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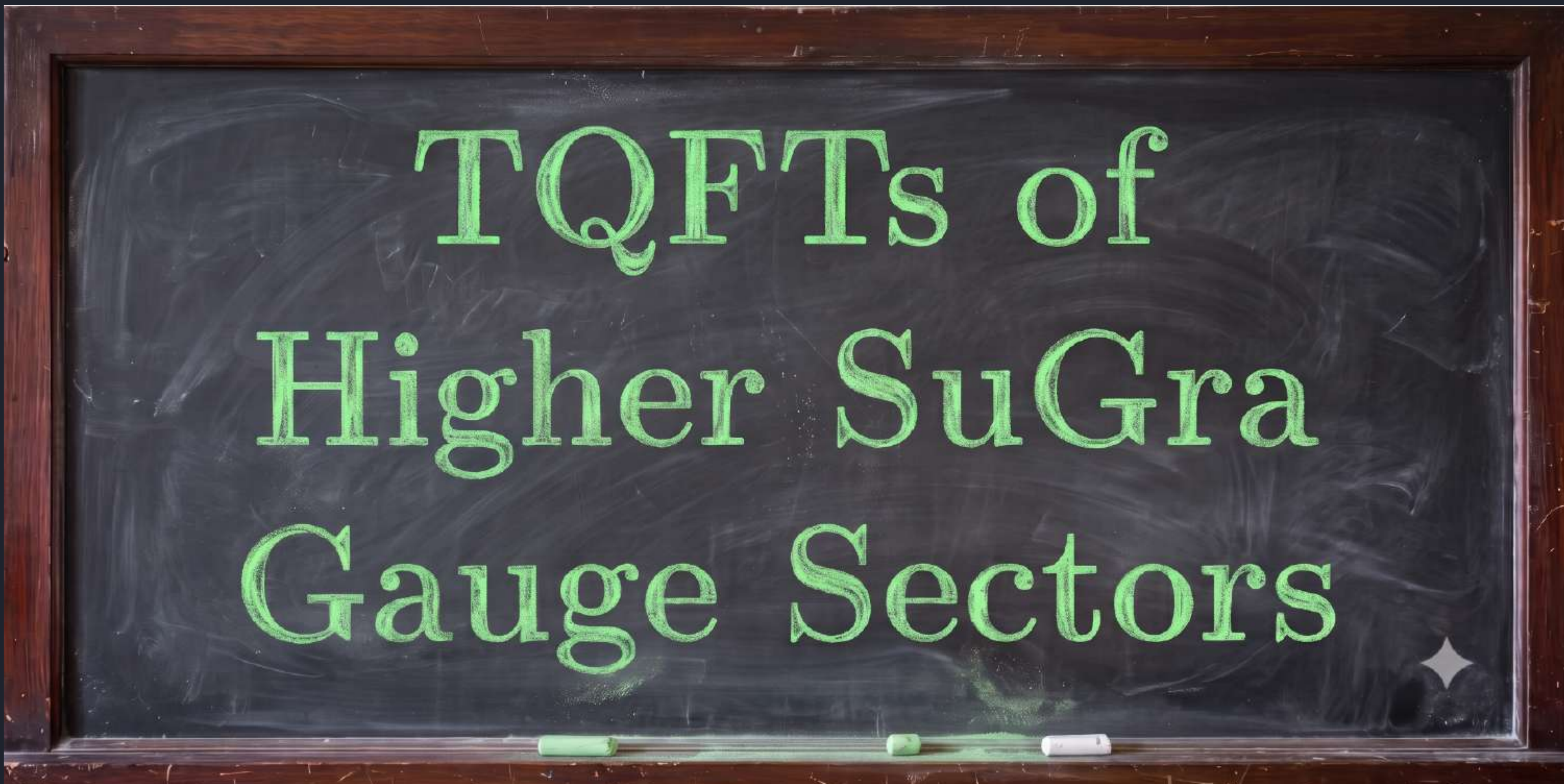
CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS



based in parts on joint work with:

D. Fiorenza, G. Giotopoulos, H. Sati

talk at *Elliptic objects, von Neumann algebras, and Functorial Field Theory*, Bernoulli Center Lausanne (16 July 2026)

Reporting on:

– *The Character Map In Nonabelian Cohomology* (esp. §9)

– *Complete Topological Quantization of Higher Gauge Fields*

– *Higher Gauge Theory via Differential Nonabelian Cohomology*

– *Higher Superspace Supergravity and its IR-Completions*

IR-Completing Higher Gauge Fields

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Quantum Fields traditionally defined incrementally:

IR-Completing Higher Gauge Fields

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UV-completion: local interactions

$\vdots$

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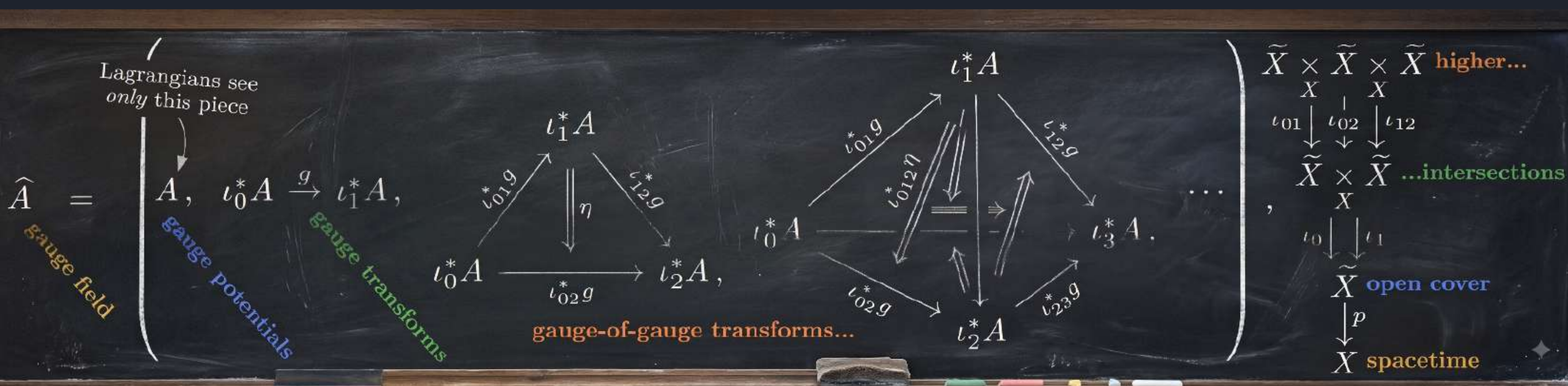

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

IR-completion: global field topology $\Rightarrow$ TQFT sector

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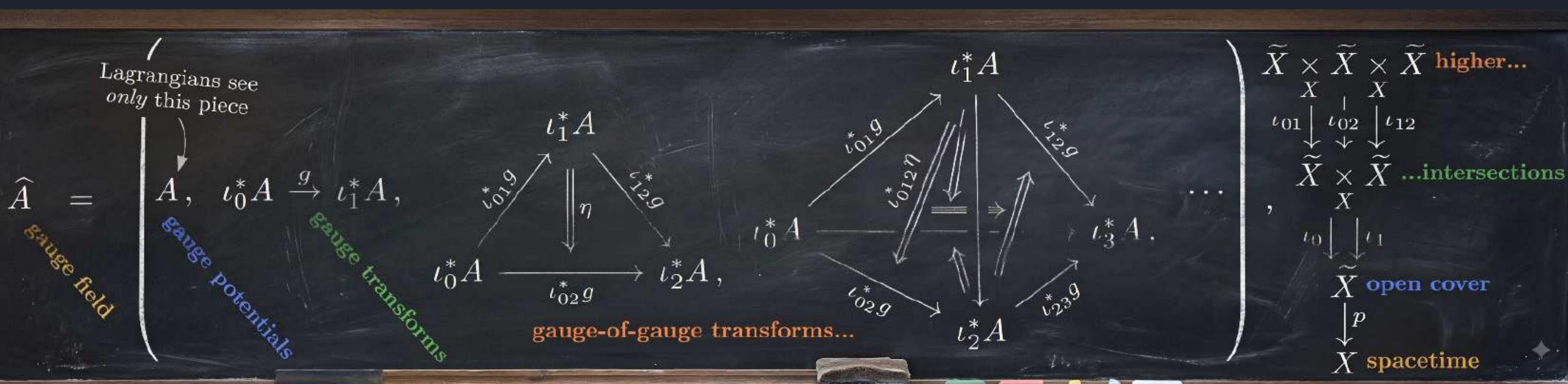

