Thus, n_W^V is the multiplicity of W in the decomposition of $V|_{N'}$, as well as the multiplicity of V in the decomposition of $\mathrm{Ind}_{N'}^N(W)$. So for any $x \in K_G^*(X)$,

$$\begin{aligned}(\mathrm{Id}\otimes \mathrm{Res}_{N'}^N)(\Psi_{G;N,H}(x)) &= \sum_{V\in\mathrm{Irr}(N)} \psi_V(x)\otimes [V|_{N'}] \\ &= \sum_{W\in\mathrm{Irr}(N')}\Big(\sum_{V\in\mathrm{Irr}(N)} n_W^V{\cdot}\psi_V(x)\Big)\otimes [W];\end{aligned}$$

and we will be done upon showing that $\psi'_W = \sum_V n_W^V{\cdot}\psi_V$ for each $W \in \mathrm{Irr}(N')$. Fix a surjection $p_0 : \mathbb{C}[N'] \twoheadrightarrow W$, and a decomposition $\mathrm{Ind}_{N'}^N(W) = \sum_{i=1}^k V_i$ (where the V_i are irreducible and $k = \sum_V n_W^V$). For each $1 \le i \le k$, let $p_i : \mathbb{C}[N] \twoheadrightarrow V_i$ be the composite of $\mathrm{Ind}_{N'}^N(p_0)$ followed by projection to V_i. Then

$$\psi_{p_0} = \bigoplus_{i=1}^k \psi_{p_i} : \big(\underline{\mathrm{Vec}}_G^{\mathbb{C}}\big)^N \longrightarrow \underline{\mathrm{Vec}}_H^{\mathbb{C}}$$

as maps of Γ-spaces, and so $\psi_W \simeq \sum_{i=1}^k \psi_{V_i}$ as maps $K_G^*(X) \to K_H^*(X)$.

It remains to show that Ψ is a homomorphism of rings. Since it is natural in N, and since $R(N)$ is detected by characters, it suffices to prove this when N is cyclic. For any $x, y \in K_G(X)$,

$$\Psi(x){\cdot}\Psi(y) = \sum_{V,W\in\mathrm{Irr}(N)} \big(\psi_V(x){\cdot}\psi_W(y)\big)\otimes [V\otimes W]$$

and

$$\Psi(xy) = \sum_{U\in\mathrm{Irr}(N)} \psi_U(xy)\otimes [U].$$

And thus $\Psi(x){\cdot}\Psi(y) = \Psi(xy)$ since

$$\psi_U\circ\mu_* = \bigoplus_{\substack{V,W\in\mathrm{Irr}(G)\\ V\otimes W\cong U}} \mu_*\circ(\psi_V\wedge\psi_W) : \big(\underline{\mathrm{Vec}}_G^{\mathbb{C}}\big)^N \wedge \big(\underline{\mathrm{Vec}}_G^{\mathbb{C}}\big)^N \longrightarrow \underline{\mathrm{Vec}}_H^{\mathbb{C}},$$

as maps of Γ-spaces, for each $U \in \mathrm{Irr}(N)$. □

4. Characters and class functions

Throughout this section, G will be a *finite* group. We prove here some results showing that certain class functions are characters; results which will be needed in the next two sections.

For any field K of characteristic zero, a *K-character* of G means a class function $G \to K$ which is the character of some (virtual) K-representation of G. Two elements $g, h \in G$ are called *K-conjugate* if g is conjugate to h^a for some a prime to $n = |g| = |h|$ such that $(\zeta \mapsto \zeta^a) \in \mathrm{Gal}(K\zeta/K)$, where $\zeta = \exp(2\pi i/n)$. For example, g and h are $\mathbb{Q}$-conjugate if $\langle g\rangle$ and $\langle h\rangle$ are conjugate as subgroups, and are $\mathbb{R}$-conjugate if g is conjugate to h or h^{-1}.

