Prequantum Field Theory

Khavkine & Schreiber

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Abstract

Modern classical local field theory, in the guise of variational calculus on jet bundles, has become a highly sophisticated theory. Nevertheless there are subtle but important global aspects that have found little attention yet, phenomena that in mechanics are known as *prequantization* and as *cancellation of classical anomalies*.

First, for *locally* variational field theories to admit an action functional, the locally defined Lagrangian forms need to be promoted to horizontal connections on p-gerbes with band $\mathbb{R}/_{\hbar}\mathbb{Z}$ over a site of partial differential equations. We develop the theory of such *Euler-Lagrange p-gerbes* and show from first principles that their curving is given by the Euler-Lagrange form whose vanishing locus is the Euler-Lagrange equation of motion.

Second, Kostant-Souriau's concept of prequantum line bundles for mechanical systems needs to be localized for local field theories such as to produce prequantizations of covariant phase spaces naturally for every choice of codimension-1 (Cauchy-)surface. We show how this is achieved by lifting Euler-Lagrange p-gerbes through a canonical codimension filtration to what we call *Lepage p-gerbes*, and then transgressing these.

Third, we highlight the little known sharp version of Noether's first theorem, which characterizes equivalence classes of conserved currents as a Lie algebra extension of infinitesimal symmetries by equivalence classes of locally trivial but globally nontrivial topological currents. Then we show that the infinitesimal symmetries of Euler-Lagrange p-gerbes promote this statement to an L_∞-algebra extension of infinitesimal symmetries by actual currents and higher currents. Correspondingly the infinitesimal symmetries of the Lepage p-gerbes give the multisymplectic refinement of the canonical Poisson bracket to an L_∞-bracket. In fact we obtain the integrated extension of the group of finite symmetries by the higher moduli stack of higher topological current symmetries, producing a Noether theorem also for discrete, finite topologically nontrivial symmetries.

Finally we apply this general theory to the case of field theories of higher dimensional parameterized WZW type that appear prominently in solid state physics (topological phases of matter) and in string/M-theory (super p-branes). These systems are only locally variational in general. Specifically for the case of the GS-WZW models for super p-branes on curved supermanifolds, we discuss how the resulting higher Noether currents promote the BPS brane charge extensions of superisometry algebras to super-L_∞-extensions by p-brane currents.

This document, with accompanying material, is kept online at
ncatlab.org/schreiber/show/Local+prequantum+field+theory.

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we discuss some aspects of applications. Finally in the appendix

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we collect statement and references for those general abstract results which we use in the main text.

For motivation, exposition and survey, see

ncatlab.org/schreiber/show/Higher+Prequantum+Geometry .

1 Classical local Lagrangian Field theory

We give here a self-contained account of the basic definitions and facts in modern variational calculus for classical local Lagrangian field theories in terms of jet geometry [Ol93, And89]. While nothing in this section is new, our review puts an emphasis on certain aspects that will be crucial below in section 2 and that are somewhat hidden in the standard literature. These aspects include:

- the comonadicity of the category of partial differential equations due to [Marvan86];
- the functoriality of the Euler-Lagrange complex over the site of differential operators, implicit in [And89].
- the sharp version of Noether's first variational theorem, due to [Vi84];

1.1 Jet bundles, Differential operators and PDEs

Throughout, let

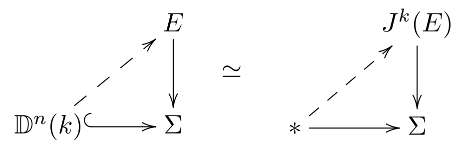
- $p \in \mathbb{N}$;
- Σ be a $(p+1)$-dimensional manifold, regarded as the *spacetime/worldvolume* of the field theory,
- $E \to \Sigma$ be a smooth bundle, called the *field bundle*, whose smooth sections $\phi \in \Gamma(E)$ are the *field configurations* of the field theory.

Definition 1.1. Any smooth bundle may be extended to a sequence of *k-jet bundles* $J^k E \to J^{k-1}E$, each an affine bundle over the preceding one, with $J^0 E = E$. The projective limit

$$J^\infty E := \varprojlim J^\bullet E \, ,$$

regarded as a bundle over Σ, is the *(∞-)jet bundle* of E.

Remark 1.2. The intuition is that a section of $J^k(E)$ over a point $x \in \Sigma$ is equivalently a section of E over the order-k infinitesimal neighbourhood $\mathbb{D}^n(k)$ of x:



This intuition becomes a precise statement [Kock80, section 2] after embedding smooth manifolds into a model for synthetic differential geometry, such as [Dubuc79, MoerdijkReyes91], where formal manifolds such as $\mathbb{D}^n(k)$ genuinely exist. We come back to this below in section 2.1. The synthetic formulation has models also in algebraic geometry, where the construction of jet bundles is known in the language of "crystals of schemes" or "$\mathcal{D}$-geometry," see for instance [Lurie09].

Remark 1.3. While $J^\infty E$ is not finite dimensional, it is nearly so, because any smooth function on it must depend only on a finite number of coordinates, with the number bounded at least locally. Technically this means that $J^\infty E$ is defined a projective limit of a tower of affine bundles over E. It follows in particular that J^∞_Σ has the same de Rham cohomology as E, $H^p(J^\infty E) \cong H^p(E)$.

Definition 1.4. Write SmoothMfd for the category of pro-finite dimensional smooth manifolds (maybe point to [GuPf13]). For Σ a smooth manifold, write $\mathrm{SmoothMfd}_{\downarrow\Sigma}$ for the category of surjective submersions of pro-finite dimensional smooth manifolds over Σ.

By remark 1.2 it is clear that we have the following (see e.g. [Marvan86]):

Definition 1.5. The jet bundle construction of def. 1.1 extends to a functor

$$J^\infty_\Sigma : \mathrm{SmoothMfd}_{\downarrow\Sigma} \longrightarrow \mathrm{SmoothMfd}_{\downarrow\Sigma} .$$

Notice the following degenerate case.

Example 1.6. If we regard $\Sigma \overset{\mathrm{id}}{\to} \Sigma$ canonically as a bundle over itself, then it coincides with its jet bundle: $J^\infty_\Sigma(\Sigma) \simeq \Sigma$.

Simple as this is, it induces the following key construction.

Definition 1.7. Given a section $\phi\colon \Sigma \to E$, $\phi \in \Gamma_\Sigma(E)$, its *jet prolongation* is its image under the jet functor, def. 1.5, regarded as a section of the jet bundle via the equivalence of example 1.6:

$$j^\infty(\phi) := \ \Sigma \xrightarrow{\ \simeq\ } J^\infty_\Sigma(\Sigma) \xrightarrow{\ J^\infty_\Sigma(\phi)\ } J^\infty_\Sigma(E) \ .$$

Remark 1.8. In terms of remark 1.2 the jet extension of def. 1.7 is the result of restricting ϕ to all order-k infinitesimal neighbourhoods of its domain.

It turns out that the construction of jet bundles has some excellent abstract properties that are useful in the classical theory and indispensable in the prequantum theory which we turn to in [**?**]. Before stating them, we briefly recall the pertinent definitions.

Proposition 1.9 ([Marvan86]). *The jet bundle endofunctor J^∞_Σ of def. 1.5, together with the canonical projection map $J^\infty_\Sigma E \to \Sigma$ as well as with the natural transformation $J^\infty_\Sigma(E) \longrightarrow J^\infty_\Sigma(\mathrm{Jet}(E))$ induced from the jet prolongation operation j^∞, def. 1.7, is a co-monad.*

Proposition 1.10. *For $E_1, E_2 \in \mathrm{SmoothMfd}_{/\Sigma}$ two bundles over Σ, then a* differential operator $D \colon \Gamma_\Sigma(E_1) \longrightarrow \Gamma_\Sigma(E_2)$ *is equivalently a map between their spaces of sections of the form $\phi \mapsto \tilde{D} \circ j^\infty(\phi)$, where j^∞ is the jet prolongation of def. 1.7, and where $\tilde{D}$ is a morphism of bundles over Σ of the form*

$$\tilde{D} : J^\infty_\Sigma(E_1) \longrightarrow E_2 \,.$$

The composite $D_2 \circ D_1$ of two differential operators is given by

$$\widetilde{D_2 \circ D_1} \colon \; J^\infty_\Sigma(E_1) \xrightarrow{p^\infty(\tilde{D}_1)} J^\infty_\Sigma(E_2) \xrightarrow{\tilde{D}_2} E_3 \;.$$

In other words, the category DiffOp_Σ of smooth bundles over Σ with morphisms the differential operators between their sections is equivalently the Kleisli category, def. 4.17, of the jet comonad of prop. 1.9.

Remark 1.11. Prop. 1.10 says in particular that the jet extension of a bundle E itself is the universal differential operator $j^\infty \colon \Gamma_\Sigma(E) \to \Gamma_\Sigma(J^\infty_\Sigma(E))$. with $\widetilde{j^\infty} = \mathrm{id}$.

Definition 1.12. In the situation of prop. 1.10, the composition

$$p^\infty(\tilde{D}) \colon \; J^\infty(E_1) \longrightarrow J^\infty(J^\infty(E_1)) \xrightarrow{J^\infty(\tilde{D})} J^\infty(E_2)$$

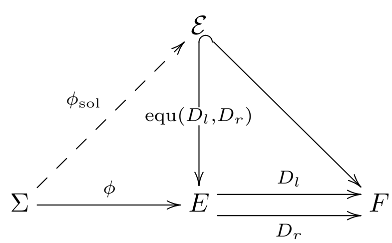
is called the *prolongation* of the map $\tilde{D}$.

Below in prop. 1.17 we give the co-monadic interpretation of p^∞, using the following generalization of prop. 1.10.

Theorem 1.13 ([Marvan86])**.** *The category of co-algebras $\mathrm{EM}(J^\infty_\Sigma)$ (def. 4.13) over the jet comonad over Σ (prop. 1.9) is equivalently the category PDE_Σ of (non-singular) partial differential equations with free variables ranging in Σ, and with solution-preserving differential operators between these [Vi80]:*

$$\mathrm{EM}(J^\infty_\Sigma) \simeq \mathrm{PDE}_\Sigma \,.$$

Remark 1.14. The identification of objects $\mathcal{E} \in \mathrm{PDE}_\Sigma$ in theorem 1.13 with (non-singular) partial differential equations works as follows. First of all, one finds that every $\mathcal{E} \in \mathrm{PDE}_\Sigma$ is the equalizer of a pair of morphisms [1] $D_l, D_r : E \longrightarrow F$ in $\mathrm{DiffOp}_\Sigma \hookrightarrow \mathrm{PDE}_\Sigma$, hence, by prop. 1.10, of two differential operators acting on sections of a field bundle E over Σ. By the universal property of equalizers, this means that the morphisms $\Sigma \xrightarrow{\phi_{\mathrm{sol}}} \mathcal{E}$ in PDE_Σ are in bijection with those morphisms $\Sigma \xrightarrow{\phi_{\mathrm{sol}}} E$ such that the two composites $\Sigma \xrightarrow{\phi_{\mathrm{sol}}} E \xrightarrow{D_{l,r}} F$ agree.



$\in \mathrm{PDE}_\Sigma \,.$

Now by example 1.16 the morphisms ϕ here are equivalently sections $\phi \in \Gamma_\Sigma(E)$, and by prop. 1.10 these equalize the morphisms D_l, D_r precisely if the action of these as differential operators acting on sections agrees

$$D_l(\phi) = D_r(\phi) \,.$$

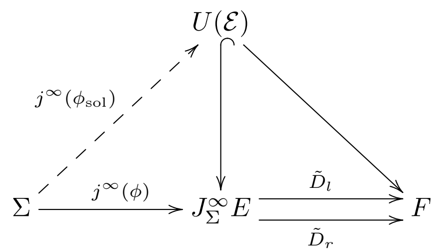
[1]It is here where the non-singularity condition comes in: If the equalizer of $\tilde{D}_l, \tilde{D}_r : J^\infty E \to F$ is not a smooth submanifold, then de facto it does not exist in PDE_Σ as defined here. This is a minor point. To deal with this one passes to an improved category of smooth manifolds where all fiber products exists. This is preferably achieved by a category of "derived" manifolds, whose formal duals are not just plain function algebras, but simplicial function algebras. In the physics literature these are known as BV-complexes. It is fairly straightforward to lift the entire discussion here from smooth manifolds to derived smooth manifolds, and once one does so the non-singular-clauses above may be omitted.

This is the explicit traditional incarnation of the differential equation embodied by the object $\mathcal{E} \in \mathrm{PDE}_\Sigma$.

Yet another way to say this is that the monomorphism $\mathcal{E} \hookrightarrow E$ in PDE_Σ maps under $U : \mathrm{PDE}_\Sigma \to \mathrm{SmoothMfd}_{\downarrow\Sigma}$ to a submanifold inclusion

$$U(\mathcal{E}) \hookrightarrow J^\infty E$$

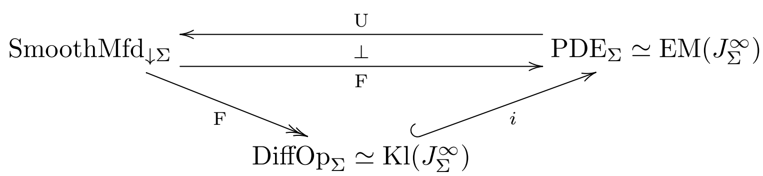
of the jet bundle of E, and that the solutions ϕ_{sol} to the differential equation are those sections $\phi \in \Gamma_\Sigma(E)$ whose jet prolongation, def. 1.7 factors through this inclusion



$\in \mathrm{SmoothMfd}_{\downarrow\Sigma}$.

It is common to notationally suppress the underlying-bundle functor U and just write $\mathcal{E} \hookrightarrow J^\infty_\Sigma E$ if the context is clear. One then also says that $\mathcal{E} \subset J^\infty_\Sigma E$ is the *dynamical shell* of the PDE.

Remark 1.15. In summary, prop. 1.10 and theorem 1.13 say, via prop. 4.16, that jet geometry constitutes the following comonadic situation:



The category of PDEs over Σ (equivalently the Eilenberg-Moore category of J^∞_Σ-coalgebras) has a forgetful functor to the category of pro-finite dimensional smooth bundles over Σ. This functor has a right adjoint, sending any bundle E to the "co-free" differential equation it defines, namely the trivial differential equation on smooth sections of E, for which every section is a solution. Even though these are trivial as differential equations, the morphism between bundles when regarded as cofree differential equations are interesting, they are precisely the differential operators. Hence the cofree functor from bundles to PDEs factors through the full inclusion of the category DiffOp_Σ of bundles with differential operators between them, which is equivalently the Kleisli category, def. 4.17, of J^∞_Σ. Finally

$$J^\infty_\Sigma \simeq \mathrm{U} \circ \mathrm{F}\,.$$

Due to the nature of the factorization through the Kleisli category, it makes sense and is convenient to leave F notationally implicit.

Example 1.16. We have

$$\mathrm{DiffOp}_\Sigma(\Sigma, E) \simeq \mathrm{PDE}_\Sigma(\Sigma, E) \simeq \Gamma_\Sigma(E)\,.$$

Proposition 1.17. *Given a morphism D in DiffOp_Σ represented as a co-Kleisli morphism (remark 4.18) $\tilde{D} : J^\infty_\Sigma E_1 \to E_2$, then its underlying bundle map is the prolongation $p^\infty(\tilde{D})$ according to def. 1.12:*

$$\mathrm{U}(D) \simeq p^\infty(\tilde{D})\,.$$

Proof. The morphism D is identified with a morphism in PDE_Σ of the form $D : \mathrm{F}(E_1) \to \mathrm{F}(E_2)$. The morphism $\tilde{D}$ is the adjunct of this under $(\mathrm{U} \dashv \mathrm{F})$, and conversely, hence, by the formula prop. 4.9 for adjuncts ,

$$D : \mathrm{F}(E_1) \overset{\eta_{\mathrm{F}(E_1)}}{\longrightarrow} \mathrm{F}(\mathrm{U}(\mathrm{F}(E_1))) \overset{\mathrm{F}(\tilde{D})}{\longrightarrow} \mathrm{F}(E_2)\,.$$

Therefore

$$\mathrm{U}(D) : \mathrm{U}(\mathrm{F}(E_1)) \overset{\mathrm{U}(\eta_{\mathrm{F}(E_1)})}{\longrightarrow} \mathrm{U}(\mathrm{F}(\mathrm{U}(\mathrm{F}(E_1)))) \overset{\mathrm{U}(\mathrm{F}(\tilde{D}))}{\longrightarrow} \mathrm{U}(\mathrm{F}(E_2)) .$$

Via $J^\infty_\Sigma \simeq \mathrm{U} \circ \mathrm{F}$ (prop. 4.16) and the formula for the coproduct via the adjunction counit (prop. 4.14) the right hand is indeed the formula for p^∞ from def. 1.12. □

1.2 Horizontal de Rham complex

A key fact of variational calculus is that the de Rham complex of a jet bundle naturally splits into a bicomplex of horizontal and vertical differentials, with the latter encoding the Euler-Lagrange variation of fields. In terms of the characterization of differential operators due to prop. 1.10, the horizontal subcomplex has the following neat formulation.

Definition 1.18 (e.g. [KrVe98, def. 3.27])**.** A *horizontal n-form* α on a jet bundle $J^\infty_\Sigma(E)$ is a differential operator[2] of the form

$$\alpha \colon E \to \wedge^n T^*\Sigma \, . \tag{1}$$

With the de Rham differential $d \colon \Omega^n(\Sigma) \to \Omega^{n+1}(\Sigma)$ on Σ regarded as a differential operator

$$d \colon \wedge^n T^* X \to \wedge^{n+1} T^* X \, , \tag{2}$$

then the *horizontal differential* of a horizontal n-form α is the composite of differential operators

$$d_H \alpha \colon F \overset{\alpha}{\longrightarrow} \wedge^n T^*\Sigma \overset{d}{\longrightarrow} \wedge^{n+1} T^* X \, . \tag{3}$$

The resulting cochain complex $(\Omega^\bullet_H(E), d_H)$ is the *horizontal de Rham complex* of the jet bundle of E.

Remark 1.19. By prop. 1.10 a horizontal n-form as in def. 1.18 is equivalently a bundle morphism of the form $\tilde{\alpha} \colon J^\infty_\Sigma(E) \to \wedge^n T^*\Sigma$. Composed with the canonical bundle morphism $\wedge^n T^*\Sigma \to \wedge^n T^* J^\infty_\Sigma(E)$ induced from the bundle projection $J^\infty_\Sigma(E) \to \Sigma$, this becomes an actual n-form $\tilde{\alpha} \in \Omega^n(J^\infty_\Sigma(E))$ on the jet bundle, whence the name. On the other hand, composed with a jet prolongation $j^\infty(\phi) \colon \Sigma \to J^\infty_\Sigma(E)$, def. 1.7, then

$$j^\infty(\phi)^* \tilde{\alpha} \;:\; \Sigma \xrightarrow{\simeq} J^\infty_\Sigma(\Sigma) \xrightarrow{J^\infty_\Sigma(\phi)} J^\infty_\Sigma(E) \xrightarrow{\tilde{\alpha}} \wedge^n T^*\Sigma$$

is a horizontal n-form on Σ, hence, by example 1.6, just a plain n-form on Σ. We use this interpretation to identify horizontal forms with a subset $\Omega^\bullet_H(E) \subset \Omega^\bullet(J^\infty(E))$. Moreover, we can actually extend the action of d_H to arbitrary forms in $\Omega^\bullet(J^\infty(E))$ as follows. As a graded commutative algebra, $\Omega^\bullet(J^\infty(E))$ is generated by $\Omega^0(J^\infty(E))$ and $d\Omega^0(J^\infty(E))$. The action of d_H on $\Omega^0(J^\infty(E))$, since any 0-form is automatically a horizontal form. Further, let $d_H df = -dd_H f$, for any $f \in \Omega^0(J^\infty(E))$. Having defined d_H on the generators, we extend it to all of $\Omega^\bullet(J^\infty(E))$ as a graded differential. Note that this definition implies the identity $d_H d + dd_H$.

