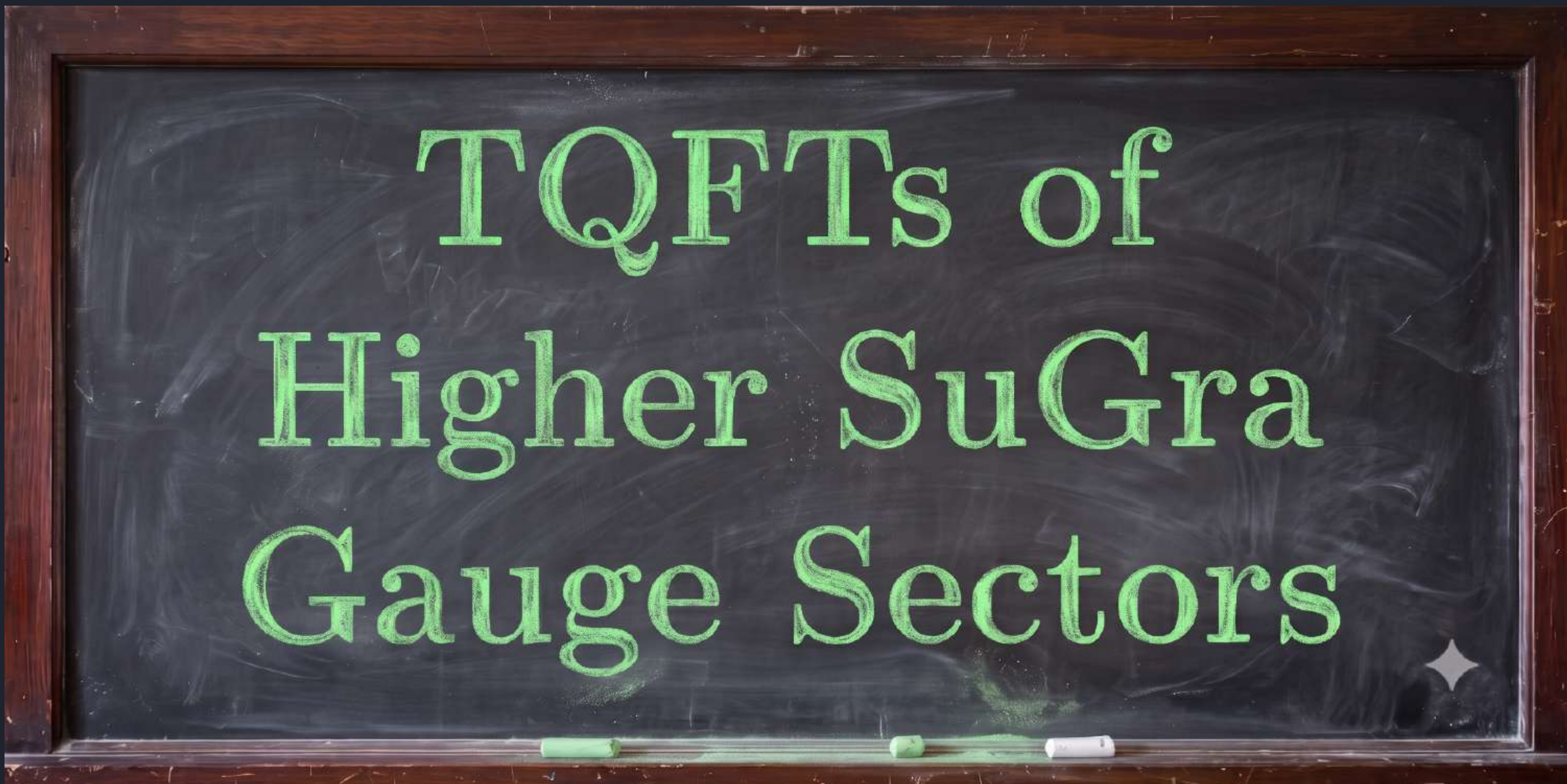
Urs Schreiber

New York University, Abu Dhabi

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS



based in parts on joint work with:

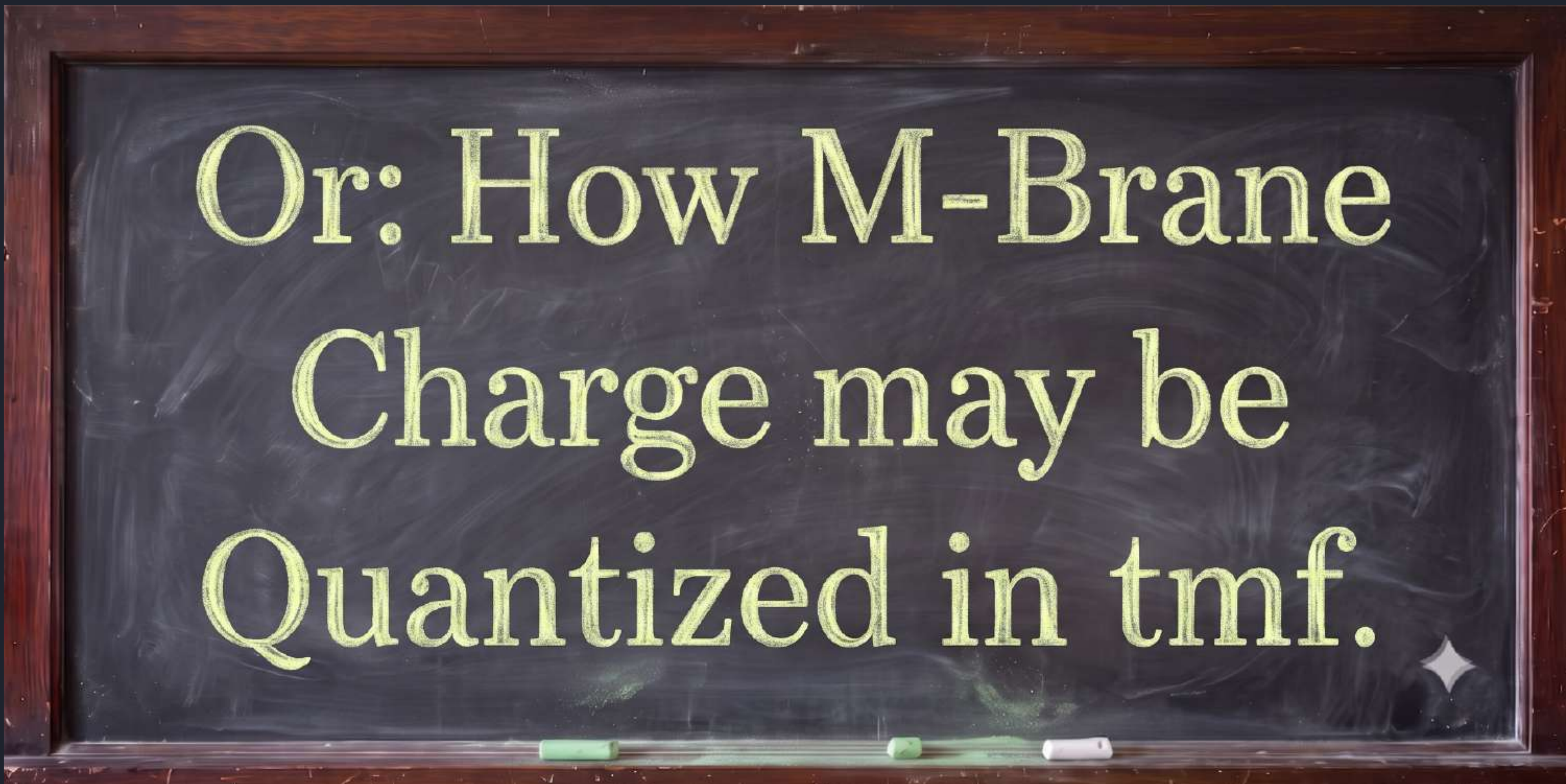
D. Fiorenza, G. Giotopoulos, H. Sati

talk at *Elliptic objects, von Neumann algebras, and Functorial Field Theory*, Bernoulli Center Lausanne (16 July 2026)

Urs Schreiber

New York University, Abu Dhabi

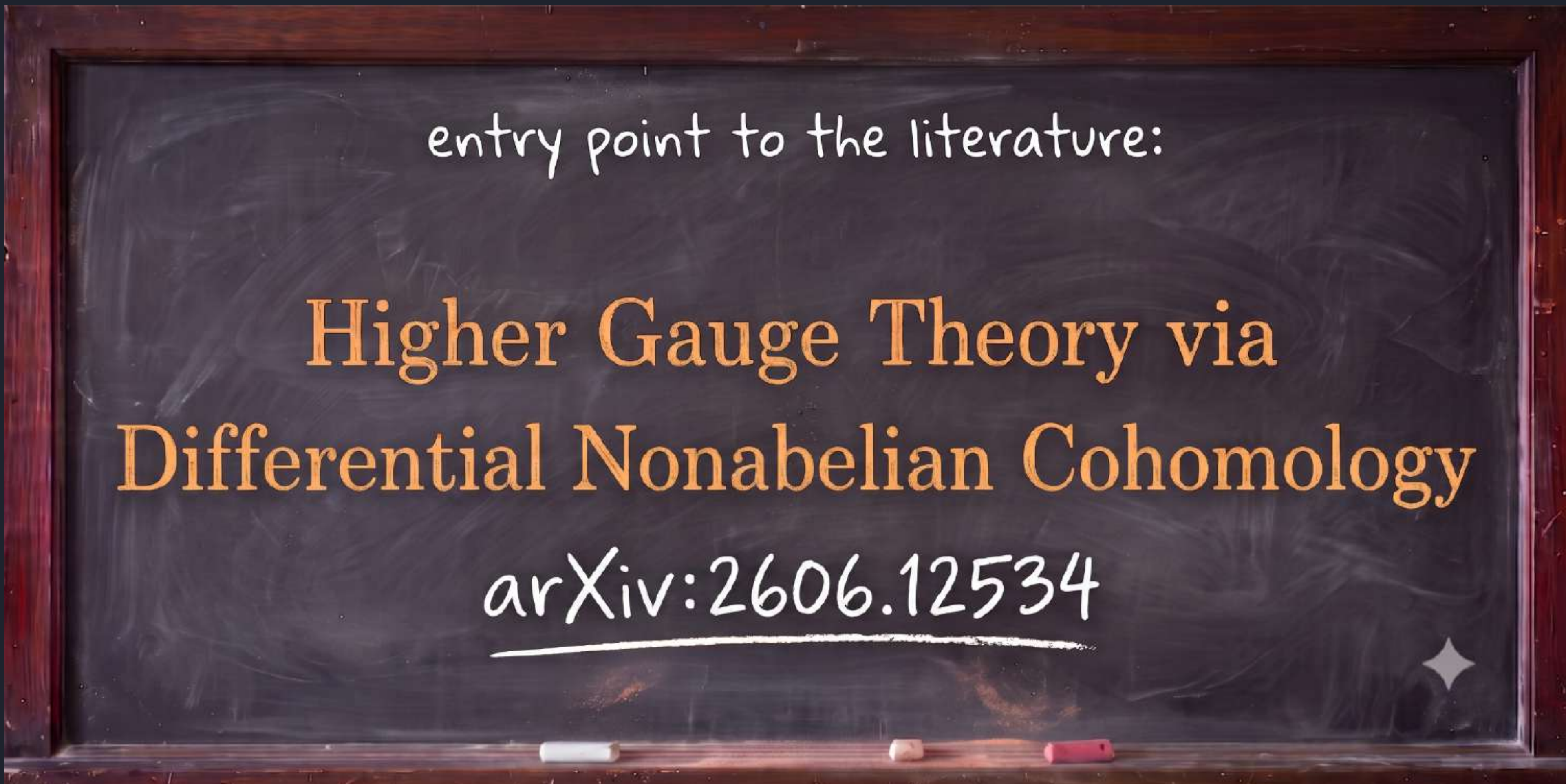
CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS



based in parts on joint work with:

D. Fiorenza, G. Giotopoulos, H. Sati

talk at *Elliptic objects, von Neumann algebras, and Functorial Field Theory*, Bernoulli Center Lausanne (16 July 2026)


entry point to the literature:
Higher Gauge Theory via
Differential Nonabelian Cohomology
arXiv:2606.12534

IR-Completing Higher Gauge Fields

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

widely considered,
but not here

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

widely considered,
but not here

IR-completion: global field topology

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

widely considered,
but not here

IR-completion: global field topology

underappreciated,
our topic here

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

widely considered, but not here

IR-completion: global field topology

flux quantization

underappreciated, our topic here

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

widely considered,
but not here

IR-completion: global field topology
flux quantization
brane charges

underappreciated,
our topic here

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

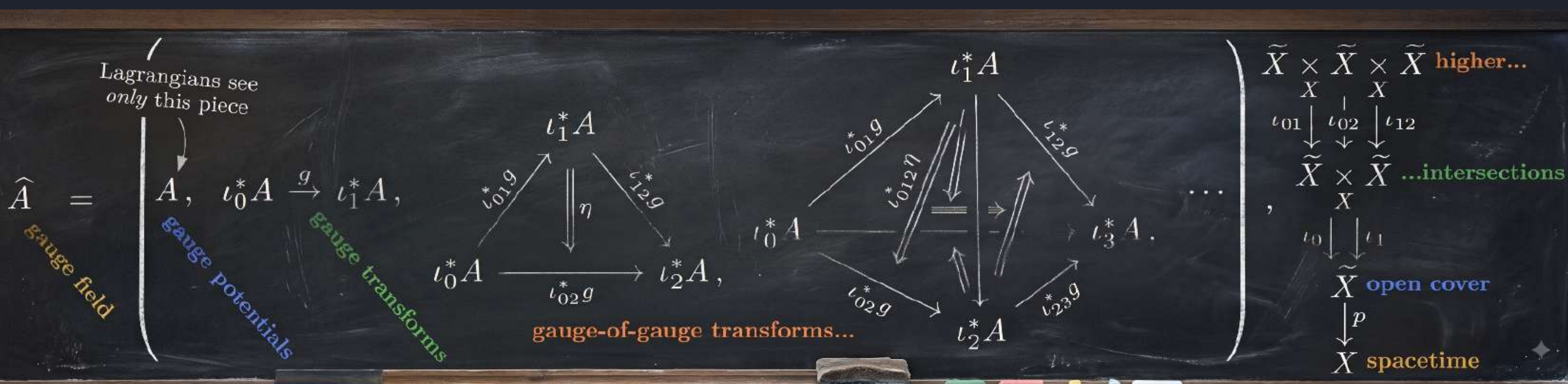
IR-completion: global field topology
flux quantization
brane charges

IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

IR-completion: global field topology
flux quantization
brane charges



IR-Completing Higher Gauge Fields

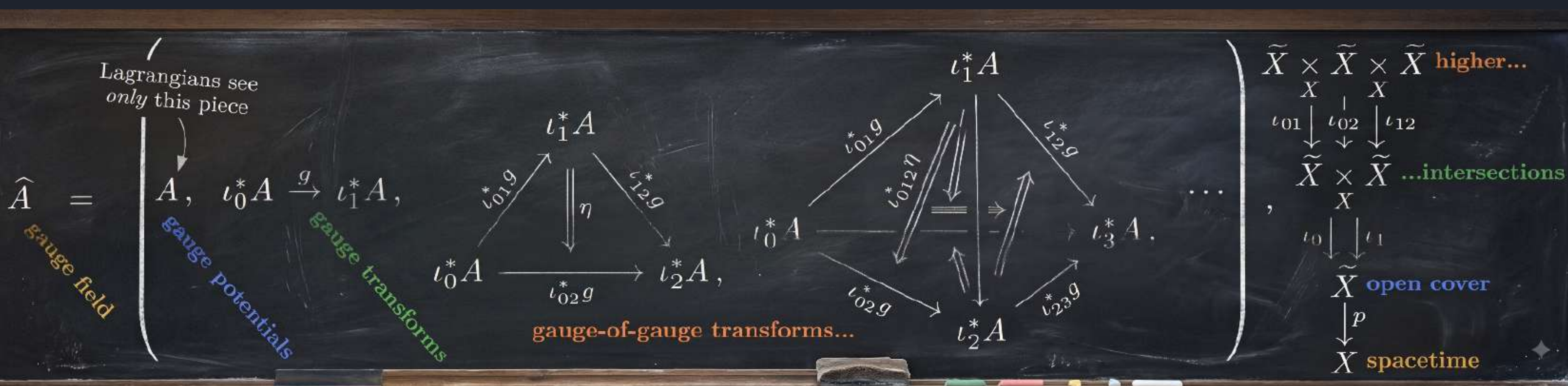
Quantum Fields traditionally defined incrementally:

UV-completion: local interactions

IR-completion: global field topology $\Rightarrow$ TQFT sector

flux quantization

brane charges



IR-Completing Higher Gauge Fields

Quantum Fields traditionally defined incrementally:

