Lemma 3.24. *Let C_n be a cyclic group of order n. The rationalized complex representation ring of C_n is isomorphic to a product of cyclotomic fields*

$$\mathbb{Q} \otimes R_{\mathbb{C}}(C_n) \cong \prod_{d|n} \mathbb{Q}(\zeta_d),$$

where ζ_d denotes a primitive d-th root of 1, and d runs over the divisors of n. Moreover, the idempotent e_{C_n} of $\mathbb{Q} \otimes R_{\mathbb{C}}(C_n)$ corresponding to $\mathbb{Q}(\zeta_n)$ has the following property. If $C_n < G$ and $Q = N_G(C_n)/C_G(C_n)$, then the fixed-point space

$$(e_{C_n}(\mathbb{Q} \otimes R_{\mathbb{C}}(C_n)))^Q = \bigoplus \mathbb{Q}$$

is a $\mathbb{Q}$-vector space of dimension equal to the number of G-conjugacy classes of elements of order n in C_n, a number which equals $\varphi(n)/|Q|$.

Proof. One first observes that as a ring,

$$\mathbb{Q} \otimes R_{\mathbb{C}}(C_n) \cong \mathbb{Q}[C_n] \cong \mathbb{Q}[x]/(x^n - 1).$$

Factoring $x^n - 1$ into $\mathbb{Q}$-irreducible polynomials yields the decomposition of the representation ring into cyclotomic fields. By identifying the Galois group $G(n)$ of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ with the automorphism group $(\mathbb{Z}/n\mathbb{Z})^\times$ of C_n, we can view Q as a subgroup of $G(n)$. Since $\mathbb{Q}(\zeta_n)$ is a free $\mathbb{Q}[G(n)]$-module of rank one, it is free as $\mathbb{Q}[Q]$-module, of rank equal the index $[G(n) : Q] = \varphi(n)/|Q|$. It follows that the Q-invariant part of $\mathbb{Q}(\zeta_n)$ has dimension $\varphi(n)/|Q|$ as a vector space over $\mathbb{Q}$; by the choice of Q this equals the number of G-conjugacy classes of generators of C_n. □

Theorem 3.25. *Let G be an arbitrary group. Then one has*

$$H^i_{\mathfrak{Fin}}(G; \mathbb{Q} \otimes R_{\mathbb{C}}) = H^i_{\mathfrak{Fin}}(\underline{E}G; \mathbb{Q} \otimes R_{\mathbb{C}}) \cong \prod_{[x] \in \mathrm{FC}(G)} H^i(C_G(x); \mathbb{Q})$$

and

$$H_i^{\mathfrak{Fin}}(G; \mathbb{Q} \otimes R_{\mathbb{C}}) = H_i^{\mathfrak{Fin}}(\underline{E}G; \mathbb{Q} \otimes R_{\mathbb{C}}) \cong \bigoplus_{[x] \in \mathrm{FC}(G)} H_i(C_G(x); \mathbb{Q}).$$

The product (resp. sum) is taken over $\mathrm{FC}(G)$*, the set of conjugacy classes of elements of finite order in G. The right hand sides denote ordinary group (co)homology of the centralizers $C_G(x)$, with constant coefficients $\mathbb{Q}$.*

Proof. We follow the ideas of Lück and Oliver [93, Section 5], which have their root in Slominska's paper [124]. We will concentrate on the case of $\mathbb{Q} \otimes R_{\mathbb{C}} = R^\sharp$; the other case is similar. Let $Z(G)$ denote the set of conjugacy classes of finite, cyclic subgroups of G. If S is a finite cyclic subgroup of G, we write $S \in [S] \in Z(G)$. For a finite subgroup $H < G$ we define the idempotent $e_{S,H} \in \mathbb{Q} \otimes R_{\mathbb{C}}(H)$ to be the restriction of the class function $e_S : G \to \mathbb{C}$, whose value on $g \in G$ is 1 if the subgroup $\langle g \rangle$ generated by g is conjugate to S, and zero otherwise. That $e_S|H$ is indeed a rational linear combination of characters (even of characters of