The formulation of jet prolongation in def. 1.7 and of the horizontal complex in def.1.18 in terms of the jet comonad structure of prop. 1.9 makes the following key property of the horizontal differential follow from general abstract reasoning that holds in general models of jet geometry as in remark 1.2.

Proposition 1.20. *Pullback of horizontal forms along jet prolongations intertwines the horizontal differential with the de Rham differential on Σ: for $\phi \in \Gamma_\Sigma(E)$ and $\alpha \in \Omega_H(E)$, we have a natural identification*

$$d_\Sigma(j^\infty(\phi)^*\tilde{\alpha}) = j^\infty(\phi)^*(d_H\tilde{\alpha}) \, .$$

[2]Lets be consistent about the notation for differential operators (maps between spaces of sections) and bundle maps. I would write the maps below as either maps between spaces of sections, or as bundle maps of the form $J^\infty(-) \to (-)$, but not as unadorned bundle maps, as written now. –IK

Proof. Unwinding the definitions, the right hand is the form given by the composite

$$\Sigma \overset{\simeq}{\to} J^\infty_\Sigma(\Sigma) \overset{J^\infty_\Sigma(\phi)}{\longrightarrow} J^\infty_\Sigma(E) \to J^\infty_\Sigma(J^\infty_\Sigma(E)) \overset{J^\infty_\Sigma(\tilde\alpha)}{\longrightarrow} J^\infty_\Sigma(\wedge^n T^*\Sigma) \overset{\tilde d_\Sigma}{\longrightarrow} \wedge^{n+1}T^*\Sigma\,.$$

Since the J^∞_Σ-coproduct is a natural transformation, we may pass $J^\infty_\Sigma(\phi)$ through the coproduct from the left to the right to obtain the equivalent morphism

$$\Sigma \overset{\simeq}{\to} J^\infty_\Sigma(\Sigma) \overset{\simeq}{\to} J^\infty_\Sigma(J^\infty_\Sigma(\Sigma)) \overset{J^\infty_\Sigma(J^\infty_\Sigma(\phi))}{\longrightarrow} J^\infty(J^\infty_\Sigma(E)) \overset{J^\infty_\Sigma(\tilde\alpha)}{\longrightarrow} J^\infty_\Sigma(\wedge^n T^*\Sigma) \overset{\tilde d_\Sigma}{\longrightarrow} \wedge^{n+1}T^*\Sigma\,.$$

By functoriality of J^∞_Σ we may compose this as

$$\Sigma \overset{\simeq}{\to} J^\infty_\Sigma(\Sigma) \overset{\simeq}{\to} J^\infty_\Sigma(J^\infty_\Sigma(\Sigma)) \overset{J^\infty_\Sigma(\tilde\alpha\circ J^\infty_\Sigma(\phi))}{\longrightarrow} J^\infty_\Sigma(\wedge^n T^*\Sigma) \overset{\tilde d_\Sigma}{\longrightarrow} \wedge^{n+1}T^*\Sigma\,.$$

This is the co-Kleisli morphism (remark 4.18) expressing the left hand side of the equation to be established. □

1.3 Variational bicomplex

Definition 1.21. Write $\Omega^\bullet_V(E) \hookrightarrow \Omega^\bullet(J^\infty_\Sigma(E))$ for the joint kernel of the pullback maps along jet prolongations, def. 1.7

$$j^\infty(\phi)^* : \Omega^\bullet(J^\infty_\Sigma(E)) \longrightarrow \Omega^\bullet(\Sigma) \tag{4}$$

along all section $\phi \in \Gamma_\Sigma(E)$. These are called the *vertical differential forms* (sometimes also *contact forms*) on the jet bundle. The vertical forms constitute a differential ideal of $(\Omega^\bullet(J^\infty(E)), d)$, known as the *contact* or *Cartan ideal*. The *vertical differential*

$$d_V \colon \Omega^\bullet(J^\infty(E)) \to \Omega^\bullet_V(E)$$

is

$$d_V := d - d_H\,.$$

Proposition 1.22. *The complex of differential forms on the jet bundle is a direct sum of the horizontal forms from def. 1.18, remark 1.19 with the vertical forms of def. 1.21*

$$\Omega^\bullet(J^\infty_\Sigma E) \simeq \Omega^\bullet_H(E) \oplus \Omega^\bullet_V(E)\,. \tag{5}$$

In fact, the quotient of the de Rham complex $(\Omega^\bullet(J^\infty(E)), d)$ by the differential ideal $\Omega_V(E)$ gives precisely the horizontal de Rham complex $(\Omega^\bullet_H(E), d_H)$.

Considering the above decomposition on 1-forms, $\Omega^1(J^\infty(E)) = \Omega^1_H(E) \oplus \Omega^1_V(E)$, we assign to elements of $\Omega^1_H(E)$ *horizontal* degree 1 and *vertical* degree 0, while to elements of $\Omega^1_V(E)$ *horizontal* degree 0 and *vertical* degree 1. Also, we assign both horizontal and vertical degree 0 to elements of $\Omega^0(J^\infty(E))$. Obviously, the sum of the horizontal and vertical degrees is the total form degree. Since all forms are generated as a graded algebra by forms of total degrees 0 and 1, we have just defined a bigrading on the forms on $J^\infty(E)$, which we denote as $\Omega^\bullet(J^\infty(E)) = \bigoplus_{h,v} \Omega^{h,v}(E)$, where h stands for the *horizontal* and v for *vertical* degrees.

Proposition 1.23. *The horizontal-vertical bigrading and the operators d_H, d_V turns the de Rham complex on $J^\infty(E)$ into a bicomplex, called the* variational bicomplex $(\Omega^{\bullet,\bullet}(E), d_H, d_V)$*, where d_H is of horizontal* *degree* 1 *and vertical degree* 0*, while* d_V *is of horizontal degree* 0 *and vertical degree* 1*.*

$$
\begin{array}{ccccccccccccc}
 & & \Omega^0_H(E) & \xrightarrow{d_H} & \Omega^1_H(E) & \xrightarrow{d_H} & \Omega^2_H(E) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^p & \xrightarrow{d_V} & \Omega^{p+1}_H(E) \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \cdots & & \downarrow{\scriptstyle d_H} & & \downarrow{\scriptstyle d_V} \\
0 & \longrightarrow & \Omega^{0,1}(E) & \xrightarrow{d_H} & \Omega^{1,1}(E) & \xrightarrow{d_H} & \Omega^{2,1}(E) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p,1}(E) & \xrightarrow{d_H} & \Omega^{p+1,1}(E) \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \cdots & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} \\
0 & \longrightarrow & \Omega^{0,2}(E) & \xrightarrow{d_H} & \Omega^{1,2}(E) & \xrightarrow{d_H} & \Omega^{2,2}(E) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{d_H} & \Omega^{p+1,2}(E) \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \cdots & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}
$$

Here the horizontal rows $(\Omega^{\bullet,v\geq 1}(E), d_H)$ *are exact, except at* $\Omega^{p+1,v}(E)$*, and also the vertical columns* $(\Omega^{h,\bullet}(E), d_V)$ *are exact, except at* $\Omega^{h,0}(E)$*.*

Proposition 1.24. *The total complex of the variational bicomplex is isomorphic to the de Rham complex* $(\Omega^\bullet(J^\infty(E)), d)$*.*

Remark 1.25. By the above proposition, the variational bicomplex must fail to be exact in some places whenever its total complex $(\Omega^\bullet(J^\infty(E)), d)$ has non-trivial cohomology, which is isomorphic to $H^\bullet(E)$, since $J^\infty(E)$ is contractible to E. In the bicomplex, these de Rham classes are concentrated in the $v = 0$ horizontal row and, in a way to be described below, in the $h = p + 1$ vertical column. In fact, all of these cohomology classes are controlled precisely by $H^n_{\mathrm{dR}}(E)$. This is captured by the *Euler-Lagrange complex*, to which we turn below in def. 1.35.

The bigraded forms in the variational bicomplex may naturally be identified with certain differential operators. This is particularly important for the $(p + 1, 1)$-forms where the following operation will serve to identify the variational derivative of a Lagrangian with the differential operator that embodies the corresponding Euler-Lagrange equations of motion.

Definition 1.26. For $n, k \in \mathbb{N}$ write

$$\widetilde{(-)} \colon \Omega^{n,k}(E) \longrightarrow \mathrm{DiffOp}_\Sigma(\wedge^k_E(VE), \wedge^n T^*\Sigma) \tag{6}$$

for the map from (n, k)-bigraded differential forms as in prop. 1.23, to differential operators, which sends $\beta \in \Omega^{n,k}(E)$ to the differential operator $\tilde{\beta}$ whose value on any $(\phi; u_1 \wedge \cdots \wedge u_k) \in \Gamma(\wedge^k_E(VE))$ is

$$\tilde{\beta}[\phi; u_1 \wedge \cdots \wedge u_k] := (j^\infty\phi)^*(\iota_{p^\infty u_1 \wedge \cdots \wedge p^\infty u_k}\beta)\,, \tag{7}$$

where the vector fields u_i have been prolonged to the evolutionary vector fields $p^\infty u_i$, as discussed in Remark **??**.

Notice that the bundle $VE \to E \to \Sigma$, or a tensor power of it, is a vector bundle over E, but may not be linear over Σ if E itself is not a vector bundle. Write $\mathrm{DiffOp}^{E\text{-lin}}_\Sigma(\wedge^k_E(VE), \wedge^n T^*\Sigma)$ for those differential operators which are linear over E.

Proposition 1.27. *The construction in def. 1.26 constitutes a linear isomorphism onto those differential operators that are linear over* E*:*

$$\widetilde{(-)} \colon \Omega^{n,k}(E) \xrightarrow{\simeq} \mathrm{DiffOp}^{E\text{-lin}}_\Sigma(\wedge^k_E(VE), \wedge^n T^*\Sigma) \tag{8}$$

Definition 1.28. For $k \geq 1$, there is a map (*formal adjoint*)

$$(-)^* \colon \mathrm{DiffOp}_\Sigma^{E\text{-lin}}(\wedge_E^k(VE), \wedge^{p+1}T^*\Sigma) \longrightarrow \mathrm{DiffOp}_\Sigma^{E\text{-lin}}(\mathbb{R} \times \wedge_E^{k-1}(VE), \wedge^{p+1}T^*\Sigma \otimes_E V^*E) \tag{9}$$

which is uniquely characterized [ViKr99, Sec.5.2.3] by the condition that for every differential operator $D \in \mathrm{DiffOp}_\Sigma^{E\text{-lin}}(\wedge_E^k(VE), \wedge^{p+1}T^*\Sigma)$ there is an

$$\omega_D \in \mathrm{DiffOp}_\Sigma^{E\text{-lin}}(\mathbb{R} \times \wedge_E^k(VE), \wedge^p T^*\Sigma) \tag{10}$$

such that for every $f \in C^\infty(\Sigma)$ and every $(\phi; u_1 \wedge \cdots \wedge u_k) \in \Gamma(\wedge_E^k(VE))$ we have

$$fD[\phi; u_1 \wedge \cdots \wedge u_k] - D^*[\phi; f, u_1 \wedge \cdots \wedge u_{k-1}] \cdot u_k = d_\Sigma \omega_D[\phi; f, u_1 \wedge \cdots u_{k-1}, u_k]\,. \tag{11}$$

1.4 Euler-Lagrange complex

Recall from prop. 1.23 that any 1-form on $J^\infty(E)$ can be uniquely decomposed into its horizontal and vertical parts.

Definition 1.29. The subspace of *order-0 vertical* 1*-forms*

$$\Omega^1_{V,0}(E) \subset \Omega^1_V(E)$$

is the image of the projection of the forms $(\pi^0_\infty)^*[\Omega^1(E)]$ onto their vertical parts, where we take the pullback along the natural projection $\pi^0_\infty \colon J^\infty(E) \to E$.

Definition 1.30. For $k \geq 1$, the subspace of *(k-vertical) source forms* is

$$\Omega_S^{p+1,k}(E) := \Omega^{p+1,k-1}(E) \wedge \Omega^1_{V,0}(E)\,.$$

Remark 1.31. The 1-vertical source forms of def.1.30 are also known as *dynamical form* or *Euler-Lagrange forms*, while 2-vertical source forms are known as *Helmholtz forms* [PRWM15].

Source forms are a subspace of $\Omega^{p+1,\bullet}(E)$ forms, but can also be obtained by means of an idempotent projection $\mathcal{I} \colon \Omega^{p+1,\bullet}(E) \to \Omega^{p+1,\bullet}(E)$, called the *interior Euler* operator.

Definition 1.32. The *interior Euler map* [And89, Sec.2.B] is the map

$$\mathcal{I} \colon \Omega^{p+1,k}(E) \to \Omega^{p+1,k}(E) \tag{12}$$

defined on any β as the equivalent differential operator (via prop. 1.27)

$$\widetilde{\mathcal{I}(\beta)}[\phi; u_1 \wedge \cdots \wedge u_k] := \frac{1}{k} \sum_{a=1}^{k} (-)^{k-a} \tilde{\beta}^*[\phi; 1, u_1 \wedge \cdots \widehat{u_a} \cdots \wedge u_k] \cdot u_a. \tag{13}$$

(where on the right we have the formal adjoint of def. 1.28 applied to the differential operator of def. 1.26). The *higher Euler operator* is the composite

$$\delta_V := \mathcal{I} \circ d_V \colon \Omega^{p+1,k}(E) \to \Omega^{p+1,k+1}(E)\,. \tag{14}$$

Remark 1.33. For $k = 0$ then δ_V is better known as the *Euler-Lagrange derivative* and for $k = 1$ and restricted to source forms, def. 1.30, then δ_V is better known as the *Helmholtz operator*.

Proposition 1.34. *The higher Euler operator is a projection, $\mathcal{I} \circ \mathcal{I} = \mathcal{I}$. Its image is the space of source forms, def. 1.30, and its kernel is the space of horizontally exact forms*

$$\mathrm{im}(\mathcal{I}) \cong \Omega_S^{p+1,k}(E), \tag{15}$$

$$\ker(\mathcal{I}) \cong \mathrm{im}(d_H). \tag{16}$$

In particular prop. 1.34 means that the Euler operators continue the complex of horizontal forms, def. 1.18, by source forms, def. 1.30:

Definition 1.35. The *Euler-Lagrange complex* of E is the chain complex

$$\Omega^\bullet_{\mathrm{EL}_\Sigma}(E) := 0 \to \Omega^0_H(E) \xrightarrow{d_H} \Omega^1_H(E) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p+1}_H(E) \xrightarrow{\delta_V} \Omega^{p+1,1}_S \xrightarrow{\delta_V} \Omega^{p+1,2}_S \xrightarrow{\delta_V} \cdots \tag{17}$$

built from the horizontal derivatives d_H of def. 1.18 and the Euler operators δ_V of def. 1.32.

Proposition 1.36. *For $k \geq 1$ we have an exact sequence*

$$0 \to \Omega^{0,k} \xrightarrow{d_H} \Omega^{1,k} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n,k} \xrightarrow{\mathcal{I}} \Omega^{n,k}_S \to 0 \tag{18}$$

formed by the horizontal differentials d_H of def. 1.18 and the interior Euler operator $\mathcal{I}$ of def. 1.32. Hence the variational bicomplex in prop. 1.23 is augmented as follows, with exact rows as shown below. The dashed morphisms indicate how the Euler-Lagrange complex (def. 1.35) sits in this bicomplex.

$$\begin{array}{ccccccccccccccc}
 & & \Omega^0_H(E) & \xrightarrow{d_H} & \Omega^1_H(E) & \xrightarrow{d_H} & \Omega^2_H(E) & \xrightarrow{d_H} \cdots \xrightarrow{d_H} & \Omega^p_H(E) & \xrightarrow{d_H} & \Omega^{p+1}_H(E) & \overset{\delta_V}{\searrow} & & & \\
 & & \downarrow d_V & & \downarrow d_V & & \downarrow d_V & \cdots & \downarrow d_V & & \downarrow d_V & & & & \\
0 & \longrightarrow & \Omega^{0,1}(E) & \xrightarrow{d_H} & \Omega^{1,1}(E) & \xrightarrow{d_H} & \Omega^{2,1}(E) & \xrightarrow{d_H} \cdots \xrightarrow{d_H} & \Omega^{p,1}(E) & \xrightarrow{d_H} & \Omega^{p+1,1}(E) & \underset{\leftarrow}{\xrightarrow{\mathcal{I}}} & \Omega^{p+1,1}_S(E) & \longrightarrow & 0 \\
 & & \downarrow d_V & & \downarrow d_V & & \downarrow d_V & \cdots & \downarrow d_V & & \downarrow d_V & \overset{\delta_V}{\searrow} & \downarrow \delta_V & & \\
0 & \longrightarrow & \Omega^{0,2}(E) & \xrightarrow{d_H} & \Omega^{1,2}(E) & \xrightarrow{d_H} & \Omega^{2,2}(E) & \xrightarrow{d_H} \cdots \xrightarrow{d_H} & \Omega^{p,2}(E) & \xrightarrow{d_H} & \Omega^{p+1,2}(E) & \underset{\leftarrow}{\xrightarrow{\mathcal{I}}} & \Omega^{p+1,1}_S(E) & \longrightarrow & 0 \\
 & & \downarrow d_V & & \downarrow d_V & & \downarrow d_V & \cdots & \downarrow d_V & & \downarrow d_V & & \downarrow \delta_V & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(For $k = 0$ we instead have theorem 1.39 below.)