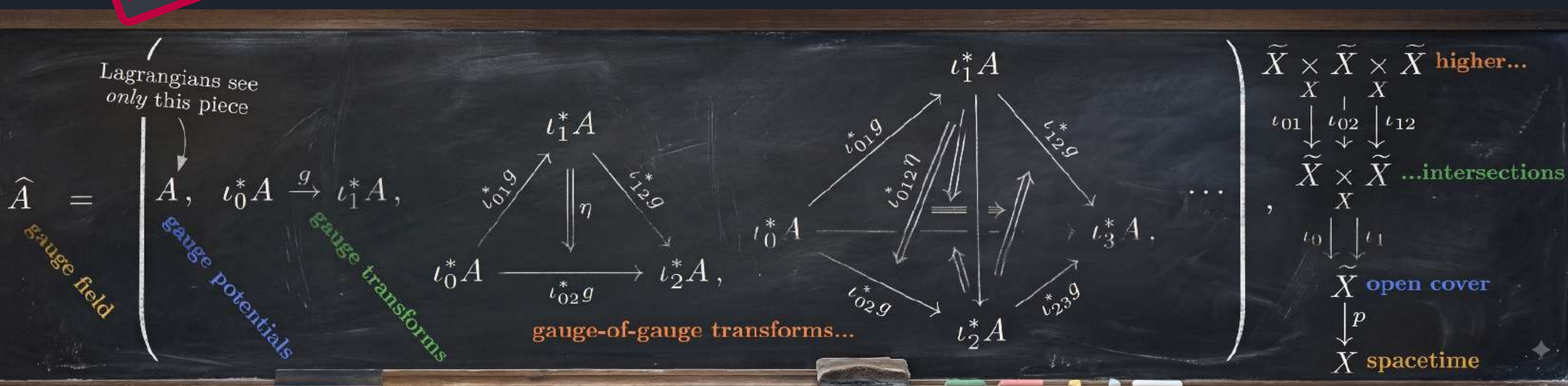
UV-completion: local interactions

IR-completion: global field topology $\Rightarrow$ TQFT sector

flux quantization

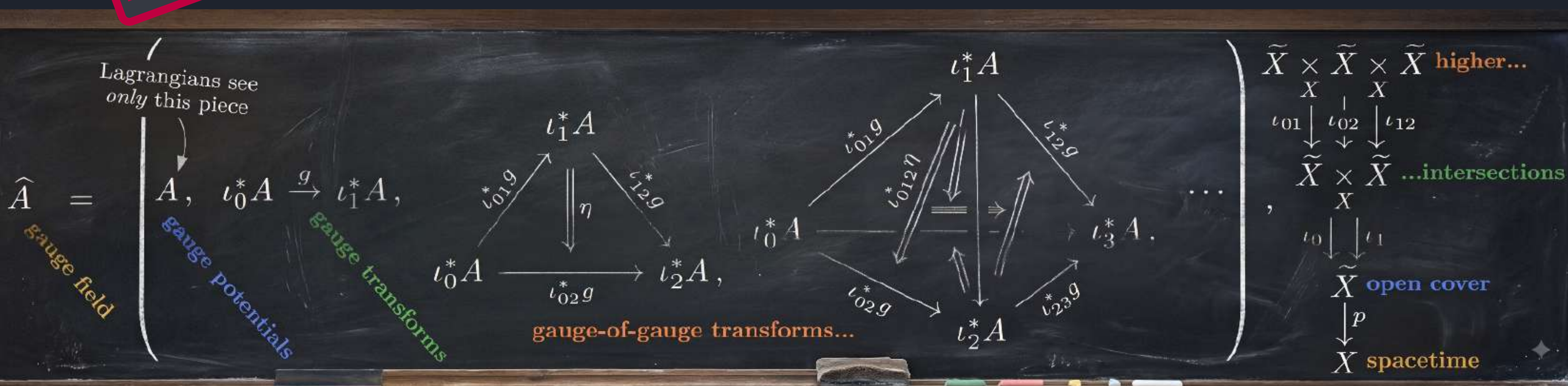
brane charges

non-perturbative



Recall: IR-Completion of Maxwell

non-perturbative



Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

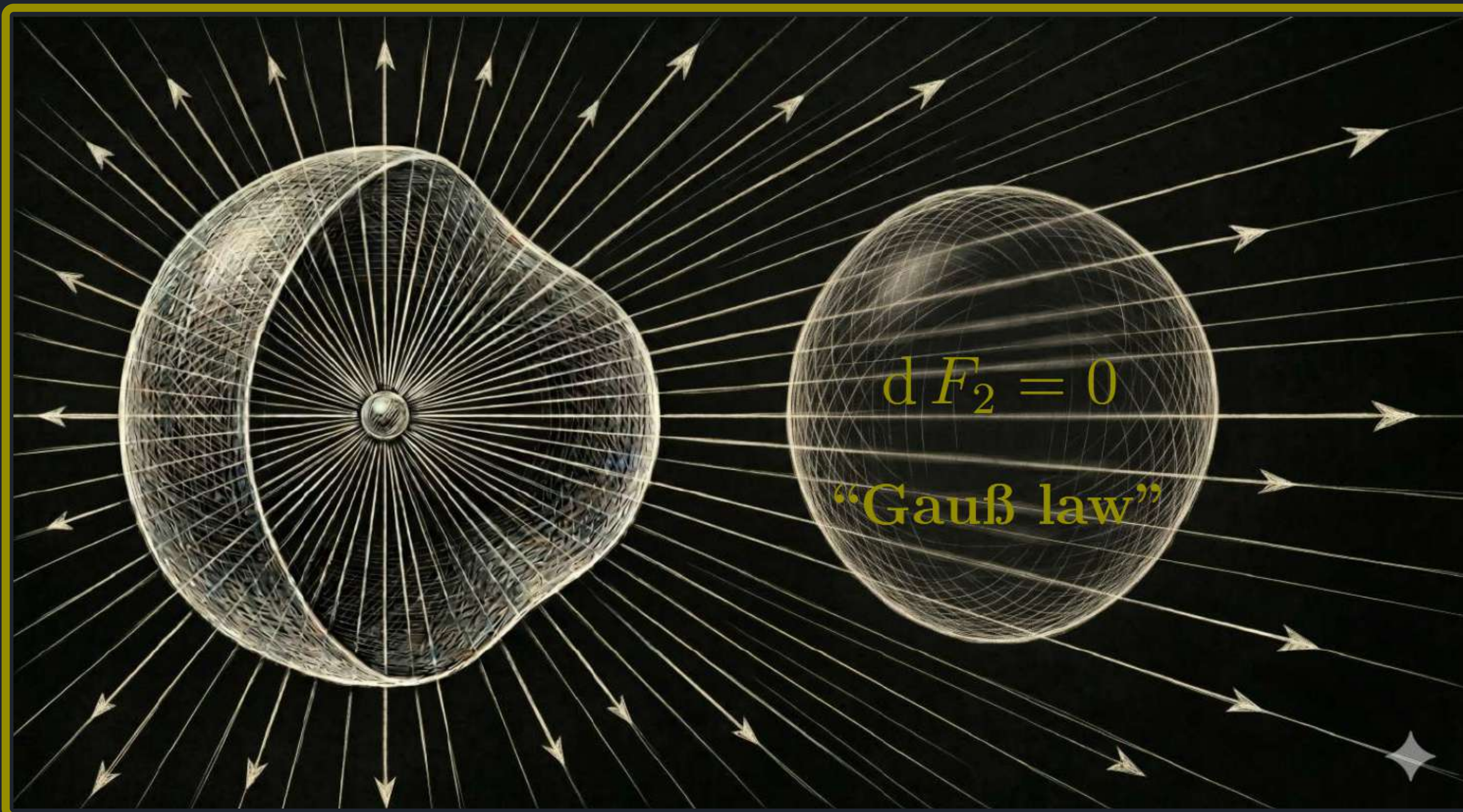
Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

Cauchy hypersurface

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:



Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \in \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \in \Omega^2_{\mathrm{cl}}(X^3)$

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \in \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \in \Omega^2_{\mathrm{cl}}(X^3)$
IR-completion after Dirac 1931	$\widehat{A}_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$	$\wr\!\wr$ $E_2 \in H^2_{\mathrm{diff}}(X^3; \mathbb{R})$

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \in \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \in \Omega^2_{\mathrm{cl}}(X^3)$
		$\wr\wr$
IR-completion after Dirac 1931	$\widehat{A}_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$	$E_2 \in H^2_{\mathrm{diff}}(X^3; \mathbb{R})$
duality-symmetric IR-completion	$\widehat{A}_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$	$\widehat{A}'_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \,\in\, \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \,\in\, \Omega^2_{\mathrm{cl}}(X^3)$
IR-completion after Dirac 1931	$\widehat{A}_1 \,\in\, H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$	$\wr\wr$ $E_2 \,\in\, H^2_{\mathrm{dff}}(X^3;\mathbb{R})$
duality-symmetric IR-completion	$\widehat{A}_1 \,\in\, H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$	$\widehat{A}'_1 \,\in\, H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$
general IR-completion	$(\widehat{A}_1, \widehat{A}'_1) \,\in\, H^0_{\mathrm{dff}}(X^3;\mathcal{A})$ for $\mathcal{A}_{\mathbb{R}} \simeq B^2\mathbb{R}^2$	

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \in \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \in \Omega^2_{\mathrm{cl}}(X^3)$
IR-completion after Dirac 1931	$\widehat{A}_1 \in H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$	$E_2 \in H^2_{\mathrm{dff}}(X^3;\mathbb{R})$
duality-symmetric IR-completion	$\widehat{A}_1 \in H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$	$\widehat{A}'_1 \in H^2_{\mathrm{dff}}(X^3;\mathbb{Z})$
general IR-completion	$(\widehat{A}_1, \widehat{A}'_1) \in H^0_{\mathrm{dff}}(X^3;\mathcal{A})$ differential nonabelian cohomology for $\mathcal{A}_{\mathbb{R}} \simeq B^2\mathbb{R}^2$	

Recall: IR-Completion of Maxwell

for vacuum electromagnetism on $X^{1,3} \simeq \mathbb{R}^{1,0} \times X^3$:

phase space	canonical **coordinate**	canonical **momentum**
naïve	$A_1 \in \Omega^1_{\mathrm{dR}}(X^3)/\mathrm{im}(\mathrm{d})$	$E_2 \in \Omega^2_{\mathrm{cl}}(X^3)$ $\wr$
IR-completion after Dirac 1931	$\widehat{A}_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$	$E_2 \in H^2_{\mathrm{diff}}(X^3; \mathbb{R})$
duality-symmetric IR-completion	$\widehat{A}_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$	$\widehat{A}'_1 \in H^2_{\mathrm{diff}}(X^3; \mathbb{Z})$
general IR-completion	$(\widehat{A}_1, \widehat{A}'_1) \in H^0_{\mathrm{diff}}(X^3; \mathcal{A})$ differential nonabelian cohomology for $\mathcal{A}_{\mathbb{R}} \simeq B^2\mathbb{R}^2$	

$$(\widehat{A}_1, \widehat{A}'_1) \in H^0_{\mathrm{dff}}(X^3; \mathcal{A})$$

differential nonabelian cohomology for $\mathcal{A}_{\mathbb{R}} \simeq B^2\mathbb{R}^2$

Differential Cohomology

$$(\widehat{A}_1, \widehat{A}'_1) \in H^0_{\mathrm{dff}}(X^3; \mathcal{A})$$

differential nonabelian cohomology for $\mathcal{A}_{\mathbb{R}} \simeq B^2\mathbb{R}^2$

Differential Cohomology

ordinary

$$H^n_{\mathrm{dff}}(X;\mathbb{Z}) \;\simeq\; H^0\big(X;\underbrace{\mathbb{Z}[n]_{\mathrm{dff}}}_{\Omega^n_{\mathrm{cl}} \underset{\mathbb{R}[n]}{\times} \mathbb{Z}[n]}\big)$$

Differential Cohomology

ordinary

$$H^n_{\mathrm{dff}}(X;\mathbb{Z}) \;\simeq\; H^0\Big(X; \underbrace{\mathbb{Z}[n]_{\mathrm{dff}}}_{\Omega^n_{\mathrm{cl}} \underset{\mathbb{R}[n]}{\times} \mathbb{Z}[n]}\Big)$$

nonabelian

$$H^0_{\mathrm{dff}}(X;\mathcal{A}) \;\equiv\; H^0\Big(X; \underbrace{\mathcal{A}_{\mathrm{dff}}}_{\Omega^1_{\mathrm{cl}}(\cdot;\mathfrak{l}\mathcal{A}) \underset{\mathcal{A}_{\mathbb{R}}}{\times} \mathcal{A}}\Big)$$

formed in $\mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{SmthMfd})$

Differential Cohomology

ordinary

$$H^n_{\mathrm{dff}}(X;\mathbb{Z}) \;\simeq\; H^0\big(X;\underbrace{\mathbb{Z}[n]_{\mathrm{dff}}\big)}_{\Omega^n_{\mathrm{cl}} \underset{\mathbb{R}[n]}{\times} \mathbb{Z}[n]}$$

nonabelian

$$H^0_{\mathrm{dff}}(X;\mathcal{A}) \;\equiv\; H^0\big(X;\underbrace{\mathcal{A}_{\mathrm{dff}}\big)}_{\Omega^1_{\mathrm{cl}}(\cdot;\,\mathfrak{l}\mathcal{A}) \underset{\mathcal{A}_{\mathbb{R}}}{\times} \mathcal{A}}$$

formed in $\mathrm{SmthGrpd}_\infty :=$ $\mathrm{Sh}_\infty(\mathrm{SmthMfd})$

$$\text{for}\;\begin{cases} \mathcal{A} \;\; \text{a} \;\; \text{(classifying) space} & \left(\begin{array}{l}\text{pointed connected \&}\\ \text{nilpotent of } \mathbb{Q}\text{-finite type}\end{array}\right) \\ \mathfrak{l}\mathcal{A} := \pi_\bullet(\Omega\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R} \;\in\; L_\infty\mathrm{Alg}_{\mathbb{R}} & \text{(Whitehead brackets)} \end{cases}$$

Differential Cohomology

ordinary

$$H^n_{\mathrm{dff}}(X;\mathbb{Z}) \;\simeq\; H^0\big(X;\underbrace{\mathbb{Z}[n]_{\mathrm{dff}}}_{\Omega^n_{\mathrm{cl}} \underset{\mathbb{R}[n]}{\times} \mathbb{Z}[n]}\big)$$

nonabelian

$$H^0_{\mathrm{dff}}(X;\mathcal{A}) \;\equiv\; H^0\big(X;\underbrace{\mathcal{A}_{\mathrm{dff}}}_{\Omega^1_{\mathrm{cl}}(\cdot;\,\mathfrak{l}\mathcal{A}) \underset{\mathcal{A}_{\mathbb{R}}}{\times} \mathcal{A}}\big)$$

formed in $\mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{SmthMfd})$

$$\text{for}\begin{cases} \mathcal{A} \;\; \text{a} \;\; \text{(classifying) space} & \text{(pointed connected \& nilpotent of } \mathbb{Q}\text{-finite type)} \\ \mathfrak{l}\mathcal{A} := \pi_\bullet(\Omega\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{R} \;\in\; L_\infty\mathrm{Alg}_{\mathbb{R}} & \text{(Whitehead brackets)} \end{cases}$$

IR-Completion of Higher Maxwell-Type

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory
of Maxwell-type:

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: $\left\{ \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \right.$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	magnetic & electric: $\mathrm{d}\,F_2 = 0$, $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\, F_2 = 0$ $\mathrm{d}\, E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$
temporal germs of on-shell fluxes
on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0/\!\!/B\mathrm{U}(1))$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l} B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l} S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$

classifying space of twisted K-theory

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: $\left\{ \begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: $\left\{\begin{array}{c}\text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}\end{array}\right.$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

gauge sector of

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$
temporal germs of on-shell fluxes
on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples: gauge sector of 4D SuGra

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

gauge sector of

4D SuGra

5D SuGra

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

$$\left\{ \begin{array}{c} \text{For } \mathit{characteristic}\ \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$$

Examples: gauge sector of

theory	Gauß law	characteristic
4D SuGra: 4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D SuGra: 5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
10D SuGra: idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ BU(1))$
11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

gauge sector of

theory	Gauß law	characteristic
4D SuGra: 4D Maxwell	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_2 = 0$	$\mathfrak{l}B^2\mathbb{Z}^2$
5D SuGra: 5D Maxwell-CS	$\mathrm{d}\,F_2 = 0$ $\mathrm{d}\,E_3 = \frac{1}{2}F_2F_2$	$\mathfrak{l}S^2$
10D SuGra: idealized 10D IIA NS/RR	$\mathrm{d}\,H_3 = 0$ $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D SuGra: 11D Maxwell-CS	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2}G_4G_4$	$\mathfrak{l}S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

Examples

theory	Gauß law	characteristic
4D Maxwell	$\mathrm{d}\, F_2 = 0$ $\mathrm{d}\, E_2 = 0$	$\mathfrak{l} B^2\mathbb{Z}^2$
5D Maxwell-CS	$\mathrm{d}\, F_2 = 0$ $\mathrm{d}\, E_3 = \frac{1}{2} F_2 F_2$	$\mathfrak{l} S^2$
idealized 10D IIA NS/RR	$\mathrm{d}\, H_3 = 0$ $\mathrm{d}\, F_{2\bullet} = F_{2\bullet - 2} H_3$	$\mathfrak{l}(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1))$
11D Maxwell-CS	$\mathrm{d}\, G_4 = 0$ $\mathrm{d}\, G_7 = \frac{1}{2} G_4 G_4$	$\mathfrak{l} S^4$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type: $\left\{\begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array}\right.$

so: for IR-completion choose space $\mathcal{A}$ s.t. $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

$$\boxed{\Omega^1_{\mathrm{cl}}(\cdot;\, \underset{\mathfrak{a}}{\underset{\wr\!\shortmid}{\mathfrak{l}\mathcal{A}}})}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

so: for IR-completion choose space $\mathcal{A}$ s.t. $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

$$\Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}})$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

then fluxes are modulated by:

$$X^d \xrightarrow{\;\vec{F}\;} \Omega^1_{\mathrm{cl}}\big(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}\big)$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

$$\left\{ \begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty \mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array} \right.$$

then fluxes are modulated by:

$$\boxed{X^d \xrightarrow[\text{fluxes}]{\vec{F}} \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}})}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

rational homotopy theory gives:

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[r, bend right, "\vec{F}", "\text{fluxes}"'] & \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:
$$
\left\{
\begin{array}{c}
\text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\
\text{temporal germs of on-shell fluxes} \\
\text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\
\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}
\end{array}
\right.
$$

rational homotopy theory gives:

$$
\boxed{
\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[r, bend left, "\vec{F}", "\text{fluxes}"'] & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian}"', "\text{de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
}
$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

flux quantization:
fluxes
are
rational
image of
charges χ
in
$\mathcal{A}$-cohomology

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[urr, bend left, "\chi"] \arrow[r, "\vec{F}", "\text{fluxes}"'] & \Omega^1_{\mathrm{cl}}(\cdot\,; \mathfrak{l}\mathcal{A}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

$\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

$$\left\{\begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array}\right.$$

flux quantization:
fluxes
are
rational
image of
charges χ
in
$\mathcal{A}$-cohomology

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rationalization}"] \\
X^d \arrow[urr, bend left, "\underset{\chi}{\text{charges}}"] \arrow[r, "\vec{F}"', "\text{fluxes}"'] & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\overset{\mathfrak{l}\mathcal{A}}{\wr}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

up to homotopy

$$\begin{tikzcd}
X^d \arrow[rr, bend left=40, "\text{charges}", "\chi"'{name=U}] \arrow[dr, "\text{fluxes}"', "\vec{F}"{name=D}] \arrow[Rightarrow, from=U, to=D, "\widehat{A}"] & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
& \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\overset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

$\left\{\begin{array}{c} \text{For } \textit{characteristic } \mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}} \\ \text{temporal germs of on-shell fluxes} \\ \text{on } X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d \text{ are:} \\ \boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})} \end{array}\right.$

up to homotopy this exhibits the gauge potentials

$$
\begin{tikzcd}
X^d \arrow[rr, bend left, "\text{charges}"] \arrow[dr, "\vec{F}", "\text{fluxes}"'] \arrow[r, phantom, "{\chi = \widehat{A} \,\Downarrow\, \text{potentials}}"] & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
& \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

up to homotopy this exhibits the gauge potentials

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[urr, bend left, "\text{charges}", ""{name=U, below}] \arrow[r, "\text{fluxes}"', "\vec{F}", ""{name=D, above}] & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\arrow[from=U, to=D, Rightarrow, "\chi = \widehat{A}"', "\text{potentials}"]
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory
of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$
temporal germs of on-shell fluxes
on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

by the
universal
property
of the
homotopy
pullback

$$\begin{tikzcd}
& \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[urr, bend left, "\text{charges}"] \arrow[r, bend right, "\text{fluxes}"'] \arrow[ur, phantom, "\chi = \widehat{A} \overset{\text{potentials}}{\Downarrow} \vec{F}"] & \Omega^1_{\mathrm{cl}}(\cdot; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty \mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

by the universal property of the homotopy pullback

$$
\begin{tikzcd}
 & \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\substack{\mathbb{R}\text{-rational-}\\ \text{ization}}"] \\
X^d \arrow[urr, bend left, "\text{charges}"] \arrow[r, bend right, "\text{fluxes}"'] \arrow[r, phantom, bend left, "\chi = \widehat{A} \overset{\text{potentials}}{\Downarrow} \vec{F}"] & \Omega^1_{\mathrm{cl}}(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}) \arrow[r, "\substack{\text{non-abelian}\\ \text{de Rham map}}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$ temporal germs of on-shell fluxes on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

the IR-completed field configurations are the cocycles in differential $\mathcal{A}$-cohomology

$$
\begin{tikzcd}
 & \mathcal{A}_{\mathrm{dff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[ur, dashed, "{\chi = \widehat{A}}"] \arrow[urr, bend left, "\text{charges}"] \arrow[r, bend right, "\text{fluxes}"'] \arrow[r, phantom, "{\widehat{A} \overset{\text{potentials}}{\Downarrow} \vec{F}}"] & \Omega^1_{\mathrm{cl}}\big(\cdot\,; \underset{\mathfrak{a}}{\underset{\wr}{\mathfrak{l}\mathcal{A}}}\big) \arrow[r, "\text{non-abelian de Rham map}"'] & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}
$$