Proposition 4.1. *Fix a finite extension K of $\mathbb{Q}$, and let $A \subseteq K$ be its ring of integers. Let $f : G \to A$ be any function which is constant on K-conjugacy classes. Then $|G|{\cdot}f$ is an A-linear combination of K-characters of G.*

Proof. Set $n = |G|$, for short. Let $V_1, \ldots, V_k$ be the distinct irreducible $K[G]$-representations, let χ_i be the character of V_i, set $D_i = \operatorname{End}_{K[G]}(V_i)$ (a division algebra over K), and set $d_i = \dim_K(D_i)$. Then by [11, Theorem 25, Cor. 2],

$$|G|{\cdot}f = \sum_{i=1}^{k} r_i\chi_i \qquad \text{where} \qquad r_i = \frac{1}{d_i}\sum_{g\in G} f(g)\chi_i(g^{-1});$$

and we must show that $r_i \in A$ for all i. This means showing, for each $i = 1, \ldots, k$, and each $g \in G$ with K-conjugacy class $\operatorname{conj}_K(g)$, that $|\operatorname{conj}_K(g)|{\cdot}\chi_i(g) \in d_iA$.

Fix i and g; and set $C = \langle g\rangle$, $m = |g| = |C|$, and $\zeta = \exp(2\pi i/m)$. Then $\operatorname{Gal}(K(\zeta)/K)$ acts freely on the set $\operatorname{conj}_K(g)$: the element $(\zeta \mapsto \zeta^a)$ acts by sending h to h^a. So $[K(\zeta){:}K]\,\big|\,|\operatorname{conj}_K(g)|$.

Let $V_i|_C = W_1^{a_1} \oplus \cdots \oplus W_t^{a_t}$ be the decomposition as a sum of irreducible $K[C]$-modules. For each j, $K_j \stackrel{\text{def}}{=} \operatorname{End}_{K[C]}(W_j)$ is the field generated by K and the r-th roots of unity for some $r|m$ ($m = |C|$), and $\dim_{K_j}(W_j) = 1$. So

$$\dim_K(W_j)\,\big|\,[K(\zeta){:}K].$$

Also, $d_i\big|\dim_K(W_j^{a_j})$, since $W_j^{a_j}$ is a D_i-module; and thus $d_i\big|a_j{\cdot}|\operatorname{conj}_K(g)|$. So if we set $\xi_j = \chi_{W_j}(g) \in A$, then

$$|\operatorname{conj}_K(g)|{\cdot}\chi_i(g) = |\operatorname{conj}_K(g)|{\cdot}\sum_{j=1}^{t} a_j\xi_j \in d_iA,$$

and this finishes the proof. □

For each prime p and each element $g \in G$, there are unique elements g_r of order prime to p and g_u of p-power order, such that $g = g_rg_u = g_ug_r$. As in [11, §10.1], we refer to g_r as the p'-component of g. We say that a class function $f : G \to \mathbb{C}$ is *p-constant* if $f(g) = f(g_r)$ for each $g \in G$. Equivalently, f is p-constant if and only if $f(g) = f(g')$ for all $g, g' \in G$ such that $[g, g'] = 1$ and $g^{-1}g'$ has p-power order.

Lemma 4.2. *Fix a finite group G, a prime p, and a field K of characteristic zero. Then a p-constant class function $\varphi : G \to K$ is a K-character of G if and only if $\varphi|_H$ is a K-character of H for all subgroups $H \subseteq G$ of order prime to p.*

Proof. Recall first that G is called K-elementary if for some prime q, $G = C_m \rtimes Q$, where C_m is cyclic of order m, $q\nmid m$, Q is a q-group, and the conjugation action of Q on $K[C_m]$ leaves invariant each of its field components. By [11, §12.6, Prop. 36], a K-valued class function of G is a K-character if and only if its restriction to any K-elementary subgroup of G is a K-character. Thus, it suffices to prove the lemma when G is K-elementary.