$\mathbb{Q}$-representations) follows from standard facts on representations of finite groups (cf. [121]). There is a natural splitting

$$R^\sharp = \prod_{[S]\in Z(G)} R^\sharp_S,$$

where $R^\sharp_S$ denotes the contravariant functor given on objects by $G/H \mapsto e_{S,H}(\mathbb{Q}\otimes R_{\mathbb{C}}(H))$. We therefore obtain a splitting

$$H^i_{\mathfrak{Fin}}(G;\mathbb{Q}\otimes R_{\mathbb{C}}) \cong \prod_{[S]\in Z(G)} H^i_{\mathfrak{Fin}}(G;R^\sharp_S).$$

Now $R^\sharp_S(G/H)$ is 0 if $[S]$ contains no representative $gSg^{-1} < H$, and in the other case is isomorphic to $\mathbb{Q}(\zeta_{|S|})^N$, with N the normalizer of some $gSg^{-1} < H$, acting via an identification of a generator of gSg^{-1} with $\zeta_{|S|}$. It follows that for any $M \in \mathrm{Mod}_{\mathfrak{Fin}}\text{-}G$

$$\mathrm{mor}(M, R^\sharp_S) \cong \mathrm{Hom}_{N_G(S)}(M(G/S), \mathbb{Q}(\zeta_{|S|}))$$

where $N_G(S)$ acts on $\mathbb{Q}(\zeta_{|S|})$ via an identification of a generator of S with $\zeta_{|S|}$. Therefore,

$$\mathrm{mor}(\underline{C_*(\underline{E}G)}, R^\sharp) \cong \prod_{[S]\in Z(G)} \mathrm{Hom}_{N_G(S)}(C_*(\underline{E}G^S), \mathbb{Q}(\zeta_{|S|}))\,.$$

Recall that for any group K and any finite subgroup $L < K$, the $\mathbb{Q}[K]$-module $\mathbb{Q}[K/L]$ is projective, since it is isomorphic to the induced module $\mathbb{Q}[K]\otimes_L \mathbb{Q}$ and $\mathbb{Q}$ is a projective $\mathbb{Q}[L]$-module. Since, in the notation above, $N_G(S)$ acts properly on the space $\underline{E}G^S$, the complex $C_*(\underline{E}G^S)$ is in each degree $* > 0$ a sum of permutation modules of the form $\mathbb{Z}[N_G(S)/H]$ with H finite. Therefore, using that $\underline{E}G^S$ is contractible, $C_*(\underline{E}G^S)\otimes\mathbb{Q}$ is a $\mathbb{Q}[N_G(S)]$-projective resolution of $\mathbb{Q}$ as a trivial $\mathbb{Q}[N_G(S)]$-module. It follows that

$$H^*(\mathrm{Hom}_{N_G(S)}(C_*(\underline{E}G^S), \mathbb{Q}(\zeta_{|S|}))) \cong H^*(N_G(S);\mathbb{Q}(\zeta_{|S|})).$$

The short exact sequence

$$C_G(S) \rightarrowtail N_G(S) \twoheadrightarrow Q(S),$$

with $Q(S)$ a finite group of order dividing $\varphi(|S|)$, yields a collapsing Serre spectral sequence, with edge isomorphism

$$H^*(N_G(S);\mathbb{Q}(\zeta_{|S|})) \cong H^*(C_G(S);\mathbb{Q}(\zeta_{|S|}))^{Q(S)}.$$

Using the previous lemma, and the fact that taking $Q(S)$-invariants commutes with taking rational homology, we see that

$$H^*(C_G(S);\mathbb{Q}(\zeta_{|S|}))^{Q(S)} \cong \prod_{\varphi(|S|)/|Q(S)|} H^*(C_G(S);\mathbb{Q}).$$

Since every conjugacy class $[S]$ of subgroups of order n in G corresponds to $\varphi(n)/|Q(S)|$ conjugacy classes of elements of order n, the result follows. □

