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Flux Quantization on M-Strings

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Abstract
The electric Gauss law in 11D SuGra is famously non-linear, whence its flux quantization must be in nonabelian cohomology. We have previously shown that the minimal admissible choice is 4-Cohomotopy, which in the presence of magnetized M5-probes takes its relative twistorial form.
Here we discuss how this situation is further refined in the presence of M-string probes on the M5-worldvolume. Based on the superspace formulation of 11D SuGra, we find the nested Bianchi identities by iterating the superembedding construction for super p-branes. The resulting probe brane hierarchy (M1 on magnetized M5 in 11D bulk) turns out to admit flux quantization in a doubly-relative form of twisted Cohomotopy, classified by the factorization of the quaternionic Hopf fibration through the twistor fibration.
The further equivariant refinement of this cohomology theory reduces on A-type singularities to a form of relative Cohomotopy which geometrically engineers Chern-insulator phases on $\mathrm{M5} \cap \mathrm{A}_n$, with the M-string playing the role of gapped nodal lines.

Keywords: M-theory, 11D Supergravity, M5-branes, M-strings, superembedding formalism, flux quantization, Cohomotopy, topological phases

1 Introduction

Despite the relevance of topological quantum effects in general ([Simon(2023)], notably in potential applications to topological quantum hardware [Nayak *et al.*(2008)Nayak, Simon, Stern, Freedman, and Das Sarma]) and in particular in string/M-theory [Duff(1999), Miemiec and Schnakenburg(2006)], the global topological completion of (higher) gauge fields coupling to branes ([Fré(2013), Lu(2025), §7]) has not received sufficient attention.

For example, the famous conjecture that RR-flux is quantized in K-theory (cf. [Moore and Witten(2000), Freed(2002), Grady and Sati(2022)] and (40) below) tacitly admits that besides magnetic flux also (Hodge dual) electric fluxes are to be charge-quantized. But this insight has been applied to other flux species only recently (review in [Sati and Schreiber(2025a)]), where it requires new mathematical methods (such as developed in [Fiorenza *et al.*(2023)Fiorenza, Sati, and Schreiber, Sati and Schreiber(2025b)]), since electric Gauss laws are generically non-linear, precluding their quantization in Whitehead-generalized abelian cohomology theories like K-theory (we briefly review this below in § 3).

Yet more pronounced subtleties appear when bulk fluxes are probed by branes that carry their own worldvolume flux densities: For instance, whichever quantization law is chosen for the 4-flux G_4 in the bulk X of 11D Supergravity (cf. [Miemiec and Schnakenburg(2006), §3.1.3]), which is already subtle ([Witten(1997), Diaconescu *et al.*(2007)Diaconescu, Freed, and Moore, Fiorenza *et al.*(2020)Fiorenza, Sati, and Schreiber]), it must affect the quantization of the (itself rather subtle) self-dual 3-form flux H_3 on M5-brane embeddings $\Sigma \xhookrightarrow{\Phi} X$ (cf. [Duff(1999), §3]), due to the *relative Gauss law*, $\mathrm{d}\, H_3 = \Phi^* G_4$ ([Howe and Sezgin(1997), (36)][Sorokin(2000), (5.75)]), that relates the two; but no real method to address this situation had been considered until recently

[Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Fiorenza *et al.*(2021a)Fiorenza, Sati, and Schreiber, Fiorenza *et al.*(2021b)Fiorenza, Sati, and Schreiber] (but cf. [Kalkkinen and Stelle(2003), Figueroa-O'Farrill and Stanciu(2001), Stanciu(2000)]). Here we explore the iteration of this subtlety, namely the global completion of 11D SuGra by flux quantization in the presence of a *sequence* of probe brane embeddings. We also include an "M-string" (cf. [Duff and Lu(1994), §11][Haghighat *et al.*(2015)Haghighat, Iqbal, Kozcaz, Lockhart, and Vafa]) probing the M5-brane probe of the 11D bulk (cf. Fig. 1):

$$\text{M-string} \overset{\phi}{\hookrightarrow} \text{M5-brane} \overset{\Phi}{\hookrightarrow} \text{11D bulk}\,. \tag{1}$$

The end result situation is summarized in Fig. 4.
Besides showcasing (in § 4.1) the algebro-topological technology which allows proper flux quantization of such rich situations, we will demonstrate (in § 4.2) that, on A-type orbi-singularities, the global completion of 11D Sugra with such sequential probe branes naturally admits *geometric engineering* (cf. [Duplij(2017)]) of anyonic topological quantum effects recently seen in experiment (in the conclusion § 5 below).

2 M-String Superembedding into M5

To start with, we apply the *super-embedding formalism* (going back to [Howe *et al.*(1998a)Howe, Sezgin, and West, Howe *et al.*(1998b)Howe, Raetzel, and Sezgin, Sorokin(2000)]) to the construction of M-string probes (cf. [Haghighat *et al.*(2015)Haghighat, Iqbal, Kozcaz, Lockhart, and Vafa] going back to [Duff and Lu(1994), §11]) of M5-brane probes of the 11D SuGra bulk. We follow the super-embedding construction of the M5-brane itself, as laid out in [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Giotopoulos *et al.*(2025)Giotopoulos, Sati, and Schreiber], following [Howe and Sezgin(1997)][Sorokin(2000), §5.2].

2.1 The Spin Geometry

Let Γ_a denote the 11D Clifford generators satisfying (cf. [Miemiec and Schnakenburg(2006), §2.5], we follow [Giotopoulos *et al.*(2024b)Giotopoulos, Sati, and Schreiber, §2.2.1])

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a \;=\; +2\,\eta_{ab}\,\mathrm{id}_{\mathbf{32}}\,, \tag{2}$$

where the tangential Minkowski metric is $\eta := \mathrm{diag}(-1, +1, \cdots, +1)$ and where $\mathbf{32}$ denotes the real 32-dimensional $\mathrm{Spin}(1,10)$-irrep with its $\mathrm{Spin}(1,10)$-invariant skew-symmetric bilinear form denoted

$$\big(\overline{(-)}(-)\big) \;:\; \mathbf{32} \otimes_{\mathbb{R}} \mathbf{32} \longrightarrow \mathbb{R}\,, \tag{3}$$

with respect to which $\overline{\Gamma_a} = -\Gamma_a$. As usual, we denote the Clifford algebra basis elements as

$$\Gamma_{a_1 \cdots a_p} \;:=\; \tfrac{1}{p!} \sum_{\sigma \in \mathrm{Sym}_p} \mathrm{sgn}(\sigma) \Gamma_{a_{\sigma(1)}} \Gamma_{a_{\sigma(2)}} \cdots \Gamma_{a_{\sigma(p)}}$$

and we will use that (cf. [Giotopoulos *et al.*(2024b)Giotopoulos, Sati, and Schreiber, Lem. 2.63])

$$\Gamma_{0123455'6789} \;=\; \mathrm{id}_{\mathbf{32}}\,, \tag{4}$$

reflecting the $\mathcal{N} = 1$ supersymmetry in 11D.
Now, adapted to the tangent geometry of an M-string embedded into an M5-brane embedded into the 11D bulk (cf. Fig. 1), we decompose the tangential basis of Clifford generators like this:

$$\begin{array}{llllllllllllll}
\multicolumn{11}{l}{\overbrace{\overbrace{\overbrace{0\;\;\;1}^{\text{M-string}}\;\;\;2\;\;\;3\;\;\;4\;\;\;5}^{\text{M5-brane}}\;\;\;\overbrace{5'}^{\text{transverse}}\;\;\;6\;\;\;7\;\;\;8\;\;\;9}^{\text{bulk}}} \\
\Gamma_0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_{5'} & \Gamma_6 & \Gamma_7 & \Gamma_8 & \Gamma_9 & \in \mathrm{Pin}^+(1,10) & \subset \mathrm{End}_{\mathbb{R}}(\mathbf{32}) \\
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & & & & & & \in \mathrm{Pin}^+(1,5) & \subset \mathrm{End}_{\mathbb{R}}(2\cdot\mathbf{8}_+ \oplus 2\cdot\mathbf{8}_-) \\
\sigma_0 & \sigma_1 & & & & & & & & & & \in \mathrm{Pin}^+(1,1) & \subset \mathrm{End}_{\mathbb{R}}(16\cdot\mathbf{1}_+ \oplus 16\cdot\mathbf{1}_-)\,.
\end{array} \tag{5}$$

Here the elements of the last two lines are defined by

$$\overline{P}\,\Gamma_a\,P + P\,\Gamma_a\,\overline{P} \;=\; \begin{cases}\gamma_a & \text{for } a \text{ tangential to M5}\\ 0 & \text{for } a \text{ transverse to M5}\end{cases} \tag{6a}$$

$$p\,\gamma_a\,p + p'\,\gamma_a\,p' \;=\; \begin{cases}\sigma_a & \text{for } a \text{ tangential to M1}\\ 0 & \text{for } a \text{ transverse to M1}\,,\end{cases} \tag{6b}$$

with respect to the following projection operators:

$$\begin{array}{lll} P \;:=\; \tfrac{1}{2}(1+\Gamma_{012345}), & & \overline{P} \;:=\; \tfrac{1}{2}(1-\Gamma_{012345}) \\ p \;:=\; \tfrac{1}{2}(1+\gamma_{015'}) \;=\; \overline{p}, & & p' \;:=\; \tfrac{1}{2}(1-\gamma_{015'})\,, \end{array} \tag{7}$$

satisfying

$$\begin{array}{ll} PP = P, & \overline{P}P = 0, \\ \overline{P}\,\overline{P} = \overline{P}, & P\overline{P} = 0 \end{array} \quad \text{and} \quad \left\{\begin{array}{lll} \Gamma_a P = \overline{P}\Gamma_a & \text{for } a \text{ tangential to M5,} \\ \Gamma_a P = P\Gamma_a & \text{for } a \text{ transverse to M5,} \end{array}\right. \tag{8a}$$

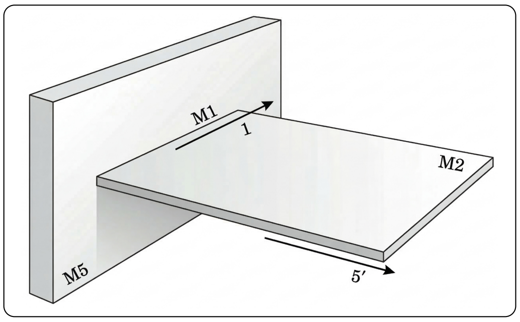
$$\begin{array}{ll} p\,p = p, & p'\,p = 0, \\ p'\,p' = p', & p\,p' = 0 \end{array} \quad \text{and} \quad \left\{\begin{array}{lll} \gamma_a\,p = p\,\gamma_a & \text{for } a \text{ tangential to M1,} \\ \gamma_a\,p = p\,\gamma_a & \text{for } a = 5', \\ \gamma_a\,p = p'\,\gamma_a & \text{for } a \neq 5' \text{ transverse to M1,} \end{array}\right. \tag{8b}$$

and

$$\begin{array}{ll} P\,p = p\,P\,, & \overline{P}\,p = p\,\overline{P}\,, \\ P\,p' = p'\,P\,, & \overline{P}\,p' = p'\,\overline{P}\,. \end{array} \tag{9}$$

The operator P in (7) projects onto that half of the spinors/susy charges which survives on the $^1\!/\!_2$-BPS M5-brane, while p projects onto the further half of that which survives on an M2-brane intersecting the M5 locally in the 01-plane and extending transversally along the $5'$-axis. This is what characterizes the intersection as an *M-string* (cf. [Haghighat *et al.*(2015)Haghighat, Iqbal, Kozcaz, Lockhart, and Vafa, (2.2)]).

