Super-Lie$_\infty$ T-Duality and M-Theory

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October 30, 2024

Abstract

Super L_∞-algebras unify extended super-symmetry with rational classifying spaces for higher flux densities: The super-invariant super-fluxes which control super p-branes and their supergravity target super-spaces are, together with their (non-linear) Bianchi identities, neatly encoded in (non-abelian) super-L_∞ cocycles. These are the rational shadows of flux-quantization laws (in ordinary cohomology, K-theory, Cohomotopy, iterated K-theory, etc).

We first review, in streamlined form and filling some previous gaps, double-dimensional reduction/oxidation and 10D superspace T-duality along higher-dimensional super-tori, tangent super-space wise. This is viewed as an instance of adjunctions (dualities) between super-L_∞-extensions and -cyclifications arising from supercocycles from first principles. This then allows for deriving the proposed laws of "topological T-duality" at the rational level from the super-L_∞ structure of type II superspace.

Then, by considering super-space T-duality along all 1+9 spacetime dimensions while retaining the 11th dimension as in F-theory, we find the M-algebra appearing as the D/NS5-brane extension of the fully T-doubled/correspondence super-spacetime. On this backdrop, we recognize the "decomposed" M-theory 3-form on the "hidden M-algebra" as an M-theoretic lift of the Poincaré super 2-form that controls superspace T-duality (as the integral kernel of the super Fourier-Mukai transform). This provides an M-theory lift of T-duality at the superspace level.

Recalling that the hidden M-algebra appears also in a higher form of rational-topological T-duality where strings are replaced by M5-branes, we end with a perspective on the M-algebra as a Kleinian local model space for U-duality-covariant superspace supergravity.

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The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

1 Intro & Overview

Open question of non-perturbative theory of strongly coupled/correlated quantum systems. In contrast to the oft-heard lament that contemporary experimental results in fundamental physics are all-too-well explained by existing theory (thus allegedly lacking desired hints of "new physics") it is a public secret that large swaths of phenomena exhibited by quantum systems are explained by existing theory at best in principle: Namely common perturbation/mean-field theory has little to say [BaSh10] about the behavior of *strongly coupled* quantum systems [FS10][Str13] (which includes no less than confined ordinary hadronic matter at non-excessive temperature [Br$^+$14, p. 6][RS20]) and hence of *strongly correlated* quantum systems [Fu12][FGSS20], which notably includes fractional quantum Hall systems [St99] and other quantum materials [KM17] expected to exhibit anyonic topological order arguably needed for scalable quantum computation (e.g. [Sau17][SS23c][MSS24]).

M-theory as a candidate solution. Hints for a hypothetical but plausible non-perturbative theory of strongly-coupled quantum systems have emerged since the 1990s [Wi95][Du96][Du99a], originating in discussion of quantum gravity and "grand unified" quantum field theory. In its "holographic" guise (e.g. [Nat15]) this approach models (possibly counter-intuitively but with remarkable success) strongly-coupled quantum systems by matching them onto the dynamics of "branes" (higher dimensional *m*embranes, whence "M-theory" [HW96a, p 1], introductions include [We12]) whose quantum fluctuations inside an auxiliary higher dimensional spacetime turn out to be usefully reflected in the ambient gravitational field.

Notably quantum critical superconductors have partially been understood holographically this way [HKSS07][GSW10a][GSW10b][GPR10][DGP13] (review in [Pi14][ZLSS15][Na17][HLS18]); but M-theory will be needed (cf. [AG$^+$00, p. 60][CP18, p. 2][Ch18, p. iii][SS23c, Fig. 4]) to complete the holographic description beyond the usual (but unrealistic) large-N limit.

For example, with more M-theoretic aspects included, anyons as in fractional quantum Hall systems can potentially be understood from first principles by careful analysis ([HS01][SS23b][SS23c][SS24d]) of $N = 1$ five-dimensional such branes ("M5-branes", review and pointers in [Du99a, §3][GSS24b]).

While the full formulation of M-theory remains an open and ambitious problem — as the community periodically reminds itself of (e.g., [Du96, §6][Mo14, p. 43][CP18][Du20] [BLOY24]) — there is a substantial web of hints as to its nature, mainly from

(i) extended super-symmetry,

(ii) hidden duality symmetry,

and our aim here and in the companion article [GSS24d] is towards the combination/unification of these two aspects (which previously has found little attention):

Towards M-theory via extended super-symmetry. The characteristic local [1] spacetime super-symmetry of higher dimensional supergravity (cf. Ex. 2.3) famously admits Spin-equivariant central extensions (e.g. [vHvP82][KQS10], cf. Def. 2.21 & Rem. 3.37 below) by Noether charges of global symmetries of branes probing super-spacetime [SS17], suggesting these extensions as the local symmetry of completions of supergravity by brane dynamics. The maximal such central extension for 11d SuGra, known as the *M-algebra* ([DF82][To95, (13)][To98, (1)][Se97], cf. Def. 3.39 below) is by charges of the very membrane (and fivebrane) that give M-theory its name, plausibly going some way towards elucidating its nature (cf. [To99]).

Towards M-theory via duality symmetry. But the founding observation of the field that came to be known as "M-theory" is that different-looking expected corners or limits of M-theory tend to be subtly equivalent to each other via "dualities" [Schw97][FL98][dWl99][Pol17], suggesting that the full M-theory could be revealed by making manifest a symmetry-principle of "U-duality symmetries". The archetypical example is *T-duality* (cf. [AAL95][Bu19][Wa24], we re-derive various ingredients in §3 below), whereby, remarkably spacetime dimensions transmute into charges of 1-dimensional branes (strings), and vice versa. Including also higher-dimensional brane charges into this picture reveals, at least super-tangent space wise, a much larger *brane rotating symmetry* ([BW00], cf. Ex. 3.45) which in turn is argued to be the shadow of a humongous *U-duality* symmetry ([HT95][OP99] [2])

[1] Supergravity is *locally* supersymmetric in direct analogy to how ordinary Einstein gravity is *locally* Poincaré-symmetric (only), namely: on each (super-)tangent space (hence in the *infinitesimal* neighbourhood of any point) but not globally (in other words: on each "super-Kleinian" local model space but globally curved by "super-Cartan geometry"). In contrast to global low-energy supersymmetry that has gotten so much attention in the last couple of decades, the phenomenolgy of local Planck-scale supersymmetry remains viable even if it has received much less attention, but see e.g. [MN18][BQ22].

[2] The term "U-duality" was introduced in [HT95] for restricted actions of integral subgroups $E_{n(n)}(\mathbb{Z}) \subset E_{n(n)}$ conjecturally enforced by charge quantization. But for the tangent-space wise Lie algebra discussion of concern here (and since the precise nature of the charge quantization needs more attention anyway, cf. also [dWN01, p. 4]) we will use "U-duality" more broadly (as is not uncommon, e.g. [HS13a], and in line with the common use of "T-duality") as shorthand for the "hidden exceptional symmetry" of toroidal compactifications of 11d SuGra.

governed ultimately by the exceptional Kac-Moody Lie algebra [3] $\mathfrak{e}_{11}$ [We01], thought to exhibit much of the hidden structure of M-theory [Ni99][dWN01][Ni24]. We will discuss general U-duality in the companion article [GSS24d], but here we first take a step back and revisit T-duality in view of M-theory.

Revisiting super-space T-duality. While T-duality is expected to be a symmetry already at the perturbative level, it carries within the seed of the expected more general U-duality symmetries of M-theory: The expected $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry of M/F-theory is the result of lifting T-duality on a single circle fiber to M-theory on a 2-torus fiber (e.g. [Schw96][Jo97]; we see the super-tangent space incarnation of this phenomenon below in Prop. 3.31.) At the same time, T-duality is by far the best-understood of the U-dualities:

In its strong (rational-)topological formulation (reviewed and in fact derived, rationally, in §3, cf. Rem. 3.17) T-duality is neatly reflected in a twisted *Poincaré line bundle* with curvature *Poincaré 2-form* P_2 (7) on a "doubled-" (physics jargon) or "correspondence-" (math jargon) spacetime (cf. Rem. 3.22). For super-flux densities on super-tangent spaces, this statement was the main result of [FSS18a], which below we review and extend in streamlined form.

Superspace T-duality and M-theory. Our central observation here is that when performing super-space T-duality on *all* spacetime dimensions at once, it has a natural lift to, plausibly, M-theory, where

- §3.3 the doubled super-space is further extended (172) by the *M-algebra* (Def. 3.39) of extended 11D super-symmetry,
- §4 the Poincaré super 2-form is the reduction of a super 3-form P_3 (173) on the M-algebra, which may be identified with the "decomposed" M-theory 3-form (194) that had originally motivated the M-algebra, way back in [DF82] (cf. [AD24]).

We suggest this as further evidence that the M-algebra plays the role of the local model space for a duality-covariant completion of superspace supergravity, as has previously been suggested, in one way or other, by [Vau07][Ba17][FSS20a, §4.6][FSS20b][FSS21a] and as we will further justify in the companion article [GSS24d].

What follows is slightly more technical overview of the discussion in the main text:

Key role of super-flux avatars on super-tangent spaces. Remarkably, higher dimensional supergravity is an instance of (super-)Cartan geometry (cf. [Sh97, §7][Ba77b][GSS24a, §2]) in a strong higher sense: The ubiquitous *supergravity torsion constraints* (e.g. [Lo90][Lo01]) say that the dynamics of supergravity super-fields on curved super-space $X^{1,d\,|\,\mathbf{N}}$ is largely controlled by the demand that the bifermionic components of higher super-flux densities restrict on each super-tangent space (the super-Kleinian model space, see Ex. 2.3)

$$\underset{\substack{\text{super-tangent}\\ \text{super-space}}}{T_x X^{1,d\,|\,\mathbf{N}}} \;\simeq\; \underset{\substack{\text{super-Kleinian}\\ \text{model super-space}}}{\mathbb{R}^{1,d\,|\,\mathbf{N}}} \;\simeq\; \underset{\substack{\text{super-symmetry}\\ \text{super-algebra}}}{\mathfrak{iso}\big(\mathbb{R}^{1,d\,|\,\mathbf{N}}\big)} \Big/ \underset{\substack{\text{super-point}\\ \text{algebra}}}{\mathfrak{so}_{1,10}} \tag{1}$$

to fixed super-invariant avatar forms. This is most pronounced for 11d SuGra (as was particularly highlighted by [Ho97]): Its typical super-tangent space $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ with its canonical super-coframe field $\big((e^a)_{a=0}^{10}, (\psi^\alpha)_{\alpha=1}^{32}\big)$ carries super-invariant avatars of the Hodge-duality symmetric C-field flux densities ([DF82, (3.26)][NOF86, (2.27-28)][CL94, (6.6,10)][CdAIP00, (8.8)], cf. Ex. 2.8):

$$\underset{\text{Duality-symmetric avatar super-flux densities of 11d SuGra}}{\left.\begin{aligned} G_4 \;&:=\; \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} \\ G_7 \;&:=\; \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \end{aligned}\right\}} \in \underset{\substack{\text{super left-invariant}\\ \text{differential forms}}}{\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{li}}} \;\simeq\; \underset{\substack{\text{Chevalley-Eilenberg alg.}\\ \text{of super-symmetry}}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)}\,, \qquad \underset{\text{Bianchi identities}}{\left\{\begin{aligned} \mathrm{d}\,G_4 \;&=\; 0 \\ \mathrm{d}\,G_7 \;&=\; \tfrac{1}{2}G_4\,G_4\,, \end{aligned}\right.} \tag{2}$$

and the equations of motion of 11d SuGra on a supermanifold $X^{1,10\,|\,\mathbf{32}}$ are *equivalent* to the demand that this situation suitably globalizes to $X^{1,10\,|\,\mathbf{32}}$ ([GSS24a, Thm. 3.1] following [BH80][CF80][CDF91, §III.8.5]).

Similarly for 10d type II supergravity, the NS&RR-flux densities and their Bianchi identities have super-invariant avatars on the respective super-tangent spaces $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ (type IIA, cf. Ex. 3.2) and $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}$ (type IIB, cf. Ex. 3.6), respectively, given [4] for type IIA by ([CdAIP00, §6.1], cf. Prop. 3.13 below) by

$$\underset{\text{Duality-symmetric avatar super-flux densities of 10d type IIA SuGra}}{\left.\begin{aligned} H_3^A \;&:=\; \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a \\ F_{2\bullet} \;&:=\; \tfrac{1}{(2k)!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{2\bullet-2}}\,\psi\big)e^{a_1}\cdots e^{a_{2\bullet-2}} \end{aligned}\right\}} \in\; \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big)\,, \qquad \underset{\text{Bianchi identities}}{\left\{\begin{aligned} \mathrm{d}\,H_3^A \;&=\; 0 \\ \mathrm{d}\,F_{2k} \;&=\; H_3^A\,F_{2\bullet-2}\,, \end{aligned}\right.} \tag{3}$$

[3] Strictly all Lie algebras considered are split real forms, so that we omit the further notational decoration: $\mathfrak{sl}_n$ is short for $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{e}_n$ is short for $\mathfrak{e}_{n(n)}$.

[4] Our undecorated Clifford generators $(\Gamma^a)_{a=0}^{10}$ are always those of $\mathrm{Pin}^+(1,10)$, reviewed in §A. In particular, under the reduction $\mathrm{Spin}(1,9) \hookrightarrow \mathrm{Pin}^+(1,10)$ the "chirality operator" often denoted "Γ_{11}" (a reminiscence of the ancient tradition of writing "γ^5" for the chirality operator on Dirac spinors in 4d) is in our notation: Γ_{10}. This and further algebraic expressions of type II spinors in 10d in terms of Majorana spinors in 11d (immediate for type IIA and a little more subtle for type IIB) are discussed in §3.1.

and for type IIB by ([Sak00, §2], cf. Prop. 3.15 below) by:

$$\underbrace{\left.\begin{array}{ll} H_3^B &:= \big(\overline{\psi}\,\Gamma^B_A\Gamma_{10}\,\psi\big)e^a \\ F_{2\bullet+1} &= \tfrac{1}{(2\bullet+1)!}\big(\overline{\psi}\,\Gamma^B_{a_1}\cdots\Gamma^B_{a_{2\bullet-1}}\Gamma_9(\Gamma_{10})^{\bullet+1}\,\psi\big)e^{a_1}\cdots e^{a_{2\bullet-1}}\end{array}\right\}}_{\text{Duality-symmetric avatar super-flux densities of 10d type IIB SuGra}} \in \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big), \qquad \underbrace{\left\{\begin{array}{ll} \mathrm{d}\,H_3^B &= 0 \\ \mathrm{d}\,F_{2\bullet+1} &= H_3^B\,F_{2\bullet-1}\,.\end{array}\right.}_{\text{Bianchi identities}} \tag{4}$$

Avatar flux densities as super-L_∞-algebra cocycles. Moreover, the Bianchi identities satisfied by these avatar super-flux densities may equivalently be understood [FSS17] as making them cocycles on super-space with coefficients in *higher* (categorified symmetry) Lie algebras (cf. [FSS19][Al24]), namely L_∞-algebras (reviewed in §2.1 below), such as the real Whitehead bracket L_∞-algebras $\mathfrak{l}(-)$ of spaces (Ex. 2.4) and of parameterized spectra (Ex. 2.11). For 11D SuGra the L_∞-coefficients are those of 4-Cohomotopy (Ex. 2.8 below):

$$\underset{\text{satisfying Bianchi in (2)}}{(G_4, G_7)} \qquad\Leftrightarrow\qquad \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;(G_4,G_7)\;} \mathfrak{l}S^4\,, \tag{5}$$

while for 10D type II SuGra the L_∞-coefficients are those of 3-twisted complex K-theory (Prop. 3.13, Prop. 3.15):

$$\begin{array}{ccc} \underset{\text{satisfying Bianchi in (3)}}{\big(H_3^A, (F_{2\bullet})\big)} & \Leftrightarrow & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \xrightarrow{\;(H_3^A,(F_{2\bullet}))\;} \mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)\,, \\ \underset{\text{satisfying Bianchi in (4)}}{\big(H_3^B, (F_{2\bullet+1})\big)} & \Leftrightarrow & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \xrightarrow{\;(H_3^B,(F_{2\bullet+1}))\;} \mathfrak{l}\big(\Sigma\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)\,. \end{array} \tag{6}$$

This is a remarkable higher Cartan-geometric aspect of higher-dimensional supergravity (which is closely related to the point of view of [DF82][CDF91][AD24]): It means that the global dynamics of global field configurations is tightly controlled by super-L_∞ algebraic avatar structures on the typical super-tangent space (1). This suggests that also deeper structures of supergravity, and thereby of M-theory and its duality symmetries, are (partially, namely, rationally [FSS19]) reflected in and hence recognizable from the local super-space L_∞-cocycle structure. [5]

Further investigation along these lines is the theme of the present article.

Duality formalized as adjunction. In fact, we find (in §3.2, following [FSS18a][FSS18b][BMSS19]) that T-duality between the type II super-flux avatars (6) is exhibited (138) by a fundamental "adjunction" (the category-theorist's term for "duality", gentle introduction in [Sc18]) which serves to neatly capture the mechanism of double-dimensional reduction/oxidation of flux densities (Prop. 2.25), the backbone of all hidden U-duality symmetry.

By passage to *homotopy fibers* [6] of the L_∞-cocycles of thus reduced super-fluxes, higher flux-extended super-spacetimes emerge (in Lem. 3.18 below, of the kind previously studied in [CdAIP00][Az05]) equipped with equivalent incarnations of the super-flux T-duality equivalence (Lem. 3.19). This brings about the *doubled* super-space, whose M-theoretic lift will concern us particularly:

Doubled tangent super-spacetime and Poincaré super-form. The *fiber product* (59) of the above IIA- and IIB- superspaces over the type II super-tangent space of 9d supergravity is the "correspondence super-space" or *doubled super-space* (Def. 3.20, as in [HKS14][Ba15][Ce16][FSS18a, §6]):

$$\begin{tikzcd}[column sep=small] & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \underset{\mathbb{R}^{1,8|\mathbf{16}\oplus\mathbf{16}}}{\times} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_A"] & \\ \underset{\text{super-space}}{\text{10D type IIA}}\;\; \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dr, ->>] & & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \;\;\underset{\text{super-space}}{\text{10D type IIB}} \arrow[dl, ->>] \\ & \underset{\substack{\text{9D type II} \\ \text{super-space}}}{\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}} & \end{tikzcd}$$

This doubled super-spacetime carries a further component $\tilde{e}^9$ to its coframe field, reflecting the string winding charges along the T-dualized coordinate axis (here: the 9th) and thus manifestly putting them on the same footing as spacetime dimensions. The wedge product of this with the original coframe field component in this direction, e^9, plays a significant role, as it may be identified with the local super-form version of what in topological T-duality is known as the twisted *Poincaré form* (Rem. 3.22) on the correspondence space. This is a coboundary for the difference of the NS super 3-flux densities of type IIA and IIB (Prop. 3.21):

$$\underbrace{P_2 \;:=\; e^9_B\, e^9_A \;\in \mathrm{CE}\Big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \underset{\mathbb{R}^{1,8|\mathbf{16}\oplus\mathbf{16}}}{\times} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\Big)}_{\text{twisted Poincaré super 2-form for doubled 10D type II SuGra}}\,, \qquad \underbrace{\mathrm{d}\,P_2 \;=\; \pi_A^* H_3^A - \pi_B^* H_3^B}_{\text{Bianchi identity}}\,. \tag{7}$$

[5] The issue of promoting L_∞-algebra valued flux densities as in (5) to globally defined higher gauge fields is the topic of *flux quantization* [SS24c][GSS24a][GSS24b] by which more subtle topological aspects of M-theory are resolved [FSS20c][GS21][FSS21b].

[6] The (rational) homotopy theoretic constructions that we allude to here are explained in some detail in [FSS23], but the reader need not further concern themselves with these foundations just for reading the present article.

In fact, the super-Poincaré form exhibits all of super-flux T-duality, in that it serves as the "integral kernel" of a Fourier-Mukai transformation (pp. 55) taking the RR super-flux densities of IIA and IIB into each other (Cor. 3.28).

Up to this point, the discussion is a streamlined and completed account of the results announced in [FSS18a]. Next, we connect these to M-theory.

Fully doubled super-spacetime and full Poincaré form. With the further super-L_∞ algebraic machinery of *toroidification* (§2.3, following [SV23][SV24]) we may analyze the analogous super-space T-duality but for reduction along all 10 space-time dimensions down to the (super-)point (Ex. 2.34), where we find (155) type IIA self-duality exhibited on a fully (i.e., along all space-time axes) doubled super-spacetime:

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{Dbl} := \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \underset{\mathbb{R}^{0\,|\,32}}{\times} \widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_{\widetilde{A}}"] & \\
\underset{\substack{\text{10D type IIA}\\ \text{super-space}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[dr, ->>] & & \underset{\substack{\text{10D type II}\widetilde{\text{A}}\\ \text{super-space}}}{\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}} \arrow[dl, ->>] \\
 & \underset{\text{super-point}}{\mathbb{R}^{0\,|\,32}} &
\end{tikzcd}$$

This fully doubled superspace $\mathfrak{Dbl}$ carries a dual coframe field $\tilde{e}_a$ for each of the coordinate axes, reflecting corresponding string winding charges, and which thus supports an analogous Poincaré 2-form exhibiting (Prop. 3.35) the T-duality between type IIA and its full T-dual super-space II$\widetilde{\text{A}}$ (which is essentially type IIA itself again, up to some fine-print, cf. Rem. 3.33):

$$\underset{\text{Twisted Poincaré super 2-form for fully doubled 10D type II SuGra}}{P_2 \;:=\; \tilde{e}_a\, e^a \;\;\in\;\; \mathrm{CE}\big(\mathfrak{Dbl}\big)\,,} \qquad\qquad \underset{\text{Bianchi identity}}{\mathrm{d}\, P_2 \;=\; \pi_A^* H_3^A - \pi_{\widetilde{A}}^* H_3^{\widetilde{A}}\,.} \tag{8}$$

Extended IIA super-symmetry algebra extends doubled super-space. Recalling at this point that the "doubling" of spacetime dimensions happening here is equivalently the adjoining of string (winding) charges, we observe that the doubled super-spacetime is an intermediate stage in the fully extended type IIA super-symmetry algebra II$\mathfrak{A}$ (Def. 3.36), which adjoins furthermore the charges of the D-branes and the NS5-brane (Rem. 3.37). To make this observation more manifest, we consider the super-algebra $\mathfrak{Brn}$ (166) which extends the super-point purely by the type IIA brane charges (string, D-branes & NS5-brane). We observe (167) that the fully extended IIA super-symmetry algebra is the fiber product (59) over the fully T-dual IIA algebra of this pure brane charge algebra with the fully doubled super-spacetime:

$$\begin{tikzcd}
 & \overset{\substack{\text{Fully extended IIA}\\ \text{super-algebra}}}{\mathrm{II}\mathfrak{A}} \arrow[dl, ->>] \arrow[dr, ->>] & \\
\underset{\substack{\text{Fully doubled}\\ \text{super-spacetime}}}{\mathfrak{Dbl}} \arrow[dr, ->>] & & \underset{\substack{\text{Pure brane}\\ \text{charge algebra}}}{\mathfrak{Brn}} \arrow[dl, ->>, "p^{\mathrm{Brn}}"', "\substack{\text{extension by}\\ \text{brane charges}}"] \\
 & \underset{\substack{\text{Fully T-dual IIA}\\ \text{super-spacetime}}}{\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} &
\end{tikzcd}$$

This relation between extended super-symmetry and T-duality doubled super-spacetime may not have been addressed before — but now we see that this brings out the M-theoretic lift of T-duality doubled super-space:

Extending double super-space to the M-algebra. We then observe that finally extending all of the above setting also along the fibration of the 11D- over the 10D type IIA super-tangent space makes the II$\mathfrak{A}$-algebra extend to the basic *M-algebra* $\mathfrak{M}$ (recalled as Def. 3.39, and makes the $\mathfrak{Dbl}$-algebra extend to the analog of the F-theory super-tangent space (153) for reduction along all spacetime directions, which, therefore, we denote $\mathfrak{F}$). This concretely exhibits the M-algebra as the M-theoretic analog of the T-duality correspondence super-space

$$\begin{tikzcd}[column sep=small]
 & & & \overset{\text{M-algebra}}{\mathfrak{M}} \arrow[dl, ->>] \arrow[dr, ->>, "p^M"] & & \\
 & & \mathfrak{F} \arrow[dl, ->>] \arrow[dr, ->>] & & \mathrm{II}\mathfrak{A} \arrow[dl, ->>, "p^{\mathrm{Brn}}"'] & \\
 & \mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[dr, ->>, "\substack{\text{extension by}\\ \text{11th dimension}}"'] & & \mathfrak{Dbl} \arrow[dl, ->>] \arrow[dr, ->>] & & \\
 & & \underset{\substack{\text{Type IIA}\\ \text{super-spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} & & \underset{\substack{\text{Type IIB}\\ \text{super-spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}} &
\end{tikzcd}$$

In particular, this diagram exhibits the dual coframe field $\tilde{e}_a$ on (hence the doubled dimensions of) $\mathfrak{Dbl}$ as the *mem*brane charges $e_{a_1 a_2}$ that wrap the 11th dimension: $\tilde{e}_a \;\leftrightarrow\; e_{10\,a}$ (169).

M-theoretic Poincaré 3-forms and the hidden decomposition of the M-theory 3-form. With this understood it becomes evident that there are super-invariant 3-forms on $\mathfrak{M}$ which dimensionally reduce (Rem. 3.42) to the Poincaré 2-form (8) controlling T-duality on 10D super-space:

$$P_3 \;:=\; \tfrac{1}{2} e^{a_1} e_{a_1 a_2} e^{a_2} \;+\; \cdots \;, \qquad\qquad \underbrace{p^{\mathrm{Brn}}_{\mathrm{bas}}\, p^M_*}_{\substack{\text{basic part of} \\ \text{11D fiber integration}}} P_3 \;=\; P_2$$

Remarkably, super-invariants with this leading term have been discussed before (194) from a rather different point of view, under the name of "decomposed" M-theory 3-forms ([DF82][BDIPV04][AD24], see (194) below) satisfying in addition

$$\underset{\text{Poincaré 3-form in M-extended 11D SuGra}}{P_3 \;\in\; \mathrm{CE}\big(\widehat{\mathfrak{M}}\big),} \qquad\qquad \underset{\text{Bianchi identity}}{\mathrm{d}\, P_3 \;=\; G_4\,.} \tag{9}$$

In the concluding section §4, we summarize the M-theoretic T-duality picture that we establish here and give an outlook on the Poincaré super 3-form P_3 as in (9) as exhibiting the rational-topological enhancement of aspects of U-duality symmetry for super-space supergravity.

We will further discuss in [GSS24d] the hidden M-algebra with its Poincaré 3-form as an M-theoretic candidate for U-duality covariant super-space supergravity. This article lays the groundwork by a comprehensive discussion of the underlying super-L_∞ algebraic T-duality mechanisms, completing and extending previous such work in [FSS18a][FSS18b][FSS20a][SS18].

Outline.

§2 discusses super-L_∞ algebraic T-duality in abstract generality,
§3 realizes this on the avatar super-flux densities on super-tangent spacetimes.
The first couple of subsections in each case are concerned with type A/B-duality along a 1-dimensional fiber, the latter couple of subsections deal with full T-duality on all 10 spacetime dimensions related to M-theory,
§4 sums up the curious picture thus obtained and gives an outlook on M-theoretic lessons.

2 Super-L_∞ theory

In this section, we discuss in abstract generality the super-L_∞-algebraic structures and phenomena which, when applied to super-flux densities on super-spacetime, in §3 below, exhibit super-space T-duality. While the super-L_∞ perspective makes various constructions nicely transparent, all our computations take place in the dual Chevalley-Eilenberg dgc-algebra picture that is familiar in the supergravity literature ("FDA"s, cf. Rem. 2.1 below).

2.1 Super-L_∞ algebra

We recall (from [FSS15, §2][FSS18a, §2][FSS19, (21)][HSS19, §3.2][Sc21, p 33, 48]) the notion of higher (meaning: categorified symmetry) super-Lie algebras (of finite type) and their identification with the "FDA"s from the supergravity literature ([vN83][CDF91, §III.6], cf. [AD24]). Our ground field is the real numbers $\mathbb{R}$, and all super-vector spaces are assumed to be finite-dimensional.

Given a finite dimensional super-Lie algebra $\mathfrak{g} \simeq \mathfrak{g}_{\mathrm{evn}} \oplus \mathfrak{g}_{\mathrm{odd}}$, the linear dual of the super-Lie bracket map

$$[\text{-},\text{-}] \;:\; \mathfrak{g} \vee \mathfrak{g} \longrightarrow \mathfrak{g}$$

may be understood to map the first to the second exterior power of the underlying dual super-vector space, and as such it extends uniquely to a $\mathbb{Z}\times\mathbb{Z}_2$-graded derivation d of degree=$(1,\mathrm{evn})$ on the exterior super-algebra (where the minus sign is just a convention)

$$\begin{array}{ccc} \wedge^1 \mathfrak{g}^* & \xrightarrow{-[\text{-},\text{-}]^*} & \wedge^2 \mathfrak{g}^* \\ \downarrow & & \downarrow \\ \wedge^\bullet \mathfrak{g}^* & \xrightarrow{\mathrm{d}} & \wedge^\bullet \mathfrak{g}^* \end{array}$$

With this, the condition $d \circ d = 0$ is equivalently the super-Jacobi identity on $[\text{-},\text{-}]$, and the resulting differential graded super-commutative algebra is know as the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$:

$$\mathrm{CE}\big(\mathfrak{g},\, [\text{-},\text{-}]\big) \;:=\; \big(\wedge^\bullet\, \mathfrak{g}^*,\, \mathrm{d}\big)\,.$$

This construction is a *fully faithful formal duality*

$$\begin{array}{ccc} \mathrm{sLieAlg} & \xhookrightarrow{\quad \mathrm{CE} \quad} & \mathrm{sDGCAlg}^{\mathrm{op}} \\ \big(\underbrace{V}_{\substack{\text{super-}\\ \text{vector space}}}, [\text{-},\text{-}]\big) & \longmapsto & \big(\wedge^\bullet V^*,\, \mathrm{d} = -[\text{-},\text{-}]^*\big)\,, \end{array} \tag{10}$$

in that

(i) for every super-vector space V a choice of such differential d on $\wedge^\bullet V^*$ uniquely comes from a super-Lie bracket $[\text{-},\text{-}]$ on V this way, and

(ii) super-Lie homomorphisms $\phi : \mathfrak{g} \to \mathfrak{g}'$ are in bijection with super-dg-algebra homomorphisms $\phi^* : \mathrm{CE}(\mathfrak{g}') \to \mathrm{CE}(\mathfrak{g})$.

More concretely, given $(T_i)_{i=1}^n$ a linear basis for $\mathfrak{g}$ with corresponding structure constants $\big(f^k_{ij} \in \mathbb{R}\big)_{i,j,k=1}^n$, then the Chevalley-Eilenberg algebra is equivalently the graded-commutative polynomial algebra

$$\mathrm{CE}\big(\mathfrak{g}, [\text{-},\text{-}]\big) \;\simeq\; \big(\mathbb{R}\big[t^1, \cdots, t^1\big],\, \mathrm{d}\big)$$

on generators of degree $(1, \sigma_i)$ with corresponding structure constants for its differential:

	Super Lie algebra	**Super dgc-algebra**
Generators	$\big(\underbrace{T_i}_{\deg = (0,\sigma_i)}\big)_{i=1}^n$	$\big(\underbrace{t^i}_{\deg = (1,\sigma_i)}\big)_{i=1}^n$
Relations	$[T_i, T_j] \;=\; f^k_{ij}\, T_k$	$\mathrm{d}\, t^k \;=\; -\tfrac{1}{2} f^k_{ij}\, t^i t^j$

(11)

This dual perspective via the CE-algebra is most convenient for passing from super-Lie to *strong homotopy super-Lie algebras*, also known as *super Lie ∞-algebra* (subsuming Lie 2-algebras, Lie 3-algebras etc., hence infinitesimal "categorified symmetry" algebras), and also known as *super-L_∞ algebras*, for short: These are obtained simply by dropping the assumption that the CE-generators are in degree 1:

Namely for a $\mathbb{Z}$-*graded* super-vector space $V_\bullet$ (degree-wise finite-dimensional by our running assumption, hence "of finite type"), a sequence of higher arity super-skew-commutative brackets is dually a map from the degreewise dual $V^\vee$ (with $V^\vee_n := (V_n)^*$) to its graded Grassmann algebra:

$$\mathrm{d} \,:\, \wedge^1 V^\vee \longrightarrow \wedge^\bullet V^\vee$$

and the higher super-Jacobi identity is dually simply the statement that this map, extended uniquely as a super-graded derivation to all of $\wedge^\bullet V^\vee$, is a differential

$$\mathrm{d} \;:\; \wedge^\bullet V^\vee \longrightarrow \wedge^\bullet V^\vee$$

in that it squares to zero: $\mathrm{d}\circ\mathrm{d} \;=\; 0$. (This is the evident super-algebraic enhancement of the characterization of finite-type L_∞-algebras in [SSS09, §6.1].)

This way, super-L_∞ algebras (of finite type) are equivalently nothing but super dgc-algebras whose underlying super-graded algebra is of the form $\wedge^\bullet V^\vee$ for some $\mathbb{Z}$-graded super-vector space, with super L_∞-homomorphisms identified as homomorphisms of these super dgc-algebras going in the *opposite* direction ("pullback"):

$$\begin{array}{ccc}
\mathrm{sLieAlg}_\infty & \xhookrightarrow{\quad\quad\quad\quad\mathrm{CE}\quad\quad\quad\quad} & \mathrm{sDGCAlg}^{\mathrm{op}} \\
\big(\underbrace{V}_{\substack{\text{graded super-}\\ \text{vector space}}}, [\text{-}], [\text{-},\text{-}], [\text{-},\text{-},\text{-}], \cdots\big) & \longmapsto & \big(\wedge^\bullet V^\vee,\, \mathrm{d} = -[\text{-}]^\ast - [\text{-},\text{-}]^\ast - [\text{-},\text{-},\text{-}]^\ast - \cdots\big)\,.
\end{array} \tag{12}$$

More concretely, by a choice of linear basis $(T_i)_{i\in I}$ for its underlying graded super vector space V, the CE-algebra of a super-L_∞-algebra may be written as:

$$\mathrm{CE}(\mathfrak{g}) \;\;\simeq\;\; \mathbb{R}_{\mathrm{d}}\big[(\underbrace{t^i}_{\mathrm{deg}=(n_i,\sigma_i)})_{i\in I}\big]\big/\big(\mathrm{d}\,t^i \;=\; P^i(\vec t\,)\big)_{i\in I}\,, \tag{13}$$

where

- $\mathrm{deg}(t^i) \;=\; \mathrm{deg}(T_i) + (1,\mathrm{evn})$
- $\mathbb{R}_{\mathrm{d}}\big[(t^i)_{i\in I}\big]$ is the free differential $(\mathbb{Z}\times\mathbb{Z}_2)$-graded symmetric algebra on these generators and their differentials (207), whose product is subject only to the sign rule (205).
- $P^i(\vec t\,)$ are graded-symmetric polynomials in the generators,
- d is extended from generators to polynomials as a super-graded derivation of degree $(1,\mathrm{evn})$,
- the consistency condition is (only) that $\mathrm{d}\circ\mathrm{d} = 0$.

Accordingly, a homomorphism of super L_∞-algebras $f : \mathfrak{g} \to \mathfrak{h}$ with dual linear basis $(e^i)_{i\in I}$ and $(t^j)_{j\in J}$ is dually given by an algebra homomorphism $f^\ast : \mathrm{CE}(\mathfrak{h}) \to \mathrm{CE}(\mathfrak{g})$ pulling back the generators t^j to polynomials $f^\ast(t^j) \in \wedge^\bullet(\mathfrak{g}^\vee)$ in the generators e^i such that the differential is respected:

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\;\;f\;\;} & \mathfrak{h} \\
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\;\;f^\ast\;\;} & \mathrm{CE}(\mathfrak{h}) \\
f^\ast(t^j) & \longleftarrow & t^j
\end{array}\;,\qquad \text{such that} \qquad \underset{j\in J}{\forall}\; \mathrm{d}f^\ast(t^j) \;=\; f^\ast(\mathrm{d}t^j)\,. \tag{14}$$

Remark 2.1 (CE-algebras are differential quotients of free differential graded-commutative algebras.). As such, we may recognize the CE-algebras (13) as the "free differential algebras" of the supergravity literature [vN83][CDF91, §III.6]. The quotient notation in (13), following [FSS23, §4], is justified by thinking of

- $\mathbb{R}_{\mathrm{d}}\big[(e^i)_{i\in I}\big]$ as the (actual) *free differential* super-graded-commutative algebra, hence with each $\mathrm{d}e^i$ being a new generator subject to no relation (except super-graded commutativity),
- $\big(\mathrm{d}\,e^i \;=\; P^i\big((e^j)_{j\in I}\big)\big)_{i\in I}$ as a differential ideal,
- the quotient hence enforcing these equations on the previously free differential.

Remark 2.2 (L_∞-jargon).
(i) Another name for L_∞-algebras is *strong homotopy Lie algebra* (which was more popular in the past), also abbreviated *sh-Lie algebra*, as in the original articles [LS93][LM95]. Our formulation (12) of L_∞-algebras via their CE-algebras (which brings out the equivalence of super L_∞-algebras with the "FDA"s in the supergravity literature, Rem. 2.1) is contained in these original articles, made explicit in [SSS09, Def. 13]. Similarly, our homomorphisms of L_∞-algebras (14) were also called *strong homotopy maps* or *sh-maps*, for short.

(ii) Or rather, our (14) subsumes the slightly larger generality known as "curved" morphisms between non-curved(!) L_∞-algebras (as in [MZ12, below (2)]): Namely, CE-algebras (13) carry a canonical *augmentation* ϵ — the homomorphism which projects out the scalar summand $\mathbb{R} \simeq \wedge^0 V^\vee \hookrightarrow \wedge^\bullet V^\vee$ (dual to a canonical base-point, see Ex. 2.5):

$$\begin{array}{ccc}
\mathfrak{l}(\ast) & \xhookrightarrow{\quad\quad 0 \quad\quad} & \mathfrak{g} \\
\mathbb{R} & \xtwoheadleftarrow{\quad\epsilon\quad} & \mathrm{CE}(\mathfrak{g}) \simeq \mathbb{R}\oplus\wedge^{\bullet\geq 1}V^\vee
\end{array}$$

and the "non-curved" morphisms $f : \mathfrak{g} \to \mathfrak{g}'$ are those preserving these base-points, hence dually preserving these augmentations:

$$
\begin{tikzcd}
 & \mathfrak{l}(*) \arrow[dl, "0"'] \arrow[dr, "0"] & \\
\mathfrak{g} \arrow[rr, "f"] & & \mathfrak{g}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & & \mathrm{CE}(\mathfrak{g}') \arrow[ll, "f^*"'] \\
 & \mathbb{R} \arrow[ul, "\epsilon"] \arrow[ur, "\epsilon'"'] &
\end{tikzcd}
$$

Hence "non-curved homomorphisms between non-curved L_∞"-algebras really means: base-point preserving homomorphisms. But we generically allow "curved homomorphisms", not required to respect the base-point.

