A synthetic approach to the formal theory of PDEs

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Observations about Jets

- Work in the category $\mathcal{F}_M$ of smooth **fibered manifolds** $E \to M$, $\dim M < \infty$, $\dim E \leq \infty$ (local dependence on *finitely* many of at most *countably* many coordinates, LocProMfd).
- **Jet bundles** define a functor $J := J^\infty \colon \mathcal{F}_M \to \mathcal{F}_M$ that preserves "sufficiently regular" limits (monos, fibered products, …).
- The **jet extension** $j^\infty f \colon M \to JE$ of a section $f \colon M \to E$:
  (a) recalls the *original section* $f = \epsilon_E \circ j^\infty f$,
  (b) and knows its *own jet extension* $j^\infty j^\infty f = \Delta_E \circ j^\infty f$.
- The natural transformations $\epsilon \colon J \to \mathrm{id}$, $\Delta \colon J \to JJ$ satisfy

$$\begin{array}{ccccc} & & J & & \\ & \swarrow^{\mathrm{id}} & \downarrow \Delta & \searrow^{\mathrm{id}} & \\ J & \xleftarrow[\epsilon(J)]{} & JJ & \xrightarrow[J(\epsilon)]{} & J \end{array} \quad \text{and} \quad \begin{array}{ccc} J & \xrightarrow{\Delta} & JJ \\ \Delta \downarrow & & \downarrow \Delta(J) \\ JJ & \xrightarrow[J(\Delta)]{} & JJJ \end{array} ,$$

the axioms of a **comonad**.

Observations about PDEs

- A sufficiently regular PDE $\mathcal{E}^k \hookrightarrow J^k E$ can be put into a **canonical first order form** $\rho^1 \colon \mathcal{E}^{k+1} \hookrightarrow J^1 \mathcal{E}^k$.
  - Introduce a new variables u for each component of $j^k \phi$ of a solution. Use $u = j^k \phi$ to solve $j^1 f(j^k \phi) = 0$ for $j^1 u = \rho^1(u)$.
- A sufficiently regular formally integrable PDE $\mathcal{E} \hookrightarrow JE$ can be put into a **canonical infinitely prolonged form** $\rho \colon \mathcal{E} \hookrightarrow J\mathcal{E}$.
- The canonical form $j^\infty u = \rho(u)$ satisfies the **universal integrability condition**

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho} & J\mathcal{E} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle J\rho} \\ J\mathcal{E} & \xrightarrow[\Delta_{\mathcal{E}}]{} & JJ\mathcal{E} \end{array} \quad \text{and of course} \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho} & J\mathcal{E} \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle \epsilon_{\mathcal{E}}} \\ & & \mathcal{E} \end{array} ,$$

the axioms of a **coalgebra** over the comonad J.

Jets & PDEs :: Comonads & Coalgebras

- Observations due to Marvan (*Proc DG&A* 1986, *PhD* 1989).
- The category of **differential operators** $\alpha[f] = \alpha \circ j^\infty f$ is equivalent to the **co-Kleisli** category of J, $\mathrm{DiffOp}(\mathcal{F}_M) \simeq \mathrm{Kl}(J)$. Follows from the composition formula

$$(\alpha \circ \beta)[f] = (\alpha \circ p^\infty \beta) \circ j^\infty f, \quad \text{where} \quad p^\infty \beta = J\beta \circ \Delta.$$

- Vinogradov's category of **PDEs** is equivalent to the **Eilenberg-Moore** category of coalgebras over J, $\mathrm{PDE}(\mathcal{F}_M) \simeq \mathrm{EM}(J)$. Morphisms of coalgebras satisfy

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\alpha} & \mathcal{E}_2 \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\rho_2} \\ J\mathcal{E}_1 & \xrightarrow[J\alpha]{} & J\mathcal{E}_2 \end{array} .$$

- Remaining questions:
  - How much can the regularity assumptions be relaxed?
  - Can $\mathcal{E}$ be a variety, orbifold, stratified, ... supermanifold, stack, ..., have boundaries, singularities, ...?

Synthetic Differential Geometry (SDG)

- **SDG** is an *axiomatic/categorical* approach to the study of smooth spaces, operations between them and their generalizations.
- We will work specifically with the **Cahiers Topos H**, introduced by Dubuc (*Cahiers T&GD* 1979).
- **H** has fully faithful embeddings of well-known categories:

$$\mathrm{Mfd} \hookrightarrow \mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd} \hookrightarrow \mathrm{DiflSp} \hookrightarrow \mathbf{H} \hookleftarrow \mathrm{FormalMfd}$$

