots i_{l+1}}(x, y). \end{aligned}$$

**Definition A.6.** (i) The category of *formal Cartesian spaces* is the full subcategory of the opposite of commutative algebras,

$$\begin{aligned} \text{FrmCrtSp} &\xleftarrow{C^\infty(-)} \text{CAlg}^{\text{op}} \\ U \times \mathbb{D} &\longmapsto C^\infty(U) \otimes_{\mathbb{R}} C^\infty(\mathbb{D}), \end{aligned}$$

on those which are tensor products of an algebra of smooth functions on a  $U \in \text{CrtSp}$  (cf. Prop. A.2) with those on a  $\mathbb{D} \in \text{FrmPnt}$  (Def. A.3).

(ii) We regard this category as a site by equipping it with the coverage (Grothendieck pre-topology) whose covering families are of the form

$$\left\{ U_i \times \mathbb{D} \xrightarrow{\iota_i \times \text{id}} U \times \mathbb{D} \right\}_{i \in I}$$

for

$$\{ U_i \xrightarrow{\iota_i} U \}_{i \in I}$$

a covering family in  $\text{CrtSp}$  (a differentiably good open cover).

By construction, we have a fully faithful inclusion of sites:

$$(A.1) \quad \begin{aligned} \text{CrtSp} &\xleftarrow{i} \text{FrmCrtSp} \\ U &\longmapsto U \times \mathbb{D}^0. \end{aligned}$$

**Definition A.7.** We say that the sheaf topos

- (i) over  $\text{CrtSp}$  is the category of *smooth sets*,
- (ii) over  $\text{FrmCrtSp}$  is the category of *formal smooth sets*,

and that the essential image of the former under the left base change  $i_!$  along the inclusion (A.1) constitute the *reduced* formal smooth sets:

$$(A.2) \quad \text{SmthSet} := \text{Sh}(\text{CrtSp}) \begin{array}{c} \xleftarrow{i_!} \longrightarrow \\ \perp \\ \xleftarrow{i^*} \longrightarrow \end{array} \text{Sh}(\text{FrmCrtSp}) =: \text{FrmSmthSet},$$

Here  $i_!$  on sheaves is the *left Kan extension*  $i_!$  of presheaves (2.5) followed by sheafification  $L$ , and we are using that  $i^*$  manifestly preserves sheaves:

$$(A.3) \quad \begin{array}{ccc} \text{PSh}(\text{CrtSp}) & \begin{array}{c} \xrightarrow{i_!} \longrightarrow \\ \perp \\ \xleftarrow{i^*} \longrightarrow \end{array} & \text{PSh}(\text{FrmCrtSp}) \\ \uparrow & & \uparrow \downarrow L \\ \text{Sh}(\text{CrtSp}) & \begin{array}{c} \xrightarrow{i_!} \longrightarrow \\ \perp \\ \xleftarrow{i^*} \longrightarrow \end{array} & \text{Sh}(\text{FrmCrtSp}). \end{array}$$

Now, basic category theory shows the following important properties:

**Lemma A.8.** *The inclusion  $i_1$  in (A.2):*

(i) *preserves representables  $V \in \text{CrtSp} \hookrightarrow \text{SmthSet}$  in that:*<sup>1</sup>

$$(A.4) \quad i_1 V \simeq V \equiv V \times \mathbb{D}^0 \in \text{FrmCrtSp} \hookrightarrow \text{FrmSmthSet},$$

(ii) *is fully faithful, hence*

$$(A.5) \quad i_1^* \circ i_1 \simeq \text{id},$$

(iii) *and such that the reduced objects  $X \in \text{SmthSet} \xrightarrow{i_1} \text{FrmSmthSet}$  are equivalently those which are isomorphic to their purely reduced aspect  $\mathfrak{R}(X)$ ,*

$$(A.6) \quad X \simeq \mathfrak{R}(X),$$

where

$$(A.7) \quad \mathfrak{R} := i_1 \circ i_1^*.$$

*Proof.* The first statement follows with the coend formula (2.5): Every pair  $(\iota, f)$  (of maps into a Cartesian space  $V$  factoring through a Cartesian space  $W$ ) is equivalent to its actual composite regarded as the pair  $(f \circ \iota, \text{id})$ , like this:

$$\begin{array}{ccccc} U \times \mathbb{D} & \xrightarrow{\iota} & W & \xrightarrow{f} & V. \\ & \searrow & \downarrow f & \nearrow & \\ & f \circ \iota & V & \xrightarrow{\text{id}} & V. \end{array}$$

This shows that  $i_1$  on presheaves preserves representables. Finally, our coverage is clearly subcanonical in that representable presheaves are already sheaves.