(iii) Note that this is an issue if and only if $\wedge^1 V^\vee$ has elements in degree 0 (hence iff V has elements in degree -1). In the interpretation of homomorphisms $\mathbb{R}^{1,d\,|\,\mathbf{N}} \to \mathfrak{g}$ as super-flux densities (in §3.2 and §3.3) this is the case of "axion fields".

Examples of super-L_∞ algebra. The base example in supergravity is the following Ex. 2.3:

Example 2.3 (Supersymmetry algebras). For $d \in \mathbb{N}$ and $\mathbf{N} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(1,d))$ a real spin-representation equipped with a $\mathrm{Spin}(1,d)$-equivariant linear map

$$
\big(\overline{(\text{-})}\Gamma(\text{-})\big) \;:\; \mathbf{N} \underset{\mathrm{sym}}{\otimes} \mathbf{N} \longrightarrow \mathbb{R}^{1,d}\,, \tag{15}
$$

the corresponding super-translation super-Lie algebra $\mathbb{R}^{1,d\,|\,\mathbf{N}}$ is given by

$$
\mathrm{CE}\big(\mathbb{R}^{1,d\,|\,\mathbf{N}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\, (\underbrace{\psi^\alpha}_{\mathrm{deg}\,=\,(1,\mathrm{odd})})_{\alpha=1}^{N},\;\; (\underbrace{e^a}_{\mathrm{deg}\,=\,(1,\mathrm{evn})})_{a=0}^{d} \Big] \Big/ \left(\begin{aligned} \mathrm{d}\,\psi^\alpha &= 0 \\ \mathrm{d}\,e^a &= \big(\overline{\psi}\,\Gamma^a\,\psi\big) \end{aligned} \right). \tag{16}
$$

Specific examples of this kind are the topic of §3.1 below.

Dually, this means that the super-Lie algebra itself is

$$
\mathbb{R}^{1,10\,|\,\mathbf{32}} \;\simeq\; \mathbb{R}\Big\langle\, \underbrace{(Q_\alpha)_{\alpha=1}^{32}}_{\mathrm{deg}\,=\,(0,\mathrm{odd})} \;,\; \underbrace{(P_a)_{a=0}^{10}}_{\mathrm{deg}\,=\,(0,\mathrm{evn})} \Big\rangle \tag{17}
$$

with the only non-trivial super-Lie brackets on basis elements being the usual [7]

$$
\big[Q_\alpha,\, Q_\beta\big] \;=\; -2\,\Gamma^a_{\alpha\beta} P_a\,. \tag{18}
$$

The assumed $\mathrm{Spin}(1,d)$-equivariance implies that the ordinary Lorentz Lie algebra $\mathfrak{so}_{1,d}$ acts automrophically on $\mathbb{R}^{1,d\,|\,\mathbf{N}}$. The corresponding semidirect product super-Lie algebra is the *super-Poincaré Lie algebra*, the full "supersymmetry algebra" in these dimensions:

$$
\mathrm{CE}\big(\mathbb{R}^{1,d\,|\,\mathbf{N}} \rtimes \mathfrak{so}_{1,d}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\, (\underbrace{\psi^\alpha}_{\mathrm{deg}\,=\,(1,\mathrm{odd})})_{\alpha=1}^{N},\;\; (\underbrace{e^a}_{\mathrm{deg}\,=\,(1,\mathrm{evn})})_{a=0}^{d},\; (\underbrace{\omega^{ab} = -\omega^{ba}}_{\mathrm{deg}\,=\,(1,\mathrm{evn})})_{a,b=0}^{d} \Big] \Big/ \left(\begin{aligned} \mathrm{d}\,\psi^\alpha &= 0 \\ \mathrm{d}\,e^a &= \big(\overline{\psi}\,\Gamma^a\,\psi\big) + \omega^a{}_b\, e^b \\ \mathrm{d}\,\omega^{ab} &= \omega^a{}_c \omega^{cb} \end{aligned} \right).
$$

Important examples among higher Lie-algebras come from

(I) topological spaces,

(II) spectra of spaces,

and generally, unifying these two cases:

(III) bundles of spectra over topological spaces.

(I) Whitehead L_∞-algebras of spaces.

Example 2.4 (Real Whitehead L_∞-algebras of topological spaces cf. [FSS23, Prop. 5.11]**).** Given a topological space X — which is (a) connected, (b) nilpotent, e.g., in that its fundamental group is abelian, and (c) whose $\mathbb{R}$-cohomology $H^\bullet(X;\mathbb{R})$ is degreewise finite-dimensional — there is an L_∞-algebra, $\mathfrak{l}X$, characterized by the following two properties:

(i) The underlying graded vector space is the $\mathbb{R}$-rationalization of the homotopy groups $\pi_\bullet(X)$ of the based loop space ΩX:

$$
\mathfrak{l}X \;\simeq\; \big(\underbrace{\pi_\bullet(\Omega X)\otimes_{\mathbb{Z}}\mathbb{R}}_{\mathrm{deg}\,=\,(\bullet,\mathrm{evn})},\; [\text{-},\text{-}], [\text{-},\text{-},\text{-}], \cdots\big)\,, \qquad \mathrm{CE}\big(\mathfrak{l}X\big) \;\simeq\; \big(\wedge^\bullet\big(\pi_\bullet(\Omega X)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee,\; \mathrm{d}\big)\,,
$$

[7] Our prefactor convention in (18) – ultimately enforced via the translation (11) by our convention for the super-torsion tensor in (233), cf. [GSS24a] and [GSS24a] – coincides with that in [DF99, (1.16)][Fr99, p. 52].

(which means, cf. below (13), that the generators of $\mathrm{CE}(\mathfrak{l}X)$ are in the degrees of the homotopy groups of X).

(ii) The cochain cohomology of its CE-algebra reproduces the ordinary cohomology of X:

$$H^\bullet\big(\mathrm{CE}(\mathfrak{l}X),\, \mathrm{d}\big) \;\simeq\; H^\bullet(X;\mathbb{R})\,.$$

In rational homotopy theory the dg-algebra $\mathrm{CE}\big(\mathfrak{l}X\big)$ is known (reviewed in [FSS23, §5]) as the *minimal Sullivan model* of the topological space X, retaining exactly the information of its rational homotopy type.

A trivial but useful example is the following:

Example 2.5 (The point). The real Whitehead L_∞-algebra (Ex. 2.4) of the point space $*$ is the 0-object in super-L_∞-algebras

$$0 \;\simeq\; \mathfrak{l}(*)$$

given by

$$\mathrm{CE}\big(0\big) \;\simeq\; \big(\wedge^\bullet 0,\, \mathrm{d}=0\big) \;=\; \big(\mathbb{R},\, \mathrm{d}=0\big)\,.$$

Of course, this is also the real Whitehead L_∞-algebra of every *contractible* topological space.

Example 2.6 (Line Lie n-algebra.). For $n \in \mathbb{N}$ and X an integral Eilenberg-MacLane space

$$X \underset{\mathrm{hmtp}}{\simeq} B^n U(1) \underset{\mathrm{hmtp}}{\simeq} K(\mathbb{Z}, n+1)$$

(classifying ordinary integral cohomology in degree $n+1$ and equivalently classifying complex line bundles, for $n=1$, line bundle gerbes, for $n=2$, and generally principal circle n-bundles, see [FSS23, Ex. 2.1]) its real Whitehead L_∞-algebra (Ex. 2.4)

$$b^n\mathbb{R} \;:=\; \mathfrak{l}\big(B^n\mathrm{U}(1)\big) \tag{19}$$

is given by

$$\mathrm{CE}\big(\mathfrak{l}B^n\mathrm{U}(1)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\big[\underbrace{\omega_{n+1}}_{\deg\,=(n+1,\mathrm{evn})}\big]\Big/\big(\mathrm{d}\,\omega_{n+1} \;=\; 0\big)\,. \tag{20}$$

This means that super-L_∞ homomorphisms (14) into these higher Lie algebras are equivalently $(n+1)$-cocycles:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;\alpha_{n+1}\;} & \mathfrak{l}\big(B^n\mathrm{U}(1)\big) \\ \alpha_{n+1} & \longleftarrow\!\!\shortmid & \omega_{n+1} \end{array} \qquad \Leftrightarrow \qquad \left\{\begin{array}{l} \alpha_{n+1} \,\in\, \mathrm{CE}(\mathfrak{g})\,, \\ \deg(\alpha_{n+1}) \;=\; (n+1,\, \mathrm{evn}), \\ \mathrm{d}\,\alpha_{n+1} \;=\; 0\,. \end{array}\right. \tag{21}$$

As an aside: For $n \in \mathbb{N}$ the classifying space $B^n\mathrm{U}(1) \,\simeq\, K(\mathbb{Z}, n+1)$ carries the structure of a higher (categorical symmetry-)group, equivalent to the based loop ∞-group of the next space in the sequence:

$$B\mathrm{U}(1) \underset{\text{as }\infty\text{-groups}}{\simeq} \Omega B^{n+1}\mathrm{U}(1)\,.$$

(The underlying homotopy equivalences make the $\big(B^n\mathrm{U}(1)\big)_{n\in\mathbb{N}}$ a *spectrum* of spaces, cf. Ex. 2.9.)

For this reason, the operation $B(-)$ is also called *de-looping*. After passage to Whitehead L_∞-algebras

$$b \circ \mathfrak{l}\,(\text{-}) \;\simeq\; \mathfrak{l} \circ B(\text{-})$$

this is given, via (20), by shifting the degree of the single generator,

Example 2.7 (Real Whitehead L_∞-algebra of the 4-sphere, cf. [FSS23, Ex. 5.3]**).** The Whitehead L_∞-algebra (Ex. 2.4) of the 4-sphere, $\mathfrak{l}S^4$, is given by

$$\mathrm{CE}\big(\mathfrak{l}S^4\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} g_4 \\ g_7 \end{bmatrix}\Big/\left(\begin{array}{lcl} \mathrm{d}\,g_4 & = & 0 \\ \mathrm{d}\,g_7 & = & \tfrac{1}{2}\,g_4\,g_4 \end{array}\right). \tag{22}$$

Namely, the generators in degree 4 and 7 reflect the fact that S^4 has such generators for non-torsion homotopy groups in these degrees, while the differential in (22) cuts down the resulting cohomology ring from $\mathbb{R}[g_4] \,\simeq\, H^4\big(K(\mathbb{Z},4);\mathbb{R}\big)$ to the correct $\mathbb{R}[g_4]/(g_2^2) \,\simeq\, H^4\big(S^4;\,\mathbb{R}\big)$. The prefactor of $1/2$ in (22) is not fixed up to isomorphism of L_∞-algebras, but is the natural choice for capturing the Bianchi identity of the C-field in the next Ex. 2.8.

Note the homomorphism (14) to the line Lie 4-algebra (Ex. 2.6)

$$\begin{array}{cc} \mathfrak{l}S^4 & g_4 \\ \downarrow & \uparrow \\ b^3\mathbb{R} & \omega_4 \end{array} \qquad \simeq \qquad \mathfrak{l}\left(\begin{array}{c} S^4 \\ \downarrow \\ B^3\mathrm{U}(1) \end{array}\right) \tag{23}$$

which rationally reflects the "1st Postnikov stage" of the 4-sphere (cf. [GS21]).

On the other hand, rationally the Eilenberg-MacLane space $B^3\mathrm{U}(1) \,\simeq\, K(\mathbb{Z},4)$ is indistinguishable from the classifying space for $\mathrm{SU}(2)$-principal bundles, $\mathfrak{l}B^3\mathrm{U}(1) \,\simeq\, \mathfrak{l}B\mathrm{SU}(2)$, so that up to choices of flux quantization laws

([SS24c]) the above map may also be thought of as the map classifying the quaternionic Hopf fibration

$$\begin{array}{ccc} S^7 & \longrightarrow & S^4 \\ & & \downarrow \\ & & B\mathrm{SU}(2)\,. \end{array} \tag{24}$$

The Lie 7-algebra $\mathfrak{l}S^4$ of Ex. 2.7 (a 6-fold "categorified symmetry" algebra) is noteworthy because it provides the correct coefficients for the duality-symmetric C-field super-flux densities in 11d supergravity (for more on this see [GSS24a]):

Example 2.8 (**4-Sphere valued super-flux of 11d SuGra** [FSS15, p 5][FSS17, Cor. 2.3][GSS24a, Ex. 2.30] following [Sa13, §2.5]). On the 11d super-Minkowski algebra $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ (Ex. 2.3) the super-invariants (2)

$$\left.\begin{array}{lcl} G_4 & := & \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} \\ G_7 & := & \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \end{array}\right\} \in \mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)\,, \qquad \begin{array}{lcl} \mathrm{d}\,G_4 & = & 0 \\ \mathrm{d}\,G_7 & = & \tfrac{1}{2}G_4\,G_4\,, \end{array}$$

are identified with a homomorphism (14) of super-L_∞-algebras from super-Minkowski space to $\mathfrak{l}S^4$ (Ex. 2.7):

$$\begin{array}{ccc} \mathbb{R}^{1,10\,|\,\mathbf{32}} & \xrightarrow{\;(G_4,\,G_7)\;} & \mathfrak{l}S^4 \\ G_4 & \longleftarrow & g_4 \\ G_7 & \longleftarrow & g_7\,. \end{array} \tag{25}$$

(II) Whitehead L_∞-algebras of spectra of spaces.

Example 2.9 (**Real Whitehead L_∞-algebras of spectra** [BMSS19, Lem. 2.25][FSS23, Ex. 5.7]). The real Whitehead L_∞-algebra of a spectrum E of topological spaces has underlying it the $\mathbb{R}$-rationalization of the stable homotopy groups of ΩE, equipped with trivial brackets / trivial differential:

$$\mathfrak{l}E \;\simeq\; \Big(\underbrace{\pi_\bullet(\Omega E)\otimes_{\mathbb{Z}}\mathbb{R}}_{\mathrm{deg}\,=\,(\bullet,\mathrm{evn})},\,[\text{-},\cdots,\text{-}]=0\Big), \qquad \mathrm{CE}(\mathfrak{l}E) \;\simeq\; \Big(\wedge^\bullet\,\big(\pi_\bullet(\Omega E)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee,\,\mathrm{d}=0\Big).$$

While the differential is trivial, the crucial difference here to the Whitehead L_∞-algebras of topological spaces (Ex. 2.4) is that there may be elements in non-positive degree.

Example 2.10 (**Real Whitehead L_∞-algebra of complex topological K-theory**). The spectrum KU of complex topological K-theory has stable homotopy groups in every even degree, hence its suspension ΣKU in every odd degree

$$\pi_k(\mathrm{KU}) \;\simeq\; \begin{cases} \mathbb{Z} & \text{for even } k \\ * & \text{otherwise} \end{cases}, \qquad \pi_k(\Sigma\mathrm{KU}) \;\simeq\; \begin{cases} \mathbb{Z} & \text{for odd } k \\ * & \text{otherwise} \end{cases}, \qquad \pi_k(\Sigma^n\mathrm{KU}) \;\simeq\; \begin{cases} \mathbb{Z} & \text{for even } k+n \\ * & \text{otherwise} \end{cases}$$

and hence its real Whitehead L_∞-algebra (Ex. 2.9) is given by

$$\mathrm{CE}\big(\mathfrak{l}(\mathrm{KU})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\big[(\underbrace{f_{2k}}_{\mathrm{deg}\,=\,(2k,\mathrm{evn})})_{k\in\mathbb{Z}}\big]\big/\big(\mathrm{d}\,f_{2\bullet}\;=\;0\big)\,, \qquad \mathrm{CE}\big(\mathfrak{l}(\Sigma\mathrm{KU})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\big[(\underbrace{f_{2k+1}}_{\mathrm{deg}\,=\,(2k+1,\mathrm{evn})})_{k\in\mathbb{Z}}\big]\big/\big(\mathrm{d}\,f_{2\bullet+1}\;=\;0\big). \tag{26}$$

Analogously to Ex. 2.6, this means that super-L_∞ homomorphisms (14) into these higher Lie algebras are equivalently sequences of cocycles in degrees $(2k)_{k\in\mathbb{Z}}$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;(F_{2k})_{k\in\mathbb{Z}}\;} & \mathfrak{l}(\mathrm{KU}) \\ (F_{2k})_{k\in\mathbb{Z}} & \longleftarrow & (f_{2k})_{k\in\mathbb{Z}} \end{array} \qquad \Leftrightarrow \qquad \begin{cases} (F_{2k})_{k\in\mathbb{Z}} \;\subset\; \mathrm{CE}(\mathfrak{g})\,, \\ \mathrm{deg}(F_{2k}) \;=\; (2k,\,\mathrm{evn}), \\ \mathrm{d}\,F_{2k} \;=\; 0\,, \end{cases}$$

or similarly, sequences of cocycles in degrees $(2k+1)_{k\in\mathbb{Z}}$ when valued instead in $\mathfrak{l}(\Sigma\mathrm{KU})$. Notice such sequences may be also thought of as even "*periodic cocycles*" of degree 0 mod 2 and 1 mod 2, respectively.

(III) Whitehead L_∞-algebras of bundles of spectra.

Example 2.11 (**Real Whitehead L_∞-algebra of bundles of spectra** [BMSS19, §2.1]). Given X, $\mathfrak{l}X$ as in Ex. 2.4 and E, $\mathfrak{l}E$ as in Ex. 2.9, the Whitehead L_∞-algebras of E-fiber ∞-bundles $E\!\sslash\!\Omega X$ over X are characterized as having underlying graded vector space that of $\mathfrak{l}(E)\oplus\mathfrak{l}(X)$ with L_∞-brackets such that the corresponding split exact sequence of graded vector spaces makes a (necessarily homotopy fiber-)sequence of L_∞-homomorphisms:

$$
\begin{tikzcd}
\mathfrak{l}E \arrow[r, "\mathrm{hofib}(p)"] \arrow[d, equal] & \mathfrak{l}(E /\!\!/ \Omega X) \arrow[r, two heads, "p"] \arrow[d, equal] & \mathfrak{l}X \arrow[d, equal] \\
\big(\pi_\bullet(\Omega E)\otimes_{\mathbb{Z}}\mathbb{R}, [\text{-},\text{-}]_E, [\text{-},\text{-},\text{-}]_E, \cdots\big) \arrow[r, hook] & \big((\pi_\bullet(\Omega E)\oplus\pi_\bullet(\Omega X))\otimes_{\mathbb{Z}}\mathbb{R}, [\text{-},\text{-}], [\text{-},\text{-},\text{-}], \cdots\big) \arrow[r, two heads] & \big(\pi_\bullet(\Omega X)\otimes_{\mathbb{Z}}\mathbb{R}, [\text{-},\text{-}]_X, [\text{-},\text{-},\text{-}]_X, \cdots\big) \\
\big(\wedge^\bullet\,(\pi_\bullet(\Omega E))^\vee, \mathrm{d}_E = 0\big) \arrow[d, equal] & \big(\wedge^\bullet\,(\pi_\bullet(\Omega E)\oplus\pi_\bullet(\Omega X))^\vee, \mathrm{d}\big) \arrow[l] \arrow[d, equal] & \big(\wedge^\bullet\,(\Omega X)^\vee, \mathrm{d}_X\big) \arrow[l] \arrow[d, equal] \\
\mathrm{CE}(\mathfrak{l}E) & \mathrm{CE}\big(\mathfrak{l}(E /\!\!/ \Omega X)\big) \arrow[l] & \mathrm{CE}\big(\mathfrak{l}(X)\big) \arrow[l]
\end{tikzcd}
$$

(The only choice is in the shaded brackets/differential in the middle).

Example 2.12 (**Real Whitehead L_∞-algebra of twisted K-theory spectrum** cf. [FSS17, §4][FSS23, Ex. 5.7, 6.6])**.** The real Whitehead L_∞-algebras (Ex. 2.4) of the classifying spectra $\Sigma^0\mathrm{KU}$ and $\Sigma^1\mathrm{KU}$ for complex topological K-theory canonically homotopy-quotiented by $\mathrm{PU}(\mathcal{H}) \simeq B\mathrm{U}(1)$ have a generator in degree 3 together with generators in every even (every odd) degree, with differential of the form known from 3-twisted de Rham cohomology

$$
\begin{aligned}
\mathrm{CE}\Big(\mathfrak{l}\big(\,\Sigma^0\mathrm{KU} /\!\!/ B\mathrm{U}(1)\,\big)\Big) \;&\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\overbrace{h_3}^{\deg\,=\,(3,\mathrm{evn})},\, \big(\underbrace{f_{2k}}_{\deg\,=\,(2k,\mathrm{evn})}\big)_{k\in\mathbb{Z}}\Big]\Big/\left(\begin{aligned}\mathrm{d}\,h_3 \;&=\; 0\\ \mathrm{d}\,f_{2k+2} \;&=\; h_3\,f_{2k}\end{aligned}\right)\\
\mathrm{CE}\Big(\mathfrak{l}\big(\,\Sigma^1\mathrm{KU} /\!\!/ B\mathrm{U}(1)\,\big)\Big) \;&\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\overbrace{h_3}^{\deg\,=\,(3,\mathrm{evn})},\, \big(\underbrace{f_{2k+1}}_{\deg\,=\,(2k+1,\mathrm{evn})}\big)_{k\in\mathbb{Z}}\Big]\Big/\left(\begin{aligned}\mathrm{d}\,h_3 \;&=\; 0\\ \mathrm{d}\,f_{2k+3} \;&=\; h_3\,f_{2k+1}\end{aligned}\right).
\end{aligned}
\tag{27}
$$

Since the general h_3 here is closed, these L_∞-algebras are canonically fibered over the line Lie 2-algebra (Ex. 2.6) with the fiber being the Whitehead L_∞-algebra (26) of the plain K-theory spectrum:

$$
\begin{tikzcd}
\mathfrak{l}\big(\Sigma^m\mathrm{KU}\big) \arrow[r] & \mathfrak{l}\big(\Sigma^m\mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \arrow[d, two heads] & h_3 \\
& \mathfrak{l}\,B^2\mathrm{U}(1) & \omega_3\,. \arrow[u, mapsto]
\end{tikzcd}
\tag{28}
$$

In rational homotopy theory this is the model for the fibration classifying 3-twisted complex-topological K-theory (cf. [FSS23, Ex. 3.4, Prop. 6.11, Prop. 10.1]).

A key application of this Ex. 2.12 is as the classifying object for 3-twisted cohomology in the familiar sense of [RW86, (23) & appndx]; in fact this is just the first example of a much more general concept of twisted real cohomology [FSS23, pp 120] as we briefly recall now:

Twisted rational cohomology. We have seen in Ex. 2.6 and Ex. 2.10 that the L_∞-algebras $b^n\mathbb{R}$ and $\Sigma^m\mathrm{KU}$ classify, respectively, ordinary $(n+1)$-cocycles and cocycles in 2-periodic degrees. Accordingly, the rational twisted K-theory spectra from Ex. 2.12 classify "3-twisted periodic cocycles" in the sense of [RW86, (23) & appndx][BCMMS02, §9.3]:

Definition 2.13 (**3-Twisted periodic Chevalley-Eilenberg complex**)**.** Let $\mathfrak{g}$ be a super-L_∞ algebra and $H_3 \in \mathrm{CE}(\mathfrak{g})$ a closed element in degree $(3, \mathrm{evn})$, to be called the "twisting 3-cocycle". The 3-*twisted Chevalley–Eilenberg complex* of $\mathfrak{g}$ with respect to H_3 is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (periodic mod 2 and super, respectively) dgca

$$
\mathrm{CE}^{\bullet+H_3}(\mathfrak{g}) \;:=\; \big(\,\mathrm{CE}(\mathfrak{g}),\, \mathrm{d}_{H_3} := \mathrm{d}_{\mathrm{CE}} - H_3\,\big)\,,
$$

where on the right-hand side we abusively write $\mathrm{CE}(\mathfrak{g})$ for the graded commutative super-algebra underlying the original Chevalley-Eilenberg (12) dgca of $\mathfrak{g}$.

It follows immediately (e.g. [FSS23, Ex. 6.6]) that $\mathfrak{l}\big(\Sigma^0\mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)$ (Ex. 2.12) serves as a classifying object for H_3-twisted (even) cocycles of degree 0 mod 2 on $\mathfrak{g}$. Indeed, maps of super-L_∞ algebras from $\mathfrak{g}$ into $\mathfrak{l}\big(\Sigma^0\mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)$, which respect the corresponding fiberings over $b^2\mathbb{R} \cong \mathfrak{l}B^2U(1)$, correspond precisely to sequences $(F_{2k})_{k\in\mathbb{Z}}$ satisfying the H_3-twisted closure condition:

$$
\begin{tikzcd}
\mathfrak{g} \arrow[rr, dashed, "{\left(H_3,\,(F_{2k})_{k\in\mathbb{Z}}\right)}"] \arrow[dr, "H_3"'] & & \mathfrak{l}\big(\Sigma^0\mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \arrow[dl, "h_3"] \\
& b^2\mathbb{R} &
\end{tikzcd}
\quad\Leftrightarrow\quad
\left\{\begin{aligned}
&(F_{2k})_{k\in\mathbb{Z}} \,\subset\, \mathrm{CE}(\mathfrak{g})\,,\\
&\deg(F_{2k}) \;=\; (2k,\,\mathrm{evn}),\\
&\mathrm{d}\,F_{2k} \;=\; H_3\cdot F_{2k-2}\,.
\end{aligned}\right.
\tag{29}
$$

Viewing equivalently such a sequence as a (0 mod 2, evn) cochain yields precisely an H_3-twisted cocycle

$$(F_{2k})_{k\in\mathbb{Z}} \in \mathrm{CE}^{0+H_3}(\mathfrak{g}) \qquad \text{s.t.} \qquad \mathrm{d}_{H_3}(F_{2k})_{k\in\mathbb{Z}} = 0\,.$$

Analogously, $\mathfrak{l}\big(\Sigma^1\mathrm{KU}\,/\!\!/ B\mathrm{U}(1)\big)$ classifies H_3-twisted cocycles of degree (1 mod 2, evn). In view of this canonical identification, we shall refer to 3-twisted cocycles equivalently as *rational twisted K-theory cocycles* – they are of the form of images of twisted K-theory classes under the twisted Chern character [FSS23, Prop. 10.1].

This situation has an evident generalization to higher degree twists: Naturally, we may consider twisting the Chevalley-Eilenberg cochain complex of a super-L_∞ algebra by any ordinary cocycle of degree $(2n+1, \mathrm{evn})$, instead, and hence also the corresponding $(2n+1)$-twisted cohomology for any $n \in \mathbb{N}$.

Definition 2.14 ((2n+1)-Twisted periodic Chevalley-Eilenberg complex). Let $\mathfrak{g}$ be a super-L_∞ algebra and $H_{2n+1} \in \mathrm{CE}(\mathfrak{g})$ a "twisting $(2n+1)$-cocycle". The $(2n+1)$-*twisted Chevalley–Eilenberg complex* of $\mathfrak{g}$ with respect to H_{2n+1} is the $\mathbb{Z}_{2n} \times \mathbb{Z}_2$-graded (periodic mod $2n$ and super respectively) dgca

$$\mathrm{CE}^{\bullet + H_{2n+1}}(\mathfrak{g}) := \big(\mathrm{CE}(\mathfrak{g}),\, \mathrm{d}_{H_{2n+1}} := \mathrm{d}_{\mathrm{CE}} - H_{2n+1}\big)\,,$$

where on the right-hand side we abusively write $\mathrm{CE}(\mathfrak{g})$ for the graded commutative super-algebra underlying the original Chevalley-Eilenberg (12) dgca of $\mathfrak{g}$.

In a similar fashion to the 3-twisted case from Eq. (29), $(2n+1)$-twisted cocycles correspond precisely to maps into certain classifying super-L_∞ algebras generalizing those of the rational twisted K-theory spectra from Ex. 2.12.

Example 2.15 ((2n+1)-twisted cocycle classifying L_∞-algebras [FSS23, Ex. 6.7, Rem. 10.1]). For any two positive integers $m, n \in \mathbb{N}$ with $m < 2n$, the classifying super-L_∞ algebra for $(2n+1)$-twisted cocycles in degree m mod $2n$ is defined by

$$\mathrm{CE}\Big(\mathfrak{l}\big(\Sigma^m\mathrm{K}^n\mathrm{U}\,/\!\!/ B^{2n-1}\mathrm{U}(1)\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\overbrace{h_{2n+1}}^{\deg\,=\,(2n+1,\mathrm{evn})},\, \big(\underbrace{f_{2nk+m}}_{\deg\,=\,(2nk+m,\mathrm{evn})}\big)_{k\in\mathbb{Z}}\Big]\Big/\left(\begin{aligned} \mathrm{d}\,h_{2n+1} &= 0 \\ \mathrm{d}\,f_{2(n+1)k+m} &= h_{2n+1}\,f_{2nk}\end{aligned}\right) \tag{30}$$

In analogy with (28), since h_{2n+1} is closed, these L_∞-algebras are canonically fibered over the line Lie $2n$-algebra (Ex. 2.6):

$$\begin{array}{ccc} \mathfrak{l}\big(\Sigma^m\mathrm{K}^n\mathrm{U}\big) \longrightarrow \mathfrak{l}\big(\Sigma^m\mathrm{K}^n\mathrm{U}\,/\!\!/ B^{2n-1}\mathrm{U}(1)\big) & & h_{2n+1} \\ \qquad\qquad\qquad\qquad \downarrow & & \uparrow \\ \qquad\qquad\qquad\qquad b^{2n}\mathbb{R} & & \omega_{2n+1}\,. \end{array} \tag{31}$$

Evidently, $\mathfrak{l}\big(\Sigma^m\mathrm{K}^n\mathrm{U}\,/\!\!/ B^{2n-1}\mathrm{U}(1)\big)$ classifies H_{2n+1}-twisted cocycles on $\mathfrak{g}$ in the sense of Def. 2.14, since maps of super-L_∞ algebras between the two, which respect the corresponding fiberings over $b^{2n}\mathbb{R}$, correspond precisely to sequences $(F_{2nk})_{k\in\mathbb{Z}}$ satisfying the H_{2n+1}-twisted closure condition

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, dashed, "{\left(H_{2n+1},\,(F_{2nk})_{k\in\mathbb{Z}}\right)}"] \arrow[dr, "H_{2n+1}"'] & & \mathfrak{l}\big(\Sigma^m\mathrm{K}^n\mathrm{U}\,/\!\!/ B^{2n-1}\mathrm{U}(1)\big) \arrow[dl, "h_{2n+1}"] \\ & b^{2n}\mathbb{R} & \end{tikzcd} \quad\Leftrightarrow\quad \left\{\begin{aligned} &(F_{2nk+m})_{k\in\mathbb{Z}} \subset \mathrm{CE}(\mathfrak{g})\,, \\ &\deg(F_{2nk+m}) = (2kn+m,\,\mathrm{evn}), \\ &\mathrm{d}\,F_{2(n+1)k+m} = H_{2n+1}\cdot F_{2nk}\,. \end{aligned}\right. \tag{32}$$

Viewing equivalently such a sequence as a (m mod $2n$, evn) cochain yields precisely an H_{2n+1}-twisted cocycle

$$(F_{2nk+m})_{k\in\mathbb{Z}} \in \mathrm{CE}^{m+H_{2n+1}}(\mathfrak{g}) \qquad \text{s.t.} \qquad \mathrm{d}_{H_{2n+1}}(F_{2nk+m})_{k\in\mathbb{Z}} = 0\,.$$

Twisted Nonabelian cohomology. The above examples of classifying cocycles in *abelian* H_{2n+1}-twisted CE algebras exhibit a clear pattern, namely: A H_{2n+1}-twisted periodic cocycle is precisely a lift of the twisting cocycle map $H_{2n+1} : \mathfrak{g} \longrightarrow b^{2n}\mathbb{R}$ along the twisting fibration of the twisted classifying space. This suggests the following general definition of what rational twisted "non-abelian" cocycles should be.

Definition 2.16 (Rational Nonabelian Twisted Cocycles [FSS23, Def. 6.7]). Let $\mathfrak{g}$ and $\mathfrak{c}$ be two super-L_∞ algebras, where we think of $\mathfrak{c}$ as a *classifying space.*

(i) We call *(rational nonabelian) $\mathfrak{c}$-cocycles on* $\mathfrak{g}$ simply the set of maps of super-L_∞ algebras

$$\mathfrak{g} \longrightarrow \mathfrak{c}\,.$$

(ii) Given a fibration

$$\mathfrak{c} \longrightarrow \widehat{\mathfrak{c}} \xrightarrow{\;h\;} \mathfrak{b}$$

and a fixed *"twisting"* $\mathfrak{b}$-cocycle

$$H \,:\, \mathfrak{g} \longrightarrow \mathfrak{b}\,,$$

we call *(rational nonabelian) H-twisted $\mathfrak{c}$-cocycles* the set of lifts along the fibration $h : \widehat{\mathfrak{c}} \longrightarrow \mathfrak{b}$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[rr, dashed, "\text{twisted cocycle}"] \arrow[dr, "\text{twist}", "H"'] & & \widehat{\mathfrak{c}} \arrow[dl, "h"] \\
& \mathfrak{b} &
\end{tikzcd}
$$

Remark 2.17 (Twisted rational cohomology). There is a notion of coboundaries between these twisted nonabelian L_∞-cocycles (given by concordance), thus yielding a corresponding notion of twisted $\mathbb{R}$-rational cohomology [FSS23, §6], which in particular subsumes the notion of twisted abelian cohomology from Def. 2.14 [FSS23, Prop. 6.13]. Here we need not further dwell on this.

Examples of this more general notion of twisting includes the following seemingly simple but important one:

Example 2.18 (Relative cocycles). Elements $c_n \in \mathrm{CE}(\mathfrak{g})$ that are "closed relative to" a fixed cocycle k_{n+1}, in that $\mathrm{d}c_n = k_{n+1}$, are classified by $eb^{n-1}\mathbb{R}$ given by

$$
\mathrm{CE}\big(eb^{n-1}\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \begin{bmatrix} c_n \\ k_{n+1} \end{bmatrix} \Big/ \left(\begin{array}{rcl} \mathrm{d}\, c_n & = & k_{n+1}, \\ \mathrm{d}\, k_{n+1} & = & 0 \end{array} \right), \tag{33}
$$

fibered as follows, this being the image under $\mathfrak{l}$ (Def. 2.4, cf. Ex. 2.6) of the universal $B^{n-1}\mathrm{U}(1)$-principal ∞-bundle (cf. [SS25]):

$$
\begin{tikzcd}
b^{n-1}\mathbb{R} \arrow[r] & eb^{n-1}\mathbb{R} \arrow[d, ->>] & k_{n+1} \\
& b^n\mathbb{R} & \omega_{n+1} \arrow[u, |->]
\end{tikzcd}
\;\simeq\; \mathfrak{l}\left(
\begin{tikzcd}
B^{n-1}\mathrm{U}(1) \arrow[r] & EB^{n-1}\mathrm{U}(1) \arrow[d, ->>] \\
& B^n\mathrm{U}(1)
\end{tikzcd}
\right).
$$

in that given $K_{n+1} \in \mathrm{CE}(\mathfrak{g})$ with $\mathrm{d}\, K_{n+1} = 0$ then

$$
\begin{tikzcd}
& eb^{n-1}\mathbb{R} \arrow[d] \\
\mathfrak{g} \arrow[ur, dashed, "C_n"] \arrow[r, "K_{n+1}"] & b^n\mathbb{R}
\end{tikzcd}
\qquad \Leftrightarrow \qquad \mathrm{d}\, C_n \;=\; K_{n+1}\,. \tag{34}
$$

An example where the twisting cocycle appears as a relative closure but instead via a higher polynomial twisting condition is the following case of $\mathfrak{l}S^4$-cocycles:

Example 2.19 ($\mathfrak{l}S^4$-cocycles as twisted $b^6\mathbb{R}$-cocycles). Rational 4-cohomotopy cocycles on a super-L_∞ algebra $\mathfrak{g}$ may equivalently be regarded as 4-twisted (in the sense of Def. 2.16) $b^6\mathbb{R}$-cocycles via the fibration

$$
\begin{tikzcd}
b^6\mathbb{R} \arrow[r] & \mathfrak{l}S^4 \arrow[d, ->>] & g_4 \\
& b^3\mathbb{R} & \omega_4 \arrow[u, |->]
\end{tikzcd}
\tag{35}
$$

or, rationally equivalently, as twisted cohomology classified by the fibration (24)

$$
\begin{tikzcd}
\mathfrak{l}S^7 \arrow[r] & \mathfrak{l}S^4 \arrow[d, ->>] & g_4 \\
& \mathfrak{l}BS^3 & \omega_4 \arrow[u, |->]
\end{tikzcd}
\tag{36}
$$

in that given G_4 with $\mathrm{d}G_4 = 0$ then

$$
\begin{tikzcd}
& \mathfrak{l}S^4 \arrow[d] \\
\mathfrak{g} \arrow[ur, dashed, "G_7"] \arrow[r, "G_4"] & b^3\mathbb{R}
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
& \mathfrak{l}S^4 \arrow[d] \\
\mathfrak{g} \arrow[ur, dashed, "G_7"] \arrow[r, "G_4"] & \mathfrak{l}BS^3
\end{tikzcd}
\quad \Leftrightarrow \quad \mathrm{d}\, G_7 \;=\; \tfrac{1}{2} G_4\, G_4\,. \tag{37}
$$

This exhibits G_7 as being a 7-cocycle twisted by G_4, cf. (2) – in somewhat subtle variation of the familiar situation of 3-twisted cohomology in (29) – as suggested in [Sa06, §3].

