PULL-PUSH CONSTRUCTIONS AND STRING TOPOLOGY OPERATIONS

A.P.M. KUPERS

In this note we will first give a informal discussion of pull-push constructions and their relation for topology quantum field theories. After that, we will show show string topology naturally fits in this framework.

1. Pull-push constructions and quantum field theories

As you may know, in general some quantum field theories are thought to arise from classical field theories by quantisation. For example, quantum mechanics (a zero-dimensional quantum field theory) of a free particle in $\mathbb{R}^3$ comes from classical Newtonian mechanics with zero potential. Another example is quantum electrodynamics, which comes from the classical theory of electromagnetism. Not all quantum field theories arise this way though. An example is spin, which is generally considered to have no classical analogue.

We therefore expect that one can define a mathematical notion of classical field theory and construct a quantum field theory from this. Many attempts to describe such a quantisation procedure exist but it seems to be very difficult to find one which applies in sufficiently general situations and has nice properties.

However, in the case of topological field theories there are some heuristics of the shape that such a construction could take. The idea is that a "classical topological field theory" can be described as a topological field theory with values in the category of correspondences $\mathsf{Corr}(\mathsf{C})$ and quantisation is induced by a functor $\int : \mathsf{Corr}(\mathsf{C}) \to \mathsf{C}$ called a pull-push construction. In this section we will give some heuristics and examples which hopefully shows why this is reasonable. In the next section we show how string topology operations similarly arise from a pull-push construction.

1.1. **Discretized quantum field theory.** Consider a classical particle on a line subject to mechanics coming from a Lagrangian L. Then the full dynamics is encoded by the following set of maps for $t \in \mathbb{R}_{>0}$:

$$\begin{tikzcd} & \mathrm{Map}_{C^\infty}([0,t],\mathbb{R}) \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\ \mathbb{R} \arrow[dr] \arrow[rr, phantom, "\overset{\exp(iS)}{\Rightarrow}"] & & \mathbb{R} \arrow[dl] \\ & BU(1) & \end{tikzcd}$$

where the maps to $BU(1)$ classify the complexified tangent bundle and $\exp(iS)$ is the homotopy induced by the phase

$$\exp(iS)(\gamma) = \exp(i \int_{[0,t]} L(\gamma, \dot{\gamma}) ds)$$

The physical solutions are recovered as those paths which are critical points of the functional $\exp(iS)$. The quantisation of this system is as follows: the spaces of fields are the spaces of section of the hermitian line bundles over $\mathbb{R}$ associated to the maps $\mathbb{R} \to BU(1)$ and the time evolution for time t is given by the path integral:

$$\psi \mapsto \left(x \mapsto \int_{\mathrm{Map}_{C^\infty}([0,t],\mathbb{R})_x} \exp(iS)(\gamma)\psi(\gamma(0))d\gamma \right)$$

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where $\mathrm{Map}_{C^\infty}([0,t],\mathbb{R})_x$ is the subspace of those paths that end at the point x at time t. It is easier to see how this is a pull-push construction when we discretize time into n steps. Then we are dealing with the correspondence

$$\begin{tikzcd} & \mathbb{R}^{n+1} \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$

Consider a function $\psi \in \Omega^0(\mathbb{R},\mathbb{C})$, then the pullback $\partial_{in}^*\psi$ to $\mathbb{R}^{n+1}$ is the function $(x_0,\ldots,x_n) \mapsto \psi(x_0)$. We multiply this with the action integral kernel

$$\exp(iS) = \exp\left(i\sum_{i=1}^n L(x_i, \frac{x_i - x_{i-1}}{t/n})\right) dx_0 \wedge \ldots \wedge dx_{n-1}$$

to get a n-form $\exp(iS)\partial_{in}^*\psi$ on $\mathbb{R}^{n+1}$. To pushforward to $\mathbb{R}$ along ∂_{out} we integrate over the fibers $\partial_{out}^{-1}(\{x_n\}) \cong \mathbb{R}^n$.

$$\partial_{out}^!(\exp(iS)\partial_{in}^*\psi) = \left(x_n \mapsto \int_{\mathbb{R}^n} \exp\left(i\sum_{i=1}^n L(x_i, \frac{x_i - x_{i-1}}{t/n})\right)\psi(x_0)dx_0 \wedge \ldots \wedge dx_{n-1}\right)$$

In the previous non-discrete context, we see the same pull-push construction appear. The path integral essentially is given by pulling back a field along ∂_{in} and pushing it forward along ∂_{out}; in other words, the function ψ on $\mathbb{R}$ is made into a function on $\mathrm{Map}_{C^\infty}([0,t],\mathbb{R})$, then multiplied with the "integral kernel" $\exp(iS)(\gamma)d\gamma$ and finally integrated over the fibers of ∂_{out}.