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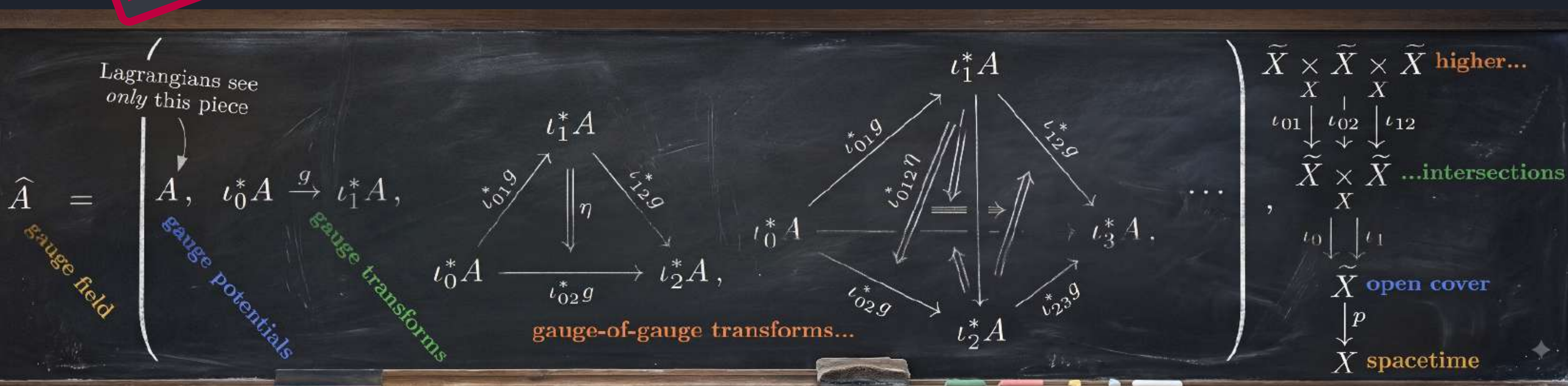
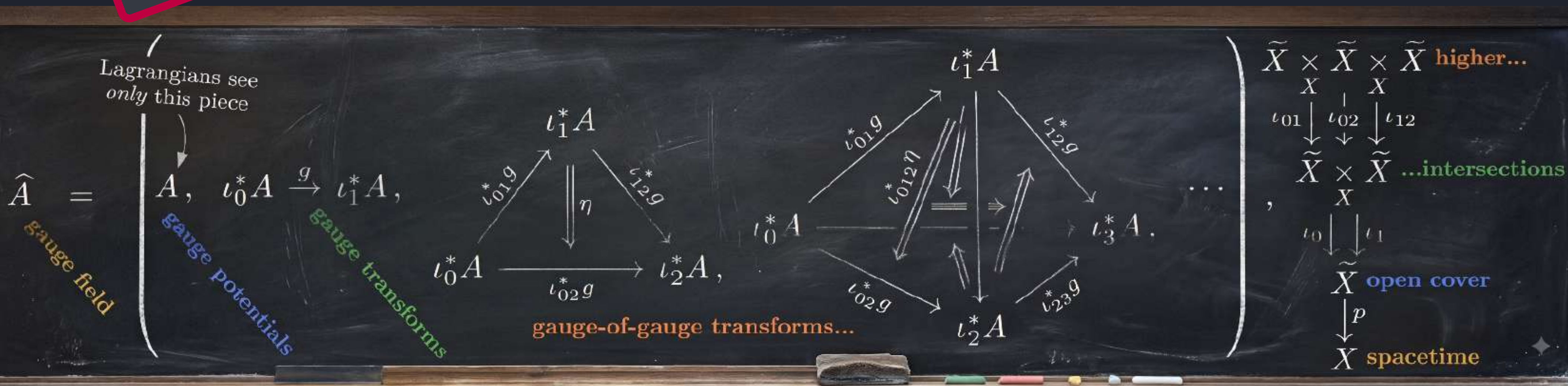
non-perturbative



Recall: IR-Completion of Maxwell

non-perturbative



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for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

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Cauchy hypersurface

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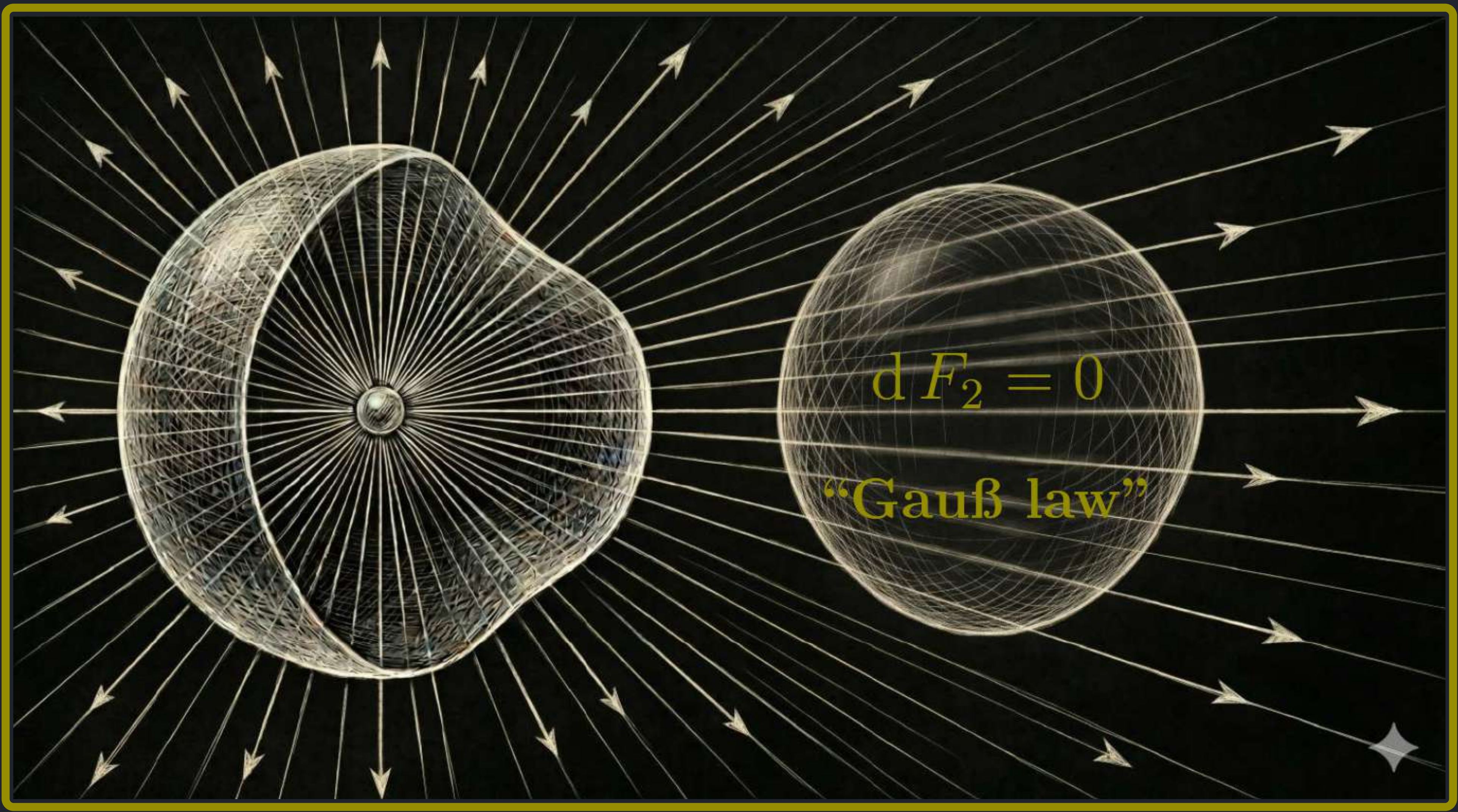