Another more subtle variant of Ex. 2.18 is:

Example 2.20 (Real Whitehead L_∞-algebra of quaternionic Hopf fibration [FSS20c, Prop. 3.20]). The relative real Whitehead L_∞-algebra of the quaternionic Hopf fibration over the 4-sphere is $\mathfrak{l}_{S^4}S^7$ given by

$$
\mathrm{CE}\big(\mathfrak{l}_{S^4}S^7\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \begin{bmatrix} g_4 \\ g_7 \\ h_3 \end{bmatrix} \Big/ \left(\begin{array}{rcl} \mathrm{d}\, g_4 & = & 0 \\ \mathrm{d}\, g_7 & = & \frac{1}{2}\, g_4\, g_4 \\ \mathrm{d}\, h_3 & = & g_4 \end{array} \right)
$$

and fibered over the 4-sphere (Ex. 2.7) as

$$
\begin{tikzcd}
\mathfrak{l}S^3 \arrow[r] & \mathfrak{l}_{S^4}S^7 \arrow[d] & g_4 & g_7 \\
 & \mathfrak{l}S^4 & g_4 \arrow[u, maps to] & g_7 \arrow[u, maps to]
\end{tikzcd}
\;\;\simeq\;\;
\mathfrak{l}\left(
\begin{tikzcd}
S^3 \arrow[r] & S^7 \arrow[d, "h_{\mathbb{H}}"] \\
 & S^4
\end{tikzcd}
\right).
$$

This means that the twisted non-abelian cocycles (Def. 2.16) classified by the quaternionic Hopf fibration (a twisted form of 3-Cohomotopy) are rationally given by 3-coboundaries of the 4-form datum (only): Given a twisting cocycle in rational 4-Cohomotopy

$$
\mathfrak{g} \xrightarrow{(G_4, G_7)} \mathfrak{l}S^4 \quad \leftrightsquigarrow \quad \left(G_4,\, G_7 \,\in\, \mathrm{CE}(\mathfrak{g}) \;\middle|\; \begin{array}{lcl} \mathrm{d}\, G_4 & = & 0 \\ \mathrm{d}\, G_7 & = & \tfrac{1}{2} G_4\, G_4 \end{array} \right),
$$

then the corresponding twisted 3-Cohomotopy cocycles are

$$
\begin{tikzcd}
\mathfrak{g} \arrow[rr, dashed, "H_3"] \arrow[dr, "(G_4, G_7)"'] & & \mathfrak{l}_{S^4}S^7 \arrow[dl] \\
 & \mathfrak{l}S^4 &
\end{tikzcd}
\quad \leftrightsquigarrow \quad \big(H_3 \,\in\, \mathrm{CE}(\mathfrak{g}) \,\big|\, \mathrm{d}\, H_3 \,=\, G_4 \big)\,.
$$

2.2 Ext/Cyc adjunction

With (extended) super-spacetimes understood — via their translational super-symmetry (Ex. 2.3) — as (higher) super-Lie algebras, fundamental constructions of super-Lie theory have (rational/infinitesimal) geometric significance. Notably the process of *central extension* (Def. 2.21) of super-L_∞ algebras by 2-cocycles corresponds in the super-geometric interpretation to the emergence of extra dimensions by 0-brane condensation ([CdAIP00, §2][FSS15, Rem. 3.11][HS18], see Ex. 3.2, 3.3 below).

One may hence ask for the (higher super) Lie-theoretic incarnation of the geometrically expected process of double [8] dimensional Kaluza-Klein reduction — and conversely: oxidation — along such extensions. Remarkably, this is given by the process of *cyclification* (passage to loop spaces homotopy-quotiented by loop rotation, as known from cyclic cohomology and from the geometric motivation for the Witten genus): On the rational-homotopy level of super-L_∞-algebras this is due to [FSS18a, §3][FSS18b, §2.6], recalled as Def. 2.23 and Prop. 2.25 below (for exposition see [Sc16, §4], for more in the context of Mysterious Triality and U-duality within a bosonic CDGA algebraic approach see [SV23][SV23], for the topological globalization see [BMSS19, §2.2][SS24a] and for its application to double-field theory see [Al20][Al21]).

Definition 2.21 (Central extension of super-L_∞ algebra by 2-cocycle). Given $\mathfrak{g} \,\in\, \mathrm{sLieAlg}_\infty$ and a 2-cocycle

$$
\omega_2 \,\in\, \mathrm{CE}(\mathfrak{g})\,, \;\; \deg(\omega_2) \,=\, (1, \mathrm{evn})\,, \;\; \mathrm{d}\,\omega_2 \,=\, 0 \qquad \Leftrightarrow \qquad \mathfrak{g} \xrightarrow{\;\;\omega_2\;\;} b\mathbb{R}
$$

then the corresponding *central extension* $\widehat{\mathfrak{g}} \,\in\, \mathrm{sLieAlg}_{\mathbb{R}}$ is that super-Lie algebra whose CE-algebra is that of $\mathfrak{g}$ with one more generator e' adjoined whose differential is ω_2:

$$
\mathrm{CE}\big(\widehat{\mathfrak{g}}\big) \;=\; \mathrm{CE}(\mathfrak{g})\big[\underbrace{\,e'\,}_{\deg = (1,\mathrm{evn})}\big]\big/\big(\mathrm{d}\,e' \;=\; \omega_2\big) \qquad \Leftrightarrow \qquad
\begin{tikzcd}
\widehat{\mathfrak{g}} \arrow[d, ->>, "p \,:=\, \mathrm{hofib}(\omega_2)"] & \\
\mathfrak{g} \arrow[r, "\omega_2"] & b\mathbb{R}\,.
\end{tikzcd}
$$

Remark 2.22 (Basic and fiber forms on a centrally extended super-L_∞ algebra).
(i) Given a central extension as in Def. 2.21, every element in its CE-algebra decomposes uniquely as the sum [9]

$$
\alpha \;=\; \alpha_{\mathrm{bas}} \,+\, e'\, p_*(\alpha) \tag{38}
$$

of a *basic form* (not involving the generator e, hence in the image of the pullback p^*)

$$
\alpha_{\mathrm{bas}} \,\in\, p^*\big(\mathrm{CE}(\mathfrak{g})\big)
$$

[8] The term “double dimensional reduction” originates with [DHIS87], referring to the fact that for Kaluza-Klein reduction of target spaces for p-branes both the target spacetime as well as the worldvolume of *wrapping* branes reduces in dimension – or, essentially equivalently, that also the corresponding flux densities decrease in degree upon integration over the fiber spaces. This is, of course, the very mechanism that underlies the emergence of fields with enhanced/exceptional symmetry in lower dimensions.

[9] Beware that [FSS18a, (1)] and [FSS20a, (21)] have a minus sign on the second summand in (38). This is, of course, a possible convention in itself, but breaks the desirable property of p_* being a graded derivation (39), that we want to retain here. With the plus sign in (38) we get the corresponding minus sign in (44) below, correspondingly differing from the sign in [FSS18a, (3)].

and the product of the generator e' with the image of α under *fiber integration* p_*, which is a super-graded derivation of degree $(-1, \mathrm{evn})$:

$$\begin{array}{ccc} \mathrm{CE}(\,\widehat{\mathfrak{g}}\,) & \xrightarrow{\;\;p_*\;\;} & \mathrm{CE}(\mathfrak{g}) \\ e' & \longmapsto & 1 \\ e^i & \longmapsto & 0 \end{array} \tag{39}$$

(where in the last line $(e^i)_{i\in I}$ denote generators for $\mathrm{CE}(\mathfrak{g})$).

(ii) The differential of a general element is given in this decomposition in terms of (the image under p^* of) the differential $\mathrm{d}_{\mathfrak{g}}$ by:

$$\begin{aligned} \mathrm{d}_{\widehat{\mathfrak{g}}}\big(\alpha_{\mathrm{bas}} + e'\,p_*\alpha\big) &= \mathrm{d}_{\widehat{\mathfrak{g}}}\,\alpha_{\mathrm{bas}} + \big(\mathrm{d}_{\widehat{\mathfrak{g}}}\,e'\big)\,p_*\alpha - e'\,\mathrm{d}_{\widehat{\mathfrak{g}}}\,p_*\alpha \\ &= \big(\mathrm{d}_{\mathfrak{g}}\alpha_{\mathrm{bas}} + \omega_2\,p_*\alpha\big) - e'\,\mathrm{d}_{\mathfrak{g}}p_*\alpha\,. \end{aligned} \tag{40}$$

Definition 2.23 (**Cyclification of super L_∞-algebras**, cf. [FSS17, Prop. 3.2][FSS18a, Def. 3.3])**.** Given $\mathfrak{h} \in \mathrm{sLieAlg}^{\mathrm{fin}}_\infty$ with presentation $\mathrm{CE}(\mathfrak{h}) \simeq \mathbb{R}_{\mathrm{d}}\big[(e^i)_{i\in I}\big]/\big(\mathrm{d}\,e^i = P^i(\vec{e}\,)\big)_{i\in I}$, its *cyclification* $\mathrm{cyc}(\mathfrak{h}) \in \mathrm{sLieAlg}_\infty$ is given by

$$\mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) \;:=\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l} \big(e^i\big)_{i\in I},\, \overbrace{\omega_2,}^{\mathrm{deg}\,=\,(2,\mathrm{evn})} \\ \big(\underbrace{\mathrm{s}e^i}_{\substack{\mathrm{deg}\,= \\ \mathrm{deg}(e^i)-(1,\mathrm{evn})}}\big)_{i\in I} \end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\omega_2 &= 0 \\ \mathrm{d}\,e^i &= \mathrm{d}_{\mathfrak{h}}\,e^i + \omega_2\,\mathrm{s}e^i \\ \mathrm{d}\,\mathrm{s}e^i &= -\mathrm{s}\big(\mathrm{d}_{\mathfrak{h}}\,e^i\big) \end{array}\right), \tag{41}$$

where in the last line on the right the shift is understood as uniquely extended to a super-graded derivation of degree $(-1, \mathrm{evn})$:

$$\begin{array}{rcl} \mathrm{s} : \mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) & \longrightarrow & \mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) \\ \omega_2 & \longmapsto & 0\,, \\ e^i & \longmapsto & \mathrm{s}e^i\,, \\ \mathrm{s}e^i & \longmapsto & 0\,. \end{array}$$

To check that this is well-defined:

Lemma 2.24 (**Differential and shift in cyclification**)**.** *In Def. 2.23 the differential* d *and shift* s *square to zero and anti-commute with each other:*

$$\mathrm{d}\,\mathrm{d} = 0\,, \qquad \mathrm{s}\,\mathrm{s} = 0\,, \qquad \mathrm{s}\,\mathrm{d} + \mathrm{d}\,\mathrm{s} = 0\,. \tag{42}$$

Proof. First, that s squares to zero is immediate from the definition. Moreover, since we are dealing with (graded) derivations and their (graded) commutator, it is sufficient to check all these statements on generators.

The anticommutativity is thus seen as:

$$\begin{aligned} \mathrm{s}\,\mathrm{d}\,\omega_2 + \mathrm{d}\,\mathrm{s}\,\omega_2 &= 0 + 0 = 0\,, \\ \mathrm{s}\,\mathrm{d}\,e^i + \mathrm{d}\,\mathrm{s}\,e^i &= \mathrm{s}\big(\mathrm{d}_{\mathfrak{h}}e^i + \omega_2\,\mathrm{s}e^i\big) - \mathrm{s}\,\mathrm{d}_{\mathfrak{h}}e^i = 0\,, \\ \mathrm{s}\,\mathrm{d}\,\mathrm{s}e^i + \mathrm{d}\,\mathrm{s}\,\mathrm{s}e^i &= -\mathrm{s}\,\mathrm{s}\,\mathrm{d}_{\mathfrak{h}}e^i = 0\,. \end{aligned}$$

For nilpotency of d we first trivially have $\mathrm{d}\,\mathrm{d}\,\omega_2 = 0$, then

$$\begin{aligned} \mathrm{d}\,\mathrm{d}\,e^i &= \mathrm{d}\big(\mathrm{d}_{\mathfrak{h}}e^i + \omega_2\,\mathrm{s}e^i\big) \\ &= \underbrace{\mathrm{d}_{\mathfrak{h}}\,\mathrm{d}_{\mathfrak{h}}}_{=0}\,e^i + \omega_2\,s\,\mathrm{d}_{\mathfrak{h}}e^i + \omega_2\,\mathrm{d}(\mathrm{s}\,e^i) + \omega_2\,\omega_2\,\underbrace{\mathrm{s}\,\mathrm{s}}_{=0}\,e^i \\ &= \omega_2\big(\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} - \mathrm{s}\,\mathrm{d}_{\mathfrak{h}}\big)e^i = 0\,. \end{aligned}$$

From this, finally:

$$\mathrm{d}\,\mathrm{d}\,\mathrm{s}\,e^i = \mathrm{s}\,\mathrm{d}\,\mathrm{d}\,e^i = 0\,.$$

□

The following statement is due to [FSS18a, Thm. 3.8], we give a streamlined proof with more details.

Proposition 2.25 (**The Ext/Cyc-adjunction**)**.** *Given* $\mathfrak{g}, \mathfrak{h} \in \mathrm{sLieAlg}_\infty$ *with a 2-cocycle* [10] $c_1 \in \mathrm{CE}(\mathfrak{g})$, *there is a bijection between:*

(i) *maps into* $\mathfrak{h}$ *out of the central extension* $\widehat{\mathfrak{g}}$ *classified by the 2-cocycle* (Def. 2.21),

[10] We usually give all algebra generators a subscript indicative of their degree. But here we write "c_1" since this is the standard symbol for the 1st Chern class of a line bundle, namely here for the Lie-theoretic line bundle $\widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$.

(ii) *maps out of* $\mathfrak{g}$ *into the cyclification of* $\mathfrak{h}$ (Def. 2.23) *that preserve the 2-cocycle:*

$$\left\{ \widehat{\mathfrak{g}} \xrightarrow{\;\;f\;\;} \mathfrak{h} \right\} \;\underset{\textbf{oxidation}\ \ \mathrm{oxd}_{c_1}}{\overset{\textbf{reduction}\ \ \mathrm{rdc}_{c_1}}{\overset{\sim}{\rightleftarrows}}}\; \left\{ \begin{tikzcd} \mathfrak{g} \arrow[rr, "\widetilde{f}"] \arrow[dr, "c_1"'] && \mathrm{cyc}(\mathfrak{h}) \arrow[dl, "\omega_2"] \\ & b\mathbb{R} & \end{tikzcd} \right\} \tag{43}$$

given by

$$\begin{array}{ccc} \widehat{\mathfrak{g}} & \xrightarrow{\;\;f\;\;} & \mathfrak{h} \\ \alpha^i_{\mathrm{bas}} + e'\, p_* \alpha^i & \longleftarrow\!\shortmid & e^i \end{array} \quad\rightsquigarrow\quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;\;\widetilde{f}\;\;} & \mathrm{cyc}(\mathfrak{h}) \\ \alpha^i_{\mathrm{bas}} & \longleftarrow\!\shortmid & e^i \\ -p_*\alpha^i & \longleftarrow\!\shortmid & \mathrm{s}e^i \\ c_1 & \longleftarrow\!\shortmid & \omega_2\,. \end{array} \tag{44}$$

Proof. The assignment (44) is manifestly a bijection of maps of underlying graded super-algebras. Hence, it suffices to show that if one of these is, moreover, a homomorphism of dg-algebras (in that it preserves the differential), then so is its image.

To that end, first note that when the map on the left of (44) is a dg-homomorphism then this implies that

$$\begin{array}{rcll} f^*(\mathrm{d}_{\mathfrak{h}} e^i) & = & \mathrm{d}_{\widehat{\mathfrak{g}}} f^*(e^i) & \text{by homomorphy} \\ & = & \mathrm{d}_{\widehat{\mathfrak{g}}}(\alpha^i_{\mathrm{bas}} + e'\, p_*\alpha^i) & \text{by (44)} \\ & = & (\mathrm{d}_{\mathfrak{g}}\alpha^i_{\mathrm{bas}} + \omega_2\, p_*\alpha^i) - e'\, \mathrm{d}_{\mathfrak{g}}\, p_*\alpha^i & \text{by (40)}\,, \end{array} \tag{45}$$

while the map on the right being an algebra homomorphism already implies (seen e.g. by expanding in generators):

$$\begin{array}{rcl} \widetilde{f}^*\big(\mathrm{d}_{\mathfrak{h}} e^i\big) & = & \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}} \\ \widetilde{f}^*(\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} e^i) & = & -p_*\big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)\,. \end{array} \tag{46}$$

If the map $\widetilde{f}$ on the right is moreover a dg-homomorphism then this implies that the map f on the left is so, as follows:

$$\begin{array}{rcll} f^*(\mathrm{d}_{\mathfrak{h}} e^i) & = & \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}} + e'\, p_* f^*(\mathrm{d}_{\mathfrak{h}} e^i) & \text{by (38)} \\ & = & \widetilde{f}^*(\mathrm{d}_{\mathfrak{h}} e^i) - e'\, \widetilde{f}^*\big(\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} e^i\big) & \text{by (46)} \\ & = & \widetilde{f}^*(\mathrm{d}_{\mathrm{cyc}(\mathfrak{h})} e^i - \omega_2\,\mathrm{s}e^i) - e'\; \widetilde{f}^*\big(\mathrm{s}\,\mathrm{d}_{\mathrm{cyc}(\mathfrak{h})} e^i\big) & \text{by (41)} \\ & = & \widetilde{f}^*(\mathrm{d}_{\mathrm{cyc}(\mathfrak{h})} e^i - \omega_2\,\mathrm{s}e^i) + e'\; \widetilde{f}^*\big(\mathrm{d}_{\mathrm{cyc}(\mathfrak{h})}\, \mathrm{s}e^i\big) & \text{by (42)} \\ & = & \mathrm{d}_{\mathfrak{g}} \widetilde{f}^*(e^i) - \widetilde{f}^*(\omega_2\,\mathrm{s}e^i) + e'\; \mathrm{d}_{\mathfrak{g}} \widetilde{f}^*\big(\mathrm{s}e^i\big) & \text{by homomorphy} \\ & = & \mathrm{d}_{\mathfrak{g}}\alpha^i_{\mathrm{bas}} + \omega_2\, p_*\alpha^i - e'\, \mathrm{d}_{\mathfrak{g}} p_*\alpha^i & \text{by (44)} \\ & = & \mathrm{d}_{\widehat{g}}\big(\alpha^i_{\mathrm{bas}} + e'\, p_*\alpha_i\big) & \text{by (40)} \\ & = & \mathrm{d}_{\widehat{g}}\, f^*(e^i) & \text{by (44).} \end{array} \tag{47}$$

Conversely, when f on the left of (44) is a dg-homomorphism, respect for the differential on the right is implied:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\widetilde{f}"] & & \mathrm{cyc}(\mathfrak{h}) \\ \alpha^i_{\mathrm{bas}} \arrow[d, mapsto, "\mathrm{d}"] & & e^i \arrow[ll, mapsto] \arrow[dd, mapsto, "\mathrm{d}"] \\ \mathrm{d}_{\mathfrak{g}}\alpha^i_{\mathrm{bas}} \arrow[dr, equal, "(45)"'] & & \\ & \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}} - \omega_2\, p_*\alpha^i & \mathrm{d}_{\mathfrak{h}} e^i + \omega_2\,\mathrm{s}e^i \arrow[l, mapsto, "(46)"] \\ -p_*\alpha^i \arrow[d, mapsto, "\mathrm{d}"] & & \mathrm{s}e^i \arrow[ll, mapsto] \arrow[dd, mapsto, "\mathrm{d}"] \\ -\mathrm{d}_{\mathfrak{g}}\, p_*\alpha^i \arrow[dr, equal, "(45)"'] & & \\ & p_* f^*(\mathrm{d}_{\mathfrak{h}} e^i) & -\mathrm{s}\big(\mathrm{d}_{\mathfrak{h}} e^i\big) \arrow[l, mapsto, "(46)"] \\ \omega_2 \arrow[d, mapsto, "\mathrm{d}"] & & \omega_2 \arrow[ll, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\ 0 & & 0\,. \arrow[ll, mapsto] \end{tikzcd}$$

□

Example 2.26 (**Cyclification of the 4-Sphere** [FSS17, Ex. 3.3])**.** The cyclification (Def. 2.23) of the real Whitehead L_∞-algebra of the 4-sphere (Ex. 2.4) is given by:

$$\mathrm{CE}\big(\mathrm{cyc}(\mathfrak{l}S^4)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} \omega_2 \\ g_4 \\ \mathrm{s}g_4 \\ g_7 \\ \mathrm{s}g_7 \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\omega_2 &=& 0 \\ \mathrm{d}\,g_4 &=& \omega_2\,\mathrm{s}g_4 \\ \mathrm{d}\,\mathrm{s}g_4 &=& 0 \\ \mathrm{d}\,g_7 &=& \tfrac{1}{2} g_4\, g_4 + \omega_2\, \mathrm{s}g_7 \\ \mathrm{d}\,\mathrm{s}g_7 &=& -g_4\,\mathrm{s}g_4 \end{array}\right). \tag{48}$$

Note that this is fibered over $b^2\mathbb{R}$ — in fact over $\mathrm{cyc}\, b^3\mathbb{R}$, via (23) — and as such, remarkably, a rational approximation to the twisted K-theory spectrum (27), via a comparison map to the 6-truncation τ_6 of its underlying space Ω^∞ (where only generators with degrees in $\{0, \cdots, 6\}$ are kept, cf. [FSS18a, Prop. 4.8]):

$$\begin{tikzcd}
 & 0 & & f_0 \arrow[ll, mapsto] & \\
 & \omega_2 & & f_2 \arrow[ll, mapsto] & \\
 & g_4 & & f_4 \arrow[ll, mapsto] & \\
 & -\mathrm{s}g_7 & & f_6 \arrow[ll, mapsto] & \\
\mathrm{s}g_4 & \mathrm{cyc}\,\mathfrak{l}S^4 \arrow[rr] \arrow[dr, ->>] & & \mathfrak{l}\big(\tau_6\Omega^\infty\,\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[dl, ->>] & h_3 \\
 & & b^2\mathbb{R} & & \\
 & & \omega_3 \arrow[uull, mapsto] \arrow[uurr, mapsto] & &
\end{tikzcd}$$

A rationale for completing $\mathrm{cyc}\,\mathfrak{l}S^4$ to all of $\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)$ is discussed in [BMSS19], namely by fiberwise *stabilization* (i.e., homotopical *linearization*) of $\mathrm{cyc}\,\mathfrak{l}S^4$ over $b^2\mathbb{R}$, as would befit a perturbative approximation to the dimensional reduction of the non-linear Bianchi identity (2). This step is relevant for a deeper understanding of the lift of T-duality into M-theory indicated in §4; but its discussion needs an article of its own.

The main example of interest here are the L_∞-algebraic cyclifications of twisted K-theory spectra (Ex. 2.28) since their structure turns out to embody the rational-topological structure of T-duality (Lem. 2.30, as made concrete in §3), whence we may speak of *L_∞-algebraic T-duality* [FSS18a, §5]:

Example 2.27 (**Cyclification of bundle gerbe classifying space**)**.** The cyclifications (Def. 2.23) of the real Whitehead L_∞-algebra of $B^2\mathrm{U}(1)$ (Ex. 2.6), $\mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1)$, is given by

$$\mathrm{CE}\Big(\mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega_2 \\ \omega_3 \\ \widetilde{\omega}_2 := \mathrm{s}\omega_3 \end{array}\right] \Big/ \left(\begin{array}{l} \mathrm{d}\,\omega_2 = 0 \\ \mathrm{d}\,\omega_3 = \omega_2\,\widetilde{\omega}_2 \\ \mathrm{d}\,\widetilde{\omega}_2 = 0 \end{array}\right)$$

being equivalently the higher central extension (Def. 2.59) of $b\mathbb{R}^2$ by its canonical 4-cocycle

$$b\mathcal{T} \;:=\; \mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1) \xrightarrow{\;\mathrm{hofib}(\omega_2\,\widetilde{\omega}_2)\;} b\mathbb{R}^2 \xrightarrow{\;\omega_2\,\widetilde{\omega}_2\;} b^3\mathbb{R} \tag{49}$$

and as such also known as (the Whitehead L_∞-algebra of) the delooping of the *T-duality Lie 2-group* [FSS13, §3.2.1][FSS18a, Rem. 7.2][NW20, §3.2]:

$$\mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1) \;\;\simeq\;\; \mathfrak{l}B\mathcal{T} \;\;\equiv\;\; \mathrm{hofib}\big(B\mathrm{U}(1)\times B\mathrm{U}(1) \xrightarrow{\;\pi_1 c_1 \cup \pi_2 c_1\;} B^3\mathrm{U}(1)\big)\,.$$

It is evident at a glance that (49) has an automorphism symmetry given by exchanging the two degree=2 generators (we may as well include a minus sign, for compatibility further below 2.28):

$$\begin{array}{ccc} b\mathcal{T} & \xleftrightarrow{\;\sim\;} & b\mathcal{T} \\ -\widetilde{\omega}_2 & \longleftarrow\!\shortmid & \omega_2 \\ -\omega_2 & \longleftarrow\!\shortmid & \widetilde{\omega}_2\,. \end{array} \tag{50}$$

This simplistic example already carries in it the seed of T-duality: The next example, recalled from [FSS18a, Prop. 5.1], shows that this automorphism lifts to an equivalence between the cyclifications of the 3-twisted K-theory spectra. This serves here as warmup for the higher toroidal super-L_∞ T-duality introduced in §2.3.

Example 2.28 (**Cyclification of twisted K-spectra and T-duality 2-group**)**.** The cyclifications (Def. 2.23) of the real Whitehead L_∞-algebra of the twisted K-theory spectra (Ex. 2.12) are identified by an isomorphism (14):

$$
\begin{array}{c}
\mathrm{CE}\Big(\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega_2 \\ h_3 \\ \mathrm{s}h_3 \\ (f_{2k})_{k\in\mathbb{Z}} \\ (\mathrm{s}f_{2k})_{k\in\mathbb{Z}} \end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\omega_2 & = 0 \\ \mathrm{d}\,h_3 & = \omega_2\,\mathrm{s}h_3 \\ \mathrm{d}\,\mathrm{s}h_3 & = 0 \\ \mathrm{d}\,f_{2k+2} & = h_3\,f_{2k} + \omega_2\,\mathrm{s}f_{2k+2} \\ \mathrm{d}\,\mathrm{s}f_{2k+2} & = -(\mathrm{s}h_3)f_{2k} + h_3\,\mathrm{s}f_{2k} \end{array}\right)
\\
\begin{array}{ccccc} -\omega_2 & -\mathrm{s}h_3 & h_3 & f_{2k} & \mathrm{s}f_{2k+2} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathrm{s}h_3 & \omega_2 & h_3 & \mathrm{s}f_{2k+1} & f_{2k+1} \end{array} \quad \Big\uparrow \wr
\\
\mathrm{CE}\Big(\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega_2 \\ h_3 \\ \mathrm{s}h_3 \\ (f_{2k+1})_{k\in\mathbb{Z}} \\ (\mathrm{s}f_{2k+1})_{k\in\mathbb{Z}} \end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\omega_2 & = 0 \\ \mathrm{d}\,h_3 & = \omega_2\,\mathrm{s}h_3 \\ \mathrm{d}\,\mathrm{s}h_3 & = 0 \\ \mathrm{d}\,f_{2k+1} & = h_3\,f_{2k-1} + \omega_2\,\mathrm{s}f_{2k+1} \\ \mathrm{d}\,\mathrm{s}f_{2k+3} & = -(\mathrm{s}h_3)f_{2k+1} + h_3\,\mathrm{s}f_{2k+1} \end{array}\right).
\end{array}
\tag{51}
$$

compatible with their fibration (28) over $\mathfrak{l}B^2\mathrm{U}(1)$ via its automorphisms (50), where the homotopy fiber of the cyclified fibration is now the direct sum of K-theory spectra in degrees 0 and 1, respectively, with the automorphism acting by swapping them ([FSS18a, Prop. 7.3]):

$$
\begin{tikzcd}
\mathfrak{l}\Sigma^{0}\mathrm{KU} \times \mathfrak{l}\Sigma^{1}\mathrm{KU} \arrow[d] \arrow[r] & \mathfrak{l}\Sigma^{1}\mathrm{KU} \times \mathfrak{l}\Sigma^{0}\mathrm{KU} \arrow[d] \\
\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[r, leftrightarrow, "\sim"] \arrow[d, ->>] & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[d, ->>] \\
\mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1) \arrow[r, leftrightarrow, "\sim"] & \mathrm{cyc}\,\mathfrak{l}B^2\mathrm{U}(1)\,.
\end{tikzcd}
\tag{52}
$$

Here we record the fact that there exist two further isomorphisms with the property of swapping swapping the 2-cocycles $\mathrm{s}h_3 \leftrightsquigarrow \omega_2$ and the 'fluxes' $f_{2k+m} \leftrightsquigarrow \mathrm{s}f_{2k+m}$, up to a consistent choice of signs. This may have been previously unnoticed.

Lemma 2.29 (All isomorphisms of cyclified twisted K-spectra). *There are in total 4 isomorphisms*

$$
\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) \xleftrightarrow{\;\sim\;} \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)
$$

with the property of swapping $\mathrm{s}h_3 \leftrightsquigarrow \omega_2$ *and* $f_{2k+m} \leftrightsquigarrow \mathrm{s}f_{2k+m}$*, while mapping* h_3 *to* h_3*, up to relative sign prefactors. Explicitly, in addition to* (51) *one has*

$$
\begin{array}{ccc}
\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) & \xleftrightarrow{\;\sim\;} & \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) \\
h_3 & \longleftarrow\!\shortmid & h_3 \\
-\mathrm{s}h_3 & \longleftarrow\!\shortmid & \omega_2 \\
-\omega_2 & \longleftarrow\!\shortmid & \mathrm{s}h_3 \\
-\mathrm{s}f_{2k+2} & \longleftarrow\!\shortmid & f_{2k+1} \\
-f_{2k} & \longleftarrow\!\shortmid & \mathrm{s}f_{2k+1}\,,
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) & \xleftrightarrow{\;\sim\;} & \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) \\
h_3 & \longleftarrow\!\shortmid & h_3 \\
\mathrm{s}h_3 & \longleftarrow\!\shortmid & \omega_2 \\
\omega_2 & \longleftarrow\!\shortmid & \mathrm{s}h_3 \\
-\mathrm{s}f_{2k+2} & \longleftarrow\!\shortmid & f_{2k+1} \\
f_{2k} & \longleftarrow\!\shortmid & \mathrm{s}f_{2k+1}\,,
\end{array}
\tag{53}
$$

and

$$
\begin{array}{ccc}
\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \xleftrightarrow{\sim} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^1\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
h_3 & \longleftarrow & h_3 \\
\mathrm{s}h_3 & \longleftarrow & \omega_2 \\
\omega_2 & \longleftarrow & \mathrm{s}h_3 \\
\mathrm{s}f_{2k+2} & \longleftarrow & f_{2k+1} \\
-f_{2k} & \longleftarrow & \mathrm{s}f_{2k+1}\,.
\end{array}
$$

Evidently, the original isomorphism (51) *and the first above are the two possible extensions of the automorphism* (50) *of* $b\mathcal{T}$ *, while the latter two isomorphisms are the two possible extensions of the "opposite" automorphism of* $b\mathcal{T}$

$$
\begin{array}{ccc}
b\mathcal{T} & \xleftrightarrow{\sim} & b\mathcal{T} \\
\mathrm{s}h_3 & \longleftarrow & \omega_2 \\
\omega_2 & \longleftarrow & \mathrm{s}h_3\,.
\end{array}
$$

Proof. This follows by direct inspection. Explicitly, starting (for instance) from the first isomorphism (51) one may ask which possible extra set of signs one can insert in the image of the generators, such that it still commutes with the differentials. The relation $\mathrm{d}h_3 = \omega_2 \mathrm{s}h_3$ restricts the map of graded commutative algebras to be of the form

$$
\begin{array}{ccc}
\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \xleftrightarrow{\sim} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^1\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
h_3 & \longleftarrow & h_3 \\
-(-1)^q\mathrm{s}h_3 & \longleftarrow & \omega_2 \\
-(-1)^q\omega_2 & \longleftarrow & \mathrm{s}h_3 \\
(-1)^{x_0}\mathrm{s}f_{2k+2} & \longleftarrow & f_{2k+1} \\
(-1)^{x_1}f_{2k} & \longleftarrow & \mathrm{s}f_{2k+1}\,,
\end{array}
$$

for some $q, x_0, x_1 \in \mathbb{N}$. Demanding that it further commutes with the corresponding differentials on f_{2k+1} (and $\mathrm{s}f_{2k+3}$) yields the condition

$$
(-1)^{q+x_0+x_1} = 1\,,
$$

whose set of solutions gives the 3 extra isomorphisms above. □

With cyclification and with this automorphism in hand, we already obtain the following general construction, which turns out to be the L_∞-algebraic template of super-space T-duality in §3 below, see (138) there, for illustration:

Lemma 2.30 (Twisted K–theory cocycles under reduction–isomorphism–reoxidation).
(i) *The composite operation of*

(a) *reducing* (43) *twisted* KU_0 *cocycles on a centrally extended super* L_∞*-algebra* $\mathfrak{g}_A$

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_A & \longrightarrow & \mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
H^3_A & \longleftarrow & h_3 \\
(F_{2k})_{k\in\mathbb{Z}} & \longleftarrow & (f_{2k})_{k\in\mathbb{Z}}
\end{array}
$$

along its fibration

$$
\widehat{\mathfrak{g}}_A \xrightarrow{\;p_A\;} \mathfrak{g} \xrightarrow{\;c_1^A\;} b\mathbb{R}\,,
$$

(b) *applying the isomorphism* (51) *on the target cyclification of twisted* KU_0*, hence viewing them instead as valued in the cyclification of twisted* KU_1*, while noticing that this swaps the "Chern class"* c_1^A *from that classifying the* $\widehat{\mathfrak{g}}_A$*-extension to that classifying a different extension*

$$
\widehat{\mathfrak{g}}_B \longrightarrow \mathfrak{g}\,,
$$

i.e., via

$$
c_B^1 := p_{A*}(H_A^3) \quad : \quad \mathfrak{g} \longrightarrow b\mathbb{R}\,,
$$

(c) *re-oxidizing* (43) *the result, but now along the new fibration*

$$
\widehat{\mathfrak{g}}_B \xrightarrow{\;p_A\;} \mathfrak{g} \xrightarrow{\;c_B^1\;} b\mathbb{R}\,,
$$

results into the twisted KU_{1} *cocycles given precisely by*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad} & \mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\ H^3_{A\,\mathrm{bas}} + e'_B \cdot c_1^A & \longleftarrow\!\!\shortmid & h_3 \\ \big(-p_{A*}F_{2k+2} - e'_B \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}} \end{array} \tag{54}$$

(ii) *Applying instead one of the isomorphisms from Lem. 2.29 in step* **(b)** *yields similar, but essentially different maps between twisted K-theory cocycles of different extensions over $\mathfrak{g}$. For instance, using the isomorphism (53) results into the twisted* KU_{1} *cocycles given by*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad} & \mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\ H^3_{A\,\mathrm{bas}} - e'_{B'} \cdot c_1^A & \longleftarrow\!\!\shortmid & h_3 \\ \big(+p_{A*}F_{2k+2} - e'_{B'} \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}} \end{array}$$

where now the extension

$$\widehat{\mathfrak{g}}_{B'} \longrightarrow \mathfrak{g}\,,$$

is instead via the opposite 2-cocycle

$$c^1_{B'} := -p_{A*}(H^3_A)\,.$$

Proof. This is a matter of carefully tracking through the (bijective) operations on the corresponding sets of L_∞-algebra morphisms. Explicitly, under the reduction (43) from Prop. 2.25 the first step yields the map of super L_∞-algebras

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\qquad\quad} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\ H^3_{A\,\mathrm{bas}} & \longleftarrow\!\!\shortmid & h_3 \\ F_{2k\,\mathrm{bas}} & \longleftarrow\!\!\shortmid & f_{2k} \\ -p_{A*}H^3_A & \longleftarrow\!\!\shortmid & \mathrm{s}h_3 \\ -p_{A*}F_{2k} & \longleftarrow\!\!\shortmid & \mathrm{s}f_{2k} \\ c_1^A & \longleftarrow\!\!\shortmid & \omega_2\,. \end{array}$$

In the second step, postcomposition of the above morphism with the first isomorphism in (51)

$$\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \xrightarrow{\ \sim\ } \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big)$$

yields

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\qquad\quad} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\ H^3_{A\,\mathrm{bas}} & \longleftarrow\!\!\shortmid & h_3 \\ -p_{A*}F_{2k+2} & \longleftarrow\!\!\shortmid & f_{2k+1} \\ -c_1^A & \longleftarrow\!\!\shortmid & \mathrm{s}h_3 \\ F_{2k\,\mathrm{bas}} & \longleftarrow\!\!\shortmid & \mathrm{s}f_{2k+1} \\ c^1_B := p_{A*}H^3_A & \longleftarrow\!\!\shortmid & \omega_2\,. \end{array}$$

Lastly, in the third step oxidizing (43) via the new 2-cocycle

$$c^1_B := p_{A*}(H^3_A) \quad : \quad \mathfrak{g} \longrightarrow b\mathbb{R}\,,$$

immediately yields precisely the morphism of super L_∞-algebras out of the corresponding central extension

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad} & \mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\ H^3_{A\,\mathrm{bas}} + e'_B \cdot c_1^A & \longleftarrow\!\!\shortmid & h_3 \\ \big(-p_{A*}F_{2k+2} - e'_B \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}}\,. \end{array}$$

The case of using instead the isomorphism (53) follows analogously. □

Using the first iso from Lem. 2.29, results instead in

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad\qquad} & \mathfrak{l}\big(\,\Sigma^1\mathrm{KU}/\!\!/BU(1)\,\big) \\
H^3_{A\,\mathrm{bas}} + e'_B \cdot c_1^A & \longleftarrow & h_3 \\
\big(+p_{A*}F_{2k+2} + e'_B \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow & (f_{2k+1})_{k\in\mathbb{Z}}\,.
\end{array}
$$

That is, same extension and twist as that of (54), but opposite fluxes (E.g. IIA/IIB fluxes and IIA/-IIB fluxes on same IIB spacetime).

Similarly, using the last iso from Lem. 2.29 results instead in

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad\qquad} & \mathfrak{l}\big(\,\Sigma^1\mathrm{KU}/\!\!/BU(1)\,\big) \\
H^3_{A\,\mathrm{bas}} - e'_{B'} \cdot c_1^A & \longleftarrow & h_3 \\
\big(-p_{A*}F_{2k+2} + e'_{B'} \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow & (f_{2k+1})_{k\in\mathbb{Z}}
\end{array}
$$

where now the extension

$$
\widehat{\mathfrak{g}}_{B'} \longrightarrow \mathfrak{g}\,,
$$

is instead via the opposite 2-cocycle

$$
c^1_{B'} \;:=\; -p_{A*}(H^3_A)\,.
$$

That is, same extension and twist as that of (ii) from Lem. 2.30, but opposite fluxes.

Remark 2.31 (Towards T-duality.). Since the isomorphism (51) swaps the Chern class ω_2 with the dimensional reduction $\widetilde{\omega}_2 \equiv s h_3$ of the 3-form, when applied over super-space this results in "swapping the spacetime extension". At the same time, the same operation swaps the "winding and non-winding modes" of the corresponding "fluxes", sending $(F_{2k\,\mathrm{bas}} + e'_A \cdot p_{A*}F_{2k})$ to $(-p_{A*}F_{2k+2} - e'_B \cdot F_{2k\,\mathrm{bas}})$, up to an overall (conventional) sign. These effects may be seen as abstract incarnations of the analogous phenomena in superspace T-duality, shown below in §3.2.

But first, we now generalize dimensional reduction from 1-dimensional fibers to n-dimensional fibers.