1.2. **Other examples of pull-push construction leading to TQFT's.** There are many other TQFT's which arise essentially in this way. In this section we give some examples.

Sigma-models: A sigma-model is a theory which is similar to the simple quantum field theory we used to motivate the idea that quantum field theories can arise from pull-push construction applied to classical field theory.

In general setup of a sigma-model, one starts with a classical d-dimensional field theory corresponding to a target space X with background n-circle field α and connection. This is a functor $\mathsf{Bord}_d^R \to \mathsf{Corr}(\mathsf{C})$ (where R denotes some type of structure on the cobordisms) of a special type in which the correspondences should look like

$$\begin{tikzcd} & \mathrm{Map}(\Sigma, X) \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\ \mathrm{Map}(\partial_{in}\Sigma, X) \arrow[dr] \arrow[rr, phantom, "\overset{\exp(iS)}{\Rightarrow}"] & & \mathrm{Map}(\partial_{out}\Sigma, X) \arrow[dl] \\ & \mathsf{C} & \end{tikzcd}$$

where the mapping spaces are considered as ∞-groupoids and C is thought to classify associated bundles to α. The maps to C should come from transgression of α and S from parallel transport.

The pull-push $\int : \mathsf{Corr}(\mathsf{C}) \to \mathsf{C}$ along the incoming and outgoing boundary with integral kernel $\exp(iS)$ – if it can be defined, which depends on C – then should give a functor $\mathsf{Bord}_d^R \to \mathsf{C}$ which is our quantized sigma-model.

The following quantum field theories are examples of a sigma-model: Chern-Simons theory, Dijkgraaf-Witten theory, the WZW-model, Rozansky-Witten theory.

Gromov-Witten theory: Another nice example of a pull-push construction giving a TQFT is Gromov-Witten theory. Given a smooth projective complex variety X and a class $a \in H_2(X)$, which should be thought as a background field, the goal here is to construct maps

$$H_*(\mathcal{M}_{g,I};\mathbb{C}) \otimes H_*(X;\mathbb{C})^{\otimes n} \to H_*(X;\mathbb{C})^{\otimes m}$$

where $\mathcal{M}_{g,I}$ is the moduli stack of stable surfaces of genus g with graph I and $n+m$ labelled marked points.

Using Poincaré duality for X and $\mathcal{M}_{g,I}$, we can equivalently construct a map $H^*(X;\mathbb{C})^{\otimes(n+m)} \to H^*(\mathcal{M}_{g,I};\mathbb{C})$. To do this, we will use the moduli stack $\mathcal{M}_{g,I,\alpha}(X)$ of maps $\Sigma \to X$ from genus g stable surfaces Σ with graph I and $n+m$ labelled marked points, which map the fundamental class $[\Sigma]$ to $\alpha \in H_2(X)$. The idea is that there are maps $\mathrm{ev} : \mathcal{M}_{g,I,\alpha}(X) \to X^{m+n}$ and $F : \mathcal{M}_{g,I,\alpha}(X) \to \mathcal{M}(g,I)$. Then we can obtain a map $\phi : H^*(X;\mathbb{C})^{\otimes(n+m)} \to H^*(\mathcal{M}_{g,I};\mathbb{C})$ as follows:

$$\phi(\omega_1,\ldots,\omega_{n+m}) = F^!(\mathrm{ev}^*(\omega_1,\ldots,\omega_{n+m}))$$

$F^!$ is integration with respect to a virtual fundamental class of the fiber. This operation is therefore obtained by a pull-push construction along the correspondence

$$\begin{tikzcd} & \mathcal{M}_{g,I,\alpha}(X) \arrow[dl] \arrow[dr] & \\ X^{m+n} & & \mathcal{M}_{g,I} \end{tikzcd}$$

2. Degree zero string operations

In this section we describe string topology operations as a pull-push construction. In this context, string topology seems to be the quantized sigma-model corresponding to a topological string with trivial background field. The trivial background says that the action integral kernel $\exp(iS)$ is trivial, so we forget about it in our correspondences.