IR-Completion of Higher Maxwell-Type

Higher Gauge Theory of Maxwell-type:

For *characteristic* $\mathfrak{a} \in L_\infty\mathrm{Alg}_{\mathbb{R}}$

temporal germs of on-shell fluxes

on $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ are:

$$\boxed{\{\text{on-shell fluxes}\} \simeq \Omega^1_{\mathrm{cl}}(X^d; \mathfrak{a})}$$

the IR-completed field configurations are cocycles in differential $\mathcal{A}$-cohomology

$$\begin{tikzcd}
 & & \mathcal{A}_{\mathrm{dff}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{A} \arrow[d, "\mathbb{R}\text{-rational-ization}"] \\
X^d \arrow[urr, dashed, "{\chi = \widehat{A}}", "\text{potentials}"'] \arrow[urrrr, bend left, "\text{charges}"] \arrow[rr, bend right, "\vec{F}", "\text{fluxes}"'] \arrow[rr, phantom, "\Downarrow", near start] & & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}\mathcal{A}) \arrow[rr, "\text{non-abelian de Rham map}"'] & & \mathcal{A}_{\mathbb{R}}
\end{tikzcd}$$

$\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

IR-Completed Phase Space

IR-Completed Phase Space

phase space: $\quad \mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$

differential cocycles
$/\!\!/$gauge symmetries

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big)$

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles // gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A})$

path ∞-groupoid,
topological realization,
left adjoint to $\mathrm{Grpd}_\infty \hookrightarrow \mathrm{SmthGrpd}_\infty$

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$

path ∞-groupoid,
topological realization,
left adjoint to $\mathrm{Grpd}_\infty \hookrightarrow \mathrm{SmthGrpd}_\infty$

IR-Completed Phase Space

phase space: $\quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\quad \int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$

by cohesion

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \equiv \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles
$/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \simeq \mathrm{Map}(X^d, \mathcal{A} \quad)$ black brane charges
$/\!\!/$global symmetries

by cohesion

IR-Completed Phase Space

$$\text{phase space:} \quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$$

differential cocycles
$/\!\!/$gauge symmetries

$$\text{its shape:} \quad \int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \simeq \mathrm{Map}\big(X^d, \mathcal{A}\quad\big)$$

black brane charges
$/\!\!/$global symmetries

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \simeq \mathrm{Map}\big(X^d, \mathcal{A} \;\;\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big)$

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles //gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \simeq \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$ black brane charges //global symmetries

solitonic: $\int \mathrm{Phs}_{\mathrm{sltnc}}\big(X^d; \mathcal{A}\big)$

compact support
for fluxes & charges

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$

compact support for fluxes & charges

by cohesion

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A} \;\;\big)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$

compact support for fluxes & charges

by cohesion

one-point compactification

IR-Completed Phase Space

$$\text{phase space:} \quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$$

differential cocycles
$/\!\!/$gauge symmetries

$$\text{its shape:} \quad \textstyle\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$$

black brane charges
$/\!\!/$global symmetries

$$\text{solitonic:} \quad \textstyle\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$$

all brane charges
$/\!\!/$global symmetries

compact support
for fluxes & charges

by cohesion

one-point
compactification

IR-Completed Phase Space

phase space: $\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$ differential cocycles //gauge symmetries

its shape: $\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A} \quad\big)$ black brane charges //global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\Big(X^d_{\cup\{\infty\}}, \mathcal{A}\Big)$ all brane charges //global symmetries

both singular branes like monopoles & solitonic branes like Abrikosov vortices

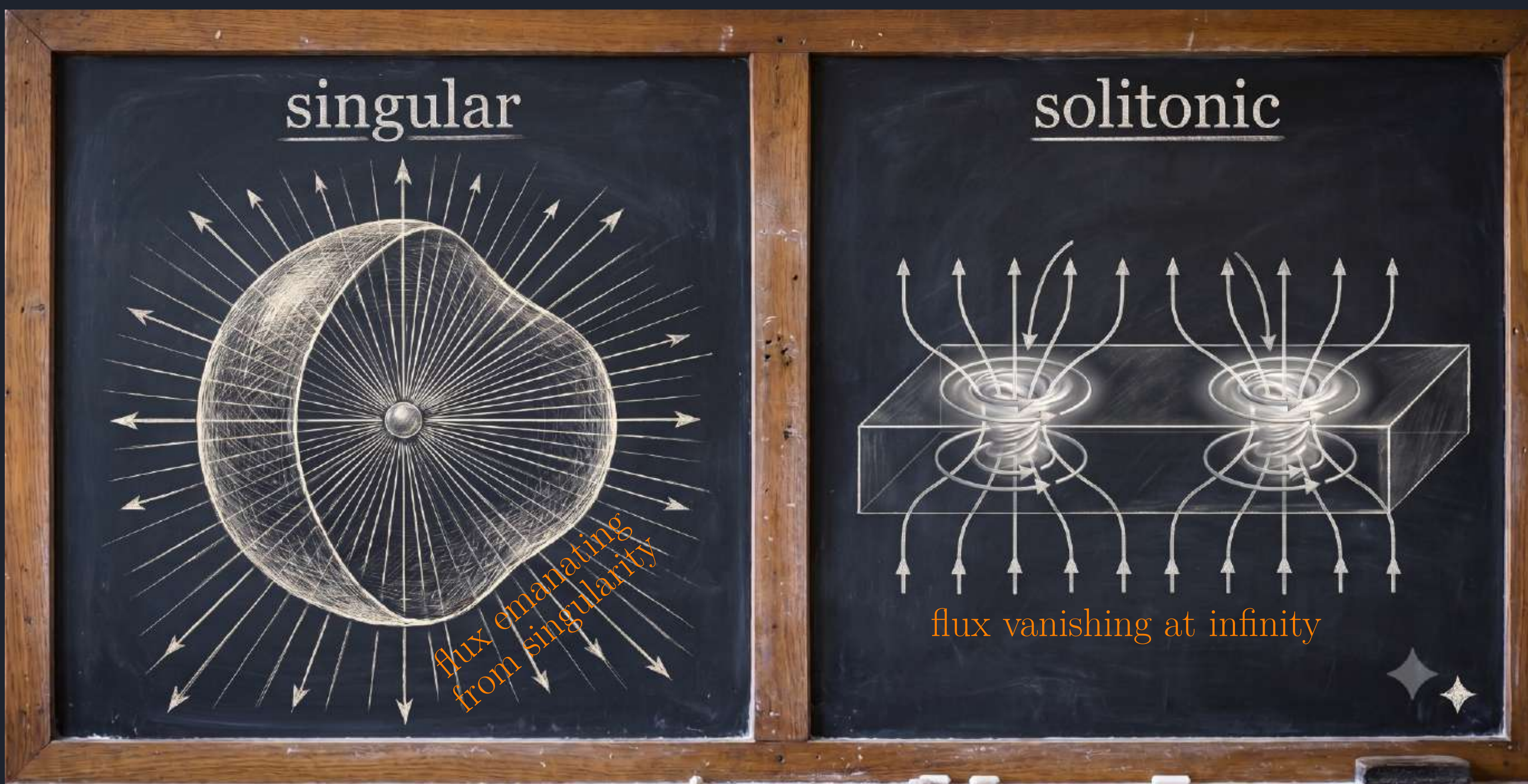
IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles //gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A} \;\;)$ black brane charges //global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ all brane charges //global symmetries

both singular branes like monopoles & solitonic branes like Abrikosov vortices



IR-Completed Phase Space

phase space:	$\mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\quad\big)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\Big(X^d_{\cup\{\infty\}}, \mathcal{A}\Big)$	all brane charges $/\!\!/$global symmetries

IR-Completed Phase Space

phase space:	$\mathrm{Phs}(X^d;\mathcal{A}) \;\equiv\; \mathrm{Map}(X^d;\mathcal{A}_{\mathrm{dff}})$	differential cocycles $/\!\!/$gauge symmetries
its shape:	$\int \mathrm{Phs}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}(X^d,\mathcal{A}\;\;)$	black brane charges $/\!\!/$global symmetries
solitonic:	$\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d;\mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}},\mathcal{A}\big)$	all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

IR-Completed Phase Space

$$\text{phase space:} \quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big)$$

differential cocycles
$/\!\!/$gauge symmetries

$$\text{its shape:} \quad \int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\;\;\big)$$

black brane charges
$/\!\!/$global symmetries

$$\text{solitonic:} \quad \int_{\mathrm{sltnc}} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^*\Big(X^d_{\cup\{\infty\}}, \mathcal{A}\Big)$$

all brane charges
$/\!\!/$global symmetries

Observables on IR-Complete Fields

$$\text{observables:} \left\{ \quad \text{general:} \quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \xrightarrow{\;\;\mathcal{O}\;\;} \mathbb{C} \right.$$

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ — differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A}\;\;)$ — black brane charges $/\!\!/$global symmetries

solitonic: $\int_{\mathrm{sltnc}} \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ — all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables: general: / topological:

$$
\begin{tikzcd}
\mathrm{Phs}(X^d; \mathcal{A}) \arrow[rr, dashed, "\mathcal{O}"] \arrow[dr, "\eta^{\int}"'] & & \mathbb{C} \\
& \int \mathrm{Phs}(X^d; \mathcal{A}) \arrow[ur, dashed, "\mathcal{O}_{\mathrm{top}}"'] &
\end{tikzcd}
$$

IR-Completed Phase Space

$$\text{phase space:} \quad \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d; \mathcal{A}_{\mathrm{dff}}\big) \quad \begin{array}{l}\text{differential cocycles}\\ /\!\!/\text{gauge symmetries}\end{array}$$

$$\text{its shape:} \quad \textstyle\int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d, \mathcal{A}\big) \quad \begin{array}{l}\text{black brane charges}\\ /\!\!/\text{global symmetries}\end{array}$$

$$\text{solitonic:} \quad \textstyle\int \underset{\mathrm{sltnc}}{\mathrm{Phs}}\big(X^d; \mathcal{A}\big) \;\simeq\; \mathrm{Map}^\ast\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big) \quad \begin{array}{l}\text{all brane charges}\\ /\!\!/\text{global symmetries}\end{array}$$

Observables on IR-Complete Fields

$$\text{observables:} \left\{ \begin{array}{rl} \text{general:} & \begin{tikzcd} \mathrm{Phs}\big(X^d; \mathcal{A}\big) \arrow[rr, dashed, "\mathcal{O}"] \arrow[dr, "\eta^{\int}"'] & & \mathbb{C} \\ & \int \mathrm{Phs}\big(X^d; \mathcal{A}\big) \arrow[ur, dashed, "\mathcal{O}_{\mathrm{top}}"'] & \end{tikzcd} \\ \text{topological:} & \\ \text{realistic:} & \text{compactly supported} \end{array} \right.$$

IR-Completed Phase Space

phase space: $\mathrm{Phs}(X^d; \mathcal{A}) \equiv \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$ differential cocycles $/\!\!/$gauge symmetries

its shape: $\int \mathrm{Phs}(X^d; \mathcal{A}) \simeq \mathrm{Map}(X^d, \mathcal{A}\;\;)$ black brane charges $/\!\!/$global symmetries

solitonic: $\int \mathrm{Phs}_{\mathrm{sltnc}}(X^d; \mathcal{A}) \simeq \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$ all brane charges $/\!\!/$global symmetries

Observables on IR-Complete Fields

observables: $\left\{ \begin{array}{rl} \text{general:} & \mathrm{Phs}(X^d; \mathcal{A}) \xrightarrow{\;\mathcal{O}\;} \mathbb{C} \\ & \quad \searrow^{\eta^{\int}} \qquad\qquad \nearrow_{\mathcal{O}_{\mathrm{top}}} \\ \text{topological:} & \qquad\qquad \int \mathrm{Phs}(X^d; \mathcal{A}) \\ \text{realistic:} & \text{compactly supported} \end{array} \right.$

$\Rightarrow$ *realistic topological solitonic observables* are homology of shape of solitonic phase space:

IR-Completed Phase Space

phase space:	$\mathrm{Phs}(X^d; \mathcal{A}) \;\equiv\; \mathrm{Map}(X^d; \mathcal{A}_{\mathrm{dff}})$	differential cocycles //gauge symmetries
its shape:	$\int \mathrm{Phs}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}(X^d, \mathcal{A}\;\;)$	black brane charges //global symmetries
solitonic:	$\int \underset{\mathrm{sltnc}}{\mathrm{Phs}}(X^d; \mathcal{A}) \;\simeq\; \mathrm{Map}^*\big(X^d_{\cup\{\infty\}}, \mathcal{A}\big)$	all brane charges //global symmetries

Observables on IR-Complete Fields

observables:

- general: $\mathrm{Phs}(X^d; \mathcal{A}) \overset{\mathcal{O}}{\dashrightarrow} \mathbb{C}$
- topological: $\mathrm{Phs}(X^d; \mathcal{A}) \xrightarrow{\eta^{\int}} \int \mathrm{Phs}(X^d; \mathcal{A}) \overset{\mathcal{O}_{\mathrm{top}}}{\dashrightarrow} \mathbb{C}$
- realistic: compactly supported

$\Rightarrow$ *realistic topological solitonic observables* are homology of shape of solitonic phase space:

$$\boxed{\mathrm{TopObs}_\bullet(X^d; \mathcal{A}) \;\simeq\; H_\bullet\Big(\mathrm{Map}^*\big(X^d_{\cup\{\infty\}}; \mathcal{A}\big), \mathbb{C}\Big)}$$

IR-Completed Phase Space

$$\text{phase space:}\quad \mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\equiv\; \mathrm{Map}\big(X^d;\mathcal{A}_{\mathrm{dff}}\big)$$

differential cocycles
$/\!\!/$gauge symmetries

$$\text{its shape:}\quad \textstyle\int\mathrm{Phs}\big(X^d;\mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(X^d,\mathcal{A}\;\;\big)$$

black brane charges
$/\!\!/$global symmetries

$$\text{solitonic:}\quad \textstyle\int\underset{\mathrm{sltnc}}{\mathrm{Phs}}\big(X^d;\mathcal{A}\big) \;\simeq\; \mathrm{Map}^\ast\Big(X^d_{\cup\{\infty\}},\mathcal{A}\Big)$$

all brane charges
$/\!\!/$global symmetries

Observables on IR-Complete Fields

observables:

general: $\mathrm{Phs}\big(X^d;\mathcal{A}\big) \xrightarrow{\;\mathcal{O}\;} \mathbb{C}$

topological:

$$\begin{tikzcd} \mathrm{Phs}\big(X^d;\mathcal{A}\big) \arrow[rr, dashed, "\mathcal{O}"] \arrow[dr, "\eta^{\int}"'] & & \mathbb{C} \\ & \int\mathrm{Phs}\big(X^d;\mathcal{A}\big) \arrow[ur, dashed, "\mathcal{O}_{\mathrm{top}}"'] & \end{tikzcd}$$

realistic: compactly supported

$\Rightarrow$ *realistic topological solitonic observables* are homology of shape of solitonic phase space:

$$\boxed{\mathrm{TopObs}_\bullet\big(X^d;\mathcal{A}\big) \;\simeq\; H_\bullet\Big(\mathrm{Map}^\ast\Big(X^d_{\cup\{\infty\}};\mathcal{A}\Big),\mathbb{C}\Big)}$$

Topological Light-Cone Quantization

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

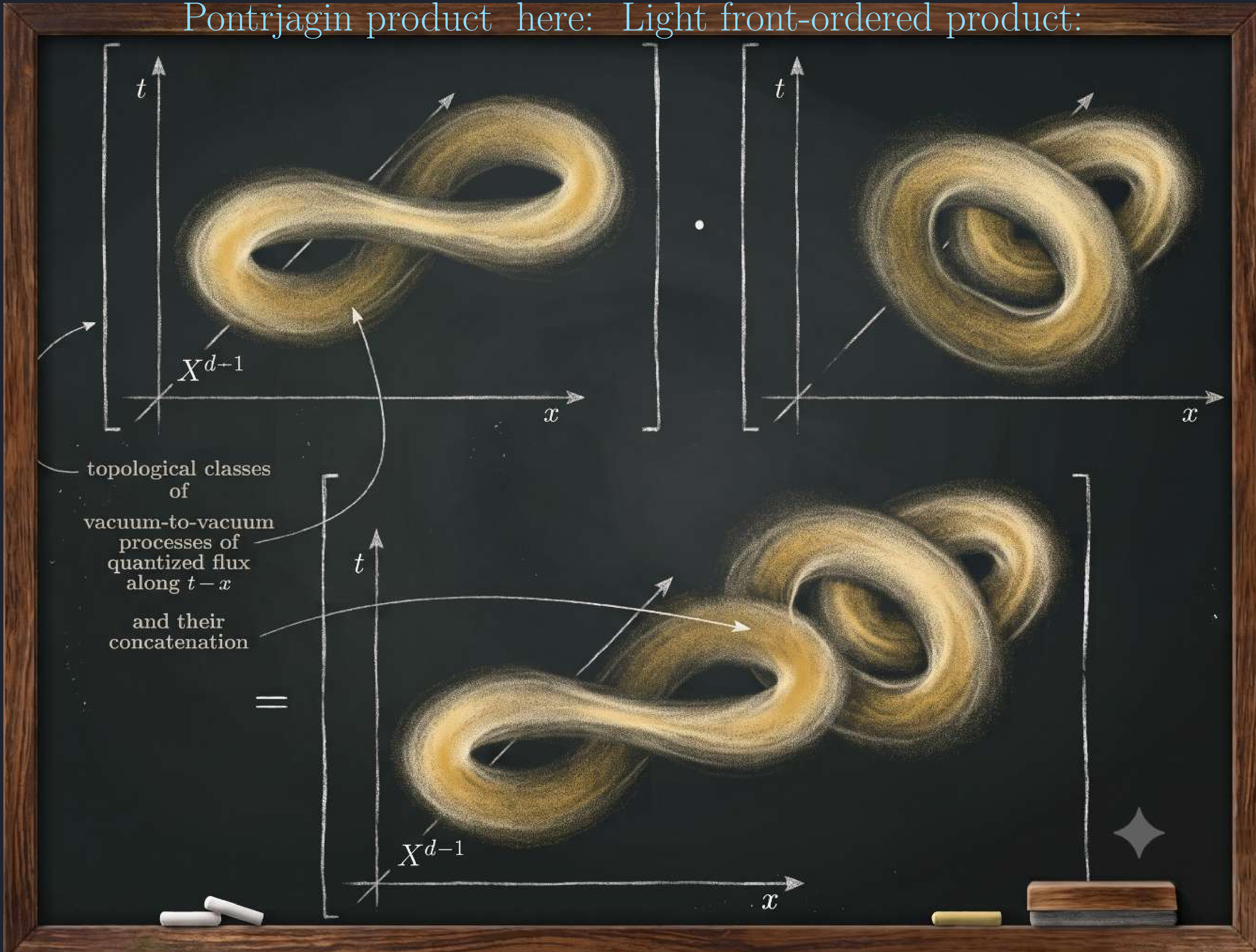
Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned}\mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big]\end{aligned}}$$