Proposition 1.37. *The definition of the variational bicomplex, prop. 1.23, and of the Euler-Lagrange complex, prop. 1.35 of a jet bundle is contravariantly functorial in differential operators mapping via their prolongation, def. 1.12, between jet bundles.*

For $E, F, F' \in \mathrm{SmoothMfd}_{\downarrow\Sigma}$ and $D : E \to F$, $D' : F \to F'$ differential operators, then:

(i) [And89, Prop.1.6] The prolongation $p^\infty \tilde{D} \colon J^\infty E \to J^\infty F$ of def. 1.12 preserves both the horizontal and vertical forms (Definitions 1.18 and 1.21, Proposition 1.22)

$$(p^\infty \tilde{D})^* \Omega^\bullet_H(F) \subseteq \Omega^\bullet_H(E) \quad \textit{and} \quad (p^\infty \tilde{D})^* \Omega^\bullet_V(F) \subseteq \Omega^\bullet_V(E). \tag{19}$$

(ii) [And89, Thm.3.15] The pullback along the prolongation $p^\infty \tilde{D} \colon J^\infty E \to J^\infty F$ (def. 1.12) is a cochain map for the variational bicomplex (Prop. 1.23), respecting both degrees and both differentials,

$$(p^\infty \tilde{D})^* \colon (\Omega^{h,v}(F), d_H, d_V) \longrightarrow (\Omega^{h,v}(E), d_H, d_V). \tag{20}$$

(iii) Considering the differential operators D and D', the composition of the pullbacks along prolongations is equal to the pullback along the composition of the prolongations, which is also equal to the pullback along the prolongation of the composition of the differential operators,

$$(p^\infty \tilde{D})^* \circ (p^\infty \tilde{D}')^* = (p^\infty \tilde{D}' \circ p^\infty \tilde{D})^* = (p^\infty \widetilde{D' \circ D})^*. \tag{21}$$

(iv) The interior Euler projected pullback along the prolongation $p^\infty \tilde{D}$ maps source forms into source forms (def. 1.30),

$$\mathcal{I} \circ (p^\infty \tilde{D})^* \Omega_S^{p+1,k}(F) \subseteq \Omega_S^{p+1,k}(E). \tag{22}$$

(v) [And89, Thm.3.21] The map between the Euler-Lagrange complexes

$$\Omega^\bullet_{\mathrm{EL}_\Sigma}(F) \longrightarrow \Omega^\bullet_{\mathrm{EL}_\Sigma}(E) \tag{23}$$

defined by the pullback $(p^\infty \tilde{D})^*$ on the horizontal forms $\Omega^{\bullet,0}(-)$ and by the interior Euler projected pullback $\mathcal{I} \circ (p^\infty \tilde{D})^*$ on source forms $\Omega_S^{p+1,\bullet}(-)$ is a cochain map, respecting all the gradings and differentials.

(vi) The composition of the interior Euler projected pullbacks along the prolongations of the differential operators D and D' is equal to the interior Euler projected pullback along the composition of the differential operators,

$$\mathcal{I} \circ (p^\infty \tilde{D})^* \circ \mathcal{I} \circ (p^\infty \tilde{D}')^* = \mathcal{I} \circ (p^\infty \widetilde{D' \circ D}). \tag{24}$$

Sketch of proof. Statement (i) is a fundamental property of horizontal and vertical forms. For horizontal forms, it follows straight from the definitions. For vertical forms, the simplest proof follows from an elementary calculation in local coordinates, which can be found in the cited reference.

Essentially, all other statements follow from (i) and basic properties of pullbacks of forms and of differential operators. For (ii), it suffices to combine with (i) the known property that pullbacks commute with the de Rham differential. For (iii), it suffices to recall the composition property of pullbacks and of prolongations of differential operators (Proposition 1.10). For (iv), it suffices to combine (ii) with the fact that source forms are defined as the image of $\mathcal{I}$. For (v), the horizontal part of $\mathrm{EL}^\bullet$ is already taken care of by (ii). Also, since source forms can be thought of as canonical representatives of equivalence classes modulo d_H, which by (ii) are preserved by the pullback, the rest of $\mathrm{EL}^\bullet$ is also covered. The same argument based on equivalence classes also covers (vi).

Applying the desired statements to 1-parameter families of differential operators, we can obtain obvious corresponding infinitesimal versions, applicable to vector fields that preserve vertical forms. However, since some of these vector fields do not come from linearizing such 1-parameter families of differential operators, they could also be proven directly by in infinitesimal form, as for example in [And89, Prop.3.17] and [And89, Thm.3.21]. □

Remark 1.38. The statements in prop. 1.37 have obvious infinitesimal versions that apply to any vector field from $\mathfrak{X}_H(E) + \mathfrak{X}_{\mathrm{ev}}(X)$ (Definition **??** and the remarks following it).

Theorem 1.39 (e.g. [And89, Thm.5.9]). *For E a bundle over Σ, there is a chain map, given degreewise by projection on horizontal forms and on vertical source forms, respectively from the Euler-Lagrange complex of E, def. 1.35, to the de Rham complex of $J^\infty_\Sigma E$:*

$$\Omega^\bullet_{\mathrm{dR}}(E) \xrightarrow{\simeq_{\mathrm{qi}}} \Omega^\bullet_{\mathrm{dR}}(J^\infty_\Sigma E) \xrightarrow{\simeq_{\mathrm{qi}}} \Omega^\bullet_{\mathrm{EL}_\Sigma}(E)\,.$$

This is a quasi-isomorphism, i.e. it induces isomorphism on all cohomology groups:

$$H^\bullet(\Omega_{\mathrm{dR}}(E)) \simeq H^\bullet(\Omega_{\mathrm{EL}_\Sigma}(E)) \tag{25}$$

Moreover, this chain map is a natural transformation with respect to the functoriality in prop. 1.37.

1.5 Equations of motion and Lagrangians

Definition 1.40. For $\omega \in \Omega_S^{p+1,1}(E)$ a source form, def. 1.30, then the partial differential equation on sections $\phi \in \Gamma_\Sigma(E)$ it induces is

$$\underset{v\in\Gamma(VE)}{\forall} j^\infty(\phi)^* \iota_v \omega = 0\,,$$

saying that for all vertical tangent vectors v, the pullback of the contracted form $\iota_v\omega$ along the jet prolongation, def. 1.7, of ϕ vanishes.

Proposition 1.41. *As an object of* PDE_Σ*, via theorem 1.13 and remark 1.14. the differential equation in def. 1.40 is the equalizer of*

1. *the differential operator*

$$\tilde{\omega} : E \longrightarrow \wedge^{p+1} T^*\Sigma \times_\Sigma V^*E$$

that corresponds to ω *under the isomorphism of prop. 1.27;*

2. *the "0-morphism"*

$$\tilde{0} : E \longrightarrow \wedge^{p+1} T^*\Sigma \times_\Sigma V^*E$$

which sends any point $(\sigma, e, j) \in J^\infty E$ *to the pair consisting of* $0 \in \wedge^{p+1} T^*_\sigma \Sigma$ *and* $0 \in V^*_e E \hookrightarrow (V^*E)_\sigma$.

Proof. By direct comparison of def. 1.26 with def. 1.40. □

Remark 1.42. Prop. 1.41 suggests that the differential equation induced by the source form ω should be thought of the *kernel* or *fiber* of $\tilde{\omega}$. However, a kernel or fiber of D would be the pullback of a point inclusion into its codomain, and preferably of the zero point in an object with abelian group structure. But this is not the case here. However, when below in section 2 we broaden the perspective from PDE_Σ to the sheaf topos over it, then source forms ω are given equivalently by maps into an abelian "moduli space" $\mathbf{\Omega}_S^{p+1,1}$, and then indeed the differential equation in question turns out to be precisely the kernel of these representing maps. This is the content of prop. 2.36 below.

Definition 1.43. Given a $(p+1)$-dimensional smooth manifold Σ and a field bundle $E \to \Sigma$, then

1. a *globally defined local Lagrangian* is a horizontal $(p+1)$-form

$$\mathrm{L} \in \Omega_H^{p+1}(E)$$

according to def. 1.18;

2. the *Euler-Lagrange form* of **L** is its image under the Euler operator, def. 1.32,

$$\mathrm{EL} := \delta_V \mathrm{L} \in \Omega_S^{p+1,1}(E)\,,$$

3. the *Euler-Lagrage equation* $\mathcal{E}$ of L is the differential equation induced by EL via prop. 1.41.

(The prequantum-analog of this definition we give in def. 2.34 below.)

Remark 1.44. Unwinding the definitions, the concise concepts in def. 1.43 reproduce more common expression found in the literature as follows.

1. The vertical derivative, def. 1.21, of the Lagrangian form L, splits uniquely into the sum of a source form EL, def. 1.30, and a horizontally exact form

$$\mathrm{d}_V \mathrm{L} = \mathrm{EL} - \mathrm{d}_H \theta\,.$$

The source form is indeed $\delta_V \mathrm{L} = \mathrm{EL}$, by prop. 1.34. This decomposition is known as the *first variation formula* in the geometric literature on the calculus of variations.

In components, EL is obtained from $d_V \mathbf{L}$ by a formal integration by parts, def. 1.28, that removes all the vertical differentials of jet coordinates involving derivatives. The boundary term picked up in this operation is $d_H\theta$. This is the classical recipe for obtaining Euler-Lagrange equations.

Notice that EL is unaffected by a change to the Lagrangian of the form $\mathrm{L} \mapsto \mathrm{L}+\mathrm{d}_H K$, for any horizontal p-form K (though θ is affected).

2. The submanifold inclusion
$$\mathcal{E} \hookrightarrow J^\infty_\Sigma E$$
that characterizes the Euler-Lagrange equation in def. 1.43 via remark 1.14 (notationally suppressing the underlying bundle functor U) is also called the *dynamical shell* or just *shell* for short.

There exist situations when, even though the equations of motion are given by a globally defined source form $\mathrm{EL} \in \Omega^{p+1,1}_S(E)$, def. 1.29, and for any contractible open $U \subset J^\infty F$ there exists a local Lagrangian L_U, according to def. 1.43, such that $\delta_V \mathrm{L}_U = \mathrm{EL}|_U$, there may not exist any globally defined local Lagrangian $\mathrm{L} \in \Omega^{n,0}(E)$ such that the same formula holds on all of $J^\infty E$. Examples include the charged point particle in an external non-exact electromagnetic field, also the usual 2-dimensional and higher-dimensional WZW models [[gawedzki?]], and higher dimensional Chern-Simons models [[XXX]]. Such equations are *locally* but not *globally variational*.

To decide whether a source form EL is locally variational, we use the local exactness of the Euler-Lagrange complex (Thm. 1.39):

Definition 1.45. A 1-vertical source form $\mathrm{EL} \in \Omega^{p+1,1}_S(E)$, def. 1.29, is called *locally variational* if the identity $\delta_V \mathrm{EL} = 0$ (which is known as the *Helmholtz condition*). The source form EL is called *globally variational* if there exists a local Lagrangian $\mathrm{L} \in \Omega^{p+1,0}(E)$ such that $\mathrm{EL} = \delta_V \mathrm{L}$.

Example 1.46. Let $\Sigma = \mathbb{R}^d$ and let E be the trivial real line bundle over Σ. Let η be the Minkowski metric on $\mathbb{R}^d$. We write dvol_Σ for the corresponding volume form. Write $\{\{x^\mu\}, \phi, \{\phi_{,\mu}\}, \{\phi_{,\mu\nu}\}, \cdots\}$ for the canonical coordinates on $J^\infty E$.

In these coordinates, the Lagrangian density for the free scalar field on Σ reads
$$L = \tfrac{1}{2}\left(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\mathrm{dvol}_\Sigma + m^2\phi^2\right) .$$

Its vertical differential is
$$d_V L = \eta^{\mu\nu}\phi_{,\mu} d_V\phi_{,\nu} \wedge \mathrm{dvol}_\Sigma + m^2 \phi d_V \phi\,.$$
In order to find EL and θ, we need to exhibit this as the sum of the form $(-) \wedge d_V\phi - d_H\theta$.

The key to find θ is to realize $d_V\phi_{,\nu} \wedge \mathrm{dvol}_\Sigma$ as a horizontal derivative. Since $d_H\phi = \phi_{,\mu}dx^\mu$ this is accomplished by
$$d_V\phi_{,\nu} \wedge \mathrm{dvol}_\Sigma = d_V d_H\phi \wedge \iota_{\partial_\nu}\mathrm{dvol}_\Sigma$$

Hence we set
$$\theta := \eta^{\mu\nu}\phi_{,\mu} d_V\phi \wedge \iota_{\partial_\nu}\mathrm{dvol}_\Sigma\,.$$
This way we have
$$d_H\theta = -\eta^{\mu\nu}\left(\phi_{,\mu\nu} d_V\phi + \eta^{\mu\nu}\phi_{,\mu} d_V\phi_{,\nu}\right) \wedge \mathrm{dvol}_\Sigma\,,$$
which is to be read as the local version of integration by parts.

In conclusion this yields the decomposition
$$d_V L = \underbrace{-\left(\eta^{\mu\nu}\phi_{,\mu\nu} + m^2\phi\right) d_V\phi \wedge \mathrm{dvol}_\Sigma}_{\mathrm{EL}} - d_H\theta\,.$$

Hence for $\Phi : \Sigma \to \mathbb{R}$ the ϕ-component of a section, its equation of motion is the Klein-Gordon equation
$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\Phi = 0\,.$$

1.6 Transgression

We review now the integration of the local Lagrangian form data over submanifolds of Σ of codimension k. This gives

- $k = 0$ – The action functional, section 1.6.1;
- $k = 1$ – The covariant phase space, section 1.6.2.

Remark 1.47. In the classical theory this looks somewhat unsystematic, as in one case one is integrating the Lagrangian form, in the other case one is fiber integrating the form θ appearing in its variational derivative. That this actually does follow a unified pattern is revealed by the prequantum theory which we turn to below in section 2.

1.6.1 Action functional

Definition 1.48. Given a smooth bundle E over Σ, write $\mathbf{\Gamma}_\Sigma(E)$ for its space of smooth sections regarded as a diffeological space.

Then jet prolongation of sections (def. 1.7) followed by evaluation of sections gives a smooth function

$$\mathrm{ev} j^\infty : \Sigma \times \Gamma_\Sigma(E) \stackrel{(\mathrm{id}, j^\infty)}{\longrightarrow} \Sigma \times \mathbf{\Gamma}_\Sigma(J^\infty_\Sigma E) \stackrel{\mathrm{ev}}{\longrightarrow} J^\infty_\Sigma E \,.$$

Notice that the space $\Sigma \times \Gamma_\Sigma(E)$, being a Cartesian product, has a canonical bicomplex structure on its de Rham complex, coming simply from the de Rham differential along Σ and along $\Gamma_\Sigma(E)$, separately.

Proposition 1.49 ([Zu87]). *Pullback of differential forms along* $\mathrm{ev} j^\infty$

$$(\mathrm{ev} j^\infty) : \Omega^\bullet(J^\infty_\Sigma E) \longrightarrow \Omega^\bullet(\Sigma \times \mathbf{\Gamma}_\Sigma(E))$$

constitutes an inclusion of bicomplexes

$$(\mathrm{ev} j^\infty) : \Omega^{\bullet,\bullet}(E) \simeq \Omega^{\bullet,\bullet}_{\mathrm{loc}}(\Sigma \times \mathbf{\Gamma}_\Sigma(E)) \hookrightarrow \Omega^{\bullet,\bullet}(\Sigma \times \mathbf{\Gamma}_\Sigma(E))$$

from the variational bicomplex, prop. 1.23, into the canonical bicomplex on the Cartesian product,

The image of the inclusion is the called the bicomplex of local *differential forms on* $\Sigma \times \Gamma_\Sigma(E)$

This implies that there is a well defined action functional associated with a horizontal $(p+1)$-form:

Definition 1.50. For compact Σ the *action functional* is the smooth function

$$S_{(-)}(-) : \Omega^{p+1}_H(E) \times \Gamma_\Sigma(E) \stackrel{(\mathrm{ev} j^\infty)^*}{\hookrightarrow} \Omega^{p+1,0}(\Sigma \times \Gamma_\Sigma(E)) \times \Gamma_\Sigma(E) \stackrel{\mathrm{ev}}{\longrightarrow} \Omega^{p+1}(\Sigma) \stackrel{\int_\Sigma}{\longrightarrow} \mathbb{R} \,.$$

1.6.2 Covariant phase space

Given a local Lagrangian $L \in \Omega^{p+1}_H(E)$, a choice of $\theta \in \Omega^{p,1}(E)$ from remark 1.44 is called a choice of *presymplectic potential current.* Its vertical derivative

$$\omega := d_V \theta$$

is called the *presymplectic current.*

Given the a choice of compact p-dimensional submanifold $\Sigma_p \hookrightarrow \Sigma$, the diffeological space $\Gamma_{\Sigma_p}(E)$ equipped with the differential form 2-form

$$\int_{\Sigma_p} j^\infty(-)^*(\omega) \in \Omega^2(\mathbf{\Gamma}_{\Sigma_p}(E))$$

is the *presymplectic off-shell covariant phase space.* Its restriction to the shell is the on-shell *presymplectic covariant phase space.* A good source is [Zu87].

The quotient of this by the kernel of ω is the *reduced symplectic covariant phase space.*

Generally $\int_{\Sigma_p} j^\infty(-)^*\theta$ won't pass to this quotient as a globally defined form, but only as a connection on a principal bundle. This is what we get to in section 2.4.

1.7 Symmetries and conserved currents

Definition 1.51 ([Marvan86, 3.2])**.** Given $\mathcal{E} \in \mathrm{PDE}_\Sigma$, corresponding under theorem 1.13 to a J^∞_Σ coalgebra given by a morphism in $\mathrm{SmoothMfd}_{/\Sigma}$ of the form $e : \mathcal{E} \longrightarrow J^\infty_\Sigma \mathcal{E}$, its *vertical tangent bundle PDE* is the object $V\mathcal{E} \in \mathrm{PDE}_\sigma$ $V\mathcal{E} \in \mathrm{PDE}_\Sigma$ for coalgebra given by the image of e under the vertical tangent bundle functor:

$$Ve : V\mathcal{E} \longrightarrow VJ^\infty_\Sigma \mathcal{E} \simeq J^\infty_\Sigma V\mathcal{E} \,.$$

An *infinitesimal symmetry* v on $\mathcal{E}$ is a section

$$\begin{array}{c} V\mathcal{E} \\ {\scriptstyle v}\nearrow \; \downarrow \\ \mathcal{E} \end{array}$$

in PDE_Σ of the canonical projection morphism.

Definition 1.52. Given a globally defined local Lagrangian $L \in \Omega^{p+1}_H(E)$, def. 1.43, then an *infinitesimal variational symmetry* is an infinitesimal symmetry v of E, def. 1.51, hence just a vertical vector field on the bundle E with its jet extension $j^\infty v$, such that there is $\Delta_v \in \Omega^p_H(E)$ with

$$\mathcal{L}_v L = d_H \Delta_v \,.$$

Definition 1.53. Given a globally defined local Lagrangian $L \in \Omega^{p+1}_H(E)$, def. 1.43, then an *on-shell consered current* for its dynamics is a horizontal p-form

$$J \in \Omega^P_H(E)$$

such that it is horizontally closed when restricted to the shell $\mathcal{E} \overset{\ker(\mathrm{EL}(L))}{\hookrightarrow} E$:

$$(d_H J)|_{\mathcal{E}} = 0 \,.$$

Proposition 1.54 (Noether's first variational theorem)**.** *Given a variational symmetry as in def. 1.52, then*

$$J_v := \iota_v \theta - \Delta_v \quad \in \Omega^p_H(E)$$

with θ from remark 1.44, is an on-shell conserved current, def. 1.53, called a Noether current *for v.*

Proof. By Cartan's formula for Lie derivatives on $J^\infty_\Sigma E$

$$\mathcal{L}_v L = \iota_v dL + \underbrace{d\iota_v L}_{=0} \,,$$

where the second summand vanishes due to v being vertical and L being horizontal. By remark 1.44 the first term is

$$\mathcal{L}_v L = \iota_v \mathrm{EL} + d_H \iota_v \theta \,,$$

where we used that the vertical contraction ι_v anti-commutes with the horizontal differential d_H. In summary this gives

$$d_H(\iota_v \theta - \Delta_v) = \iota_v \mathrm{EL} \,.$$

The claim follows since $\mathrm{El}|_{\mathcal{E}} = 0$ by the very definition of $\mathcal{E}$.