The idea is as follows: we first fix a commutative ring spectrum E and a compact E-oriented d-dimensional manifold M. A manifold is called E-oriented if the fiberwise smash product $M^{TM} \wedge_M E$ of the fiberwise Thom spectrum M^{TM} of the tangent space with E is weakly equivalent as a parametrized spectrum to a trivial parametrized spectrum $\Sigma^\infty_M M \wedge_M \Sigma^d E$. Some examples:

(1) If $E = H\mathbb{Z}$ this notion coincides with the classical notion of orientation. If $E = H\mathbb{Z}_2$, then every manifold is E-oriented, as in the classical case.
(2) If $E = KO$, then a choice of orientation is a choice of spin structure. If $E = TMF$, then an orientation is a string structure.

Then we want to start with the 2-dimensional TQFT Z_M of correspondences with values in Top given as follows: on objects it is given by:

$$Z_M\left(\coprod_n S^1\right) = \mathrm{Map}(S^1, M)^n$$

On a 2-dimensional cobordism Σ with n incoming boundary circles and m outgoing boundary circles, it is given by the following naive correspondence $Z_M(\Sigma)$ of mapping spaces

$$\begin{tikzcd} & \mathrm{Map}(\Sigma, M) \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\ \mathrm{Map}(S^1, M)^n & & \mathrm{Map}(S^1, M)^m \end{tikzcd}$$

Our hope is produce a TQFT with values in the homotopy category of spectra $\mathsf{HSpectra}$ by smashing with a commutative ring spectrum E. This requires a pull-push construction for E-module spectra.

$$\begin{tikzcd} & \mathrm{Map}(\Sigma, M) \wedge E \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\ \mathrm{Map}(S^1, M)^n \wedge E & & \mathrm{Map}(S^1, M)^m \wedge E \end{tikzcd}$$

In the next section we describe a method which produces umkehr maps for two classes of maps between mapping spaces of compact oriented manifolds. After that we will describe how to modify the above correspondence in such a way that wrong-way maps fall into one of these classes.

2.1. **Umkehr functors for mapping spaces.** In this section we describe an advanced version of the Pontryagin-Thom collapse construction which allows one to construct umkehr maps for some maps between mapping spaces. We will start by describing the classical Pontryagin-Thom collapse construction.

Let $i : N \hookrightarrow M$ be an embedding of E-oriented compact manifolds. Then N has a normal bundle $\nu = TM|_N/TN$ in M. Because both TM and TN are E-oriented, this normal bundle will be as well. The tubular neighborhood theorem tells us that there exists a map $f : \nu \to M$ which is a homeomorphism onto its image. One should think of it as a thickening of i in the normal directions. Then we can construct a collapse map $\bar{f}$ from M to the one-point compactification $\mathrm{Thom}(\nu)$ of ν as follows:

$$\bar{f} : M \to M/(M\backslash f(\nu)) \cong \mathrm{Thom}(\nu)$$

Since ν is a E-oriented, the Thom isomorphism for E-homology tells us that $\mathrm{Thom}(\nu) \wedge E \cong N \wedge \Sigma^d E$, where d is the dimension of ν or equivalently the codimension of i. Then the umkehr map $i^! : M \wedge E \to N \wedge \Sigma^d E$ is the composite

$$M \wedge E \xrightarrow{\bar{f}\wedge id_E} \mathrm{Thom}(\nu) \wedge E \cong N \wedge \Sigma^d E$$

We now want to repeat this construction with mapping spaces $\mathrm{Map}(X, M)$ in the place of the compact manifolds. The only thing we need to prove is a tubular neighborhood theorem for such spaces, because nothing else of the discussion used the fact that we working with manifolds.