2.3 Torus extensions

We discuss a higher dimensional analog of the Ext/Cyc-adjunction of §2.2 corresponding to double-dimensional reduction/oxidation along products of rational circles. This *toroidification* construction may conceptually be understood via rational homotopy theory, see [SV24, p. 10]. Here, we give an analogous discussion without explicitly passing through algebraic topology.

Toroidal central extensions. In evident generalization of Def. 2.21 we may consider super-L_∞ extension by a whole sequence of 2-cocyles:

Definition 2.32 (Central n-torus extension [FSS20a, §3.1]). For $\mathfrak{g} \in \mathrm{sLieAlg}_\infty$ equipped with $n \in \mathbb{N}$ 2-cocycles

$$
\underbrace{\overset{1}{c}_1, \cdots, \overset{n}{c}_1}_{\deg = (2,\mathrm{evn})} \;\in\; \mathrm{CE}(\mathfrak{g})\,, \qquad \underset{k}{\forall}\; \mathrm{d}\,\overset{k}{c}_1 \;=\; 0
$$

we say that the n-*toroidal* central extension classified thereby is $\overset{1\cdots n}{\widehat{\mathfrak{g}}} \in \mathrm{sLieAlg}_\infty$ given by

$$
\mathrm{CE}\Big(\overset{1\cdots n}{\widehat{\mathfrak{g}}}\Big) \;\simeq\; \mathrm{CE}(\mathfrak{g})\big[\underbrace{\overset{1}{e}, \cdots \overset{n}{e}}_{\deg = (1,\mathrm{evn})}\big]\big/\big(\mathrm{d}\,\overset{k}{e} \;=\; \overset{k}{c}_1\big)_{k=1}^n\,.
$$

The terminology n-*torus extension* in Def. 2.32 refers to the following phenomenon:

Example 2.33 (The real Whitehead L_∞-algebra of the n-torus). Consider on $0 \in \mathrm{sLieAlg}_\infty$ (Ex. 2.5) n copies of (necessarily) the vanishing 2-cocycle

$$
\overset{k}{c}_1 \;=\; 0 \;\in\; \mathrm{CE}(0)\,, \qquad k \in \{1, \cdots, n\}\,.
$$

The corresponding n-toroidal extension (Def. 2.32) is the real Whitehead L_∞-algebra (Ex. 2.4) of the actual n-torus

$$
\overset{1\cdots n}{\widehat{0}} \;\simeq\; \mathbb{R}^n \;\simeq\; \mathfrak{l}\big(\mathbb{T}^n\big) \;\equiv\; \mathfrak{l}\big((\mathbb{R}/\mathbb{Z})^n\big)\,, \tag{55}
$$

given by

$$
\mathrm{CE}\big(\mathfrak{l}T^n\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\big[\,(\overset{k}{e})_{k=1}^n\,\big]\big/\big(\mathrm{d}\,\overset{k}{e} \;=\; 0\big)_{k=1}^n\,.
$$

Incidentally this shows also that the delooping

$$b(\mathbb{R}^n) \;\simeq\; b\big(\mathfrak{l}(\mathbb{R}^n)\big) \;\simeq\; \mathfrak{l}\big(B\mathbb{R}^n\big)$$

given by

$$\mathrm{CE}\big(b\mathbb{R}^n\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\big(\underbrace{\overset{k}{\omega}_2}_{\mathrm{deg}\,=\,(2,\mathrm{evn})}\big)_{k=1}^n\Big]\Big/\big(\mathrm{d}\,\overset{k}{\omega}_2 \;=0\big)_{k=1}^n$$

is the classifying L_∞-algebra for n-toroidal extensions, in that an n-tuple of 2-cocycles is equivalently an L_∞-homomorphism (14) into it, this being the image under passage to real Whitehead L_∞-algebras of the classification of n-torus principal bundles P by the classifying space $B\mathbb{T}^n$, via pullback of the universal n-torus bundle $E\mathbb{T}^n$:

$$\begin{tikzcd} \overset{1\cdots n}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\mathrm{hofib}\left(\overset{1\cdots n}{c}_1\right)"'] & \\ \mathfrak{g} \arrow[r, "\overset{1\cdots n}{c}_1"] & b\mathbb{R}^n \\ \overset{k}{c}_1 & \overset{k}{\omega}_2 \arrow[l, mapsto] \end{tikzcd} \quad \underset{\text{if } \mathfrak{g}\,\simeq\,\mathfrak{l}(X)}{\simeq} \quad \mathfrak{l}\left(\begin{tikzcd} P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & E\mathbb{T}^n \arrow[d, two heads] \\ X \arrow[r, "\vec{c}_1"] & B\mathbb{T}^n \end{tikzcd}\right).$$

The following example of toroidal super-extensions is noteworthy (maybe first highlighted in [CdAIP00, §2.1], see also [HS18]):

Example 2.34 (**Super-Minkowski spacetime as toroidal extension of a super-point).** Every super-Minkowski spacetime super-Lie algebra $\mathbb{R}^{1,d\,|\,\mathbf{N}}$ (16) is a $(1+d)$-toroidal central super-L_∞ extension (Def. 2.32) of a superpoint:

$$\mathbb{R}^{1,d\,|\,\mathbf{N}} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}^{0\,|\,N} \xrightarrow{\;(\overline{\psi}\,\Gamma\,\psi)\;} b\mathbb{R}^{1+d}\,, \tag{56}$$

where the super-point super-Lie algebra $\mathbb{R}^{0|N}$ is given simply by

$$\mathrm{CE}\big(\mathbb{R}^{0\,|\,N}\big) \;\;\simeq\;\; \mathbb{R}_{\mathrm{d}}\big[(\psi^\alpha)_{\alpha=1}^N\big]\big/\big(\mathrm{d}\,\psi^\alpha \;=\; 0\big)_{\alpha=1}^N\,. \tag{57}$$

Remark 2.35 (**Decomposing n-torus extensions into circle-extensions).**
(i) A 1-torus extension in the sense of Def. 2.32 is evidently the same as a central extension according to Def. 2.21, namely a "circle extension" (cf. Ex. 2.33)

$$\widehat{\mathfrak{g}} \;\simeq\; \overset{1}{\widehat{\mathfrak{g}}}\,.$$

Any n-torus extension may equivalently be obtained as a sequence of k_j-torus extensions, for any partitioning $k_j \in \mathbb{N}$, $\sum_j k_j = n$, e.g.:

$$\begin{tikzcd} \overset{1\cdots 3}{\widehat{\mathfrak{g}}} \arrow[dddd, "\mathrm{hofib}\left(\overset{1\cdots 3}{c}_1\right)"] & \\ & \\ & \\ & \\ \mathfrak{g} \arrow[r, "\overset{1\cdots 3}{c}_1"] & b\mathbb{R}^3 \end{tikzcd} \qquad \begin{tikzcd} \overset{1\cdots 3}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\mathrm{hofib}(\overset{3}{c}_1)"] & \\ \overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[r, "\overset{3}{c}_1"] \arrow[ddd, "\mathrm{hofib}(\overset{1\cdots 2}{c}_1)"] & b\mathbb{R} \\ & \\ & \\ \mathfrak{g} \arrow[r, "\overset{1\cdots 2}{c}_1"] & b\mathbb{R}^2 \end{tikzcd} \qquad \begin{tikzcd} \overset{1\cdots 3}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\mathrm{hofib}(\overset{3}{c}_1)"] & \\ \overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[r, "\overset{3}{c}_1"] \arrow[d, two heads, "\mathrm{hofib}(\overset{2}{c}_1)"] & b\mathbb{R} \\ \overset{1}{\widehat{\mathfrak{g}}} \arrow[r, "(\overset{2}{c}_1)"] \arrow[d, two heads, "\mathrm{hofib}(\overset{1}{c}_1)"] & b\mathbb{R} \\ \mathfrak{g} \arrow[r, "\overset{1}{c}_1"] & b\mathbb{R}\,. \end{tikzcd} \tag{58}$$

(ii) Here the order of the extensions does not matter, up to isomorphism, in that the following diagram commutes:

$$\begin{tikzcd} & & \overset{1\cdots 2}{\widehat{g}} \arrow[dl, two heads, "\mathrm{hofib}\left(\overset{1}{p}^*(\overset{2}{c}_1)\right)"'] \arrow[dr, two heads, "\mathrm{hofib}\left(\overset{2}{p}^*(\overset{1}{c}_1)\right)"] & & \\ b\mathbb{R} & \overset{1}{\widehat{\mathfrak{g}}} \arrow[l, "\overset{1}{p}^*(\overset{2}{c}_1)"'] \arrow[dr, two heads, "\overset{1}{p}\,=\,\mathrm{hofib}\left(\overset{1}{c}_1\right)"'] & & \overset{2}{\widehat{\mathfrak{g}}} \arrow[r, "\overset{2}{p}^*(\overset{1}{c}_1)"] \arrow[dl, two heads, "\overset{2}{p}\,=\,\mathrm{hofib}\left(\overset{2}{c}_1\right)"] & b\mathbb{R} \\ & b\mathbb{R} & \mathfrak{g} \arrow[l, "\overset{2}{c}_1"] \arrow[r, "\overset{1}{c}_1"'] & b\mathbb{R}\,. & \end{tikzcd} \tag{59}$$

This is ultimately due to the fact that all cocycles are (by assumption) defined down on $\mathfrak{g}$, hence each independent of the extension classified by the others, which we may express as the statement that the fiber integration (39) vanishes of the next cocycle over the extension fiber brought about by the previous cocycle:

$$\overset{1}{p}_* \overset{2}{p}{}^* \overset{k}{c}_1 \;=\; 0\,, \qquad \overset{2}{p}_* \overset{1}{p}{}^* \overset{k}{c}_1 \;=\; 0\,. \tag{60}$$

This relation becomes crucial below in specializing iterated *cyclification* along n-torus fibrations to *toroidifictionation.*

(iii) Stated more abstractly, the commuting square in (59) is "Cartesian", exhibiting $\overset{1\cdots 2}{\widehat{\mathfrak{g}}}$ as the **fiber product** (in fact as the *homotopy fiber product*) of $\overset{1}{\widehat{\mathfrak{g}}}$ with $\overset{2}{\widehat{\mathfrak{g}}}$ over their common base $\mathfrak{g}$.

(iv) Note that the property of commuting squares to exhibit fiber products is closed under "pasting" these squares together: In a commuting diagram of the form

$$\begin{tikzcd}[column sep=small]
 & & \overset{1\cdots 3}{\widehat{\mathfrak{g}}} \arrow[dl, two heads] \arrow[dr, two heads] & \\
 & \overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[dl, two heads] \arrow[dr, two heads] & & \overset{3}{\widehat{\mathfrak{g}}} \arrow[dl, two heads] \\
\overset{1}{\widehat{\mathfrak{g}}} \arrow[dr, two heads] & & \overset{2}{\widehat{\mathfrak{g}}} \arrow[dl, two heads] & \\
 & \mathfrak{g} & &
\end{tikzcd} \tag{61}$$

if the bottom diamond exhibits a fiber product, then the total diamond does, too, iff the top diamond does. (This is a general abstract fact known as the "pasting law", but here in our context of n-toroidal central extensions of super-L_∞ algebras it is also readily checked by inspecting generators.)

(v) This property clearly iterates over consecutive pastings of fiber product diamonds. As such it controls the final picture in (190) below.

2-Toroidification. We need the following simple observation:

Lemma 2.36 (**Quotient of L_∞-algebra by abelian anti-ideal).** *Given an L_∞-algebra $\mathfrak{g}$ whose CE-algebra has a closed generator ω, then discarding that generator yields the CE-algebra of a sub-L_∞-algebra:*

$$\mathbb{R}_{\mathrm{d}}\big[\,(e^i)_{i \in I}\,\big] \big/ \big(\mathrm{d}\, e^i \;=\; P^i_{(0)}(\vec e)\big) \longleftarrow\!\!\!\!\!\leftarrow \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega \\ (e^i)_{i\in I}\end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\omega & = 0 \\ \mathrm{d}\, e^i & = \sum_k P^i_{(k)}(\vec e) \underbrace{\omega \cdots \omega}_{k \text{ factors}} \end{array}\right).$$

Proof. The claim is that the operator d on the left still satisfies $\mathrm{d} \circ \mathrm{d} = 0$ if that on the right does. But on the right, the condition is

$$\mathrm{d}^2\, e^i \;=\; 0 \qquad \Leftrightarrow \qquad \underset{k \in \mathbb{N}}{\forall} \big(\mathrm{d}P^i_{(k)}(\vec e) = 0\big),$$

hence immediately implies the claim for $k = 0$. □

Example 2.37 (**Double cyclification).** For $\mathfrak{h} \in \mathrm{sLieAlg}_\infty$ with presentation as in Def. 2.23, applying cyclification (Def. 2.23) twice yields the L_∞-algebra $\mathrm{cyc}^2\mathfrak{h}$ given by

$$\mathrm{CE}\big(\mathrm{cyc}^2(\mathfrak{h})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \overset{2}{\omega}_2, \\ \overset{1}{\omega}_2, \\ \overset{2\,1}{\mathrm{s}\omega}_2, \\ (e_i)_{i\in I}, \\ (\overset{2}{\mathrm{s}}e^i)_{i \in I} \\ (\overset{1}{\mathrm{s}}e^i)_{i\in I}, \\ (\overset{2\,1}{\mathrm{ss}}e^i)_{i\in I} \end{array}\right] \Bigg/ \left(\begin{array}{lcc} \mathrm{d}\,\overset{2}{\omega}_2 & = & 0 \\ \mathrm{d}\,\overset{1}{\omega}_2 & = & \overset{2}{\omega}_2\, \overset{2\,1}{\mathrm{s}\omega}_2 \\ \mathrm{d}\,\overset{2\,1}{\mathrm{s}\omega}_2 & = & 0 \\ \mathrm{d}\,e^i & = & \mathrm{d}_{\mathfrak{h}} e^i + \overset{1}{\omega}_2\, \overset{1}{\mathrm{s}}e^i + \overset{2}{\omega}_2\, \overset{2}{\mathrm{s}}e^i \\ \mathrm{d}\,\overset{2}{\mathrm{s}}e^i & = & -\overset{2}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}} e^i - (\overset{2\,1}{\mathrm{s}\omega}_2)(\overset{1}{\mathrm{s}}e^i) - \overset{1}{\omega}_2\, \overset{2\,1}{\mathrm{ss}}e^i \\ \mathrm{d}\,\overset{1}{\mathrm{s}}e^i & = & -\overset{1}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}} e^i + \overset{2}{\omega}_2\, \overset{2\,1}{\mathrm{ss}}e^i \\ \mathrm{d}\,\overset{2\,1}{\mathrm{ss}}e^i & = & \overset{2\,1}{\mathrm{ss}}\,\mathrm{d}_{\mathfrak{h}} e^i \end{array}\right). \tag{62}$$

Definition 2.38 (**2-Toroidification).** Since the generator $\overset{2\,1}{\mathrm{s}\omega}_2$ in the CE-algebra (62) of a double cyclification is closed, discarding this generator yields (by Lem. 2.36) a sub-L_∞-algebra, to be called the the *toroidification* of the given $\mathfrak{h} \in \mathrm{sLieAlg}_\infty$

$$\mathrm{tor}^2(\mathfrak{h}) \lhook\joinrel\longrightarrow \mathrm{cyc}^2(\mathfrak{h})\,,$$

and given by

$$
\mathrm{CE}\big(\mathrm{tor}^2\mathfrak{h}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}
\overset{2}{\omega}_2, \\
\overset{1}{\omega}_2, \\
(e_i)_{i\in I}, \\
(\overset{2}{\mathrm{s}}e^i)_{i\in I} \\
(\overset{1}{\mathrm{s}}e^i)_{i\in I}, \\
(\overset{2\,1}{\mathrm{ss}}e^i)_{i\in I}
\end{array}\right] \Big/ \left(\begin{array}{rcl}
\mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\
\mathrm{d}\,\overset{1}{\omega}_2 &=& 0 \\
\mathrm{d}\,e^i &=& \mathrm{d}_{\mathfrak{h}} e^i + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}e^i + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}e^i \\
\mathrm{d}\,\overset{2}{\mathrm{s}}e^i &=& -\overset{2}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}} e^i - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}e^i \\
\mathrm{d}\,\overset{1}{\mathrm{s}}e^i &=& -\overset{1}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}} e^i + \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}e^i \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}e^i &=& \overset{2\,1}{\mathrm{ss}}\,\mathrm{d}_{\mathfrak{h}} e^i
\end{array}\right). \tag{63}
$$

We may regard this as fibered over $b\mathbb{R}^2$ via:

$$
\begin{array}{c}
\mathrm{tor}^2(\mathfrak{h}) \\
\big\downarrow {\scriptstyle \overset{1\cdots 2}{\omega}_2 \,:=\, (\overset{1}{\omega}_2, \overset{2}{\omega}_2)} \\
b\mathbb{R}^2\,.
\end{array} \tag{64}
$$

Example 2.39 (**Double cyclification of the 4-sphere** [SV23, Ex. 2.6])**.** The double cyclification (Ex. 2.37) of (the real Whitehead L_∞-algebra of) the 4-sphere (Ex. 2.7) is given by

$$
\mathrm{CE}\big(\mathrm{cyc}^2(\mathfrak{l}S^4)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}
\overset{2}{\omega}_2 \\
\overset{1}{\omega}_2 \\
\overset{2\,1}{\mathrm{s}\omega}_2 \\
g_4 \\
\overset{2}{\mathrm{s}}g_4 \\
\overset{1}{\mathrm{s}}g_4 \\
\overset{2\,1}{\mathrm{ss}}g_4 \\
g_7 \\
\overset{2}{\mathrm{s}}g_7 \\
\overset{1}{\mathrm{s}}g_7 \\
\overset{2\,1}{\mathrm{ss}}g_7
\end{array}\right] \Big/ \left(\begin{array}{rcl}
\mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\
\mathrm{d}\,\overset{1}{\omega}_2 &=& \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}\omega_2 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}\omega_2 &=& 0 \\
\mathrm{d}\,g_4 &=& \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}g_4 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}g_4 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}g_4 &=& -(\overset{2\,1}{\mathrm{s}\omega}_2)(\overset{1}{\mathrm{s}}g_4) - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_4 \\
\mathrm{d}\,\overset{1}{\mathrm{s}}g_4 &=& \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_4 \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}g_4 &=& 0 \\
\mathrm{d}\,g_7 &=& \tfrac{1}{2}g_4 g_4 + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}g_7 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}g_7 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}g_7 &=& -g_4\,\overset{2}{\mathrm{s}}g_4 - (\overset{2}{\mathrm{s}}\omega_2)(\overset{1}{\mathrm{s}}g_7) - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_7 \\
\mathrm{d}\,\overset{1}{\mathrm{s}}g_7 &=& -g_4\,\overset{1}{\mathrm{s}}g_4 + \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_7 \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}g_7 &=& (\overset{2}{\mathrm{s}}g_4)(\overset{1}{\mathrm{s}}g_4) + g_4\,\overset{2\,1}{\mathrm{ss}}g_4
\end{array}\right)
$$

and its toroidification (Ex. 2.38) is given by

$$
\mathrm{CE}\big(\mathrm{tor}^2(\mathfrak{l}S^4)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}
\overset{2}{\omega}_2 \\
\overset{1}{\omega}_2 \\
g_4 \\
\overset{2}{\mathrm{s}}g_4 \\
\overset{1}{\mathrm{s}}g_4 \\
\overset{2\,1}{\mathrm{ss}}g_4 \\
g_7 \\
\overset{2}{\mathrm{s}}g_7 \\
\overset{1}{\mathrm{s}}g_7 \\
\overset{2\,1}{\mathrm{ss}}g_7
\end{array}\right] \Big/ \left(\begin{array}{rcl}
\mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\
\mathrm{d}\,\overset{1}{\omega}_2 &=& 0 \\
\mathrm{d}\,g_4 &=& \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}g_4 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}g_4 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}g_4 &=& -\overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_4 \\
\mathrm{d}\,\overset{1}{\mathrm{s}}g_4 &=& \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_4 \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}g_4 &=& 0 \\
\mathrm{d}\,g_7 &=& \tfrac{1}{2}g_4 g_4 + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}g_7 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}g_7 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}g_7 &=& -g_4\,\overset{2}{\mathrm{s}}g_4 - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_7 \\
\mathrm{d}\,\overset{1}{\mathrm{s}}g_7 &=& -g_4\,\overset{1}{\mathrm{s}}g_4 + \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}g_7 \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}g_7 &=& (\overset{2}{\mathrm{s}}g_4)(\overset{1}{\mathrm{s}}g_4) + g_4\,\overset{2\,1}{\mathrm{ss}}g_4
\end{array}\right).
$$

In generalization of Rem. 2.22 we have:

Remark 2.40 (**Basic and fiber forms on a 2-toroidially extended super-L_∞ algebra**)**.** Given a 2-toroidal central extension as in Def. 2.32, every element in its CE-algebra decomposes uniquely as a sum of the form

$$
\alpha \;=\; \alpha_{\mathrm{bas}} \,+\, \overset{1}{e}\,\alpha_1 \,+\, \overset{2}{e}\,\alpha_2 \,+\, \overset{2}{e}\,\overset{1}{e}\,\alpha_{2\,1} \tag{65}
$$

with all the coefficients in the image of the total pullback operation

$$
\alpha_{\mathrm{bas}}, \alpha_1, \alpha_2, \alpha_{2\,1} \;\in\; \overset{2}{p}{}^*\,\overset{1}{p}{}^*\,\mathrm{CE}(\mathfrak{g})\,.
$$

Proposition 2.41 (**Universal property of 2-toroidification**)**.** *Consider a central extension* (Ex. 2.21)

$$
\overset{1}{\widehat{\mathfrak{g}}} \xrightarrow{\;\overset{1}{p} \,=\, \mathrm{hofib}(\overset{1}{c}_1)\;} \mathfrak{g} \xrightarrow{\;\overset{1}{c}_1\;} b\mathbb{R}
$$

carrying a $\mathrm{cyc}(\mathfrak{h})$*-valued cocycle* $\tilde{f} : \overset{1}{\widehat{\mathfrak{g}}} \to \mathrm{cyc}(\mathfrak{h})$ *whose Chern class*

$$\overset{2}{c}_1 \;:=\; \tilde{f}^* \omega_2$$

has trivial fiber integration along $\overset{1}{p}$*. Then the reduction* $\tilde{\tilde{f}}$*(43) of* $\tilde{f}$ *along* $\overset{1}{p}$ *has factors uniquely through the toroidification (63), which has an inverse oxidation when regarded as sliced over* $b\mathbb{R}^2$ *via (64):*

$$
\begin{tikzcd}
\overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[r, "f"] \arrow[d, two heads, "\overset{2}{p}"'] & \mathfrak{h} \\
\overset{1}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\overset{1}{p}"'] & \\
\mathfrak{g} &
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
\overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\overset{2}{p}"'] & & \\
\overset{1}{\widehat{\mathfrak{g}}} \arrow[rr, "\tilde{f}"] \arrow[dr, "\overset{2}{c}_1"'] \arrow[dd, two heads, "\overset{1}{p}"'] & & \mathrm{cyc}(\mathfrak{h}) \arrow[dl, "\omega_2"] \\
& b\mathbb{R} & \\
\mathfrak{g} & &
\end{tikzcd}
\quad
\underset{\overset{1}{p}_* f^* \overset{1}{\omega}_2 \;=\; 0}{\leftrightsquigarrow}
\quad
\begin{tikzcd}
\overset{1\cdots 2}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\overset{2}{p}"'] & & & \\
\overset{1}{\widehat{\mathfrak{g}}} \arrow[d, two heads, "\overset{1}{p}"'] & & & \\
\mathfrak{g} \arrow[rr, dashed, "\tilde{\tilde{f}}"] \arrow[dr, "\overset{1\cdots 2}{c}_1"'] & & \mathrm{tor}^2(\mathfrak{h}) \arrow[r, hook] \arrow[dl, "\overset{1\cdots 2}{\omega}_1"] & \mathrm{cyc}^2(\mathfrak{h}) \\
& b\mathbb{R}^2 & &
\end{tikzcd}
$$

given by

$$
\begin{array}{ccc}
\overset{1\cdots 2}{\widehat{\mathfrak{g}}} & \xrightarrow{\;\; f \;\;} & \mathfrak{h}
\\
\begin{array}{l}
\alpha^i_{\mathrm{bas}} + \overset{1}{e}\,\alpha^i_1 + \\
\overset{2}{e}\,\alpha^i_2 \,+\, \overset{2}{e}\overset{1}{e}\,\alpha^i_{2\,1}
\end{array}
& \longmapsfrom & e^i
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{ccc}
\overset{1}{\widehat{\mathfrak{g}}} & \xrightarrow{\;\; \tilde{f} \;\;} & \mathrm{cyc}(\mathfrak{h})
\\
c_2 & \longmapsfrom & \omega_2
\\
\alpha^i_{\mathrm{bas}} + \overset{1}{e}\,\alpha^i_1 & \longmapsfrom & e^i
\\
-\alpha^i_2 \,-\, \overset{1}{e}\,\alpha^i_{2\,1} & \longmapsfrom & \mathrm{s}e^i\,,
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{ccc}
\overset{1\cdots 2}{\widehat{\mathfrak{g}}} & \xrightarrow{\;\; \tilde{\tilde{f}} \;\;} & \mathrm{tor}^2(\mathfrak{h})
\\
c_2 & \longmapsfrom & \overset{2}{\omega}_2
\\
c_1 & \longmapsfrom & \overset{1}{\omega}_2
\\
\alpha^i_{\mathrm{bas}} & \longmapsfrom & e^i
\\
-\alpha^i_1 & \longmapsfrom & \overset{1}{\mathrm{s}}e^i
\\
-\alpha^i_2 & \longmapsfrom & \overset{2}{\mathrm{s}}e^i
\\
-\alpha^i_{2\,1} & \longmapsfrom & \overset{2}{\mathrm{s}}\overset{1}{\mathrm{s}}e^i .
\end{array}
\tag{66}
$$

In view of (59), this says equivalently that

- while double-cyclification cyc^2 classifies reductions along arbitrary 2-dimensional central extensions,
- toroidification tor^2 classifies among these the reductions along 2-torus extensions.

Proof. This follows from the universal property of cyclification (43) by observing that the datum needed for re-oxidation which is forgotten by factoring through $\mathrm{tor}^2 \hookrightarrow \mathrm{cyc}^2$, namely $\overset{1}{p}_* \overset{1}{c}_1$, is exactly the datum which vanishes by the assumption that the 2-dimensional extension is a 2-torus extension, via (60). ☐

Definition 2.42 (Higher dimensional toroidification). We say that

(i) *1-toroidification* is the same as *cyclification* cyc from Def. 2.23:

$$\mathrm{tor}^1(\mathfrak{h}) \;:=\; \mathrm{cyc}(\mathfrak{h})\,.$$

(ii) *2-toroidification* is the same as the toroidification tor^2 from Ex. 2.38:

$$\mathrm{tor}^2(\mathfrak{h}) \longhookrightarrow \mathrm{cyc}^2(\mathfrak{h})\,.$$

(iii) $(n+1)$*-toroidification* for $n \geq 2$ is the sub-L_∞-algebra of the cyclification of, recursively, the n-toroidification obtained by discarding (via Lem. 2.36) all generators $\overset{n+1}{\mathrm{s}}\,\overset{k}{\omega}_2$ for $k \in \{1, \cdots, n\}$:

$$\mathrm{tor}^{n+1}(\mathfrak{h}) \;:=\; \mathrm{tor}\,\mathrm{tor}^n(\mathfrak{h}) \xhookrightarrow{\qquad} \mathrm{cyc}\,\mathrm{tor}^n(\mathfrak{h})\,. \tag{67}$$

The following statement is due to [SV24, Thm. 2.6], there argued via rational homotopy theory. We give a direct proof.

Proposition 2.43 (Explicit n-Toroidification). *For* $n \in \mathbb{N}$ *the* n*-toroidification* (Def. 2.42) *of* $\mathfrak{h} \in \mathrm{sLieAlg}_\infty$*, with presentation as in Def. 2.23, is given by*

$$
\mathrm{CE}\big(\mathrm{tor}^n(\mathfrak{h})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\big(\underbrace{\overset{k}{\omega}_2}_{\substack{\mathrm{deg}\,= \\ (2,\mathrm{evn})}} \big)_{k=1}^{n},\; \big(\underbrace{\overset{i_k}{\mathrm{s}} \cdots \overset{i_2}{\mathrm{s}}\overset{i_1}{\mathrm{s}} e^i}_{\substack{\mathrm{deg}\,= \\ \mathrm{deg}(e^i) - (k,\mathrm{evn})}} \big)_{\substack{i \in I,\quad 0 \leq k \leq n, \\ n \geq i_k > \cdots > i_2 > i_1 \geq 1}} \Big] \Big/ \left(\begin{array}{l} \mathrm{d}\,\overset{k}{\omega}_2 \;=\; 0 \\ \mathrm{d}\,e^i \;=\; \mathrm{d}_{\mathfrak{h}} e^i \,+\, \sum_{k=1}^n \overset{k}{\omega}_2\, \overset{k}{\mathrm{s}}e^i \\ \mathrm{d} \circ \overset{k}{\mathrm{s}} \;=\; -\overset{k}{\mathrm{s}} \circ \mathrm{d}, \;\; \overset{k}{\mathrm{s}} \circ \overset{k'}{\mathrm{s}} \;=\; -\overset{k'}{\mathrm{s}} \circ \overset{k}{\mathrm{s}} \end{array} \right). \tag{68}
$$

Here on the right of (68) we mean that the differential d is extended to the shifted generators by the rule that it graded-commutes with all the shift operators $\overset{k}{\mathrm{s}}$, which in turn are regarded as uniquely extended to graded derivations of degree $\mathrm{deg} = -1$, anti-commuting among each other — in evident generalization of (42).

Proof. This follows readily by induction on k: For $k = 0$ the statement is trivial, for $k = 1$ the statement is that of Lem. 2.24. Hence for the induction step, assume that the statement is true for some $n \geq 1$. Then we find the CE-differential on $\mathrm{tor}\big(\mathrm{tor}^n(\mathfrak{h})\big)$ to be given on generators as claimed:

$$
\begin{array}{rcll}
\mathrm{d}_{\mathrm{tor}\,\mathrm{tor}^n}\, \overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}e^i & = & \mathrm{d}_{\mathrm{tor}^n}\, \overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{d}}e^i \;+\; \overset{n+1}{\omega_2}\,\overset{n+1}{\mathrm{s}}\,\overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}e^i & \text{by (67) \& (63)} \\
& = & (-1)^k\, \overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\big(\mathrm{d}_{\mathfrak{h}}e^i \;+\; \sum_{r=1}^{n} \overset{r}{\omega_2}\,\overset{r}{\mathrm{s}}e^i\big) \;+\; \overset{n+1}{\omega_2}\,\overset{n+1}{\mathrm{s}}\,\overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}e^i & \text{by indctn assmpt \& (68)} \\
& = & (-1)^k\, \overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\big(\mathrm{d}_{\mathfrak{h}}e^i \;+\; \sum_{r=1}^{n+1} \overset{r}{\omega_2}\,\overset{r}{\mathrm{s}}e^i\big) & \\
& = & \mathrm{d}_{\mathrm{tor}^{n+1}}\, \overset{i_k}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}e^i & \text{according to (68)}
\end{array}
$$

and trivially:

$$
\begin{array}{rcll}
\mathrm{d}_{\mathrm{tor}\,\mathrm{tor}^n}\, \overset{n+1}{\omega_2} & = & 0 & \text{by (63)} \\
& = & \mathrm{d}_{\mathrm{tor}^{n+1}}\, \overset{n+1}{\omega_2} & \text{by (68)}\;.
\end{array}
$$

□

Example 2.44 (3-Toroidification). The 3-toroidification $\mathrm{tor}^3(\mathfrak{h})$ (Def. 2.42) of a super-L_∞-algebra with generators as in Def. 2.23 is given by:

$$
\mathrm{CE}\big(\mathrm{tor}^3(\mathfrak{h})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\overset{3}{\omega}_2 \\
\overset{2}{\omega}_2 \\
\overset{1}{\omega}_2 \\
(e^i)_{i\in I} \\
(\overset{1}{\mathrm{s}}e^i)_{i\in I}) \\
(\overset{2}{\mathrm{s}}e^i)_{i\in I} \\
(\overset{3}{\mathrm{s}}e^i)_{i\in I} \\
(\overset{2\,1}{\mathrm{ss}}e^i)_{i\in I} \\
(\overset{3\,1}{\mathrm{ss}}e^i)_{i\in I}) \\
(\overset{3\,2}{\mathrm{ss}}e^i)_{i\in I} \\
(\overset{3\,2\,1}{\mathrm{sss}}e^i)_{i\in I}
\end{array}
\right]
\Big/
\left(
\begin{array}{rcc}
\mathrm{d}\,\overset{3}{\omega}_2 & = & 0 \\
\mathrm{d}\,\overset{2}{\omega}_2 & = & 0 \\
\mathrm{d}\,\overset{1}{\omega}_2 & = & 0 \\
\mathrm{d}\,e^i & = & \mathrm{d}_{\mathfrak{h}}e^i + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}e^i + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}e^i + \overset{3}{\omega}_2\,\overset{3}{\mathrm{s}}e^i \\
\mathrm{d}\,\overset{1}{\mathrm{s}}e^i & = & -\overset{1}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}}e^i + \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}e^i + \overset{3}{\omega}_2\,\overset{3\,1}{\mathrm{ss}}e^i \\
\mathrm{d}\,\overset{2}{\mathrm{s}}e^i & = & -\overset{2}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}}\,e^i - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}e^i + \overset{3}{\omega}_2\,\overset{3\,2}{\mathrm{ss}}e^i \\
\mathrm{d}\,\overset{3}{\mathrm{s}}e^i & = & -\overset{3}{\mathrm{s}}\,\mathrm{d}_{\mathfrak{h}}e^i - \overset{1}{\omega}_2\,\overset{3\,1}{\mathrm{ss}}e^i - \overset{2}{\omega}_2\,\overset{3\,2}{\mathrm{ss}}e^i \\
\mathrm{d}\,\overset{2\,1}{\mathrm{ss}}e^i & = & \overset{2\,1}{\mathrm{ss}}\,\mathrm{d}_{\mathfrak{h}}e^i + \overset{3}{\omega}_2\,\overset{3\,2\,1}{\mathrm{sss}}e^i \\
\mathrm{d}\,\overset{3\,1}{\mathrm{ss}}e^i & = & \overset{3\,1}{\mathrm{ss}}\,\mathrm{d}_{\mathfrak{h}}e^i - \overset{2}{\omega}_2\,\overset{3\,2\,1}{\mathrm{sss}}e^i \\
\mathrm{d}\,\overset{3\,2}{\mathrm{ss}}e^i & = & \overset{3\,2}{\mathrm{ss}}\,\mathrm{d}_{\mathfrak{h}}e^i + \overset{1}{\omega}\,\overset{3\,2\,1}{\mathrm{sss}}e^i \\
\mathrm{d}\,\overset{3\,2\,1}{\mathrm{sss}}e^i & = & -\overset{3\,2\,1}{\mathrm{sss}}\,\mathrm{d}_{\mathfrak{h}}e^i
\end{array}
\right).
$$

Remark 2.45 (Universal property of n-toroidification [SV24, Thm. 2.13]). In evident generalization of Prop. 2.41, the n-toroidification operation (Def. 2.42) provides the L_∞-coefficients for (double-dimensionally) reducing L_∞-cocycles along n-toroidal extensions (Def. 2.32):

$$
\mathfrak{g} \,\in\, \mathrm{sLieAlg}_\infty\,, \quad \overset{1}{c}_1, \cdots \overset{n}{c}_1 \,\in\, \mathrm{CE}(\mathfrak{g})\,, \quad \mathrm{d}\,\overset{k}{c}_1 \,=\, 0
\qquad
\begin{tikzcd}
\overset{1\cdots n}{\widehat{\mathfrak{g}}} \arrow[d] & \\
\mathfrak{g} \arrow[r, "\overset{1\cdots n}{c_1}"] & b\mathbb{R}^n
\end{tikzcd}
$$

in that we have bijections of sets of L_∞-homomorphisms, in generalization of (43):

$$
\left\{ \overset{1\cdots n}{\widehat{g}} \xrightarrow{\quad f \quad} \mathfrak{h} \right\}
\underset{\textbf{oxidation}\ \ \mathrm{oxd}_{\overset{1\cdots n}{c_1}}}{\overset{\textbf{reduction}\ \ \mathrm{rdc}_{\overset{1\cdots n}{c_1}}}{\underset{\sim}{\rightleftarrows}}}
\left\{
\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\widetilde{f}"] \arrow[dr, "\overset{1\cdots n}{c_1}"'] & & \mathrm{tor}^n(\mathfrak{h}) \arrow[dl, "\overset{1\cdots n}{\omega_2}"] \\
& b\mathbb{R}^n &
\end{tikzcd}
\right\}. \tag{69}
$$

Moreover, along the lines of Rem. 2.35, these reductions/oxidations may be applied in any sequence of steps:

$$
\left(
\begin{tikzcd}
\widehat{\widehat{\widehat{\widehat{\mathfrak{g}}}}} \arrow[r] \arrow[d, two heads] & \mathfrak{h} \\
\widehat{\widehat{\widehat{\mathfrak{g}}}} \arrow[d, two heads] & \\
\widehat{\widehat{\mathfrak{g}}} \arrow[d, two heads] & \\
\widehat{\mathfrak{g}} \arrow[d, two heads] & \\
\mathfrak{g} &
\end{tikzcd}
\right)
\;\leftrightsquigarrow\;
\left(
\begin{tikzcd}
\widehat{\widehat{\widehat{\widehat{\mathfrak{g}}}}} \arrow[d, two heads] & & \\
\widehat{\widehat{\widehat{\mathfrak{g}}}} \arrow[rr] \arrow[dr] \arrow[d, two heads] & & \mathrm{tor}(\mathfrak{h}) \arrow[dl] \\
\widehat{\widehat{\mathfrak{g}}} \arrow[d, two heads] & b\mathbb{R} & \\
\widehat{\mathfrak{g}} \arrow[d, two heads] & & \\
\mathfrak{g} & &
\end{tikzcd}
\right)
\;\leftrightsquigarrow\;
\left(
\begin{tikzcd}
\widehat{\widehat{\widehat{\widehat{\mathfrak{g}}}}} \arrow[d, two heads] & & \\
\widehat{\widehat{\widehat{\mathfrak{g}}}} \arrow[d, two heads] & & \\
\widehat{\widehat{\mathfrak{g}}} \arrow[rr] \arrow[dr] \arrow[d, two heads] & & \mathrm{tor}^2(\mathfrak{h}) \arrow[dl] \\
\widehat{\mathfrak{g}} \arrow[d, two heads] & b\mathbb{R}^2 & \\
\mathfrak{g} & &
\end{tikzcd}
\right)
\;\leftrightsquigarrow\;
\left(
\begin{tikzcd}
\widehat{\widehat{\widehat{\widehat{\mathfrak{g}}}}} \arrow[d, two heads] & & \\
\widehat{\widehat{\widehat{\mathfrak{g}}}} \arrow[d, two heads] & & \\
\widehat{\widehat{\mathfrak{g}}} \arrow[d, two heads] & & \\
\widehat{\mathfrak{g}} \arrow[rr] \arrow[dr] \arrow[d, two heads] & & \mathrm{tor}^3(\mathfrak{h}) \arrow[dl] \\
\mathfrak{g} & b\mathbb{R}^3 &
\end{tikzcd}
\right).
$$

Toroidification of twisted K-theory spectra. We turn to the main example of interest here, the toroidifications of twisted K-theory spectra carrying archetypes of, as we will see in §3.3, toroidal T-duality of flux densities.