For the second statement cf. [Kel82, Prop. 4.23] with [Mac98, §IV.3 Thm. 1]. From this the last statement follows.  $\square$

A classical notion in infinite-dimensional differential geometry are *Fréchet manifolds* (cf. [Ham82, §I.4]). These are faithfully subsumed here:

**Lemma A.9.** *The evident inclusion of Fréchet manifolds among smooth sets (and hence further among formal smooth sets Def. A.7)*

$$(A.8) \quad \begin{array}{ccc} \text{FrchtMfd} & \hookrightarrow & \text{SmthSet} \\ X & \mapsto & (U \mapsto C^\infty(U, X)) \end{array}$$

*is fully faithful.*

*Proof.* This is essentially the statement of [Los94, Thm. 3.1.1], cf. [KS26, Prop. 2.22].  $\square$

## A.2. Formal embeddings.

**Definition A.10.** A map  $i : U \times \mathbb{D} \rightarrow V$  in  $\text{FrmCrtSp}$  (Def. A.6) is a *formal embedding* if its restriction to  $U \times \{0\}$  is an embedding of smooth manifolds and its restriction to each  $\{u\} \times \mathbb{D}$  is a monomorphism. Moreover, this is a *proper* formal embedding if the restriction to  $U \times \{0\}$  is also a proper map.

The following lemma is immediate but still worth recording:

<sup>1</sup>We are leaving the Yoneda embedding notationally implicit.

**Lemma A.11.** *Every  $U \times \mathbb{D} \xrightarrow{i} V \xrightarrow{f} X$  is equivalent (2.3) to  $(\iota, f')$ , where  $\iota$  is a formal proper embedding (Def. A.10) of arbitrarily high codimension.*

*Proof.* Choose a monomorphism (where  $d$  may be arbitrarily large) of the form  $\delta : U \times \mathbb{D} \hookrightarrow U \times \mathbb{R}^d$  (cf. [GS25b, Cor. 4.3(iii)]), note that this makes the following diagram commute:

$$\begin{array}{ccccc} U \times \mathbb{D} & \xrightarrow{i} & V & \xrightarrow{f} & X, \\ & \searrow^{\iota := (i, \delta)} & \uparrow \text{pr}_V & \nearrow_{f' := f \circ \text{pr}_V} & \\ & & V \times (U \times \mathbb{R}^d) & & \end{array}$$

and finally observe that  $\iota := (i, \delta)$  is a formal proper embedding.  $\square$

**Lemma A.12.** *For every proper embedding  $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+c}$  of sufficiently high codimension  $c$ , there exists a diffeomorphism of  $\mathbb{R}^{n+c}$  under which the embedding is a rectilinear inclusion:  $i \simeq (\text{id}, 0)$ .*

*Proof.* First we claim that there exists a proper smooth homotopy from  $i$  to  $(\text{id}, 0)$ . This may be seen by observing that proper maps extend to pointed maps between one-point compactifications (cf. [Jam84, p. 80]). In our case the extended maps are of the form  $S^n \rightarrow S^{n+c}$  and hence pointed null-homotopic for  $c \geq 1$ . This pointed null-homotopy comes from a proper homotopy between the original maps as desired.

From this it follows that  $i$  is related to  $(\text{id}, 0)$  by ambient isotopy (with [Hir76, §8.1 Exc. 11], which requires  $c > 1 + n$ ) and thus by diffeotopy (with [Hir76, §8.1 Thm. 1.3]).  $\square$

**A.3. Jet Bundles.** For traditional background on jet bundle geometry cf. [Sau89; Tak79].

A finite-rank *jet bundle*  $J_{\Sigma}^k E$  is the bundle of all possible partial derivatives, of order  $\leq k$ , of sections of a given surjective submersion  $p : E \rightarrow \Sigma$  of smooth manifolds.

This traditional notion has a slick synthetic differential formulation ([KS26, §3.3][GS25b, §2.4]) in terms of the maximal infinitesimal disks  $\mathbb{D}_s(k) \rightarrow \Sigma$  (Ex. A.4) around any point  $s \in \Sigma$ : The fiber of  $J_{\Sigma}^k E$  over  $s$  consists of the sections of  $p$  over  $\mathbb{D}_s(k)$ :

$$(J_{\Sigma}^k E)_s \simeq \text{FrmSmthSet}_{/\Sigma}(\mathbb{D}_s(k), E) \equiv \left\{ \begin{array}{c} \begin{array}{ccc} & \nearrow & E \\ & \text{---} & \downarrow p \\ \mathbb{D}_s(k) & \hookrightarrow & \Sigma \end{array} \end{array} \right\},$$

and  $J_{\Sigma}^k E$  canonically inherits the structure of a smooth manifold this way.

Now, the evident inclusions  $\mathbb{D}_s(k) \hookrightarrow \mathbb{D}_s(k+1)$  thus induce natural projections  $J_{\Sigma}^{k+1} E \twoheadrightarrow J_{\Sigma}^k E$ , and the *infinite rank* jet bundle (cf. [Sau89, §7]) is the projective limit over these projections:

$$(A.9) \quad J_{\Sigma}^{\infty} E := \varprojlim_{k \in \mathbb{N}} J_{\Sigma}^k E.$$

Traditionally, this limit is understood in the category of Fréchet manifolds (cf. Lem. A.9), and it is not *a priori* clear how this relates to the corresponding limit formed in formal smooth sets, since the inclusion  $i_!$  (A.2) does not preserve limits in general

That  $i_!$  does preserve the particular limits (A.9) is the content of our main result Thm. 2.1.

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