A synthetic approach to the formal theory of PDEs
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Observations about Jets

- Work in the category $\mathcal{F}_M$ of smooth fibered manifolds $E \to M$, $\dim M < \infty$, $\dim E \leq \infty$ (local dependence on finitely many of at most countably many coordinates, LocProMfd).

- **Jet bundles** define a functor $J := J^\infty : \mathcal{F}_M \to \mathcal{F}_M$ that preserves “sufficiently regular” limits (monos, fibered products, . . .).

- The jet extension $j^\infty f : M \to JE$ of a section $f : M \to E$:
  
  (a) recalls the original section $f = \epsilon_E \circ j^\infty f$,
  
  (b) and knows its own jet extension $j^\infty j^\infty f = \Delta_E \circ j^\infty f$.

- The natural transformations $\epsilon : J \to \text{id}, \Delta : J \to JJ$ satisfy

\[
\begin{array}{ccc}
J & \overset{\text{id}}{\leftarrow} & \overset{\epsilon(J)}{\rightarrow} & \overset{\Delta}{\rightarrow} & \overset{\text{id}}{\leftarrow} & JJ \\
J & \overset{\Delta}{\rightarrow} & JJ & \overset{\Delta(J)}{\downarrow} & \overset{\Delta(J)}{\downarrow} & JJJ
\end{array}
\]

the axioms of a **comonad**.
Observations about PDEs

- A sufficiently regular PDE $\mathcal{E}^k \rightarrow J^k E$ can be put into a **canonical first order form** $\rho^1 : \mathcal{E}^{k+1} \rightarrow J^1 \mathcal{E}^k$.
  - Introduce a new variables $u$ for each component of $j^k \phi$ of a solution. Use $u = j^k \phi$ to solve $j^1 f(j^k \phi) = 0$ for $j^1 u = \rho^1 (u)$.

- A sufficiently regular formally integrable PDE $\mathcal{E} \rightarrow JE$ can be put into a **canonical infinitely prolonged form** $\rho : \mathcal{E} \rightarrow JE$.

- The canonical form $j^\infty u = \rho(u)$ satisfies the **universal integrability condition**

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\rho} & JE \\
\rho \downarrow & & \downarrow J\rho \\
JE & \Delta_{\mathcal{E}} & JJE
\end{array}
\quad \text{and of course}
\quad \begin{array}{ccc}
\mathcal{E} & \xrightarrow{\rho} & JE \\
\downarrow \text{id} & & \downarrow \epsilon_{\mathcal{E}} \\
\mathcal{E} & \xrightarrow{\epsilon_{\mathcal{E}}} & E
\end{array}
\]

the axioms of a **coalgebra** over the comonad $J$. 

The category of *differential operators* $\alpha[f] = \alpha \circ j^\infty f$ is equivalent to the *co-Kleisli* category of $J$, $\text{DiffOp}(\mathcal{F}_M) \simeq \text{Kl}(J)$. Follows from the composition formula

$$(\alpha \circ \beta)[f] = (\alpha \circ p^\infty \beta) \circ j^\infty f,$$

where $p^\infty \beta = J\beta \circ \Delta$.

Vinogradov’s category of *PDEs* is equivalent to the *Eilenberg-Moore* category of coalgebras over $J$, $\text{PDE}(\mathcal{F}_M) \simeq \text{EM}(J)$. Morphisms of coalgebras satisfy

$$
\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{\alpha} & \mathcal{E}_2 \\
\rho_1 \downarrow & & \downarrow \rho_2 \\
J\mathcal{E}_1 & \xrightarrow{J\alpha} & J\mathcal{E}_2
\end{array}
$$

Remaining questions:

- How much can the regularity assumptions be relaxed?
- Can $\mathcal{E}$ be a variety, orbifold, stratified, ... supermanifold, stack, ... , have boundaries, singularities, ... ?
Synthetic Differential Geometry (SDG)

- **SDG** is an axiomatic/categorical approach to the study of smooth spaces, operations between them and their generalizations.
- We will work specifically with the **Cahiers Topos H**, introduced by Dubuc (*Cahiers T&GD* 1979).
- **H** has fully faithful embeddings of well-known categories:

  \[
  \text{Mfd} \hookrightarrow \text{LocProMfd} \hookrightarrow \text{FrMfd} \hookrightarrow \text{DiflSp} \hookrightarrow \text{H} \hookleftarrow \text{FormalMfd}
  \]