Or rather, the T-duality must be carried by a sub-algebra of the toroidification, as brought out by the following variant of Ex. 2.27.

Example 2.46 (**Geometric 2-Toroidification of bundle gerbe classifying space).** It is readily checked that the 2-cyclification $\mathrm{cyc}^2 b^2\mathbb{R}$ of the line Lie 3-algebra (19) no longer enjoys an automorphism of the kind (50) that the 1-cyclification did, namely by swapping $\overset{i}{\mathrm{s}}h_3 \leftrightsquigarrow \overset{i}{\omega}_2$ *for both* $i = 1, 2$, as would befit a toroidal T-duality classifying algebra. In fact, nor does its toroidified subalgebra (Def. 2.38),

$$\mathrm{tor}^2(b^2\mathbb{R}) \hookrightarrow \mathrm{cyc}^2(b^2\mathbb{R}),$$

given by

$$\mathrm{CE}\big(\mathrm{tor}^2 b^2\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \overset{2}{\omega}_2, \\ \overset{1}{\omega}_2, \\ h_3 \\ \overset{2}{\mathrm{s}}h_3 \\ \overset{1}{\mathrm{s}}h_3 \\ \overset{21}{\mathrm{ss}}h_3 \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\ \mathrm{d}\,\overset{1}{\omega}_2 &=& 0 \\ \mathrm{d}\,h_3 &=& \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 \\ \mathrm{d}\,\overset{2}{\mathrm{s}}h_3 &=& -\overset{1}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3 \\ \mathrm{d}\,\overset{1}{\mathrm{s}}h_3 &=& +\overset{2}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3 \\ \mathrm{d}\,\overset{21}{\mathrm{ss}}h_3 &=& 0 \end{array}\right), \tag{70}$$

via which one immediately identifies the obstruction for such a swapping automorphism to exist to be the double shift of the generator h_3

$$\overset{21}{\mathrm{ss}}h_3 \;\in\; \mathrm{CE}\big(\mathrm{tor}^2\, b^2\, \mathbb{R}\big).$$

However, since this generator is closed, this implies at once that there exists (via Lem. 2.36) the L_∞-subalgebra obtained by discarding this term

$$\mathrm{tor}^{2'}(b^2\mathbb{R}) \hookrightarrow \mathrm{tor}^2(b^2\mathbb{R}) \hookrightarrow \mathrm{cyc}^2(b^2\mathbb{R}),$$

given by

$$\mathrm{CE}\big(\mathrm{tor}^{2'} b^2\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \overset{2}{\omega}_2, \\ \overset{1}{\omega}_2, \\ h_3 \\ \overset{2}{\mathrm{s}}h_3 \\ \overset{1}{\mathrm{s}}h_3 \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\ \mathrm{d}\,\overset{1}{\omega}_2 &=& 0 \\ \mathrm{d}\,h_3 &=& \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 \\ \mathrm{d}\,\overset{2}{\mathrm{s}}h_3 &=& 0 \\ \mathrm{d}\,\overset{1}{\mathrm{s}}h_3 &=& 0 \end{array}\right), \tag{71}$$

being equivalently the higher central extension (Def. 2.59) of $b\mathbb{R}^2 \times b\mathbb{R}^2$ by its canonical 4-cocycle

$$b\mathcal{T}^2 \;:=\; \mathrm{tor}^{2'} b^2\mathbb{R} \xrightarrow{\;\mathrm{hofib}\left(\overset{1}{\omega}_2\,\overset{1}{\tilde{\omega}}_2 + \overset{2}{\omega}_2\,\overset{2}{\tilde{\omega}}_2\right)\;} b\mathbb{R}^2 \times b\mathbb{R}^2 \xrightarrow{\;\overset{1}{\omega}_2\,\overset{1}{\tilde{\omega}}_2 + \overset{2}{\omega}_2\,\overset{2}{\tilde{\omega}}_2\;} b^3\mathbb{R}, \tag{72}$$

does have an automorphism symmetry given by swapping the two degree=2 generators as:

$$\begin{array}{ccc} b\mathcal{T}^2 & \overset{\sim}{\longleftrightarrow} & b\mathcal{T}^2 \\ \overset{1}{\mathrm{s}}h_3 & \longleftarrow\!\shortmid & \overset{1}{\omega}_2 \\ \overset{1}{\omega}_2 & \longleftarrow\!\shortmid & \overset{1}{\mathrm{s}}h_3 \\ \overset{2}{\mathrm{s}}h_3 & \longleftarrow\!\shortmid & \overset{2}{\omega}_2 \\ \overset{2}{\omega}_2 & \longleftarrow\!\shortmid & \overset{2}{\mathrm{s}}h_3 \end{array} \tag{73}$$

Remark 2.47 (**Geometric T-duality). (i)** When it comes to toroidal T-duality in §3.3, the vanishing of second contractions $\overset{21}{\mathrm{ss}}h_3$ of the 3-flux H_3 — that is reflected by the restricted toroidifications (71) and (75), and the vanishing of higher contractions that is further reflected below in (85) — is known in the literature as that corresponding to "geometric-" or "F^2-" T-duality backgrounds (e.g. [KS22, p. 6]).
(ii) In our context of super-space T-duality, this is the case realized by the fixed form of the avatar super-flux densities H_3^A (133) and H_3^B (136), which turn out to have vanishing higher order contraction with bosonic vector fields. Aspects of T-duality beyond this "geometric" case have been discussed (as "non-geometric-backgrounds" such as "T-folds" or yet more exotic structures) but their possible relation to actual supergravity may not have found attention.

Now the K-theoretic enhancement of Ex. 2.46 and thus the 2-dimensional analog of Ex 2.28 is:

Example 2.48 (2-Toroidification of twisted K-theory). The toroidification (Def. 2.38) of the Whitehead L_∞-algebras of twisted K-theory spectra (Ex. 2.12) is given by

$$
\begin{aligned}
&\mathrm{CE}\Big(\mathrm{tor}^2\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)\Big)\\
&\simeq\; \mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\overset{1}{\omega}_2\\
\overset{2}{\omega}_2\\
h_3\\
\overset{1}{\mathrm{s}}h_3\\
\overset{2}{\mathrm{s}}h_3\\
\overset{21}{\mathrm{ss}}h_3\\
f_{2\bullet+m}\\
\overset{1}{\mathrm{s}}f_{2\bullet+m}\\
\overset{2}{\mathrm{s}}f_{2\bullet+m}\\
\overset{21}{\mathrm{ss}}f_{2\bullet+m}
\end{array}
\right]
\Big/
\left(
\begin{aligned}
\mathrm{d}\,\overset{1}{\omega}_2 &= 0\\
\mathrm{d}\,\overset{2}{\omega}_2 &= 0\\
\mathrm{d}\,h_3 &= 0\\
\mathrm{d}\,\overset{1}{\mathrm{s}}h_3 &= +\overset{2}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3\\
\mathrm{d}\,\overset{2}{\mathrm{s}}h_3 &= -\overset{1}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3\\
\mathrm{d}\,\overset{21}{\mathrm{ss}}h_3 &= 0\\
\mathrm{d}\,f_{2k+m} &= h_3\, f_{2(k-1)+m}\\
\mathrm{d}\,\overset{1}{\mathrm{s}}f_{2k+m} &= -\overset{1}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{2}{\omega}_2\,\overset{21}{\mathrm{ss}}f_{2k+m}\\
\mathrm{d}\,\overset{2}{\mathrm{s}}f_{2k+m} &= -\overset{2}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} - \overset{1}{\omega}_2\,\overset{21}{\mathrm{ss}}f_{2k+m}\\
\mathrm{d}\,\overset{21}{\mathrm{ss}}f_{2k+m} &= \overset{21}{\mathrm{ss}}h_3\, f_{2(k-1)+m} - \overset{2}{\mathrm{s}}h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{1}{\mathrm{s}}h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} + h_3\,\overset{21}{\mathrm{ss}}f_{2(k-1)+m}
\end{aligned}
\right).
\end{aligned}
\tag{74}
$$

Since here the generator $\overset{21}{\mathrm{ss}}h_3$ is closed, discarding it (cf. Rem. 2.47) yields (by Lem. 2.36) the *geometric* 2-toroidification sub-L_∞-algebra

$$
\mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \hookrightarrow \mathrm{tor}^2\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)
\tag{75}
$$

given by

$$
\begin{aligned}
&\mathrm{CE}\Big(\mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)\Big)\\
&\simeq\; \mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\overset{1}{\omega}_2\\
\overset{2}{\omega}_2\\
h_3\\
\overset{1}{\mathrm{s}}h_3\\
\overset{2}{\mathrm{s}}h_3\\
f_{2\bullet+m}\\
\overset{1}{\mathrm{s}}f_{2\bullet+m}\\
\overset{2}{\mathrm{s}}f_{2\bullet+m}\\
\overset{21}{\mathrm{ss}}f_{2\bullet+m}
\end{array}
\right]
\Big/
\left(
\begin{aligned}
\mathrm{d}\,\overset{1}{\omega}_2 &= 0\\
\mathrm{d}\,\overset{2}{\omega}_2 &= 0\\
\mathrm{d}\,h_3 &= \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3\\
\mathrm{d}\,\overset{1}{\mathrm{s}}h_3 &= 0\\
\mathrm{d}\,\overset{2}{\mathrm{s}}h_3 &= 0\\
\mathrm{d}\,f_{2k+m} &= +h_3\, f_{2(k-1)+m} + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}f_{2k+m} + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}f_{2k+m}\\
\mathrm{d}\,\overset{1}{\mathrm{s}}f_{2k+m} &= -\overset{1}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{2}{\omega}_2\,\overset{21}{\mathrm{ss}}f_{2k+m}\\
\mathrm{d}\,\overset{2}{\mathrm{s}}f_{2k+m} &= -\overset{2}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} - \overset{1}{\omega}_2\,\overset{21}{\mathrm{ss}}f_{2k+m}\\
\mathrm{d}\,\overset{21}{\mathrm{ss}}f_{2k+m} &= -\overset{2}{\mathrm{s}}h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{1}{\mathrm{s}}h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} + h_3\,\overset{21}{\mathrm{ss}}f_{2(k-1)+m}
\end{aligned}
\right).
\end{aligned}
\tag{76}
$$

Lemma 2.49 (T-Automorphism of geometric 2-toroidified twisted K-theory). *The CE-algebra (76) of the geometric 2-toroidified twisted K-theory spectrum has an automorphism given by*

$$
\begin{array}{ccc}
\mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) & \xrightarrow[\sim]{\;T^2\;} & \mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)\\
h_3 & \longmapsfrom & h_3\\
\overset{1}{\omega}_2 & \longmapsfrom & \overset{1}{\mathrm{s}}h_3\\
\overset{2}{\omega}_2 & \longmapsfrom & \overset{2}{\mathrm{s}}h_3\\
\overset{1}{\mathrm{s}}h_3 & \longmapsfrom & \overset{1}{\omega}_2\\
\overset{2}{\mathrm{s}}h_3 & \longmapsfrom & \overset{2}{\omega}_2\\
-\overset{21}{\mathrm{ss}}f_{2(k+1)+m} & \longmapsfrom & f_{2k+m}\\
-\overset{2}{\mathrm{s}}f_{2k+m} & \longmapsfrom & \overset{1}{\mathrm{s}}f_{2k+m}\\
+\overset{1}{\mathrm{s}}f_{2k+m} & \longmapsfrom & \overset{2}{\mathrm{s}}f_{2k+m}\\
+f_{2(k-1)+m} & \longmapsfrom & \overset{21}{\mathrm{ss}}f_{2k+m}\,.
\end{array}
\tag{77}
$$

Proof. It is clear that the assignment uniquely extends to an automorphism of the underlying graded superalgebra. What remains to be seen is that this respects the differential (76). By unwinding the definitions, we check this explicitly on all generators:

$$
\begin{tikzcd}
-\overset{1}{\mathrm{s}}h_3 \arrow[d, mapsto, "\mathrm{d}"] & \overset{1}{\omega}_2 \arrow[l, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
-\overset{2}{\mathrm{s}}h_3 \arrow[d, mapsto, "\mathrm{d}"] & \overset{2}{\omega}_2 \arrow[l, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
-\overset{1}{\omega}_2 \arrow[d, mapsto, "\mathrm{d}"] & \overset{1}{\mathrm{s}}h_3 \arrow[l, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
-\overset{2}{\omega}_2 \arrow[d, mapsto, "\mathrm{d}"] & \overset{2}{\mathrm{s}}h_3 \arrow[l, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
h_3 \arrow[d, mapsto, "\mathrm{d}"] & & h_3 \arrow[ll, mapsto] \arrow[dd, mapsto, "\mathrm{d}"] \\
\overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 \arrow[d, equal] & & \\
+\overset{1}{\mathrm{s}}h_3\,\overset{1}{\omega}_2 + \overset{2}{\mathrm{s}}h_3\,\overset{2}{\omega}_2 & & \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 \arrow[ll, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
-\overset{2\,1}{\mathrm{ss}}f_{2(k+1)+m} \arrow[d, mapsto] & f_{2k+m} \arrow[l, mapsto] \arrow[dd, mapsto] \\
-\overset{2\,1}{\mathrm{ss}}(h_3\, f_{2k+m}) \arrow[d, equal] & \\
-h_3\,\overset{2\,1}{\mathrm{ss}}f_{2k+m} - \overset{1}{\mathrm{s}}h_3\,\overset{2}{\mathrm{s}}f_{2k+m} + \overset{2}{\mathrm{s}}h_3\,\overset{1}{\mathrm{s}}f_{2k+m} & h_3\, f_{2(k-1)+m} + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}f_{2k+m} + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}f_{2k+m} \arrow[l, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
-\overset{2}{\mathrm{s}}f_{2k+m} \arrow[d, mapsto] & \overset{1}{\mathrm{s}}f_{2k+m} \arrow[l, mapsto] \arrow[dd, mapsto] \\
+\overset{2}{\mathrm{s}}(h_3\, f_{2(k-1)+m} + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}f_{2k+m}) \arrow[d, equal] & \\
+\overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}f_{2k+m} - h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} - \overset{2}{\mathrm{s}}h_3\, f_{2(k-1)+m} & -\overset{1}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}f_{2k+m} \arrow[l, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
+\overset{1}{\mathrm{s}}f_{2k+m} \arrow[d, mapsto] & \overset{2}{\mathrm{s}}f_{2k+m} \arrow[l, mapsto] \arrow[dd, mapsto] \\
-\overset{1}{\mathrm{s}}(h_3\, f_{2(k-1)+m} + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}f_{2k+m}) \arrow[d, equal] & \\
+\overset{2}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}f_{2k+m} + h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} - \overset{1}{\mathrm{s}}h_3\, f_{2(k-1)+m} & -\overset{2}{\mathrm{s}}h_3\, f_{2(k-1)+m} + h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} - \overset{1}{\omega}_2\,\overset{2\,1}{\mathrm{ss}}f_{2k+m} \arrow[l, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
f_{2(k-1)+m} \arrow[d, mapsto] & \overset{2\,1}{\mathrm{ss}}f_{2k+m} \arrow[l, mapsto] \arrow[dd, mapsto] \\
+h_3\, f_{2(k-2)+m} + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} \arrow[d, equal] & \\
+\overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} + \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + h_3\, f_{2(k-2)+m} & -\overset{2}{\mathrm{s}}h_3\,\overset{1}{\mathrm{s}}f_{2(k-1)+m} + \overset{1}{\mathrm{s}}h_3\,\overset{2}{\mathrm{s}}f_{2(k-1)+m} + h_3\,\overset{2\,1}{\mathrm{ss}}f_{2(k-1)+m}\,. \arrow[l, mapsto]
\end{tikzcd}
$$

□

Remark 2.50 (Automorphism of geometric 2-toroidified K-theory fibrations). In analogy with Ex. 2.28, the automorphism (77) from Lem. 2.49 is compatible with the automorphism (73) from Ex. 2.46, and as such constitutes an automorphism of geometric 2-toroidifed K-theory fibrations, with fibers given by 2 even copies and 2 odd copies respectively. The automorphism acts by swapping appropriately the even fluxes among themselves, and the odd fluxes among themselves, respectively. The diagram from Ex. 2.28 is modified appropriately to

$$
\begin{tikzcd}
\mathfrak{l}\Sigma^{0}\mathrm{KU} \arrow[ddr] & \mathfrak{l}\Sigma^{0}\mathrm{KU} \\
\times & \times \\
\mathfrak{l}\Sigma^{0}\mathrm{KU} \arrow[uur] & \mathfrak{l}\Sigma^{0}\mathrm{KU} \\
\times & \times \\
\mathfrak{l}\Sigma^{1}\mathrm{KU} \arrow[ddr] & \mathfrak{l}\Sigma^{1}\mathrm{KU} \\
\times & \times \\
\mathfrak{l}\Sigma^{1}\mathrm{KU} \arrow[uur] \arrow[d] & \mathfrak{l}\Sigma^{1}\mathrm{KU} \arrow[d] \\
\mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^{0}\mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) \arrow[r, leftrightarrow, "\sim"] \arrow[d, ->>] & \mathrm{tor}^{2'}\,\mathfrak{l}\big(\Sigma^{0}\mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) \arrow[d, ->>] \\
b\mathcal{T}^2 \arrow[r, leftrightarrow, "\sim"] & b\mathcal{T}^2 \,.
\end{tikzcd}
$$

Here the first cross over map corresponds to the exchange (up to signs) of generators with even number of shifts

$$f_{2k} \;\longleftrightarrow\; \mathrm{s}_1\mathrm{s}_2 f_{2k+2}\,,$$

while the second one corresponds to the exchange (up to signs) of generators with an odd number of shifts

$$\mathrm{s}_2 f_{2k} \;\longleftrightarrow\; \mathrm{s}_1 f_{2k}\,.$$

The above observations generalize appropriately to the case of a toroidification along 3 rational circles.

Example 2.51 (**Geometric 3-Toroidification of bundle gerbe classifying space).** In completely analogous manner to Ex. 2.46, it can be seen that neither the 3-cyclification $\mathrm{cyc}^3(b^2\mathbb{R})$ nor the 3-toroidification algebra

$$\mathrm{tor}^3(b^2\mathbb{R}) \;:=\; \mathrm{tor}\,\mathrm{tor}^2(b^2\mathbb{R}) \hookrightarrow \mathrm{cyc}\,\mathrm{tor}^2(b^2\mathbb{R})\,. \tag{78}$$

enjoys an automorphism with the property that of swapping $\overset{i}{\mathrm{s}}h_3 \leftrightsquigarrow \overset{i}{\omega}_2$ *for all* $i=1,2,3$, as would befit a 3-toroidal T-duality classifying algebra. The latter 3-toroidification is given by the CE-algebra

$$
\mathrm{CE}\big(\mathrm{tor}^3 b^2\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\overset{2}{\omega}_2, \\
\overset{1}{\omega}_2, \\
h_3 \\
\overset{3}{\mathrm{s}}h_3 \\
\overset{2}{\mathrm{s}}h_3 \\
\overset{1}{\mathrm{s}}h_3 \\
\overset{21}{\mathrm{ss}}h_3 \\
\overset{31}{\mathrm{ss}}h_3 \\
\overset{32}{\mathrm{ss}}h_3 \\
\overset{321}{\mathrm{sss}}h_3
\end{array}
\right]
\Big/
\left(
\begin{array}{lcc}
\mathrm{d}\,\overset{3}{\omega}_2 &=& 0 \\
\mathrm{d}\,\overset{2}{\omega}_2 &=& 0 \\
\mathrm{d}\,\overset{1}{\omega}_2 &=& 0 \\
\mathrm{d}\,h_3 &=& \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 + \overset{3}{\omega}_2\,\overset{3}{\mathrm{s}}h_3 \\
\mathrm{d}\,\overset{3}{\mathrm{s}}h_3 &=& -\overset{1}{\omega}_2\,\overset{31}{\mathrm{ss}}h_3 - \overset{3}{\omega}_2\,\overset{32}{\mathrm{ss}}h_3 \\
\mathrm{d}\,\overset{2}{\mathrm{s}}h_3 &=& -\overset{1}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3 + \overset{3}{\omega}_2\,\overset{32}{\mathrm{ss}}h_3 \\
\mathrm{d}\,\overset{1}{\mathrm{s}}h_3 &=& +\overset{2}{\omega}_2\,\overset{21}{\mathrm{ss}}h_3 + \overset{3}{\omega}_2\,\overset{31}{\mathrm{ss}}h_3 \\
\mathrm{d}\,\overset{21}{\mathrm{ss}}h_3 &=& +\overset{3}{\omega}_2\,\overset{321}{\mathrm{sss}}h_3 \\
\mathrm{d}\,\overset{31}{\mathrm{ss}}h_3 &=& -\overset{2}{\omega}_2\,\overset{321}{\mathrm{sss}}h_3 \\
\mathrm{d}\,\overset{32}{\mathrm{ss}}h_3 &=& +\overset{1}{\omega}_2\,\overset{321}{\mathrm{sss}}h_3 \\
\mathrm{d}\,\overset{321}{\mathrm{sss}}h_3 &=& 0
\end{array}
\right), \tag{79}
$$

via which the obstructions for such a swapping automorphism to exist are seen to be the triply shifted 0-cocycle and the doubly shifted (twisted) cocycle 1-generators

$$\overset{321}{\mathrm{sss}}h_3,\quad \overset{32}{\mathrm{ss}}h_3,\quad \overset{31}{\mathrm{ss}}h_3,\quad \overset{21}{\mathrm{ss}}h_3\,.$$

This implies that the *geometric* 3-toroidification L_∞-subalgebra (cf. Rem. 2.47) obtained by : **1)** firstly discarding the 3-shifted 0-cocycle $\overset{321}{\mathrm{sss}}h_3$ and **2)** secondly discarding the (resulting untwisted) 2-shifted 1-cocycles $\{\overset{ij}{\mathrm{ss}}h_3\}_{i,j=1,2,3}$ (Lem. 2.36),

$$\mathrm{tor}^{3'}(b^2\mathbb{R}) \hookrightarrow \mathrm{tor}^3(b^2\mathbb{R}) \hookrightarrow \mathrm{cyctor}^2(b^2\mathbb{R})\,,$$

given by

$$
\mathrm{CE}\big(\mathrm{tor}^{3'} b^2\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \overset{3}{\omega}_2, \\ \overset{2}{\omega}_2, \\ \overset{1}{\omega}_2, \\ h_3 \\ \overset{3}{\mathrm{s}}h_3 \\ \overset{2}{\mathrm{s}}h_3 \\ \overset{1}{\mathrm{s}}h_3 \end{array}\right] \Big/ \left(\begin{array}{ccc} \mathrm{d}\,\overset{3}{\omega}_2 & = & 0 \\ \mathrm{d}\,\overset{2}{\omega}_2 & = & 0 \\ \mathrm{d}\,\overset{1}{\omega}_2 & = & 0 \\ \mathrm{d}\,h_3 & = & \overset{1}{\omega}_2\,\overset{1}{\mathrm{s}}h_3 + \overset{2}{\omega}_2\,\overset{2}{\mathrm{s}}h_3 + \overset{3}{\omega}_2\,\overset{3}{\mathrm{s}}h_3 \\ \mathrm{d}\,\overset{3}{\mathrm{s}}h_3 & = & 0 \\ \mathrm{d}\,\overset{2}{\mathrm{s}}h_3 & = & 0 \\ \mathrm{d}\,\overset{1}{\mathrm{s}}h_3 & = & 0 \end{array}\right), \tag{80}
$$

being equivalently the higher central extension (Def. 2.59) of $b\mathbb{R}^2 \times b\mathbb{R}^2 \times b\mathbb{R}^2$ by its canonical 4-cocycle

$$
b\mathcal{T}^3 \;:=\; \mathrm{tor}^{3'} b^2\mathbb{R} \xrightarrow{\mathrm{hofib}(\overset{1}{\omega}_2\,\overset{1}{\widetilde{\omega}}_2 + \overset{2}{\omega}_2\,\overset{2}{\widetilde{\omega}}_2 + \overset{3}{\omega}_2\,\overset{3}{\widetilde{\omega}}_2)} b\mathbb{R}^2 \times b\mathbb{R}^2 \times b\mathbb{R}^2 \xrightarrow{\overset{1}{\omega}_2\,\overset{1}{\widetilde{\omega}}_2 + \overset{2}{\omega}_2\,\overset{2}{\widetilde{\omega}}_2 + \overset{3}{\omega}_2\,\overset{3}{\widetilde{\omega}}_2} b^3\mathbb{R}\,, \tag{81}
$$

does have an automorphism symmetry given by swapping the two degree=2 generators as:

$$
\begin{array}{ccc} b\mathcal{T}^3 & \xleftrightarrow{\sim} & b\mathcal{T}^3 \\ -\overset{1}{\mathrm{s}}h_3 & \longleftarrow & \overset{1}{\omega}_2 \\ -\overset{1}{\omega}_2 & \longleftarrow & \overset{1}{\mathrm{s}}h_3 \\ -\overset{2}{\mathrm{s}}h_3 & \longleftarrow & \overset{2}{\omega}_2 \\ -\overset{2}{\omega}_2 & \longleftarrow & \overset{2}{\mathrm{s}}h_3 \\ -\overset{3}{\mathrm{s}}h_3 & \longleftarrow & \overset{3}{\omega}_2 \\ -\overset{3}{\omega}_2 & \longleftarrow & \overset{3}{\mathrm{s}}h_3 \end{array} \tag{82}
$$

This geometric 3-toroidification construction extends to the twisted K-theory spectra along the lines of Ex. 2.48, and so does the swapping automorphism (82) to an isomorphism of the geometric 3-toroidified K-theory spectra fibrations

$$
\begin{tikzcd}
\mathrm{tor}^{3'}\,\iota\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[r, leftrightarrow, "\sim"] \arrow[d, ->>] & \mathrm{tor}^{3'}\,\iota\big(\Sigma^1\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[d, ->>] \\
b\mathcal{T}^3 \arrow[r, leftrightarrow, "\sim"] & b\mathcal{T}^3\,.
\end{tikzcd}
$$

in analogy to Rem. 2.50, which now acts by swapping appropriately the 4 even $\Sigma^0\mathrm{KU}$ and the 4 odd $\Sigma^1\mathrm{KU}$ fibers. The extended automorphism can be seen to act by (up to signs)

$$
\begin{array}{ccc} \overset{3}{\mathrm{s}}f_{2k} & \longleftrightarrow & \overset{1\,2}{\mathrm{s}\mathrm{s}}f_{2k+1} \\ \overset{1}{\mathrm{s}}f_{2k} & \longleftrightarrow & \overset{2\,3}{\mathrm{s}\mathrm{s}}f_{2k+1} \\ \overset{2}{\mathrm{s}}f_{2k} & \longleftrightarrow & \overset{3\,1}{\mathrm{s}\mathrm{s}}f_{2k+1} \\ f_{2(k-1)} & \longleftrightarrow & \overset{1\,2\,3}{\mathrm{s}\mathrm{s}\mathrm{s}}f_{2k+1}\,, \end{array}
$$

hence swapping the even and odd copies.

Rather than writing out the analogous explicit formulas and proofs for this 3-toroidified case, we do this more generally and concisely for the (geometric) n-toroidified twisted K-theory spectra.

n-Toroidification of twisted K-theory spectra. With the case of 2- and 3-toroidification thus understood, we next give more abstract but general formulas for the situation of n-toroidification.

Example 2.52 (Geometric n-Toroidification of bundle gerbe classifying space). The geometric n-toroidification (cf. Rem. 2.47) of the line Lie 3-algebra

$$
b\mathcal{T}^n \;\equiv\; \mathrm{tor}^{n'}(b^2\mathbb{R}) \hookrightarrow \mathrm{tor}^n(b^2\mathbb{R}) \hookrightarrow \mathrm{cyctor}^{n-1}(b^2\mathbb{R})\,,
$$

is the L_∞-algebra obtained by (successively) discarding all shifted generators $\mathrm{s}_{i_1}\cdots\mathrm{s}_{i_k}h_3$ for $k \in \{2,\cdots,n\}$ and $i_1,\cdots,i_k \in \{1,\cdots,n\}$ (those shifted more than once), hence given by

$$
\mathrm{CE}\big(b\mathcal{T}^n\big) \;\;\simeq\;\; \left(\begin{array}{ll} \mathrm{d}\,\overset{r}{\omega}_2 & = 0 \\ \mathrm{d}\,h_3 & = \sum_r \overset{r}{\omega}_2\,\overset{r}{\mathrm{s}}h_3 \\ \mathrm{d}\,\overset{r}{\mathrm{s}}h_3 & = 0 \end{array}\right). \tag{83}
$$

Crucially, the geometric n-toroidification of $b^2\mathbb{R}$ enjoys the swapping automorphism

$$\begin{array}{ccc} b\mathcal{T}^n & \xleftrightarrow{\;\sim\;} & b\mathcal{T}^n \\ (-1)^n\, \overset{r}{\mathrm{s}}h_3 & \longmapsfrom & \overset{r}{\omega}_2 \\ (-1)^n\, \overset{r}{\omega}_2 & \longmapsfrom & \overset{r}{\mathrm{s}}h_3\,, \end{array} \tag{84}$$

which extends appropriately to a swapping isomorphism of the (geometric) n-toroidification of twisted K-theory spectra.

Example 2.53 (Geometric n-Toroidification of twisted K-theory). The geometric n-toroidification (cf. Rem. 2.47) of (the Whitehead L_∞-algebra of) the twisted K-theory spectra is:

$$\mathrm{CE}\Big(\mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big)\Big) \;\simeq\; \left(\begin{array}{rcl} \mathrm{d}\,\overset{r}{\omega}_2 & = & 0 \\ \mathrm{d}\,h_3 & = & \sum_r \overset{r}{\omega}_2\, \overset{r}{\mathrm{s}}h_3 \\ \mathrm{d}\,\overset{r}{\mathrm{s}}h_3 & = & 0 \\ \mathrm{d}\,\overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}f_{2k+m} & = & (-1)^r\, \overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\big(h_3\, f_{2(k-1)+m} + \sum_{r'} \overset{r'}{\omega}_2\, \overset{r'}{\mathrm{s}}f_{2k+m}\big) \end{array}\right). \tag{85}$$

For analyzing this, it will be convenient to make explicit the following:

Definition 2.54 (Fiberwise Hodge duality in n-toroidification). Consider the Hodge duality-like operation on iterated shift operators $\overset{i_r}{\mathrm{s}}\cdots\overset{i_2}{\mathrm{s}}\overset{i_1}{\mathrm{s}}$ in the CE-algebra of the n-toroidified twisted K-theory from Ex. 2.53, given by:

$$\big(\star\, \overset{i_r}{\mathrm{s}}\cdots\overset{i_2}{\mathrm{s}}\overset{i_1}{\mathrm{s}}\big)\alpha \;:=\; \tfrac{1}{(n-r)!}\,\epsilon_{i_n\cdots i_{r+1} i_r\cdots i_2 i_1}\overset{i_n}{\mathrm{s}}\cdots\overset{i_{r+1}}{\mathrm{s}}\alpha\,, \quad (\epsilon_{n\cdots 21} := 1) \quad . \tag{86}$$

Lemma 2.55 (Properties of Hodge operator on winding modes). *The operation (86) satisfies the usual properties of a Hodge star operator:*

$$(-1)^{r(n-1)} \star\star \;=\; \mathrm{id} \tag{87}$$

$$(-1)^{rn-1} \star \overset{k}{\mathrm{s}}\star \;=\; \text{derivation that removes } \overset{k}{\mathrm{s}} \tag{88}$$

Proof. Since

$$\begin{aligned} \big(\star\star\overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\big)\alpha &= \tfrac{1}{(n-r)!}\,\epsilon_{i_n\cdots i_{r+1} i_r\cdots i_1}\big(\star\,\overset{i_n}{\mathrm{s}}\cdots\overset{i_{r+1}}{\mathrm{s}}\big)\alpha \\ &= \tfrac{1}{(n-r)!}\,\epsilon_{i_n\cdots i_{r+1} i_r\cdots i_1}\tfrac{1}{r!}\epsilon^{j_r\cdots j_1 i_n\cdots i_{r+1}}\,\overset{j_r}{\mathrm{s}}\cdots\overset{j_1}{\mathrm{s}}\alpha \\ &= (-1)^{r(n-r)}\,\delta^{j_r\cdots j_1}_{i_r\cdots i_1}\overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha \\ &= (-1)^{r(n-1)}\,\overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha \end{aligned}$$

and

$$\begin{aligned} \big(\star\,\overset{k}{\mathrm{s}}\star\overset{i_r}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\big)\alpha &= \tfrac{1}{(n-r)!}\,\epsilon_{i_n\cdots i_{r+1} i_r\cdots i_2 i_1}\big(\star\,\overset{k}{\mathrm{s}}\overset{i_n}{\mathrm{s}}\cdots\overset{i_{r+1}}{\mathrm{s}}\big)\alpha \\ &= \tfrac{1}{(n-r)!}\,\tfrac{1}{(r-1)!}\,\epsilon_{i_n\cdots i_{r+1} i_r\cdots i_1}\epsilon^{j_{r-1}\cdots j_1 k i_n\cdots i_{r+1}}\overset{j_{r-1}}{\mathrm{s}}\cdots\overset{j_1}{\mathrm{s}}\alpha \\ &= (-1)^{r(n-r)+r-1}\,r\,\delta^{k\, i_{r-1}\cdots i_1}_{i_r\cdots i_1}\overset{i_{r-1}}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha \\ &= (-1)^{rn-1}\,r\,\delta^{k\, i_{r-1}\cdots i_1}_{i_r\cdots i_1}\overset{i_{r-1}}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha \\ &= \left\{\begin{array}{cl} (-1)^{rn-1}\overset{i_{r-1}}{\mathrm{s}}\overset{i_{r-2}}{\mathrm{s}}\overset{i_{r-3}}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha & \text{if } k = i_r \\ -(-1)^{rn-1}\,\overset{i_r}{\mathrm{s}}\;\overset{i_{r-2}}{\mathrm{s}}\overset{i_{r-3}}{\mathrm{s}}\cdots\overset{i_1}{\mathrm{s}}\alpha & \text{if } k = i_{r-1} \\ \vdots & \vdots \\ (-1)^{r-1}(-1)^{rn-1}\,\overset{i_r}{\mathrm{s}}\;\overset{i_{r-1}}{\mathrm{s}}\overset{i_{r-2}}{\mathrm{s}}\cdots\overset{i_2}{\mathrm{s}}\alpha & \text{if } k = i_1 \\ 0 & \text{otherwise}\,. \end{array}\right. \end{aligned}$$

$\square$

Using the Hodge duality notation of Rem. 2.54, we may generalize Lem. 2.49 as follows:

Proposition 2.56 (T-Automorphism of n-torified twisted K-theory). *For any $n \in \mathbb{N}$, there is an isomorphism between the geometric n-toroidified twisted spectra (85) of degree m and degree $m + (n \bmod 2)$ given* *by*

$$
\begin{array}{ccc}
\mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) & \xrightarrow[\sim]{\;\;T^n\;\;} & \mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^{m+(n \,\mathrm{mod}\, 2)}\mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) \\
(-1)^n\, \overset{r}{\mathrm{s}}h_3 & \longmapsfrom & \overset{r}{\omega}_2 \\
(-1)^n\, \overset{r}{\omega}_2 & \longmapsfrom & \overset{r}{\mathrm{s}}h_3 \\
(-1)^{(n+r+1+r(r+1)/2)}\, \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f & \longmapsfrom & \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f\,,
\end{array}
\tag{89}
$$

where in the last line of (89) and in the following we suppress the indices on the generators just for notational brevity. Notably, this is an automorphism for even $n = 2k$, $k \in \mathbb{N}$.