Pontrjagin product here: Light front-ordered product:



Topological Light-Cone Quantization

For spacetimes of the form

$$
\begin{aligned}
X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\
X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1}
\end{aligned}
$$

observables are Pontrjagin $*$-algebra

$$
\boxed{
\begin{aligned}
\mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\
&\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big]
\end{aligned}
}
$$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1} \times \Sigma^2$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1} \times \Sigma^2$

Lie-Poisson algebra of fluxes: $\underbrace{C^\infty\big(\Sigma^2, \mathfrak{g}_\hbar\big)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty\big(\Sigma^2, \mathfrak{g}_0\big)}_{\text{magnetic}}$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1} \times \Sigma^2$

Lie-Poisson algebra of fluxes: $\underbrace{C^\infty\big(\Sigma^2, \mathfrak{g}_\hbar\big)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty\big(\Sigma^2, \mathfrak{g}_0\big)}_{\text{magnetic}}$

nonperturb. quantization: $\mathbb{C}\Big[\underbrace{C^\infty\big(\Sigma^2, G\big)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty\big(\Sigma^2, \mathfrak{g}_0/\Lambda\big)}_{\text{magnetic}}\Big]$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned}\mathrm{TopObs}_0(X^d; \mathcal{A}) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big]\end{aligned}}$$

Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1} \times \Sigma^2$

Lie-Poisson algebra of fluxes: $\underbrace{C^\infty(\Sigma^2, \mathfrak{g}_\hbar)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2, \mathfrak{g}_0)}_{\text{magnetic}}$

nonperturb. quantization: $\mathbb{C}\Big[\underbrace{C^\infty(\Sigma^2, G)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2, \mathfrak{g}_0/\Lambda)}_{\text{magnetic}}\Big]$

generally elusive, works here
due to Lie-Poisson structure

topological sector: $\mathbb{C}[\pi_0(-)] \simeq \mathbb{C}\Big[\pi_1\, \mathrm{Map}\big(\Sigma^2, \underbrace{BG}_{\text{el}} \times \underbrace{B^2\Lambda}_{\text{mag}}\big)\Big]$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned}\mathrm{TopObs}_0(X^d;\mathcal{A}) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}},\mathcal{A}\big);\mathbb{C}\Big)\\ &\simeq \mathbb{C}\Big[\pi_1\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}},\mathcal{A}\big)\Big]\end{aligned}}$$

Example/Theorem [arXiv:2312.13037]: $\mathfrak{g}$-Yang-Mills on $\mathbb{R}^{1,1}\times\Sigma^2$

Lie-Poisson algebra of fluxes: $\underbrace{C^\infty(\Sigma^2,\mathfrak{g}_\hbar)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2,\mathfrak{g}_0)}_{\text{magnetic}}$

nonperturb. quantization: $\mathbb{C}\Big[\underbrace{C^\infty(\Sigma^2,G)}_{\text{electric}} \rtimes_{\mathrm{ad}} \underbrace{C^\infty(\Sigma^2,\mathfrak{g}_0/\Lambda)}_{\text{magnetic}}\Big]$

generally elusive, works here
due to Lie-Poisson structure

topological sector: $\mathbb{C}\big[\pi_0(-)\big] \simeq \mathbb{C}\Big[\pi_1\,\mathrm{Map}\big(\Sigma^2,\underbrace{BG}_{\text{el}}\times\underbrace{B^2\Lambda}_{\text{mag}}\big)\Big]$

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Light-Cone Quantization

For spacetimes of the form
$$\begin{aligned} X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\ X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1} \end{aligned}$$

observables are Pontrjagin $*$-algebra

$$\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}$$

Topological Quantum States

Topological Light-Cone Quantization

For spacetimes of the form
$$\begin{aligned} X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\ X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1} \end{aligned}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Quantum States

quantum states form Obs-modules

Topological Light-Cone Quantization

For spacetimes of the form
$$\begin{aligned} X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\ X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1} \end{aligned}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Quantum States

quantum states form Obs-modules

here: $\pi_1(-)$-representations!

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d;\mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}},\mathcal{A}\big);\mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}},\mathcal{A}\big)\Big] \end{aligned}}$$

Topological Quantum States

quantum states form Obs-modules

here: $\pi_1(-)$-representations!

hence local systems on the mapping space:

Topological Light-Cone Quantization

For spacetimes of the form
$$X^{1,d} \simeq \mathbb{R}^{1,1} \times X^{d-1}$$
$$X^{d} \simeq \mathbb{R}^{1} \times X^{d-1}$$

observables are Pontrjagin $*$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) & \simeq H_0\Big(\Omega\, \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ & \simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Quantum States

quantum states form Obs-modules

here: $\pi_1(-)$-representations!

hence local systems on the mapping space:

$$\boxed{\mathcal{H} \in \mathrm{TopObs}\,\mathrm{Mod} \subset \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)}$$

Topological Light-Cone Quantization

For spacetimes of the form
$$\begin{aligned} X^{1,d} &\simeq \mathbb{R}^{1,1} \times X^{d-1} \\ X^{d} &\simeq \mathbb{R}^{1} \times X^{d-1} \end{aligned}$$

observables are Pontrjagin $\ast$-algebra

$$\boxed{\begin{aligned} \mathrm{TopObs}_0\big(X^d; \mathcal{A}\big) &\simeq H_0\Big(\Omega\,\mathrm{Map}^\ast\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq \mathbb{C}\Big[\pi_1 \mathrm{Map}^\ast\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big] \end{aligned}}$$

Topological Quantum States

quantum states form Obs-modules

here: $\pi_1(-)$-representations!

hence local systems on the mapping space:

$$\boxed{\mathcal{H} \in \mathrm{TopObs}\,\mathrm{Mod} \subset \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^\ast\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)}$$

Modular Functor

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\Big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\Big)\Big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

here: boundary theory $\left\{ \begin{array}{ccccc} & & \mathrm{Mod}_{\mathbb{C}} & \longrightarrow & \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\Big) \\ X^{d-1} & \vdash & \mathbb{1} & \mapsto & \mathcal{H}\big(X^{d-1}\big) \end{array} \right.$

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

here: boundary theory $\left\{ \begin{array}{ccccc} & & \mathrm{Mod}_{\mathbb{C}} & \longrightarrow & \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\Big) \\ X^{d-1} & \vdash & \mathbb{1} & \mapsto & \mathcal{H}\big(X^{d-1}\big) \end{array} \right.$

$\Rightarrow$ over X^{d-1} these boundary state spaces are:

$$\mathcal{H} \in \varprojlim_{\substack{X^{d-1} \in \\ B\mathrm{Diff}(X^{d-1})}} \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)$$

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

here: boundary theory $\begin{cases} \qquad\qquad \mathrm{Mod}_{\mathbb{C}} \longrightarrow \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\big) \\ X^{d-1} \;\vdash\; \mathbb{1} \;\mapsto\; \mathcal{H}\big(X^{d-1}\big) \end{cases}$

$\Rightarrow$ over X^{d-1} these boundary state spaces are:

$$\mathcal{H} \in \varprojlim_{X^{d-1} \in B\mathrm{Diff}(X^{d-1})} \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)$$

$$\simeq \quad \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}, \mathcal{A}\big) /\!\!/ \mathrm{Diff}\big(X^{d-1}\big)\Big)$$

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

here: boundary theory $\begin{cases} \qquad\qquad \mathrm{Mod}_{\mathbb{C}} \longrightarrow \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\Big) \\ X^{d-1} \;\vdash\; \mathbb{1} \;\mapsto\; \mathcal{H}(X^{d-1}) \end{cases}$

$\Rightarrow$ over X^{d-1} these boundary state spaces are:

$$\mathcal{H} \in \underset{\substack{X^{d-1} \in \\ B\mathrm{Diff}(X^{d-1})}}{\underleftarrow{\lim}} \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\Big)$$

$$\simeq \quad \mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}^*\big(X^{d-1}, \mathcal{A}\big) /\!\!/ \mathrm{Diff}\big(X^{d-1}\big)\Big)$$

"general covariance"

Modular Functor

Stefanich 2025 suggests: $\begin{cases} X^{d-1} \mapsto \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\big) \\ \text{extends to an extended } \mathrm{TQFT}_d \end{cases}$

here: boundary theory $\left\{ \begin{array}{ccccc} & & \mathrm{Mod}_{\mathbb{C}} & \longrightarrow & \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big((-)_{\cup\{\infty\}}; \mathcal{A}\big)\big) \\ X^{d-1} & \vdash & \mathbb{1} & \mapsto & \mathcal{H}(X^{d-1}) \end{array} \right.$

$\Rightarrow$ over X^{d-1} these boundary state spaces are:

$$\mathcal{H} \in \varprojlim_{X^{d-1} \in BDiff(X^{d-1})} \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big(X^{d-1}_{\cup\{\infty\}}, \mathcal{A}\big)\big)$$

this relates to actual experiment! cf. below

$$\simeq \mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}^*\big(X^{d-1}, \mathcal{A}\big) /\!\!/ \mathrm{Diff}\big(X^{d-1}\big)\big)$$

"general covariance"

IR-Completion with Sources & Branes

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \big\downarrow\!\!\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty \mathrm{Alg}_{\mathbb{R}})$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \big\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty \mathrm{Alg}_{\mathbb{R}})$

Examples

theory	Gauß law	characteristic

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\mathrm{d}\, J_3^{\mathrm{mg}} = 0$ $\mathrm{d}\, J_3^{\mathrm{el}} = 0$	$X^3_{\cup\{\infty\}} \xrightarrow{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})} \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}BB^2\mathbb{Z}^2)$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\mathrm{d}\, J_3^{\mathrm{mg}} = 0$ $\mathrm{d}\, J_3^{\mathrm{el}} = 0$	$X^3_{\cup\{\infty\}} \xrightarrow{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})} \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l} B B^2 \mathbb{Z}^2)$

electron number = current cohomology
with compact spatial support!

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \big\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	F_2 E_2 $\mathrm{d}\, J_3^{\mathrm{mg}} = 0$ $\mathrm{d}\, J_3^{\mathrm{el}} = 0$	$\begin{array}{c} X^3 \\ \big\downarrow{\scriptstyle \iota} \\ X^3_{\cup\{\infty\}} \xrightarrow{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})} \Omega^1_{\mathrm{cl}}(\cdot;\, \mathfrak{l}BB^2\mathbb{Z}^2) \end{array}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\mathrm{d}\,F_2 = \iota^* J_3^{\mathrm{mg}}$ $\mathrm{d}\,E_2 = \iota^* J_3^{\mathrm{el}}$ $\mathrm{d}\,J_3^{\mathrm{mg}} = 0$ $\mathrm{d}\,J_3^{\mathrm{el}} = 0$	$\begin{array}{ccc} X^3 & \xrightarrow{(F_2, E_2)} & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}EB^2\mathbb{Z}^2) \\ \big\downarrow \iota & & \big\downarrow \\ X^3_{\cup\{\infty\}} & \xrightarrow{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})} & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}BB^2\mathbb{Z}^2) \end{array}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \big\downarrow \\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\begin{aligned}\mathrm{d}\,F_2 &= \iota^* J_3^{\mathrm{mg}}\\ \mathrm{d}\,E_2 &= \iota^* J_3^{\mathrm{el}}\\ \mathrm{d}\,J_3^{\mathrm{mg}} &= 0\\ \mathrm{d}\,J_3^{\mathrm{el}} &= 0\end{aligned}$	$\begin{tikzcd} X^3 \arrow[r, "{(F_2, E_2)}"] \arrow[d, hook, "\iota"] & \Omega^1_{\mathrm{cl}}(\cdot;\, \mathfrak{l}EB^2\mathbb{Z}^2) \arrow[d, two heads] \\ X^3_{\cup\{\infty\}} \arrow[r, "{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})}"] & \Omega^1_{\mathrm{cl}}(\cdot;\, \mathfrak{l}BB^2\mathbb{Z}^2) \end{tikzcd}$

M5-probe
of 11D SUGra

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration} \quad \begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}\big(L_\infty \mathrm{Alg}_{\mathbb{R}}\big)$$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\mathrm{d}\,F_2 = \iota^* J_3^{\mathrm{mg}}$ $\mathrm{d}\,E_2 = \iota^* J_3^{\mathrm{el}}$ $\mathrm{d}\,J_3^{\mathrm{mg}} = 0$ $\mathrm{d}\,J_3^{\mathrm{el}} = 0$	$\begin{tikzcd} X^3 \arrow[r, "{(F_2, E_2)}"] \arrow[d, hook, "\iota"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}EB^2\mathbb{Z}^2) \arrow[d, two heads] \\ X^3_{\cup\{\infty\}} \arrow[r, "{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})}"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}BB^2\mathbb{Z}^2) \end{tikzcd}$
M5-probe of 11D SUGra	$\mathrm{d}\,G_4 = 0$ $\mathrm{d}\,G_7 = \frac{1}{2} G_4 G_4$	$X^{10} \xrightarrow{(G_4, G_7)} \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}S^4)$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\begin{aligned} \mathrm{d}\,F_2 &= \iota^* J_3^{\mathrm{mg}} \\ \mathrm{d}\,E_2 &= \iota^* J_3^{\mathrm{el}} \\ \mathrm{d}\,J_3^{\mathrm{mg}} &= 0 \\ \mathrm{d}\,J_3^{\mathrm{el}} &= 0 \end{aligned}$	$\begin{array}{ccc} X^3 & \xrightarrow{(F_2,E_2)} & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}EB^2\mathbb{Z}^2) \\ \Big\downarrow{\scriptstyle\iota} & & \Big\downarrow \\ X^3_{\cup\{\infty\}} & \xrightarrow{(J_3^{\mathrm{mg}},J_3^{\mathrm{el}})} & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}BB^2\mathbb{Z}^2) \end{array}$
M5-probe of 11D SUGra	$\begin{aligned} \mathrm{d}\,G_4 &= 0 \\ \mathrm{d}\,G_7 &= \tfrac{1}{2}G_4G_4 \end{aligned}$	$\begin{array}{ccc} \Sigma^5 & & \\ \Big\downarrow{\scriptstyle\phi} & & \\ X^{10} & \xrightarrow{(G_4,G_7)} & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}S^4) \end{array}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \big\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$$

Examples

M5-probe of 11D SUGra

$$\begin{aligned} \mathrm{d}\, G_4 &= 0 \\ \mathrm{d}\, G_7 &= \tfrac{1}{2} G_4 G_4 \end{aligned}$$

$$\begin{array}{c} \Sigma^5 \\ \big\downarrow \phi \\ X^{10} \xrightarrow{(G_4, G_7)} \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l} S^4) \end{array}$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

Examples

M5-probe of 11D SUGra

$$\begin{aligned} \mathrm{d}\, H_3 &= \phi^* G_4 \\ \mathrm{d}\, G_4 &= 0 \\ \mathrm{d}\, G_7 &= \tfrac{1}{2} G_4 G_4 \end{aligned}$$

$$\begin{tikzcd} \Sigma^5 \arrow[r, "H_3"] \arrow[d, hook, "\phi"] & \Omega^1_{\mathrm{cl}}(\cdot;\, \mathfrak{l}_{S^4} S^7) \arrow[d, "\mathfrak{l}h_{\mathbb{H}}"] \\ X^{10} \arrow[r, "(G_4, G_7)"] & \Omega^1_{\mathrm{cl}}(\cdot;\, \mathfrak{l}S^4) \end{tikzcd}$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty \mathrm{Alg}_{\mathbb{R}})$$

Examples

theory	Gauß law	characteristic
4D Maxwell with sources	$\begin{aligned} \mathrm{d}\, F_2 &= \iota^* J_3^{\mathrm{mg}} \\ \mathrm{d}\, E_2 &= \iota^* J_3^{\mathrm{el}} \\ \mathrm{d}\, J_3^{\mathrm{mg}} &= 0 \\ \mathrm{d}\, J_3^{\mathrm{el}} &= 0 \end{aligned}$	$\begin{tikzcd} X^3 \arrow[r, "{(F_2, E_2)}"] \arrow[d, hook, "\iota"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}EB^2\mathbb{Z}^2) \arrow[d] \\ X^3_{\cup\{\infty\}} \arrow[r, "{(J_3^{\mathrm{mg}}, J_3^{\mathrm{el}})}"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}BB^2\mathbb{Z}^2) \end{tikzcd}$
M5-probe of 11D SUGra	$\begin{aligned} \mathrm{d}\, H_3 &= \phi^* G_4 \\ \mathrm{d}\, G_4 &= 0 \\ \mathrm{d}\, G_7 &= \tfrac{1}{2} G_4 G_4 \end{aligned}$	$\begin{tikzcd} \Sigma^5 \arrow[r, "H_3"] \arrow[d, hook, "\phi"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}_{S^4} S^7) \arrow[d, "\mathfrak{l}h_{\mathbb{H}}"] \\ X^{10} \arrow[r, "{(G_4, G_7)}"] & \Omega^1_{\mathrm{cl}}(\cdot; \mathfrak{l}S^4) \end{tikzcd}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration } \begin{array}{c}\mathfrak{a}\\ \big\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty \mathrm{Alg}_{\mathbb{R}})$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

classifying space now becomes a classifying fibration $\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle \wp}\\ \mathcal{B}\end{array}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

$$\text{a } \textit{characteristic fibration} \quad \begin{array}{c} \mathfrak{a} \\ \downarrow \\ \mathfrak{b} \end{array} \;\in\; \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$$

$$\text{classifying space now becomes a classifying fibration} \quad \begin{array}{c} \mathcal{A} \\ \downarrow{\scriptstyle \wp} \\ \mathcal{B} \end{array}$$

$$\text{subject to} \qquad \mathfrak{l}\left(\begin{array}{c} \mathcal{A} \\ \downarrow{\scriptstyle \wp} \\ \mathcal{B} \end{array} \right) \;\simeq\; \begin{array}{c} \mathfrak{a} \\ \downarrow \\ \mathfrak{b} \end{array}$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\!\downarrow \\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

classifying space now becomes a classifying fibration $\begin{array}{c}\mathcal{A}\\ \downarrow{\scriptstyle \wp} \\ \mathcal{B}\end{array}$

subject to $\quad \mathfrak{l}\left(\begin{array}{c}\mathcal{A}\\ \downarrow{\scriptstyle \wp} \\ \mathcal{B}\end{array}\right) \simeq \begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\!\downarrow \\ \mathfrak{b}\end{array}$

meaning that its Sullivan relative minimal model is $\mathrm{CE}\left(\begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\!\downarrow \\ \mathfrak{b}\end{array}\right)$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

classifying space now becomes a classifying fibration $\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle \wp}\\ \mathcal{B}\end{array}$

subject to $\quad \mathfrak{l}\left(\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle \wp}\\ \mathcal{B}\end{array}\right) \simeq \begin{array}{c}\mathfrak{a}\\ \downarrow\\ \mathfrak{b}\end{array}$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\!\downarrow\\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

classifying space now becomes a classifying fibration $\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle\wp}\\ \mathcal{B}\end{array}$

subject to $\mathfrak{l}\left(\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle\wp}\\ \mathcal{B}\end{array}\right) \simeq \begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\!\downarrow\\ \mathfrak{b}\end{array}$