- Objects in **H** may have algebraic or orbifold singularities, may have boundaries and corners, could be infinite dimensional, and may have infinitesimal directions.
- **Infinitesimal** spaces are particularly well-adapted to the **formal theory** of PDEs.
- **Literature:**
  - R. Lavendhome: *Basic Concepts of SDG* (Springer 1996)
  - U. Schreiber: *dcct* [arXiv:1310.7930]
Generalized Smooth Spaces

- $M \in \text{Mfd}$, dim $M = n$; $\text{Atlas}(M) \subset C^\infty(\mathbb{R}^n, M)$.
- $\text{CartSp}$ — category of all $\mathbb{R}^k \to \mathbb{R}^m$ smooth; $\text{CartSp}_{\text{diff}}(n)$ — all diffeomorphisms onto image $\mathbb{R}^n \to \mathbb{R}^n$.
- Functor $\text{CartSp}_{\text{diff}}(n)^{\text{op}} \to \text{Set}$, $\mathbb{R}^n \mapsto \text{Atlas}(M)$, satisfies gluing:
  
  (illustration)

- No harm in extending $\text{Atlas}(M) \subset C^\infty(\_, M) : \text{CartSp}^{\text{op}} \to \text{Set}$.
- Now $C^\infty(\_, M) \in \text{Sh}(\text{CartSp}, \text{Set})$ is a sheaf with respect to the “open cover” Grothendieck topology on CartSp.
- Fully faithful $\text{Mfd} \leftrightarrow \text{SmthSp}$ (Generalized Smooth Spaces):
  
  (Yoneda) $\text{SmthSp} \ni M \leftrightarrow "C^\infty(\_, M)" \in \text{Sh}(\text{CartSp}, \text{Set})$
Cahiers Topos

Sheaves (“$C^\infty(\mathbb{R}^k)$”) on test spaces ($\mathbb{R}^k$) are generalized spaces ($M$).

- **FormalCartSp** := $\langle \mathbb{R}^k, \mathbb{D}^k(m), \times \rangle$ — opposite to the full subcategory

$$\langle C^\infty(\mathbb{R}^k), C^\infty(\mathbb{R}^k)/(x^1, \ldots, x^k)^{m+1}, \otimes \rangle \hookrightarrow \text{CAlg}^{\mathbb{R}}$$

of commutative $\mathbb{R}$-algebras, closed under products.

- **Ex:** $f \in C^\infty(\mathbb{R}^n \times \mathbb{D}^k(m))$ is a formal power series

$$f(x^1, \ldots, x^n, \varepsilon^1, \ldots, \varepsilon^k) = \sum_{|I|\leq m} f_I(x^1, \ldots, x^n) \varepsilon^I.$$

- **Cahiers topos** — $H := \text{Sh}(\text{FormalCartSp}, \text{Set})$:
  - closed under all small $\text{limg}(\_)$, $\text{lim}(\_)$ and internal $\text{Hom}(\_\_, \_\_)$;
  - fully faithful embedding of many categories of “smooth spaces”;
  - access to infinitesimals without leaving the category, e.g., formal disks $\mathbb{D}^k(\infty) := \text{limg}_m \mathbb{D}^k(m)$. 

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Infinitesimals, Formal Disks, Jets

- Take \( M \in \text{Mfd} \hookrightarrow \mathbb{H} \), \( \dim M = n \) (independent variables); take \((E \to M) \in \mathbb{H}_M\) (dependent variables, no extra regularity!).

- Every \( x \in M \) has **formal disk neighborhood** \( T^\infty_x \simeq \mathbb{D}^n(\infty) \to M \).

- Formal neighborhoods functor \( T^\infty : \mathbb{H}_M \to \mathbb{H}_M \), \( T^\infty E := T^\infty M \times_M E \).

- In “coordinates” \( f \in \text{Hom}_{\mathbb{H}_M}(T^\infty E, F) \) is of the form

\[
f(x, u, \varepsilon) = \sum_{|I| < \infty} f_I(x, u) \varepsilon^I \quad \text{(formal series)}.
\]

- **Jets** — right adjoint of \( T^\infty \dashv J^\infty : \mathbb{H}_M \to \mathbb{H}_M \) (exists in topos!):

\[
\text{Hom}_{\mathbb{H}_M}(T^\infty E, F) \simeq \text{Hom}_{\mathbb{H}_M}(E, J^\infty F) \quad \text{naturally } \forall E, F \in \mathbb{H}_M;
\]

\[
f(x, u, \varepsilon) \dashv f(x, u, -) \simeq \tilde{f}(x, u) = (f_I(x, u))_{|I| < \infty}.
\]

- **Right adjoints** automatically preserve all limits (monos, fibered products, . . .). No need for Marvan’s “sufficient regularity”.