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Differential Cohomology

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Differential Cohomology

ordinary

$$H^n_{\mathrm{dff}}(X;\mathbb{Z}) \;\simeq\; H^0\big(X;\underbrace{\mathbb{Z}[n]_{\mathrm{dff}}}_{\Omega^n_{\mathrm{cl}} \underset{\mathbb{R}[n]}{\times} \mathbb{Z}[n]}\big)$$

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IR-Completion of Higher Maxwell-Type

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Higher Gauge Theory
of Maxwell-type:

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IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

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4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,G_2 = 0$	$\mathfrak{l} B^2 \mathbb{Z}^2$

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classifying space of twisted K-theory

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so: for IR-completion choose space $\mathcal{A}$ s.t. $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

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then fluxes are modulated by:

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rational homotopy theory gives:

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flux quantization:
fluxes
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rational
image of
charges χ
in
$\mathcal{A}$-cohomology

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IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

$$\left\{ \begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$$

flux quantization:
fluxes
are
rational
image of
charges χ
in
$\mathcal{A}$-cohomology

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-}\atop\text{ization}"] \\
X^d \arrow[urr, bend left, "\chi", "\text{charges}"'] \arrow[r, "\vec{F}", "\text{fluxes}"'] & \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian}\atop\text{de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

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up to homotopy

$$
\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[urr, bend left, "\text{charges}", "\chi"'] \arrow[r, bend right, "\text{fluxes}"', "\vec{F}"] \arrow[ur, phantom, "\chi = \widehat{A} \Downarrow"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}\mathcal{A}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
$$

$\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

IR-Completion of Higher Maxwell-Type

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up to homotopy this exhibits the gauge potentials

$$
\begin{tikzcd}
& & \mathcal{A} \arrow[dd, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[urr, bend left, "\text{charges}"] \arrow[drr, phantom, "\chi \overset{\widehat{A}}{\Longrightarrow} \vec{F}\ \text{potentials}"] \arrow[dr, bend right, "\text{fluxes}"'] & & \\
& \Omega^1_{\mathrm{cl}}\big(\cdot;\, \underset{\overset{\wr}{\mathfrak{a}}}{\mathfrak{l}\mathcal{A}}\big) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: $\left\{ \begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$

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$$\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[urr, bend left, "\text{charges}"] \arrow[urr, phantom, "\chi = \widehat{A} \,\Downarrow\, \text{potentials}"'] \arrow[r, "\text{fluxes}"', "\vec{F}"] & \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\substack{\text{non-abelian} \\ \text{de Rham map}}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

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$$\left\{ \begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$$

by the universal property of the homotopy pullback

$$\begin{tikzcd}
 & \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[urr, bend left, "\text{charges}"] \arrow[r, "\text{fluxes}"'] \arrow[ur, phantom, "\chi = \widehat{A} \Downarrow_{\text{potentials}} \vec{F}"] & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

by the universal property of the homotopy pullback

$$\begin{tikzcd}
X^d \arrow[rr, bend left, "\text{charges}"] \arrow[dr, bend right, "\text{fluxes}"'] \arrow[r, phantom, "\chi = \widehat{A} \overset{\text{potentials}}{\Downarrow} \vec{F}"] & \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
& \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
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the IR-completed field configurations are cocycles in differential $\mathcal{A}$-cohomology

$$\begin{tikzcd}
 & & \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-}\\ \text{ization}"] \\
X^d \arrow[urr, dashed, "\chi = \widehat{A} \Downarrow \vec{F}", "\text{potentials}"'] \arrow[urrr, bend left, "\text{charges}"] \arrow[drr, bend right, "\text{fluxes}"'] & & & \\
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\end{tikzcd}$$

$\mathfrak{l}\mathcal{A} \cong \mathfrak{a}$

IR-Completion of Higher Maxwell-Type

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the IR-completed field configurations are cocycles in differential $\mathcal{A}$-cohomology

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X^d \arrow[ur, dashed, "{\chi = \widehat{A}}", "\text{potentials}"'] \arrow[dr, "\text{fluxes}"'] \arrow[urrr, bend left, "\text{charges}"] \arrow[r, phantom, "\Downarrow \vec{F}"] & & & \\
 & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[rr, "\text{non-abelian de Rham map}"'] & & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}}$$

IR-Completed Phase Space

IR-Completed Phase Space

phase space: $\quad \mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles
$/\!\!/$gauge symmetries

IR-Completed Phase Space

phase space: $\quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\quad \int \mathrm{Phs}\big(X^d; \mathcal{A}\big)$

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d;\mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d;\mathcal{A}\big)$

path ∞-groupoid,
topological realization,
left adjoint to $\mathrm{Grpd}_\infty \hookrightarrow \mathrm{SmthGrpd}_\infty$

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

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its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$

by cohesion

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \equiv \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \simeq \mathrm{Map}(X^d, \mathcal{A} \quad)$ black brane charges $/\!\!/$global symmetries

by cohesion

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phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

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its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A} \quad\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big)$

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \simeq \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int \mathrm{Phs}_{\mathrm{sltnc}}\big(X^d; \mathcal{A}\big)$

compact support
for fluxes & charges

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles //gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$ black brane charges //global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$

compact support for fluxes & charges

by cohesion

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A} \;\;)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*(X^d_{\cup\{\infty\}}, \mathcal{A})$

compact support for fluxes & charges

by cohesion

one-point compactification

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ — differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$ — black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^\ast\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ — all brane charges $/\!\!/$global symmetries

compact support for fluxes & charges

by cohesion

one-point compactification

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\quad\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ all brane charges $/\!\!/$global symmetries

both singular branes like monopoles & solitonic branes like Abrikosov vortices

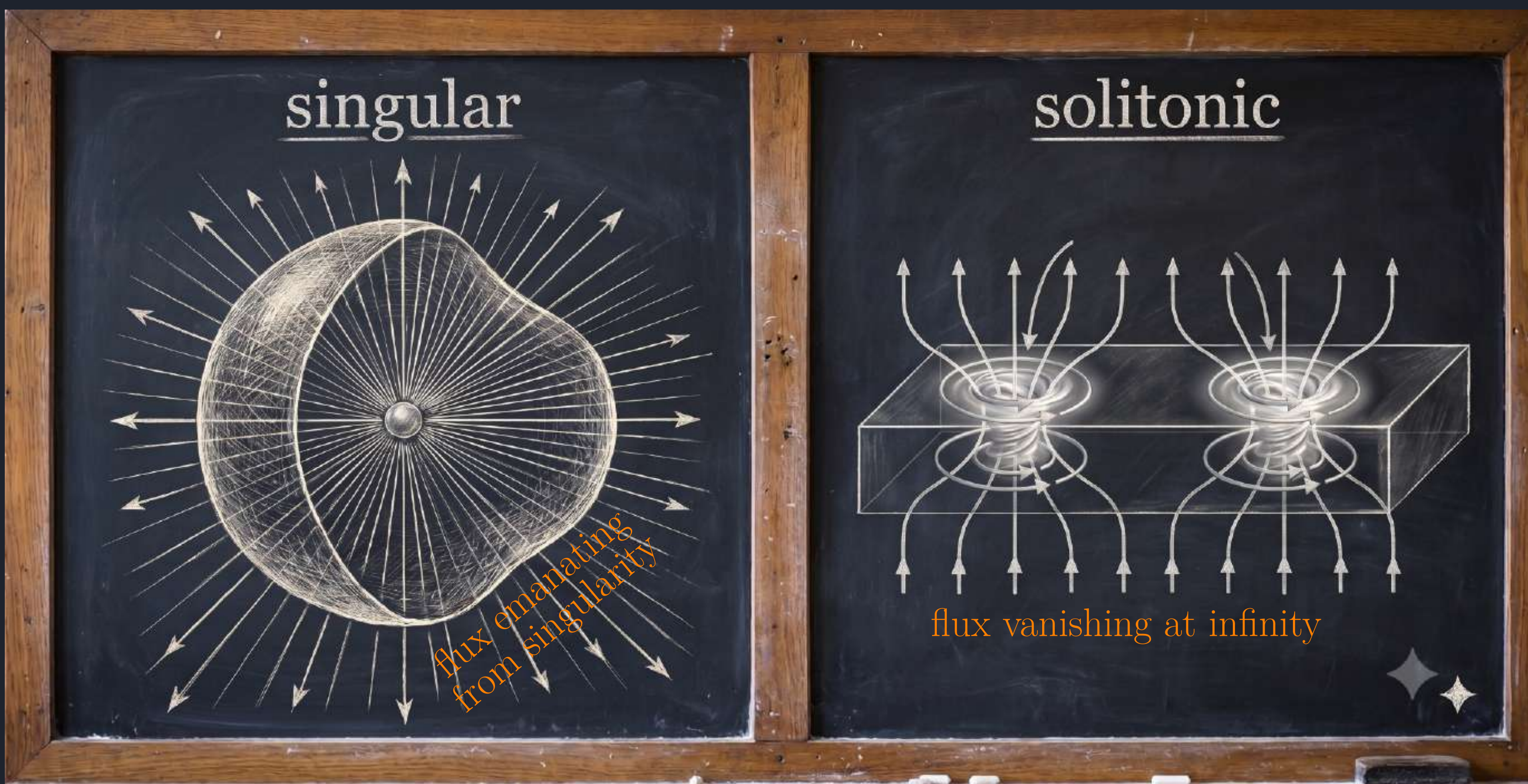
IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ — differential cocycles // gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A}\;\;)$ — black brane charges // global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ — all brane charges // global symmetries