& charges are in *twisted relative* nonabelian cohomology:

$$[\chi] \in \tilde{H}^0(\Phi_{\cup\{\infty\}};\wp) \equiv \pi_0\left\{\begin{tikzcd} \Sigma_{\cup\{\infty\}} \arrow[r, dashed] \arrow[d, "\Phi_{\cup\{\infty\}}"] & \mathcal{A} \arrow[d, "\wp"] \\ X_{\cup\{\infty\}} \arrow[r, dashed] & \mathcal{B} \end{tikzcd}\right\}$$

IR-Completion with Sources & Branes

higher Maxwell-type fluxes with sources have

a *characteristic fibration* $\begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\downarrow \\ \mathfrak{b}\end{array} \in \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}})$

classifying space now becomes a classifying fibration $\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle \wp} \\ \mathcal{B}\end{array}$

subject to $\quad \mathfrak{l}\left(\begin{array}{c}\mathcal{A}\\ \big\downarrow{\scriptstyle \wp} \\ \mathcal{B}\end{array}\right) \simeq \begin{array}{c}\mathfrak{a}\\ \downarrow\!\!\downarrow \\ \mathfrak{b}\end{array}$

& charges are in *twisted relative* nonabelian cohomology:

$$[\chi] \in \tilde{H}^0(\Phi_{\cup\{\infty\}};\wp) \equiv \pi_0\left\{\begin{tikzcd} \Sigma_{\cup\{\infty\}} \arrow[r, dashed] \arrow[d, "\Phi_{\cup\{\infty\}}"] & \mathcal{A} \arrow[d, "\wp"] \\ X_{\cup\{\infty\}} \arrow[r, dashed] & \mathcal{B} \end{tikzcd}\right\}$$

IR-Completion on Orbifolds

IR-Completion on Orbifolds

moreover, higher Maxwell-type fluxes on G-orbifolds

have an *equivariant* characteristic fibration

IR-Completion on Orbifolds

moreover, higher Maxwell-type fluxes on G-orbifolds
have an *equivariant* characteristic fibration

$$
\begin{array}{ccc}
\mathrm{Orb}(G)^{\mathrm{op}} & \longrightarrow & \mathrm{Fib}(L_\infty\mathrm{Alg}_{\mathbb{R}}) \\
 & & \underline{\mathfrak{a}}_H \\
G/H & \mapsto & \big\downarrow \\
 & & \underline{\mathfrak{b}}_H
\end{array}
$$

IR-Completion on Orbifolds

moreover, higher Maxwell-type fluxes on G-orbifolds have an *equivariant* characteristic fibration

$$\mathrm{Orb}(G)^{\mathrm{op}} \longrightarrow \mathrm{Fib}\big(L_\infty \mathrm{Alg}_{\mathbb{R}}\big)$$

$$G/H \quad \mapsto \quad \begin{array}{c} \underline{\mathfrak{a}}_H \\ \big\downarrow \\ \underline{\mathfrak{b}}_H \end{array}$$

& charges in *equivariant* nonabelian cohomology:

$$[\chi] \in \tilde{H}^0_G(\Phi_{\cup\{\infty\}}; \wp) \equiv \pi_0 \left\{ \begin{tikzcd} G \circlearrowright \Sigma_{\cup\{\infty\}} \arrow[r, dashed] \arrow[d, "\Phi_{\cup\{\infty\}}"] & G \circlearrowright \mathcal{A} \arrow[d, "\wp"] \\ G \circlearrowright X_{\cup\{\infty\}} \arrow[r, dashed] & G \circlearrowright \mathcal{B} \end{tikzcd} \right\}$$

IR-Completion on Orbifolds

moreover, higher Maxwell-type fluxes on G-orbifolds have an *equivariant* characteristic fibration

$$\mathrm{Orb}(G)^{\mathrm{op}} \longrightarrow \mathrm{Fib}\big(L_\infty \mathrm{Alg}_{\mathbb{R}}\big)$$

$$G/H \quad \mapsto \quad \begin{array}{c} \underline{\mathfrak{a}}_H \\ \big\downarrow \\ \underline{\mathfrak{b}}_H \end{array}$$

& charges in *equivariant* nonabelian cohomology:

$$[\chi] \;\in\; \tilde{H}^0_G(\Phi_{\cup\{\infty\}};\wp) \;\equiv\; \pi_0 \left\{ \begin{array}{ccc} G \circlearrowright \Sigma_{\cup\{\infty\}} & \dashrightarrow & G \circlearrowright \mathcal{A} \\ \big\downarrow{\scriptstyle \Phi_{\cup\{\infty\}}} & & \big\downarrow{\scriptstyle \wp} \\ G \circlearrowright X_{\cup\{\infty\}} & \dashrightarrow & G \circlearrowright \mathcal{B} \end{array} \right\}$$

IR-Complete C-Field & M5 Model

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\
\int S^4
\end{tikzcd}
$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{array}{c}
\mathfrak{l}_{S^4} S^7 \\
\big\downarrow \mathfrak{l} h_{\mathbb{H}} \\
\mathfrak{l} S^4
\end{array}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{array}{c}
S^7 \\
\big\downarrow h_{\mathbb{H}} \\
S^4
\end{array}
$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \Big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \Big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \text{ʃ}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \text{ʃ} h_{\mathbb{H}}} \\ \text{ʃ} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- *minimal* completion in number of cells

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow \mathfrak{l} h_{\mathbb{H}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- *minimal* completion in number of cells

- implies that M5's Hopf-WZ term is globally defined, like Dirac CQ implied electron WZ term well defined

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \wr_{S^4} S^7 \\ \big\downarrow \wr h_{\mathbb{H}} \\ \wr S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- *minimal* completion in number of cells
- implies that M5's Hopf-WZ term is globally defined, like Dirac CQ implied electron WZ term well defined
- *Hypothesis H*: "This completion corresponds to M-theory."

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{array}{c}
\textstyle\int_{S^4} S^7 \\
\big\downarrow \textstyle\int h_{\mathbb{H}} \\
\textstyle\int S^4
\end{array}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{array}{c}
S^7 \\
\big\downarrow h_{\mathbb{H}} \\
S^4
\end{array}
$$

- *minimal* completion in number of cells

- implies that M5's Hopf-WZ term is globally defined, like Dirac CQ implied electron WZ term well defined

- *Hypothesis H*: "This completion corresponds to M-theory."

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- *minimal* completion in number of cells
- implies that M5's Hopf-WZ term is globally defined, like Dirac CQ implied electron WZ term well defined
- *Hypothesis H*: "This completion corresponds to M-theory."
- entails that M-brane charges are in *CoHomotopy*
- but variants may be possible

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\ ʃ S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

- *minimal* completion in number of cells

- implies that M5's Hopf-WZ term is globally defined, like Dirac CQ implied electron WZ term well defined

- *Hypothesis H*: "This completion corresponds to M-theory."

- entails that M-brane charges are in *CoHomotopy*

- but variants may be possible

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow{\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by: $\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration $\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\ \int S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\ ʃ S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow \mathfrak{l} h_{\mathbb{H}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow{\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\left.\begin{array}{r} H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; S^4) \\ \simeq \pi_{9-p}(S^4) \end{array}\right\} \simeq$$

$p =$	0	1	2	3	4	5	6	7	8	9	
	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}_{/12}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \Big\downarrow \int h_{\mathbb{H}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \Big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

near horizon of a p-brane

	$p =$	0	1	2	3	4	5	6	7	8	9	
$H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; S^4)$	$\simeq$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
		$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
$\simeq \pi_{9-p}(S^4)$		$\mathbb{Z}^2_{/2}$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}_{/12}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{array}{c}
ʃ_{S^4} S^7 \\
\big\downarrow ʃ h_{\mathbb{H}} \\
ʃ S^4
\end{array}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{array}{c}
S^7 \\
\big\downarrow h_{\mathbb{H}} \\
S^4
\end{array}
$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

M2-brane

$p =$	0	1	2	3	4	5	6	7	8	9	
$H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; S^4) \simeq \pi_{9-p}(S^4) \simeq$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}_{/12}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\
\int S^4
\end{tikzcd}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{tikzcd}
S^7 \arrow[d, "h_{\mathbb{H}}"] \\
S^4
\end{tikzcd}
$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

M5-brane

$p =$	0	1	2	3	4	5	6	7	8	9	
$H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; S^4)$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
$\simeq$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
$\simeq \pi_{9-p}(S^4)$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}_{/12}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \wr_{S^4} S^7 \\ \big\downarrow \wr h_{\mathbb{H}} \\ \wr S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$p =$	0	1	2	3	4	5	6	7	8	9	
$H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; S^4)$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
$\simeq \pi_{9-p}(S^4)$ $\Big\} \simeq$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}^2_{/2}$	$\mathbb{Z}_{/12}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow \mathfrak{l} h_{\mathbb{H}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\big\downarrow$$

hence stably $\mathbb{S}^4(X^{10})$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} ʃ_{S^4} S^7 \\ \big\downarrow {\scriptstyle ʃ h_{\mathbb{H}}} \\ ʃ S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\big\downarrow$$

hence stably $\mathbb{S}^4(X^{10})$

$p =$	0	1	2	3	4	5	6	7	8	9	
$H^0(\mathbb{R}^{1,10} - \mathbb{R}^{1,p}; \mathbb{S}^4) \simeq \pi_{9-p}(\mathbb{S}^4) \simeq$	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	(integral)
	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	
	0	0	$\mathbb{Z}_{/24}$	$\mathbb{Z}_{/2}$	$\mathbb{Z}_{/2}$	0	0	0	0	0	(fractional)

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow \mathfrak{l} h_{\mathbb{H}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\big\downarrow$$

hence stably $\mathbb{S}^4(X^{10})$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\ \int S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\begin{tikzcd} \pi^4(X^{10}) \arrow[d] \\ \mathbb{S}^4(X^{10}) \end{tikzcd}$$

hence stably $\mathbb{S}^4(X^{10})$

exactly in this (co)dimension
the sphere spectrum $\mathbb{S}$
is approximated by tmf:

$$\mathbb{S} \xrightarrow{\simeq_{\leq 6}} \mathrm{tmf}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\begin{array}{c} \pi^4(X^{10}) \\ \big\downarrow \\ \mathbb{S}^4(X^{10}) \\ \big\downarrow {\scriptstyle \wr} \\ \mathrm{tmf}^4(X^{10}) \end{array}$$

hence stably $\mathbb{S}^4(X^{10})$

in tmf $\mathrm{tmf}^4(X^{10})$

exactly in this (co)dimension the sphere spectrum $\mathbb{S}$ is approximated by tmf:

$$\mathbb{S} \xrightarrow{\simeq_{\leq 6}} \mathrm{tmf}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by: $\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow{\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration $\begin{array}{c} S^7 \\ \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

$$\begin{array}{rc} & \pi^4(X^{10}) \\ & \big\downarrow \\ \text{hence stably} & \mathbb{S}^4(X^{10}) \\ & \big\downarrow{\scriptstyle \wr} \\ \text{in tmf} & \mathrm{tmf}^4(X^{10}) \end{array}$$

IR-Complete C-Field & M5 Model

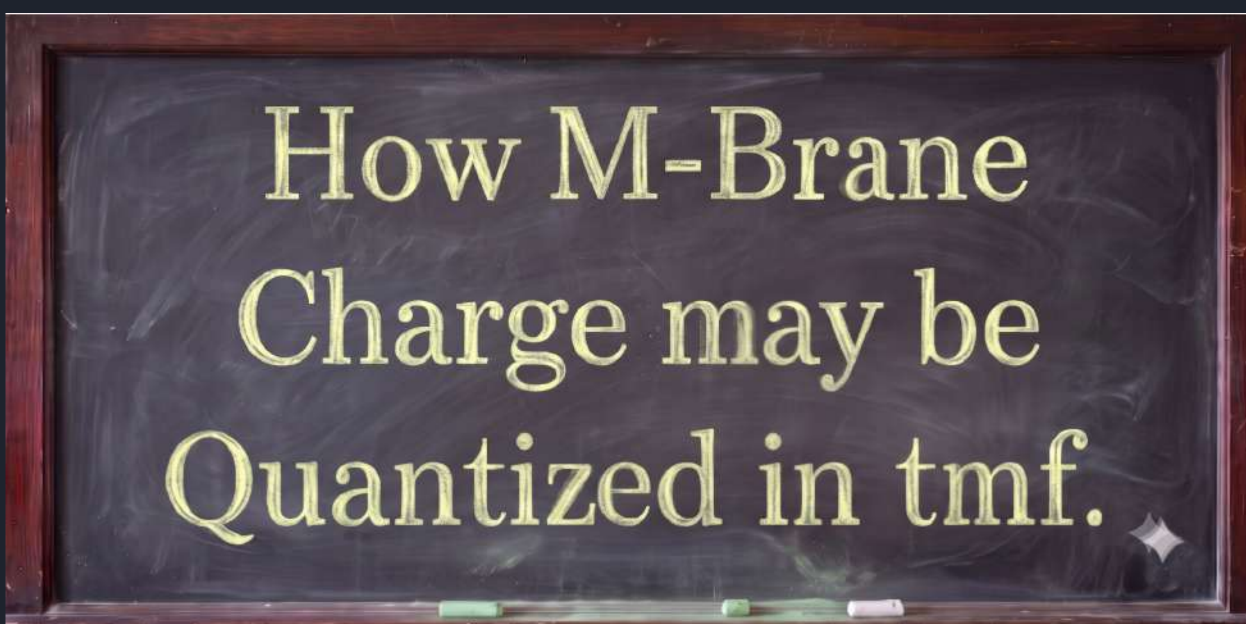
recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow{\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$



$$\begin{array}{lr} & \pi^4(X^{10}) \\ & \big\downarrow \\ \text{hence stably} & \mathbb{S}^4(X^{10}) \\ & \big\downarrow{\scriptstyle \wr} \\ \text{in tmf} & \mathrm{tmf}^4(X^{10}) \end{array}$$

IR-Complete C-Field & M5 Model

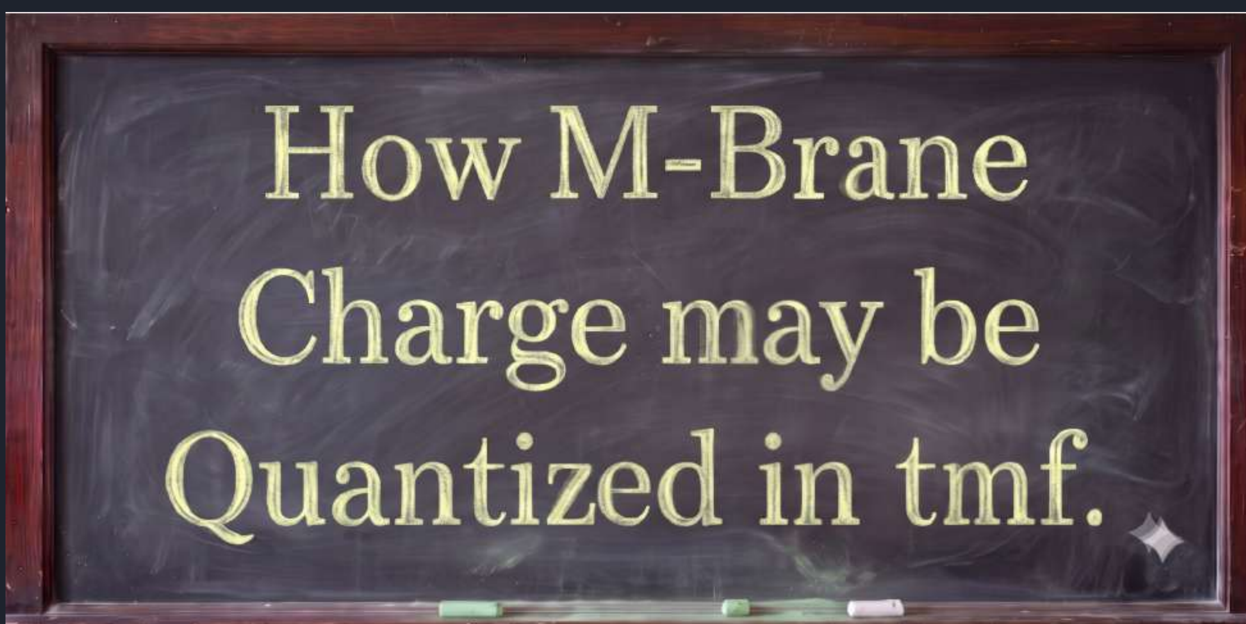
recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\
ʃ S^4
\end{tikzcd}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{tikzcd}
S^7 \arrow[d, "h_{\mathbb{H}}"] \\
S^4
\end{tikzcd}
$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$



and M-string charges in *twisted relative* CoHomotopy:

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \natural_{S^4} S^7 \\ \big\downarrow \natural h_{\mathbb{H}} \\ \natural S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy* $\pi^4(X^{10})$

and M-string charges in *twisted relative* CoHomotopy:

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

and M-string charges in *twisted relative* CoHomotopy:

$$\mathbb{R}^1 \times X^9 \dashrightarrow S^4$$

M2/M5 charges in π^4

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow \scriptstyle \int h_{\mathbb{H}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow \scriptstyle h_{\mathbb{H}} \\ S^4 \end{array}$$

- entails that M-brane charges are in *CoHomotopy*

and M-string charges in *twisted relative* CoHomotopy:

$$\underset{\text{M2/M5 charges}}{\mathbb{R}^1 \times X^9} \dashrightarrow \underset{\text{in } \pi^4}{S^4} \xrightarrow{\quad \Sigma^4 1^E \quad} \underset{\text{seen in stable } E\text{-cohomology}}{E_4}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \natural_{S^4} S^7 \\ \big\downarrow {\scriptstyle \natural h_{\mathbb{H}}} \\ \natural S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

$$\begin{tikzcd} \mathbb{R}^1 \times \Sigma^4 \arrow[d, hook, "\phi"'] & & \\ \mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[r, "\Sigma^4 1^E"] & E_4 \end{tikzcd}$$

M5-probe

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

M1-charges in twisted π^3

$$\begin{tikzcd}
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[d, hook, "\phi"', "\text{M5-probe}"'] & S^7 \arrow[d, "h_{\mathbb{H}}"] & \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[r, "\Sigma^4 1^E"] & E_4
\end{tikzcd}$$

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \natural_{S^4} S^7 \\ \Big\downarrow \natural h_{\mathbb{H}} \\ \natural S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \Big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

M1-charges in twisted π^3 measured in E-cohomology

$$\begin{tikzcd}
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[d, hook, "\phi"', "\text{M5-probe}"'] & S^7 \arrow[r] \arrow[d, "h_{\mathbb{H}}"] & {*} \arrow[d, "0"] \arrow[dl, Rightarrow, "h_3^E"'] \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[r, "\Sigma^4 1^E"] & E_4
\end{tikzcd}$$

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\ ʃ S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