2 Prequantum local Lagrangian Field Theory

We now go beyond the existing literature and set up prequantum local Lagrangian field theory in terms of higher Euler-Lagrange gerbes over a site of differential operators.

2.1 PDE Homotopy theory

We discuss how the category PDE_Σ (theorem 1.13) of partial differential equations on sections of smooth bundles sits inside a homotopy theory (an ∞-category) $\mathrm{PDE}_\Sigma(\mathbf{H})$ that contains complexes of sheaves over PDE_Σ, as well as PDEs on sections of stacky bundles. This is used below in 2.3 to construct Euler-Lagrange p-gerbes, which constitute globally defined prequantum local Lagrangian field theories. Moreover, considering such EL p-gerbes on genuinely stacky bundles means to consider such prequantum local Lagrangian theories with gauge symmetries and higher gauge symmetries-of-symmetries.

(The following definitions and statements are with more detail in [dcct].)

Definition 2.1. Write SmoothCartSp for the category of smooth manifolds of the form $\mathbb{R}^n$, for $n \in \mathbb{N}$, regarded as a site with the standard coverage by open covers. Similarly, write FormalSmoothCartSp for the site of formal Cartesian spaces. This is the full subcategory

$$\mathrm{FormalSmoothCartSp} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of that of commutative $\mathbb{R}$-algebras on those of the form $C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{D})$, where $C^\infty(\mathbb{R}^n)$ is the algebra of smooth functions on $\mathbb{R}^n$ for any $n \in \mathbb{N}$, and where $C^\infty(\mathbb{D}) \simeq \mathbb{R} \oplus V$ with V nilpotent. We regard this as a site by taking the coverings to be of the form $\{U_i \times \mathbb{D} \stackrel{(\phi_i, \mathrm{id})}{\longrightarrow} X \times \mathbb{D}\}$, for $\{U_i \stackrel{\phi_i}{\longrightarrow} X\}$ an ordinary open cover.

We consider now the sheaf toposes and ∞-toposes over these sites (4.2).

Definition 2.2. Write

$$\mathrm{SmoothSet} := \mathrm{Sh}(\mathrm{SmoothCartSp})$$

for the sheaf topos over the site of smooth manifolds from def. 2.1. Write

$$\mathrm{Smooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{SmoothCartSp})$$

for the homotopy theory of simplicial sheaves over this site. Similarly, write

$$\mathrm{FormalSmoothSet} := \mathrm{Sh}(\mathrm{FormalSmoothCartSp})$$

and

$$\mathrm{FormalSmooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{FormalSmoothCartSp})$$

Proposition 2.3. *There is a system of fully faithful inclusions of categories and ∞-categories of spaces as follows*

$$\begin{array}{ccccccccc}
\mathrm{Set} & \hookrightarrow & & & & \longrightarrow & \infty\mathrm{Grpd} & & \\
\downarrow & & & & & & \downarrow & & \\
\mathrm{SmoothMfd} & \hookrightarrow & \mathrm{SmoothMfd} & \hookrightarrow & \mathrm{SmoothSet} & \hookrightarrow & \mathrm{Smooth}\infty\mathrm{Grpd} & =: & \mathbf{H}_{\Re} \\
\downarrow{\scriptstyle \iota} & & & & \downarrow{\scriptstyle \iota_!} & & \downarrow{\scriptstyle \iota_!} & & \downarrow{\scriptstyle \iota_!} \\
\mathrm{FormalSmoothMfd} & \hookrightarrow & & \longrightarrow & \mathrm{FormalSmoothSet} & \hookrightarrow & \mathrm{FormalSmooth}\infty\mathrm{Grpd} & =: & \mathbf{H}
\end{array}$$

Moreover, the canonical embedding of the category of smooth Cartesian spaces into that of formal smooth Cartesian spaces is coreflective, i.e. it has a right adjoint (given by forgetting the infinitesimal thickening)

$$\mathrm{SmoothCartSp} \underset{\longleftarrow}{\overset{\iota}{\hookrightarrow}} \mathrm{FormalSmoothCartSp} \; .$$

This adjoint pair induces an adjoint quadruple of functors and compatibly of ∞-functors

$$
\begin{array}{ccc}
\mathrm{SmoothSet} & \begin{array}{c} \xrightarrow{\iota_!} \\ \xleftarrow{\iota^*} \\ \xrightarrow{\iota_*} \\ \xleftarrow{\iota^!} \end{array} & \mathrm{FormalSmoothSet} \\
\downarrow \uparrow \tau_0 & & \downarrow \uparrow \tau_0 \\
\mathrm{Smooth}\infty\mathrm{Grpd} & \begin{array}{c} \xrightarrow{\iota_!} \\ \xleftarrow{\iota^*} \\ \xrightarrow{\iota_*} \\ \xleftarrow{\iota^!} \end{array} & \mathrm{FormalSmooth}\infty\mathrm{Grpd}
\end{array} .
$$

Definition 2.4. We write

$$(\Re \dashv \Im) := (\iota_! \circ \iota^* \dashv \iota_* \circ \iota^*) : \mathbf{H} \longrightarrow \mathbf{H}$$

for the induced adjoint pair of an ∞-comonad $\Re$ and ∞-monad $\Im$ acting on $\mathbf{H}$.

Example 2.5. For $X \times \mathbb{D} \in \mathbf{H}$ represented by a formal smooth manifold, then $\Re(X \times \mathbb{D}) \simeq X$, hence $\Re$ is the operation of *reduction* of infinitesimal thickening. Accordingly, by adjointness, a space of the form $\Im X$ is characterized by the property that probing it by any formal smooth manifold is equivalent to probing it just by the underlying reduced manifold

$$\frac{U \times \mathbb{D} \longrightarrow \Im X}{U \longrightarrow X} .$$

Hence $\Im X$ may be thought of as obtained from X by "identifying all infinitesimal close points". From this perspective the adjunction unit

$$\eta_\Sigma : \Sigma \longrightarrow \Im\Sigma$$

has the interpretation of sending all infinitesimal neighbours of a global point $x : * \to X$ to that global point.

Proposition 2.6. *For all $\Sigma \in \mathbf{H}$, the $\Im$-unit is an epimorphism*

$$\eta_\Sigma : \ \Sigma \twoheadrightarrow \Im\Sigma \ .$$

Proof. We need to check that η_Σ becomes a surjection of sets of connected components of stalk ∞-groupoids. But in fact for any simplicial presheaf representing Σ, η_Σ is already an epimorphism in simplicial degree 0 over all objects in the site FormalSmoothMfd, by example 2.5. This implies the claim. □

Corollary 2.7. *For all $\Sigma \in \mathbf{H}$, pullback along the $\Im$-unit*

$$(\eta_\Sigma)^* : \mathbf{H}_{/\Im\Sigma} \longrightarrow \mathbf{H}_{/\Sigma}$$

is a conservative functor, def. 4.19.

Proof. By using prop. 2.6 in prop. 4.21. □

Definition 2.8. For any $\Sigma \in \mathbf{H}$, write

$$(T^\infty_\Sigma \dashv J^\infty_\Sigma) := ((\eta_\Sigma)^* \circ (\eta_\Sigma)_! \dashv (\eta_\Sigma)^*(\eta_\Sigma)_*) : \mathbf{H}_{/\Im\Sigma} \longrightarrow \mathbf{H}_{/\Im\Sigma}$$

for the adjoint pair of a monad and comonad that is induced, via example 4.15, from the base change adjoint triple, def. 4.11 along the unit η_Σ of the monad $\Im$, def. 2.4.

Proposition 2.9. *For $\Sigma \in$ SmoothMfd $\hookrightarrow \mathbf{H}$, the comonad J^∞_Σ of def. 2.8 restricts to pro-finite dimensional smooth bundles along the canonical inclusion*

$$\mathrm{SmoothMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$$

and coincides there with the jet comonad 1.9 of prop. 1.9.

Proof. It is straightforward to analyze the action of the left adjoint $T^\infty_\Sigma : (\eta_\Sigma)^* \circ (\eta_\Sigma)_!$. One finds that this sends any open $U \hookrightarrow \Sigma$ to the infinitesimal disk bundle $T^\infty U$. By adjunction it follows that the sections $U \to J^\infty E$ over Σ are equivalently maps $T^\infty U \to E$ over Σ. These pick over each point $\sigma \in U \hookrightarrow \Sigma$ a section of E over the infinitesimal neighbourhood $\mathbb{D}_\sigma$, hence a jet at that point. $\square$

By theorem 1.13 this means that the coalgebras of J^∞_Σ whose underlying objects are in $\mathrm{SmoothMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$ form the category of partial differential equations with free variables in Σ. In the present context it makes sense and is convenient to slightly generalize this traditional category by allowing the solution bundles to these differential equations to be not just smooth manifolds, but formal smooth manifolds.

Proposition 2.10. *For every* $\Sigma \in \mathrm{FormalSmoothSet}$*, there is an equivalence of categories between the category of coalgebras, def. 4.13 of the jet comonad* J^∞_Σ *on formal smooth sets, and the slice category of formal smooth sets over* Σ*:*

$$\mathrm{EM}(J^\infty_\Sigma|_{\mathrm{FormalSmoothSet}}) \simeq \mathrm{FormalSmoothSet}_{/\Im\Sigma} \,.$$

Proof. Via prop. 2.6 this follows from comonadic descent, prop. 4.23. $\square$

From this we get the following refinement of the classical situation summarized in remark 1.15.

Corollary 2.11. *For* $\Sigma \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{FormalSmoothSet}$*, there are canonical inclusions of categories*

$$\mathrm{DiffOp}_\Sigma \hookrightarrow \mathrm{PDE}_\Sigma \hookrightarrow \mathrm{FormalSmoothSet}_{/\Im\Sigma} \,.$$

Here:

1. PDE_Σ *is equivalently the preimage under* $(\eta_\Sigma)^*$ *of the category of pro-finite dimensional smooth bundles over* Σ*:*

$$\begin{array}{ccc} \mathrm{FormalSmoothMfd}_{\downarrow\Sigma} & \underset{F}{\overset{U}{\rightleftarrows}} & \mathrm{PDE}_\Sigma \simeq ((\eta_\Sigma)^*)^{-1}(\mathrm{FormalSmoothMfd}_{\downarrow\Sigma}) \\ \downarrow & & \downarrow \\ \mathrm{FormalSmoothSet}_{/\Sigma} & \underset{(\eta_\Sigma)_*}{\overset{(\eta_\Sigma)^*}{\rightleftarrows}} & \mathrm{FormalSmoothSet}_{/\Im\Sigma} \end{array}$$

2. *the total inclusion of the category* DiffOp_Σ *of bundles and differential operators over* Σ *is equivalently the full subcategory of* $\mathrm{FormalSmoothSet}_{/\Im\Sigma}$ *on the objects in the direct image of the base change along the counit of the jet comonad*

$$\mathrm{FormalSmoothMfd}_{\downarrow\Sigma} \hookrightarrow \mathrm{FormalSmoothSet}_{/\Sigma} \overset{(\eta_\Sigma)_*}{\longrightarrow} \mathrm{FormalSmoothSet}_{/\Im\Sigma} \,.$$

Proof. By theorem 1.13, proposition 2.10 and prop. 1.10. $\square$

The analog of proposition 2.10 still holds for the full ∞-category

Proposition 2.12. *For any* $\Sigma \in \mathrm{SmoothMfd} \hookrightarrow \mathbf{H}$ *there is an equivalence of* ∞*-categories between that of* ∞*-coalgebras over the jet* ∞*-comonad over* Σ*, and the slice over* $\Im\Sigma$*:*

$$\mathrm{EM}(J^\infty_\Sigma) \simeq \mathbf{H}_{/\Im\Sigma} \,.$$

Proof. Via prop. 2.6 this follows from ∞-comonadic descent, prop. 4.23. $\square$

Remark 2.13. In view of theorem 1.13 we may think for any $\Sigma \in \mathrm{SmoothMfd} \hookrightarrow \mathbf{H}$ of an ∞-coalgebra over $J^\infty_\Sigma : \mathbf{H}_\Sigma \to \mathbf{H}_{/\Sigma}$ as a higher stacky partial differential equation with variables in Σ. Hence we also write

$$\mathrm{PDE}_\Sigma(\mathbf{H}) := \mathrm{EM}(J^\infty_\Sigma) \,.$$

We connect now the traditional theory of PDEs to that of homotopy PDEs by establishing how the latter is presented by homotopy colimits of the former.

Lemma 2.14. *Let $K \in \mathrm{FormalSmoothMfd}$ and $f : K \longrightarrow \Im\Sigma$ a morphism in $\mathrm{FormalSmoothSet}$. Then the pullback $(\eta_\Sigma)^* K$ is still in $\mathrm{FormalSmoothMfd} \hookrightarrow \mathrm{FormalSmoothSet}$.*

Proof. We may check this on a local chart U of K. For this the pullback is $U \times \mathbb{D}^{p+1}$, where $\mathbb{D}^{p+1}$ is the formal disk in Σ. □

Definition 2.15. Hence by corollary 2.11 there is a canonical subcategory inclusion

$$\mathrm{FormalSmoothMfd}/\Im\Sigma \hookrightarrow \mathrm{PDE}_\Sigma\,.$$

We consider PDE_Σ as equipped with the pre-topology that makes this the inclusion of a dense subsite, hence we consider a presheaf on PDE_Σ to be a sheaf if its restriction along this site inclusion is.

Proposition 2.16. *For any $\Sigma \in \mathbf{H}$, a small site of definition for the ∞-topos $\mathrm{PDE}_\Sigma(\mathbf{H})$ is given by the comma-category $\mathrm{FormalSmoothCartSp}/\Im\Sigma$ equipped with the coverage that regards a collection of morphisms over $\Im\Sigma$ as covering if they are covering in $\mathrm{FormalSmoothCartSp}$ after forgetting the maps to $\Im\Sigma$. Similarly a large site of definition is given by the slice category $\mathrm{FormalSmoothSet}_{/\Im\Sigma}$*

$$\mathrm{PDE}_\Sigma(\mathbf{H}) \simeq \mathrm{Sh}_\infty(\mathrm{FormalSmoothCartSp}_{/\Im\Sigma}) \simeq \mathrm{Sh}_\infty(\mathrm{FormalSmoothSet}_{/\Im\Sigma})\,.$$

Proof. See the proof in the nLab entry on slice ∞-toposes. □

We now have the following homotopy theoretic version of the classical situation in remark 1.15.

Proposition 2.17. *We have*

$$\begin{array}{ccccc}
\mathbf{H} & \xrightarrow{\Sigma^*} & \mathbf{H}_{/\Sigma} & \underset{\mathrm{F}}{\overset{\mathrm{U}}{\rightleftarrows}} & \mathrm{PDE}_\Sigma(\mathbf{H}) \\
\uparrow{\simeq} & & \uparrow{\simeq} & & \uparrow{\simeq} \\
\mathrm{Sh}_\infty(\mathrm{FormalSmoothMfd}) & \xrightarrow{(\Sigma_!)^* \simeq (\Sigma^*)_!} & \mathrm{Sh}_\infty(\mathrm{FormalSmoothMfd}_{/\Sigma}) & \underset{\mathrm{U}^* \simeq \mathrm{F}_!}{\overset{\mathrm{U}_!}{\rightleftarrows}} & \mathrm{Sh}_\infty(\mathrm{PDE}_\Sigma)
\end{array}$$

2.2 Differential cohomology on PDEs

We discuss a canonical lift of ordinary differential cohomology from $\mathbf{H}$ to $\mathrm{PDE}_\Sigma(\mathbf{H})$. We show that the classical Euler-Lagrange complex, def. 1.35, is what provides a well-adapted Hodge filtration on constant real coefficients in this case.

But first recall the standard Poincaré lemma in its stacky incarnation (where DK denotes the Dold-Kan correspondence, prop. 4.4).

Definition 2.18. Write

$$\flat \mathbf{B}^{p+2}\mathbb{R} \simeq \mathbf{B}^{p+2}\flat\mathbb{R} := \mathrm{DK}(\mathbb{R}[p+2]) \in \mathbf{H}$$

and

$$\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR,cl}} := \mathrm{DK}(\mathbf{\Omega}^0 \stackrel{d}{\to} \mathbf{\Omega}^1 \stackrel{d}{\to} \cdots \to \mathbf{\Omega}^{p+2}_{\mathrm{cl}}) \in \mathbf{H}\,.$$

Proposition 2.19 (Poincaré lemma)**.** *The canonical inclusion of chain complexes induces an equivalence*

$$\mathbf{B}^{p+2}\flat\mathbb{R} \xrightarrow{\simeq} \mathbf{\Omega}^{\bullet\leq p+2}_{\mathrm{dR,cl}}$$

in $\mathbf{H}$.

Proof. A map of sheaves of chain complexes is such an equivalence if when evaluated on any object in the site, there is a covering of that object such that when pulled back to any member of the covering, the morphism becomes a quasi-isomorphism of chain complexes. Here we may cover any manifold by a good open cover whose elements are diffeomorphic to a Cartesian space $\mathbb{R}^n$, and the traditional statement of the Poincaré lemma then gives that all closed elements in $\Omega^{\bullet\geq 1}_{\mathrm{dR}}(\mathbb{R}^n)$ are exact, hence that the cohomology of $\Omega^\bullet(\mathbb{R}^n)$ is concentrated in degree 0 on $\Omega^0_{\mathrm{dR}}(\mathbb{R}^n)_{\mathrm{cl}} \simeq \mathbb{R}$, hence that the canonical inclusion of this cohomology group is a quasi-isomorphism. □

The Poincaré lemma, prop. 2.19, induces a filtration on $\flat \mathbf{B}^{p+2}\mathbb{R}$. In the complex-analytic case this is called the Hodge filtration, and so we will just call it that here, too.

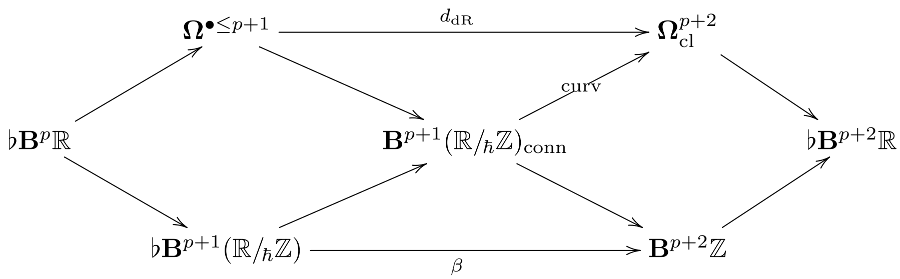
Definition 2.20. The Hodge filtration induces a morphism

$$\mathbf{\Omega}^{p+2}_{\mathrm{cl}} \longrightarrow \flat \mathbf{B}^{p+2}\mathbb{R}$$

in $\mathbf{H}$.