Figure 1. The *M-string* (M1) is the transversal intersection of an M2-brane incident onto an M5-brane. In a Darboux frame adapted to the intersection, the tangent spaces of the M5 may be taken to extend along the 012345 axes, that of the M2 along 0125', hence that of the M1 along 01.



First, recall (following [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, §3.2]) the spinor representation on the M5-brane, projected out by the operator P. We observe that P decomposes as

$$\begin{aligned} P \;&\equiv\; \tfrac{1}{2}(1+\Gamma_{012345}) \;=\; \tfrac{1}{2}(1+\Gamma_{5'6789}) \\ &=\; \underbrace{\tfrac{1}{2}(1+\Gamma_{5'})\,\tfrac{1}{2}(1+\Gamma_{6789})}_{P_+} + \underbrace{\tfrac{1}{2}(1-\Gamma_{5'})\,\tfrac{1}{2}(1-\Gamma_{6789})}_{P_-}\,, \end{aligned} \tag{10}$$

with the two summands exhibiting the irrep decomposition of the M5 spinor representation as [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (91)]:

$$P(\mathbf{32}) \underset{\mathrm{Spin}(1,5)}{\simeq} 2\cdot\mathbf{8}_+\,, \qquad \overline{P}(\mathbf{32}) \underset{\mathrm{Spin}(1,5)}{\simeq} 2\cdot\mathbf{8}_-\,, \tag{11}$$

where $\mathbf{8}_\pm$ denote the two real 8-dimensional irreps of $\mathrm{Spin}(1,5)$. In addition, $P(\mathbf{32})$ and $\overline{P}(\mathbf{32})$ inherit a transversal Spin(5)-action (acting in 5'6789, being the *R-symmetry* group from the M5's worldvolume perspective). With respect to that, they are both isomorphic to the $4\cdot\mathbf{4}$ of Spin(5) [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (98)]:

$$2\cdot\mathbf{8}_+ \underset{\mathrm{Spin}(1,5)}{\simeq} P(\mathbf{32}) \underset{\mathrm{Spin}(5)}{\simeq} 4\cdot\mathbf{4} \underset{\mathrm{Spin}(5)}{\simeq} \overline{P}(\mathbf{32}) \underset{\mathrm{Spin}(1,5)}{\simeq} 2\cdot\mathbf{8}_-\,.$$

Our task here is the analogous analysis of the induced spinor rep on the M-string. To that end, notice that, since the group $\mathrm{Spin}(1,1)$ is generated by exponentiating γ_{01}, it has two 1-dimensional spinorial irreps $\mathbf{1}_{\pm}$, characterized, as Clifford modules, by the two possible eigenvalues

$$\gamma_{01}|_{\mathbf{1}_\pm} \;=\; \pm \mathrm{id}_{\mathbf{1}_\pm}\,. \tag{12}$$

With this, we find that:

$$pP(\mathbf{32}) \underset{\mathrm{Spin}(1,1)}{\simeq} 4\cdot \mathbf{1}_+ \oplus 4\cdot\mathbf{1}_- \underset{\mathrm{Spin}(1,1)}{\simeq} p'P(\mathbf{32})\,, \tag{13}$$

For $pP(\mathbf{32})$ this may be seen as follows:

$$\begin{aligned} \Gamma_{01}\,|_{pP(\mathbf{32})} &\underset{(7)}{=} \Gamma_{01}\Gamma_{015'}\Gamma_{012345}\,|_{pP(\mathbf{32})} \\ &\underset{(2)}{=} \Gamma_{0123455'}\,|_{pP(\mathbf{32})} \\ &\underset{(4)}{=} \Gamma_{6789}\,|_{pP(\mathbf{32})} \\ &\underset{(10)}{=} +\mathrm{id}\,|_{pP_+(\mathbf{32})} - \mathrm{id}\,|_{pP_-(\mathbf{32})}\,, \end{aligned} \tag{14}$$

and analogously for $p'P(\mathbf{32})$:

$$\begin{aligned} \Gamma_{01}\,|_{p'P(\mathbf{32})} &\underset{(7)}{=} -\Gamma_{01}\Gamma_{015'}\Gamma_{012345}\,|_{p'P(\mathbf{32})} \\ &= -\cdots\,. \end{aligned} \tag{15}$$

Moreover, the spinor subspaces $pP(\mathbf{32})$ and $p'P(\mathbf{32})$ inherit compatible representations of $\mathrm{Spin}(4)_L$ operating in 2345 and of $\mathrm{Spin}(4)_R$ operating in 6789 (cf. Fig. 2).

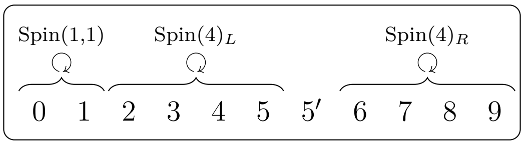


Figure 2. The presence of an M-string on an M5-brane breaks the local Lorentz spin symmetry from $\mathrm{Spin}(1,5)\times\mathrm{Spin}(5)$ to $\mathrm{Spin}(1,1)\times\mathrm{Spin}(4)_L\times\mathrm{Spin}(4)_R$.

In order to analyze this transversal group action, observe that the projectors p and p' decompose as:

$$\begin{aligned} p &= \tfrac{1}{2}(1+\Gamma_{015'}) = \tfrac{1}{2}(1+\Gamma_{23456789}) \\ &= \tfrac{1}{2}(1+\Gamma_{2345})\,\tfrac{1}{2}(1+\Gamma_{6789}) - \tfrac{1}{2}(1-\Gamma_{2345})\,\tfrac{1}{2}(1-\Gamma_{6789})\,, \\ p' &= \tfrac{1}{2}(1-\Gamma_{015'}) = \tfrac{1}{2}(1-\Gamma_{23456789}) \\ &= \tfrac{1}{2}(1-\Gamma_{2345})\,\tfrac{1}{2}(1+\Gamma_{6789}) - \tfrac{1}{2}(1+\Gamma_{2345})\,\tfrac{1}{2}(1-\Gamma_{6789})\,, \end{aligned} \tag{16}$$

respectively. Combined with (10) this yields:

$$\begin{aligned} pP &= \tfrac{1}{2}(1+\Gamma_{2345})\,\tfrac{1}{2}(1+\Gamma_{5'})\,\tfrac{1}{2}(1+\Gamma_{6789}) + \tfrac{1}{2}(1-\Gamma_{2345})\,\tfrac{1}{2}(1-\Gamma_{5'})\,\tfrac{1}{2}(1-\Gamma_{6789})\,, \\ p'P &= \tfrac{1}{2}(1-\Gamma_{2345})\,\tfrac{1}{2}(1+\Gamma_{5'})\,\tfrac{1}{2}(1+\Gamma_{6789}) + \tfrac{1}{2}(1+\Gamma_{2345})\,\tfrac{1}{2}(1-\Gamma_{5'})\,\tfrac{1}{2}(1-\Gamma_{6789})\,. \end{aligned} \tag{17}$$

We see that $pP(\mathbf{32})$ is $\mathrm{Spin}(4)_L$-equivariantly isomorphic to $p'P(\mathbf{32})$, for instance via the action of $\Gamma_{5'6}$, using that $\Gamma_{5'6}\,p\,P = p'\,P\,\Gamma_{5'6}$ (8):

$$pP(\mathbf{32}) \underset{\mathrm{Spin}(4)_L}{\simeq} p'P(\mathbf{32})\,. \tag{18}$$

On the other hand, the two spinor spaces are *not* $\mathrm{Spin}(4)_L\times\mathrm{Spin}(4)_R$-equivariantly isomorphic. This is because any such isomorphism has to commute with $\Gamma_{2345}\Gamma_{6789}$, while (17) shows that this product operator acts differently on the two spaces:

$$pP(\mathbf{32}) \underset{\mathrm{Spin}(4)_L\times\mathrm{Spin}(4)_R}{\not\simeq} p'P(\mathbf{32})\,. \tag{19}$$

More concretely, with

$$\pi := \tfrac{1}{2}(1+\Gamma_{2345}), \qquad \pi' := \tfrac{1}{2}(1-\Gamma_{2345}),$$

we have

$$\begin{aligned} \mathrm{Spin}(4)_L &\simeq \mathrm{Spin}(3)\times\mathrm{Spin}(3) \\ &\simeq \exp\big(\langle \pi\Gamma_{23}, \pi\Gamma_{24}, \pi\Gamma_{34}\rangle\big)\exp\big(\langle \pi'\Gamma_{23}, \pi'\Gamma_{24}, \pi'\Gamma_{34}\rangle\big), \end{aligned} \tag{20}$$

whence $\mathrm{Spin}(4)_L$-representations on which Γ_{2345} has positive or negative eigenvalue are left-chiral, $(\mathbf{n},\mathbf{1})$, or right-chiral, $(\mathbf{1},\mathbf{n})$, respectively, as representations of $\mathrm{Spin}(3)\times\mathrm{Spin}(3)$. But since the analogous statement holds also for $\mathrm{Spin}(4)_R$, it follows with (17) (using also $\Gamma_{5'}|_{pP(\mathbf{32})} = \Gamma_{01}|_{pP(\mathbf{32})}$ and $\Gamma_{5'}|_{p'P(\mathbf{32})} = -\Gamma_{01}|_{p'P(\mathbf{32})}$) that (for the first line, cf. [Haghighat *et al.*(2015)Haghighat, Iqbal, Kozcaz, Lockhart, and Vafa, (2.4)]): [1]

$$\begin{aligned} pP(\mathbf{32}) &\underset{\mathrm{Spin}(1,1)\times\mathrm{Spin}(4)_L\times\mathrm{Spin}(4)_R}{\simeq} \mathbf{1}_+\boxtimes(\mathbf{2},\mathbf{1})\boxtimes(\mathbf{2},\mathbf{1}) \;\oplus\; \mathbf{1}_-\boxtimes(\mathbf{1},\mathbf{2})\boxtimes(\mathbf{1},\mathbf{2}), \\ p'P(\mathbf{32}) &\underset{\mathrm{Spin}(1,1)\times\mathrm{Spin}(4)_L\times\mathrm{Spin}(4)_R}{\simeq} \mathbf{1}_+\boxtimes(\mathbf{2},\mathbf{1})\boxtimes(\mathbf{1},\mathbf{2}) \;\oplus\; \mathbf{1}_-\boxtimes(\mathbf{1},\mathbf{2})\boxtimes(\mathbf{2},\mathbf{1}). \end{aligned} \tag{21}$$

In particular, this confirms (19), which is the relevant statement for the following superembedding construction.