M1-charges in twisted π^3 measured in E-cohomology

$$\begin{tikzcd}[column sep=large] \mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[dd, hook, "\phi"', "\text{M5-probe}"] & S^7 \arrow[rr] \arrow[dd, "h_{\mathbb{H}}"] \arrow[dr, phantom, "\lrcorner", very near end] & & * \arrow[dl] \arrow[dd, "0"] \\ & & \mathbb{H}P^2 \arrow[dr, "h_3^E"] & \\ \mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[ur, hook] \arrow[rr, "\Sigma^4 1^E"] & & E_4 \end{tikzcd}$$

is equivalently quaternionic E-orientation over 10D spaces

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\ ʃ S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

M1-charges in twisted π^3 measured in E-cohomology

$$\begin{tikzcd}[column sep=large]
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[dd, hook, "\phi"', "\text{M5-probe}"'] & S^7 \arrow[rr] \arrow[dd, "h_{\mathbb{H}}"] & & * \arrow[dl] \arrow[dd, "0"] \\
& & \mathbb{H}P^2 \arrow[dr, "h_3^E"] \arrow[ul, phantom, "\ulcorner"] & \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[ur, hook] \arrow[rr, "\Sigma^4 1^E"'] & & E_4
\end{tikzcd}$$

is equivalently quaternionic E-orientation over 10D spaces

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:
$$\begin{tikzcd} \int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\ \int S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration
$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

M1-charges in twisted π^3 measured in E-cohomology

$$\begin{tikzcd}
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[d, hook, "\phi"] & S^7 \arrow[rr] \arrow[d, "h_{\mathbb{H}}"] & & * \arrow[dl] \arrow[d, "0"] \\
 & & \mathbb{H}P^2 \arrow[dr, "h_3^E"] & \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[ur, hook] \arrow[rr, "\Sigma^4 1^E"] & & E_4
\end{tikzcd}$$

M5-probe

e.g. for E complex-oriented

M2/M5 charges in π^4 seen in stable E-cohomology

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \mathbf{\int}_{S^4} S^7 \arrow[d, "\mathbf{\int} h_{\mathbb{H}}"] \\ \mathbf{\int} S^4 \end{tikzcd}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{tikzcd} S^7 \arrow[d, "h_{\mathbb{H}}"] \\ S^4 \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[d, hook, "\phi"', "\text{M5-probe}"] & S^7 \arrow[rr] \arrow[d, "h_{\mathbb{H}}"] & & * \arrow[dl] \arrow[d, "0", "\text{e.g. for } E \text{ complex-oriented}"'] \\
 & & \mathbb{H}P^2 \arrow[dr, dashed, "h_3^E"] & \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[ur, hook] \arrow[rr, "\Sigma^4 1^E"] & & E_4
\end{tikzcd}$$

$\Rightarrow$ M5-branes on 4-manifold Σ^4 $\Big\}$ give classes in E^4 of spacetime relative worldvolume

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\
\int S^4
\end{tikzcd}
$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$
\begin{tikzcd}
S^7 \arrow[d, "h_{\mathbb{H}}"] \\
S^4
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{R}^1 \times \Sigma^4 \arrow[r, dashed] \arrow[dd, hook, "\phi"', "\text{M5-probe}"'] & S^7 \arrow[rr] \arrow[dd, "h_{\mathbb{H}}"] & & \ast \arrow[dl] \arrow[dd, "0"', "\text{e.g. for } E \text{ complex-oriented}"] \\
& & \mathbb{H}P^2 \arrow[dr, dashed, "h_3^E"] & \\
\mathbb{R}^1 \times X^9 \arrow[r, dashed] & S^4 \arrow[ur, hook] \arrow[rr, "\Sigma^4 1^E"] & & E_4
\end{tikzcd}
$$

$\Rightarrow$ $\left.\begin{array}{l}\text{M5-branes on}\\ \text{4-manifold } \Sigma^4\end{array}\right\}$ give classes in E^4 of spacetime relative worldvolume

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow \int h_{\mathbb{H}} \\ \int S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow h_{\mathbb{H}} \\ S^4 \end{array}$$

for E complex-oriented

$\Rightarrow$ $\left.\begin{array}{c}\text{M5-branes on} \\ \text{4-manifold } \Sigma^4\end{array}\right\}$ give classes in E^4 of spacetime relative worldvolume

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow{\scriptstyle \mathfrak{l} h_{\mathbb{H}}} \\ \mathfrak{l} S^4 \end{array}$$

$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow{\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

in particular:

for C an elliptic curve and E_C its elliptic cohomology

$\Rightarrow$ $\left.\begin{array}{l}\text{M5-branes on} \\ \text{4-manifold } \Sigma^4\end{array}\right\}$ give classes in E_C^4 of spacetime relative worldvolume

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

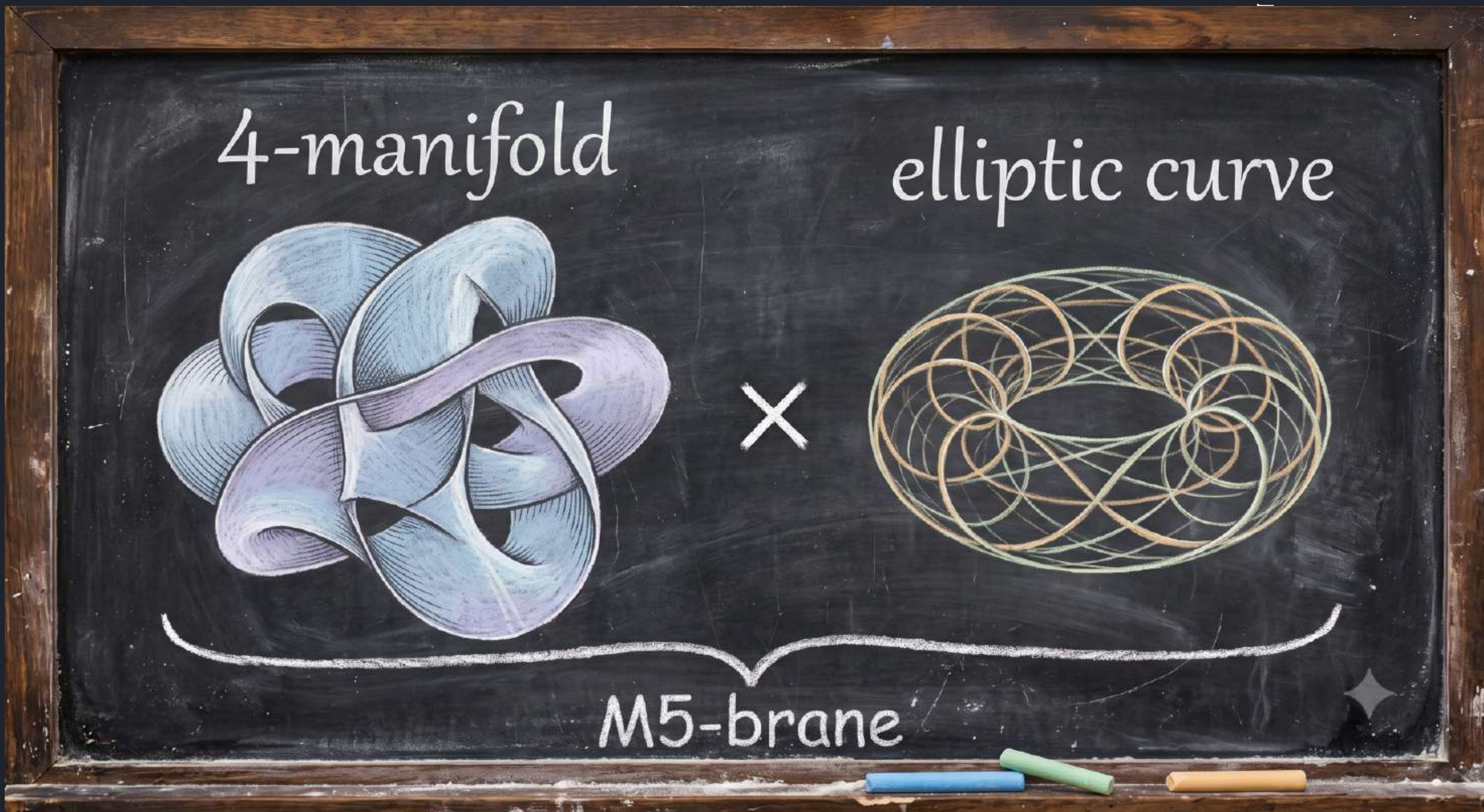
$\Rightarrow$ admits IR-completion classified by quaternionic Hopf fibration

$$\begin{array}{c} S^7 \\ \big\downarrow {\scriptstyle h_{\mathbb{H}}} \\ S^4 \end{array}$$

for C an elliptic curve and E_C its elliptic cohomology

$\Rightarrow$ $\left.\begin{array}{c}\text{M5-branes on}\\ \text{4-manifold } \Sigma^4\end{array}\right\}$ give classes in E_C^4 of spacetime relative worldvolume

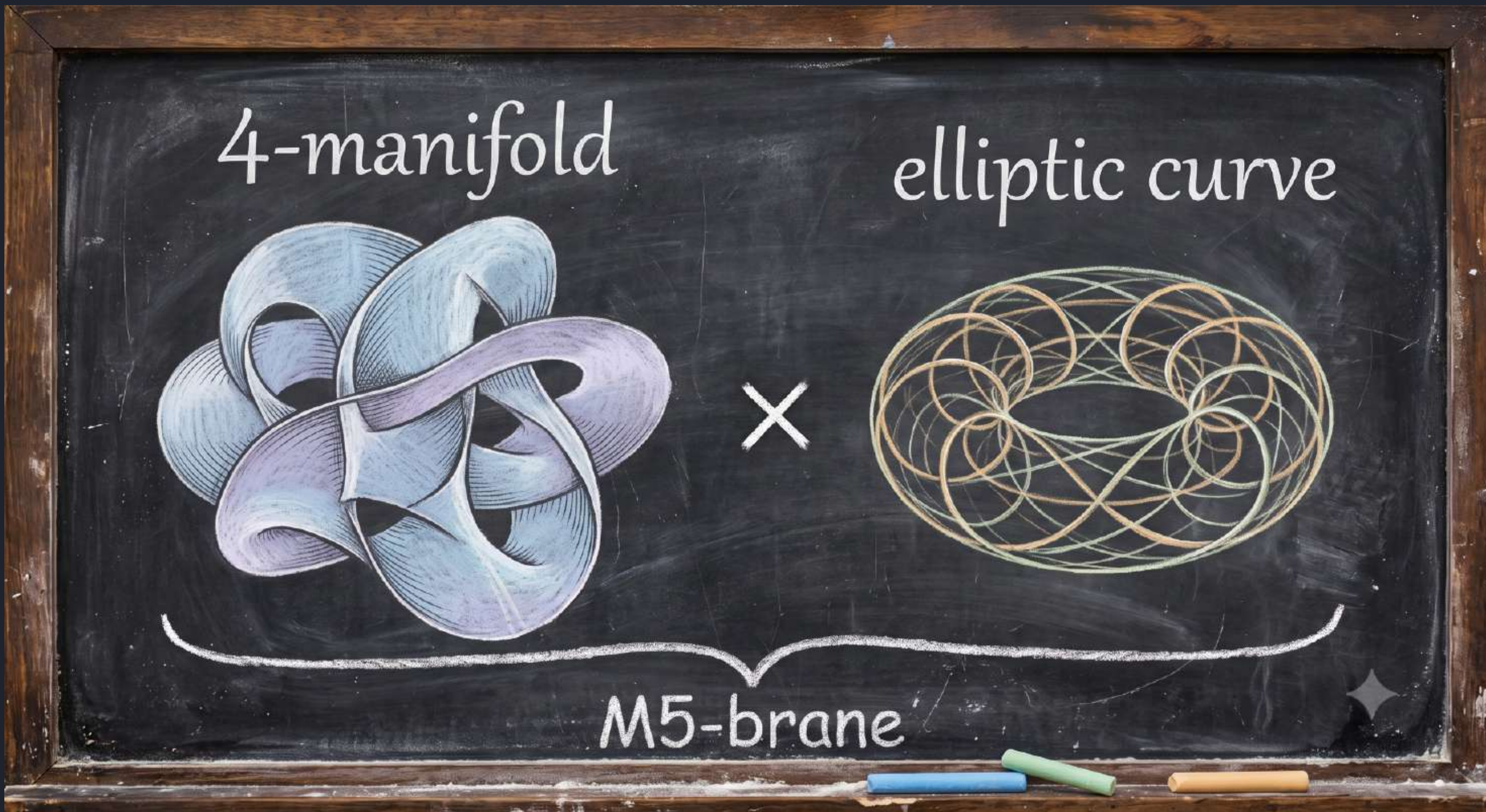
IR-Complete C-Field & M5 Model



for C an elliptic curve and E_C its elliptic cohomology

$$\Rightarrow \left.\begin{array}{c}\text{M5-branes on}\\ \text{4-manifold } \Sigma^4\end{array}\right\} \text{give classes in } E_C^4 \text{ of spacetime relative worldvolume}$$

IR-Complete C-Field & M5 Model

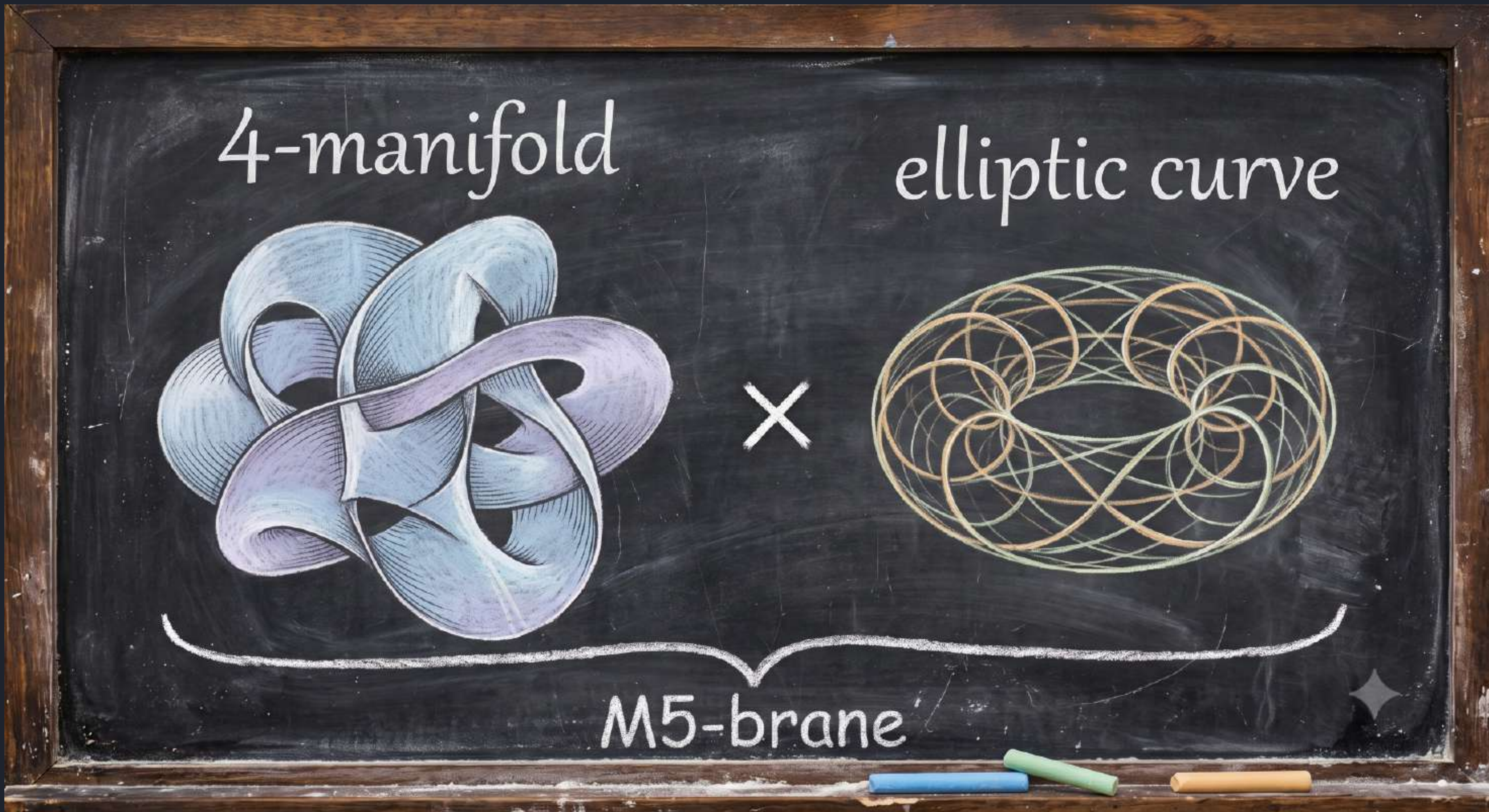


similar statements are expected in string folklore:

for C an elliptic curve and E_C its elliptic cohomology

$$\Rightarrow \left.\begin{array}{c}\text{M5-branes on}\\ \text{4-manifold } \Sigma^4\end{array}\right\} \text{give classes in } E_C^4 \text{ of spacetime relative worldvolume}$$

IR-Complete C-Field & M5 Model



similar statements are expected in string folklore:

for C an elliptic curve and E_C its elliptic cohomology

$$\Rightarrow \left.\begin{array}{c}\text{M5-branes on}\\ \text{4-manifold } \Sigma^4\end{array}\right\} \text{give classes in } E_C^4 \text{ of spacetime relative worldvolume}$$