It turns out that one can construct a tubular neighborhood for an embedding of mapping spaces $M^g : \mathrm{Map}(Y, M) \to \mathrm{Map}(X, M)$ of "finite codimension". With finite codimension we mean that we want $\mathrm{Map}(Y, M)$ to be obtained by imposing a finite number of restrictions to values of maps in $\mathrm{Map}(X, M)$. This is best made precise as follows: a semisimplicial complex is a space obtained by glueing a finite number of simplices along their faces. Then $g : X \to Y$ should be a map between semisimplicial complexes which identifies a finite number of 0-simplices but no higher-dimensional simplices. If X_0 and Y_0 denote the subspaces of 0-simplices, then we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Map}(Y, M) \arrow[r, "M^g"] \arrow[d, "\mathrm{ev}_Y"'] & \mathrm{Map}(X, M) \arrow[d, "\mathrm{ev}_X"] \\
\mathrm{Map}(Y_0, M) \arrow[r, "M^{g_0}"'] & \mathrm{Map}(X_0, M)
\end{tikzcd}$$

where the lower map is now a embedding of E-oriented compact manifolds, because X_0 consists of a finite number of points and Y_0 is obtained from X_0 by identifying some of the 0-simplices. The technique of propagating flows then allows one to lift a tubular neighborhood for M^{g_0} downstairs to an essentially unique one upstairs.

Proposition 2.1. *Let ν denote the normal bundle for M^{g_0} and $f_0 : \nu \to \mathrm{Map}(X_0, M)$ be a tubular neighborhood. Then there is a map*

$$f : \mathrm{ev}_Y^* \nu \to \mathrm{ev}_X^{-1}(f_0(\nu))$$

which is a homeomorphism onto its image. It depends on a contractible space of choices.

The idea is roughly as follows: the tubular neighborhoods downstairs gives one the information how the 0-simplices of Y can be moved in normal directions. The natural parametrisations of the higher-dimensional simplices in a semisimplicial space allow one to smoothly extend to this to the entire space Y. The conclusion is that we can construct umkehr maps for mapping spaces where we glue points together:

$$(M^g)^! : \mathrm{Map}(X, M) \wedge E \to \mathrm{Map}(Y, M) \wedge \Sigma^{rd} E$$

where r is the number $\#X_0 - \#Y_0$.

There is a second thing we can do is create new points: the idea is that if $Y = X \sqcup *$, then $M^g : \mathrm{Map}(Y, M) \to \mathrm{Map}(X, M)$ may not be embedding, but if W is an Euclidean space with an embedding $M \hookrightarrow W$ then

$$\mathrm{Map}(Y, M) \cong \mathrm{Map}(X, M) \times M \to \mathrm{Map}(X, M) \times W \simeq \mathrm{Map}(X, M)$$

is one, with codomain homotopy equivalent to the original codomain. We can their use the identity on the $\mathrm{Map}(X, M)$-component and the classical Pontryagin-Thom collapse for the other component to produce an umkehr map

$$(M^g)^! : \mathrm{Map}(X, M) \wedge E \to \mathrm{Map}(Y, M) \wedge \Sigma^{-d} E$$

We can summarize this as follows: we can define umkehr maps for maps between mapping spaces if these maps are induced by maps of the domain of one of the following two types:

(1) A map which identifies a finite number of points.
(2) A map which creates a finite number of isolated points.

2.2. **Correspondences of graphs.** Our goal will be to modify our original naive correspondence to one which has the following properties:

(1) All domains of the mapping spaces are semisimplicial spaces.
(2) All wrong-way maps between the mapping spaces are induced by maps of the domains for which an umkehr map exists, i.e. they must be of one of the two types describe above.

To solve the first problem we use the following well-known statement, which says that we can use graphs as models for cobordisms.

Proposition 2.2. *Every surface Σ such that each connected component has non-empty boundary deformation retracts onto an embedded graph Γ.*

Proof. A theorem of Strebel says that for each a surface there is a unique quadratic differential such that its horizontal trajectories are either circles of length 1, fixed points or non-closed trajectories. The union of the fixed points and the non-closed trajectories is the embedded graph of the statement of the proposition. □

We also want to know how the incoming and outgoing boundary of our cobordism are deformed by this retraction. In particular, the proposition can be improved to the following one for cobordisms.