both singular branes like monopoles & solitonic branes like Abrikosov vortices



IR-Completed Phase Space

phase space:	$\mathrm{Phs}\big(X^d; \mathcal{A}\big)$	$\equiv$	$\mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int \mathrm{Phs}\big(X^d; \mathcal{A}\big)$	$\simeq$	$\mathrm{Map}\big(X^d, \mathcal{A}\quad\big)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\int\limits_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big)$	$\simeq$	$\mathrm{Map}^*\Big(X^d_{\cup\{\infty\}}, \mathcal{A}\Big)$	all brane charges $/\!\!/$global symmetries

IR-Completed Phase Space

phase space:	$\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$		differential cocycles $/\!\!/$gauge symmetries
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Observables on IR-Complete Fields

IR-Completed Phase Space

phase space:	$\mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d;\mathcal{A}_{\mathrm{dff}}\big)$		differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int\mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d,\mathcal{A}\ \ \big)$		black brane charges $/\!\!/$global symmetries
solitonic:	$\int_{\mathrm{sltnc}}\mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\Big(X^d_{\cup\{\infty\}},\mathcal{A}\Big)$		all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables: $\left\{ \text{general:} \quad \mathrm{Phs}\big(X^d;\mathcal{A}\big) \xdashrightarrow{\ \mathcal{O}\ } \mathbb{C} \right.$

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \equiv \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles //gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \simeq \mathrm{Map}(X^d, \mathcal{A}\ \ )$ black brane charges //global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d; \mathcal{A}) \simeq \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ all brane charges //global symmetries

Observables on IR-Complete Fields

observables:
- general:
- topological:

$$
\begin{tikzcd}
\mathrm{Phs}(X^d; \mathcal{A}) \arrow[rr, dashed, "\mathcal{O}"] \arrow[dr, "\eta^{\int}"'] & & \mathbb{C} \\
& \int \mathrm{Phs}(X^d; \mathcal{A}) \arrow[ur, dashed, "\mathcal{O}_{\mathrm{top}}"'] &
\end{tikzcd}
$$

IR-Completed Phase Space

phase space:	$\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$	all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables:

- general: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \overset{\mathcal{O}}{\dashrightarrow} \mathbb{C}$
- topological: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \xrightarrow{\eta^{\int}} \int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \overset{\mathcal{O}_{\mathrm{top}}}{\dashrightarrow} \mathbb{C}$
- realistic: compactly supported

IR-Completed Phase Space

phase space:	$\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A}\;\;)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\int \underset{\mathrm{sltnc}}{\mathrm{Phs}}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$	all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables: $\left\{\begin{array}{rl} \text{general:} & \mathrm{Phs}(X^d; \mathcal{A}) \xrightarrow{\;\;\mathcal{O}\;\;} \mathbb{C} \\ & \quad \searrow^{\eta^{\int}} \qquad\qquad \nearrow_{\mathcal{O}_{\mathrm{top}}} \\ \text{topological:} & \qquad\qquad \int \mathrm{Phs}(X^d; \mathcal{A}) \\ \text{realistic:} & \text{compactly supported} \end{array}\right.$

$\Rightarrow$ *realistic topological solitonic observables* are
homology of shape of solitonic phase space:

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d;\mathcal{A}) \;\equiv\; \mathrm{Map}(X^d;\mathcal{A}_{\mathrm{dff}})$ — differential cocycles //gauge symmetries

its shape: $\int\mathrm{Phs}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}(X^d,\mathcal{A}\ \ )$ — black brane charges //global symmetries

solitonic: $\int_{\mathrm{sltnc}}\mathrm{Phs}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}},\mathcal{A}\big)$ — all brane charges //global symmetries

Observables on IR-Complete Fields

observables:

- general: $\mathrm{Phs}(X^d;\mathcal{A}) \xrightarrow{\ \mathcal{O}\ } \mathbb{C}$
- topological:

$$
\begin{tikzcd}
\mathrm{Phs}(X^d;\mathcal{A}) \arrow[rr, dashed, "\mathcal{O}"] \arrow[dr, "\eta^{\int}"'] & & \mathbb{C} \\
& \int\mathrm{Phs}(X^d;\mathcal{A}) \arrow[ur, dashed, "\mathcal{O}_{\mathrm{top}}"'] &
\end{tikzcd}
$$

- realistic: compactly supported

$\Rightarrow$ *realistic topological solitonic observables* are homology of shape of solitonic phase space:

$$
\boxed{\mathrm{TopObs}_\bullet(X^d;\mathcal{A}) \;\simeq\; H_\bullet\Big(\mathrm{Map}^*\big(X^d_{\cup\{\infty\}};\mathcal{A}\big),\mathbb{C}\Big)}
$$

IR-Completed Phase Space

phase space:	$\mathrm{Phs}(X^d;\mathcal{A}) \;\equiv\; \mathrm{Map}(X^d;\mathcal{A}_{\mathrm{dff}})$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int\mathrm{Phs}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}(X^d,\mathcal{A}\;\;)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\underset{\mathrm{sltnc}}{\int\mathrm{Phs}}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}},\mathcal{A}\big)$	all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables: $\left\{ \begin{array}{l} \text{general: } \mathrm{Phs}(X^d;\mathcal{A}) \xrightarrow{\;\mathcal{O}\;} \mathbb{C} \\ \text{topological: } \mathrm{Phs}(X^d;\mathcal{A}) \xrightarrow{\eta^{\int}} \int\mathrm{Phs}(X^d;\mathcal{A}) \xrightarrow{\;\mathcal{O}_{\mathrm{top}}\;} \mathbb{C} \\ \text{realistic: compactly supported} \end{array} \right.$

$\Rightarrow$ *realistic topological solitonic observables* are homology of shape of solitonic phase space:

$$\boxed{\mathrm{TopObs}_\bullet(X^d;\mathcal{A}) \;\simeq\; H_\bullet\Big(\mathrm{Map}^*\big(X^d_{\cup\{\infty\}};\mathcal{A}\big),\mathbb{C}\Big)}$$