2.2 *The Iterated Embedding*

Given an 11D SuGra spacetime $X^{1,10}$, we consider a sequence of 1/2-BPS superembeddings [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, §2.2]: first of a super M5 worldvolume into the 11D bulk, and then of a string worldsheet into the M5. We use the following notation to describe this situation (cf. again Fig. 1):

$$\begin{array}{lccccc} & \textbf{M-string} & & \textbf{M5-brane} & & \textbf{11D bulk} \\ \text{Worldvolume} & N^{1,1} & \overset{\phi}{\hookrightarrow} & \Sigma^{1,5} & \overset{\Phi}{\hookrightarrow} & X^{1,10} \\ \text{Super-coframe} & (\varepsilon,\chi) & & (e,\psi) & & (E,\Psi) \\ \text{Spin connection} & \varpi & & \omega & & \Omega \\ \text{Clifford algebra} & \sigma & & \gamma & & \Gamma\,. \end{array} \tag{22}$$

Since our goal is the analysis of Gauss laws/Bianchi identities for flux densities on these objects, it will be sufficient to determine the torsion constraints, which we obtain as (28) and (31) below. However, to determine this, we first need to analyze the iterated superembedding in more detail.

In (22), the coframes are understood to be adapted to the superembeddings (*Darboux coframes*, cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, §2.1]) in that

$$\phi^* e^a = \begin{cases} \varepsilon^a & \text{for } a \text{ tangential to M1} \\ 0 & \text{for } a \text{ transversal to M1} \end{cases}, \qquad \Phi^* E^a = \begin{cases} e^a & \text{for } a \text{ tangential to M5} \\ 0 & \text{for } a \text{ transversal to M5.} \end{cases} \tag{23}$$

But the supergeometric generalization of this Darboux condition allows, *a priori*, for an odd component of the superembedding to involve *fermionic shear* operators ([Howe and Sezgin(1997), (15)], cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (106)]),

$$\tilde{\not{H}}_1 := (\tilde{H}_1)_a \sigma^a \Gamma_{5'6}, \qquad \tilde{\not{H}}_3 := \tfrac{1}{3}(\tilde{H}_3)_{a_1a_2a_3}\gamma^{a_1a_2a_2}, \tag{24}$$

---

[1] Pedantically, the "$\boxtimes$" in (21) denotes the correct *external tensor product* (cf. [Fulton and Harris(1991), Ex. 2.36][Gruson and Serganova(2018), p. 3]) of representations of different groups G_i, which is the precomposition of the ordinary tensor product "$\otimes$", of representations of the product group $G_1\times G_2 \xrightarrow{\mathrm{pr}_i} G_i$, with the operation of extension (pullback pr_i^*) of reps of the factor groups to the product group:

$$\begin{tikzcd} \mathrm{Rep}(G_1)\times\mathrm{Rep}(G_2) \arrow[r, "\mathrm{pr}_1^*\times\mathrm{pr}_2^*"] \arrow[rr, bend left, "\boxtimes"] & \mathrm{Rep}(G_1\times G_2)\times\mathrm{Rep}(G_1\times G_2) \arrow[r, "\otimes"] & \mathrm{Rep}(G_1\times G_2)\,. \end{tikzcd}$$

appearing in the pullback of the ambient fermion coordinates like this:

$$\begin{tikzcd} & \overbrace{2\cdot\mathbf{8}_+ \oplus 2\cdot\mathbf{8}_-}^{\mathbf{32}} \arrow[d, ->>] \\ T\Sigma \arrow[ur, "\Phi^*\Psi = (\psi,\, \tilde{\not{H}}_3\psi)"] \arrow[r, "\psi", "\text{odd coframe on M5}"'] & 2\cdot\mathbf{8}_+\,, \end{tikzcd} \tag{25a}$$

$$\begin{tikzcd} & \overbrace{(4\cdot\mathbf{1}_+ \oplus 4\cdot\mathbf{1}_-)\oplus(4\cdot\mathbf{1}_- \oplus 4\cdot\mathbf{1}_+)}^{2\cdot\mathbf{8}_+} \arrow[d, ->>] \\ TN \arrow[ur, "\phi^*\psi = (\chi,\, \tilde{\not{H}}_1\chi)"] \arrow[r, "\chi", "\text{odd coframe on M1}"'] & 4\cdot\mathbf{1}_+ \oplus 4\cdot\mathbf{1}_-\,. \end{tikzcd} \tag{25b}$$

The shear operator $\tilde{H}_3$ in (24), which is necessarily self-dual (cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Lem. 3.8]), is the origin of the non-linearly self-dual 3-form flux H_3 on the M5 (cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Prop. 3.18, Rm. 3.19] and (49) below). Its square (cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (70)])

$$\big(\tilde{H}_3{}^2\big)_a^b \;:=\; \big(\tilde{H}_3\big)_{ac_1c_2}\big(\tilde{H}_3\big)^{bc_1c_2} \tag{26}$$

will play a key role shortly (28).
The shear operator $\tilde{\not{H}}_1$ in (24) is superficially an analogous source of a 1-form flux on the M-string (the factor $\Gamma_{5'6}$ is needed to make it actually shear into the orthogonal spin representation, by above (18)). We will see shortly (30) that $\tilde{\not{H}}_1$ is dynamically constrained to vanish — and the remainder of the article, in § 3 and § 4, is concerned with explaining how vanishing M-brane flux may still have topologically nontrivial and noteworthy gauge potentials.
However, we want to highlight that $\tilde{\not{H}}_1$ should be taken to vanish already on more fundamental grounds: Namely, the conceptual origin of the fermionic shears (25) among super-embeddings is (cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Rem. 2.22]) the possibility that half of the spinors/susy charges which survive on the embedded brane is *equivariantly* isomorphic to the broken orthogonal half, with respect to the transversal Lorentz spin symmetry.
This traditional rule of superembedding needs refinement in our case of iterated superembeddings: While $\tilde{\not{H}}_1$ in (25) is $\mathrm{Spin}(4)_L$-equivariant (18), as befits a string superembedding into an 5-brane *seen in isolation*, it is *not* (19) equivariant with respect to the actual larger transversal symmetry group $\mathrm{Spin}(4)_L \times \mathrm{Spin}(4)_R$, which takes into account that the M5-brane itself is embedded into a larger bulk (cf. Fig. 2).
So if we demand, as appears reasonable, that the rule for *iterated superembeddings* must be their equivariance under the full iterated transversal symmetry group, then $\tilde{\not{H}}_1$ in (24) must vanish already on these grounds, and with it also any other shear component $\tilde{\not{H}}_0$ and $\tilde{\not{H}}_2$, which could otherwise be present for an M-string embedding.

2.3 The Torsion Constraint

The dynamics of supergravity is largely controlled by *torsion constraints* (cf. [Lott(1990)]). In 11D, this constraint just says that the bosonic components of the *super*-torsion tensor of the super-spin geometry (E, Ψ, Ω) in (22) vanish (cf. [Howe(1997)]):

$$\mathrm{d}E^a + \Omega^a{}_b E^b - \big(\,\overline{\Psi}\Gamma^a\Psi\big) \;=\; 0\,. \tag{27}$$

Indeed, the equations of motion of 11D SuGra are already equivalent ([Giotopoulos *et al.*(2024b)Giotopoulos, Sati, and Schreiber, Thm. 3.1]) to this super-torsion constraint combined with the Gauss/Bianchi identity on the super-flux densities (51), we come back to this remarkable phenomenon in §3.3 below.
Pulling back the 11D torsion constraint (27) along the super-embedding Φ (22), of an M5-brane probe with a fermionic shear $\tilde{\not{H}}_3$ (25), yields (among other equations of motion,

cf. [Sorokin(2000), §5.2]) the *torsion constraint on the M5-brane* [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (116)]:

$$\mathrm{d}\, e^a + \omega^a{}_b\, e^b \;=\; M^a_{a'}\big(\overline{\psi}\,\gamma^{a'}\,\psi\big)\,, \quad \text{where} \quad M^a_{a'} \;:=\; \big(\delta^a_{a'} - 2(\tilde{H}_3{}^2)^a_{a'}\big)\,. \tag{28}$$

In this vein, the further pullback of this M5-torsion constraint (28) along an M-string super-embedding ϕ (22) yields the equations of motion of the M-string. Here we focus on the following two components:

(i) Since the pullback of the M1-transversal component of (28) along ϕ has no (ψ^2)-terms on the left, by (23), specifically the pullback of the $5'$ component of (28) gives the following condition, by (8a) and as in [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (118)]: [2]

$$\begin{aligned} 0 \;&=\; \phi^*\big(\overline{\psi}\gamma^{5'}\psi\big) \\ &=\; \big(\overline{\chi}\,\overline{P}\,(1 - \tilde{\not{H}}_1)\,\gamma^{5'}\,(1 + \tilde{\not{H}}_1)P\,\chi\big) \\ &=\; 2\big(\overline{\chi}\,\tilde{\not{H}}_1\gamma^{5'}\,\chi\big)\,. \end{aligned} \tag{29}$$

This means that (cf. the discussion on p. 6):

$$\tilde{\not{H}}_1 \;\equiv\; 0\,. \tag{30}$$

(ii) With that, the pullback of the M1-tangential component of (28) immediately gives the *torsion constraint on the M-string*:

$$\mathrm{d}\,\varepsilon^a + \varpi^a{}_b\,\varepsilon^b \;=\; (\phi^* M)^a_{a'}\big(\overline{\chi}\,\sigma^{a'}\,\chi\big)\,. \tag{31}$$

(Here all indices are in $\{0, 1\}$, including those being summed over.)