Local Systems in the Real World

Local Systems in the Real World

Topologically Ordered Quantum Matter

Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

gapped ground states: $\mathcal{H}_p \in \mathrm{Mod}_{\mathbb{C}}$

Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

gapped ground states: $\mathcal{H}_p \in \mathrm{Mod}_{\mathbb{C}}$

Berry connection: $\mathcal{H}_\bullet \in \mathrm{Loc}_{\mathbb{C}}(P)$

Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

gapped ground states: $\mathcal{H}_p \in \mathrm{Mod}_{\mathbb{C}}$

Berry connection: $\mathcal{H}_{\bullet} \in \mathrm{Loc}_{\mathbb{C}}(P)$

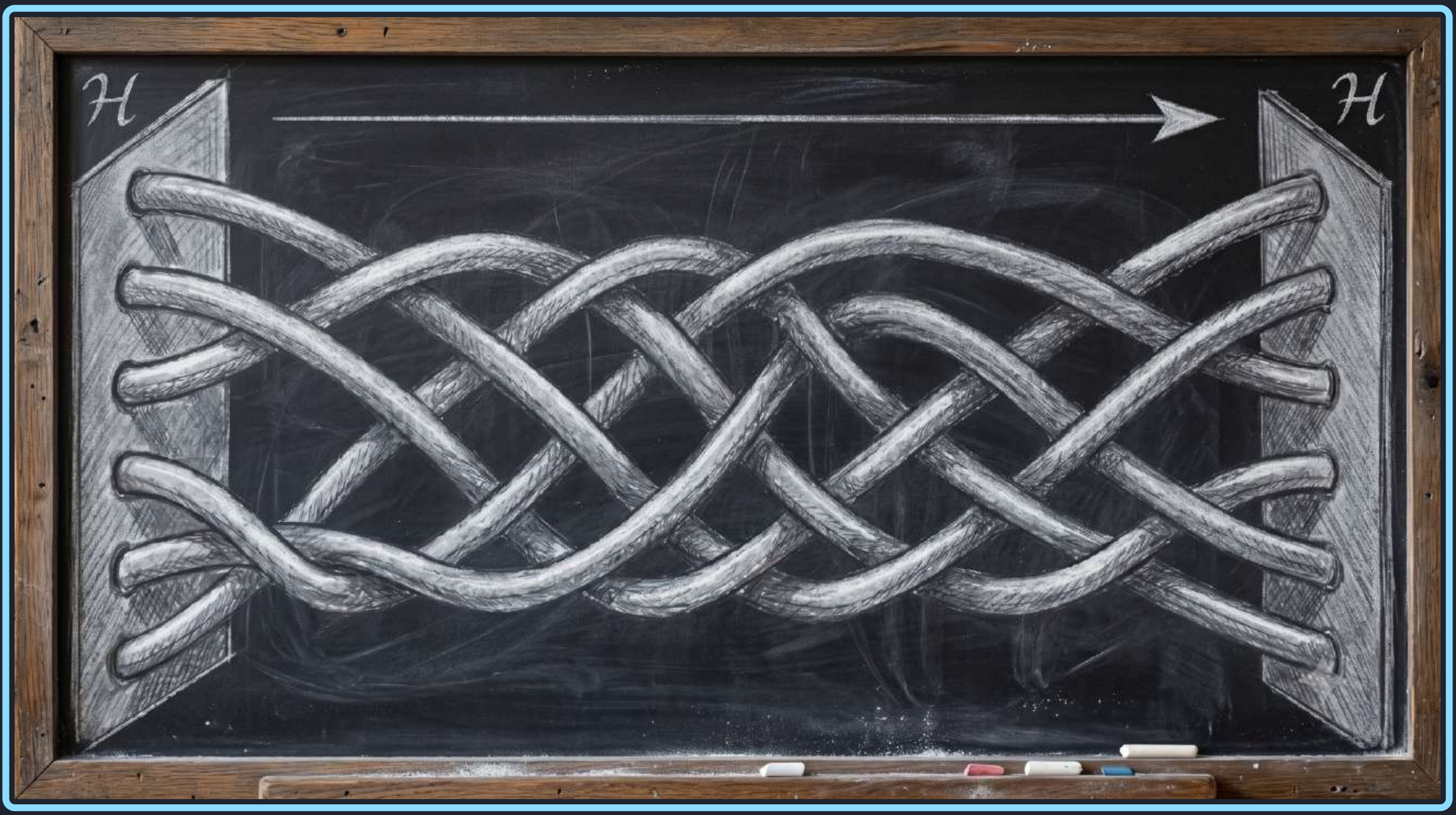
Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

gapped ground states: $\mathcal{H}_p \in \mathrm{Mod}_{\mathbb{C}}$

Berry connection: $\mathcal{H}_\bullet \in \mathrm{Loc}_{\mathbb{C}}(P)$



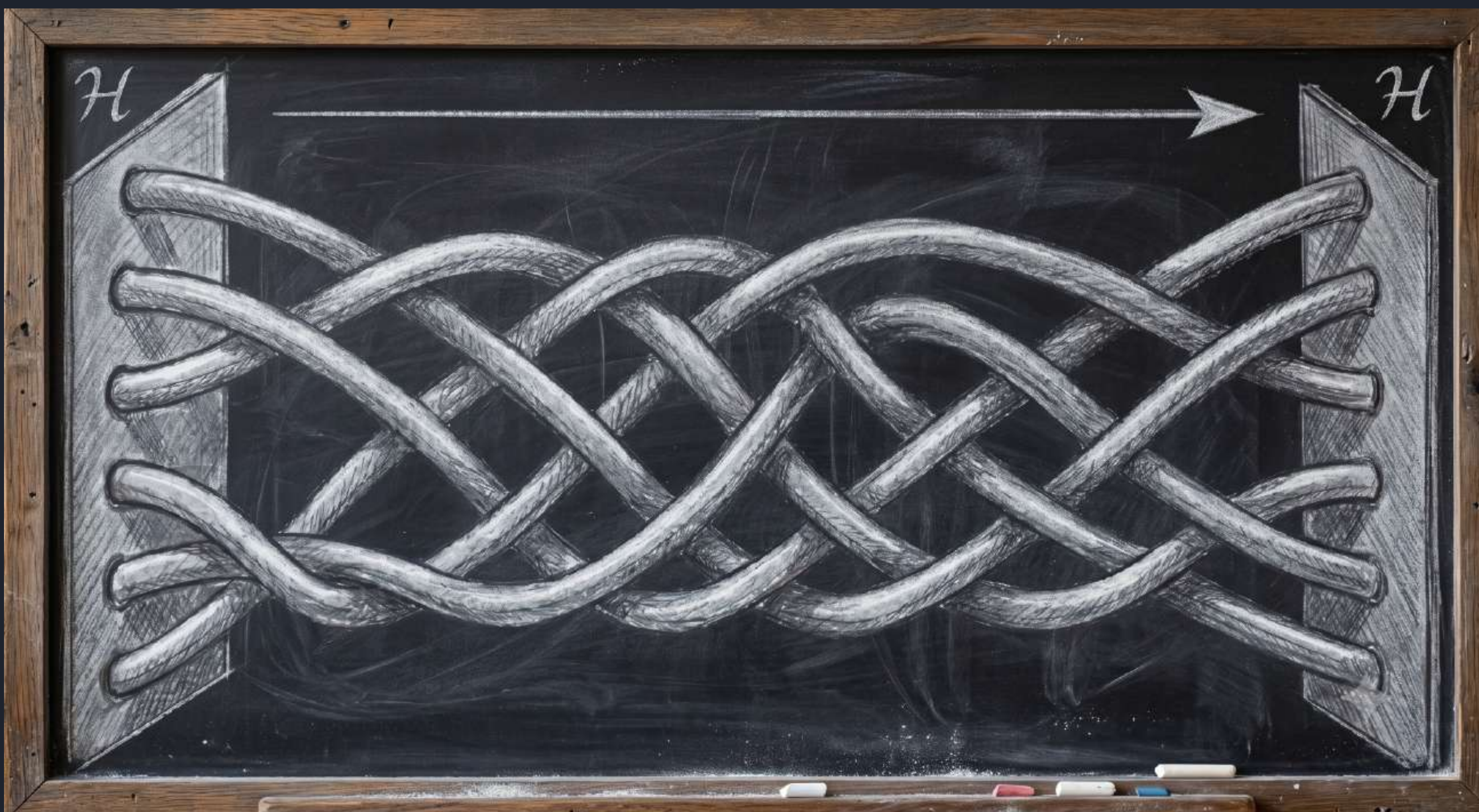
Local Systems in the Real World

Topologically Ordered Quantum Matter

external parameters: $p \in P$

gapped ground states: $\mathcal{H}_p \in \mathrm{Mod}_{\mathbb{C}}$

Berry connection: $\mathcal{H}_\bullet \in \mathrm{Loc}_{\mathbb{C}}(P)$



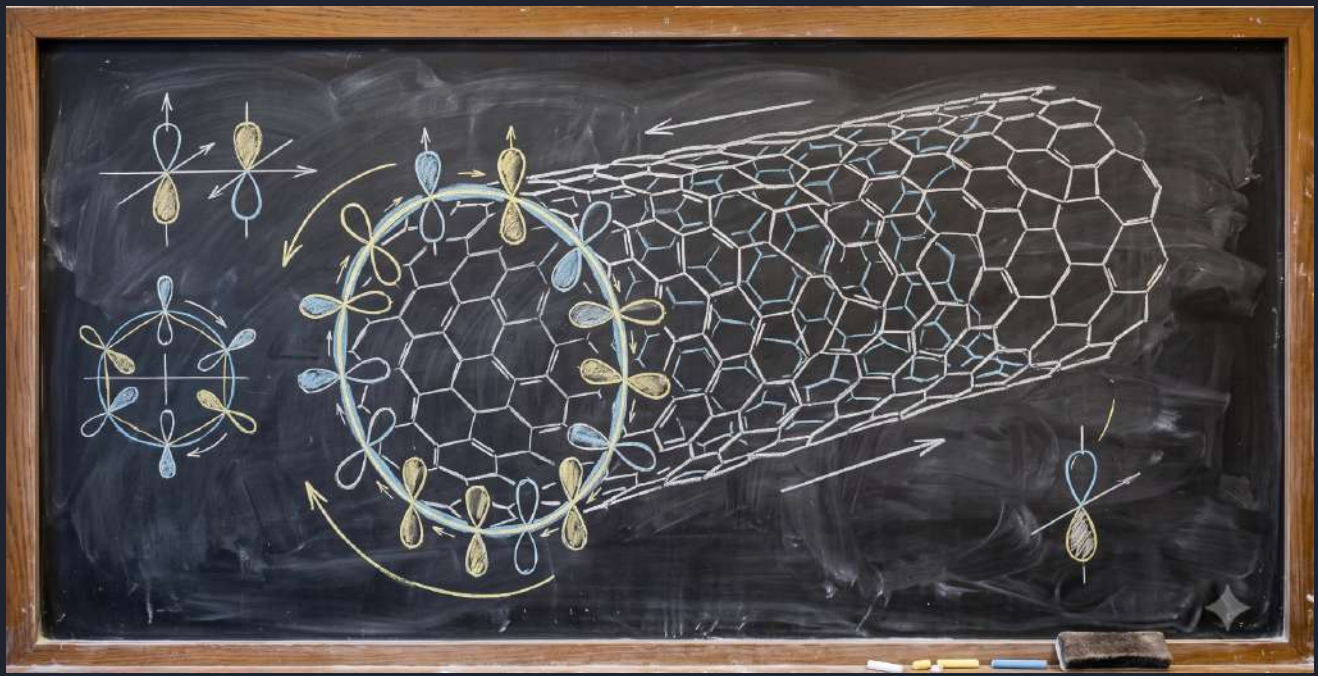
Local Systems in the Real World

Specifically: Crystalline Phases

Local Systems in the Real World

Specifically: Crystalline Phases

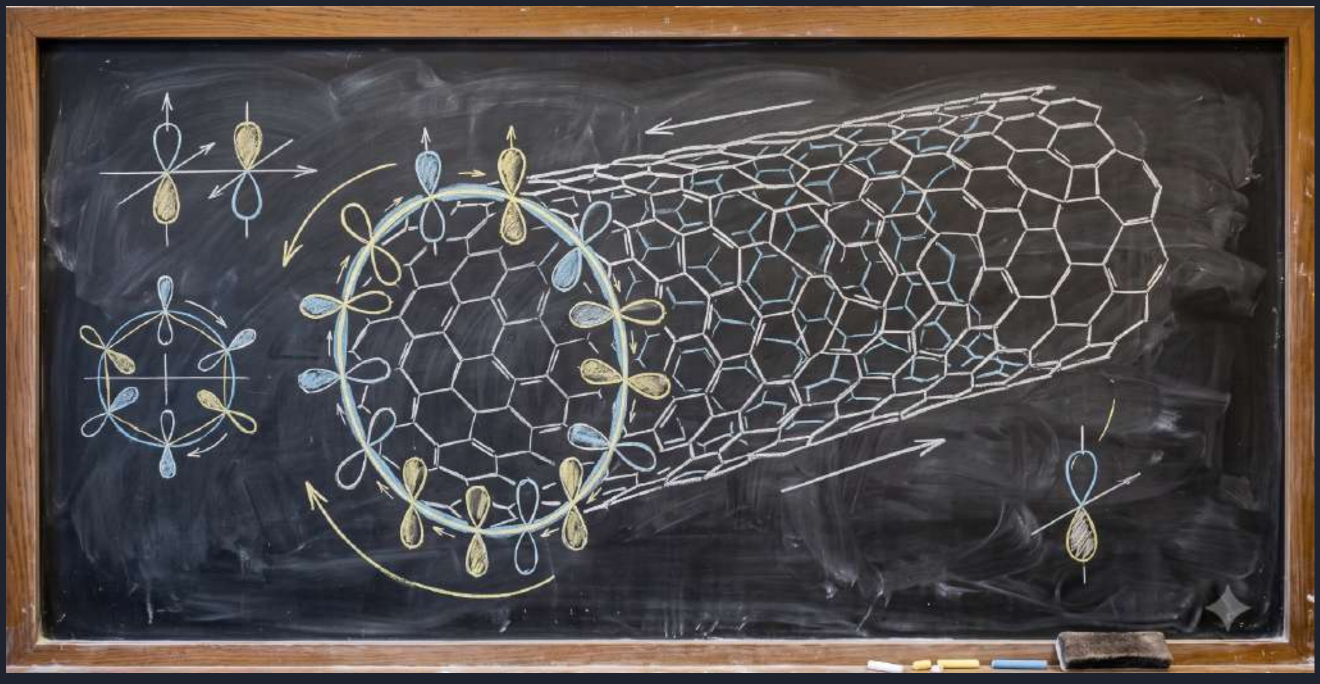
external parameter: $\left\{ \begin{array}{l} \text{crystal lattice} \\ \\ \\ \\ \end{array} \right.$



Local Systems in the Real World

Specifically: Crystalline Phases

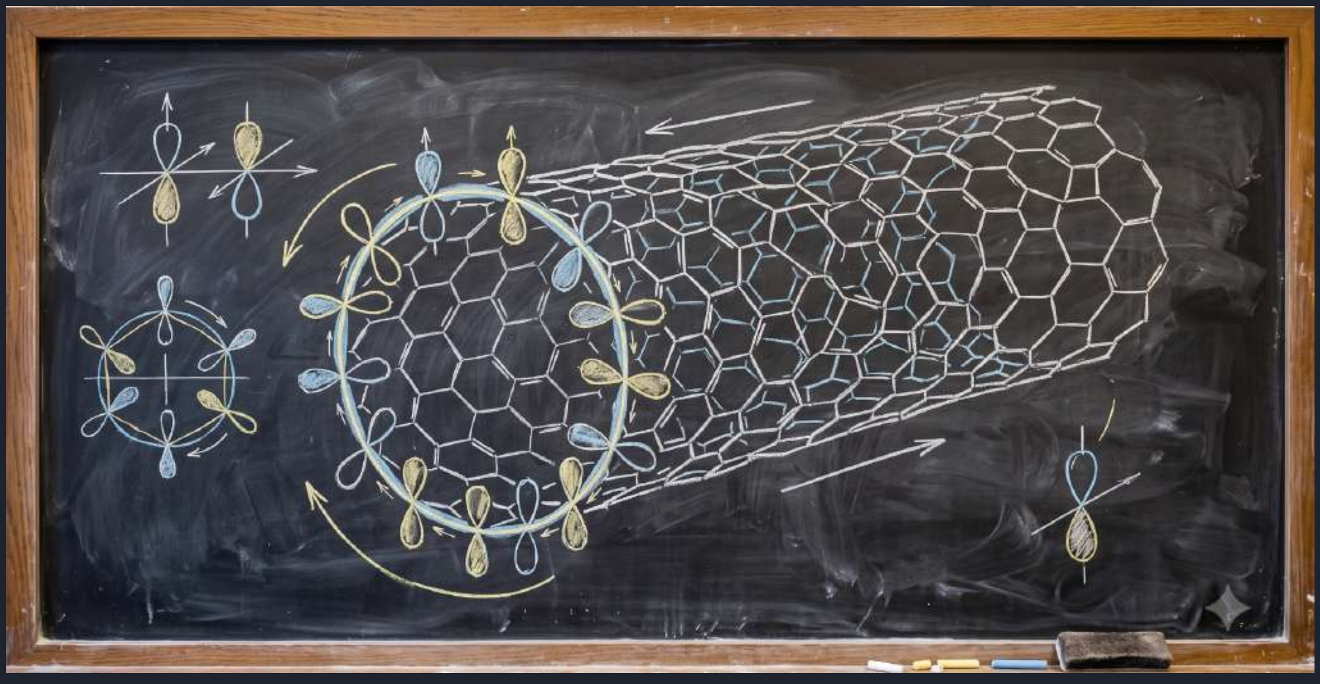
external parameter: $\left\{ \begin{array}{c} \text{crystal lattice} \\ \Updownarrow \\ \text{1-electron Hamiltonian} \end{array} \right.$



Local Systems in the Real World

Specifically: Crystalline Phases

$$\text{external parameter:}\ \left\{\begin{array}{c}\text{crystal lattice}\\ \Updownarrow \\ \text{1-electron Hamiltonian}\\ \Updownarrow \\ \begin{array}{c}\text{Brillouin}\\ \text{torus}\end{array} \longrightarrow \left\{\begin{array}{c}\text{Bloch}\\ \text{Hamiltonians}\end{array}\right\}\end{array}\right.$$



Local Systems in the Real World

Specifically: Crystalline Phases

$$\text{external parameter:}\ \left\{ \begin{array}{c} \text{crystal lattice} \\ \Updownarrow \\ \text{1-electron Hamiltonian} \\ \Updownarrow \\ \begin{array}{c}\text{Brillouin}\\ \text{torus}\end{array} \longrightarrow \left\{ \begin{array}{c}\text{Bloch}\\ \text{Hamiltonians}\end{array} \right\} \end{array} \right.$$

hence: $$P \;=\; \mathrm{Map}(\,\Sigma\,,\,\mathcal{B}\,)$$

Local Systems in the Real World

Specifically: Crystalline Phases

$$\text{external parameter:} \left\{ \begin{array}{c} \text{crystal lattice} \\ \Updownarrow \\ \text{1-electron Hamiltonian} \\ \Updownarrow \\ \begin{array}{c}\text{Brillouin}\\ \text{torus}\end{array} \longrightarrow \left\{ \begin{array}{c}\text{Bloch}\\ \text{Hamiltonians}\end{array} \right\} \end{array} \right.$$

hence: $$P = \mathrm{Map}(\Sigma, \mathcal{B})$$

$$\boxed{\mathcal{H}_\bullet \in \mathrm{Loc}_{\mathbb{C}}(\mathrm{Map}(\Sigma, \mathcal{B}))}$$

Local Systems in the Real World

Specifically: Crystalline Phases

$$\text{external parameter:} \left\{ \begin{array}{c} \text{crystal lattice} \\ \Updownarrow \\ \text{1-electron Hamiltonian} \\ \Updownarrow \\ \begin{array}{c}\text{Brillouin} \\ \text{torus}\end{array} \longrightarrow \left\{ \begin{array}{c}\text{Bloch} \\ \text{Hamiltonians}\end{array} \right\} \end{array} \right.$$

hence: $P = \mathrm{Map}(\Sigma, \mathcal{B})$

$$\boxed{\mathcal{H}_\bullet \in \mathrm{Loc}_{\mathbb{C}}(\mathrm{Map}(\Sigma, \mathcal{B}))}$$