Topological Quantum States

Note: Topological observables are
locally constant functions on (reduced) phase space
as such have trivial Poisson brackets $\Rightarrow$
quantum algebra nonperturbative beyond formal quantization

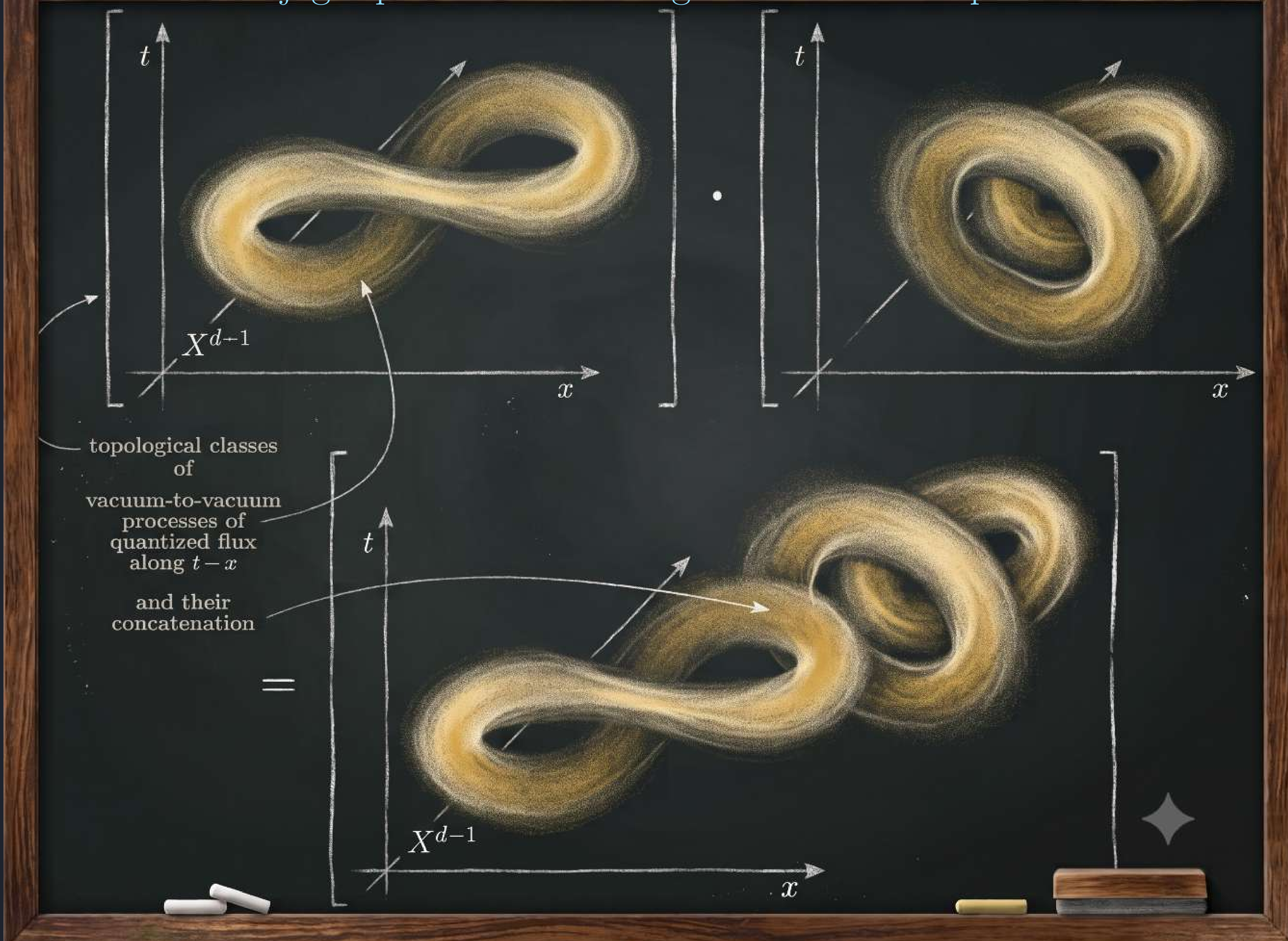
Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Pontrjagin product here: Light front-ordered product:



Topological Light-Cone Quantization

For spacetimes of the form
$$\begin{aligned} X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\ X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1} \end{aligned}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Light-Cone Quantization

For spacetimes of the form
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Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1} \times \Sigma^2$

Lie-Poisson algebra of fluxes: $\underbrace{C^\infty(\Sigma^2, \mathfrak{g}_\hbar)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2, \mathfrak{g}_0)}_{\text{magnetic}}$

nonperturb. quantization: $\mathbb{C}\Big[\underbrace{C^\infty(\Sigma^2, G)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2, \mathfrak{g}_0/\Lambda)}_{\text{magnetic}}\Big]$

generally elusive, works here
due to Lie-Poisson structure

topological sector: $\mathbb{C}\big[\pi_0(-)\big] \simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}\big(\Sigma^2, \underbrace{BG}_{\text{el}} \times \underbrace{B^2\Lambda}_{\text{mag}}\big)\Big]$

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Topological Quantum States

quantum states form Obs-modules

here: $\pi_1(-)$-representations!

hence local systems on the mapping space:

$$\boxed{\mathcal{H} \in \mathrm{TopObs\,Mod} \subset \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)}$$

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \;\mapsto\; \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big) \\ \text{extends to an extended TQFT} \end{cases}$

here: boundary theory $\begin{cases} \qquad\qquad \mathrm{Mod}_{\mathbb{C}} \longrightarrow \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\Big) \\ X^{d-1} \;\vdash\; \mathbb{1} \;\mapsto\; \mathcal{H}\big(X^{d-1}\big) \end{cases}$

$\Rightarrow$ over X^{d-1} these boundary state spaces are:

$$\mathcal{H} \;\in\; \underset{\substack{X^{d-1} \in \\ B\mathrm{Diff}(X^{d-1})}}{\underleftarrow{\lim}} \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)$$

$$\simeq \quad \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}, \mathcal{A}\big) /\!\!/ \mathrm{Diff}\big(X^{d-1}\big)\Big)$$

"general covariance"

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Topologically Ordered Quantum Matter

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external parameters: $p \in P$