We shall assume in the following that the 3-flux density on the M5-brane is non-critical on the M-string, meaning that the coefficient matrix on the right of (31) is invertible: [3]

$$\det(\phi^* M) \;\overset{!}{\neq}\; 0\,. \tag{32}$$

This is the case on an open neighborhood around $\tilde{H}_3 = 0$, hence in particular in the commonly considered *small field limit* in which the corresponding 3-flux H_3 density (which is a quadratic function of $\tilde{H}_3$) is approximately Hodge self-dual, constituting the famous self-dual tensor field in the $D = 6$, $\mathcal{N} = (2, 0)$ worldvolume SCFT (cf. [Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Rem. 3.19]).
Beyond (30) and (31), there are further constraints implied by the pullback of the torsion constraint (28) along the M-string superembedding, and these translate to the equations of motion of the M-string inside the M5-brane. Here, we do not further dwell on this local dynamics but turn attention now to global topological aspects that have not previously found the attention they deserve.
Crucially, we want to highlight that if with (30) we identify a 1-form flux density on the M-string worldvolume which is *on-shell vanishing*,

$$H_1 \,\in\, \Omega^1_{\mathrm{dR}}\big(N^{1,1}\big)\,, \qquad H_1 \;=\; 0\,, \tag{33}$$

then this is *only locally* redundant as field content. Globally, there may be nontrivial topological quantum observables associated with the global completion of the H_1 field. This phenomenon is familiar from (abelian) Chern-Simons theory (whose flux density F_2 also vanishes on-shell, $F_2 = 0$), but it may remain underappreciated in the context of M-branes. Our next goal is to work out a topological global completion of the field content of 11D SuGra probed by M5-branes probed by M-strings, including the 1-form flux (33).

[2] The other transverse components with $a' \in \{2, 3, 4, 5, 6, 7, 8, 9\}$ give no further constraints, since for them the term analogous to (29) vanishes identically, now using the properties of p (8b):

$$\begin{aligned} \phi^*\big(\overline{\psi}\gamma^{a'}\psi\big) \;&=\; \big(\overline{\chi}\,p\,(1 - \tilde{\not{H}}_1)\,\gamma^{a'}\,(1 + \tilde{\not{H}}_1)p\,\chi\big) \\ &=\; \big(\overline{\chi}\,pp'\,(1 - \tilde{\not{H}}_1)\,\gamma^{a'}\,(1 + \tilde{\not{H}}_1)\,\chi\big) \;=\; 0\,. \end{aligned}$$

[3] An exclamation mark over a relational symbol means (as usual, cf. [Greiner(2004)]) that this is an imposed condition instead of a consequence.

3 Proper Flux Quantization

Beyond local equations of motion on flux densities, the global definition of (higher) gauge fields requires a choice of *flux/charge quantization* law [Sati and Schreiber(2025a)]. This is in principle well-known, but in practice most discussions still focus on just the field content visible on a single chart of spacetime. We highlight the rich space of choices involved in defining a theory's field content globally.

In the generality that we need here, which includes electric fluxes satisfying non-linear Gauss laws, flux quantization takes place not just in Whitehead-generalized cohomology (such as ordinary cohomology or K-theory, cf. [Freed(2002)]) but in *nonabelian cohomology* [Sati and Schreiber(2025a), Sati and Schreiber(2024), Fiorenza *et al.*(2023)Fiorenza, Sati, and Schreiber].

We now give a brief review, leading from Dirac's original insight into charge quantization in Maxwell theory to the proper flux quantization of magnetized M5-brane probes, which is the starting point for flux quantization on the M-string in § 4 below.

Aimed at a theoretical physics audience, the following assumes some familiarity with basic concepts of algebraic topology, but will sacrifice mathematical fine print for ease of readability.

3.1 Flux Quantization

The notion of flux/charge quantization goes back to Dirac's foundational observation that, in modern paraphrase, the ordinary magnetic Gauss law

$$\mathrm{d}F_2 = 0 \qquad \text{magnetic Gauss law} \tag{34}$$

entails that globally the electromagnetic field consists, beyond its *flux density* F_2, also of a map from spacetime $X^{1,3}$ to a *classifying space*, like [4]

$$X^{1,3} \dashrightarrow B\mathrm{U}(1) \simeq K(\mathbb{Z},2) \qquad \text{global charge}\,, \tag{35}$$

representing a class in the ordinary cohomology $H^2\big(X^{1,3};\mathbb{Z}\big)$: the *magnetic charge* which is sourcing the magnetic flux.

More recently it is understood that also the vacuum electric Gauss law $\mathrm{d}F_2' = 0$ (for $F_2' := \star_4 F_2$) is potentially quantized, an admissible classifying space being the product space $K(\mathbb{Z},2)\times K(\mathbb{Z},2)$.

The general rule for flux quantization is essentially the following: [5] Given a set of (electric & magnetic) flux densities $F^{(i)}$ (differential forms of some degree $\deg(i)$) satisfying Gauss/Bianchi identities

$$\mathrm{d}F^{(i)} = P^{(i)}\big(F^{(1)}, F^{(2)}, \cdots\big) \tag{36}$$

for graded-symmetric polynomials $P^{(i)}$, then the admissible classifying spaces $\mathcal{A}$ are those whose real cohomology is computed by the cohomology of these differential relations when the $F^{(i)}$ are regarded as abstract algebra generators $f^{(i)}$:

$$\text{Admissible global charge} \quad X \dashrightarrow \mathcal{A} \quad \text{if} \quad H^\bullet(\mathcal{A}) \overset{!}{\simeq} \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})}\,, \quad \text{where } \mathrm{d}f^{(i)} = P^{(i)}\big(f^{(1)}, f^{(2)}, \cdots\big)\,. \tag{37}$$

For example, the real cohomology of the higher classifying space $B^n\mathrm{U}(1)$ (the *Eilenberg-MacLane space* $K(\mathbb{Z}, n+1)$) is the graded polynomial algebra on single closed generator f_{n+1}, $\mathrm{d}f_{n+1} = 0$; whence this is the standard candidate classifying space for plain n-form gauge fields:

$$\text{magnetic Gauss law} \quad \mathrm{d}F_{n+1} = 0\,, \qquad X^{1,d} \dashrightarrow K(\mathbb{Z}, n+1) \quad \text{global charge}\,. \tag{38}$$

This gets doubled when also the electric fluxes $F'_{d-n} := \star F_{n+1}$ are quantized:

$$\text{electro/magnetic Gauss laws} \quad \begin{aligned} \mathrm{d}F_{n+1} &= 0 \\ \mathrm{d}F'_{d-n} &= 0\,, \end{aligned} \qquad X^{1,d} \dashrightarrow \begin{array}{c} K(\mathbb{Z}, n+1) \\ \times K(\mathbb{Z}, d-n) \end{array} \quad \text{global charge}\,. \tag{39}$$

[4] We denote by dashed arrows maps that represent unspecified physical field configurations, in contrast to solid arrows which denote fixed background structure, such as brane embeddings, cf the overview Fig. 4.

[5] The math behind (36) is that of *minimal Sullivan models* in rational homotopy theory (cf. [Fiorenza *et al.*(2023)Fiorenza, Sati, and Schreiber, §5], as reviewed in [Sati and Schreiber(2025a), §3.1]), closely related to the "FDAs" of the SuGra literature (cf. [Fiorenza *et al.*(2019)Fiorenza, Sati, and Schreiber]).

For another example, the real cohomology of the classifying space $\mathrm{KU}_0 := \cup_n BU(n) \times \mathbb{Z}$ for *K-theory* is the graded polynomial algebra on closed generators $f_{2\bullet}$, $\mathrm{d}f_{2\bullet} = 0$ in all even degrees (the *Chern classes*); whence this is a famously conjectured candidate classifying space for the RR-field in type IIA supergravity (cf. [Sati and Schreiber(2025a), §4.1]):

$$\underset{\text{Gauss laws}}{\text{electro/magnetic}} \quad \mathrm{d}F_{2\bullet} = 0\,, \qquad X^{1,d} \dashrightarrow \mathrm{KU} \quad \underset{\text{charge}}{\text{global}}\,. \tag{40}$$

Notice how in this case the electric flux densities $F_{2\bullet>5}$ are quantized by the same classifying space as the magnetic flux densities $F_{2\bullet<5}$, not splitting off via a product space as in (39).
So far, these examples all involve vanishing differentials, hence *linear* Gauss laws whose solutions form a vector space. This is the case when the classifying space is an (infinite) *loop space* like a stage in a *spectrum of spaces* representing a Whitehead-generalized abelian cohomology theory like K-theory, elliptic cohomology or stable cobordism. Flux quantization in this linear/abelian case has been discussed in [Freed(2002)] (cf. also [Sati and Schreiber(2023)]).
However, this is not the general case. Notably the Gauss law for the electric flux $G_7 := \star G_4$ in 11D supergravity is famously quadratic, $\mathrm{d}\,\mathrm{G}_7 = \frac{1}{2} G_4 \wedge G_4$. This means that the common idea that the magnetic 4-flux G_4 can be quantized in $K(\mathbb{Z}, 4)$ (and be it a "shifted" version of that as proposed in [Hopkins and Singer(2005)], cf. [Fiorenza *et al.*(2015)Fiorenza, Sati, and Schreiber, Prop. 3.1.1]), as in (38), necessarily fails to account for any electric flux quantization as in (39).
But from (37) one sees how to fix this: We need a variant $\mathcal{A}$ of $K(\mathbb{Z}, 4)$ which still carries a cohomology class f_4, but the square f_4^2 of which vanishes in real cohomology. This is well-known (cf. [Menichi(2015), §1.2][Fiorenza *et al.*(2023)Fiorenza, Sati, and Schreiber, Ex. 5.3][Sati and Schreiber(2025a), (17,27)]) to be the case on the 4-sphere $S^4 \subset K(\mathbb{Z}, 4)$ which hence is a candidate classifying space for C-field charge (cf. [Fiorenza *et al.*(2017)Fiorenza, Sati, and Schreiber, §2][Sati(2018), §2.5][Fiorenza *et al.*(2019)Fiorenza, Sati, and Schreiber, §7]):

$$\underset{\text{Gauss laws}}{\text{electro/magnetic}} \quad \begin{aligned} \mathrm{d}\,G_7 &= \tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\,G_4 &= 0\,, \end{aligned} \qquad X^{1,d} \dashrightarrow S^4 \quad \underset{\text{charge}}{\text{global}}\,. \tag{41}$$