- \( J : \mathbb{H}_M \to \mathbb{H}_M \) is a **comonad** for abstract reasons, due to the differential cohesion (Schreiber 2013) of \( \mathbb{H}_M \).
Synthetic Geometry of PDEs

- Generalized PDE \( \mathcal{E} \leftrightarrow J^\infty F \) in \( H/M \); \textit{mono} is the only regularity condition needed.

- A \( Y \)-\textit{family} of formal sections \( \sigma : T^\infty Y \to JF \) is \textit{holonomic} if
  \[
  \sigma(x, u, \varepsilon)\sim = \sigma(x, u, \varepsilon + \sim)(0).
  \]

- A \( Y \)-\textit{family} \( s \) such that \( T^\infty Y \to \mathcal{E} \leftrightarrow JF \) is \textit{holonomic} is a \( Y \)-\textit{family} of \textit{formal solutions}.

- If \( \mathcal{E}_1 \to \mathcal{E}_2 \) preserves all families of formal solutions, it is a \textit{morphism} of generalized PDEs ("prolonged differential operator").

- There always exists a \textit{universal family of formal solutions} \( T^\infty \mathcal{E}^\infty \to \mathcal{E} \) such that \( \mathcal{E}^\infty \to T^\infty \mathcal{E}^\infty \to \mathcal{E} \) is a \textit{mono}:

\[
\begin{array}{ccc}
\mathcal{E}^\infty & \longrightarrow & JF \\
\downarrow \rho & & \downarrow \Delta_F \\
\mathcal{E} & \leftarrow \mathcal{E}^\infty & \longrightarrow \mathcal{E} & \longrightarrow & JF \\
\end{array}
\]

\( \mathcal{E}^\infty \to \mathcal{E} \to JF \) is exactly the \textit{(infinite) prolongation} of \( \mathcal{E} \leftrightarrow JF \).
Main Results

- When $\mathcal{E}^\infty \simeq \mathcal{E}$, the PDE is **formally integrable**, has intrinsic presentation $\rho: \mathcal{E} \simeq \mathcal{E}^\infty \hookrightarrow J\mathcal{E}$.

- Category $\text{PDE}(H/M)$: **objects** — formally integrable PDEs, **morphisms** — preserve all families of formal solutions. Logically independent from Vinogradov’s definition.

- **Thm**: $\text{DiffOp}(H/M) \simeq \text{Kl}(J)$ and $\text{PDE}(H/M) \simeq EM(J)$; for each formally integrable PDE, $\rho: \mathcal{E} \hookrightarrow J\mathcal{E}$ is a $J$-coalgebra; a morphism preserving families of formal solutions is a morphism of coalgebras.
  - Because of different definitions/hypotheses, the proof is *logically independent* from (but inspired by) Marvan’s.
  - The fully faithful embedding $\mathcal{F}_M \hookrightarrow H/M$ and Marvan’s original equivalence *imply* $\text{PDE}(\mathcal{F}_M) \simeq \text{PDE}_{\text{Vinogradov}}(\mathcal{F}_M)$.

- **Thm**: $\text{PDE}(H/M)$ is also a **topos** (hence has all small limits). More concretely, **all finite limits** in $\text{PDE}(H/M)$ can be **computed in** $H/M$.

- **Thm**: $\text{Sol}(\mathcal{E}) \simeq \text{Hom}_{\text{PDE}(H/M)}(M, \mathcal{E})$. 
Discussion

- Jets and PDEs **internal** to Cahiers Topos H:
  - Maximally relaxed (within smooth geometry) *regularity conditions* on spaces of dependent variables and PDEs.
  - *Infinitesimals*, *formal sections* give intrinsic and intuitive notion of a PDE category. No need to appeal to Cartan distribution as proxy for formal solutions. When comparable, coincides with Vinogradov’s.
  - All constructions inherently *independent* of (even the *existence* of) choices of local coordinates.

- \( J \)-comonad and \( J \)-coalgebra structure of PDEs suggests *natural generalization* to more general contexts of Synthetic Differential Geometry (super-, derived-, stack-, . . . manifolds).

- **Future**: study symmetries; non-integrable infinitesimal symmetries, could be *truly infinitesimal* in SDG.

- **Future**: study PDEs on derived higher super-stacks . . .

- **Future**: How to compare with Beilinson-Drinfeld’s \( \mathcal{D} \)-schemes?
Thank you for your attention!