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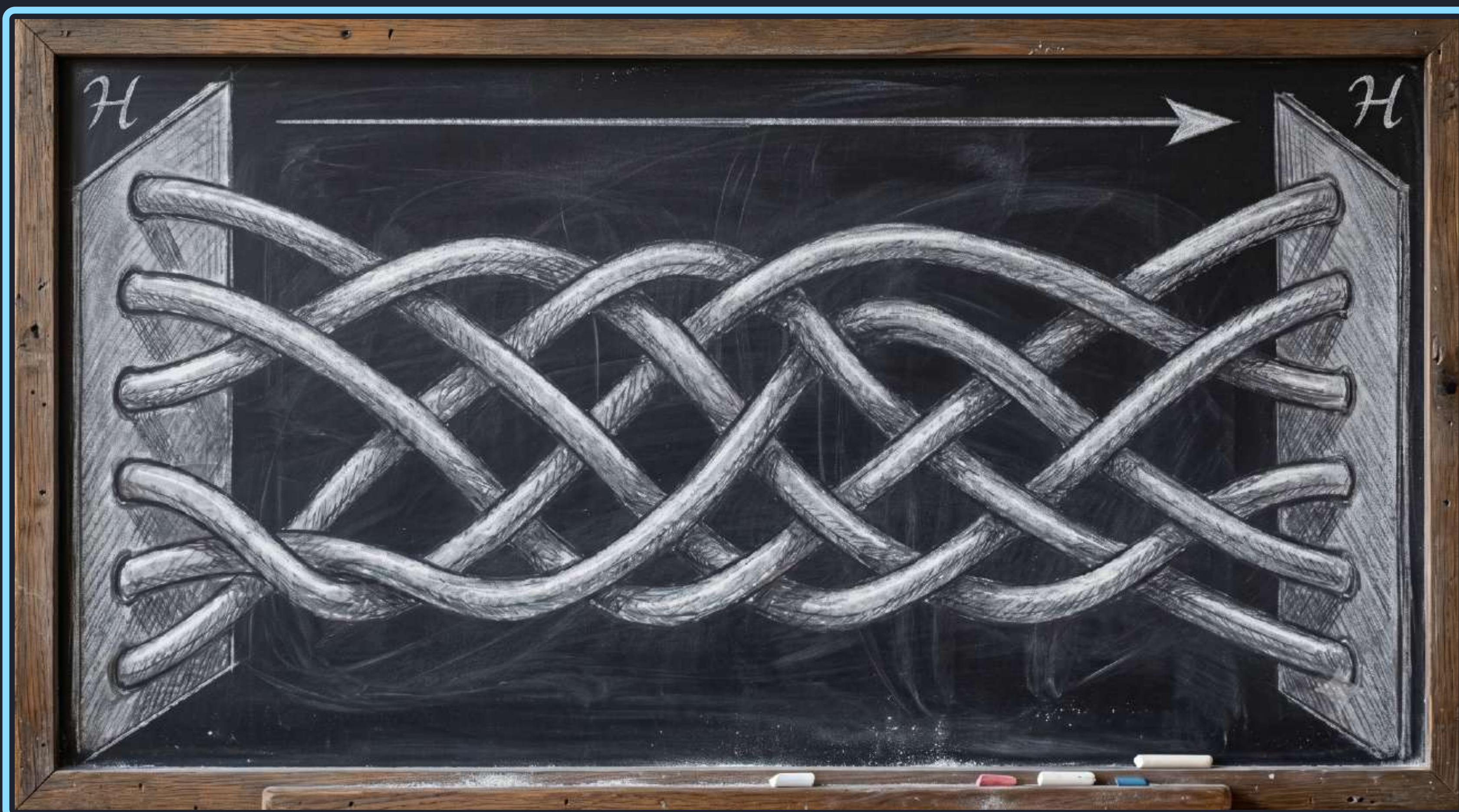

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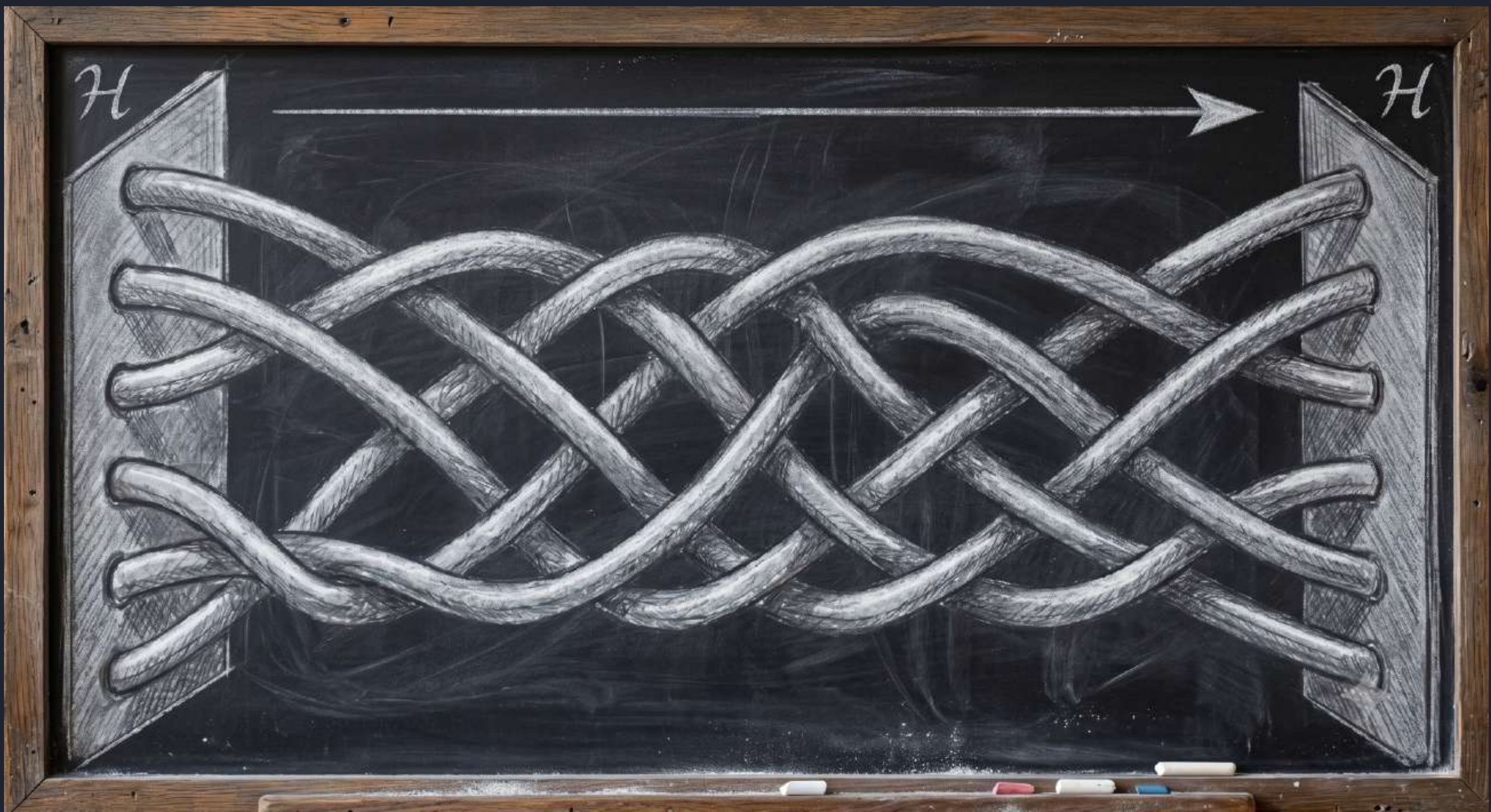

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crystal lattice

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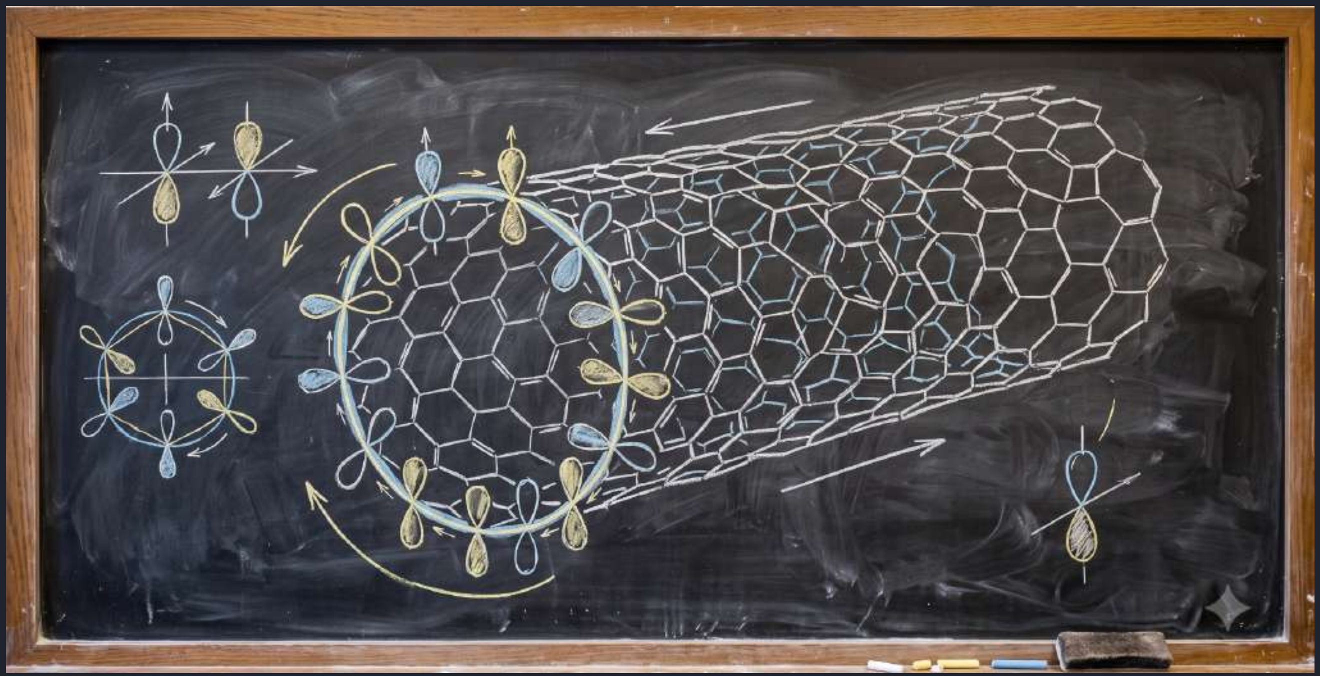