As with all flux quantizations, there are infinitely many other choices one could make for $\mathcal{A}$. For instance, with any finite group K, also $\mathcal{A} := S^4 \times BK$ satisfies the condition (37). But S^4 is in a precise sense the minimal choice of classifying space (having a single "cell") for 11D SuGra.
Since the nonabelian cohomology theory classified by spheres is called *co-Homotopy* ([Hu(1959), §VII][Fiorenza *et al.*(2023)Fiorenza, Sati, and Schreiber, Ex. 2.7], dual to the *homotopy* groups of maps out of spheres), the hypothesis that this is the "correct" choice for the global completion of 11D SuGra has been called *Hypothesis H* [Fiorenza *et al.*(2020)Fiorenza, Sati, and Schreiber, Fiorenza *et al.*(2021a)Fiorenza, Sati, and Schreiber, Sati and Schreiber(2023)]. This hypothesis is supported by how it provably implies subtle topological effects that are expected in the completion of 11D SuGra to "M-theory". In particular, the tangentially twisted version of 4-Cohomotopy (41) (which we disregard here just for brevity) does imply [Fiorenza *et al.*(2020)Fiorenza, Sati, and Schreiber, Prop. 3.13] the subtle half-integral shifting (by $\frac{1}{4}p_1$) of the de Rham class of G_4 that had famously been argued for in [Witten(1997), (1.2)].

3.2 Relative Flux Quantization

More generally, Gauss laws hold *relatively*. For instance, in general the ordinary electric Gauss law for $F_2' := \star F_2$ is of course

$$\mathrm{d}\,F_2' = J_3\,. \tag{42}$$

Here, the *electric current density* J_3, which deforms the vacuum Gauss law (39), is spatially compactly supported. As such, it is defined on the spatial 1-point compactification $X^{1,3}_{\mathrm{cpt}}$, while the law (42) holds on $X^{1,3} \hookrightarrow X^{1,3}_{\mathrm{cpt}}$. This way, (42) does not imply that the total electric charge

$$Q := -e \int_{X^3_{\mathrm{cpt}}} J_3 \tag{43}$$

vanishes, and in fact we have that $-Q/e \in \mathbb{Z}$ is the net number of electrons.
But this means that the current density J_3 is classified by $K(\mathbb{Z},3)$. To trivialize on $X^{1,3}$ the corresponding classifying map equivalently means to lift through the *path fibration* $PK(\mathbb{Z},3) \twoheadrightarrow K(\mathbb{Z},3)$ like this:

$$\begin{tikzcd} X^{1,3} \arrow[r, dashed] \arrow[d, hook] & PK(\mathbb{Z},3) \arrow[d, two heads] \\ X^{1,3}_{\mathrm{cpt}} \arrow[r, "-Q/e"] & K(\mathbb{Z},3)\,. \end{tikzcd} \tag{44}$$

This is hence the general relative form of electric flux quantization. Notice that the fiber of the path fibration is a $K(\mathbb{Z},2)$, so that (44) indeed reduces to (35) where $Q = 0$.
The general rule for relative flux quantization, in generalization of (37), is essentially the following: Consider an embedding $Y \overset{\Phi}{\hookrightarrow} X$ with flux densities $F^{(i)}$ on Y satisfying

$$\mathrm{d}\,F^{(i')} = P^{(i')}\big(F^{(1)}, F^{(2)} \cdots, \Phi^* J^{(1)}, \Phi^* J^{(2)}, \cdots\big)\,, \tag{45}$$

for *background flux densities* $J^{(i)}$ on X that themselves are subject to differential equations $\mathrm{d}J^{(i)} = P^{(i)}\big(J^{(1)}, , \cdots, J^{(\sharp_J)}\big)$ as before for F in (36). Then, given an admissible classifying space $\mathcal{B}$ for the J-s according to the corresponding version of (37), the admissible *classifying fibrations* $\mathcal{A} \twoheadrightarrow \mathcal{B}$ are those whose real cohomology of their total spaces is computed by the cohomology of these differential relations when the $F^{(i')}$ and $J^{(i)}$ are regarded as abstract algebra generators $f^{(i')}$ and $j^{(i)}$, respectively:

$$\begin{array}{c}\text{admissible}\\ \text{relative}\\ \text{global charge}\end{array} \quad \begin{tikzcd} Y \arrow[r, dashed] \arrow[d, hook, "\Phi"] & \mathcal{A} \arrow[d, two heads, "\mathcal{P}"] \\ X \arrow[r] & \mathcal{B} \end{tikzcd} \quad \text{if} \quad H^\bullet(\mathcal{A}) \overset{!}{\simeq} \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})}\,, \; \begin{cases} \mathrm{d} f^{(i')} = P^{(i')}\big(f^{(1)}, \cdots, j^{(1)}, \cdots\big) \\ \mathrm{d}j^{(i)} = P^{(i)}\big(j^{(1)}, j^{(2)}, \cdots\big)\,. \end{cases} \tag{46}$$

For example, in generalization of (40), now with h_3, $\mathrm{d}h_3 = 0$ denoting the generator of the cohomology of $K(\mathbb{Z},3)$ as in (38), consider the real cohomology of the classifying fibration $\mathrm{KU}/\!\!/K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z},3)$ for *twisted K-theory*. This is the h_3-twisted cohomology on even degree generators $f_{2\bullet}$, whence this qualifies as a classifying fibration for RR-flux $F_{2\bullet}$ twisted by N units of NS-flux H_3 in type IIA supergravity:

$$\begin{array}{c}\text{electro/magnetic}\\ \text{twisted}\\ \text{Gauss laws}\end{array} \quad \begin{aligned} \mathrm{d}\,F_{2\bullet+2} &= H_3 \wedge F_{2\bullet} \\ \mathrm{d}H_3 &= 0\,, \end{aligned} \qquad \begin{tikzcd} X^{1,9} \arrow[r, dashed] \arrow[d, hook] & \mathrm{KU}/\!\!/K(\mathbb{Z},2) \arrow[d, two heads] \\ X^{1,9}_{\mathrm{cpt}} \arrow[r, "N"] & K(\mathbb{Z},3) \end{tikzcd} \quad \begin{array}{c}\text{relative}\\ \text{global}\\ \text{charge}\end{array}. \tag{47}$$

The example of interest in the following is this: To obtain the cohomology of the total space of the *quaternionic Hopf fibration* (cf. Fig. 3)

$$\begin{tikzcd} S^3 \arrow[r] & S^7 \arrow[d, two heads] \\ & S^4 \end{tikzcd} \tag{48}$$

(which is concentrated in degree 7), starting with the generators g_4 and g_7 which present the cohomology of the 4-sphere in (41), we need to add a generator h_3 which removes g_4 from cohomology. But this models just the Gauss law for the self-dual 3-form flux density on probe M5-brane worldvolumes $\Sigma^{1,5} \hookrightarrow X^{1,10}$, which thus has admissible flux quantization in relative Cohomotopy-twisted Cohomotopy, as shown on the right here ([Fiorenza *et al.*(2020)Fiorenza, Sati, and Schreiber, §7.3][Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, (20)][Fiorenza *et al.*(2021a)Fiorenza, Sati, and Schreiber, §3]):

$$\begin{array}{c}\text{electro/magnetic}\\ \text{twisted}\\ \text{Gauss laws}\end{array} \quad \begin{aligned} \mathrm{d}\,H_3 &= \Phi^* G_4 \\ \mathrm{d}\,G_7 &= \tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\,G_4 &= 0\,, \end{aligned} \qquad \begin{tikzcd} \Sigma^{1,5} \arrow[r, dashed] \arrow[d, hook, "\Phi"] & S^7 \arrow[d, two heads] \\ X^{1,10} \arrow[r] & S^4 \end{tikzcd} \quad \begin{array}{c}\text{relative}\\ \text{global}\\ \text{charge}\end{array}. \tag{49}$$

Figure 3. The *quaternionic Hopf fibration* $h_{\mathbb{H}}$ and its factorization through the *twistor fibration* $t_{\mathbb{C}}$ is given by sending $\mathbb{R}_+$-lines in quaternionic 2-space $\mathbb{H}^2$ first to the $\mathbb{C}$-lines which they span, and then further to the $\mathbb{H}$ lines which these span, using the canonical inclusions $\mathbb{R}_+ \subset \mathbb{C} \subset \mathbb{H}$ of the positive real numbers into the complex numbers and further into the quaternions (cf. [Fiorenza *et al.*(2022)Fiorenza, Sati, and Schreiber, §2][Sati and Schreiber(2026a), Rem. 5.2.16]).

$$
\begin{tikzcd}
S^7 \arrow[rr, two heads, "h_{\mathbb{C}}"] \arrow[rrrr, bend left, "h_{\mathbb{H}}"] \arrow[d, equal] && \mathbb{C}P^3 \arrow[rr, two heads, "t_{\mathbb{C}}"] \arrow[d, equal] && \overbrace{\mathbb{H}P^1}^{S^4} \arrow[d, equal] \\
\big(\mathbb{H}^2 - \{0\}\big)/\mathbb{R}_+ \arrow[rr, two heads] && \big(\mathbb{H}^2 - \{0\}\big)/\mathbb{C}^\times \arrow[rr, two heads] && \big(\mathbb{H}^2 - \{0\}\big)/\mathbb{H}^\times
\end{tikzcd}
$$

There are various further variants of the situation (49): For example, one may consider the *double dimensional reduction* (via *cyclification* of classifying spaces, [Braunack-Mayer *et al.*(2019)Braunack-Mayer, Sati, and Schreiber, §2.2]), of these flux-quantized M5s probing 11D SuGra, to D4-branes probing type IIA 10D SuGra (cf. [Banerjee(2026)]). Another variant is to adjoin "topological fluxes" to the system; we consider this next.