- Objects in **H** may have algebraic or orbifold singularities, may have boundaries and corners, could be infinite dimensional, and may have infinitesimal directions.
- **Infinitesimal** spaces are particularly well-adapted to the **formal theory** of PDEs.
- Literature:
  - A. Kock: *SDG* (CUP 1981), *SGM* (CUP 2009)
  - R. Lavendhome: *Basic Concepts of SDG* (Springer 1996)
  - U. Schreiber: *dcct* [arXiv:1310.7930]

Generalized Smooth Spaces

- $M \in \mathrm{Mfd}$, $\dim M = n$; $\mathrm{Atlas}(M) \subset C^\infty(\mathbb{R}^n, M)$.
- CartSp — category of **all** $\mathbb{R}^k \to \mathbb{R}^m$ smooth;
  $\mathrm{CartSp}_{\mathrm{diff}}(n)$ — **all diffeomorphisms** onto image $\mathbb{R}^n \to \mathbb{R}^n$.
- Functor $\mathrm{CartSp}_{\mathrm{diff}}(n)^{op} \to \mathrm{Set}$, $\mathbb{R}^n \mapsto \mathrm{Atlas}(M)$, satisfies **gluing**:

(illustration)

- No harm in **extending** $\mathrm{Atlas}(M) \subset C^\infty(-, M) \colon \mathrm{CartSp}^{op} \to \mathrm{Set}$.
- Now $C^\infty(-, M) \in \mathrm{Sh}(\mathrm{CartSp}, \mathrm{Set})$ is a **sheaf** with respect to the "open cover" Grothendieck topology on CartSp.
- **Fully faithful** $\mathrm{Mfd} \hookrightarrow \mathrm{SmthSp}$ (*Generalized Smooth Spaces*):

(Yoneda) $\quad \mathrm{SmthSp} \ni M \leftrightarrow \text{“}C^\infty(-, M)\text{”} \in \mathrm{Sh}(\mathrm{CartSp}, \mathrm{Set})$

Cahiers Topos

Sheaves ("$C^\infty(-, M)$") on test spaces ($\mathbb{R}^k$) are generalized spaces (M).

- $\mathrm{FormalCartSp} := \langle \mathbb{R}^k, \mathbb{D}^k(m), \times \rangle$ — **opposite** to the *full subcategory*

$$\langle C^\infty(\mathbb{R}^k), C^\infty(\mathbb{R}^k)/(x^1, \dots, x^k)^{m+1}, \otimes \rangle \hookrightarrow \mathrm{CAlg}^{\mathbb{R}}$$

  of commutative $\mathbb{R}$-algebras, closed under products.
- **Ex:** $f \in C^\infty(\mathbb{R}^n \times \mathbb{D}^k(m))$ is a *formal power series*

$$f(x^1, \dots, x^n, \varepsilon^1, \dots, \varepsilon^k) = \sum_{|\mathrm{I}| \le m} f_{\mathrm{I}}(x^1, \dots, x^n) \varepsilon^{\mathrm{I}}.$$

- **Cahiers topos — H** $:= \mathrm{Sh}(\mathrm{FormalCartSp}, \mathrm{Set})$:
  - *closed* under all small $\varprojlim(-)$, $\varinjlim(-)$ and internal $\underline{\mathrm{Hom}}(-,-)$;
  - *fully faithful embedding* of many categories of "smooth spaces";
  - *access to infinitesimals* without leaving the category, e.g., formal disks $\mathbb{D}^k(\infty) := \varprojlim_m \mathbb{D}^k(m)$.

Infinitesimals, Formal Disks, Jets

- Take $M \in \mathrm{Mfd} \hookrightarrow \mathbf{H}$, $\dim M = n$ (*independent variables*); take $(E \to M) \in \mathbf{H}_{/M}$ (*dependent variables*, no extra regularity!).
- Every $x \in M$ has **formal disk neighborhood** $T^\infty_x \simeq \mathbb{D}^n(\infty) \to M$.
- Formal neighborhoods **functor** $T^\infty \colon \mathbf{H}_{/M} \to \mathbf{H}_{/M}$, $T^\infty E := T^\infty M \times_M E$.
- In "coordinates" $f \in \mathrm{Hom}_{\mathbf{H}_{/M}}(T^\infty E, F)$ is of the form

$$f(x,u,\varepsilon) = \sum_{|\mathrm{I}|<\infty} f_{\mathrm{I}}(x,u)\varepsilon^{\mathrm{I}} \quad \text{(formal series)}.$$

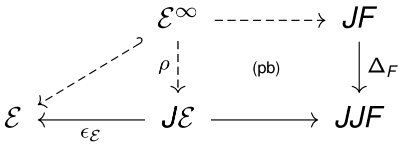
- **Jets** — *right adjoint* of $T^\infty \dashv J^\infty \colon \mathbf{H}_{/M} \to \mathbf{H}_{/M}$ (exists in topos!):

$$\mathrm{Hom}_{\mathbf{H}_{/M}}(T^\infty E, F) \simeq \mathrm{Hom}_{\mathbf{H}_{/M}}(E, J^\infty F) \quad \text{naturally } \forall E, F \in \mathbf{H}_{/M};$$
$$f(x,u,\varepsilon) \dashv f(x,u,-) \simeq \tilde{f}(x,u) = (f_{\mathrm{I}}(x,u))_{|\mathrm{I}|<\infty}.$$