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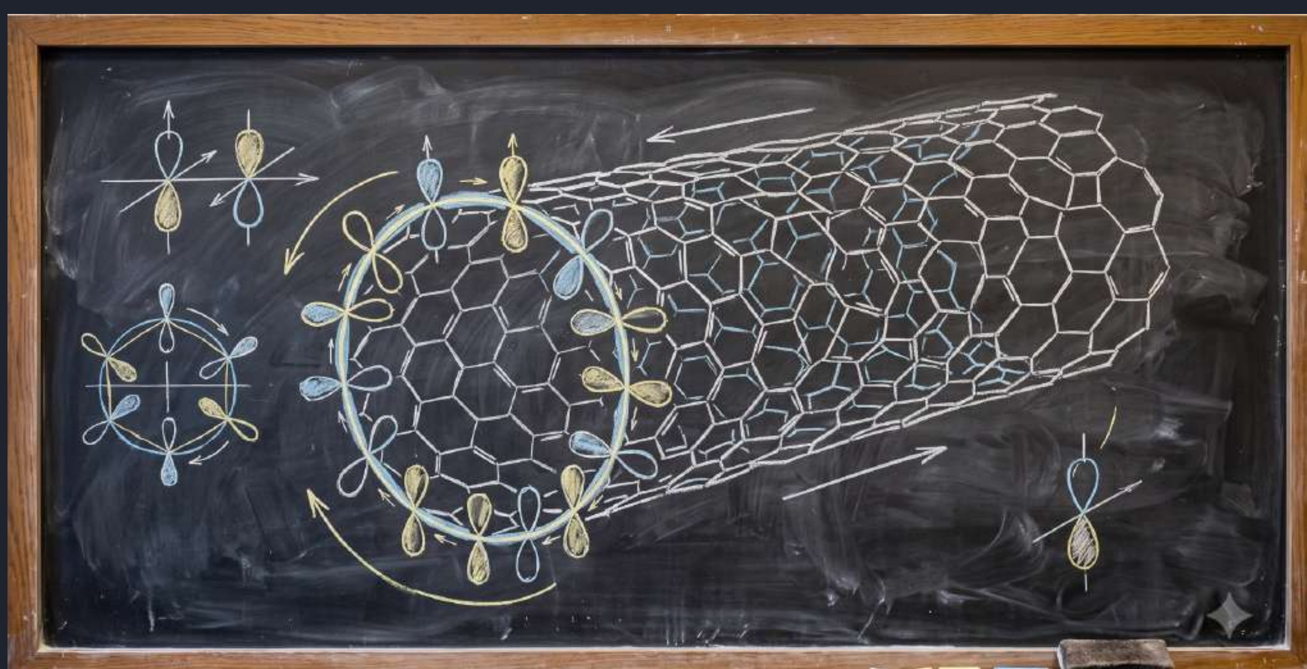

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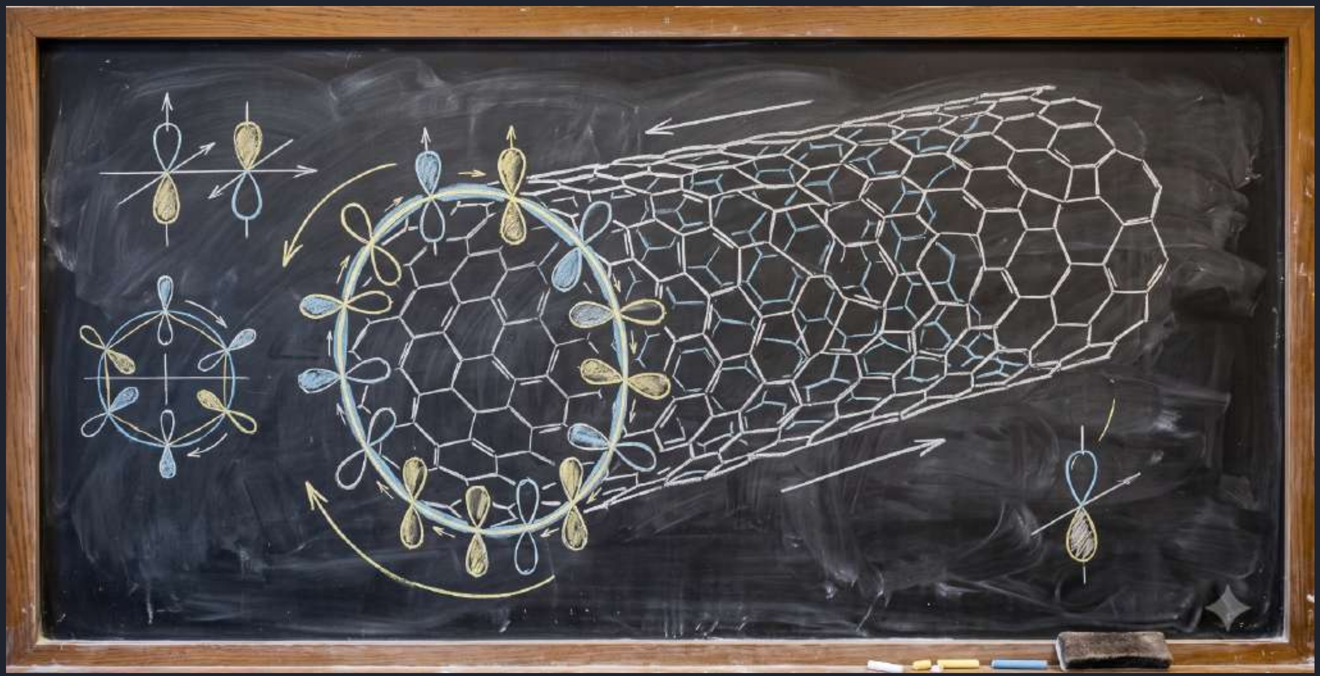