This induces ordinary differential cohomology:

Proposition 2.21. *There is a homotopy exact hexagon in* $\mathrm{Stab}(\mathbf{H})$ *of the form*



where the top right morphism is that of def. 2.20.

Proof. The general structure is amplified in [BNV13]; a detailed derivation for this case of ordinary differential cohomology is in the nLab entry on the Deligne complex. □

Next we consider cohomology in $\mathrm{PDE}_\Sigma(\mathbf{H})$ with coefficients in objects of $\mathbf{H}$ that are canonically lifted as follows:

Definition 2.22. Write

$$(-)_\Sigma : \mathbf{H} \xrightarrow{\Sigma^*} \mathbf{H}_{/\Sigma} \xrightarrow{F} \mathrm{PDE}_\Sigma(\mathbf{H}) \,.$$

Example 2.23. The object $(\mathbf{\Omega}^k)_\Sigma \in \mathrm{PDE}_\Sigma(\mathbf{H})$ modulates differential forms on the underlying bundles of PDEs.

$$\begin{array}{ccl} \mathcal{E} & \longrightarrow & (\mathbf{\Omega}^k)_\Sigma = \\ & & F(\Sigma^*(\mathbf{\Omega}^k)) \\ \hline \Sigma_!(U(\mathcal{E})) & \longrightarrow & \mathbf{\Omega}^k \end{array}$$

Specifically for cofree PDEs this gives the differential forms on the jet bundle:

$$(\mathbf{\Omega}^k)_\Sigma(E) \simeq \Omega^k(J^\infty(E)) \,.$$

Remark 2.24. From example 2.23 it follows that sending the heaxgon in prop. 2.21 along $(-)_\Sigma$ to $(\mathbf{B}^{p+1}(\mathbb{R}/_\hbar\mathbb{Z}))_\Sigma$ exhibits ordinary differential cohomology on the underlying bundles of PDEs , in particular on the jet bundles of cofree PDEs.

However, below in sections 2.3 and 2.5 we are interested in differential cocycles on jet bundles only via all their pullbacks along sections. By def. 1.21 and prop. 1.22 is is precisely only the horizontal component of differential forms which matters under these pullbacks.

This means that the standard Hodge filtration, under prolongation to PDEs, produces differential cocycles with redundant information.

We now observe that after prolonging to PDEs, there is a *different* Hodge filtration which accurately picks the non-redundant horizontal components.

Definition 2.25. By proposition 1.37, the functorial construction of Euler-Lagrange complexes, def. 1.35 constitutes a presheaf of chain complexes on DiffOp_Σ

$$E \mapsto \mathrm{DK}[\Omega^0_H(J^\infty E) \xrightarrow{d_H} \Omega^1_H(J^\infty E) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p+1}_H(J^\infty E) \xrightarrow{\delta_V} \Omega^{p+1,1}_S(J^\infty E)_{\mathrm{cl}}]\,.$$

When regarded in degrees 0 to $p+2$ we denote this by

$$\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{EL}_\Sigma,\mathrm{cl}} \in \mathrm{PSh}(\mathrm{DiffOp}_\Sigma, \mathrm{Ch}_\bullet) \xrightarrow{\mathrm{DK}} \mathrm{PSh}_\infty(\mathrm{DiffOp}_\Sigma) \xrightarrow{i_!} \mathrm{Sh}_\infty(\mathrm{PDE}_\Sigma)\,.$$

Proposition 2.26 (variational Poincaré lemma)**.** *There is an equivalence*

$$(\mathbf{B}^{p+2}\flat\mathbb{R})_\Sigma \simeq \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{EL}_\Sigma,\mathrm{cl}}$$

in $\mathrm{PDE}_\Sigma(\mathbf{H}_\Re)$ *between the constant* $\mathbb{R}$*-coefficients in degree* $(p+2)$ *prolonged to homotopy PDEs via def. 2.22), and the Euler-Lagrange complex according to def. 2.25. .*

Proof. By prop.2.19 we may equivalently show that

$$(\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}})_\Sigma \simeq \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{EL}_\Sigma,\mathrm{cl}}\,.$$

Then by prop. 2.17 it is sufficient to show that

$$\mathrm{U}^*((\Sigma_!)^*(\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}})) \simeq \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{EL}_\Sigma,\mathrm{cl}}$$

in $\mathrm{Sh}_\infty(\mathrm{PDE}_\Sigma)$. There is an implicit ∞-sheafification in these expressions, by definition, but since the precomposition maps $(\Sigma_!)^* \simeq (\Sigma^*)_!$ and $\mathrm{U}^* \simeq \mathrm{F}_!$ come from left Quillen functors given by corollary 4.6, they commute with the left Bousfield localization that presents this ∞-sheafification. Therefore it is sufficient that we prove this equivalence already at the level of ∞-presheaves.

Now by adjunction, the presheaf on the left evaluates on a representable $\mathrm{F}(E)$ as follows:

$$\begin{aligned}\mathrm{Hom}(\mathrm{F}(E), \mathrm{U}^*((\Sigma_!)^*(\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}}))) &\simeq \mathrm{Hom}(\mathrm{U}(\mathrm{F}(E)), (\Sigma_!)^*(\mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}})) \\ &\simeq \mathrm{Hom}(\Sigma_!(\mathrm{U}(\mathrm{F}(E))), \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}}) \\ &\simeq \mathrm{Hom}(J^\infty_\Sigma E, \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}}) \\ &\simeq \mathbf{\Omega}^{\bullet \leq p+2}_{\mathrm{dR},\mathrm{cl}}(J^\infty_\Sigma E)\end{aligned}$$

With this we may reduce to classical statements about the variational bicomplex: Prop. 1.17 says that the functoriality of the above assignment is the same as that in prop. 1.37, hence the claim is now given by theorem 1.39. ∎

Definition 2.27. The resolution in prop. 2.26 induces a morphism

$$\mathbf{\Omega}^{p+1,1}_{S,\mathrm{cl}} \longrightarrow (\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma\,.$$

We consider the homotopy pullback of that morphism along the morphism $(\flat \mathbf{B}^{p+2}\mathbb{Z})_\Sigma \to (\flat \mathbf{B}^{p+2}\mathbb{R})_\Sigma$ from coefficients for integral cohomology to coefficients for real cohomology.

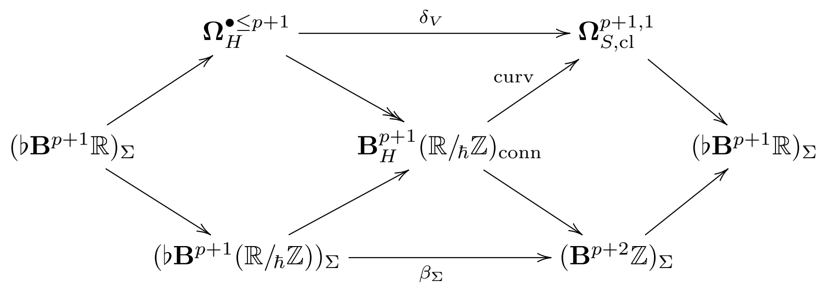
Definition 2.28. For $p+1 \in \mathbb{N}$ write

$$\mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \in \text{Sh}_\infty(\text{DiffOp}_\Sigma) \xrightarrow{i_!} \text{Sh}_\infty(\text{PDE}_\Sigma) \simeq \text{PDE}_\Sigma(\mathbf{H})$$

for the ∞-stack which is the image under the Dold-Kan correspondence DK, prop. 4.4, of the left Kan extension, def. 4.9, along the inclusion $i : \text{DiffOp}_\Sigma \hookrightarrow \text{PDE}_\Sigma$ (remark 1.15) of the Euler-Lagrane complex of def. 2.25, directly truncated after the horizontal $p+1$-form and with a copy of $\mathbb{Z}$ injected into the horizontal 0-forms:

$$\mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \simeq \text{DK}[\mathbb{Z} \overset{2\pi\hbar}{\hookrightarrow} \mathbf{\Omega}_H^0 \xrightarrow{d_H} \mathbf{\Omega}_H^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathbf{\Omega}_H^{p+1}]\,.$$

Theorem 2.29. *In* $\text{Stab}(\text{PDE}_\Sigma(\mathbf{H}_\Re))$ *there is an exact hexagon of the form*



where the top right morphism is that of def. 2.27.

Proof. In view of the variational Poincaré lemma, prop. 2.26, we obtain this hexagon in $\text{Stab}(\text{Sh}_\infty(\text{DiffOp}_\Sigma))$ from the Euler-Lagrange complex in direct analogy to the corresponding hexagon for the ordinary Deligne complex, prop. 2.21. Sending it by the Yoneda extension $\text{Sh}_\infty(\text{DiffOp}_\Sigma) \to \text{PDE}_\Sigma(\mathbf{H})$ preserves the homotopy pushouts, hence, by stability, the full homotopy exactness. □

We may also characterize this choice of differential refinement more abstractly, not presupposing that we already know about the Euler-Lagrange complex:

Proposition 2.30. *The morphism*

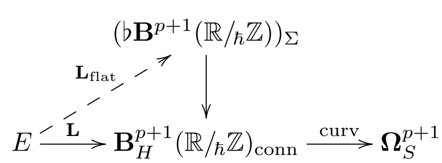
$$\mathbf{\Omega}_V^{\bullet \leq p+1} \longrightarrow (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma$$

is universally characterized by the fact that for every $E \in \text{SmoothMfd}_{/\Sigma}$ *and every section* $\phi : \Sigma \to E$ *there is a homotopy fiber sequence of the form*

$$\Omega_V^{\bullet \leq p+1}(E) \longrightarrow (\Omega_\Sigma^{\bullet \leq p+1}(E) \simeq \Omega^{\bullet \leq p+1}(J^\infty E)) \xrightarrow{\phi^*} \Omega^{\bullet \leq p+1}(\Sigma)\,.$$

Proof. First observe that the map on the right is degreewise a surjection (since ϕ is a section, the pullback of a form on Σ to the jet bundle along the canonical projection is a preimage of the form under ϕ^*). Therefore the homotopy fiber is presented by the 1-categorical fiber. To see that this is precisely the vertical forms use prop. 1.49. □

Remark 2.31. By the universal property of homotopy fibers, the exactness of the right square in the hexagon in prop. 2.29 means in particular that the curving of the Euler-Lagrange p-gerbe is precisely the obstruction to it being flat, in that the dashed morphism in the following diagram



exists, and then uniquely so up to a contractible space of choices of equivalences, precisely if the horizontal composite is zero.

Proposition 2.32. *There are equivalences*

$$\mathrm{PDE}_\Sigma(\mathbf{H})(\Sigma, \mathbf{B}^{p+1}_H(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}) \xrightarrow{\simeq} \mathrm{PDE}_\Sigma(\mathbf{H})(\Sigma, (\flat \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z}))_\Sigma) \xrightarrow{\simeq} \mathbf{H}(\Sigma, \flat \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z}))\,.$$

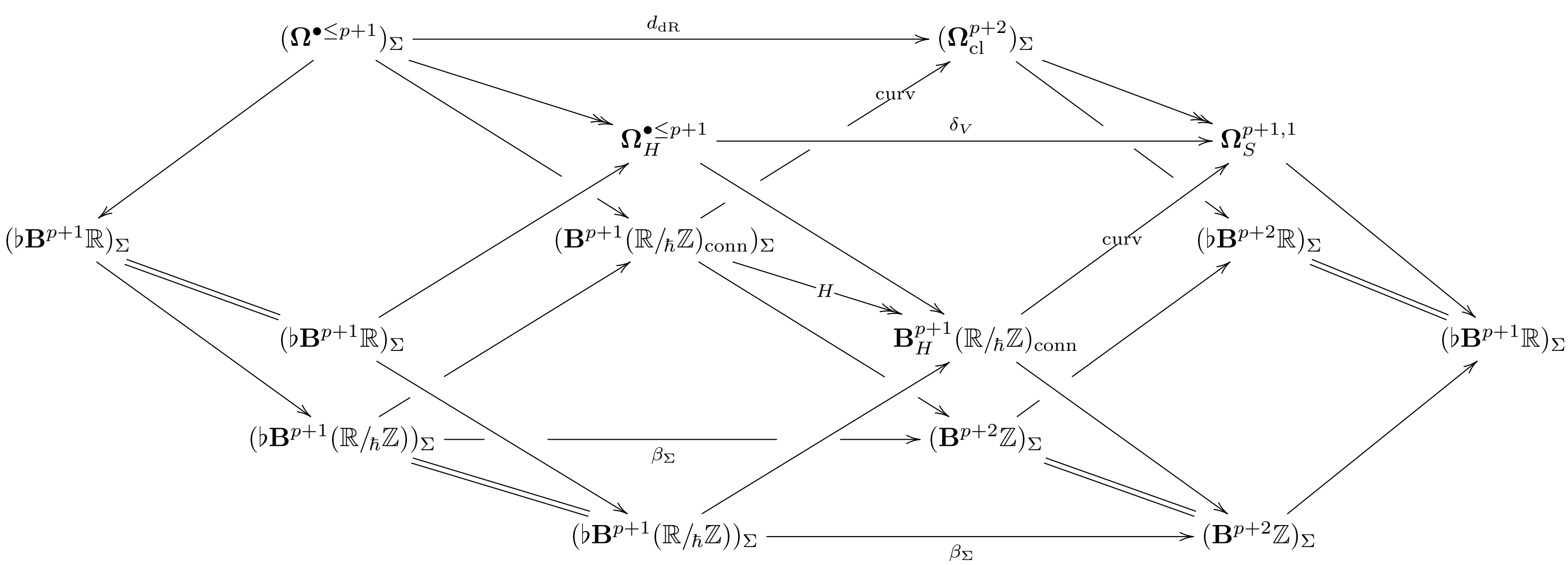
Proof. The first equivalence is obtained via remark 2.31 from the fact that every morphism $\Sigma \to \mathbf{\Omega}^{p+1,1}_S$ is zero. The second equivalence is the combined hom-equivalence of the adjunctions $(U \dashv F)$ and $(\Sigma_! \dashv \Sigma^*)$ in view of def. 2.22. □

We have a canonical comparison map between ordinary differential cohomology, prop. 2.21, prolonged to PDEs, and the Euler-Lagrange differential cohomology of prop. 2.29:

Definition 2.33. By example 2.23, projection of differential forms on jet bundles to the horizontal and their source form part, which is natural over DiffOp_Σ by prop. 1.37, constitutes projection operations that intertwine the de Rham differential with the variational Euler differential:

$$\begin{array}{ccc} (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma & \xrightarrow{(d_{\mathrm{dR}})_\Sigma} & (\mathbf{\Omega}^{p+2}_{\mathrm{cl}}) \\ \downarrow & & \downarrow \\ \mathbf{\Omega}^{\bullet \leq p+1}_H & \xrightarrow{\delta_V} & \mathbf{\Omega}^{p+1,1}_S \end{array} .$$

Via the universal properties of the exactness of the hexagons in prop. 2.21 and prop. 2.29 this induces a projection of differential cohomology coefficients,



which we denote

$$H : (\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}})_\Sigma \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}\,.$$

2.3 Prequantum Lagrangians and Equations of motion

The following is the prequantum analog of def. 1.43.

Definition 2.34. Given $E \in \mathbf{H}_{/\Sigma}$ then

1. a *pre-quantum local Lagrangian* on E is a morphism in $\mathrm{Sh}_\infty(\mathrm{DiffOp}_\Sigma)$ of the form

$$\mathbf{L} : E \longrightarrow \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} ,$$

2. the *Euler-Lagrange form* of such $\mathbf{L}$ is the curvature

$$\mathrm{EL} := \delta_V \mathbf{L} : E \xrightarrow{\mathbf{L}} \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \xrightarrow{\delta_V} \mathbf{\Omega}_S^{p+1,1} .$$

3. the *Euler-Lagrange equations* of $\mathbf{L}$ is the homotopy fiber of EL

$$\mathcal{E} := \mathrm{fib}(\mathrm{EL}) .$$

Remark 2.35 (terminology)**.** We also say that the pair $(E, \mathbf{L})$ is (or defines) a *prequantum field theory.*

Given a source form $\mathrm{EL} : E \longrightarrow \mathbf{\Omega}_S^{p+1}$ we also say that a prequantum Lagrangian $\mathbf{L} : E \to \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{\mathrm{conn}}$ is a *prequantization* of EL if $\delta_V \mathbf{L} \simeq \mathrm{EL}$, i.e. if $\mathbf{L}$ is a lift of EL through the curvature map:

$$\begin{array}{ccc} & & \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{\mathrm{conn}} \\ & \overset{\mathbf{L}}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{curv}} \\ E & \xrightarrow[\mathrm{EL}]{} & \mathbf{\Omega}_H^{p+1} \end{array}$$

Proposition 2.36. *For $\omega \in \Omega_S^{p+1,1}(E)$ a source form, then the partial differential equation $\mathcal{E}$ induced by it via def. 1.40 is equivalently the kernel in $\mathrm{PDE}_\Sigma(\mathbf{H}) \simeq \mathrm{Sh}_\infty(\mathrm{PDE}_\Sigma)$ of the representing morphism $\omega : E \longrightarrow \mathbf{\Omega}_S^{p+1,1}$:*

$$\begin{array}{ccc} \mathcal{E} & & \\ \big\downarrow{\scriptstyle \ker(\omega)} & & \\ E & \xrightarrow{\omega} & \mathbf{\Omega}_S^{p+1,1} \end{array}$$

Proof. Since this is a statement about a limit of 0-truncated objects in $\mathrm{PDE}_\Sigma(\mathbf{H}) \simeq \mathrm{Sh}_\infty(\mathrm{PDE}_\Sigma)$, we may consider the question equivalently in the sheaf 1-topos $\mathrm{Sh}(\mathrm{PDE}_\Sigma)$. Now unwinding the definitions, one sees that for a representable $\mathcal{F} \in \mathrm{PDE}_\Sigma$ to map through the kernel of ω is equivalent to it mapping through the equalizer of the differential operator $\widetilde{\omega}$ that corresponds to it under the isomorphism in prop. 1.27 with the 0-morphism, as in prop. 1.41:

$$\begin{array}{ccc} \mathcal{F} & & \\ \big\downarrow{\scriptstyle i} & \overset{0 = i^*\omega}{\searrow} & \\ E & \xrightarrow[\omega]{} & \mathbf{\Omega}_S^{p+1,1} \end{array} \quad \simeq \quad \begin{array}{ccc} \mathcal{F} & & \\ \big\downarrow{\scriptstyle i} & \overset{i^*\tilde{0} = i^*\tilde{\omega}}{\searrow} & \\ E & \overset{\tilde{0}}{\underset{\widetilde{\omega}}{\rightrightarrows}} & \wedge^{p+1} T^*\Sigma \times_\Sigma V^* E \end{array}$$

But since these equalized morphisms are morphism in the site PDE_Σ, and since the Yoneda embedding $\mathrm{PDE}_\Sigma \hookrightarrow \mathrm{Sh}(\mathrm{PDE}_\Sigma)$ preserves limits, we may compute the fiber equivalently in PDE_Σ as this equalizer. With this the statement is given by prop. 1.41. □

2.4 Prequantum covariant phase space

We discuss the prequantum version of of the (pre-)symplectic covariant phase space from section 1.6.2.