3.3 Super Flux Quantization

Since the key to quantizing (higher) flux densities is – as we have recalled – their electro/magnetic Gauss laws, it is desirable to have a good grasp on how examples of these come about. A profound observation here is that, when seen on superspace, the Gauss laws imposed on super-flux densities are close to the full equations of motions.
For an 11D super-spacetime with supertorsion-free super-vielbein (E, Ψ), consider the following super-flux lift of the ordinary flux densities (G_4, G_7), where (E, Ψ) is a super-coframe (22) satisfying the torsion constraint (27):

$$
\begin{aligned}
G_4^s &:= \tfrac{1}{4!}(G_4)_{a_1\cdots a_4} E^{a_1} \cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{ab}\,\Psi\big) E^a E^b \\
G_7^s &:= \tfrac{1}{7!}(G_7)_{a_1\cdots a_7} E^{a_1} \cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big) E^{a_1} \cdots E^{a_5} \,.
\end{aligned}
\tag{50}
$$

The form of the Gauss law (41) imposed on (50) is *equivalent* to the equations of motion of 11D supergravity ([Giotopoulos *et al.*(2024b)Giotopoulos, Sati, and Schreiber, Thm. 3.1]):

$$
\left.
\begin{aligned}
\mathrm{d}\, G_7^s &= \tfrac{1}{2} G_4^s \wedge G_4^s \\
\mathrm{d}\, G_4^s &= 0
\end{aligned}
\right\}
\quad \Leftrightarrow \quad \text{on-shell 11D Sugra.}
\tag{51}
$$

Similarly, with (e, ψ) the induced super-coframe on the M5-brane worldvolume, (23), the twisted Gauss law (49) imposed on H_3 regarded on super-space,

$$
H_3^s := \tfrac{1}{3!}(H_3)_{a_1 a_2 a_3} e^{a_1} e^{a_2} e^{a_3} + 0 \,,
\tag{52}
$$

gives the (full non-linear) self-duality constraint and the equation of motion on the H_3 flux ([Giotopoulos *et al.*(2024a)Giotopoulos, Sati, and Schreiber, Prop. 3.18]):

$$
\left. \mathrm{d}\, H_3^s = \Phi^* G_4^s \right\} \quad \Leftrightarrow \quad \text{on-shell self-dual tensor field.}
\tag{53}
$$

While largely classical, this superspace perspective on supergravity perhaps offers an underappreciated powerful perspective on *non-Lagrangian* theory building:
We may define theories by imposing Gauss laws on super-flux, followed by flux quantization.
For example, consider adjoining, in this manner, to the H_3^s-flux on the M5 super-worldvolume a super 2-flux of the form

$$
F_2^s := \tfrac{1}{2}(F_2)_{a_1 a_2} e^{a_1} e^{a_2} \,.
\tag{54}
$$

Now the non-degeneracy (32) of the torsion constraint on the M5-brane (28) implies (cf. [Sati and Schreiber(2025c), p. 7]) that imposing on this super-flux the ordinary magnetic Gauss law (does not affect the previous equations of motion but otherwise) is equivalent to $F_2 = 0$, hence:

$$
\left. \mathrm{d}\, F_2^s = 0 \right\} \quad \Leftrightarrow \quad \text{on-shell abelian Chern-Simons}\,.
\tag{55}
$$

However, this has the remarkable consequence that we may add polynomials in a closed super-flux F_2^s to the Gauss law (53) for the self-dual tensor flux on superspace, *without changing the local theory*, notably like this:

$$\left.\begin{aligned} \mathrm{d}\, H_3^s \;&=\; \Phi^* G_4^s - \theta\, F_2^s F_2^s \\ \mathrm{d}\, F_s^2 \;&=\; 0 \end{aligned}\right\} \quad\Leftrightarrow\quad \text{on-shell self-dual tensor field,} \tag{56}$$

for any *theta angle* θ.
This modified Gauss/Bianchi identity for H_3 systematically implies ([Sati and Schreiber(2025d), p. 38][Banerjee(2025), (83)] following [Giotopoulos *et al.*(2024b)Giotopoulos, Sati, and Schreiber, p. 22]) that it is locally, on an open cover $\widehat{X} \overset{\iota}{\twoheadrightarrow} X$ of spacetime, expressed in terms of *gauge potential* forms C_3, B_2, A_1 on $\widehat{X}$ as

$$\iota^* H_3 \;=\; C_3 + \mathrm{d}B_2 + \theta\, \mathrm{CS}(A_1)\,, \tag{57}$$

(where $\mathrm{CS}(A_1) \;=\; A_1 \wedge \mathrm{d}A_1$ is the Chern-Simons form, here in the abelian case).
Expressions of this form for the M-theory 3-form have previously been considered in [Donagi and Wijnholt(), (2.1)][Fiorenza *et al.*(2014)Fiorenza, Sati, and Schreiber, (2.1.15)][Fiorenza *et al.*(2015)Fiorenza, Sati, and Schreiber, §4.1], following [Diaconescu *et al.*(2007)Diaconescu, Freed, and Moore], and earlier in [Evslin and Sati(2003), (3.3)].
But while the local dynamics of flux densities does not change under this addition, when the Chern-Simons equations of motion hold (55), the flux-quantized global theory does change substantially when $\theta \neq 0$, because the flux quantization changes, since $S^7 \twoheadrightarrow S^4$ (49) is no longer an admissible classifying fibration.
To see how the flux quantization law gets modified for $\theta \neq 0$, notice that when imposed on abstract algebra generators g_4 and f_2, the law in (56) then says that g_4 represents the same cohomology class as f_2^2, whence the trivialization of g_4^2 in cohomology (by g_7) is then equivalent to the trivialization of f_2^4 in cohomology. But that is the relation on the cohomology of complex projective 3-space (cf. [Menichi(2015), §5.3]), which is fibered over S^4 via the *twistor fibration* ([Atiyah(1978), §III.1], cf. [Fiorenza *et al.*(2022)Fiorenza, Sati, and Schreiber, p. 6]):

$$\begin{array}{ccc} S^2 & \longrightarrow & \mathbb{C}P^3 \\ & & \big\downarrow \\ & & S^4. \end{array} \tag{58}$$

Therefore this is, by (46), an admissible relative flux quantization law for 11D SuGra with such "magnetized" M5-brane probes, for $\theta \neq 0$ ([Sati and Schreiber(2025c), §4] following [Fiorenza *et al.*(2022)Fiorenza, Sati, and Schreiber], cf. [Sati and Schreiber(2025d), §2]):

$$\underset{\substack{\text{twisted}\\ \text{Gauss laws}}}{} \begin{aligned} \mathrm{d}\, H_3 \;&=\; \Phi^* G_4 - F_2 \wedge F_2 \\ \mathrm{d}\, F_2 \;&=\; 0 \\ \mathrm{d}\, G_7 \;&=\; \tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\, G_4 \;&=\; 0\,, \end{aligned} \qquad \begin{array}{ccc} \Sigma^{1,5} & \dashrightarrow & \mathbb{C}P^3 \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow \\ X^{1,10} & \dashrightarrow & S^4 \end{array} \;\; \underset{\substack{\text{relative}\\ \text{global}\\ \text{charge}}}{} . \tag{59}$$

We next bring the M-string into this picture.

4 M-String Flux Quantization

4.1 The Law

In the manner of (54), we may consider adjoining to the super-flux densities in the 11D bulk (50) and on a M5-probe (52) also a super-flux on the M-string super-worldsheet (22), of the form

$$H_1^s \;:=\; (H_1)_a \varepsilon^a + 0 \tag{60}$$

and subjected to the Gauss/Bianchi law

$$\mathrm{d}\, H_1^s \;=\; \phi^* F_2^s\,. \tag{61}$$

Recall the subtle purely global/topological effect on the SuGra/brane dynamics that is controlled by such a further Gauss law: We have already seen in (55) that the Bianchi identity for F_2^s forces it to vanish on-shell. Therefore, the presence of F_2^s does not affect the on-shell dynamics at the level of flux densities, but it does crucially change the global/ topological behavior of the theory. Concretely, a flux density F_2 with $F_2 = 0$ induces a topological gauge field of abelian Chern-Simons type.
Analogously, the above Gauss law (61) is, in turn, equivalent to the condition that H_1 vanishes, compatible with the superembedding result (30) and (33):

$$\left.\begin{aligned} 0 \;&\underset{(61)}{=}\; \mathrm{d}\,H_1^s - \phi^* F_2^s \\ &\underset{(55)}{=}\; \mathrm{d}\,H_1^s \\ &\underset{(60)}{=}\; \mathrm{d}\big((H_1)_a \varepsilon^a\big) \\ &\underset{(31)}{=}\; \big(\nabla_b (H_1)_a\big)\varepsilon^b\varepsilon^a \\ &\quad + (H_1)_a (\phi^* M)^a_{a'} \big(\overline{\chi}\sigma^{a'}\chi\big) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} \big(\nabla_b (H_1)_a\big)\varepsilon^b\varepsilon^a \;&=\; 0 \\ (H_1)_a (\phi^* M)^a_{a'} \big(\overline{\chi}\sigma^{a'}\chi\big) \;&=\; 0 \end{aligned}\right\} \underset{(32)}{\Leftrightarrow} \big\{ H_1 \;=\; 0\,. \tag{62}$$

In particular, adjoining the Gauss law (61) again does not change the equations of motion of the underlying $\{$11D SuGra + branes$\}$-system, at the level of field strengths.
But it does introduce topological effects. We may already see a hint of this at the level of the local gauge potentials, for notice that gauge potential 0-form λ corresponding to H_1 in (55) is characterized, on general grounds [Banerjee(2025), (97)] by the law

$$\mathrm{d}\lambda \;=\; H_1 - \phi^*(A_1)\,, \tag{63}$$

where A_1 is the Chern-Simons field from (57). But for $H_1 = 0$ (62) this becomes exactly the defining equation [Wen(1992), above (2.62)]