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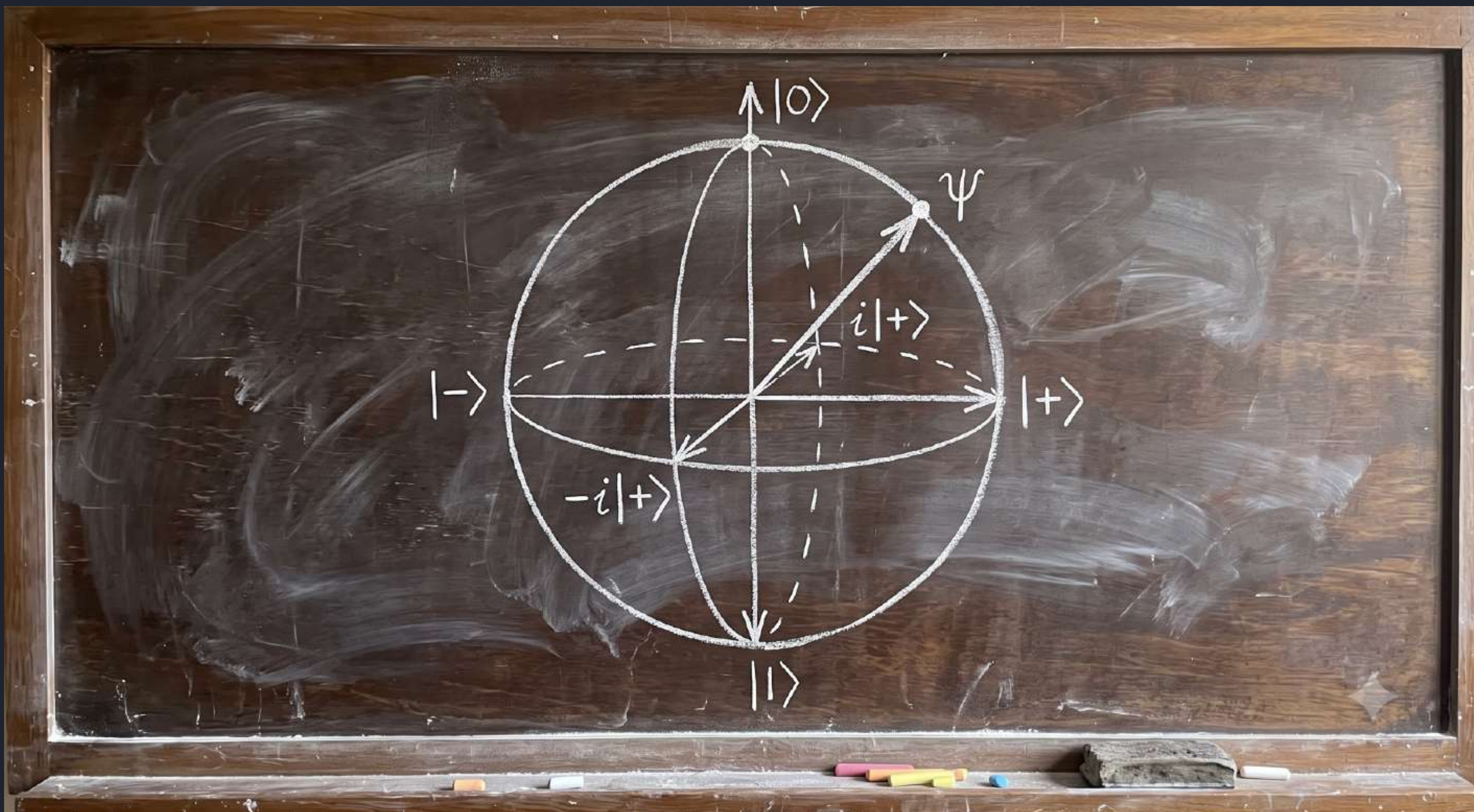

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forced by

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seen in experiment!

[Nakamura et al. 2019]

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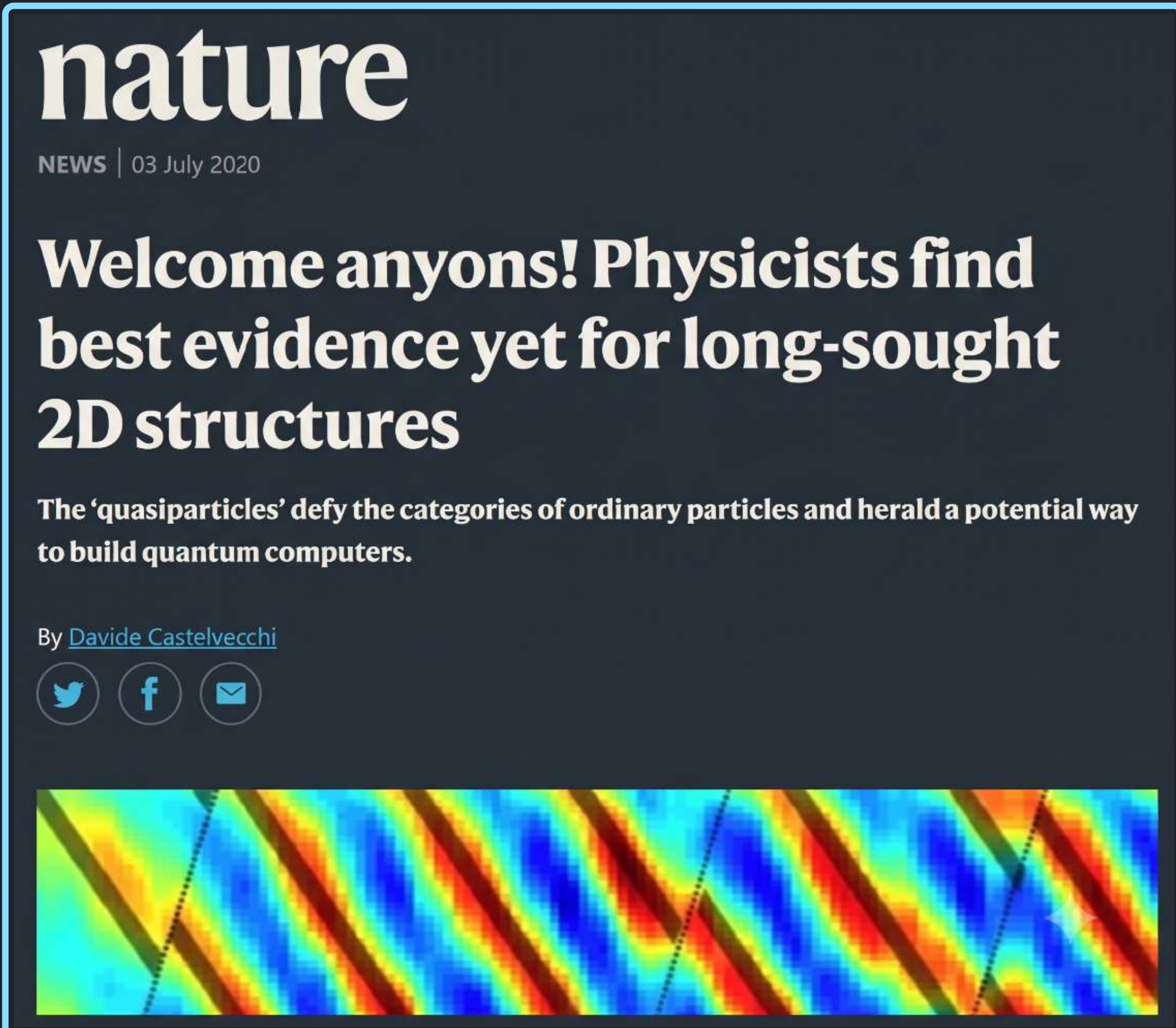

nature

NEWS | 03 July 2020

Welcome anyons! Physicists find best evidence yet for long-sought 2D structures

The 'quasiparticles' defy the categories of ordinary particles and herald a potential way to build quantum computers.

By Davide Castelvecchi

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[Nakamura et al. 2019]

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IR-Completion with Sources & Branes

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higher Maxwell-type fluxes with sources have

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electron number = current cohomology
with compact spatial support!

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M5-probe of 11D SUGra

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& charges are in *twisted relative* nonabelian cohomology:

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IR-Complete C-Field & M5 Model

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recall: 11D SuGra with M5-probes is characterized by:

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hence stably $\mathbb{S}^4(X^{10})$

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$$\begin{array}{ccc} \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{\int}_{S^4} S^7\big) & \xrightarrow{\;(\mathit{\int} h_{\mathbb{C}})_*\;} & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{\int}_{S^4} \mathbb{C}P^3\big) \\ \big\downarrow (\mathit{\int} h_{\mathbb{H}})_* & & \big\downarrow (\mathit{\int} t_{\mathbb{C}})_* \\ \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{\int} S^4\big) & = & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{\int} S^4\big) \end{array}$$

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$\Rightarrow$ *magentized* M5-brane model

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coefficient for FQH orders!

Geometrically Engineers FQH Order

putting this at an A_n orbi-singularity
engineers FQH topological order in fine detail
(...)

The End

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For more see:

– *The Character Map In Nonabelian Cohomology* (esp. §9)

– *Complete Topological Quantization of Higher Gauge Fields*

– *Higher Gauge Theory via Differential Nonabelian Cohomology*

– *Higher Superspace Supergravity and its IR-Completions*