Local Systems in the Real World

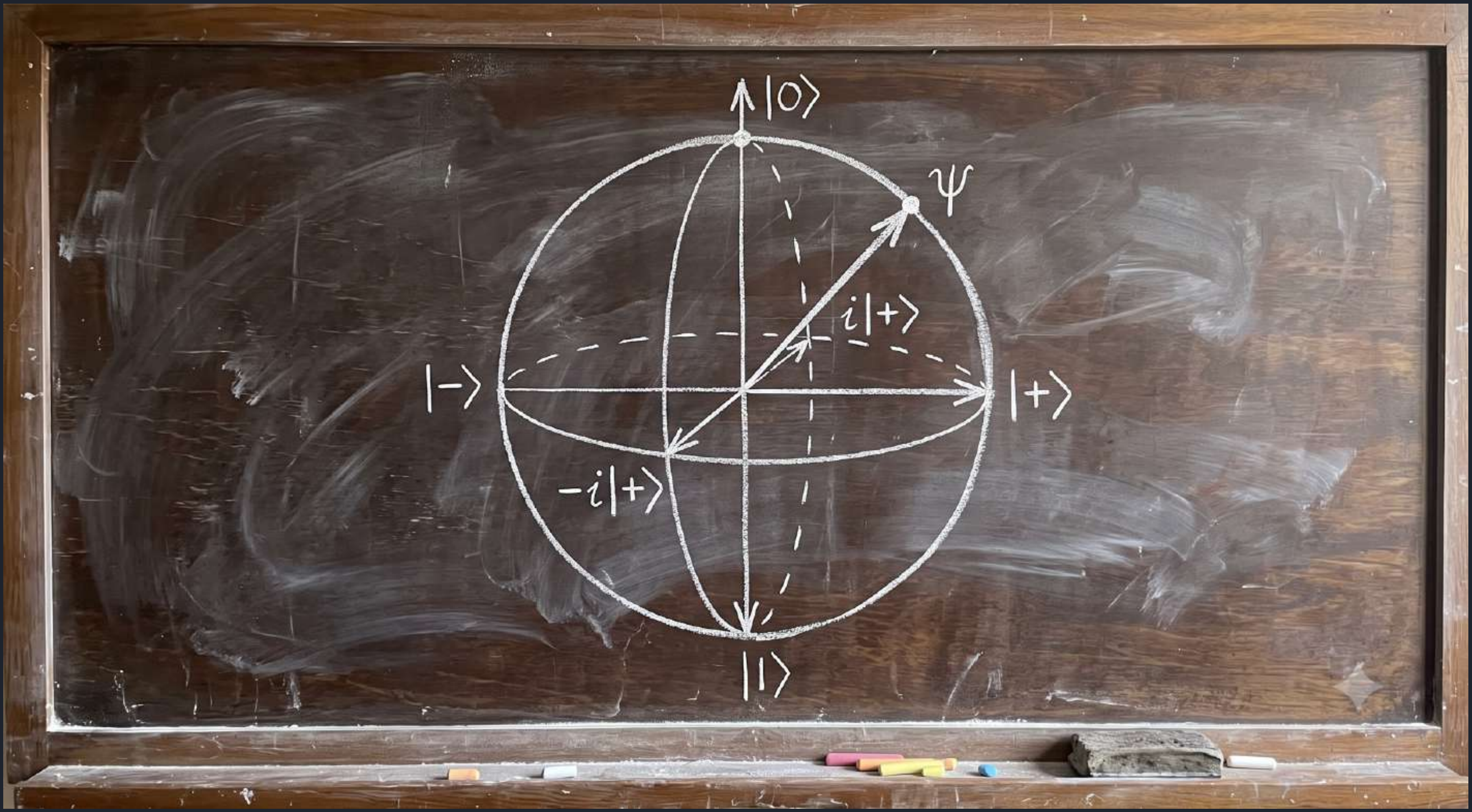
Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Local Systems in the Real World

Concretely:
2-band Chern Insulators
have: $\mathcal{B} \sim \mathbb{C}P^1$



Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$\mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}(T^2, \mathbb{C}P^1) /\!\!/ \mathrm{Diff}(T^2)\big)$

$\simeq$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$$\mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}\big(T^2, \mathbb{C}P^1\big) /\!\!/ \mathrm{Diff}(T^2)\Big)$$

$$\simeq \left\{ \begin{array}{c} \text{representations of} \\ \text{integer Heisenberg group} \\ \text{with root-of-unity centers} \end{array} \right\}$$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$$\mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}\big(T^2, \mathbb{C}P^1\big) /\!\!/ \mathrm{Diff}(T^2)\Big)$$

$$\simeq \left\{ \begin{array}{c} \text{representations of} \\ \text{integer Heisenberg group} \\ \text{with root-of-unity centers} \end{array} \right\}$$

$$\simeq \left\{ \begin{array}{c} \text{anyon quantum states of} \\ \text{FQ(A)H Phases} \end{array} \right\}$$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$$\mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}\big(T^2, \mathbb{C}P^1\big) /\!\!/ \mathrm{Diff}(T^2)\Big)$$

$$\simeq \left\{ \begin{array}{c} \text{representations of} \\ \text{integer Heisenberg group} \\ \text{with root-of-unity centers} \end{array} \right\}$$

$$\simeq \left\{ \begin{array}{c} \text{anyon quantum states of} \\ \text{FQ(A)H Phases} \end{array} \right\}$$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$$\mathrm{Loc}_{\mathbb{C}}\Big(\mathrm{Map}\big(T^2, \mathbb{C}P^1\big) /\!\!/ \mathrm{Diff}(T^2)\Big)$$

$$\simeq \left\{ \begin{array}{c} \text{representations of} \\ \text{integer Heisenberg group} \\ \text{with root-of-unity centers} \end{array} \right\}$$

diff equivariance
forced by

$$\simeq \left\{ \begin{array}{c} \text{anyon quantum states of} \\ \text{FQ(A)H Phases} \end{array} \right\}$$

Local Systems in the Real World

Concretely:

2-band Chern Insulators

have: $\mathcal{B} \sim \mathbb{C}P^1$

Theorem [arXiv:2505.22144]**:**

$$\mathrm{Loc}_{\mathbb{C}}\big(\mathrm{Map}(T^2, \mathbb{C}P^1) /\!\!/ \mathrm{Diff}(T^2)\big)$$

$$\simeq \left\{ \begin{array}{c} \text{representations of} \\ \text{integer Heisenberg group} \\ \text{with root-of-unity centers} \end{array} \right\}$$

$$\simeq \left\{ \begin{array}{c} \text{anyon quantum states of} \\ \text{FQ(A)H Phases} \end{array} \right\}$$

seen in experiment!

[Nakamura et al. 2019]

Local Systems in the Real World

Concretely:

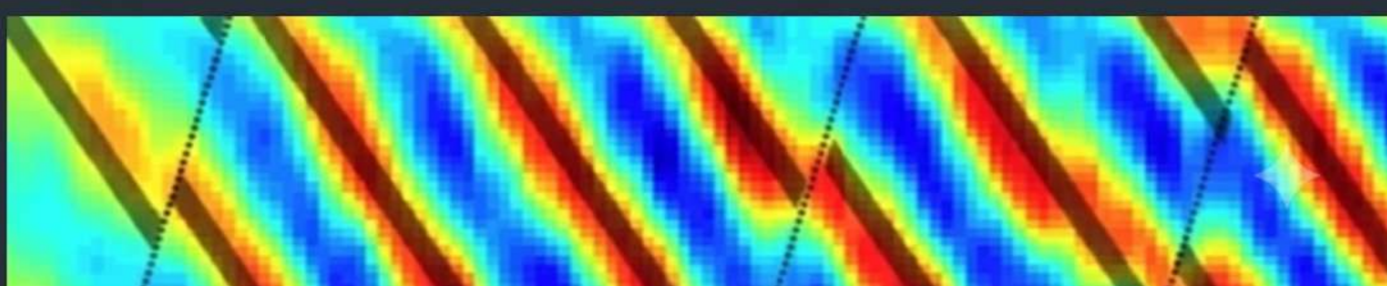
nature

NEWS | 03 July 2020

Welcome anyons! Physicists find best evidence yet for long-sought 2D structures

The 'quasiparticles' defy the categories of ordinary particles and herald a potential way to build quantum computers.

By Davide Castelvecchi



seen in experiment!

[Nakamura et al. 2019]

Local Systems in the Real World

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry
2. supersymmetry

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry
2. supersymmetry

(!!)

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry
2. supersymmetry

(!!)

same as for p-branes

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. W_∞-symmetry
2. supersymmetry

(!!)

same as for p-branes

(!!!)

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. $\underline{W_\infty\text{-symmetry}}$
2. $\underline{\text{supersymmetry}}$

(!!)

same as for p-branes
(!!!)

motivates:
geometric engineering
of FQH phases on
$N = 1$ M5-brane

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. $\underline{W_\infty\text{-symmetry}}$
2. <u>supersymmetry</u>

(!!)

same as for p-branes
(!!!)

motivates:
geometric engineering
of FQH phases on
$N = 1$ M5-brane

requires:
IR-completion of
M5 @ 11D SuGra

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. $\underline{W_\infty\text{-symmetry}}$
2. <u>supersymmetry</u>

(!!)

same as for p-branes
(!!!)

motivates:
geometric engineering
of FQH phases on
$N = 1$ M5-brane

requires:
IR-completion of
M5 @ 11D SuGra

providing global
model of the C-field
& M5-brane model.

Local Systems in the Real World

above topological sector,
FQH excitations have
two IR-symmetries
seen in experiment:

1. $\underline{W_\infty\text{-symmetry}}$
2. $\underline{\text{supersymmetry}}$

(!!)

same as for p-branes
(!!!)

motivates:
geometric engineering
of FQH phases on
$N=1$ M5-brane

requires:
IR-completion of
M5 @ 11D SuGra

providing global
model of the C-field
& M5-brane model.

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\
ʃ S^4
\end{tikzcd}
$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\text{ɩ}_{S^4} S^7 \arrow[d, "\text{ɩ} h_{\mathbb{H}}"] \\
\text{ɩ} S^4
\end{tikzcd}
$$

IR-completions may be modified along maps of characteristics.

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\
\int S^4
\end{tikzcd}
$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \mathfrak{l}_{S^4} S^7 \arrow[d, "\mathfrak{l} h_{\mathbb{H}}"] \\ \mathfrak{l} S^4 \end{tikzcd}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

$$\begin{tikzcd} S^7 \arrow[r, two heads, "h_{\mathbb{C}}"] \arrow[d, "h_{\mathbb{H}}"] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ S^4 \arrow[r, equal] & S^4 \end{tikzcd}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \Big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

this introduces a CS-like field (with vanishing on-shell flux):

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathit{ʃ}_{S^4} S^7 \\ \big\downarrow {\scriptstyle \mathit{ʃ} h_{\mathbb{H}}} \\ \mathit{ʃ} S^4 \end{array}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

this introduces a CS-like field (with vanishing on-shell flux):

$$\big\{ \mathrm{d}\, H_3 \,=\, \phi^* G_4 \,\big\} \hookrightarrow \left\{ \begin{array}{l} \mathrm{d}\, F_2 \,=\, 0 \\ \mathrm{d}\, H_3 \,=\, \phi^* G_4 - F_2 F_2 \end{array} \right\}$$

$$\begin{array}{ccc} \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{ʃ}_{S^4} S^7\big) & \xrightarrow{\quad (\mathit{ʃ} h_{\mathbb{C}})_* \quad} & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{ʃ}_{S^4} \mathbb{C}P^3\big) \\ \big\downarrow {\scriptstyle (\mathit{ʃ} h_{\mathbb{H}})_*} & & \big\downarrow {\scriptstyle (\mathit{ʃ} t_{\mathbb{C}})_*} \\ \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{ʃ} S^4\big) & =\!=\!=\!=\!= & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathit{ʃ} S^4\big) \end{array}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \mathfrak{l}_{S^4} S^7 \\ \big\downarrow{\scriptstyle \mathfrak{l}h_{\mathbb{H}}} \\ \mathfrak{l}S^4 \end{array}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

this introduces a CS-like field (with vanishing on-shell flux):

$$\left\{ \mathrm{d}\, H_3 \;=\; \phi^* G_4 \right\} \hookrightarrow \left\{ \begin{aligned} \mathrm{d}\, F_2 &\;=\; 0 \\ \mathrm{d}\, H_3 &\;=\; \phi^* G_4 - F_2 F_2 \end{aligned} \right\}$$

$$\begin{array}{ccc} \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathfrak{l}_{S^4} S^7\big) & \xrightarrow{\quad (\mathfrak{l}h_{\mathbb{C}})_* \quad} & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathfrak{l}_{S^4} \mathbb{C}P^3\big) \\ \big\downarrow{\scriptstyle (\mathfrak{l}h_{\mathbb{H}})_*} & & \big\downarrow{\scriptstyle (\mathfrak{l}t_{\mathbb{C}})_*} \\ \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathfrak{l}S^4\big) & = & \Omega^1_{\mathrm{cl}}\big(\cdot;\, \mathfrak{l}S^4\big) \end{array}$$

$\Rightarrow$ *magentized* M5-brane model

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{array}{c} \int_{S^4} S^7 \\ \big\downarrow {\scriptstyle \int h_{\mathbb{H}}} \\ \int S^4 \end{array}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

this introduces a CS-like field (with vanishing on-shell flux):

$$\Big\{ \mathrm{d}\, H_3 \;=\; \phi^* G_4 \Big\} \hookrightarrow \left\{ \begin{aligned} \mathrm{d}\, F_2 &= 0 \\ \mathrm{d}\, H_3 &= \phi^* G_4 - F_2 F_2 \end{aligned} \right\}$$

$$\begin{array}{ccc} \Omega^1_{\mathrm{cl}}\big(\cdot\,; \int_{S^4} S^7\big) & \xrightarrow{\;(\int h_{\mathbb{C}})_*\;} & \Omega^1_{\mathrm{cl}}\big(\cdot\,; \int_{S^4} \mathbb{C}P^3\big) \\ \big\downarrow {\scriptstyle (\int h_{\mathbb{H}})_*} & & \big\downarrow {\scriptstyle (\int t_{\mathbb{C}})_*} \\ \Omega^1_{\mathrm{cl}}\big(\cdot\,; \int S^4\big) & = & \Omega^1_{\mathrm{cl}}\big(\cdot\,; \int S^4\big) \end{array}$$

$\Rightarrow$ *magentized* M5-brane model

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \mathfrak{l}_{S^4} S^7 \arrow[d, "\mathfrak{l} h_{\mathbb{H}}"] \\ \mathfrak{l} S^4 \end{tikzcd}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

$$\begin{tikzcd} S^7 \arrow[r, ->>, "h_{\mathbb{C}}"] \arrow[d, "h_{\mathbb{H}}"] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ S^4 \arrow[r, equal] & S^4 \end{tikzcd}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\ \int S^4 \end{tikzcd}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

$$\begin{tikzcd} S^7 \arrow[r, two heads, "h_{\mathbb{C}}"] \arrow[d, "h_{\mathbb{H}}"] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ S^4 \arrow[r, equal] & S^4 \end{tikzcd}$$

This has a $\mathbb{Z}_{/n}$-equivariant version with fixed locus S^2:

$$\begin{tikzcd} S^2 \arrow[r, hook] \arrow[d] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ * \arrow[r, hook] & S^4 \end{tikzcd}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes
is characterized by:

$$\begin{tikzcd} ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\ ʃ S^4 \end{tikzcd}$$

IR-completions may be modified along maps of characteristics.

Here we may consider factorization through *twistor fibration*,

$$\begin{tikzcd} S^7 \arrow[r, two heads, "h_{\mathbb{C}}"] \arrow[d, "h_{\mathbb{H}}"] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ S^4 \arrow[r, equal] & S^4 \end{tikzcd}$$

This has a $\mathbb{Z}_{/n}$-equivariant version with fixed locus S^2:

coefficient for FQH orders!

$$\begin{tikzcd} S^2 \arrow[r, hook] \arrow[d] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ * \arrow[r, hook] & S^4 \end{tikzcd}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$\begin{tikzcd} \int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\ \int S^4 \end{tikzcd}$$

engineers FQH Topological Order

coefficient for FQH orders!

$$\begin{tikzcd} S^2 \arrow[r, hook] \arrow[d] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\ * \arrow[r, hook] & S^4 \end{tikzcd}$$

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\mathfrak{l}_{S^4} S^7 \arrow[d, "\mathfrak{l} h_{\mathbb{H}}"] \\
\mathfrak{l} S^4
\end{tikzcd}
$$

engineers FQH Topological Order

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
\int_{S^4} S^7 \arrow[d, "\int h_{\mathbb{H}}"] \\
\int S^4
\end{tikzcd}
$$

engineers FQH Topological Order

brane configuration

A_n-orbi-singularity

$$
X^{10} \;:=\; \Sigma^3 \;\times\; \underset{\curvearrowright G}{\mathbb{C}} \;\times\; \underset{\curvearrowright G}{\overline{\mathbb{C}}} \;\times\; X^3 \dashrightarrow \underset{\curvearrowright G}{S^4}
$$

classifying fibration

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\
ʃ S^4
\end{tikzcd}
$$

engineers FQH Topological Order

brane configuration

$$
\begin{tikzcd}[column sep=small]
\Sigma^5 \arrow[d, hook, "\phi"] & := & \Sigma^3 \arrow[d, equal] & \times & \mathbb{C} \arrow[loop above, "G"] \arrow[d, equal, "A_n\text{-orbi-singularity}"] \\
X^{10} & := & \Sigma^3 & \times & \mathbb{C} \arrow[loop below, "G"] & \times & \overline{\mathbb{C}} \arrow[loop below, "G"] & \times & X^3 \arrow[r, dashed] & S^4 \arrow[loop below, "G"]
\end{tikzcd}
$$

classifying fibration

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\
ʃ S^4
\end{tikzcd}
$$

engineers FQH Topological Order

brane configuration

$$
\begin{tikzcd}
 & & & & G \curvearrowright & & & & G \curvearrowright \\
\Sigma^5 \arrow[d, hook, "\phi"] & := & \Sigma^3 \arrow[d, equal] & \times & \mathbb{C} \arrow[d, equal, "A_n\text{-orbi-singularity}"] \arrow[rrrr, dashed, "\text{anyon charges}"] & & & & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{C}}"] \\
X^{10} & := & \Sigma^3 & \times & \mathbb{C} & \times & \overline{\mathbb{C}} & \times \; X^3 \arrow[r, dashed] & S^4 \\
 & & & & \curvearrowleft G & & \curvearrowleft G & & \curvearrowleft G
\end{tikzcd}
$$

classifying fibration

IR-Complete C-Field & M5 Model

recall: 11D SuGra with M5-probes is characterized by:

$$
\begin{tikzcd}
ʃ_{S^4} S^7 \arrow[d, "ʃ h_{\mathbb{H}}"] \\
ʃ S^4
\end{tikzcd}
$$

engineers FQH Topological Order

brane configuration

$$
\begin{tikzcd}
\Sigma^5 \arrow[d, hook, "\phi"] & := & \Sigma^3 \arrow[d, equal] & \times & \mathbb{C} \arrow[loop above, "G"] \arrow[d, equal, "A_n\text{-orbi-singularity}"] \arrow[rrrr, dashed, "\text{anyon charges}"] & & & & \mathbb{C}P^3 \arrow[loop above, "G"] \arrow[d, "t_{\mathbb{C}}"] \\
X^{10} & := & \Sigma^3 & \times & \mathbb{C} \arrow[loop below, "G"] & \times \;\; \overline{\mathbb{C}} \arrow[loop below, "G"] & \times & X^3 \arrow[r, dashed] & S^4 \arrow[loop below, "G"]
\end{tikzcd}
$$

classifying fibration

The End

The End

For more see:

– *The Character Map In Nonabelian Cohomology* (esp. §9)

– *Complete Topological Quantization of Higher Gauge Fields*

– *Higher Gauge Theory via Differential Nonabelian Cohomology*

– *Higher Superspace Supergravity and its IR-Completions*