Proof. The map is manifestly invertible on generators, so we need to check that the differential is respected. On the generators corresponding to the $b\mathcal{T}^n$ L_∞-subalgebra this is immediate (cf. 83):

$$
\begin{tikzcd}
(-1)^n \overset{r}{\mathrm{s}}h_3 \arrow[d, |->] & \overset{r}{\omega}_2 \arrow[l, |->] \arrow[d, |->] \\
0 & 0 \arrow[l, |->]
\end{tikzcd}
\qquad
\begin{tikzcd}
(-1)^n \overset{r}{\omega}_2 \arrow[d, |->] & \overset{r}{\mathrm{s}}h_3 \arrow[l, |->] \arrow[d, |->] \\
0 & 0 \arrow[l, |->]
\end{tikzcd}
\qquad
\begin{tikzcd}
h_3 \arrow[d, |->] & & h_3 \arrow[ll, |->] \arrow[dd, |->] \\
\sum_r \overset{r}{\omega}_2\, \overset{2}{\mathrm{s}}h_3 \arrow[d, equal] & & \\
\sum_r \overset{2}{\mathrm{s}}h_3\, \overset{r}{\omega}_2 & & \sum_r \overset{r}{\omega}_2\, \overset{2}{\mathrm{s}}h_3 \arrow[ll, |->]
\end{tikzcd}
$$

To see it on the remaining generators, abbreviate

$$
\sigma(r) := (-1)^{r(r+1)/2}
$$

and note that

$$
\begin{aligned}
\sigma(r+1) &= -(-1)^r \cdot \sigma(r) \\
\sigma(r-1) &= (-1)^r \cdot \sigma(r)\,.
\end{aligned}
$$

With this we compute as follows:

$$
\begin{tikzcd}
(-1)^{n+r+1}\sigma(r)\, \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \arrow[d, |->, "\mathrm{d}"] & & \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \arrow[ll, |->] \arrow[dd, |->, "\mathrm{d}"] \\
(-1)^{n+r+1}\sigma(r) \cdot (-1)^r \big(\star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}}\big)\big(h_3\, f + \sum_{r'} \overset{r'}{\omega}_2\, \overset{r'}{\mathrm{s}} f\big) \arrow[d, equal, "(88)"] & & \\
\begin{array}{l}
(-1)^{n+r+1}\sigma(r)\, h_3 \,\star\, \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \\
+(-1)^{nr+r+1}\sigma(r)\, \sum_{r'} \overset{r'}{\mathrm{s}}h_3 \,\star\, \overset{r'}{\mathrm{s}} \star \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \\
+(-1)^{r+1}\sigma(r)\, \sum_{r'} \overset{r'}{\omega}_2\, \overset{r'}{\mathrm{s}} \,\star\, \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f
\end{array} \arrow[d, equal, "(87)"] & & (-1)^r\, \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}}\big(h_3\, f + \sum_{r'} \overset{r'}{\omega}_2\, \overset{r'}{\mathrm{s}} f\big) \arrow[d, equal, "(88)"] \\
\begin{array}{l}
(-1)^{n+r+1}\sigma(r)\, h_3 \,\star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \\
-(-1)^{r+1+rn}\sigma(r-1)\, \sum_{r'} \overset{r'}{\omega}_2\, (\star\star\, \overset{r'}{\mathrm{s}} \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f) \\
-(-1)^{r+1}\sigma(r+1) \sum_{r'} \overset{r'}{\mathrm{s}}h_3 \,\star \overset{r'}{\mathrm{s}}\overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f
\end{array} & & \begin{array}{l}
h_3\, \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f \\
+(-1)^{rn} \sum_{r'} \overset{r'}{\mathrm{s}}h_3\, (\star \overset{r'}{\mathrm{s}} \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f) \\
+\sum_{r'} \overset{r'}{\omega}_2\, \overset{r'}{\mathrm{s}}\overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f
\end{array} \arrow[ll, |->]
\end{tikzcd}
$$

□

Remark 2.57 (**Further isomorphisms of n-toroidified twisted K-spectra).** As with the $n = 1$ case (Rem. 2.29), there exist two further isomorphisms apart from (89), which differ only by a consistent choice of signs for the image of the generators. Explicitly, one has

$$
\begin{array}{ccc}
\mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) & \xrightarrow[\sim]{\;\;T^n\;\;} & \mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^{m+(n \,\mathrm{mod}\, 2)}\mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) \\
(-1)^{n+1}\, \overset{r}{\mathrm{s}}h_3 & \longmapsfrom & \overset{r}{\omega}_2 \\
(-1)^{n+1}\, \overset{r}{\omega}_2 & \longmapsfrom & \overset{r}{\mathrm{s}}h_3 \\
(-1)^{(n+r(r+1)/2)}\, \star \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f & \longmapsfrom & \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f\,,
\end{array}
\tag{90}
$$

and

$$
\begin{array}{ccc}
\mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^m \mathrm{KU}/\!\!/ B\mathrm{U}(1)\big) & \xrightarrow[\sim]{\;T^n\;} & \mathrm{tor}^{n'}\,\mathfrak{l}\big(\Sigma^{m+(n \,\mathrm{mod}\, 2)} \mathrm{KU}/\!\!/ B\mathrm{U}(1)\big)
\\
(-1)^n \overset{r}{\mathrm{s}} h_3 & \longleftarrow\!\shortmid & \overset{r}{\omega}_2
\\
(-1)^n \overset{r}{\omega}_2 & \longleftarrow\!\shortmid & \overset{r}{\mathrm{s}} h_3
\\
(-1)^{(n+r+r(r+1)/2)} \,\star\, \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f & \longleftarrow\!\shortmid & \overset{i_r}{\mathrm{s}} \cdots \overset{i_1}{\mathrm{s}} f\,.
\end{array}
$$

Lemma 2.58 (**Twisted K–theory cocycles under toroidal reduction–isomorphism–reoxidation).** *Under construction/modification* **(i)** *The composite operation of*

(a) *n-toroidally reducing (69) twisted KU_m cocycles on a toroidally extended super L_∞-algebra $\overset{1\cdots n}{\widehat{g}}_A$*

$$
\begin{array}{ccc}
\overset{1\cdots n}{\widehat{g}}_A & \longrightarrow & \mathfrak{l}\big(\,\Sigma^m \mathrm{KU}/\!\!/ B\mathrm{U}(1)\,\big)
\\
H^3_A & \longleftarrow\!\shortmid & h_3
\\
(F_{2k+m})_{k \in \mathbb{Z}} & \longleftarrow\!\shortmid & (f_{2k+m})_{k\in\mathbb{Z}}\,,
\end{array}
$$

with geometric twists (Rem. 2.47) of the form

$$
H^3_A \;=\; H^3_{\mathfrak{g}} + \sum_{k=1}^{n} \overset{k}{e}_A \cdot \overset{k}{p}_{A*}(H^3_A)\,,
$$

along its fibration

$$
\overset{1\cdots n}{\widehat{g}}_A \xrightarrow{\;\overset{1\cdots n}{p_A}\;} \mathfrak{g} \xrightarrow{\;\overset{1\cdots n}{c_A}\;} b\mathbb{R}^n\,,
$$

(b) *applying the isomorphism (89) on the target (geometric) torodification of twisted KU_m, hence viewing them instead as valued in the toroidification of twisted $\mathrm{KU}_{m+(n\,\mathrm{mod}\,2)}$, while noticing that this swaps the "product Chern class" $\overset{1\cdots n}{c_A}$ from that classifying the $\overset{1\cdots n}{\widehat{\mathfrak{g}}}_A$-extension to that classifying a different extension*

$$
\overset{1\cdots n}{\widehat{\mathfrak{g}}}_B \longrightarrow \mathfrak{g}\,,
$$

i.e., via

$$
\overset{1\cdots n}{c_B} \;:=\; (-1)^n \overset{1\cdots n}{p}_{A*}(H^3_A) \equiv (-1)^n \big(\overset{1}{p}_{A*}(H^3_A), \cdots, \overset{n}{p}_{A*}(H^3_A)\big) \quad : \quad \mathfrak{g} \longrightarrow b\mathbb{R}^n\,,
$$

(c) *re-oxidizing (43) the result, but now along the new fibration*

$$
\widehat{\mathfrak{g}}_B \xrightarrow{\;p_A\;} \mathfrak{g} \xrightarrow{\;c^1_B\;} b\mathbb{R}\,,
$$

results in the twisted KU_1 cocycles given precisely by

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \longrightarrow & \mathfrak{l}\big(\,\Sigma^1 \mathrm{KU}/\!\!/ B\mathrm{U}(1)\,\big)
\\
H^3_{A\,\mathrm{bas}} + e'_B \cdot c^A_1 & \longleftarrow\!\shortmid & h_3
\\
\big(-p_{A*}F_{2k+2} - e'_B \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}}
\end{array}
\tag{91}
$$

(ii) *Applying instead the second isomorphism (53) in step* **(b)***, results in the twisted KU_1 cocycles given by*

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \longrightarrow & \mathfrak{l}\big(\,\Sigma^1 \mathrm{KU}/\!\!/ B\mathrm{U}(1)\,\big)
\\
H^3_{A\,\mathrm{bas}} - e'_{B'} \cdot c^A_1 & \longleftarrow\!\shortmid & h_3
\\
\big(+p_{A*}F_{2k+2} - e'_{B'} \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}}
\end{array}
$$

where now the extension

$$
\widehat{\mathfrak{g}}_{B'} \longrightarrow \mathfrak{g}\,,
$$

is instead via the opposite 2-cocycle

$$
c^1_{B'} \;:=\; -p_{A*}(H^3_A)\,.
$$

Proof. This is a matter of carefully tracking through the (bijective) operations on the corresponding sets of L_∞-algebra morphisms. Explicitly, under the reduction (43) from Prop. 2.25 the first step yields the map of super

L_∞-algebras

$$
\begin{array}{ccc}
\mathfrak{g} & \longrightarrow & \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)
\\
H^3_{A\,\mathrm{bas}} & \longleftarrow\!\shortmid & h_3
\\
F_{2k\,\mathrm{bas}} & \longleftarrow\!\shortmid & f_{2k}
\\
-p_{A*}H^3_A & \longleftarrow\!\shortmid & \mathrm{s}h_3
\\
-p_{A*}F_{2k} & \longleftarrow\!\shortmid & \mathrm{s}f_{2k}
\\
c_1^A & \longleftarrow\!\shortmid & \omega_2\,.
\end{array}
$$

In the second step, postcomposition of the above morphism with the first isomorphism in (51)

$$
\mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big) \xrightarrow{\;\sim\;} \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)
$$

yields

$$
\begin{array}{ccc}
\mathfrak{g} & \longrightarrow & \mathrm{cyc}\,\mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)
\\
H^3_{A\,\mathrm{bas}} & \longleftarrow\!\shortmid & h_3
\\
-p_{A*}F_{2k+2} & \longleftarrow\!\shortmid & f_{2k+1}
\\
-c_1^A & \longleftarrow\!\shortmid & \mathrm{s}h_3
\\
F_{2k\,\mathrm{bas}} & \longleftarrow\!\shortmid & \mathrm{s}f_{2k+1}
\\
c_B^1 \;:=\; p_{A*}H^3_A & \longleftarrow\!\shortmid & \omega_2\,.
\end{array}
$$

Lastly, in the third step oxidizing (43) via the new 2-cocycle

$$
c_B^1 \;:=\; p_{A*}(H^3_A) \quad : \quad \mathfrak{g} \longrightarrow b\mathbb{R}\,,
$$

immediately yields precisely the morphism of super L_∞-algebras out of the corresponding central extension

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \longrightarrow & \mathfrak{l}\big(\,\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\,\big)
\\
H^3_{A\,\mathrm{bas}} + e'_B \cdot c_1^A & \longleftarrow\!\shortmid & h_3
\\
\big(-p_{A*}F_{2k+2} - e'_B \cdot F_{2k\,\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{2k+1})_{k\in\mathbb{Z}}\,.
\end{array}
$$

The case of using instead the isomorphism (53) follows analogously. □

2.4 Higher extensions

We have discussed central extensions classified by 2-cocycles (Def. 2.21, Def. 2.32). Traditionally, these are the only central extensions considered in ordinary Lie theory. However, in L_∞-theory we may also extend by higher cocycles:

Definition 2.59 (Higher central extensions [FSS15, Prop. 3.5]**).** For $\mathfrak{g} \in \mathrm{sLie}_\infty$ equipped with an ordinary $(n+1)$-cocycle (21)

$$
\mathfrak{g} \xrightarrow{\omega_{n+1}} b^n\mathbb{R} \qquad \leftrightsquigarrow \qquad \left\{ \begin{array}{l} \omega_{n+1} \,\in\, \mathrm{CE}(\mathfrak{g}) \\ \deg(\omega_{n+1}) \,=\, (n+1, \mathrm{evn}) \\ \mathrm{d}\,\omega_{n+1} \,=\, 0 \end{array} \right.
$$

the *higher central extension* it classifies is its homotopy fiber $\widehat{\mathfrak{g}}$ given by

$$
\mathrm{CE}(\mathfrak{g})\big[\underbrace{\,b_n\,}_{\deg = (n,\mathrm{evn})}\big]/(\mathrm{d}\,b_n \;=\; \omega_{n+1}) \tag{92}
$$

fitting into a fiber sequence

$$
\widehat{\mathfrak{g}} \xrightarrow{\;p:=\mathrm{hofib}(\omega_{n+1})\;} \!\!\!\!\rightarrow \mathfrak{g} \xrightarrow{\omega_{n+1}} b^n\mathbb{R}\,.
$$

Examples of higher central extension. The base examples of relevance for super p-branes are the following:

Example 2.60 (The string-extensions of II super-space). The higher central extension (Def. 2.59) of 10D type II superspace (Ex. 3.2, Ex. 3.5) by the NS super 3-flux densities H_3^A (128) and H_3^B (136), respectively, are super-space analogs of the *string Lie 2-algebra* (cf. [FSS14, Apnd]) and as such called $\mathfrak{string}_{\mathrm{IIA/B}}$ or similar [BH11, Thm. 21][Huer12, §3.1.3][FSS15, Def. 4.2][HSS19, pp 13], cf. [CdAIP00, (6.12)].

$$
\begin{aligned}
\mathfrak{string}_{\mathrm{IIA}} &\xrightarrow{\;\mathrm{hofib}\;} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \xrightarrow{\;H_3^A\;} b^2\mathbb{R} \\
\mathfrak{string}_{\mathrm{IIB}} &\xrightarrow{\;\mathrm{hofib}\;} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \xrightarrow{\;H_3^B\;} b^2\mathbb{R}
\end{aligned}
$$

given by

$$
\mathrm{CE}\big(\mathfrak{string}_{\mathrm{IIA/B}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{9} \\ b_2 \end{array}\right] \Big/ \left(\begin{array}{l} \mathrm{d}\,\psi = 0 \\ \mathrm{d}\,e^a = \big(\overline{\psi}\,\Gamma^a_{A/B}\,\psi\big) \\ \mathrm{d}\,b_2 = \underbrace{\big(\overline{\psi}\,\Gamma_a^{A/B}\Gamma_{10}\,\psi\big)e^a}_{H_3^{A/B}} \end{array}\right). \tag{93}
$$

Analogously:

Example 2.61 (The M2-brane extension of 11D superspace). The higher central extension (Def. 2.59) of the 11D super-spacetime (Ex. 2.3) by the super 4-flux density G_4 (2) is the higher analogue of the super-string Lie 2-algebra from Ex. 2.60 and as such called the *super membrane algebra* $\mathfrak{m2brane}$ or similar [BH11, Thm. 22][FSS15, §4.4][Huer17]. Its CE-algebra is the one originally considered in [DF82, (3.15)], rediscovered in [CdAIP00, (105)]:

$$
\mathrm{CE}\big(\mathfrak{m2brane}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=1}^{10} \\ c_3 \end{array}\right] \Big/ \left(\begin{array}{l} \mathrm{d}\,\psi = 0 \\ \mathrm{d}\,e^a = \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,c_3 = \underbrace{\tfrac{1}{2}\big(\overline{\psi}\Gamma_{ab}\psi\big)e^a e^b}_{G_4} \end{array}\right),
$$

so that

$$
\mathfrak{m2brane} \xrightarrow{\;\mathrm{hofib}\;} \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;G_4\;} b^3\mathbb{R}\,.
$$

Example 2.62 (Parity isomorphism of 11d SuGra). The canonical reflection action of $\Gamma_{10} \in \mathrm{Pin}^+(1,10)$ on $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ lifts to its M2-brane extension of Ex. 2.61 by flipping the sign of the generator c_3:

$$
\begin{array}{ccl}
\mathfrak{m2brane} & \xrightarrow[\sim]{\;\mathrm{par}\;} & \mathfrak{m2brane} \\
\Gamma_{10}\psi & \longleftarrow\!\shortmid & \psi \\
+e^a & \longleftarrow\!\shortmid & e^a \quad (a<10) \\
-e^{10} & \longleftarrow\!\shortmid & e^{10} \\
-c_3 & \longleftarrow\!\shortmid & c_3\,.
\end{array} \tag{94}
$$

This is because G_4 (2) changes sign under a super-reflection:

$$
\begin{array}{lll}
\mathrm{par}^* G_4 & = \mathrm{par}^*\Big(\tfrac{1}{2}\big(\overline{\psi}\Gamma_{ab}\psi\big)e^a e^b\Big) & \text{by (2)} \\
& = \sum_{a,b<10} \tfrac{1}{2}\big(\overline{\Gamma_{10}\psi}\,\Gamma_{ab}\,\Gamma_{10}\psi\big)e^a e^b \;-\; \sum_{a<10}\big(\overline{\Gamma_{10}\psi}\,\Gamma_{a10}\,\Gamma_{10}\psi\big)e^a e^{10} & \text{by (94)} \\
& = -\sum_{a,b<10} \tfrac{1}{2}\big(\overline{\psi}\Gamma_{10}\,\Gamma_{ab}\,\Gamma_{10}\psi\big)e^a e^b \;+\; \sum_{a<10}\big(\overline{\psi}\Gamma_{10}\,\Gamma_{a10}\,\Gamma_{10}\psi\big)e^a e^{10} & \text{by (220)} \\
& = -\sum_{a,b<10} \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{ab}\,\Gamma_{10}\Gamma_{10}\psi\big)e^a e^b \;-\; \sum_{a<10}\big(\overline{\psi}\,\Gamma_{a10}\,\Gamma_{10}\Gamma_{10}\psi\big)e^a e^{10} & \text{by (214)} \\
& = -\sum_{a,b<10} \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{ab}\,\psi\big)e^a e^b \;-\; \sum_{a<10}\big(\overline{\psi}\,\Gamma_{a10}\,\psi\big)e^a e^{10} & \text{by (214)} \\
& = -\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{ab}\,\psi\big)e^a e^b \;=\; -G_4\,. &
\end{array}
$$

This transformation (94), of super-spacetime reflection together with sign-inversion on c_3, is known [DNP86, (2.2.29)] as a symmetry of the Lagrangian density for 11D SuGra, and it controls (e.g. [Fa99, (3.1)]) the behaviour of the M-theory 3-form near a Hořava-Witten MO9-brane (i.e. near the fixed locus of the super-reflection in).

Remark 2.63 (Basic and fiber forms on a higher centrally extended super-L_∞ algebra). The decomposition in terms of basic and fiber forms follows for higher central extensions in complete analogy to the case of standard central extensions (Rem. 2.22):

(i) Given a higher central extension as in Def. 2.59, every element in its CE-algebra decomposes uniquely as the sum

$$
\alpha \;=\; \alpha_{\mathrm{bas}} \,+\, b_n\, p_*(\alpha) \tag{95}
$$

of a *basic form*

$$
\alpha_{\mathrm{bas}} \;\in\; p^*\big(\mathrm{CE}(\mathfrak{g})\big)
$$

and the product of the generator b_n with the image of α under *fiber integration* p_*, which is a super-graded derivation of degree $(-n, \mathrm{evn})$:

$$
\begin{array}{ccc}
\mathrm{CE}(\widehat{\mathfrak{g}}) & \xrightarrow{p_*} & \mathrm{CE}(\mathfrak{g}) \\
b_n & \longmapsto & 1 \\
e^i & \longmapsto & 0\,.
\end{array}
\tag{96}
$$

(ii) The differential of a general element is given in this decomposition in terms of the differential $\mathrm{d}_{\mathfrak{g}}$ by:

$$
\begin{aligned}
\mathrm{d}_{\widehat{\mathfrak{g}}}\big(\alpha_{\mathrm{bas}} + b_n\, p_*\alpha\big) & = \mathrm{d}_{\widehat{\mathfrak{g}}}\,\alpha_{\mathrm{bas}} + \big(\mathrm{d}_{\widehat{g}}\, b_n\big)\, p_*\alpha + (-1)^n\, b_n\, \mathrm{d}_{\widehat{\mathfrak{g}}}\, p_*\alpha \\
& = \big(\mathrm{d}_{\mathfrak{g}}\alpha_{\mathrm{bas}} + \omega_{n+1}\, p_*\alpha\big) + (-)^n\, b_n\, \mathrm{d}_{\mathfrak{g}} p_*\alpha\,.
\end{aligned}
\tag{97}
$$

While aspects of T-duality for such higher extensions were discussed in [FSS20a] (just) in terms of the higher analog Fourier-Mukai transform, next here we develop the full story of the higher Ext/Cyc-adjunction and the automorphisms of the higher cyclified twisted higher cocycles:

Higher Cyclification along rational odd spheres. Hereon we focus on central extensions where $n = 2t-1 \in \mathbb{N}$ is an odd positive integer, hence on central extensions along even higher cocycles $\omega_{2t} : \mathfrak{g} \longrightarrow b^{2t-1}\mathbb{R}$. Notice the target L_∞-algebra here may be thought of equivalently as:

(i) the rationalization odd iterated delooping of the circle $B^{2t-1}S^1$, or

(ii) perhaps more suggestively the rationalization of the an odd sphere S^{2t-1}.

In this scenario, there is a straightforward higher generalization of the cyclification functor, which in view of **(ii)** may also be referred to as an *odd spherification* functor.

Definition 2.64 (Higher Cyclification/Odd Spherification of super L_∞-algebras). Given $\mathfrak{h} \in \mathrm{sLieAlg}_\infty^{\mathrm{fin}}$ with presentation

$$
\mathrm{CE}(\mathfrak{h}) \;\simeq\; \mathbb{R}_{\mathrm{d}}\big[(e^i)_{i\in I}\big]\big/\big(\mathrm{d}\, e^i = P^i(\vec{e})\big)_{i\in I}\,,
$$

and $t \in \mathbb{N}$ its $(2t-1)$-*cyclification* $\mathrm{cyc}_{2t-1}(\mathfrak{h}) \in \mathrm{sLieAlg}_\infty$ is given by

$$
\mathrm{CE}\big(\mathrm{cyc}_{2t-1}(\mathfrak{h})\big) \;:=\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (e^i)_{i\in I}, \overbrace{\omega_{2t},}^{\deg = (2t,\mathrm{evn})} \\ \big(\underbrace{\mathrm{s}_t e^i}_{\substack{\deg = \\ \deg(e^i) - (2t-1,\mathrm{evn})}}\big)_{i\in I} \end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\omega_{2t} & = 0 \\ \mathrm{d}\, e^i & = \mathrm{d}_{\mathfrak{h}}\, e^i + \omega_{2t}\, \mathrm{s}_t e^i \\ \mathrm{d}\,\mathrm{s}_t e^i & = -\mathrm{s}_t\big(\mathrm{d}_{\mathfrak{h}}\, e^i\big) \end{array}\right),
\tag{98}
$$

where in the last line on the right the shift is understood as extended to a super-graded derivation of degree $(-2t+1, \mathrm{evn})$:

$$
\begin{array}{rcl}
\mathrm{s}_t : \mathrm{CE}\big(\mathrm{cyc}_{2t-1}(\mathfrak{h})\big) & \longrightarrow & \mathrm{CE}\big(\mathrm{cyc}_{2t-1}(\mathfrak{h})\big) \\
\omega_{2t} & \longmapsto & 0\,, \\
e^i & \longmapsto & \mathrm{s}_t e^i\,, \\
\mathrm{s}_t e^i & \longmapsto & 0\,.
\end{array}
$$

The fact that this is well-defined, namely

$$
\mathrm{d}\,\mathrm{d} \;=\; 0\,, \qquad \mathrm{s}_t\, \mathrm{s}_t \;=\; 0\,, \qquad \mathrm{s}_t\, \mathrm{d} + \mathrm{d}\, \mathrm{s}_t \;=\; 0\,.
\tag{99}
$$

follows precisely as in Lem. 2.24.

Remark 2.65 (Rationalizing topological spherification). The case $t = 1$ in Def. 2.59 recovers the standard cyclification (Def. 2.23), which may be seen [VPB85][BMSS19][SV23][SV24] as the rationalization of a loop space homotopy-quotiented by loop rotation, namely for $\mathfrak{h} \cong \mathfrak{l}X$

$$
\mathrm{cyc}(\mathfrak{h}) \;\cong\; \mathfrak{l}\big(\,[S^1, X]/\!/S^1\,\big)\,.
$$

We expect the proof of [BMSS19] should hold for the $t = 2$ case with minor modifications, where the odd 3-sphere $S^3 \cong SU(2)$ still has proper group-structure, so that

$$
\mathrm{cyc}_3(\mathfrak{h}) \;\cong\; \mathfrak{l}\big(\,[S^3, X]/\!/S^3\,\big)\,.
$$

For $t \geq 3$, this pattern obviously breaks down since the higher odd topological spheres from S^7 onwards admit no group structure. Nevertheless, the rational higher cyclification (Def. 2.64) still makes complete sense, and so does the corresponding higher version of the Ext/Cyc adjunction of Prop. 2.25.

Proposition 2.66 (The Higher Ext/Cyc-adjunction). *Given $\mathfrak{g}, \mathfrak{h} \in \mathrm{sLieAlg}_\infty$ with a 2t-cocycle $\widehat{\omega}_{2t} \in \mathrm{CE}(\mathfrak{g})$, there is a bijection between:*

(i) *maps into $\mathfrak{h}$ out of the higher central extension $\widehat{\mathfrak{g}}$ classified by the 2t-cocycle* (Def. 2.59),
(ii) *maps out of $\mathfrak{g}$ into the higher cyclification of $\mathfrak{h}$* (Def. 2.64) *that preserve the 2t-cocycle:*

$$\left\{ \widehat{\mathfrak{g}} \xrightarrow{\;f\;} \mathfrak{h} \right\} \underset{\textbf{oxidation}\;\; \mathrm{oxd}_{\widehat{\omega}_{2t}}}{\overset{\textbf{reduction}\;\; \mathrm{rdc}_{\widehat{\omega}_{2t}}}{\rightleftarrows}} \left\{ \begin{tikzcd} \mathfrak{g} \arrow[rr, "\widetilde{f}"] \arrow[dr, "\widehat{\omega}_{2t}"'] & & \mathrm{cyc}_{2t-1}(\mathfrak{h}) \arrow[dl, "\omega_{2t}"] \\ & b^{2t-1}\mathbb{R} & \end{tikzcd} \right\} \tag{100}$$

given by

$$\begin{array}{ccc} \widehat{\mathfrak{g}} & \xrightarrow{\;f\;} & \mathfrak{h} \\ \alpha^i_{\mathrm{bas}} + b_{2t-1}\, p_* \alpha^i & \longleftarrow & e^i \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;\widetilde{f}\;} & \mathrm{cyc}(\mathfrak{h}) \\ \alpha^i_{\mathrm{bas}} & \longleftarrow & e^i \\ -p_*\alpha^i & \longleftarrow & \mathrm{s}_t e^i \\ \widehat{\omega}_{2t} & \longleftarrow & \omega_{2t}\,. \end{array} \tag{101}$$

Proof. The proof follows essentially verbatim as in that of Prop. 2.25, by adjusting the degrees of the even cocycle and extra odd generator accordingly. □

Example 2.67 (**Higher cyclification of higher bundle gerbe classifying space).** The higher cyclifications (Def. 2.64) of the real Whitehead L_∞-algebra of $B^{2t}\mathrm{U}(1)$ (Ex. 2.6), $\mathrm{cyc}_{2t-1} b^{2t}\mathbb{R}$, is given by

$$\mathrm{CE}\big(\mathrm{cyc}_{2t-1} b^{2t}\mathbb{R}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} \omega_{2t} \\ \omega_{4t-1} \\ \widetilde{\omega}_{2t} := \mathrm{s}_t \omega_{4t-1} \end{array} \right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\omega_{2t} & = 0 \\ \mathrm{d}\,\omega_{4t-1} & = \omega_{2t}\,\widetilde{\omega}_{2t} \\ \mathrm{d}\,\widetilde{\omega}_{2t} & = 0 \end{array} \right)$$

being equivalently the higher central extension (Def. 2.59) of $b^{2t-1}\mathbb{R}^2$ by its canonical $4t$-cocycle

$$b\mathcal{T}_t \;:=\; \mathrm{cyc}_{2t-1} b^{2t}\mathbb{R} \xrightarrow{\;\mathrm{hofib}(\omega_{2t}\,\widetilde{\omega}_{2t})\;} b^{2t-1}\mathbb{R}^2 \xrightarrow{\;\omega_{2t}\,\widetilde{\omega}_{2t}\;} b^{4t-1}\mathbb{R} \tag{102}$$

and as such may be called (the Whitehead L_∞-algebra of) the delooping of the *higher T-duality Lie group* (cf. [FSS18a, Def. 3.14]):

$$\mathrm{cyc}\,\mathfrak{l} B^{2t}\mathrm{U}(1) \;\simeq\; \mathfrak{l}\Big(\mathrm{hofib}\big(B^{2t-1}\mathrm{U}(1) \times B^{2t-1}\mathrm{U}(1) \xrightarrow{\;\pi_1\omega_{2t} \cup \pi_2\omega_{2t}\;} B^{4t-1}\mathrm{U}(1)\big)\Big)\,.$$

Similarly to the $t = 1$ case, it is evident that (102) has an automorphism symmetry given by exchanging the two degree$= 2t$ generators (we again include a minus sign, for compatibility below in Ex. 2.68):

$$\begin{array}{ccc} b\mathcal{T}_t & \overset{\sim}{\longleftrightarrow} & b\mathcal{T}_t \\ -\widetilde{\omega}_{2t} & \longleftarrow & \omega_{2t} \\ -\omega_{2t} & \longleftarrow & \widetilde{\omega}_{2t}\,. \end{array} \tag{103}$$

This already carries in it the seed of *higher* T-duality, with the next example lifting this automorphism to an equivalence between the higher $(2t-1)$-cyclifications of the $(4t-1)$-twisted cocycle classifying L_∞-algebras from Ex. 2.15, generalizing the situation of Ex. 2.28.

Example 2.68 (**Higher cyclification of higher twisted cocycle classifying L_∞-algebras and higher T-duality group).** The $(2t-1)$-cyclifications (Def. 2.64) of the $(4t-1)$-twisted, $(4t-2)$-periodic cocycle classifying L_∞-algebras from Ex. 2.15 are identified by an isomorphism (14). In order to ease the notation below, we abbreviate

$n_t := 2t - 1$:

$$
\begin{array}{l}
\mathrm{CE}\Big(\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^m \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)\Big)
\\[6pt]
\simeq\;\;
\mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\omega_{n_t+1}
\\
h_{2n_t+1}
\\
\mathrm{s}_t h_{2n_t+1}
\\
\big(f_{2kn_t+m}\big)_{k\in\mathbb{Z}}
\\
\big(\mathrm{s}_t f_{2kn_t+m}\big)_{k\in\mathbb{Z}}
\end{array}
\right]
\Big/
\left(
\begin{array}{rcl}
\mathrm{d}\,\omega_{n_t+1} &=& 0
\\
\mathrm{d}\,h_{2n_t+1} &=& \omega_{n_t+1}\,\mathrm{s}_t h_{2n_t+1}
\\
\mathrm{d}\,\mathrm{s}_t h_{2n_t+1} &=& 0
\\
\mathrm{d}\,f_{2(k+1)n_t+m} &=& h_{2n_t+1}\,f_{2kn_t+m} + \omega_{n_t+1}\,\mathrm{s}_t f_{2(k+1)n_t+m}
\\
\mathrm{d}\,\mathrm{s}_t f_{2(k+1)n_t+m} &=& -(\mathrm{s}_t h_{2n_t+1}) f_{2kn_t+m} + h_{2n_t+1}\,\mathrm{s}_t f_{2kn_t+m}
\end{array}
\right)
\\[6pt]
\qquad
\begin{array}{ccccc}
\omega_{n_t+1} & \mathrm{s}_t h_{2n_t+1} & h_{2n_t+1} & f_{2kn_t+m} & \mathrm{s}_t f_{(2k+1)n_t+m}
\\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\\
-\mathrm{s}_t h_{2n_t+1} & -\omega_{n_t+1} & h_{2n_t+1} & \mathrm{s}_t f_{2(k+1)n_t+m} & f_{2kn_t+m}
\end{array}
\qquad \Big\uparrow \wr
\\[6pt]
\;\;
\mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\omega_{n_t+1}
\\
h_{2n_t+1}
\\
\mathrm{s}_t h_{2n_t+1}
\\
\big(f_{(2k-1)n_t+m}\big)_{k\in\mathbb{Z}}
\\
\big(\mathrm{s}_t f_{(2k-1)n_t+m}\big)_{k\in\mathbb{Z}}
\end{array}
\right]
\Big/
\left(
\begin{array}{rcl}
\mathrm{d}\,\omega_{n_t+1} &=& 0
\\
\mathrm{d}\,h_{2n_t+1} &=& \omega_{n_t+1}\,\mathrm{s}_t h_{2n_t+1}
\\
\mathrm{d}\,\mathrm{s}_t h_{2n_t+1} &=& 0
\\
\mathrm{d}\,f_{(2k+1)n_t+m} &=& h_{2n_t+1}\,f_{(2k-1)n_t+m} + \omega_{n_t+1}\,\mathrm{s}_t f_{(2k+1)n_t+m}
\\
\mathrm{d}\,\mathrm{s}_t f_{(2k+1)n_t+m} &=& -(\mathrm{s}_t h_{2n_t+1}) f_{(2k-1)n_t+m} + h_{2n_t+1}\,\mathrm{s}_t f_{(2k-1)n_t+m}
\end{array}
\right)
\\[6pt]
\simeq\;\;
\mathrm{CE}\Big(\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^{m-n_t} \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)\Big)
\end{array}
\tag{104}
$$

compatible with their fibration (31) over $b\mathcal{T}_t \cong \mathrm{cyc}_{n_t} b^{n_t+1}\mathbb{R}$ via its automorphisms (103), where the homotopy fiber of the cyclified fibration is now the direct sum of $2n_t$-periodic cocycle spectra in degrees m and $(m - n_t)$, respectively, with the isomorphism acting by swapping their order:

$$
\begin{tikzcd}
\begin{array}{c}\mathfrak{l}\Sigma^{m}\mathrm{K}^{n_t}\mathrm{U} \\ \times \\ \mathfrak{l}\Sigma^{m-n_t}\mathrm{K}^{n_t}\mathrm{U}\end{array}
\arrow[r, "\text{swap}"]
\arrow[d]
&
\begin{array}{c}\mathfrak{l}\Sigma^{m-n_t}\mathrm{K}^{n_t}\mathrm{U} \\ \times \\ \mathfrak{l}\Sigma^{m}\mathrm{K}^{n_t}\mathrm{U}\end{array}
\arrow[d]
\\
\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{m}\mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\big)
\arrow[r, leftrightarrow, "\sim"]
\arrow[d, ->>]
&
\mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{m-n_t}\mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\big)
\arrow[d, ->>]
\\
b\mathcal{T}_t
\arrow[r, leftrightarrow, "\sim"]
&
b\mathcal{T}_t\,.
\end{tikzcd}
\tag{105}
$$

Lemma 2.69 (All isomorphisms of higher cyclified twisted cocycle classifying L_∞-algebras). *In analogy to Lem. 2.29, there exist in total 4 isomorphisms*

$$
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^m \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)
\xleftrightarrow{\;\sim\;}
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^{m-n_t} \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)
$$

with the property of swapping $\mathrm{s}h_{2n_t+1} \leftrightsquigarrow \omega_{2n_t}$ *and* $f_{2kn_t+m} \leftrightsquigarrow \mathrm{s}f_{2(k+1)n_t+m}$*, while mapping* h_{2n_t+1} *to* h_{2n_t+1}*, up to relative sign prefactors. Explicitly, the extra 3 isomorphisms are given by*

$$
\begin{array}{ccc}
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^m \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)
&\xleftrightarrow{\;\sim\;}&
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\,\Sigma^{m-n_t} \mathrm{K}^{n_t}\mathrm{U}/\!\!/ B^{2n_t-1}\mathrm{U}(1)\,\big)
\\
h_{2n_t+1} &\longleftarrow\!\shortmid & h_{2n_t+1}
\\
-\mathrm{s}_t h_{2n_t+1} &\longleftarrow\!\shortmid & \omega_{n_t+1}
\\
-\omega_{n_t+1} &\longleftarrow\!\shortmid & \mathrm{s}_t h_{2n_t+1}
\\
-\mathrm{s}_t f_{(2k+1)n_t+m} &\longleftarrow\!\shortmid & f_{2kn_t+m}
\\
-f_{2kn_t+m} &\longleftarrow\!\shortmid & \mathrm{s}_t f_{2(k+1)n_t+m}\,,
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\Sigma^m \mathrm{K}^{n_t}\mathrm{U}/\!\!/B^{2n_t-1}\mathrm{U}(1)\big) & \xleftrightarrow{\;\sim\;} & \mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\Sigma^{\color{purple}m-n_t}\mathrm{K}^{n_t}\mathrm{U}/\!\!/B^{2n_t-1}\mathrm{U}(1)\big)
\\
h_{2n_t+1} & \longleftarrow\!\shortmid & h_{2n_t+1}
\\
\mathrm{s}_t h_{2n_t+1} & \longleftarrow\!\shortmid & \omega_{n_t+1}
\\
\omega_{n_t+1} & \longleftarrow\!\shortmid & \mathrm{s}_t h_{2n_t+1}
\\
-\mathrm{s}_t f_{(2k+1)n_t+m} & \longleftarrow\!\shortmid & f_{2kn_t+m}
\\
f_{2kn_t+m} & \longleftarrow\!\shortmid & \mathrm{s}_t f_{2(k+1)n_t+m}\,,
\end{array}
\tag{106}
$$

and

$$
\begin{array}{ccc}
\mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\Sigma^m \mathrm{K}^{n_t}\mathrm{U}/\!\!/B^{2n_t-1}\mathrm{U}(1)\big) & \xleftrightarrow{\;\sim\;} & \mathrm{cyc}_{n_t}\,\mathfrak{l}\big(\Sigma^{\color{purple}m-n_t}\mathrm{K}^{n_t}\mathrm{U}/\!\!/B^{2n_t-1}\mathrm{U}(1)\big)
\\
h_{2n_t+1} & \longleftarrow\!\shortmid & h_{2n_t+1}
\\
\mathrm{s}_t h_{2n_t+1} & \longleftarrow\!\shortmid & \omega_{n_t+1}
\\
\omega_{n_t+1} & \longleftarrow\!\shortmid & \mathrm{s}_t h_{2n_t+1}
\\
\mathrm{s}_t f_{(2k+1)n_t+m} & \longleftarrow\!\shortmid & f_{2kn_t+m}
\\
-f_{2kn_t+m} & \longleftarrow\!\shortmid & \mathrm{s}_t f_{2(k+1)n_t+m}\,.
\end{array}
$$

Evidently, the original isomorphism (104) and the first above are the two possible extensions of the automorphism (103) of $b\mathcal{T}_t$, while the latter two isomorphisms are the two possible extensions of the "opposite" automorphism of $b\mathcal{T}_t$

$$
\begin{array}{ccc}
b\mathcal{T}_t & \xleftrightarrow{\;\sim\;} & b\mathcal{T}_t
\\
\mathrm{s}h_{2t+1} & \longleftarrow\!\shortmid & \omega_{2t}
\\
\omega_{2t} & \longleftarrow\!\shortmid & \mathrm{s}h_{2t+1}\,.
\end{array}
$$

Proof. By direct inspection, completely analogously to that of Lem. 2.29. ☐

With the higher cyclification and with the above isomorphism in hand, we obtain a higher generalized template for T-dualization along higher dimensional odd (rational) spheres, generalizing directly the standard case from Lem. 2.30.