The coefficient system $\mathbb{Q}\otimes R(-)$, and hence its cohomology, splits in a natural way as a product indexed over cyclic subgroups of G of finite order. For any cyclic group S of order $n<\infty$, we let $\mathbb{Z}[\zeta_S]\subseteq\mathbb{Q}(\zeta_S)$ denote the cyclotomic ring and field generated by the n-th roots of unity; but regarded as quotient rings of the group rings $\mathbb{Z}[S^*]\subseteq\mathbb{Q}[S^*]$ ($S^*=\mathrm{Hom}(S,\mathbb{C}^*)$). In other words, we fix an identification of the n-th roots of unity in $\mathbb{Q}(\zeta_S)$ with the irreducible characters of S. The kernel of the homomorphism $R(S)\cong\mathbb{Z}[S^*]\twoheadrightarrow\mathbb{Z}[\zeta_S]$ is precisely the ideal of elements whose characters vanish on all generators of S.

Lemma 5.6. *Fix a discrete group G, and let $\mathcal{S}(G)$ be a set of conjugacy class representatives for the cyclic subgroups $S\subseteq G$ of finite order. Then for any proper G-complex X, there is an isomorphism of rings*

$$H^*_G(X;\mathbb{Q}\otimes R(-))\cong\prod_{S\in\mathcal{S}(G)}\left(H^*(X^S/C_G(S);\mathbb{Q}(\zeta_S))\right)^{N(S)},$$

where $N(S)$ acts via the conjugation action on $\mathbb{Q}(\zeta_S)$ and via translation on $X^S/C_G(S)$. If, furthermore, the isotropy subgroups on X have bounded order, then the homomorphism of rings

$$H^*_G(X;R(-))\longrightarrow\prod_{S\in\mathcal{S}(G)}H\Big(\big(C^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])\big)^{N(S)}\Big)$$
$$\longrightarrow\prod_{S\in\mathcal{S}(G)}\left(H^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])\right)^{N(S)},\quad(1)$$

induced by restriction to cyclic subgroups and by the projections $R(S)\twoheadrightarrow\mathbb{Z}[\zeta_S]$, has kernel and cokernel of finite exponent.

Proof. By (5.2),

$$C^*_G(X;R(-))\cong\mathrm{Hom}_{\mathrm{Or}_f(G)}(\underline{C}_*(X),R(-))\cong\mathrm{Hom}_{\mathrm{Sub}_f(G)}(\underline{C}^{\mathrm{qt}}_*(X),R(-)).$$

For each $S\in\mathcal{S}(G)$, let $\chi_S\in\mathrm{Cl}(G)$ be the idempotent class function: $\chi_S(g)=1$ if $\langle g\rangle$ is conjugate to S, and $\chi_S(g)=0$ otherwise. By Proposition 4.1, for each finite subgroup $H\subseteq G$, $(\chi_S)|_H$ is the character of an idempotent $e^H_S\in\mathbb{Q}\otimes R(H)$. Set $\mathbb{Q}R_S(H)=e^H_S\cdot(\mathbb{Q}\otimes R(H))$, and let $R_S(H)\subseteq\mathbb{Q}R_S(H)$ be the image of $R(H)$ under the projection. This defines a splitting $\mathbb{Q}\otimes R(-)=\prod_{S\in\mathcal{S}(G)}\mathbb{Q}R_S(-)$ of the coefficient system. For each S and H,

$$\mathbb{Q}R_S(S)=\mathbb{Q}(\zeta_S)\quad\text{and so}\quad\mathbb{Q}R_S(H)\cong\mathrm{map}_{N(S)}\Big(\mathrm{Mor}_{\mathrm{Sub}_f(G)}(S,H),\,\mathbb{Q}(\zeta_S)\Big).$$

It follows that

$$\begin{aligned}C^*_G(X;\mathbb{Q}R_S(-))&\cong\mathrm{Hom}_{\mathrm{Sub}_f(G)}(\underline{C}^{\mathrm{qt}}_*(X),\mathbb{Q}R_S(-))\\&\cong\mathrm{Hom}_{\mathbb{Q}[N(S)]}\big(C_*(X^S/C_G(S)),\mathbb{Q}(\zeta_S)\big);\end{aligned}$$

and hence $H^*_G(X;\mathbb{Q}R_S(-))\cong\big(H^*(X^S/C_G(S));\mathbb{Q}(\zeta_S)\big)^{N(S)}$.