- **Right adjoints** automatically preserve all limits (monos, fibered products, ...). No need for Marvan's "sufficient regularity".
- $J \colon \mathbf{H}_{/M} \to \mathbf{H}_{/M}$ is a **comonad** for abstract reasons, due to the **differential cohesion** (Schreiber 2013) of $\mathbf{H}_{/M}$.

Synthetic Geometry of PDEs

- **Generalized PDE —** $\mathcal{E} \hookrightarrow J^\infty F$ in $\mathbf{H}_{/M}$; *mono* is the only regularity condition needed.
- A *Y-family* of formal sections $\sigma\colon T^\infty Y \to JF$ is **holonomic** if $\sigma(x,u,\varepsilon)(\sim) = \sigma(x,u,\varepsilon+\sim)(0)$.
- A *Y-family* s such that $T^\infty Y \xrightarrow{s} \mathcal{E} \hookrightarrow JF$ is *holonomic* is a Y-family of **formal solutions.**
- If $\mathcal{E}_1 \to \mathcal{E}_2$ *preserves all families of formal solutions*, it is a **morphism** of generalized PDEs ("prolonged differential operator").
- There always exists a **universal family of formal solutions** $T^\infty\mathcal{E}^\infty \to \mathcal{E}$ such that $\mathcal{E}^\infty \overset{\bar\epsilon}{\hookrightarrow} T^\infty\mathcal{E}^\infty \to \mathcal{E}$ is a *mono*:



$\mathcal{E}^\infty \hookrightarrow \mathcal{E} \hookrightarrow JF$ is exactly the *(infinite) prolongation* of $\mathcal{E} \hookrightarrow JF$.

Main Results

- When $\mathcal{E}^\infty \simeq \mathcal{E}$, the PDE is **formally integrable**, has intrinsic presentation $\rho\colon \mathcal{E} \simeq \mathcal{E}^\infty \hookrightarrow J\mathcal{E}$.
- Category $\mathrm{PDE}(\mathbf{H}_{/M})$: **objects** — formally integrable PDEs, **morphisms** — preserve all families of formal solutions. Logically independent from Vinogradov's definition.
- **Thm:** $\mathrm{DiffOp}(\mathbf{H}_{/M}) \simeq \mathrm{Kl}(J)$ and $\mathrm{PDE}(\mathbf{H}_{/M}) \simeq EM(J)$; for each formally integrable PDE, $\rho\colon \mathcal{E} \hookrightarrow J\mathcal{E}$ is a J-coalgebra; a morphism preserving families of formal solutions is a morphism of coalgebras.
  - Because of different definitions/hypotheses, the proof is *logically independent* from (but inspired by) Marvan's.
  - The fully faithful embedding $\mathcal{F}_M \hookrightarrow \mathbf{H}_{/M}$ and Marvan's original equivalence *imply* $\mathrm{PDE}(\mathcal{F}_M) \simeq \mathrm{PDE}_{\text{Vinogradov}}(\mathcal{F}_M)$.
- **Thm:** $\mathrm{PDE}(\mathbf{H}_{/M})$ is also a *topos* (hence has all small limits). More concretely, **all finite limits** in $\mathrm{PDE}(\mathbf{H}_{/M})$ can be **computed in $\mathbf{H}_{/M}$**.
- **Thm:** $\mathrm{Sol}(\mathcal{E}) \simeq \mathrm{Hom}_{\mathrm{PDE}(\mathbf{H}_{/M})}(M, \mathcal{E})$.

Discussion

- Jets and PDEs **internal** to Cahiers Topos **H**:
  - Maximally relaxed (within smooth geometry) *regularity conditions* on spaces of dependent variables and PDEs.
  - *Infinitesimals*, *formal sections* give intrinsic and intuitive notion of a PDE category. No need to appeal to Cartan distribution as proxy for formal solutions. When comparable, coincides with Vinogradov's.
  - All constructions inherently *independent* of (even the *existence* of) choices of local coordinates.
- J-comonad and J-coalgebra structure of PDEs suggests **natural generalization** to more general contexts of Synthetic Differential Geometry (super-, derived-, stack-, ... manifolds).
- **Future:** study symmetries; non-integrable infinitesimal symmetries, could be *truly infinitesimal* in SDG.
- **Future:** study PDEs on derived higher super-stacks...
- **Future:** How to compare with Beilinson-Drinfeld's $\mathcal{D}$-schemes?

Thank you for your attention!