Lemma 2.70 (Higher twisted cocycles under reduction–isomorphism–reoxidation).
(i) *The composite operation of*

(a) *reducing (100) twisted $\mathfrak{l}\Sigma^{\color{purple}m}\mathrm{K}^{n_t}\mathrm{U}$ cocycles on a higher centrally extended super L_∞-algebra $\mathfrak{g}_A$*

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_A & \longrightarrow & \mathfrak{l}\big(\Sigma^{\color{purple}m}\mathrm{K}^{n_t}U/\!\!/B^{2n_t-1}\mathrm{U}(1)\big)
\\
H_A^{2n_t+1} & \longleftarrow\!\shortmid & h_{2n_t+1}
\\
(F_{2kn_t+m})_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{2kn_t+m})_{k\in\mathbb{Z}}
\end{array}
$$

along its fibration

$$
\widehat{\mathfrak{g}}_A \xrightarrow{\;p_A\;} \mathfrak{g} \xrightarrow{\;\omega^A_{n_t+1}\;} b^{n_t}\mathbb{R}\,,
$$

(b) *applying the isomorphism (104) on the target cyclification of $\mathfrak{l}\big(\Sigma^{\color{purple}m}\mathrm{K}^{n_t}U/\!\!/B^{2n_t-1}\mathrm{U}(1)\big)$, hence viewing them instead as valued in the cyclification of $\mathfrak{l}\big(\Sigma^{\color{purple}m-n_t}\mathrm{K}^{n_t}U/\!\!/B^{2n_t-1}\mathrm{U}(1)\big)$, while noticing that this swaps the higher cocycle $\omega^A_{n_t+1}$ from that classifying the $\widehat{\mathfrak{g}}_A$-extension to that classifying a different extension*

$$
\widehat{\mathfrak{g}}_B \longrightarrow \mathfrak{g}\,,
$$

i.e., via

$$
\omega^B_{n_t+1} := p_{A*}(H_A^{2n_t+1}) \quad : \quad \mathfrak{g} \longrightarrow b^{n_t}\mathbb{R}\,,
$$

(c) *re-oxidizing (100) the result, but now along the new fibration*

$$
\widehat{\mathfrak{g}}_B \xrightarrow{\;p_A\;} \mathfrak{g} \xrightarrow{\;\omega^B_{n_t+1}\;} b^{n_t}\mathbb{R}\,,
$$

results in the twisted $\mathfrak{l}\Sigma^{\color{purple}m-n_t}\mathrm{K}^{n_t}\mathrm{U}$ cocycles given precisely by

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \longrightarrow & \mathfrak{l}\big(\Sigma^{\color{purple}m-n_t}\mathrm{K}^{n_t}U/\!\!/B^{2n_t-1}\mathrm{U}(1)\big).
\\
H^{2n_t+1}_{A\,\mathrm{bas}} + b_B^{n_t}\cdot\omega^A_{n_t+1} & \longleftarrow\!\shortmid & h_{2n_t+1}
\\
\big(-p_{A*}F_{2kn_t+m} - b_B^{n_t}\cdot(F_{2(k-1)n_t+m})_{\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{(2k-1)n_t+m})_{k\in\mathbb{Z}}
\end{array}
\tag{107}
$$

(ii) *Applying instead one of the isomorphisms from Lem. 106 in step* **(b)** *yields similar, but essentially different maps between higher twisted cocycles of different extensions over* $\mathfrak{g}$. *For instance, using the isomorphism* (106) *results in the twisted* $\mathfrak{l}\Sigma^{m-n_t}\mathrm{K}^{n_t}\mathrm{U}$ *cocycles given by*

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_B & \xrightarrow{\qquad\qquad\qquad\qquad\qquad\qquad} & \mathfrak{l}\big(\Sigma^{m-n_t}\mathrm{K}^{n_t}U /\!\!/ B^{2n_t-1}\mathrm{U}(1)\big) \\
H^{2n_t+1}_{A\,\mathrm{bas}} - b^{n_t}_{B'} \cdot \omega^A_{n_t+1} & \longleftarrow\!\shortmid & h_{2n_t+1} \\
\big(+p_{A*}F_{2kn_t+m} - b^{n_t}_{B'} \cdot (F_{2(k-1)n_t+m})_{\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow\!\shortmid & (f_{(2k-1)n_t+m})_{k\in\mathbb{Z}}
\end{array}
$$

where now the extension

$$\widehat{\mathfrak{g}}_{B'} \longrightarrow \mathfrak{g}\,,$$

is instead via the opposite (n_t+1)*-cocycle*

$$\omega^{B'}_{n_t+1} := -p_{A*}(H^{2n_t+1}_A)\,.$$

Proof. The formulas have been set up such that the proof follows essentially verbatim with the proof of Lem. 2.30, by adjusting the degrees accordingly. ☐

Remark 2.71 (**Towards Higher T-duality.**)**.** In direct generalization of Lem. 2.30, the composite operation of Lem. 2.70 swaps the extending cocycle $\omega^A_{n_t+1}$ with the n_t-dimensional reduction $\omega^B_{n_t+1} = p_{A*}H^{2n_t+1}_A$ of the original twisting cocycle. At the same time, it swaps "wrapped and non-wrapped modes" of the corresponding would be "higher" fluxes, now along the corresponding higher odd n_t-extensions. This should plausibly capture rational aspects of the generalized cohomology perspective in [LSW16].

Remark 2.72 (**Reductions along products of higher spheres**)**.** At this point, it is straightforward to develop an analogous story along the lines of §2.3 using the notion of a "higher toroidification". That is, there is an immediate generalization of reducing and oxidifying along products of rational odd spheres (cf. Rem. 2.45) and a corresponding automorphism of higher toroidified twisted cocycle classifying spaces (cf. Prop. 2.56). We leave this potentially interesting extension for future works.

3 Superspace T-Duality

We discuss how the abstract L_∞-algebraic T-duality of §2.2 (Lem. ??) is realized (§3.2) on the super-invariant super-flux densities intrinsically carried by the 10D type IIA/B super-spacetimes fibered over 9D super-spacetime (§3.1).

Then in §3.3 we consider the analogous discussion of n-toroidal L_∞-algebraic T-duality (from §2.3) realizing it as the full 1+9D reduction of type IIA super-space to the (super-)point, and we observe that the resulting Poincaré super 2-form on 10D-doubled super-space lifts to a 3-form on the "M-algebra".

3.1 Super-space-times

We discuss the translational supersymmetry algebras (Ex. 2.3) for $D = 10$ and of "type II", in terms of the algebraic data provided by the $D = 11$ supersymmetry algebra. This is immediate for type IIA, but for type IIB it requires a little bit of finesse. With that in hand, though, the superspace T-duality in §3.2 flows very naturally.

Dimensional reduction of 11d super Minkowski spacetime. Consider the projection operators

$$\left.\begin{array}{rcl} P & := & \tfrac{1}{2}\big(1+\Gamma_{10}\big) \\ \overline{P} & := & \tfrac{1}{2}\big(1-\Gamma_{10}\big) \end{array}\right\} \in \mathrm{End}_{\mathbb{R}}(\mathbf{32}) \tag{108}$$

satisfying the following immediate but consequential relations:

$$\begin{array}{rcl} P\,P &=& P, \\ \overline{P}\,\overline{P} &=& \overline{P}, \\ P\,\overline{P} &=& 0, \\ \overline{P}\,P &=& 0, \end{array} \qquad \begin{array}{rcccl} \Gamma_{\leq 9}\,P &=& \overline{P}\,\Gamma_{\leq 9}, && \\ \Gamma_{\leq 9}\,\overline{P} &=& P\,\Gamma_{\leq 9}, && \\ \Gamma_{10}\,P &=& P\,\Gamma_{10} &=& +P, \\ \Gamma_{10}\,\overline{P} &=& \overline{P}\,\Gamma_{10} &=& -\overline{P}, \end{array} \qquad P+\overline{P} = \mathrm{id}, \qquad \begin{array}{rcl} \overline{P\psi} &=& \overline{\psi}\,\overline{P}, \\ \overline{\overline{P}\psi} &=& \overline{\psi}\,P\,. \end{array} \tag{109}$$

Canonically identifying actions of spin subgroups on $\mathbf{32}$

$$\mathrm{Spin}(1,8) \hookrightarrow \mathrm{Spin}(1,9) \hookrightarrow \mathrm{Spin}(1,10)$$

by restriction of the Clifford algebra to products of its first $1+d$ generators $\Gamma_0, \Gamma_1, \cdots \Gamma_d$, the projectors (108) carve out two $\mathrm{Spin}(1,9)$-representations from the $\mathrm{Spin}(1,10)$-rep $\mathbf{32}$:

$$\begin{array}{rcccl} \mathbf{16} &:=& P(\mathbf{32}) &\in& \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,9)\big) \\ \overline{\mathbf{16}} &:=& \overline{P}(\mathbf{32}) &\in& \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,9)\big)\,, \end{array} \tag{110}$$

and hence we have a branching of representations of this form:

$$\begin{array}{ccc} \mathrm{Spin}(1,10) & \hookleftarrow & \mathrm{Spin}(1,9) \\ \mathbf{32} & \longrightarrow & \mathbf{16} \oplus \overline{\mathbf{16}} \\ \psi & \longmapsto & \underbrace{P\psi}_{\psi_1} + \underbrace{\overline{P}\psi}_{\psi_2}\,. \end{array} \tag{111}$$

Under this branching and decomposition ($\psi = \psi_1 + \psi_2 = P(\psi_1) + \overline{P}(\psi_2)$), the spinor pairing $\big(\overline{(-)}(-)\big)$ on $\mathbf{32}$ translates into pairings on $\mathbf{16} = P(\mathbf{32})$ and on $\overline{\mathbf{16}} = \overline{P}(\mathbf{32})$, by evaluating on pairs of spinors belonging in each of the projected subspaces respectively. In terms of the two projected representations the vector-valued spinor pairing $\big(\overline{(-)}\Gamma(-)\big)$ decomposes as follows:

Lemma 3.1 (**Decomposed vectorial spinor pairing**).

$$\begin{array}{rcll} \big(\overline{\psi}\,\Gamma_a\,\phi\big) &=& \big(\overline{\psi}_1\,\Gamma_a\,\phi_1\big) + \big(\overline{\psi}_2\,\Gamma_a\,\phi_2\big)\,, & \text{for } a \neq 10, \\ \big(\overline{\psi}\,\Gamma_{10}\,\phi\big) &=& \big(\overline{\psi}_2\,\phi_1\big) - \big(\overline{\psi}_1\,\phi_2\big). & \end{array} \tag{112}$$

Proof. The first line in (112) follows since the mixed terms vanish due to the decomposition (111) and using the relations (109):

$$\begin{array}{rcl} \big(\overline{\psi}_1\,\Gamma_a\,\phi_2\big) &=& \big(\overline{P\psi_1}\,\Gamma_a\,\overline{P}\phi_2\big) \\ &=& \big(\overline{\psi}_1\,\overline{P}\,\Gamma_a\,\overline{P}\phi_2\big) \\ &=& \big(\overline{\psi}_1\,\Gamma_a\,P\overline{P}\phi_2\big) \\ &=& 0\,, \end{array} \qquad \begin{array}{rcl} \big(\overline{\psi}_2\,\Gamma_a\,\phi_1\big) &=& \big(\overline{\overline{P}\psi_2}\,\Gamma_a\,P\phi_1\big) \\ &=& \big(\overline{\psi}_1\,P\,\Gamma_a\,P\phi_2\big) \\ &=& \big(\overline{\psi}_1\,\Gamma_a\,\overline{P}P\phi_2\big) \\ &=& 0\,, \end{array} \qquad \text{for } a \neq 10.$$

Similarly but complementarily, for the second line these relations give:

$$
\begin{aligned}
\big(\overline{\psi}\,\Gamma_{10}\,\phi\big) & = \big(\overline{\psi}_1\,\Gamma_{10}\,\phi_2\big) + \big(\overline{\psi}_2\,\Gamma_{10}\,\phi_1\big) + \big(\overline{\psi}_1\,\Gamma_{10}\,\phi_1\big) + \big(\overline{\psi}_2\,\Gamma_{10}\,\phi_2\big)
\\
& = \big(\overline{P\psi_1}\,\Gamma_{10}\,\overline{P}\phi_2\big) + \big(\overline{\overline{P}\psi_2}\,\Gamma_{10}\,P\phi_1\big) + \big(\overline{P\psi_1}\,\Gamma_{10}\,P\phi_1\big) + \big(\overline{\overline{P}\psi_2}\,\Gamma_{10}\,\overline{P}\phi_2\big)
\\
& = \big(\overline{\psi_1}\,\overline{P}\Gamma_{10}\,\overline{P}\phi_2\big) + \big(\overline{\psi_2}\,P\Gamma_{10}\,P\phi_1\big) + \big(\overline{\psi_1}\,\overline{P}\Gamma_{10}\,P\phi_1\big) + \big(\overline{\psi_2}\,P\Gamma_{10}\,\overline{P}\phi_2\big)
\\
& = -\big(\overline{\psi_1}\,\overline{P}\,\overline{P}\phi_2\big) + \big(\overline{\psi_2}\,PP\phi_1\big) + \big(\overline{\psi_1}\,\overline{P}P\phi_1\big) + \big(\overline{\psi_2}\,P\overline{P}\phi_2\big)
\\
& = -\big(\overline{\psi_1}\,\phi_2\big) + \big(\overline{\psi_2}\,\phi_1\big).
\end{aligned}
$$

□

This gives:

Example 3.2 (**The M/IIA super-space extension**). We have a $\mathrm{Spin}(1,9)$-equivariant isomorphism

$$
\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \big(\psi_1^\alpha\big)_{\alpha=1}^{16}, \\ \big(\psi_2^\alpha\big)_{\alpha=1}^{16}, \\ \big(e^a\big)_{a=0}^{9}, \\ e^{10} \end{array}\right] \Big/ \left(\begin{array}{rcll} \mathrm{d}\,\psi_1 & = & 0 & \\ \mathrm{d}\,\psi_2 & = & 0 & \\ \mathrm{d}\,e^a & = & \big(\overline{\psi}_1\,\Gamma_a\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma_a\,\psi_2\big) & \text{for } a \neq 10 \\ \mathrm{d}\,e^{10} & = & \underbrace{\big(\overline{\psi}_2\,\psi_1\big) - \big(\overline{\psi}_1\,\psi_2\big)}_{\big(\overline{\psi}\,\Gamma_{10}\,\psi\big)} & \end{array}\right), \tag{113}
$$

which exhibits $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ as a central extension (Def. 2.21) of $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$:

$$
\underset{\substack{\text{D = 11} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,10|\mathbf{32}}} \xrightarrow[\text{extension of}]{\text{hofib}} \underset{\substack{\text{D = 10 type IIA} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \xrightarrow[\text{classified by first Chern class}]{c_1^M \,:=\, (\overline{\psi}\,\Gamma_{10}\,\psi)} b\mathbb{R}\,. \tag{114}
$$

Reducing one dimension further, the two $\mathrm{Spin}(1,9)$-reps (110) in turn become isomorphic when restricted to $\mathrm{Spin}(1,8)$-representations, the isomorphism given by acting with Γ_9:

$$
\begin{tikzcd}
\mathbf{16} \;\equiv\; P(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[r, "\Gamma_9", "\sim"'] \arrow[d, "\Gamma_{ab}"'] & \overline{P}(\mathbf{32}) \;\equiv\; \overline{\mathbf{16}} \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[d, "\Gamma_{ab}"'] \\
P(\mathbf{32}) \arrow[r, "\Gamma_9", "\sim"'] & \overline{P}(\mathbf{32})
\end{tikzcd}
\qquad a, b \leq 8. \tag{115}
$$

This gives:

Example 3.3 (**The IIA/9D super-space extension**). We have a $\mathrm{Spin}(1,8)$-equivariant isomorphism

$$
\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi_1^\alpha)_{\alpha=1}^{16}, \\ (\psi_2^\alpha)_{\alpha=1}^{16}, \\ (e^a)_{a=0}^{8}, \\ e^9 \end{array}\right] \Big/ \left(\begin{array}{rcll} \mathrm{d}\,\psi_1 & = & 0 & \\ \mathrm{d}\,\psi_2 & = & 0 & \\ \mathrm{d}\,e^a & = & \big(\overline{\psi}_1\,\Gamma^a\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma^a\,\psi_2\big) & \text{for } a \neq 9 \\ \mathrm{d}\,e^9 & = & \underbrace{\big(\overline{\psi}_1\,\Gamma^9\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma^9\,\psi_2\big)}_{\big(\overline{\psi}\,\Gamma^9\,\psi\big)} & \end{array}\right)
$$

which exhibits $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ as a central extension (Def. 2.21) of $\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}$ by a 2-cocycle to be denoted c_1^A (cf. footnote 10):

$$
\underset{\substack{\text{D = 10 type IIA} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \xrightarrow[\text{extension of}]{\text{hofib}} \underset{\substack{\text{D = 9 type II} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}} \xrightarrow[\text{classified by first Chern class}]{c_1^A \,:=\, (\overline{\psi}\,\Gamma_9\,\psi)} b\mathbb{R}\,. \tag{116}
$$

The type IIB super-spacetime. We need a presentation of the type IIB super-spacetime analogous to the IIA case above, namely expressed in terms of the 11d spinors in $\mathbf{32}$ and their spinor pairing $\big(\overline{(-)}(-)\big) : \mathbf{32}\otimes\mathbf{32} \to \mathbb{R}$. However, since along $\mathfrak{so}_{1,9} \hookrightarrow \mathfrak{so}_{1,10}$ this representation branches as $\mathbf{32} \mapsto \mathbf{16}\oplus\overline{\mathbf{16}}$ (111) we have to re-express the given $\overline{\mathbf{16}}$ as a $\mathbf{16}$.

Observe that this may be achieved by "compensating with a group automorphism": The same diagram (115) which shows how $P(\mathbf{32})$ and $\overline{P}(\mathbf{32})$ are isomorphic as Spin(1, 8)-representations also shows which transformation on the Lie algebra generators occurs when comparing them as Spin(1, 9)-representations, namely the Lorentz-generators with an index=9 pick up a minus sign:

$$
\begin{tikzcd}
\mathbf{16} \equiv & P(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[r, "\Gamma_9", "\sim"'] \arrow[d, shift right, "\Gamma_{ab}"'] \arrow[d, shift left, "\Gamma_{a9}"] & \overline{P}(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[d, shift right, "\Gamma_{ab}"'] \arrow[d, shift left, "-\Gamma_{a9}"] & \equiv \overline{\mathbf{16}} \\
& P(\mathbf{32}) \arrow[r, "\Gamma_9", "\sim"'] & \overline{P}(\mathbf{32})\,.
\end{tikzcd}
\qquad a, b \leq 8 \tag{117}
$$

This means that the $\mathbf{16}$ of Spin(1, 9) is the pullback of $\overline{\mathbf{16}}$ along the group homomorphism which on Lie algebras is given by this sign change (115):

$$
\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\sim"] \arrow[rr, bend left, "\mathbf{16}"] & \mathfrak{so}_{1,9} \arrow[r, "\overline{\mathbf{16}}"] & \mathfrak{gl}_{16} \\
J_{ab<9} \arrow[r, mapsto] & J_{ab} \arrow[r, mapsto] & \tfrac{1}{2}\Gamma_{ab}\vert_{\overline{P}(\mathbf{32})} \\
J_{a9} \arrow[r, mapsto] & -J_{a9} \arrow[r, mapsto] & -\tfrac{1}{2}\Gamma_{a9}\vert_{\overline{P}(\mathbf{32})} \\
J_{ab} \arrow[rr, mapsto] & & \tfrac{1}{2}\Gamma^B_{ab}\vert_{\overline{P}(\mathbf{32})}\,.
\end{tikzcd}
\qquad (a, b < 9) \tag{118}
$$

Here in the last line we have summarized this situation by introducing the following notation – recalling that our undecorated "Γ_a" are always those of the 11d Clifford algebra (208):

$$
\begin{array}{rcll}
\Gamma^B_a & := & \Gamma_a & \text{for } a < 9 \\
\Gamma^B_9 & := & \Gamma_9 \Gamma_{10} & \\
\Gamma^B_{ab} & := & \Gamma^B_a \Gamma^B_b & \text{for } a < b \leq 9 \\
\Gamma^B_{ba} & := & -\Gamma^B_a \Gamma^B_b & \text{for } a < b \leq 9\,,
\end{array}
\tag{119}
$$

which works since $\Gamma_{10}\vert_{\overline{P}(\mathbf{32})} = -\mathrm{id}$, by (109). But by the same relation also $\Gamma_{10}\vert_{P(\mathbf{32})} = +\mathrm{id}$, so that the Γ^B operators reduce to the original Clifford generators Γ (208) when restricted on $\mathbf{16} \equiv P(\mathbf{32})$ and hence encoding also a copy of the original representation

$$
\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\mathbf{16}"] & \mathfrak{gl}_{16} \\
J_{ab} \arrow[r, mapsto] & \tfrac{1}{2}\Gamma^B_{ab}\vert_{P(\mathbf{32})} \equiv \tfrac{1}{2}\Gamma_{ab}\vert_{P(\mathbf{32})}
\end{tikzcd}
$$

In total we produced the type IIB spinor representation $\mathbf{16} \oplus \mathbf{16}$ of Spin(1, 9) as a pullback of the type IIA spinor representation $\mathbf{16} \oplus \overline{\mathbf{16}}$, in terms of the 11d Clifford algebra expression (119) as:

$$
\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\sim"] \arrow[rr, bend left, "\mathbf{16}\oplus\mathbf{16}"] & \mathfrak{so}_{1,9} \arrow[r, "\mathbf{16}\oplus\overline{\mathbf{16}}"] & \mathfrak{gl}_{32} \\
J_{ab} \arrow[rr, mapsto] & & \tfrac{1}{2}\Gamma^B_{ab}
\end{tikzcd}
\qquad (a < b \leq 9) \tag{120}
$$

It is this transformation which turns out to make manifest superspace T-duality below.

Remark 3.4 (**Subtleties with type IIB Clifford elements.**).
(i) Beware that the operators Γ^B in (119) do not generate a Clifford algebra (and not a Pin(1, 9)-group), but they do generate the correct Spin(1, 9)-group and -representation.
(ii) As we will see in a moment, this defect is in a sense compensated by another defect, namely that Γ^B_9 is not skew-self-adjoint as the other Clifford generators (220):

$$
\overline{\Gamma^B_{ab}} = \begin{cases} -\Gamma^B_{ab} & \text{for } a, b < 9 \\ +\Gamma^B_{ab} & \text{if } b = 9\,, \end{cases} \tag{121}
$$

where the second line comes about as

$$
\overline{\Gamma^B_{a9}} \equiv \overline{\Gamma_a \Gamma_9 \Gamma_{10}} = \overline{\Gamma_{10}}\,\overline{\Gamma_9}\,\overline{\Gamma_a} = (-1)^3 \Gamma_{10}\Gamma_9\Gamma_a = (-1)^{3+1}\Gamma_a\Gamma_9\Gamma_{10} = \Gamma^B_{a9} \quad \text{for } a < 9\,.
$$

With this notation, we may set:

Definition 3.5 (**Type IIB super-Minkowski super-Lie algebra.**). The 10d type IIB super-Minkowski Lie algebra is given by

$$\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big) \;=\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\psi_1^\alpha)_{\alpha=1}^{16},\\ (\psi_2^\alpha)_{\alpha=1}^{16},\\ (e^a)_{a=0}^{9}\end{array}\right] \Big/ \left(\begin{array}{lcl}\mathrm{d}\,\psi_1 &=& 0\\ \mathrm{d}\,\psi_2 &=& 0\\ \mathrm{d}\,e^a &=& \big(\overline{\psi}\,\Gamma_B^a\,\psi\big)\,,\end{array}\right) \tag{122}$$

where the pairing is that of spinors in the **32** of 11d (!) under the identification (111) and where, to compensate this, Γ_B^a is from (119).

Since it is evident that the differential in (122) is at least $\mathrm{Spin}(1,8)$-equivariant, we have the following analog of Ex. 3.3:

Example 3.6 (**The IIB/9D super-space extension**). We have a $\mathrm{Spin}(1,8)$-equivariant isomorphism

$$\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}(\psi_1^\alpha)_{\alpha=1}^{16},\\ (\psi_2^\alpha)_{\alpha=1}^{16},\\ (e^a)_{a=0}^{8},\\ e^9\end{array}\right] \Big/ \left(\begin{array}{lcll}\mathrm{d}\,\psi_1 &=& 0 & \\ \mathrm{d}\,\psi_2 &=& 0 & \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}_1\,\Gamma^a\,\psi_1\big)+\big(\overline{\psi}_2\,\Gamma^a\,\psi_2\big) & \text{for } a<9\\ \mathrm{d}\,e^9 &=& \underbrace{\big(\overline{\psi}_1\,\Gamma^9\,\psi_1\big)-\big(\overline{\psi}_2\,\Gamma^9\,\psi_2\big)}_{(\overline{\psi}\,\Gamma_B^9\,\psi)\,=\,(\overline{\psi}\,\Gamma^9\Gamma_{10}\,\psi)} & \end{array}\right)$$

which exhibits $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}$ as a central extension (Def. 2.21) of $\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}$ classified by a 2-cocycle to be denoted c_1^B (cf. footnote 10):

$$\underset{\substack{\textbf{D = 10 type IIB}\\ \textbf{super-Minkowski}\\ \textbf{spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}} \xrightarrow[\text{extension of}]{\text{hofib}} \underset{\substack{\textbf{D = 9 type II}\\ \textbf{super-Minkowski}\\ \textbf{spacetime}}}{\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}} \xrightarrow[\text{classified by first Chern class}]{c_1^B\,:=\,(\overline{\psi}\,\Gamma_B^9\,\psi)\,=\,(\overline{\psi}\,\Gamma^9\Gamma_{10}\,\psi)} b\mathbb{R}\,. \tag{123}$$

What is less evident is that (122) is also $\mathrm{Spin}(1,9)$-equivariant under the action (120), since the $\big\{\Gamma_B^a\big\}_{a=0}^{9} \subset \mathrm{Cl}(1,10)$ by themselves do not generate a Clifford sub-algebra, by Rem. 3.4. However, the failure of Γ_B^9 to be skew-self-adjoint (121) compensates this defect, as follows:

Lemma 3.7 (**Lorentz-equivariance of the type IIB spacetime**). *The differential in* (122) *is indeed equivariant* (15) *under the* $\mathrm{Spin}(1,9)$*-action* (120).

Checking this is straightforward, but we spell out the proof because this statement was omitted in [FSS18a]:

Proof. For a Lie action of $\mathfrak{so}_{1,10}$ on ψ by

$$J^{ab}\psi \;=\; -\tfrac{1}{2}\Gamma_B^{ab}\psi$$

we need to check that the term $\big(\overline{\psi}\,\Gamma_B\,\psi\big)$ transforms in the vector representation, namely that

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;\equiv\; \big(\overline{J^{ab}\psi}\,\Gamma_B^c\,\psi\big)+\big(\overline{\psi}\,\Gamma_B^c\,J^{ab}\psi\big) \;=\; \eta^{bc}\big(\overline{\psi}\,\Gamma_B^a\,\psi\big)-\eta^{ac}\big(\overline{\psi}\,\Gamma_B^b\,\psi\big)\,.$$

In the case where $a,b,c<9$ this is, by (119), just the ordinary case which, just to recall, works out as usual:

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{ab},\Gamma^c]\,\psi\big) \;=\; \big(\overline{\psi}(\eta^{bc}\Gamma^a-\eta^{ac}\Gamma^b)\psi\big)\,, \qquad \text{for } a,b,c<9\,,$$

where in the first step we use – via the first line in (121) – that $\overline{\tfrac{1}{2}\Gamma_{ab}} \;=\; -\tfrac{1}{2}\Gamma_{ab}$, which in the second step gives rise to the commutator [-,-] (in the 11d Clifford algebra).

Next, the case where $a,b<9$ but $c=9$ does involve the modified $\Gamma_B^9=\Gamma^9\Gamma^{10}$, but gives the correct answer trivially since the same kind of commutator appears and evidently vanishes:

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{ab},\Gamma^9\Gamma^{10}]\,\psi\big) \;=\; 0\,, \qquad \text{for } a,b<9.$$

The interesting effect is in the next case, where one of the first two indices take the value 9. Here the modified adjointness relation in the second line of (121) makes instead an anti-commutator {-,-} appear, which however becomes a commutator after pulling out the factor of Γ^{10} that comes with Γ_B^9, and that again gives the correct result:

$$J^{a9}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;=\; -\big(\overline{\psi}\,\{\tfrac{1}{2}\Gamma^{a9}\Gamma^{10},\Gamma^c\}\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{a9},\Gamma^c]\Gamma^{10}\,\psi\big) \;=\; -\eta^{ac}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \qquad \text{for } a,c<9\,.$$

Finally, a similar computation passing through an anti-commutator also confirms the last remaining case:

$$J^{a9}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \;=\; -\big(\overline{\psi}\{\tfrac{1}{2}\Gamma^{a9}\Gamma^{10},\Gamma^9\Gamma^{10}\}\psi\big) \;=\; \big(\overline{\psi}\,\Gamma_B^a\,\psi\big) \qquad \text{for } a<9. \qquad \square$$

Observing that the Clifford elements

$$\begin{array}{ccc} \sigma_1 & := & \Gamma_9 \\ \sigma_2 & := & \Gamma_{10} \end{array} \quad \text{(whose product we denote } \sigma_3 := \sigma_1 \sigma_2 \;=\; \Gamma_9 \Gamma_{10}) \tag{124}$$

anti-commute with *all* the $(\Gamma^B_a)_{a=0}^9$ (119) we also have:

Proposition 3.8 (**R-Symmetry of type IIB** [FSS18a, Rem. 2.11]). *The elements* (124) *generate a* $\mathrm{Pin}(2)$*-action on* $\mathbf{16} \oplus \mathbf{16}$*, which commutes with the* $\mathrm{Spin}(1,9)$*-action* (120)*, making a direct product action of* $\mathrm{Spin}(1,9) \times \mathrm{Pin}(2)$*.*

(This effectively 12-dimensional spin-action is seen to be related to the "F-theory"-perspective on type IIB, in Def. 3.30 and Prop. 3.31.)

Putting these pieces together, we have more generally that:

Proposition 3.9 (**Lorentz-invariants on type IIB super-spacetime**). *Setting*

$$\Gamma^B_{a_1 \cdots a_p} \;:=\; \begin{cases} (-1)^{\mathrm{sgn}(\sigma)}\, \Gamma^B_{\sigma(a_1)} \cdots \Gamma^B_{\sigma(a_p)} & \big|\;\; \exists\, \sigma \in \mathrm{Sym}(n) \text{ s.t. } a_{\sigma(1)} < \cdots < a_{\sigma(n)} \\ 0 & \big|\;\; \text{otherwise} \end{cases} \tag{125}$$

the following expressions are invariants for the $\mathrm{Spin}(1,9)$*-action* (120) *type IIB super-spacetime* (122)*:*

$$\left.\begin{array}{l} \big(\overline{\psi}\, \Gamma^B_{a_1 \cdots a_p}\, \psi\big) \quad e^{a_1} \cdots e^{a_p} \\ \big(\overline{\psi}\, \Gamma^B_{a_1 \cdots a_p}\, \sigma_i\, \psi\big) e^{a_1} \cdots e^{a_p} \end{array}\right\} \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16} \oplus \mathbf{16}}\big) \tag{126}$$

for σ_i *from* (124)*.*

(Of course, many of these expressions vanish by (225).)

Proof. The statement for the second line follows immediately from that for the first line by Prop. 124. We proceed to prove the statement for the first line.

For $p = 0$ the statement holds trivially, since the given expression vanishes, by (225).

From $p = 1$ on we shall prove the stronger statement that $\big(\overline{\phi}\, \Gamma_{a_1 \cdots a_9}\, \psi\big) e^{a_1} \cdots e^{a_p}$ is invariant for ϕ possibly any other element transforming as $J^{ab}\phi \;=\; -\tfrac{1}{2}\Gamma^{ab}_B \phi$:

The case $p = 1$ follows verbatim as in Lem. 3.7, with the first factor of "ψ" there generalized to "ϕ".

For the remaining cases we may, by (125), assume without restriction of generality that $a_1 < \cdots < a_{p+1}$, hence in particular that $a_n < 9$ whenever $n \leq p$, and we need to show that the following is invariant:

$$\begin{aligned} \big(\overline{\phi}\, \Gamma^B_{a_1 \cdots a_9}\, \psi\big) e^{a_1} \cdots e^{a_p} \quad &= \quad \big(\overline{\phi}\, \Gamma^B_{a_1 \cdots a_p} \Gamma^B_{a_{p+1}}\, \psi\big) e^{a_1} \cdots e^{a_p} e^{a_{p+1}} && \text{by assmptn} \\ &= \quad \pm \big(\overline{\Gamma^B_{a_1 \cdots a_p} \phi}\, \Gamma^B_{a_{p+1}}\, \psi\big) e^{a_1} \cdots e^{a_p}\, e^{a_{p+1}} && \text{by (221)}. \end{aligned}$$

Now observe that the expression

$$\phi' \;:=\; \Gamma^B_{a_1 \cdots a_p} \phi\, e^{a_1} \cdots e^{a_p}$$

transforms as a spinor under any J^{ab}: For $a, b < 9$ this is the standard situation, and then for J^{a9} it follows since (125) makes any Γ_{10}-factor "stay on the right". But with this we are reduced to seeing that $\big(\overline{\phi'}\, \Gamma_{a_{p+1}}\, \psi\big) e^{a_{p+1}}$ is invariant, which is the case $p = 1$ already proven. □

3.2 Super-flux T-duality

We give a streamlined review of the core part of the formulation (and in fact a derivation) from [FSS18a] of T-duality along a single isometry between type IIA and type IIB super-flux densities on super-Minkowski spacetime.[11]

The key observations driving this are that:

(i) The type IIA super-flux densities on super-Minkowki spacetime satisfying their Bianchi identities are equivalently (Prop. 3.13) super-L_∞ cocycles with coefficients in the real Whitehead L_∞-algebra of the twisted K-theory spectrum KU (Ex. 2.12).

[11] Note that the focus on Minkowski super-spacetime is only superficially a specialization: The torsion constraints that govern supergravity theories say that – in the manner of Cartan geometry – the actual super-flux densities on on-shell super-spacetimes are tangent space-wise constrained to have fermionic components given by these Minkowskian super-invariants – and in fact in 11d SuGra that condition is equivalent to the supergravity equations of motion [GSS24a, Thm. 3.1]. In this, the super-invariants on super-Minkowski spacetimes are the archetypes that govern full on-shell supergravity theories.

(ii) Double dimensional reduction of super-flux densities on super-Minkowski spacetime is equivalently given by *cyclifying* (as in *cyclic cohomology*) their coefficient L_∞-algebra (Prop. 2.25).

(iii) The cyclifications of twisted KU is equivalent to that of twisted ΣKU by swapping the Chern class with the wrapping mode of the 3-form (Ex. 2.28).

(iv) The type IIA flux densities are equivalent to the IIB flux densities after reduction via cyclification to 9d, whereby the type IIA spacetime is swapped for the type IIB spacetime (Prop. 3.15).

M/IIA duality. Recall from Ex. 2.8 that the C-field super-flux densities on 11D super-Minkowski spacetime are encoded by a super-L_∞ homomorphism of the form:

$$
\begin{array}{rcl}
 & \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;\;(G_4,\,G_7)\;\;} & \mathfrak{l}S^4 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} \;=:\; G_4 & \longleftarrow\!\shortmid & g_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \;=:\; G_7 & \longleftarrow\!\shortmid & g_7
\end{array}
\tag{127}
$$

Example 3.10 (**IIA-Reduction of C-field super-flux** [FSS17, pp 11], cf. [SS24c, Ex. 2.13])**.** Under reduction via the Ext/Cyc-adjunction (Prop. 2.25) with respect to the M/IIA extension (Ex. 3.2)

$$
\mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;(G_4,\,G_7)\;} \mathfrak{l}S^4
\qquad \leftrightarrow \qquad
\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "\mathrm{rdc}_{c_1^M}(G_4,G_7)"] \arrow[dr, "(\overline{\psi}\Gamma_{10}\,\psi)"'] & & \mathrm{cyc}(\mathfrak{l}S^4) \arrow[dl, "\omega_2"] \\
 & b\mathbb{R} &
\end{tikzcd}
$$

the flux densities from 11D (127) become:

$$
\begin{array}{rcl}
 & \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;(G_4,\,G_7)\;} & \mathfrak{l}S^4 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} & \longleftarrow\!\shortmid & g_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} & \longleftarrow\!\shortmid & g_7
\end{array}
\;\;\rightsquigarrow\;\;
\begin{array}{rcl}
 & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \xrightarrow[\;(F_2,F_4,F_6,H_3^A,H_7^A)\;]{\;\mathrm{rdc}_{c_1^M}(G_4,G_7)\,=\;} & \mathrm{cyc}\big(\mathfrak{l}S^4\big) \\
\big(\overline{\psi}\,\Gamma_{10}\psi\big) \;=:\; \mathrm{F}_2 & \longleftarrow\!\shortmid & \omega_2 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}\,e^{a_2} \;=:\; \mathrm{F}_4 & \longleftarrow\!\shortmid & g_4 \\
\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a \;=:\; H_3^A & \longleftarrow\!\shortmid & \mathrm{s}g_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \;=:\; H_7^A & \longleftarrow\!\shortmid & g_7 \\
-\tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_4} \;=:\; -F_6 & \longleftarrow\!\shortmid & \mathrm{s}g_7
\end{array}
\tag{128}
$$

satisfying:

$$
\left.
\begin{array}{rcl}
\mathrm{d}\,G_4 & = & 0 \\
\mathrm{d}\,G_7 & = & \tfrac{1}{2}\,G_4\,G_4
\end{array}
\right\}
\;\;\rightsquigarrow\;\;
\left\{
\begin{array}{rcl}
\mathrm{d}\,F_2 & = & 0 \\
\mathrm{d}\,F_4 & = & H_3^A\,F_2 \\
\mathrm{d}\,F_6 & = & H_3^A\,F_4 \\
\mathrm{d}\,H_3^A & = & 0 \\
\mathrm{d}\,H_7^A & = & \tfrac{1}{2}\,F_4\,F_4 \,-\, F_2\,F_6\,.
\end{array}
\right.
\tag{129}
$$

However, on the type IIA super-spacetime there appear further/higher super-invariants satisfying analogous differential equations — this observation is essentially due to [CdAIP00, §6.1], except that we also consider F_{12} [12].

Definition 3.11 (**Higher type IIA super-flux densities.**)**.** Consider the following super-invariants, beyond those appearing via reduction from 11d in Ex. 3.10 [13]

$$
\left.
\begin{array}{rcl}
F_8 & := & +\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6} \\
F_{10} & := & +\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_8} \\
F_{12} & := & +\tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}}
\end{array}
\right\}
\;\in\; \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big).
\tag{130}
$$

[12] A 12-form term like F_{12} in (130) – nominally the WZW term for "D10-branes" – is rarely considered in the literature (an exception is [CS09, p 30]) since it is evidently invisible on ordinary (bosonic) spacetimes. But on super-space it is non-vanishing and must be considered [BMSS19, Rem. 4.3] to complete the flux densities $F_{2\bullet}$ to an $\mathfrak{l}$KU-valued cocycle. On the other hand, the yet higher degree fluxes of this form $F_{2k+2} = \frac{1}{(2k)!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{2k}}\,\psi\big)e^{a_1}\cdots e^{a_{2k}}$ do vanish on $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ and hence need not be further considered.

[13] In the string theory lore the higher flux densities related to the higher super-invariants (130) and corresponding to (1.) the D6-brane, (2.) the D8-brane and (3.) the "D10-brane", are meant to have 11d ancestors given, more or less informally, by (1.) the 11d Kaluza-Klein monopole, (2.) a Scherk-Schwarz compactification to massive type IIA theory, respectively, while the M-theory lift of (3.) the "D10-brane" seems not to have been discussed (cf. footnote 12).