Now assume there is a bound on the orders of isotropy subgroups on X, and let m be the least common multiple of the $|G_x|$. By Proposition 4.1 again,

$me_S^H \in R(H)$ for each $S \in \mathcal{S}(G)$ and each isotropy subgroup H. So there are homomorphisms of functors

$$R(-) \underset{j}{\overset{i}{\rightleftarrows}} \prod_{S \in \mathcal{S}(G)} R_S(-),$$

where i is induced by the projections $R(H) \twoheadrightarrow R_S(H)$ and j by the homomorphisms $R_S(H) \xrightarrow{me_S^H \cdot} R(H)$ (regarding $R_S(H)$ as a quotient of $R(H)$); and $i \circ j$ and $j \circ i$ are both multiplication by m. For each S, the monomorphism

$$C_G^*(X; R_S(-)) \cong \mathrm{Hom}_{\mathbb{Z}[N(S)]}\big(C_*(X^S/C_G(S)), \mathbb{Z}[\zeta_S]\big) \longrightarrow C^*(X^S/C_G(S); \mathbb{Z}[\zeta_S])$$

is split by the norm map for the action of $N(S)/C_G(S)$, and hence the kernel and cokernel of the induced homomorphism

$$H_G^*(X; R_S(-)) \longrightarrow \big(H^*(X^S/C_G(S); \mathbb{Z}[\zeta_S])\big)^{N(S)}$$

have exponent dividing $\varphi(m)$ (since $|N(S)/C_G(S)| \big| |\operatorname{Aut}(S)| \big| \varphi(m)$). The composite in (1) thus has kernel and cokernel of exponent $m \cdot \varphi(m)$. □

By the first part of Proposition 5.6, the equivariant Chern character can be regarded as a homomorphism

$$\mathrm{ch}_X^* : K_G^*(X) \longrightarrow \prod_{S \in \mathcal{S}(G)} \big(H^*(X^S/C_G(S); \mathbb{Q}(\zeta_S))\big)^{N(S)},$$

where $\mathcal{S}(G)$ is as above. This is by construction a product of ring homomorphisms.

We now apply the splitting of Lemma 5.6 to construct a second version of the equivariant rational Chern character: one which takes values in $\mathbb{Q} \otimes H_G^*(X; R(-))$ rather than in $H_G^*(X; \mathbb{Q} \otimes R(-))$. The following lemma handles the nonequivariant case.

Lemma 5.7. *There is a homomorphism $n!\mathrm{ch} : K^*(X) \to H^{\le 2n}(X; \mathbb{Z})$, natural on the category of CW-complexes, whose composite to $H^*(X; \mathbb{Q})$ is $n!$ times the usual Chern character truncated in degrees greater than $2n$. Furthermore, $n!\mathrm{ch}$ is natural with respect to suspension isomorphisms $K^*(X) \cong \widetilde{K}^{*+m}(\Sigma^m(X_+))$, and is multiplicative in the sense that $\big(n!\mathrm{ch}(x)\big) \cdot \big(n!\mathrm{ch}(y)\big) = n! \cdot \big(n!\mathrm{ch}(xy)\big)$ for all $x, y \in K(X)$ (in both cases after restricting to the appropriate degrees).*

Proof. Define $n!\mathrm{ch} : K^0(X) \to H^{\mathrm{ev}, \le 2n}(X; \mathbb{Z})$ to be the following polynomial in the Chern classes:

$$n! \cdot \sum_{i=1}^{n} \Big(1 + x_i + \frac{x_i^2}{2!} + \cdots + \frac{x_i^n}{n!}\Big) \in \mathbb{Z}[c_1, \ldots, c_n] = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{\Sigma_n}.$$

Here, as usual, c_k is the k-th elementary symmetric polynomial in the x_i. This is extended to $K^{-1}(X) \cong \widetilde{K}(\Sigma(X_+))$ in the obvious way. The relations all follow from the usual relations between Chern classes in the rings $H^*(BU(m))$. □

We are now ready to construct the integral Chern character. What this really means is that under certain restrictions on X, some multiple of the rational Chern