Since the covariant phase space consists of fields in codimension-1, hence on p-dimensional submanifolds $\Sigma_p \hookrightarrow \Sigma$, we produce yet another Hodge filtration of $(\flat \mathbf{B}^{p+2}\mathbb{R})_\Sigma$, now the one which has minimal kernel when pulled back along sections in dimension p.

Definition 2.37. The *Lepage complex* is the chain complex (of presheaves on DiffOp_Σ)

$$\mathbf{\Omega}^\bullet_{\mathrm{L}} := \left[\begin{array}{ccccccccc} \cdots & \xrightarrow{d_H} & \mathbf{\Omega}^{p-1}_H & \xrightarrow{d_H} & \mathbf{\Omega}^p_H & \xrightarrow{d_H} & \mathbf{\Omega}^{p+1}_H & & \\ & \searrow^{d_V} & \oplus & \searrow^{d_V} & \oplus & \searrow^{d_V} & \oplus_{\mathrm{coim}(\mathcal{I}(d_V \oplus d_H))} & \searrow^{d_V} & \\ \cdots & \xrightarrow{d_H} & \mathbf{\Omega}^{p-2,1}_{\ker(d_V)} & \xrightarrow{d_H} & \mathbf{\Omega}^{p-1,1}_{\ker(d_V)} & \xrightarrow{d_H} & \mathbf{\Omega}^{p,1} & \xrightarrow{d_H} & \mathbf{\Omega}^{p+1,1}_{S,\ker(\delta_V)} \\ & & & & & & & \searrow^{d_V} & \oplus \\ & & & & & & & & \mathbf{\Omega}^{p,2}_{\mathrm{cl},\ker(d_V)} \end{array}\right],$$

which is the total complex of the "2-term outer rim" of the augmented variational bicomplex, prop. 1.36.

This constitutes yet another Hodge filtration for $(\flat \mathbf{B}^{p+2}\mathbb{R})_\Sigma$ and further factors the projection in def. 2.33

$$\begin{array}{ccc} (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma & \xrightarrow{(d_{\mathrm{dR}})_\Sigma} & (\mathbf{\Omega}^{p+2}_{\mathrm{cl}})_\Sigma \\ \downarrow & & \downarrow \\ \mathbf{\Omega}^{\bullet \leq p+1}_L & \longrightarrow & \mathbf{\Omega}^{p+1,1}_{S,\ker(\delta_V)} \oplus \mathbf{\Omega}^{p,2}_{\ker(d_V)} \\ \downarrow & & \downarrow \\ \mathbf{\Omega}^{\bullet \leq p+1}_H & \xrightarrow{\delta_V} & \mathbf{\Omega}^{p+1,1}_{S,\ker(\delta_V)} \end{array} .$$

Accordingly, induced from this is the corresponding differential coefficients $\mathbf{B}^{p+1}_L(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}}$ in direct analogy to the big diagram in def. 2.33.

Given a globally defined local Lagrangian

$$L : E \longrightarrow \mathbf{\Omega}^{\bullet \leq p+1}_H$$

then a lift of this through the Lepage complex such that the curvatures commute

$$\begin{array}{ccccc} & & \mathbf{\Omega}^{\bullet \leq p+1}_L & \longrightarrow & \mathbf{\Omega}^{p+1,1}_{S,\ker(\delta_V)} \oplus \mathbf{\Omega}^{p,2}_{\ker(d_V)} \\ & {}^{L+\theta}\nearrow & \downarrow & & \downarrow \\ E & \xrightarrow{L} & \mathbf{\Omega}^{\bullet \leq p+1}_H & \xrightarrow{\delta_V} & \mathbf{\Omega}^{p+1,1}_{S,\ker(\delta_V)} \end{array} .$$

is a choice of θ in $dL = \mathrm{EL} + d_H\theta$ (remark 1.44). Indeed, the lifted curvature coefficients are precisely so as to ask for a Lepage form for L of vertical degree ≤ 2.

Now by the yoga of the big diagram in def. 2.33 this gives us the right "Lepage gerbes" as lifts

$$\begin{array}{ccc} & & \mathbf{B}^{p+1}_L(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} \\ & {}^{\mathbf{\Theta}}\nearrow & \downarrow \\ E & \xrightarrow{\mathbf{L}} & \mathbf{B}^{p+1}_H(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} \end{array}$$

Pulling these Lepage gerbes back along sections on a codimension-1 Cauchy surface $\Sigma_p \hookrightarrow \Sigma$ (which makes the contribution of L disappear and retains only the contribution of θ) is precisely a prequantization for the canonical symplectic structure on the covariant phase space (even off-shell).

Definition 2.38. Given a morphism $f : X \longrightarrow Y$ in $\mathbf{H}$, we say that the formal normal bundle $N^\infty_Y X \in \mathbf{H}_{/Y}$ of X in Y is the formal étalification of f, hence the homotopy pullback in

$$\begin{array}{ccc} N^\infty_Y X & \longrightarrow & \Im X \\ {}^{\mathrm{et} f}\downarrow & (\mathrm{pb}) & \downarrow^{\Im f} \\ Y & \longrightarrow & \Im Y \end{array}$$

Proposition 2.39. *Jet bundles are preserved by pullback along inclusions of formal normal bundles, def. 2.38, i.e. for $f : X \to Y$ a morphism and $E \in \mathbf{H}_{/Y}$ a bundle, then*

$$(\mathrm{et} f)^* J^\infty_Y E \simeq J^\infty_X f^* E \,.$$

Proof. The homotopy pullback in def. 2.38 induces a square of base change operations

$$\begin{array}{ccc} \mathbf{H}_{/N^\infty_Y X} & \underset{(\eta^{\Im}_{N^\infty_Y X})_*}{\overset{(\eta^{\Im}_{N^\infty_Y X})^*}{\rightleftarrows}} & \mathbf{H}_{/\Im N^\infty_Y X} \\ {}^{(\mathrm{et} f)_*}\downarrow\uparrow^{(\mathrm{et} f)^*} & & {}^{(\Im \mathrm{et} f)_*}\downarrow\uparrow^{(\Im \mathrm{et} f)^*} \\ \mathbf{H}_{/Y} & \underset{(\eta^{\Im}_Y)_*}{\overset{(\eta^{\Im}_Y)^*}{\rightleftarrows}} & \mathbf{H}_{/\Im Y} \end{array} \,.$$

By Beck-Chevalley this implies that

$$(\Im \mathrm{et} f)^* (\eta^{\Im}_Y)_* \simeq (\eta^{\Im}_{N^\infty_Y X})_* (\mathrm{et} f)^* \,.$$

Using this we find

$$\begin{aligned} (\mathrm{et} f)^* J^\infty_Y E &:= (\mathrm{et} f)^* (\eta^{\Im}_Y)^* (\eta^{\Im}_Y)_* E \\ &\simeq (\eta^{\Im}_{N^\infty_Y X})^* (\Im \mathrm{et} f)^* (\eta^{\Im}_Y)_* E \\ &\simeq (\eta^{\Im}_{N^\infty_Y X})^* (\eta^{\Im}_{N^\infty_Y X})_* (\mathrm{et} f)^* E \\ &=: J^\infty_Y (\mathrm{et} f)^* E \,. \end{aligned}$$

□

Definition 2.40. Given a Lepage p-gerbe $\mathbf{\Theta} : \mathcal{E} \longrightarrow \mathbf{B}^{p+1}_L (\mathbb{R}/_\hbar \mathbb{Z})_{\mathrm{conn}}$, then given a codimension-1 submanifold $\Sigma_p \hookrightarrow \Sigma$ of spacetime/worldvolume, the corresponding covariant phase space is the transgression

$$\int_{\Sigma_p} [N^\infty_\Sigma \Sigma_p, \mathbf{\Theta}] \;:\; [N^\infty_\Sigma \Sigma_p, \mathcal{E}] \longrightarrow \mathbf{B}(\mathbb{R}/_\hbar \mathbb{Z})_{\mathrm{conn}} \,.$$

2.5 Globally defined local action functionals

Assume here that Σ is a *closed* $(p+1)$-dimensional smooth manifold.

Proposition 2.41. *The connected components of the hom-space from Σ into the $(p+1)$-fold delooping of the discrete circle group is isomorphic to that same discrete circle group*

$$\tau_0 \mathbf{H}_{\Re}(\Sigma, \flat \mathbf{B}^{p+1}(\mathbb{R}/_\hbar \mathbb{Z})) \simeq \flat \mathbb{R}/_\hbar \mathbb{Z} \,.$$

Moreover, under this identification and the Poincaré lemma, prop. 2.19, the 0-truncation map coincides with $(p+1)$-volume holonomy of p-gerbes on Σ:

$$\begin{array}{ccc} \mathbf{H}_{\Re}(\Sigma, \flat\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})) & \overset{\pi_0}{\searrow} & \\ \downarrow{\scriptstyle\simeq} & & \flat(\mathbb{R}/_{\hbar}\mathbb{Z}) \\ \mathbf{H}_{\Re}(\Sigma, \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}) & \underset{\int_{\Sigma}}{\nearrow} & \end{array}$$

Definition 2.42. Given a prequantum local Lagrangian $\mathbf{L} : E \longrightarrow \mathbf{B}^{p+1}_{\Sigma}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}$ (def. 2.34), and given a section $\phi : \Sigma \longrightarrow E$, then the *action function* induced by $\mathbf{L}$ at ϕ is

$$\exp(\tfrac{i}{\hbar}S_{\mathbf{L}}(-)) : \ \Gamma_{\Sigma}(E) \xrightarrow{\simeq} \mathrm{PDE}_{\Sigma}(\mathbf{H})(\Sigma, E) \xrightarrow{(-)^*\mathbf{L}} \mathrm{PDE}_{\Sigma}(\mathbf{H})(\Sigma, \mathbf{B}^{p+1}_{H}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}) \xrightarrow{\simeq} \mathbf{H}_{\Re}(\Sigma, \flat\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})) \xrightarrow{\pi_0} \flat(\mathbb{R}/_{\hbar}\mathbb{Z}) \ ,$$

where the first equivalence is as in example 1.16, the second equivalence is from prop. 2.32, and the last map is from prop. 2.41.

A smooth function

$$\mathbf{\Gamma}_{\Sigma}(E) \longrightarrow \mathbb{R}/_{\hbar}\mathbb{Z}$$

(on the diffeological space of smooth sections, def. 1.48) is called a (globally defined) *local action functional* if its restriction to points (forgetting the smooth structure) arises from a prequantum Lagrangian in this fashion.

2.6 Sigma-models

Definition 2.43. A prequantum field theory, def. 2.34, $\mathbf{L} : E \to \mathbf{B}^{p+1}_{H}(\mathbb{R}/_{\hbar}\mathbb{Z})$, is a *sigma model* if E is in the image of $(-)_{\Sigma} : \mathbf{H} \longrightarrow \mathrm{PDE}_{\Sigma}(\mathbf{H})$, def. 2.22, for some $X \in \mathbf{H}$. In this case X is called the *target space* of the sigma model.

Remark 2.44. By adjointness, field configurations of sigma-models are equivalently maps from Σ to X:

$$\begin{array}{rcl} \Sigma & \longrightarrow & (X)_{\Sigma} = F(\Sigma^*(X)) \\ \hline \Sigma \simeq U(\Sigma) & \longrightarrow & \Sigma^*(X) \\ \hline \Sigma \simeq \Sigma_!\Sigma & \longrightarrow & X \end{array}$$

As such, sigma-models may be thought of as describing the dynamics of trajectories of shape Σ in X. In practice this arises in two guises:

1. Σ models spacetime and X is a moduli space of certain scalar fields on Σ.
2. X models spacetime and Σ models the worldvolume of a p-brane propagating in X.

Definition 2.45. A *WZW-type Lagrangian* $\mathbf{L}_{\mathrm{WZW}}$ for a sigma-model, def. 2.43, with target space X is a prequantum Lagrangian, def. 2.34, which is the image under

$$\mathbf{H}_{/\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}} \xrightarrow{(-)_{\Sigma}} \mathbf{H}_{/(\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}})_{\Sigma}} \xrightarrow{H_!} (\mathrm{PDE}_{\Sigma}(\mathbf{H}))_{/\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}} \ ,$$

(where the first morphism is def. 2.22, the second is postcomposition with the projection from def. 2.33), of some principal connection $X \xrightarrow{\nabla} \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}$:

$$\left(\begin{array}{c} \Sigma \times X \\ \downarrow{\scriptstyle \mathbf{L}_{\mathrm{WZW}}} \\ \mathbf{B}^{p+1}_{H}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \end{array} \right) = H_! \circ F \circ \Sigma^* \left(\begin{array}{c} X \\ \downarrow{\scriptstyle \nabla} \\ \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \end{array} \right) .$$

2.7 Lie n-algebras of higher Noether currents

We discuss here how the ∞-group (higher group stack) of symmetries of a prequantum local Lagrangian $\mathbf{L}$ as in def. 2.34 forms a higher extension of the group of symmetries of its Euler-Lagrange equations of motion, and how after restricting attention to infinitesimal symmetries and after truncating away the higher homotopy information, this reproduces the classical sharp Noether theorem from section **??**.

The observation that is at the heart of the relation between the classical form of Noether's theorem, prop. 1.54, and symmetries of Euler-Lagrange gerbes is this:

Remark 2.46. Let $\mathbf{L} := L : E \longrightarrow \mathbf{\Omega}_H^{p+1}$ be a globally defined local Lagrangian, regarded as a prequantum local Lagrangian, def. 2.34. Then the datum of a diagram in $\mathrm{PDE}_\Sigma(\mathbf{H})$ of the form

$$\begin{array}{ccc} E & \xrightarrow{\quad\phi\quad} & E \\ & {}_{\mathbf{L}}\searrow \;\; \overset{\alpha}{\Swarrow} \;\; \swarrow_{\mathbf{L}} & \\ & \mathbf{B}_H^{p+1}(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} & \end{array}$$

is equivalent to a pair $(\phi, \alpha) \in \mathrm{Diff}_\Sigma(J^\infty_\Sigma E) \times \Omega^p_H(E))$ such that

$$\phi^* L - L = d\alpha\,.$$

Now given a smooth 1-parameter trajectory of such, $t \mapsto (\phi_t, \alpha_t)$

$$\begin{array}{ccc} E & \xrightarrow{\quad\phi_t\quad} & E \\ & {}_{\mathbf{L}}\searrow \;\; \overset{\alpha_t}{\Swarrow} \;\; \swarrow_{\mathbf{L}} & \\ & \mathbf{B}_H^{p+1}(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} & \end{array}$$

then the derivative at $t = 0$, being a pair

$$(v, \Delta_v)$$

consisting of an infinitesimal off-shell symmetry v and a horizontal p-form Δ_v, is a variational symmetry, def. 1.54

$$\mathcal{L}_v L = d_H \Delta_v$$

with on shell conserved Noether current $J_v := \iota_v \theta - \Delta_v$, prop. 1.54.

Now we discuss how to promote this local analysis to a global and homotopy theoretic statement (...).
Given

$$\mathbf{L} : E \longrightarrow \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}\,,$$

the ∞-group of conserved currents is the differentially concretification of the ∞-group of auto-equivalence of $\mathbf{L}$ over $\mathbf{B}^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn},H}$. In the notation of [FRS13a] this is

$$\mathbf{Cur}(\mathbf{L}) := \mathbf{QuantMorph}(\mathbf{L})\,.$$

Remark 2.47. A single element of this ∞-group is a diagram in $\mathrm{Sh}_\infty(\mathrm{DiffOp}_\Sigma)$ of the form

$$\begin{array}{ccc} E & \xrightarrow{\quad\simeq\quad} & E \;. \\ & {}_{\mathbf{L}}\searrow \;\; \overset{\simeq}{\Swarrow} \;\; \swarrow_{\mathbf{L}} & \\ & \mathbf{B}_H^{p+1}(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} & \end{array}$$

Proposition 2.48. *Given a prequantized locally variational theory as above, then there is a homotopy fiber sequence of the form*

$$\mathbf{B}^p_H(\mathbb{R}/_\hbar\mathbb{Z})_{\text{flat}}(E) \longrightarrow \text{Cur}(\mathbf{L}) \longrightarrow \text{Aut}(E)$$

Proof. By the main lemma in [FRS13a]. □

Example 2.49. If here E is an ordinary bundle in the category of smooth manifolds over Σ (instead of a more stacky bundle involving gauge symmetries)

$$E \in \text{DiffOp}_{/\Sigma}$$

then

$$\mathbf{B}^p_H(\mathbb{R}/\mathbb{Z})_{\text{flat}}(E) \simeq \text{DK}[\mathbb{Z} \hookrightarrow \Omega^0_H(E) \stackrel{d_H}{\to} \cdots \stackrel{d_H}{\to} \Omega^p_H(E)_{\text{cl}}]$$

(Notice that in degree 0 we now have the horizontally closed forms.) In particular in cohomology this is

$$H^p(E, U(1)) .$$

If we do remember the smooth structure then we obtain $H^p(E, U(1))$ as an abelian Lie group, with its smooth structure induced from that of $U(1)$. The Lie algebra of that is $H^p(E,\mathbb{R}) \simeq \text{H}^p_{\text{dR}}(E)$. Hence in that case that above homotopy fiber sequence gives an exact sequence

$$H^p_{\text{dR}}(E) \longrightarrow \text{Lie}(\mathbf{Cur}(E)) \longrightarrow \text{Vect}(E) .$$

For the special case of point symmetries of fields theories of WZW type, this was discussed in [SaSc15] in terms of gerbes on target spacetime. We now turn to discussion of these WZW models.

3 Application to field theories of higher WZW type

We consider here the prequantum field theory of the Wess-Zumino-Witten (WZW) model and its higher dimensional and parameterized analogs.

Fully generally, WZW-type models may be taken to be sigma-model field theories, def. 2.43, whose prequantum Lagrangian has a summand that is induced via a p-gerbe on target spacetime as in def. 2.45.

For instance target space X may be a smooth manifold that is equipped with a closed differential $(p+2)$-form $\omega \in \Omega^{p+2}(X)$, such that one summand in the Lagrangian, called the *WZW term* $\mathbf{L}_{\text{WZW}}$, is locally the horizontal projection of the pullback of local form potentials for ω. Globally this means that there is a $(\mathbb{R}/_\hbar)$-p-gerbe on X with curvature ω, and that $\mathbf{L}_{\text{WZW}}$ is the Euler-Lagrange p-gerbe induced by that.

Phrased in this generality, then for instance the Lorentz force coupling for an electron (hence $p = 1$) propagating in a spacetime X with Faraday tensor 2-form ω is a WZW-term. Conversely, WZW terms are generalizations of the Lorentz-force electromagnetic coupling term.