Proposition 2.3. *Every cobordism Σ such that each connected component has non-empty outgoing boundary deformation retracts onto an embedded graph Γ such that the retraction is injective on incoming boundary.*

The fact that the incoming boundary injects into the graph will be necessary when we factor our correspondence to have wrong-way arrows for which the umkehr maps exist. The previous proposition implies that we have a diagram as follows:

$$\begin{tikzcd}
\partial_{in}\Sigma \arrow[r, dotted] \arrow[d, "\simeq"] & \Sigma \arrow[d, "\simeq"] & \partial_{out}\Sigma \arrow[l, dotted] \arrow[d, "\simeq"] \\
\partial_{in}\Gamma \arrow[r, dotted] & \Gamma & \partial_{out}\Gamma \arrow[l]
\end{tikzcd}$$

where the dotted arrows are embeddings. This means that if we are going to the homotopy category anyway, we might as well replace our correspondence with the following one:

$$\begin{tikzcd}
 & \mathrm{Map}(\Gamma, M) \wedge E \arrow[dl, "\partial_{in}"'] \arrow[dr, "\partial_{out}"] & \\
\mathrm{Map}(\partial_{in}\Gamma, M) \wedge E & & \mathrm{Map}(\partial_{out}\Gamma, M) \wedge E
\end{tikzcd}$$

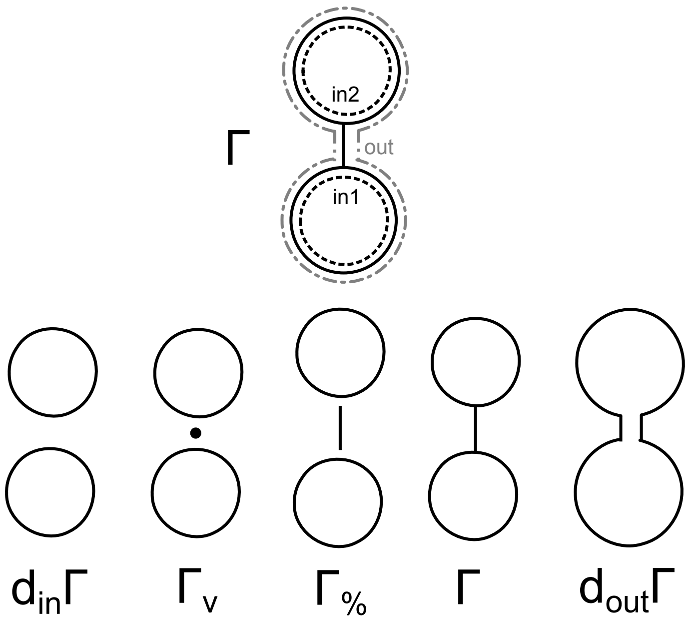

FIGURE 1. A graph Γ and the different graphs in the associated correspondence.

The problem is that the map ∂_{in} pointing in the wrong direction might not be of one of the two types for which we can construct an umkehr map. However, suppose that we could. Any good construction of umkehr maps must be natural. This means that if a correspondence

$$X \wedge E \leftarrow Y \wedge E \rightarrow Z \wedge E$$

could be factored as a larger correspondence

$$X \wedge E \leftarrow Y_1 \wedge E \rightarrow Y_2 \wedge E \leftarrow Y_3 \wedge E \rightarrow Z \wedge E$$

then the pull-push construction should give the same up to homotopy. We will therefore try to factor our naive correspondence into simpler ones. This will be done by breaking up the graphs into smaller pieces in a natural way.

First note that the fact that $\partial_{in}\Gamma$ embeds in Γ means that we can write Γ as the union of $\partial_{in}\Gamma$ and its complement $\Gamma \backslash \partial_{in}\Gamma$. We essentially have to create this complement and glue it to $\partial_{in}\Gamma$. However, as we can only create points, we have a problem when the complement $\Gamma \backslash \partial_{in}\Gamma$ is not contractible. To solve this, we break the complement apart into its constituent edges.

Definition 2.4. For a graph Γ with embedded incoming part $\partial_{in}\Gamma$ we define following two graphs.

(1) The graph Γ_v is given by disjoint union of $\partial_{in}\Gamma$ and a single vertex for each edge of $\Gamma \backslash \partial_{in}\Gamma$.
(2) The graph $\Gamma_\div$ is given by disjoint union of $\partial_{in}\Gamma$ and a disconnected edge for each edge of $\Gamma \backslash \partial_{in}\Gamma$.