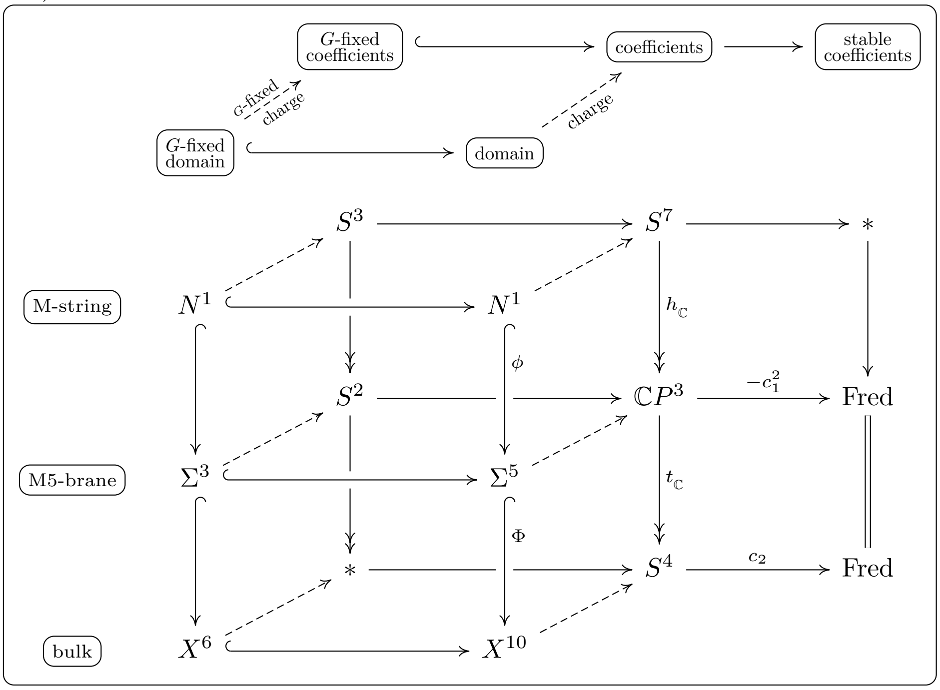
$$\mathrm{d}\lambda \;=\; -\phi^*(A_1) \tag{64}$$

for an abelian WZW field expected to emerge as a boundary effect of our abelian Chern-Simons field (55) (we come back to this at the end, § 4.3).
Indeed, adjoining H_1-flux as in (61) with (62) does affect the admissible global topological completions of the field content on the sequence of brane embeddings, as follows.
Namely, with the above Gauss law (60) adjoined to the system of Gauss laws (59) on the magnetized M5 and in the bulk, the twistor fibration (58) by itself is no longer an admissible relative flux quantization law, as in (59). This is due to the general rule in (46): An admissible classifying fibration that accounts for (60) needs its real cohomology to be computed on abstract algebra generators involving, besides the previous f_2, also a further generator h_1 whose differential is $\mathrm{d}h_1 = f_2$. This removes the class of f_2 from cohomology, contrary to the situation on $\mathbb{C}P^3$.
In fact, the only cohomology class that survives in this case is that of g_7, which may be understood as representing the unit cohomology class on the 7-sphere. Indeed, an admissible relative flux quantization of (60) *relative* to the twistorial situation (59) is the sequence of fibrations which is obtained by factoring the quaternionic Hopf fibration (48) through the twistor fibration (58) (cf. Fig. 3) – this follows by
[Fiorenza *et al.*(2022)Fiorenza, Sati, and Schreiber, Lem. 2.13]:

$$\underset{\text{Gauss laws}}{\text{twisted}}\quad \begin{aligned} \mathrm{d}\,H_1 \;&=\; \phi^* F_2 \\ \mathrm{d}\,H_3 \;&=\; \Phi^* G_4 - F_2 \wedge F_2 \\ \mathrm{d}\,F_2 \;&=\; 0 \\ \mathrm{d}\,G_7 \;&=\; \tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\,G_4 \;&=\; 0\,, \end{aligned} \qquad \begin{tikzcd} N^{1,1} \arrow[r, dashed] \arrow[d, hook, "\phi"] & S^7 \arrow[d] \\ \Sigma^{1,5} \arrow[r, dashed] \arrow[d, hook, "\Phi"] & \mathbb{C}P^3 \arrow[d] \\ X^{1,10} \arrow[r, dashed] & S^4 \end{tikzcd} \quad \underset{\text{charge}}{\overset{\text{relative}}{\text{global}}}\,. \tag{65}$$

[7] For further discussion of Fig. 4 see, for the right part: [Sati and Schreiber(2023), (97, 232)], [Sati and Schreiber(2025e), Fig. 6] for the top left part: [Sati and Schreiber(2025e), Fig. 5], for the bottom rear part: [Sati and Schreiber(2025c)], and for the bottom middle part: [Sati and Schreiber(2025c), (25)], [Sati and Schreiber(2025d), p. 13].

Figure 4. Overview[7] of the system of brane embeddings, of classifying fibrations and of charge classifying maps between these, for the M-brane on magnetized M5-branes in 11D Sugra, properly flux-quantized according to (65). Solid maps are given (brane embeddings and classifying fibrations) while dashed maps are dynamical data (charge sectors of fields).



Let us highlight the consequences for the topological charges carried by these fields: Recall that for ordinary Dirac charge quantization (35), the total magnetic charge is given by the homotopy class of the classifying map from spacetime to $B\mathrm{U}(1)$:

$$\text{magnetic charges} \;\in\; \pi_0\Big(\mathrm{Map}\big(X^{1,3}, B\mathrm{U}(1)\big)\Big) \;\simeq\; H^2\big(X^{1,3};\mathbb{Z}\big)\,. \tag{66}$$

Generally, for $\mathcal{A}$ the classifying space of a flux quantization law (37), we have:

$$\text{charges} \;\in\; \mathrm{Map}(X, \mathcal{A})\,. \tag{67}$$

As we pass to relative flux quantization (46), the single classifying map is generalized to a pair of maps forming a commuting square with the domain embedding Φ and the classifying fibration $\mathcal{P}$. If we denote the topological space of such compatible pairs of classifying maps as[8]

$$\mathrm{Map}(\Phi, \mathcal{P}) \;:=\; \left\{ \begin{array}{ccc} Y & \dashrightarrow & \mathcal{A} \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}} \\ X & \dashrightarrow & \mathcal{B} \end{array} \right\}, \tag{68}$$

then the sets of relative charges of bulk/brane systems Φ whose fluxes are classified by the fibration $\mathcal{P}$ form the connected components of this relative mapping space:

$$\text{relative charges} \;\in\; \pi_0\,\mathrm{Map}(\Phi, \mathcal{P})\,. \tag{69}$$

In this manner, if we next have a sequence $\xhookrightarrow{\phi}\,\xhookrightarrow{\Phi}$ of brane embeddings, as in (22), and a sequence $\overset{\wp}{\twoheadrightarrow}\,\overset{\mathcal{P}}{\twoheadrightarrow}$ of classifying fibrations for their relative flux quantization, as in (65),

[8]More precisely, the relative mapping space (68) is the *fiber product* of $\mathrm{Map}(X, \mathcal{B})$ with $\mathrm{Map}(Y, \mathcal{A})$ with respect to their joint fibration over $\mathrm{Map}(Y, \mathcal{B})$ (the former by precomposition with Φ, the latter by postcomposition with $\mathcal{P}$). That this topological space has the correct homotopy type follows (by the existence of the Quillen model structure on topological spaces) from $\mathcal{P}$ being a (Serre-) fibration and Φ a cofibration. An analogous comment also holds for the doubly relative mapping space (70) below.

then the connected components of the "doubly relative mapping space"

$$
\mathrm{Map}\big((\phi,\Phi),(\wp,\mathcal{P})\big) \;:=\; \left\{ \begin{tikzcd} {} \arrow[r, dashed] \arrow[d, hook, "\phi"] & {} \arrow[d, "\wp"] \\ {} \arrow[r, dashed] \arrow[d, "\Phi"] & {} \arrow[d, "\mathcal{P}"] \\ {} \arrow[r, dashed] & {} \end{tikzcd} \right\}
\tag{70}
$$

are the doubly relative charges that may be carried by this system:

$$
\text{doubly relative charges} \;\in\; \pi_0\,\mathrm{Map}\big((\phi,\Phi),(\wp,\mathcal{P})\big)\,.
\tag{71}
$$

We highlight that all this is the topological incarnation of iterated relative Gauss/Bianchi identities (45).

4.2 Orbifolding

To see the implications of this doubly relative flux quantization on M-branes (65), it is expedient to first consider the situation on A-type orbifold singularities, where the setup simplifies and can be related to more familiar topological effects (cf. §4.3).
The *orbifold cohomology* (cf. [Sati and Schreiber(2026a)]) of global orbifold quotients that we are concerned with here is modeled by equipping both the domain spaces and the classifying spaces with continuous actions of a group G and then constraining the classifying maps c to be *G-equivariant*

$$
G\text{-orbifold charges} \;\;\in\;\; \pi_0\,\mathrm{Map}(X,\mathcal{A})^G \;:=\; \pi_0\left\{ \begin{tikzcd} X \arrow[loop above, "G"] \arrow[r, dashed, "c"] & \mathcal{A} \arrow[loop above, "G"] \end{tikzcd} \right\},
\tag{72}
$$

in that for all $x \in X$ and $g \in G$ we have:

$$
c(g\cdot x) \;=\; g\cdot c(x)\,.
\tag{73}
$$

An important consequence of this equivariance condition is that *G-fixed points*

$$
X^G \;:=\; \big\{ x \in X \,\big|\, \forall_g \,:\, g\cdot x \,=\, x \big\} \,\subset\, X
\tag{74}
$$

must be mapped to G-fixed points:

$$
\begin{tikzcd} X \arrow[loop above, "G"] \arrow[r, dashed, "c"] & \mathcal{A} \arrow[loop above, "G"] \\ X^G \arrow[u, hook] \arrow[r, dashed] & \mathcal{A}^G \arrow[u, hook] \end{tikzcd}\,.
\tag{75}
$$

In particular, the (iterated) mapping spaces (70) between G-spaces become themselves G-spaces by the evident G-conjugation action, and their G-fixed loci are the (iterated) spaces of equivariant maps:

$$
\mathrm{Map}\big((\phi,\Phi),(\wp,\mathcal{P})\big)^G \;:=\; \left\{ \begin{tikzcd} {} \arrow[loop left, "G"] \arrow[r, dashed] \arrow[d, hook, "\phi"] & {} \arrow[loop right, "G"] \arrow[d, "\wp"] \\ {} \arrow[loop left, "G"] \arrow[r, dashed] \arrow[d, "\Phi"] & {} \arrow[loop right, "G"] \arrow[d, "\mathcal{P}"] \\ {} \arrow[loop left, "G"] \arrow[r, dashed] & {} \arrow[loop right, "G"] \end{tikzcd} \right\}
\tag{76}
$$