The original WZW model describes a string (hence $p = 2$) propagating on a compact simple Lie group $X = G$, and coupled to a higher gauge field (often called the Kalb-Ramond B-field) given by the "higher Faraday tensor" 3-form $\omega := \mu_3(\Theta \wedge \Theta \wedge \Theta)$ which is the canonical Lie algebra 3-cocycle, μ_3 left-invariantly extended to a 3-form on G.

Due to the high symmetry of the group manifold G this model enjoys special properties, and when speaking more specifically one may want to mean by field theories of WZW type those that share some of these properties.

These would first of all be $(p+1)$-dimensional sigma models with target space a group manifold and ω coming from a Lie algebra $(p+2)$-cocycle. Examples for this are the Green-Schwarz type sigma models that describe the propagation of super-p-branes on super-Minkowski spacetimes, regarded as super-translation groups.

In between the fully general notion of WZW terms and those coming specifically from cocycles on Lie groups G are the *parameterized* higher WZW models [Sc15]. For these target space is a manifold X that is locally (tangent-space-wise) modeled on G (a Cartan geometry) and equipped with a form $\omega \in \Omega^{p+2}(X)$ that on each tangent space is equivalent to the given cocycle. Examples for this are the Green-Schwarz type sigma-models that describe the propagation of super-p-branes on curved super-spacetimes that are solutions to the Einstein equations of suitable supergravity theories.

Finally, one may consider all this in higher differential geometry and allow X to be a higher étale stack which is locally modeled on a higher group stack G that is equipped with a $(p+2)$-cocycle on its L_∞-algebra. Examples for this are the Green-Schwarz type sigma-models for those super p-branes that have higher gauge fields on their worldvolume, the super-D-branes and the M5-brane [FSS13b][FSS13b].

3.1 The setup

The Lie group G carries a canonical $\mathfrak{g}$-valued 1-form, with $\mathfrak{g}$ is the Lie algebra of G, the *Maurer-Cartan form* $\Theta \in \Omega^1(G, \mathfrak{g}) = \Omega^1(G) \otimes \mathfrak{g}$. The Maurer-Cartan form is determined by the requirement of being invariant with respect to the left-multiplication action of G on itself, with $\Theta(e)\colon T_e G \xrightarrow{\cong} \mathfrak{g}$, where we identify the Lie algebra $\mathfrak{g} \cong T_e G$ with the tangent space at the identity element $e \in G$. The Maurer-Cartan form satisfies the*Maurer-Cartan equation*, the differential identity $\mathrm{d}\Theta = -\frac{1}{2}[\Theta \wedge \Theta]$.

Let us extend the wedge product to $\mathfrak{g}$-valued forms according to the rule $(\alpha \otimes S) \wedge (\beta \otimes T) = \alpha \wedge \beta \otimes (T \otimes T)$. We also extend the commutator and Killing bilinear maps $[-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $\langle - \rangle\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ to $\mathfrak{g}^{\otimes 2}$-valued forms using the identities $[\alpha \otimes (S \otimes T)] = \alpha \otimes [S, T]$ and $\langle \alpha \otimes (S \otimes T) \rangle = \alpha \langle S, T \rangle$. Specializing to forms on $J^\infty E$, we extend the horizontal Hodge-$*$ and the differentials d_H, d_V to $\mathfrak{g}$-valued forms on $J^\infty F$ in the obvious ways, $*(\alpha \otimes T) = (*\alpha) \otimes T$, $\mathrm{d}_H(\alpha \otimes T) = (\mathrm{d}_H \alpha) \otimes T$ and $\mathrm{d}_V(\alpha \otimes T) = (\mathrm{d}_V \alpha) \otimes T$. [XXX: say something about abelian factors like $U(1)$ or $\mathbb{R}$?]

Consider then a degree-$(p+2)$ cocycle $\mu\colon \mathfrak{g}^{\otimes(p+2)} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$, equivalently a closed left-invariant form $\mu(\Theta^{\wedge(p+2)}) \in \Omega^{p+2}_{\mathrm{cl}}(G)$. The interest is of course in cohomologically non-trivial such cocycles, i.e., such that $\mu(\Theta^{\wedge(p+2)})$ is not globally the differential of a left-invariant $(p+1)$-form. While all of the following also holds true for cohomologically trivial cocycles, the crux is not to restrict the discussion to that case.

Definition 3.1. Let then Σ be a smooth manifold of dimension $(p+1)$ and consider the field bundle

$$E := G \times \Sigma \longrightarrow \Sigma$$

so that fields are equivalently smooth G-valued functions on Σ. Write

$$\pi : J^\infty(G \times \Sigma) \longrightarrow G \times \Sigma \longrightarrow G$$

for the composite of the jet bundle projection followed by the projection onto the field fiber.

Definition 3.2. Write

$$\begin{aligned} \Theta^\infty &:= \pi^* \Theta \in \Omega^1(J^\infty(G \times \Sigma), \mathfrak{g}) \\ \mu^\infty &:= \pi^* \mu(\Theta^{\wedge(p+2)}) \in \Omega^{p+2}(J^\infty(G \times \Sigma)) \end{aligned}$$

for the pullback of the Maurer-Cartan form and of the cocycle along the projection π from definition 3.1. These forms decompose into horizontal and vertical summands according to prop. 1.23, and for the decompositon of the Maurer-Cartan form we write

$$\Theta^\infty = \Theta_H + \Theta_V \,.$$

Since by assumption on the dimension of Σ all horizontal $(p+2)$-forms on $J^\infty(G \times \Sigma)$ vanish, the form μ^∞ decomposes as

$$\mu^\infty = \langle \Theta_V \wedge \tilde{\mu}_H \rangle + \mu_V$$

where $\tilde{\mu}_H \in \Omega^n_H(G \times \Sigma, \mathfrak{g})$ is a $\mathfrak{g}$-valued horizontal $(p+1)$-form, uniquely defined by this decomposition, and γ_V is of vertical degree ≥ 2.

Remark 3.3. The form $\langle \Theta_V \wedge \tilde{\mu}_H \rangle$ is an order-0 vertical degree-1 $(p+2)$-form according to def. 1.29,

Remark 3.4. Since $H^k(J^\infty E) \cong H^k(E) \cong H^k(\Sigma \times G)$ and pull back intertwines the relevant de Rham differentials, μ^∞ represents a non-trivial cohomology class in $H^{p+2}(J^\infty E)$ whenever μ is cohomologically nontrivial.

Example 3.5. When $\Sigma = \mathbb{R}$ with canonical coordinate function denoted σ and $G = \mathbb{R}$ with canonical coordinate function denoted u, so that $J^\infty(G \times \Sigma)$ has the canonical coordinates $\{\sigma, u, u_\sigma, u_{\sigma\sigma}, \cdots\}$ then

$$\Theta = du$$

and

$$\Theta^\infty = du = \underbrace{u_\sigma d\sigma}_{\Theta_H} + \underbrace{d_V u}_{\Theta_V} .$$

Example 3.6. When $\dim \Sigma = 2$ (hence $p = 1$) and $\mu = \langle -, [-,-] \rangle$ is the *Cartan* 3-*cocycle*, the pullback of the corresponding *Cartan* 3-*form* $\mu(\Theta^{\wedge(p+2)}) = \langle \Theta \wedge [\Theta \wedge \Theta] \rangle$ decomposes as

$$\mu^\infty = \langle [\Theta_H \wedge \Theta_H] \wedge \Theta_V \rangle + \langle [\Theta_V \wedge \Theta_V] \wedge \Theta_H \rangle + \langle [\Theta_V \wedge \Theta_V] \wedge \Theta_V \rangle . \tag{26}$$

(This uses that $\langle - \wedge [- \wedge -] \rangle$ is cyclically invariant in its three arguments.) Hence here the $\tilde{\mu}_H$ from definition 3.2 is

$$\tilde{\mu}_H = [\Theta_H \wedge \Theta_H] .$$

In particular the Maurer-Cartan equation

$$d\Theta + [\Theta \wedge \Theta] = 0$$

decomposes into the equations

$$d_V \Theta_H = -d_H \Theta_V - 2[\Theta_V \wedge \Theta_H]$$

and

$$d_V \Theta_V = -[\Theta_V \wedge \Theta_V]$$

and

$$d_H \Theta_H = -[\Theta_H \wedge \Theta_H]$$

Definition 3.7. Given G and $\langle -,- \rangle$, then the *Polyakov kinetic Lagrangian for the* $(p+1)$*-dimensional sigma-model with target* G is the local Lagrangian, def. 1.43, given by

$$\mathbf{L}_{\text{kin}} := -\tfrac{1}{2} \langle \Theta_H \wedge {*\Theta_H} \rangle \quad \in \mathbf{\Omega}_H^{p+1}(J^\infty(G \times \Sigma)) .$$

Proposition 3.8. *The Euler-Lagrange operator, def. 1.43, of the Polyakov kinetic Lagrangian, def. 3.7, is*

$$\mathbf{E}_{\text{kin}} = \langle \Theta_V \wedge d_H {*\Theta_H} \rangle .$$

Proof. Since, by example 3.6, the Maurer-Cartan equation in mixed vertical/horizontal degree says that

$$d_V \Theta_H = -d_H \Theta_V - 2[\Theta_V \wedge \Theta_H]$$

and since d_V and $\Theta_V \wedge (-)$ graded-commute with the horizontal Hodge operator we get

$$\begin{aligned} d_V \langle \Theta_H \wedge {*\Theta_H} \rangle &= \langle d_V \Theta_H \wedge {*\Theta_H} \rangle + (-1)^p \langle \Theta_H \wedge {*d_V \Theta_H} \rangle \\ &= -2 \langle d_H \Theta_V \wedge {*\Theta_H} \rangle + \underbrace{-\langle [\Theta_V, \Theta_H] \wedge {*\Theta_H} \rangle - (-1)^p \langle \Theta_H \wedge *[\Theta_V, \Theta_H] \rangle}_{=0} . \\ &= -2 \left(\langle \Theta_V \wedge d_H {*\Theta_H} \rangle + d_H \langle \Theta_V \wedge {*\Theta_H} \rangle \right) , \end{aligned}$$

where in the second step we use the symmetry and the ad-invariance of $\langle -,- \rangle$. ◻

3.2 The Lepage form

The Lepage curvature is

$$\underbrace{\langle [\Theta_H \wedge \Theta_H] \wedge \Theta_V \rangle}_{\in \Omega^{2,1}_S} + \underbrace{\langle \Theta_H \wedge [\Theta_V \wedge \Theta_V] \rangle}_{\in \Omega^{1,2}}$$

Let $\Sigma = S^1 \times \mathbb{R}$.

Hence the (pre-)symplectic 2-form on $[S^1, \Sigma \times G]_\Sigma = \mathcal{L}G$ is

$$\begin{aligned}
\omega &= \int_{S^1} \langle \Theta_H \wedge [\Theta_V \wedge \Theta_V] \rangle \\
&= \int_{S^1} \langle \Theta_V \wedge [\Theta_V \wedge \Theta_H] \rangle \\
&- \tfrac{1}{2} \int_{S^1} \langle \Theta_V \wedge d_V \Theta_H \rangle - \tfrac{1}{2} \int_{S^1} \langle \Theta_V \wedge d_H \Theta_V \rangle \\
&=
\end{aligned}$$

3.3 The model restricted to small field configurations

We may consider all constructions in 3.1 restricted to any contractible open subset $U \hookrightarrow G$. The field bundle $U \times \Sigma \longrightarrow \Sigma$ then parameterizes fields of the WZW model the variation of whose values is constrained not to be too large. While we must not be content with this restriction, for discussion of the general case it is useful to consider this case first.

Since U is assumed contractible, by the Poincaré lemma we may choose $B \in \Omega^{p+1}(U)$ a differential form such that

$$dB = \mu(\Theta^{\wedge(p+2)})|_U = \mu(\Theta^{\wedge(p+2)}|_U)\,.$$

Not to overburden the notation, for the remainder of this subsection we will leave the restriction of Θ and μ to U notationally implicit.

Analogous to def. 3.2, write

$$B^\infty := \pi^* B$$

for the pullback of B to the jet bundle, and consider there its decomposition into horizontal and vertical summands

$$B^\infty = B_H + B_V\,.$$

Since the first summand here is a horizontal $(p+1)$-form, it may serve as a local Lagrangian according to def. 1.43. To indicate this usage, we write

$$\mathbf{L}_{\mathrm{top}} := B_H\,.$$

Proposition 3.9. *The Euler-Lagrange operator, def. 1.43, of* $\mathbf{L}_{\mathrm{top}} = B_H$ *is*

$$\mathbf{E}_{\mathrm{top}} = \langle \Theta_V \wedge \tilde{\mu}_H \rangle$$

(where the form on the right is from def. 3.2)

Proof. Notice that $\mathrm{d}_H B_H = 0$ for dimensional reasons and write $B_V = -B_V^1 + B_V^{\geq 2}$ for the decomposition of B_V into vertical degrees. Unwinding the definitions, we get

$$\begin{aligned}
d_V B_H &= dB_H \\
&= \underbrace{dB_H + dB_V}_{\mu^\infty} - dB_V \\
&= \langle \tilde{\mu}_H \wedge \Theta_V \rangle + \mu_V - d_H B_V - d_V B_V \\
&= \langle \tilde{\mu}_H \wedge \Theta_V \rangle + d_H(B_V^1) + \underbrace{\mu_V - d_H B_V^{\geq 2} - d_V B_V}_{=0}
\end{aligned}$$

where the first equality is due to Σ being $(p+1)$-dimensional. In the last line we use that $d_V B_H$ is of vertical degree 1, while the terms over the braces are all terms of vertical degree greater than 1. The vanishing of the latter terms essentially gives us the explicit formula for μ_V in terms of B_V. □

Remark 3.10. While $\mathbf{L}_{\text{top}} = B_H$ is only defined locally on U, the form of the Euler-Lagrange form $\mathbf{E}_{\text{top}}$ from prop. 3.9 makes sense globally.

3.4 The variational analysis

We now consider the *classical* higher dimensional WZW models with fields varying over all of G, defined by their equations of motion, via remark 3.10.

Definition 3.11. Given a Lie group G equipped with ad-invariant metric $\langle -,-\rangle$ and with a $(p+2)$-cocycle μ, then the *classical WZW model* defined by this data is the equations of motion defined over any smooth $(p+1)$-manifold Σ on the jet bundle $J^\infty(\Sigma\times G)$ by the order-0 vertical $(p+2)$-form $\mathbf{E}\in\Omega_V^{p+2}(\Sigma\times G)$ which is the sum of the kinetic Euler-Lagrange operator of prop. 3.8 with the topological EL operator of the form as in prop. 3.9:

$$\mathbf{E} = \mathbf{E}_{\text{kin}} - \mathbf{E}_{\text{top}}, \tag{27}$$

$$\text{where}\quad \mathbf{E}_{\text{kin}} = \langle \Theta_V \wedge \mathrm{d}_H{*}\Theta_H\rangle, \tag{28}$$

$$\mathbf{E}_{\text{top}} = \langle \Theta_V \wedge \tilde{\mu}_H\rangle. \tag{29}$$

Proposition 3.12. *Both terms are at least locally variational, def. 1.45, in that there exist vertical $(p+2)$-forms $\omega_{\text{kin}},\omega_{\text{top}}\in\Omega_V^{p+2}(E)$ of vertical degree ≥ 2 such that $\mathrm{d}(\mathbf{E}_{\text{kin}}+\omega_{\text{kin}})=0$ and $\mathrm{d}(\mathbf{E}_{\text{top}}+\omega_{\text{top}})=0$.*

Proof. The kinetic term is actually globally variational, since by prop. 3.8 it comes from the globally defined local Lagrangian $\mathbf{L}_{\text{kin}} = -\frac{1}{2}\langle\Theta_H\wedge{*}\Theta_H\rangle$, or equivalently $\mathrm{d}(\mathbf{L}_{\text{kin}}+\theta_{\text{kin}}) = \mathbf{E}_{\text{kin}}+\omega_{\text{kin}}$, where $\theta_{\text{kin}} = \langle\Theta_V\wedge{*}\Theta_H\rangle$ and $\omega_{\text{kin}} = \langle[\Theta_V\wedge\Theta_V]\wedge{*}\Theta_H\rangle$ (XXX: prefactor of 2?).

Local variationality of the topological term follows from prop. 3.9. It is exhibited by setting $\omega_{\text{top}} = \mu_V$ (from def. 3.2) and noting that $\mathbf{E}_{\text{top}}+\omega_{\text{top}} = \mu^\infty$, which is closed on $J^\infty E$. □

Example 3.13. For the case of the 2d WZW model from example 3.6 we get (using the horizontal Maurer-Cartan equation from example 3.6)

$$\begin{aligned}\mathbf{E} &= \langle\Theta_V\wedge(d_H * \Theta_H + [\Theta_H\wedge\Theta_H])\rangle \\ &= \langle\Theta_V\wedge(d_H * \Theta_H - d_H\Theta_H)\rangle\end{aligned}.$$

Hence the equation of motion is

$$d_H * \Theta_H - d_H\Theta_H$$

hence in terms of the canonical coordinates $\{x,t\}$ on $\Sigma = S^1\times\mathbb{R}$

$$\partial_x g^{-1}\partial_x g - \partial_t g^{-1}\partial_t g - \partial_t g^{-1}\partial_x g + \partial_x g^{-1}\partial_t g = \partial_{x-t}(g^{-1}\partial_{x+t} g)$$

Proposition 3.14. *We claim that Lagrangian of the 2d WZW model is invariant under transformations by the loop group.*

Proof. For let γ be a generator for an evolutionary vector field. Then because

$$L_{\gamma_{\text{ev}}}\Theta_H = \ldots = d_H\gamma - 2[\Theta_H\wedge\gamma]$$

we have

$$L_{\gamma_{\text{ev}}}(L_{\text{kin}}+L_{\text{top}}) = \langle\Theta_H\wedge(\star\pm\mathrm{id})d_H\gamma\rangle\,.$$

Notice that

$$\tfrac{1}{2}(\mathrm{id}\pm *)$$

is the projector on $dx\pm dt$. So if we specify γ on any spatial slice $S^1\hookrightarrow S^1\times\mathbb{R}$ then it has a unique extension over $S^1\times\mathbb{R}$ satisfying $(\star\pm\mathrm{id})d_H\gamma = 0$, and by the above this is a symmetry of the Lagrangian. □

Now, we establish that the generalized WZW model satisfies the hypotheses needed for the application of Proposition **??** and Theorem **??**.

Proposition 3.15. *When (M, h) is a globally hyperbolic manifold, the WZW model of def. 3.11 is a regular non-gauge theory (Definition* **??***) that is topologically neutral (Definition* **??***) and Noether consistent (Definition* **??***).*

Proof. These are all well known facts, often even taken for granted. Hence we only sketch the main arguments. The key technical observation about the equations of motion $\mathbf{E}[\phi] = 0$ (Equation (27)) of WZW theory is that they constitute a non-linear wave equation [Ga02, Sec.3.1]. More precisely, it is a quasi-linear (the highest derivatives appear linearly, with coefficients that may depend on lower order derivatives), second order, hyperbolic equation of the *wave map* type (see [Mi78, ChBr87], also the more recent [Ta04] and the references there in). The kinetic Lagrangian $\mathbf{L}_{\text{kin}}$ is precisely the wave map Lagrangian for maps from the Lorentzian manifold (M, h) to the Riemannian manifold $(G, \langle -, - \rangle)$. The topological term $\mathbf{E}_{\text{top}}[\phi]$, as is clear from its construction, contains only first order derivatives and thus does not change the type of the equation.