There are several maps between these graphs and Γ, $\partial_{in}\Gamma$ and $\partial_{out}\Gamma$. There is a map $i_{in} : \partial_{in}\Gamma \to \Gamma_v$ which includes in the incoming boundary into Γ_v. Next there is a map $s_v : \Gamma_\div \to \Gamma_{in}$ which collapses the disconnected edges to points. There is a map $s_\div : \Gamma_\div \to \Gamma$ which attaches the disconnected edges and finally there is a map $i_{out} : \partial_{out}\Gamma \to \Gamma$ which includes the outgoing boundary.

These fits together into a diagram of graphs

$$\partial_{in}\Gamma \xrightarrow{i_{in}} \Gamma_v \xleftarrow{s_v} \Gamma_\div \xrightarrow{s_\div} \Gamma \xleftarrow{i_{out}} \partial_{out}\Gamma$$

Mapping this into M gives us a correspondence which refines our naive one, because we broke up the wrong-way map in the naive correspondence (displayed dotted in the diagram) into three maps:

$$\begin{tikzcd}
 & \mathrm{Map}(\Gamma_v, M) \arrow[dl, "M^{i_{in}}"', "(2)"] \arrow[dr, "M^{s_v}", "\simeq"'] & & \mathrm{Map}(\Gamma, M) \arrow[dl, "M^{s_\div}"', "(1)"] \arrow[dr, "M^{i_{out}}"] \arrow[dlll, dotted, bend right] & \\
\mathrm{Map}(\partial_{in}\Gamma, M) & & \mathrm{Map}(\Gamma_\div, M) & & \mathrm{Map}(\partial_{out}\Gamma, M)
\end{tikzcd}$$

Both of the maps pointing in the wrong way are of a type for which we can construct the umkehr maps after smashing with E: the first is induced by creating a finite number of points, hence is of type (2), while the second is induced by identifying a finite number of points, hence is of type (1).

The result is a well-defined map in the homotopy category of spectra

$$(M^{i_{out}})_* \circ (M^{s_\div})^! \circ (M^{s_v})_* \circ (M^{i_{in}})^! : \mathrm{Map}(\partial_{in}\Gamma, M) \wedge E \to \mathrm{Map}(\partial_{out}\Gamma, M) \wedge \Sigma^h E$$

One may wonder what the integer h is. Recall that $d = \dim M$. Each vertex corresponding to an edge that i_{in} creates contributes $-d$ and each point, corresponding to a vertex in Γ where we identifying $\mathrm{val}(v)$ points contributes $\mathrm{val}(v)d$. We can think obtain these number in a different way. We can think of each edge of the complement contributing d, each half-edge of the complement $-d$ and each vertex of the complement $\mathrm{val}(v)d$. Then the contributions of half-edges and vertices cancel if the vertex is contained in the incoming boundary, but they contribute $-d$ if they are not contained in the incoming boundary. From this we see that $h = -d(\dim H_0(\Sigma, \partial_{in}\Sigma) - \dim H_1(\Sigma, \partial_{in}\Sigma))$, in other words $-d$ times the relative Euler characteristic of Σ relative to its incoming boundary.

Now the hard work starts: (1) prove that this is independent of the choice of Γ, (2) prove that this is compatible with disjoint union and composition. The previous arguments about naturality say that morally this should be true:

(1) The construction shouldn't depend on the choice of Γ, since any Γ gives a refinement of a correspondence homotopy equivalent to the naive correspondence and by naturality should be equal to.
(2) The construction should be compatible with composition because there is a covering correspondence

$$\begin{tikzcd}
 & & \mathrm{Map}(\Sigma_2 \circ \Sigma_1, M) \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Map}(\Sigma_1, M) \arrow[dl] \arrow[dr] & & \mathrm{Map}(\Sigma_2, M) \arrow[dl] \arrow[dr] & \\
\mathrm{Map}(\partial_{in}\Sigma_1, M) & & \begin{array}{c}\mathrm{Map}(\partial_{out}\Sigma_1, M) = \\ \mathrm{Map}(\partial_{in}\Sigma_2, M)\end{array} & & \mathrm{Map}(\partial_{out}\Sigma_2, M)
\end{tikzcd}$$

However, because currently we do not have the tools to construct umkehr maps for the correspondences used in these arguments, there really is something to prove.