The joint homotopy classes of such equivariant maps are the (relative) orbifold charges (cf. [Sati and Schreiber(2020), (3)][Sati and Schreiber(2025e), Fig. 7][Sati and Schreiber(2026a)]), in generalization of (71):

$$
\text{Doubly relative orbifold charges} \;\in\; \pi_0\,\mathrm{Map}\big((\phi,\Phi),(\wp,\mathcal{P})\big)^G.
\tag{77}
$$

4.3 A-Singularities

Specifically for A_{n-1}-type orbi-singularities, the equivariance group is finite cyclic

$$G := \mathbb{Z}_{/n},$$

and the action on the domain space is locally a product with $\mathbb{C}^2$ on which $[k] \in \mathbb{Z}_{/n}$ acts as follows:

$$\overset{\curvearrowright G}{\mathbb{H}} \simeq_{\mathbb{R}} \overset{\curvearrowright G}{\mathbb{C}} \times \overset{\curvearrowright G}{\overline{\mathbb{C}}} \quad : \quad [k]\cdot(z_1, z_2) := \big(e^{2\pi \mathrm{i} k/n} z_1,\, e^{-2\pi \mathrm{i} k/n} z_2\big). \tag{78}$$

As indicated, this action is in fact right $\mathbb{H}$-linear with respect to the identification $\mathbb{C}^2 \simeq_{\mathbb{R}} \mathbb{H}$, being the left multiplication action with unit quaternions in the image of the inclusion

$$\mathrm{U}(1) \subset \mathrm{SU}(2) \simeq S(\mathbb{H}). \tag{79}$$

Therefore we may take the G-action on the classifying fibrations (65) to be "of the same form" as on spacetime, namely with G acting as in (78) on one of the two $\mathbb{C}^2 \simeq \mathbb{H}$-factors:

$$\begin{tikzcd}
\overset{\curvearrowright G}{S^7} \arrow[rr, ->>, "t_{\mathbb{C}}"] \arrow[d, equal] && \overset{\curvearrowright G}{\mathbb{C}P^3} \arrow[rr, ->>, "t_{\mathbb{H}}"] \arrow[d, equal] && \overset{\curvearrowright G}{S^4} \arrow[d, equal] \\
\underset{\circlearrowright G}{(\mathbb{C}^2 \times \mathbb{C}^2 - \{0\})/\mathbb{R}_+} \arrow[rr, ->>] && \underset{\circlearrowright G}{(\mathbb{C}^2 \times \mathbb{C}^2 - \{0\})/\mathbb{C}^\times} \arrow[rr, ->>] && \underset{\circlearrowright G}{(\mathbb{C}^2 \times \mathbb{C}^2 - \{0\})/\mathbb{H}^\times}\,.
\end{tikzcd} \tag{80}$$

It is manifest from this definition that the corresponding fibration of fixed loci (75) is the map from S^3 to the point, factored through the ordinary complex Hopf fibration $h_{\mathbb{C}}$:

$$\begin{tikzcd}
(S^7)^G \arrow[r, ->>, "h_{\mathbb{C}}^G"] \arrow[d, equal] & (\mathbb{C}P^3)^G \arrow[r, ->>, "t_{\mathbb{C}}^G"] \arrow[d, equal] & (S^4)^G \arrow[d, equal] \\
S^3 \arrow[r, ->>, "h_{\mathbb{C}}"] \arrow[d, equal] & S^2 \arrow[r, ->>] \arrow[d, equal] & * \arrow[d, equal] \\
\big(\mathbb{C}^2 - \{0\}\big)/\mathbb{R}_+ \arrow[r, ->>] & \big(\mathbb{C}^2 - \{0\}\big)/\mathbb{C}^\times \arrow[r, ->>] & \big(\mathbb{C}^2 - \{0\}\big)/\mathbb{H}^\times\,.
\end{tikzcd} \tag{81}$$

Consider then (as in [Couzens *et al.*(2025)Couzens, Lüscher, and Sparks, p. 5]) an M5-brane worldvolume wrapping the half-space

$$\overset{\curvearrowright G}{\mathbb{C}} \subset \overset{\curvearrowright G}{\mathbb{C}^2}$$

of the A-type spacetime singularity (78), with the M-string embedded transverse to the singularity:

$$\begin{tikzcd}
N^{1,1} \arrow[d, hook, "\phi"] & \equiv & N^{1,1} \arrow[d, hook] \\
\Sigma^{1,5} \arrow[d, hook, "\Phi"] & := & \Sigma^{1,3} \arrow[d, equal] & \times & \overset{\curvearrowright G}{\mathbb{C}} \arrow[d, equal] \\
X^{1,10} & := & \Sigma^{1,3} & \times & \underset{\circlearrowright G}{\mathbb{C}} & \times & \underset{\circlearrowright G}{\overline{\mathbb{C}}} & \times & X^3.
\end{tikzcd}$$

This configuration is evidently G-equivariantly homotopic to its G-fixed locus (74) (by linearly contracting the non-compact singularity to the point, $\mathbb{C}^2 \sim *$). Since the (iterated) equivariant mapping spaces (76) are homotopy-invariant under equivariant homotopy equivalence, the orbifold charges (77) reduce, by (75), to plain charges (71) on the fixed locus with G-fixed coefficients (81), cf. [Sati and Schreiber(2025c), p.

7][Sati and Schreiber(2025b), §4]:

$$
\left\{
\begin{tikzcd}
N^{1,1} \arrow[loop left, "G"] \arrow[r, dashed] \arrow[d, hook, "\phi"] & S^7 \arrow[loop right, "G"] \arrow[d, "h_{\mathbb{C}}"] \\
\Sigma^{1,5} \arrow[loop left, "G"] \arrow[r, dashed] \arrow[d, hook, "\Phi"] & \mathbb{C}P^3 \arrow[loop right, "G"] \arrow[d, "t_{\mathbb{C}}"] \\
X^{1,10} \arrow[loop left, "G"] \arrow[r, dashed] & S^4 \arrow[loop right, "G"]
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
N^{1,1} \arrow[r, dashed] \arrow[d, hook, "\phi"] & S^3 \arrow[d, "h_{\mathbb{C}}"] \\
\Sigma^{1,3} \arrow[r, dashed] & S^2
\end{tikzcd}
\right\}. \tag{82}
$$

But this is, on the right, exactly the *Hopfion*-like anyonic charge structure of Fractional Quantum (Anomalous) Hall (FQ(A)H) systems (by [Sati and Schreiber(2025f), Sati and Schreiber(2026b), Sati and Schreiber(2025g)]), with the M-string now playing the role of a gapped nodal line [Sati and Schreiber(2026a), §2.2, Fig. 2.3] on which (64) exhibits edge mode excutations [Wen(1992), §2.5][Tong(2016), §6.1.2]. This phenomenon, that anyonic topological order may be *geometrically engineered* (cf. [Duplij(2017)]) on M5-branes wrapping (here: orbisingular) Seifert fibrations, was also argued (by very different, informal Lagrangian means) in [Cho *et al.*(2020)Cho, Gang, and Kim].

5 Conclusion

Discussions of field theories, notably in the context of supergravity with brane probes, traditionally tend to restrict attention to the local field content given by flux densities defined on a single chart of spacetime, ignoring the global topological nature of the (higher) gauge fields. We have recalled in §3 how global completion requires a choice of *flux quantization* given by (fibrations of) topological *classifying spaces* for the charges. This involves a great freedom of choice, reflecting a rich and largely unexplored space of flux-quantized Gauss laws/Bianchi identities. Ordinary electromagnetic Dirac charge quantization as well as K-theory quantization of RR-flux are just two examples of a general theory.

With a general theory of flux quantization in hand, we may in particular treat more subtle situations such as *relative flux quantization* where bulk fluxes are probed by branes that carry their own worldvolume flux, a crucial step that has traditionally not found attention. The key example here is the M5-brane with its self-dual worldvolume flux probing the bulk of 11D supergravity with its C-field flux.

In this vein, here we went one step further and considered the situation of sequences of iterated brane species, concretely the case where the M5-brane probing the 11D bulk is itself probed by an M-string (M1), for which we developed relevant aspects of its superembedding construction in §2.

This development is logically independent from the web of conjectures in the string theory folklore: We are using the on-shell superspace formulation of supergravity with super p-branes, and consider its systematic global completion by flux quantization; all our statements stand rigorously on their own feet.

Concretely, in §4 we have explored one possible flux quantization law, (65), for M1-probes of M5-probes in 11D, highlighting its resemblance to previous conjectures about the structure of M-theory 3-forms (57) and its implication of previously conjectured topological order engineered on M5-branes wrapping Seifert fibers (§4.3) — now enhanced by the presence of gapped nodal lines represented by the M-string.

This choice crucially involves another underappreciated aspect of global field completion: Since the non-Lagrangian on-shell superspace construction of 11D SuGra and its branes prescribes only flux densities and not their gauge potentials (which instead are brought into existence by compatible flux quantization), there is actually the freedom to adjoin Chern-Simons-type gauge fields, whose flux densities identically vanish on-shell (55)(62). These are invisible to the local physics but their topological effect shows up in the global completion of the theory. This phenomenon is implicit in previous proposals, cf. (57), but remains underappreciated.

We have used this freedom in global completion to exhibit, in §4.1, a Chern-Simons-like 1-form flux H_1 on the M-string, which is not visible from the usual local superembedding construction but is consistent with it. This choice is motivated by and appears rather

suggestive due to the fact (§4.3) that it implies on A-type orbisingularities, where the M5-brane engineers FQAH-type topological order, that the M-string engineers the gapped nodal lines of the system.
However, we emphasize that this is just one choice out of an infinite space of other admissible choices. For instance, if we omitted the (on-shell vanishing) source term on the right of (61) then instead of the classifying space S^7 in (65), non-trivially Hopf-fibered over $\mathbb{C}P^3$, an admissible classifying space at that stage would instead be the trivial circle bundle space $\mathbb{C}P^3 \times S^1$, with less interesting topological consequences.
Hence, independent of the concrete choice of flux quantization on the M-string explored here, our discussion highlights that *some* such choice is crucially to be made when passing from the traditional local discussion of fluxes on M-branes to their globally completed theories. Exploring this realm of globally completed brane-probed supergravity remains a wide-open field, to which we suggest we have opened the door a little.

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Data availability
No new data were created or analysed in this study.

Conflict of interest
The authors declare that they have no conflict of interest.

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