Remark 3.26. The splitting $R_\sharp = \bigoplus R_{\sharp,S}$ (and similarly for $R^\sharp$) can also be used to decompose $H_*^{\mathfrak{F}\mathrm{in}}(X; R_\mathbb{C} \otimes \mathbb{Q})$ (resp. $H^*_{\mathfrak{F}\mathrm{in}}(X; R_\mathbb{C} \otimes \mathbb{Q})$) for an arbitrary proper G-CW-complex X. One finds

$$\underline{C_*(X)} \otimes_{\mathfrak{F}} R_\sharp \cong \bigoplus_{[S]\in Z(G)} C_*(X^S) \otimes_{N_G(S)} R_{\sharp,S}(S)$$

and with $W_G(S) = N_G(S)/C_G(S)$, as $C_G(S)$ acts properly on X^S, the obvious map

$$\bigoplus_{[S]\in Z(G)} C_*(X^S) \otimes_{N_G(S)} R_{\sharp,S}(S) \to \bigoplus_{[S]\in Z(G)} C_*(X^S/C_G(S)) \otimes_{W_G(S)} R_{\sharp,S}(S)$$

is a homology isomorphism, showing that

$$H_i^{\mathfrak{F}\mathrm{in}}(X; R_\mathbb{C} \otimes \mathbb{Q}) \cong \bigoplus_{[S]\in Z(G)} H_i(X^S/C_G(S); \mathbb{Q}) \otimes_{W_G(S)} R_{\sharp,S}(S).$$

We used here that $R_{\sharp,S}(S)$ is a projective $\mathbb{Q}[W_G(S)]$-module, because $W_G(S)$ is a finite group. As we have seen, the $\mathbb{Q}$-vector space

$$R_{\sharp,S}(S)^{W_G(S)} \cong \mathbb{Q} \otimes_{W_G(S)} R_{\sharp,S}(S)$$

has dimension $\phi(|S|)/|Q(S)|$, which is the number of G-conjugacy classes of generators of S. This implies the following.

Theorem 3.27. *Let X be a proper G-CW-complex. In the notation of Remark 3.26 we have*

$$H_i(X^S/C_G(S); \mathbb{Q}) \otimes_{W_G(S)} R_{\sharp,S} \cong \bigoplus_{[g]\in[S;G]} H_i(X^g/C_G(g); \mathbb{Q}),$$

where the sum is taken over the set $[S; G]$ of G-conjugacy classes of generators of the cyclic group S, and there is an isomorphism

$$H_i^{\mathfrak{F}\mathrm{in}}(X; R_\mathbb{C} \otimes \mathbb{Q}) \cong \bigoplus_{[g]\in \mathrm{FC}(G)} H_i(X^g/C_G(g); \mathbb{Q})$$

where the sum is taken over all conjugacy classes $\mathrm{FC}(G)$ of elements of finite order in G.

The following is a simple example.

Lemma 3.28. *For $G = S\ell(2,\mathbb{Z})$ one has*

$$H_*^{\mathfrak{F}\mathrm{in}}(S\ell(2,\mathbb{Z}); \mathbb{Q} \otimes R_\mathbb{C}) = H_*^{\mathfrak{F}\mathrm{in}}(\underline{E}S\ell(2,\mathbb{Z}); \mathbb{Q} \otimes R_\mathbb{C}) \cong \begin{cases} 0, & \text{for } * > 0 \\ \mathbb{Q}^8, & \text{for } * = 0. \end{cases}$$

Proof. Since $S\ell(2,\mathbb{Z})$ admits a decomposition of the form $C_4 *_{C_2} C_6$, all finite subgroups of $S\ell(2,\mathbb{Z})$ are conjugate to a subgroup of one of the subfactors C_4 resp. C_6. It follows that the centralizers of finite subgroups have the following form: for $\{e\}$ and C_2 the centralizer is all of $S\ell(2,\mathbb{Z})$, whereas the centralizers of the other finite subgroups are all finite. Since the Mayer–Vietoris sequence of the