Equations of this type (quasi-linear hyperbolic) can be written in Cauchy-Kovalevskaya form (solved for the highest time derivatives with respect to a foliation of M by Cauchy surfaces), which implies that the PDE submanifold $\mathcal{E} \subset J^\infty E$ is regular and that the initial data can be specified freely, in this case up to first order derivatives. In other words, the projection $\mathcal{E} \to J^1 E$ is surjective, which by quasi-linearity also forms an affine bundle over $J^1 E$. Thus, topologically, the PDE submanifold may be contracted to $J^1 E$ and hence also to $J^0 E \cong M \times G$, showing that it is topologically neutral. Finally, since hyperbolic equations of this type are well known to have a locally well-posed initial value formulation (essentially given by the Cauchy-Kovalevskaya form mentioned above), this model is a non-gauge theory and by being (locally) variational its phase space has a Poisson structure that is Noether compatible [BSF89, HeTe94]. ☐

Remark 3.16. Theorem **??** applies to the complete Lie algebra of globally variational local symmetries $\mathcal{S} = \mathrm{Sym}_{\text{glob.var}}(E, \mathbf{E})$ (def. **??**) of the generalized WZW model, def. 3.11. However, it may be a highly non-trivial task to identify all local symmetries of a given model. For example, some interesting special models possess infinitely many linearly independent local symmetries (so-called *integrable systems*). On the other hand, even if we only know a certain sub-algebra $\mathcal{S}' \subset \mathcal{S}$, such as the point symmetries, def. **??**, the central extension **??** easily restricts to a central extension

$$0 \to \mathcal{T} \to \mathcal{Q}' \to \mathcal{S}' \to 0 \tag{30}$$

of the known algebra by the same topological charges.

It remains to precisely characterize the space $\mathcal{T} \subset C^\infty(\mathfrak{P})$ of topological charges (Definition **??**) on the phase space (i.e. the solution space) $\mathfrak{P} \subset \Gamma(E)$ of the theory. By global hyperbolicity, we can identify $\Sigma \cong \mathbb{R} \times \Sigma_p$, where by hypothesis Σ_p is a compact p-dimensional manifold, with each Σ_p-level set a Cauchy surface in (Σ, h), all belonging to the same homology class. Topological charges constitute the image of the map $\int_{\Sigma_p} : H^p(E) \to \mathcal{T} \subset C^\infty(\mathfrak{P})$ which associates to each topological current $\tau \in H^p(\mathcal{E}) \cong H^p(E)$ (by topological neutrality) the integrated charge $t \in \mathcal{T}$, whose value for any $\phi \in \mathfrak{P}$ is given by integration $t(\phi) = \int_{\Sigma_p} (j^\infty \phi)^* \tau$ over any cycle in the Cauchy surface homology class, which we also denote by Σ_p. As was mentioned earlier, topological charges $t \in \mathcal{T}$ are locally constant functions on $\mathfrak{P}$. Thus, $\mathcal{T}$ has non-trivial structure only in the case when $\mathfrak{P}$ has more than one connected component. Also, by the invariance of the integrated charge formula defining $t(\phi)$ under continuous deformations of $\phi \in \mathfrak{P}$ or even $\Gamma(E)$ (topological neutrality), each topological charge will be constant on $\mathfrak{P}_\sigma$, where $\sigma \in [\Sigma_p, E]$ is the homotopy class of the the restriction $\sigma = [\phi|_{\Sigma_p}]$ of a section $\phi \colon M \to F$ to a Cauchy surface Σ_p. While $\mathfrak{P} = \bigsqcup_{\sigma \in [\Sigma_p, E]} \mathfrak{P}_\sigma$, that does not automatically imply that each of $\mathfrak{P}_\sigma$ is itself connected, though that may indeed follow from other considerations. [XXX: What is a precise characterization of $\mathcal{T}$ in terms of the topology of $E \to M$?]

[XXX: go a step further, by making the extension cocycle explicit]

4 Category theory

This section collects the basics of category theory that we need in the main text. All statements here have direct ∞-categorical analogs, and hence we state it in this generality.

4.1 Categories

Definition 4.1. Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}$, then the *slice category* $\mathcal{C}_{/c}$ has as objects the morphisms of $\mathcal{C}$ into c, and as morphisms between these the commuting triangles in $\mathcal{C}$ of the form

$$\begin{array}{ccc} a_1 & \longrightarrow & a_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & c & \end{array} .$$

Example 4.2. If $* \in \mathcal{C}$ is a terminal object, then there is an equivalence of categories

$$\mathcal{C}_{/*} \simeq \mathcal{C} .$$

Proposition 4.3. *The hom-spaces in a slice category $\mathcal{C}_c$, def. 4.1 are equivalently given by the fiber product:*

$$\mathcal{C}_{/c}(f_1, f_2) \simeq \mathcal{C}(a_1, a_2) \underset{\mathcal{C}(a_2,c)}{\times} \{f_2\}$$

of hom-spaces in $\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C}_{/c}(f_1, f_2) & \longrightarrow & \mathcal{C}(a_1, a_2) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\tilde{f}_2} & \mathcal{C}(a_2, c) \end{array} .$$

where $\tilde{f}_2$ picks the element f_2 in $\mathcal{C}(a_2, c)$.

Proposition 4.4 (Dold-Kan correspondences)**.** *The functor forming normalized chain complexes from simplicial abelian groups constitutes an equivalence of categories*

$$\mathrm{Ch}_{\bullet \geq 0} \underset{\Gamma}{\overset{N}{\underset{\simeq}{\rightleftarrows}}} \mathrm{sAb}$$

We write

$$\mathrm{DK} : \mathrm{Ch}_\bullet \xrightarrow{\simeq} \mathrm{sAb} \xrightarrow{\text{forget}} \mathrm{KanCplx} \hookrightarrow \mathrm{sSet}$$

for the functor that first applies the Dold-Kan correspondence and then forgets the abelian group structure on the resulting Kan complexes.

4.2 Toposes

(...)

Given a site C, there are model category structures on the categories $\mathrm{PSh}(C, \mathrm{sSet})$ and $\mathrm{PSh}(C, \mathrm{sSet})$ of simplicial (pre-)sheaves over C whose weak equivalences are the local (with respect to forming covers in the site) weak homotopy equivalences of simplicial sets. Similarly, on the categories $\mathrm{PSh}(C, \mathrm{KanCplx})$ and $\mathrm{Sh}(C, \mathrm{KanCplx})$ of Kan-complex valued (pre-)sheaves there are structures of categories of fibrant objects with such weak equivalences. Under simplicial localization, all of these homotopical structures present the (hypercomplete) ∞-topos over C, which we denote $\mathrm{Sh}_\infty(C)$

(...)

Proposition 4.5 ([L-Topos, cor. A.3.7.2]). *If $\mathcal{C}$ and $\mathcal{D}$ are simplicial model categories and $\mathcal{D}$ is a left proper model category, then for an* sSet*-enriched adjunction*

$$(L \dashv R) : \mathcal{C} \underset{}{\overset{}{\rightleftarrows}} \mathcal{D}$$

to be a Quillen adjunction it is already sufficient that L preserves cofibrations and R preserves fibrant objects.

Corollary 4.6. *Let $f : C \to D$ be a functor between sites that sends covers to covers. Then its left Kan extension adjunction (prop. 4.10) extends to an adjunction of ∞-toposes*

$$(f_! \dashv f^*) : \mathrm{Sh}_\infty(C) \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{Sh}_\infty(D) \ .$$

Proof. Consider the 1-categorical adjunction on categories of simplicial presheaves

$$(f_! \dashv f^*) : \mathrm{sPSh}(C) \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathrm{sPSh}(D)$$

This is naturally a simplicial adjunction, and it is clearly a Quillen adjunction with respect to the global projective model structure on both sides, since f^* manifestly preserves fibrations and equivalences in this case, hence it follows that $f_!$ preserves cofibrations. Now passing to the local projective model structure, since this has the same cofibrations as the global structure $f_!$ still preserves cofibrations, and by the assumption that f preserves covers, f^* now still manifestly preserves fibrant objects. With the the statement follows by prop. 4.5. ☐

4.3 Universal constructions

Definition 4.7. A pair of *adjoint functors*, denoted $(L \dashv R)$, is a pair of functors of the form

$$\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$$

such that there is a natural isomorphism ("forming adjuncts")

$$\mathrm{Hom}_{\mathcal{C}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{D}}(-, R(-)) \, .$$

Here L is called *left adjoint to* R and R is called *right adjoint to* L. The image η_d of id_{Ld} under this isomorphism is called the *unit* of the adjunction at $d \in \mathcal{D}$

$$\eta_d : d \longrightarrow R(L(d)) \, ,$$

while, conversely, the image ϵ_d of id_{Rc} is called the *counit*

$$\epsilon_d : L(R(d)) \longrightarrow d \, .$$

(Unit and counit are themselves natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \longrightarrow R \circ L$ and $\epsilon : L \circ R \longrightarrow \mathrm{id}$.)

One also writes horizontal lines for indicating these bijections between sets of adjunct morphisms:

$$\frac{d \longrightarrow Rc}{Ld \longrightarrow c}$$

Proposition 4.8. *A right adjoint function (def. 4.7) preserves all small limits. Dually, a left adjoint functor preserves all small colimits.*

Proposition 4.9. *Given an adjunction $(L \dashv R)$ as in def. 4.7, then*

- *the adjunct of a morphism of the form* $f : d \longrightarrow Rc$ *is equivalently the composite*

$$Ld \xrightarrow{L(f)} LRc \xrightarrow{\epsilon_c} c\ ;$$

- *the adjunct of a morphism of the form* $g : Lc \longrightarrow d$ *is equivalently the composite*

$$c \xrightarrow{\eta_c} RLc \xrightarrow{R(g)} Rd\ .$$

Key examples of adjoint pairs and adjoint triples are Kan extensions and dependent sums and products:

Proposition 4.10 (Kan extension)**.** *Given a functor* $f : C \longrightarrow D$ *between small categories, then the induced functor on categories of presheaves* $f^* : \mathrm{PSh}(D) \longrightarrow \mathrm{PSh}(C)$ *(given by precomposing a presheav with* f*) has both a left and a right adjoint (def. 4.7), denoted* $f_!$ *and* f_* *respectively, and called the operations of left and right Kan extension along* f*.*

$$(f_! \dashv f^* \dashv f_*) :\ \mathrm{PSh}(C) \underset{f_*}{\overset{f_!}{\underset{\longleftarrow f^* \longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}}} \mathrm{PSh}(D)\ .$$

Moreover, the left Kan extension of a presheaf $A \in \mathrm{PSh}(C)$ *is equivalently the presheaf which to any object* $d \in D$ *assigns the set expressed by the coend formula*

$$(f_! A)(d) \simeq \int^{c \in C} \mathrm{Hom}_D(d, f(c)) \times \mathrm{Hom}_{\mathrm{PSh}(C)}(c, A)\,,$$

where on the right we are identifying c *with the presheaf that it represents. Explicitly, this coend gives the set of equivalence classes of pairs of morphisms*

$$(d \to f(c), c \to A)$$

where two such pairs are regarded as equivalent if there is a morphism $\phi : c_1 \to c_2$ *in* C *such that the following two triangles commute*

$$\begin{array}{ccccc} & \swarrow & d & \searrow & \\ f(c_1) & & \xrightarrow{f(\phi)} & & f(c_2) \\ c_1 & & \xrightarrow{\phi} & & c_2 \\ & \searrow & & \swarrow & \\ & & A & & \end{array}$$

Proposition 4.11 (base change)**.** *For* $\mathbf{H}$ *a topos and* $f : X \longrightarrow Y$ *any morphism, then the functor*

$$f^* : \mathbf{H}_{/Y} \longrightarrow \mathbf{H}_{/X}$$

between slice categories, def. 4.1, given by pullback along f *has both a left and a right adjoint, the* base change adjoint triple *along* f

$$(f_! \dashv f^* \dashv f_*) :\ \mathbf{H}_{/X} \underset{f_*}{\overset{f_!}{\underset{\longleftarrow f^* \longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}}} \mathbf{H}_Y\ .$$

Here $f_!$ *is the operation of postcomposition with* f*.*

Where adjunctions map back and forth between two categories, (co-)monads act on a single category:

Definition 4.12. For $\mathcal{C}$ a category, then a *monad* on $\mathcal{C}$ is an endofunctor

$$J \colon \mathcal{C} \to \mathcal{C}$$

equipped with natural transformations

- $\nabla : J \circ J \longrightarrow J$ (product)
- $\eta : J \longrightarrow \mathrm{id}_{\mathcal{C}}$ (unit)

such that these satisfy the evident associativity and unitalness properties.

Dually, a *comonad* on $\mathcal{C}$ is an endofunctor J equipped with natural transformations

- $\Delta : J \longrightarrow J \circ J$ (coproduct)
- $\epsilon : \mathrm{id}_{\mathcal{C}} \longrightarrow J$ (counit)

such that these satisfy the evident co-associativity and co-unitalness properties.

Definition 4.13. Given a comonad (J, ϵ, Δ) on $\mathcal{C}$, def. 4.12, then a *coalgebra* over the comonad is an object $E \in \mathcal{C}$ equipped with a morphism

$$\rho : E \longrightarrow JE$$

that satisfies the evident axioms of a co-action. A homomorphism of coalgebras $f(E_1, \rho_1) \longrightarrow (E_2, \rho_2)$ is a morphism $f : E_1 \longrightarrow E_2$ in $\mathcal{C}$ which respects these coaction morphisms. The resulting category of coalgebras is denoted $\mathrm{EM}(J)$.

Proposition 4.14. *For* $(L \dashv R) : \mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ *an adjunction, def. 4.7, then the endofunctor*

$$T := L \circ R : \mathcal{C} \to \mathcal{C}$$

becomes a comonad on $\mathcal{C}$*, def. 4.12, with counit the adjunction counit* $L \circ R \to \mathrm{id}_{\mathcal{C}}$ *(def. 4.7), and with coproduct induced from the unit of the adjunction by*

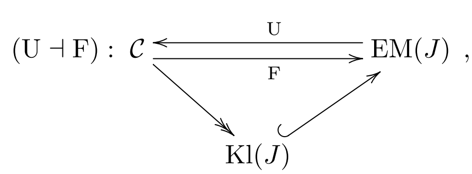
$$\Delta_T := L(\eta_{R(-)}) \, .$$

Dually $R \circ L$ *is canonically equipped with the structure of a monad.*

Example 4.15. Given an adjoint triple $(L \dashv C \dashv R)$ then the monad $C \circ L$ and the comonad $C \circ R$ induced via prop. 4.14 themselves form an adjoint pair:

$$(C \circ L \dashv C \circ R) : \mathcal{C} \longrightarrow \mathcal{C} \, .$$

Proposition 4.16. *The category of coalgebras over a comonad on a category* $\mathcal{C}$*, def. 4.13, is related to* $\mathcal{C}$ *by a pair of adjoint functors, def. 4.7, of the form*



where the left adjoint U *("underlying") forgets the coalgebra structure,* $U : (E, \rho) \mapsto E$*, while the right adjoint* F *("co-free") sends an object* $c \in \mathcal{C}$ *to to the object* Jc *with coaction given by the coproduct* Δ_J*. The comonad induced from this adjunction via prop. 4.14 coincides with* J*:*

$$J \simeq \mathrm{U} \circ \mathrm{F} \, .$$

Definition 4.17. The full subcategory of EM(J) on the cofree coalgebras, i.e. on the objects in the image of F, prop. 4.16, is also called the coKleisli category Kl(J).

Remark 4.18. Given objects $c_1, c_2 \in \mathcal{C}$, then by adjunction we have a bijection of morphisms of the form

$$\frac{\dfrac{Fc_1 \longrightarrow Fc_2}{UFc_1 \longrightarrow c_2}}{Jc_1 \longrightarrow c_2}$$

Hence morphisms f in the coKleisli category Kl(J), def. 4.17, are equivalently morphisms in $\mathcal{C}$ of the form $\tilde{f} : Jc_1 \longrightarrow c_2$. Under this identification the composition of morphisms $g \circ f$ in Kl(J) is given by the "co-Kleisli composite"

$$\widetilde{g \circ f} : Jc_1 \xrightarrow{\Delta_c} JJc_1 \xrightarrow{J(\tilde{f})} Jc_1 \xrightarrow{\tilde{g}} c_2 \,.$$

Definition 4.19. A functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ is called *conservative* if it reflects equivalences, hence if for a morphism f in $\mathcal{D}$ we have that if $F(f)$ is an equivalence then already f was an equivalence.

Theorem 4.20 (Beck monadicity theorem)**.** *Sufficient conditions for an adjunction* $(L \dashv R)$*, def. 4.7, to be equivalent to a comonadic adjunction* (U $\dashv$ F) *as in prop. 4.16 is that*

1. U *is conservative, def. 4.19;*
2. U *preserves certain limits (...).*

Proof. For 1-category theory this may be found e.g. in [Bor, vol. 4 sect. 2]. For ∞-category theory this is [L-Alg, theorem 4.7.4.5]. ☐

Hence it is useful to record some facts about conservative functors:

Proposition 4.21. *For* $\mathbf{H}$ *a topos and* $f : X \longrightarrow Y$ *an epimorphism in* $\mathbf{H}$*, then the pullback functor* $f^* : \mathbf{H}_{/Y} \longrightarrow \mathbf{H}_{/X}$ *is conservative, def. 4.19.*

Proof. For 1-category theory this is for instance a special case of [Joh02, lemma 1.3.2]. For ∞-category theory see the nLab entry on conservative ∞-functors. ☐

Proposition 4.22. *A conservative functor reflects all the limits and colimits which it preserves.*

Corollary 4.23 (comonadic descent)**.** *Given an epimorphism* $X \stackrel{f}{\longrightarrow\!\!\!\!\to} Y$ *in a topos* $\mathbf{H}$*, with induced base change comonad*

$$J := f^* f_* : \mathbf{H}_{/X} \to \mathbf{H}_{/X}$$

(via prop. 4.11, prop. 4.14), then there is an equivalence of categories

$$\mathrm{EM}(J) \simeq \mathbf{H}_Y$$

between the J*-coalgebras in* $\mathbf{H}_{/X}$*, def. 4.13, and the slice* $\mathbf{H}_{/Y}$*. Moreover, under this identification the comonadic adjunction* $(U_J \dashv F_J)$ *from prop. 4.16 coincides with the base change adjunction* $(f^* \dashv f_*)$ *of prop. 4.11:*

$$(U_J \dashv F_J) \simeq (f^* \dashv f_*) \,.$$

Proof. Since f is assumed to be epi, prop. 4.21 says that f^* is conservative. Moreover, since f^* is right adjoint to $f_!$ by prop. 4.11, it preserves all small limit, by prop. 4.8. Therefore the conditions in the monadicity theorem 4.20 are satisfied. This yields the statement. ☐

See also for instance [JaTh, 2.4].

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