More in the spirit of the rigorous derivations here, we have shown in [BMSS19] that the relevant higher generators appear when the 4-sphere coefficient (Ex. 2.8) for the fluxes in 11d are subjected to fiberwise "stabilization" over the 3-sphere (in the sense of stable homotopy theory).

Lemma 3.12 (**Bianchi identities for higher IIA super-fluxes**). *The higher super-flux densities* (130) *satisfy*

$$\begin{aligned} \mathrm{d}\, F_8 &= H_3^A\, F_6 \\ \mathrm{d}\, F_{10} &= H_3^A\, F_8 \\ \mathrm{d}\, F_{12} &= H_3^A\, F_{10} \\ 0 &= H_3^A\, F_{12} \end{aligned} \tag{131}$$

with H_3^A *from* (128).

Proof. The proof for the first equation in (131) is also given in [CdAIP00, §B], which we follow. First to note that the closure of H_3^A from (129)

$$\begin{aligned} 0 &= \mathrm{d}\, H_3^A \\ &= \mathrm{d}\Big(\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big) e^a \Big) \qquad\qquad a < 10 \\ &= \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^a\,\psi\big) \end{aligned}$$

means in components that

$$\begin{aligned} 0 &= \big(\Gamma_a\Gamma_{10}\big)_{(\alpha\beta}\Gamma^a_{\gamma\delta)} \\ &= \tfrac{2\cdot 3!}{4!}\Big(\big(\Gamma_a\Gamma_{10}\big)_{(\alpha\beta}\Gamma^a_{\delta)\gamma} \,+\, \big(\Gamma_a\Gamma_{10}\big)_{\gamma(\alpha}\Gamma^a_{\beta\delta)} \Big)\,, \end{aligned} \qquad a < 10 \tag{132}$$

where in the second line we used (223) that $(\Gamma_a\Gamma_{10})_{\alpha\beta}$ and $(\Gamma_a)_{\gamma\delta}$ both already are symmetric in their spinor indices.

With this in hand we compute as follows:

$$\begin{aligned} \mathrm{d}\, F_8 &= \mathrm{d}\, \tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big) e^{a_1}\cdots e^{a_6} && \text{by (130)} \\ &= -\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5\, b}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big) e^{a_1}\cdots e^{a_5} && \text{by (16)} \\ &= -\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big) e^{a_1}\cdots e^{a_5} && \text{by (215) \& (225)} \\ &= +\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_{10}\,\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big) e^{a_1}\cdots e^{a_5} && \text{by (214)} \\ &= +\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_5} && \text{matrix multip.} \\ &= -\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_5} && \text{by (132)} \\ &= +\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_b\Gamma_{10}\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_5} && \text{matrix multip.} \\ &= +\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_5} \\ &= +\tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_4}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_5]}\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_5} && \text{by (215) \& (225)} \\ &= +\, F_6\, H_3^A && \text{by (128)}\,. \end{aligned}$$

The remaining two cases (and in fact all cases) work analogously:

$$\begin{aligned} \mathrm{d}\, F_{10} &= \mathrm{d}\, \tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big) e^{a_1}\cdots e^{a_8} && \text{by (130)} \\ &= -\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7\, b}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big) e^{a_1}\cdots e^{a_7} && \text{by (16)} \\ &= -\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big) e^{a_1}\cdots e^{a_7} && \text{by (215) \& (225)} \\ &= -\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_7} && \text{matrix multip.} \\ &= +\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_7} && \text{by (132)} \\ &= +\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\Gamma_b\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\; \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_7} && \text{matrix multip.} \\ &= +\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_7} \\ &= +\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_6}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_7]}\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_7} && \text{by (215) \& (225)} \\ &= +F_8\, H_3^A && \text{by (128)} \end{aligned}$$

and:

$$
\begin{aligned}
\mathrm{d}\, F_{12} &= \mathrm{d}\, \tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}} && \text{by (130)}\\
&= -\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9\, b}\,\psi\big)\big(\overline{\psi}\,\Gamma_b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (16)}\\
&= -\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (215) \& (225)}\\
&= +\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_{10}\,\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (214)}\\
&= +\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_9} && \text{matrix multip.}\\
&= -\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_9} && \text{by (132)}\\
&= +\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_b\Gamma_{10}\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\, e^{a_1}\cdots e^{a_9} && \text{matrix multip.}\\
&= +\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_9} && \\
&= +\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_8}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_9]}\Gamma_{10}\,\psi\big)\, e^{a_1}\cdots e^{a_9} && \text{by (215) \& (225)}\\
&= +F_{10}\, H_3^A && \text{by (128)}\,.
\end{aligned}
$$

□

It is worth summarizing this state of affairs in super-L_∞ algebraic language:

Proposition 3.13 (**The type IIA super-cocycles** [FSS18a, Prop. 4.8])**.** *On the type IIA super-Minkowski spacetime* $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ *we have the following super-invariants*

$$
\left.
\begin{aligned}
H_3^A &:= \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a\\
F_{-2k} &:= 0\,,\quad k\in\mathbb{N}\\
F_2 &:= \big(\overline{\psi}\,\Gamma_{10}\,\psi\big)\\
F_4 &:= \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2}\\
F_6 &:= \tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_4}\,\psi\big)e^{a_1}\cdots e^{a_4}\\
F_8 &:= \tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6}\\
F_{10} &:= \tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_8}\,\psi\big)e^{a_1}\cdots e^{a_8}\\
F_{12} &:= \tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}}\\
F_{14+2k} &= 0\,,\quad k\in\mathbb{N}
\end{aligned}
\right\}
\;\in\; \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big)
\quad \text{s.t.} \quad
\begin{cases}
\mathrm{d}\, H_3^A \;=\; 0\\
\mathrm{d}\, F_{2\bullet+2} \;=\; H_3^A\, F_{2\bullet}\,,
\end{cases}
\tag{133}
$$

hence equivalently constituting a super-L_∞ homomorphism (14) *to the real Whitehead L_∞-algebra of twisted* KU_0 (27)*:*

$$
\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "{(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}})}"] \arrow[dr, "H_3^A"'] && \mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)\,. \arrow[dl, "h_3"] \\
& \mathfrak{l}B^2\mathrm{U}(1) &
\end{tikzcd}
\tag{134}
$$

Example 3.14 (**Reduction of IIA super-cocycles to 9d**)**.** The reduction of of the type IIA cocycles (Prop. 3.13) via the Ext/Cyc-adjunction (Prop. 2.25) along the IIA/9D extension (Ex. 3.3)

$$
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \xrightarrow{(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}})} \mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)
\quad\leftrightsquigarrow\quad
\begin{tikzcd}
\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{\mathrm{rdc}_{c_1^A}(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}})}"] \arrow[dr, "{c_1^A = (\overline{\psi}\,\Gamma^9\,\psi)}"'] && \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[dl, "\omega_2"] \\
& b\mathbb{R} &
\end{tikzcd}
$$

gives the following system of super-invariants in 9D, where on the right we show their equivalent incarnation as having coefficients either in the cyclification of twisted KU_0 or of twisted KU_1, via the first isomorphism in (51):

$$
\begin{array}{rcccccc}
 & \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} & \xrightarrow[(134)]{\;\mathrm{rdc}_{c_1^A}\left(H_3^A,(F_{2k})_{k\in\mathbb{Z}}\right)\;} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{0}\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \xrightarrow[(51)]{\;\sim\;} & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^{1}\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
+c_1^A \;= & \big(\overline{\psi}\,\Gamma_9\,\psi\big) & \longleftarrow & \omega_2 & \longleftarrow & -\mathrm{s}h_3 \\
 & \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a & \longleftarrow & h_3 & \longleftarrow & h_3 \\
-c_1^B \;= & -\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big) & \longleftarrow & \mathrm{s}h_3 & \longleftarrow & -\omega_2 \\
 & 0 & \longleftarrow & f_{\leq 0} & \longleftarrow & \mathrm{s}f_{\leq 1} \\
 & 0 & \longleftarrow & \mathrm{s}f_{\leq 0} & \longleftarrow & f_{\leq 1} \\
 & \big(\overline{\psi}\,\Gamma_{10}\,\psi\big) & \longleftarrow & f_2 & \longleftarrow & \mathrm{s}f_3 \\
 & 0 & \longleftarrow & \mathrm{s}f_2 & \longleftarrow & f_1 \\
 & \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2} & \longleftarrow & f_4 & \longleftarrow & \mathrm{s}f_5 \\
 & \big(\overline{\psi}\,\Gamma_a\Gamma_9\,\psi\big)e^a & \longleftarrow & \mathrm{s}f_4 & \longleftarrow & f_3 \\
 & \tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_4} & \longleftarrow & f_6 & \longleftarrow & \mathrm{s}f_7 \\
 & \tfrac{1}{3!}\big(\overline{\psi}\Gamma_{a_1a_2a_3}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}e^{a_2}e^{a_3} & \longleftarrow & \mathrm{s}f_6 & \longleftarrow & f_5 \\
 & \tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6} & \longleftarrow & f_8 & \longleftarrow & \mathrm{s}f_9 \\
 & \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_5} & \longleftarrow & \mathrm{s}f_8 & \longleftarrow & f_7 \\
 & \tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_8} & \longleftarrow & f_{10} & \longleftarrow & \mathrm{s}f_{11} \\
 & \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_7} & \longleftarrow & \mathrm{s}f_{10} & \longleftarrow & f_9 \\
 & \tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}} & \longleftarrow & f_{12} & \longleftarrow & \mathrm{s}f_{13} \\
 & \tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_9} & \longleftarrow & \mathrm{s}f_{12} & \longleftarrow & f_{11} \\
 & 0 & \longleftarrow & f_{\geq 14} & \longleftarrow & \mathrm{s}f_{\geq 15} \\
 & 0 & \longleftarrow & \mathrm{s}f_{\geq 14} & \longleftarrow & f_{\geq 13}
\end{array}
\tag{135}
$$

T-dualization. Using the super-L_∞ machinery, we now obtain with mechanical ease the T-dual version of Prop. 3.13 (the latter was claimed also in [Sak00, §2]):

Proposition 3.15 (**The type IIB super-cocycles** [FSS18a, Prop. 4.10]). *On the type IIB super-Minkowski spacetime* $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}$ (Def. 3.5) *we have the following super-invariants*

$$
\left.
\begin{array}{rcl}
H_3^B &=& \big(\overline{\psi}\,\Gamma_a^B\,\Gamma_{10}\,\psi\big)e^a \\
F_{\leq 1} &=& 0 \\
F_3 &=& \big(\overline{\psi}\,\Gamma_a^B\Gamma_9\,\psi\big)e^a \\
F_5 &=& \tfrac{1}{3!}\big(\overline{\psi}\,\Gamma^B_{a_1a_2a_3}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}e^{a_2}e^{a_3} \\
F_7 &=& \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_5}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_5} \\
F_9 &=& \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_7} \\
F_{11} &=& \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_9}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_9} \\
F_{13+2k} &=& 0\,, \quad k\in\mathbb{N}
\end{array}
\right\}
\in \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big)
\quad s.t. \quad
\left\{
\begin{array}{rcl}
\mathrm{d}\,H_3^B &=& 0 \\
\mathrm{d}\,F_{2\bullet+1} &=& H_3^B\,F_{2\bullet-1}\,,
\end{array}
\right.
\tag{136}
$$

where the $\Gamma^B_{a_1\cdots a_p}$ *are from* (125)*, hence equivalently constituting a super-*L_∞ *homomorphism* (14) *to the real Whitehead* L_∞*-algebra of twisted* $\Sigma^1\mathrm{KU}$ (27)*:*

$$
\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{\left(H_3^B,(F_{2k+1})_{k\in\mathbb{Z}}\right)}"] \arrow[dr, "H_3^B"'] & & \mathfrak{l}\big(\Sigma^1\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)\,. \arrow[dl, "h_3"] \\
& \mathfrak{l}\big(B^2\mathrm{U}(1)\big) &
\end{tikzcd}
\tag{137}
$$

Proof. This is the *T-dual* statement of Prop. 3.13 in that the super-invariants in (136) are the result of an application of the composite operation from Lem. **??**, namely:

(i) reducing (43) the IIA super-invariants (134) along the type IIA fibration (116) to 9d,

(ii) equivalently re-regarding their coefficients in the cyclification of twisted KU_1 instead of twisted KU_0, via (51), noticing that this swaps the Chern class from that classifying the type IIA extension to that classifying the IIB extension (123),

(iii) re-oxidizing (43) the result, but now along the IIB fibration (123):

$$
\begin{tikzcd}
 & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rrr, "\mathrm{oxd}_{c_1^B} T\left(\mathrm{rdc}_{c_1^A}(H_3^A, (F_{2k})_{k\in\mathbb{Z}})\right)", "= (H_3^B, (F_{2k+1})_{k\in\mathbb{Z}})"'] \arrow[d, two heads] & & & \mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rrr, "(H_3^A, (F_{2k})_{k\in\mathbb{Z}})"] \arrow[dr, two heads] & & & \mathfrak{l}\big(\Sigma^0 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \\
 & \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rrr, "T\left(\mathrm{rdc}_{c_1^A}(H_3^A, (F_{2k})_{k\in\mathbb{Z}})\right)"] \arrow[drr, "\mathrm{rdc}_{c_1^A}(H_3^A, (F_{2k})_{k\in\mathbb{Z}})"'] & & & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^1 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
 & & & \mathrm{cyc}\,\mathfrak{l}\big(\Sigma^0 \mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[ur, "T", "\sim"'] &
\end{tikzcd}
\tag{138}
$$

By the Ext/Cyc-adjunction (Prop. 2.25) the result of this process is guaranteed to be a super-L_∞ homomorphism of the form shown in the top right of the above diagram, which implies the claimed Bianchi identities (137).

Hence, all that remains to be shown is that the super-invariants produced by this process are indeed those shown in (136). This is a straightforward matter of plugging the IIA super-invariants (134) into the "winding / non-winding swapping" formula (54), or equivalently plugging the reduced 9d super-invariants (135) into the oxidation formula (44):

$$
\begin{aligned}
F_1 &= \overbrace{0}^{f_1} - e^9 \overbrace{0}^{\mathrm{sf}_1} \\
&= 0\,, \\
F_3 &= \overbrace{\sum_{a<9} \big(\overline{\psi}\,\Gamma_a \Gamma_9\,\psi\big) e^a}^{f_3} - e^9 \overbrace{\underbrace{\big(\overline{\psi}\,\Gamma_{10}\,\psi\big)}_{-\big(\overline{\psi}\,\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\Gamma_9\,\psi\big)}}^{\mathrm{sf}_3} \\
&= \big(\overline{\psi}\,\Gamma^B_a \Gamma_9\,\psi\big) e^a, \\
F_5 &= \overbrace{\sum_{a<9} \tfrac{1}{3!}\big(\overline{\psi}\,\Gamma_{a_1a_2a_2}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1} e^{a_2} e^{a_3}}^{f_5} - e^9 \overbrace{\sum_{a<9} \tfrac{1}{2}\underbrace{\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1a_2}\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\Gamma_9\Gamma_{10}\,\psi\big)} e^{a_1} e^{a_2}}^{\mathrm{sf}_5} \\
&= \tfrac{1}{3!}\big(\overline{\psi}\Gamma^B_{a_1a_2a_3}\Gamma_9\Gamma_{10}\psi\big) e^{a_1} e^{a_2} e^{a_3}, \\
F_7 &= \overbrace{\sum_{a<9} \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_5}}^{f_7} - e^9 \overbrace{\tfrac{1}{4!}\underbrace{\sum_{a<9}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\Gamma_9\,\psi\big)} e^{a_1}\cdots e^{a_4}}^{\mathrm{sf}_7} \\
&= \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_5}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_5}, \\
F_9 &= \overbrace{\sum_{a_i<9} \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1}\cdots e^{a_7}}^{f_9} - e^9 \overbrace{\sum_{a_i<9} \tfrac{1}{6!}\underbrace{\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\Gamma_9\Gamma_{10}\,\psi\big)} e^{a_1\cdots a_6}}^{\mathrm{sf}_9} \\
&= \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1}\cdots e^{a_7}, \\
F_{11} &= \overbrace{\sum_{a_i<9} \tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_9}}^{f_{11}} - e^9 \overbrace{\sum_{a_i<9} \tfrac{1}{8!}\underbrace{\big(\Gamma_{a_1\cdots a_8}\Gamma_{10}\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\Gamma_9\,\psi\big)} e^{a_1}\cdots e^{a_8}}^{\mathrm{sf}_{11}} \\
&= \tfrac{1}{9!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_9}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_9}.
\end{aligned}
$$

$\square$

Remark 3.16 (**Lorentz invariance of IIB super-fluxes**)**.** While the proof of Prop. 3.15 does not make manifest that the resulting super-translation invariants (136) are also $\mathrm{Spin}(1,9)$-invariant, this is immediate by Prop. 3.9.

Remark 3.17 (**T-Dual NS-Flux and topological T-duality** [FSS18a, Rem. 5.4])**.**
(i) The action of the T-duality operation from Prop. 3.15 on the NS-flux densities $H_3^{A/B}$ is particularly interesting: Note that both these fluxes come out as the sum of (1.) the basic H_3-flux in 9d (pulled back to 10d along the corresponding fibration) with (2.) the fiber form e^9 times the Chern class classifying the *other* extension:

$$
\begin{array}{ccccc}
\begin{aligned} H_3^{A} &= \overbrace{H_3}^{h_3} + e^9_A \overbrace{\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big)}^{-sh_3} \\ &= H_3 + e^9 \quad c_1^{B} \end{aligned}
& & \xrightarrow{\quad\text{superspace T-duality}\quad} & &
\begin{aligned} H_3^{B} &= H_3 + e^9\big(\overline{\psi}\,\Gamma_9\,\psi\big) \\ &= H_3 + e^9_B \quad c_1^{A} \end{aligned}
\\
& \xmapsto[\text{reduction along IIA extension}]{\mathrm{rdc}_{c_1^A}} &
\begin{aligned} H_3 &= \sum_{a<9} e^a\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big) \\ c_1^A &= \big(\overline{\psi}\,\Gamma_9\,\psi\big) \\ -c_1^B &= -\big(\overline{\psi}\,\Gamma_9^B\,\psi\big)\,, \end{aligned}
& \xleftarrow[\text{oxidation along IIB extension}]{\mathrm{oxd}_{c_1^B}} &
\end{array}
\tag{139}
$$

where by the closure of H_3^A (or that of H_3^B) from (128) the basic H_3-flux satisfies

$$
\mathrm{d}\, H_3 \;=\; -c_1^A \cdot c_1^B\,. \tag{140}
$$

(ii) In particular, this says that the fiber integration of H_3^A from the type IIA spacetime down to 9d is the Chern class classifying the type IIB extension, and vice versa:

$$
p^A_*\, H_3^A \;=\; c_1^B \qquad \text{and} \qquad p^B_*\, H_3^B \;=\; c_1^A\,. \tag{141}
$$

(iii) The analogous phenomenon in ordinary T-duality (i.e., not on super-flux densities over super-spacetime as considered here) was originally proposed in [BEM04, (1.8)] and gave rise to the mathematical notion of "topological T-duality".
(iv) While the formalism of topological T-duality has worked wonders, its actual relation to string/M-theory rests on educated guesses (though much progress was recently made when [Wa24] related it to the Buscher rules). Here it is interesting that we find (139) this relation being hard-coded in the DNA of supergravity.
(v) Also note that the form of the 3-flux in (139) is analogous, up to degrees, to the form of the 11D 7-flux in its closed Page-charge form, $\widetilde{G}_7 \,=\, G_7 \,-\, \tfrac{1}{2} H_3\, G_4$. This analogy is made precise by the notion of higher T-duality discussed in §3.4.

Next we obtain a maybe more vivid perspective on this super-space T-duality by turning attention from the dimensionally reduced L_∞-cocycles to the higher spacetime extensions that these classify, which reveals the appearance of *doubled* superspace:

Doubled super-spacetime and the Poincaré form. The following Prop. 3.18 speaks of the *homotopy fiber* of non-abelian L_∞-algebra cocycles. This concept – standard in (rational) homotopy theory – is explained in some detail in [FSS23], but the reader not to be bothered by such notions may take the following (143) as a definition. In any case, either abstractly or by inspection, one sees that the following serves to express an equivalent point of view on the above T-duality isomorphism.

Lemma 3.18 (**Extended doubled super-space as homotopy fiber of reduced 3-flux** [FSS18a, Prop. 7.5])**.** *The homotopy fibers of the A/B-reduced (43) $H_3^{A/B}$-flux cocycle (139), to be denoted*

$$
\mathbb{R}^{1,\widehat{8+(1+1)}\,|\,32}_{A/B} \xrightarrow{\quad\mathrm{hofib}\quad} \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \xrightarrow{\quad \mathrm{red}_{c_1^{A/B}}\big(H_3^{A/B}\big) \quad} \mathrm{cycl}B^2\mathrm{U}(1) \tag{142}
$$

are given by

$$
\mathrm{CE}\Big(\mathbb{R}^{1,\widehat{8+(1+1)}\,|\,32}_{A/B}\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{8} \\ e^9_A \\ e^9_B \\ b_2 \end{array}\right] \Big/ \left(\begin{array}{l} \mathrm{d}\,\psi \;= 0 \\ \mathrm{d}\,e^a \;= \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,e^9_A = c_1^A \\ \mathrm{d}\,e^9_B = c_1^B \\ \mathrm{d}\,b_2 = H_3^{A/B} \end{array}\right) \tag{143}
$$

and as such are equivalently further extensions of the type A/B string-extended super-spacetime (Ex. 2.60) by a further ("doubled") copy of the fiber coframe e^9.

Lemma 3.19 (**T-duality of string-extended doubled super-spacetimes** [FSS18a, Prop. 6.2])**.** *The equivalence that is induced by the T-duality isomorphism* (155) *between the doubled & extended homotopy-fiber spaces* (142) *sends all generators to the generators of the same name, except for b_2 (the avatar of the string's "B-field"), which is instead shifted by the "Poincaré form"* (cf. Rem. 3.22 below) $P_2 \;:=\; e^9_B e^9_A$*:*

$$
\begin{tikzcd}
b_2 + \overbrace{e^9_B\, e^9_A}^{P_2} & & & b_2 \arrow[lll, mapsto] \\
\mathbb{R}_{B}^{\widehat{1,8+(1+1)}\,|\,32} \arrow[rrr, leftrightarrow, "\sim"] \arrow[drr, "\mathrm{hofib}"] & & & \mathbb{R}_{A}^{\widehat{1,8+(1+1)}\,|\,32} \arrow[dl, "\mathrm{hofib}"'] \\
& & \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[dl, "\mathrm{red}_{c_1^A}(H_3^A,\, F_{2\bullet})"'] \arrow[d, "\mathrm{red}_{c_1^B}(H_3^B,\, F_{2\bullet+1})"] & \\
& \mathrm{cycl}\big(\Sigma^0 \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \arrow[r, leftrightarrow, "\sim"] \arrow[dl] & \mathrm{cycl}\big(\Sigma^1 \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \arrow[dr] & \\
\mathrm{cycl} B^2\mathrm{U}(1) \arrow[rrr, "\sim", "(52)"'] & & & \mathrm{cycl} B^2\mathrm{U}(1)\,.
\end{tikzcd}
\tag{144}
$$

This makes us to turn attention to the "doubled" or "correspondence space" of the IIA/B superspacetimes:

Definition 3.20 ($\mathbb{R}^1$**-Doubled super-space** [FSS18a, Def. 6.1])**.** Write $\mathbb{R}^{1,8=(1+1)\,|\,32}$ for the super-Lie algebra given by extending the 9d super type II spacetime by *both* the Chern class classifying the IIA extension as well as that classifying the IIB extension:

$$
\mathrm{CE}\big(\mathbb{R}^{1,8+(1+1)\,|\,32}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=0}^{32} \\ (e^a)_{a=0}^{8} \\ e^9_A, e^9_B \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\psi &=& 0 \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,e^9_{A/B} &=& \big(\overline{\psi}\,\Gamma^9_{A/B}\,\psi\big) \end{array}\right). \tag{145}
$$

This may be understood as the fiber product (59) of the IIA- with the IIB extension, making a Cartesian square of super-Lie algebras, as follows:

$$
\begin{tikzcd}
& \mathbb{R}^{1,8+(1+1)\,|\,32} \arrow[dl, ->>, "\pi_A"'] \arrow[dr, ->>, "\pi_B"] \arrow[dd, phantom, "(\mathrm{pb})"] & \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dr, ->>] & & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[dl, ->>] \\
& \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} &
\end{tikzcd}
$$

We say that

$$
P_2 \;:=\; e^9_B\, e^9_A \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,8+(1+1)\,|\,32}\big) \tag{146}
$$

is the *twisted Poincaré super 2-form* (or just *Poincaré form*, for short) on the doubled super spacetime.

With this we may concisely re-state Rem. 3.17 as follows:

Proposition 3.21 (**Poincaré form is coboundary for difference of T-dual NS super-fluxes** [FSS18a, Prop. 6.2])**.** *The pullbacks of the NS super-flux densities to the doubled super-spacetime* (145) *differ by the differential of the twisted Poincaré 2-form* (146)

$$
\mathrm{d}P_2 \;=\; \pi_A^* H_3^A - \pi_B^* H_3^B\,. \tag{147}
$$

Proof. Via Prop. 3.15 and Rem. 3.17 this follows straightforwardly:

$$
\begin{aligned}
\pi_A^* H_3^A - \pi_B^* H_3^B \;&=\; \Big(\sum_{a<9} e^a\big(\overline{\psi}\,\Gamma_a\,\psi\big) + e^9_A\,\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big)\Big) \;-\; \Big(\sum_{a<9} e^a\big(\overline{\psi}\,\Gamma_a\,\psi\big) + e^9_B\,\big(\overline{\psi}\,\Gamma_9\,\psi\big)\Big) \\
&=\; e^9_A\,\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big) \;-\; e^9_B\,\big(\overline{\psi}\,\Gamma_9\,\psi\big) \\
&=\; \mathrm{d}\big(e^9_B\, e^9_A\big) \;\equiv\; \mathrm{d}\,P_2\,.
\end{aligned}
$$

□

Remark 3.22 (**Poincaré 2-form and Buscher rules in T-duality literature** [FSS18a, Rem. 6.3])**.**
(i) That the analog of the relation (147) should hold for ordinary T-duality (i.e., disregarding super-flux densities on super-spacetimes as considered here) was originally proposed by [BEM04, (1.12)]. As previously in Rem. 3.17, here it is interesting to find these phenomena hard-coded in the DNA of supergravity.

(ii) In fact, understanding the super-Lie algebraic content of Prop. 3.21 through the lens of (super-)rational homotopy theory (essentially via Ex. 2.4), it reproduces the image under rationalization of topological T-duality in the form proposed in [BRS06, Def. 2.8] in the sense of Def. 3.24 and Thm. 3.27 (in their purely even form).
(iii) A transparent understanding of how the (twisted) Poincaré 2-form and its Bianchi identity (147) controls the *Buscher rules* of T-duality was more recently obtained in [Wa24, Lem. 3.3.1(c)].

Remark 3.23 (**Classifier for the Poincaré 2-form).** The L_∞-algebra which classifies the Bianchi identity (147) of the Poincaré 2-form is the homotopy fiber of the universal map that forms the difference of a pair of degree=3 classes:

$$
\begin{array}{ccccc}
\mathfrak{poin}_2 & \xrightarrow{\mathrm{hofib}} & \mathfrak{l}B^2\mathrm{U}(1) \times \mathfrak{l}B^2\mathrm{U}(1) & \longrightarrow & \mathfrak{l}B^2\mathrm{U}(1) \\
& & \pi_L^*\omega_3 - \pi_R^*\omega_3 & \longleftarrow\!\!\shortmid & \omega_3
\end{array}
\tag{148}
$$

(where $\pi_{L/R}$ are the two projections out of the direct product). This is given by

$$
\mathrm{CE}\big(\mathfrak{poin}_2\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} \omega_3^A \\ \omega_3^B \\ p_2 \end{array} \right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\omega_3^A & = 0 \\ \mathrm{d}\,\omega_3^B & = 0 \\ \mathrm{d}\,p_2 & = \omega_3^A - \omega_3^B \end{array} \right)
\tag{149}
$$

and in that the Bianchi identity (147) on P_2 characterizes dashed maps, making the following diagram commute:

$$
\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \underset{\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}}{\times} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, dashed, "P_2"] \arrow[d, "{(\pi_A,\pi_B)}"'] & & \mathfrak{poin}_2 \arrow[d] \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \times \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "{(H_3^A,\, H_3^B)}"] & & \mathfrak{l}B^2\mathrm{U}(1) \times \mathfrak{l}B^2\mathrm{U}(1)\,.
\end{tikzcd}
\tag{150}
$$

T-duality as a Fourier-Mukai transform. The above doubled super-space picture (146) coupled with the observations from (141) and (147) lead to an alternative but equivalent formulation of the T-duality phenomenon within (super-)rational homotopy theory in terms of correspondences and an induced Fourier-Mukai integral-transform ([Ho99, (1.1)][BEM04, (1.9)][GS14, §4.1]), here on 3-twisted Chevalley–Eilenberg cochain complexes (Def. 2.13).

Definition 3.24 (**T-duality Correspondence).** Pairs $(\widehat{\mathfrak{g}}_A, H_A)$ and $(\widehat{\mathfrak{g}}_B, H_B)$ of centrally extended super L_∞-algebras (Def. 2.21) over $\mathfrak{g}$ via even 2-cocycles $c_A, c_B \in \mathrm{CE}(\mathfrak{g})$, supplied with 3-cocycle twists, respectively, are said to be in *T-duality correspondence* if:

(i) The respective fiber integration of the twists $h_{A/B}$ yields the opposite extension cocycles $c_{B/A}$ (cf. Eq. (141) of Rem. 3.17)

$$
(p_{A/B})_* H_{A/B} \;=\; c_{B/A} \qquad \in \qquad \mathrm{CE}(\mathfrak{g})\,.
$$

(ii) On the doubly extended space $\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B$ (cf. Def. 3.20)

$$
\begin{tikzcd}
& \widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] \arrow[d, phantom, "\text{(pb)}"] & \\
\widehat{\mathfrak{g}}_A \arrow[r, two heads, "p_A"'] & \mathfrak{g} & \widehat{\mathfrak{g}}_B\,, \arrow[l, two heads, "p_B"]
\end{tikzcd}
$$

defined dually via

$$
\mathrm{CE}\big(\,\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B\big) \quad = \quad \mathrm{CE}(\mathfrak{g})\big[e_A, e_B\big]\big/\big(\mathrm{d}e_{A/B} = c_{A/B}\big)\,,
$$

the *Poincaré form* (cf. (146))

$$
P \;:=\; e_B \cdot e_A \qquad \in \qquad \mathrm{CE}\big(\,\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B\big)
$$

is a coboundary of the difference between the pullbacks of the two twisting cocycles (cf. Prop. 3.21)

$$
\mathrm{d}P \;=\; \pi_A^* H_A - \pi_B^* H_B\,.
$$

The observation for the decomposed structure of the twisting NS-fluxes from Rem. 3.17 yields in fact an equivalent characterization of such a T-duality correspondence.

Lemma 3.25 (**T-duality conditions on the base).** *Two pairs* $(\widehat{\mathfrak{g}}_A, H_A)$ *and* $(\widehat{\mathfrak{g}}_B, H_B)$ *are in T-duality correspondence (Def. 3.51) if and only if the twists are of the form*

$$
H_{A/B} \;=\; H_{\mathfrak{g}} + e_{A/B} \cdot c_{B/A}
$$

for a common basic 3*-cochain* $H_{\mathfrak{g}} \in \mathrm{CE}(\mathfrak{g})$ *whose differential trivializes the product of the corresponding extending* 2*-cocycles*

$$
\mathrm{d}H_{\mathfrak{g}} \;=\; -c_A \cdot c_B\,.
$$

Proof. The first condition from Def. 3.51 yields immediately that

$$H_{A/B} = H_{\mathfrak{g}}^{A/B} + e_{A/B} \cdot c_{B/A}$$

for potentially different basic $H_{\mathfrak{g}}^A, H_{\mathfrak{g}}^B \in \mathrm{CE}(\mathfrak{g})$. The second condition then expands as

$$\begin{aligned} e_A \cdot c_B - e_B \cdot c_A = \mathrm{d}P &= \pi_A^* H_A - \pi_B^* H_B \\ &= H_{\mathfrak{g}}^A + e_A \cdot c_B - H_{\mathfrak{g}}^B - e_B \cdot c_A \end{aligned}$$

which holds if and only if

$$H_{\mathfrak{g}}^A = H_{\mathfrak{g}}^B .$$

Strictly speaking, this is an equation on $\mathrm{CE}(\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B)$ of (doubly) basic forms via $\pi_A^* \circ p_A^*$ and $\pi_B^* \circ p_B^*$, but these two *injective* morphisms actually coincide as maps $\mathrm{CE}(\mathfrak{g}) \hookrightarrow \mathrm{CE}(\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B)$, by construction, as can be seen immediately by their action on generators. Hence the equation holds equivalently on $\mathrm{CE}(\mathfrak{g})$.

Finally since the twists $H_{A/B}$ are by assumption closed on the respective extensions $\widehat{\mathfrak{g}}_{A/B}$, it follows in particular that

$$0 = \mathrm{d}H_A = \mathrm{d}H_{\mathfrak{g}} + c_A \cdot c_B$$

as an equation on $\mathfrak{g}_A$, which also holds as an equation on $\mathfrak{g}$ since both $H_{\mathfrak{g}}$ and $c_A \cdot c_B$ are implicitly pullbacks of basic forms via the dgca morphism $p_A^* : \mathrm{CE}(\mathfrak{g}) \hookrightarrow \mathrm{CE}(\widehat{\mathfrak{g}}_A)$, which is injective. The closure of h_B yields the same condition.

The reverse implication follows by the same computations. □

Corollary 3.26 (T-duality correspondence classifying space). *It follows that the T-duality L_∞-algebra $b\mathcal{T} \cong$ $\mathrm{cyc}\, b^2\mathbb{R}$ from Ex. 2.27 classifies the set of T-duality correspondences over any super-L_∞ algebra $\mathfrak{g}$, in that morphisms of super L_∞-algebras*

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & b\mathcal{T} \\ c_A & \longleftarrow\!\shortmid & \omega_2 \\ c_B & \longleftarrow\!\shortmid & -\widetilde{\omega}_2 \\ H_{\mathfrak{g}} & \longleftarrow\!\shortmid & h_3\,, \end{array}$$

are in canonical bijection with the set of T-duality correspondences over $\mathfrak{g}$ in terms of Lem. 3.25 (hence equivalently in terms of Def. 3.24).

For any such T-duality correspondence, the following natural "pull-push" homomorphism of 3-twisted cochain complexes

$$T_{\mathrm{FM}} \quad := \quad \pi_{B*} \circ e^{-P} \circ \pi_A^* \equiv \pi_{B*} \circ (1 + e_A \cdot e_B) \circ \pi_A^* \tag{151}$$

is an isomorphism of degree $(-1 \bmod 2, \mathrm{evn})$. In particular, it descends to an isomorphism on cohomology.

Theorem 3.27 (T-duality/Fourier-Mukai isomorphism). *Let $(\widehat{\mathfrak{g}}_A, H_A)$ and $(\widehat{\mathfrak{g}}_B, H_B)$ be in T-duality correspondence. Then the pull-push homomorphism (151) is in fact an isomorphism of 3-twisted cochain complexes (Def. 2.13) and acts explicitly as*

$$\begin{array}{ccc} T_{\mathrm{FM}} \;:\; \mathrm{CE}^{\bullet + H_A}(\widehat{\mathfrak{g}}_A) & \xrightarrow{\;\sim\;} & \mathrm{CE}^{(\bullet - 1) + H_B}(\widehat{\mathfrak{g}}_B) \\ \alpha = \alpha_{\mathrm{bas}} + e_A \cdot p_{A*}\alpha & \longmapsto & p_{A*}\alpha + e_B \cdot \alpha_{\mathrm{bas}}\,, \end{array} \tag{152}$$

thereby swapping the "winding" and "non-winding" modes.

In particular, it descends to a "Fourier-Mukai" isomorphism of twisted cocycles and furthermore twisted cohomologies

$$T_{\mathrm{FM}} \;:\; H_{\mathrm{CE}}^{\bullet + H_A}(\widehat{\mathfrak{g}}_A) \xrightarrow{\;\sim\;} H_{\mathrm{CE}}^{(\bullet - 1) + H_B}(\widehat{\mathfrak{g}}_B)\,.$$

Proof. The explicit form of the mapping follows immediately as

$$\begin{aligned} \pi_{B*}\big((1 + e_A \cdot e_B) \cdot (\alpha_{\mathrm{bas}} + e_A \cdot p_{A*}\alpha)\big) &= \pi_{B*}\big(\alpha_{\mathrm{bas}} + e_A \cdot e_B \cdot \alpha_{\mathrm{bas}} + e_A \cdot p_{A*}\alpha + 0\big) \\ &= p_{A*}\alpha + e_B \cdot \alpha_{\mathrm{bas}} \end{aligned}$$

where we absorbed the explicit mention of the injective morphism π_A^*, and used the fact that the fiber integration along $\pi_B : \widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B \to \mathfrak{g}_B$ is the derivation that takes the value 1 on the e_A generator, and 0 on the rest.

The fact that T_{FM} is a linear isomorphism of cochains follows by the existence of the explicit inverse

$$T_{\mathrm{FM}}^{-1} \quad := \quad \pi_{A*} \circ e^{P} \circ \pi_B^* \equiv \pi_{A*} \circ (1 + e_B \cdot e_A) \circ \pi_B^*$$

acting by "swapping back" the winding and non-winding modes,

$$\begin{array}{ccc} T_{\mathrm{FM}}^{-1} \;:\; \mathrm{CE}^{\bullet + H_B}(\widehat{\mathfrak{g}}_B) & \xrightarrow{\;\sim\;} & \mathrm{CE}^{(\bullet - 1 + H_A)}(\widehat{\mathfrak{g}}_A) \\ \tilde{\alpha} = \tilde{\alpha}_{\mathrm{bas}} + e_B \cdot p_{B*}\tilde{\alpha} & \longmapsto & +p_{B*}\tilde{\alpha} + e_B \cdot \tilde{\alpha}_{\mathrm{bas}}\,, \end{array}$$

precisely by the same calculation under the exchange of indices $A \leftrightarrow B$.

Lastly, to see that the linear isomorphism T_{FM} interwines with the twisted differentials, up to a sign (due to the odd degree of T_{FM} and that of the differentials), we compute

$$\begin{aligned}(\mathrm{d}_{\widehat{\mathfrak{g}}_B} - H_B) \circ T_{\mathrm{FM}}(\alpha) &= (\mathrm{d}_{\widehat{\mathfrak{g}}_B} - H_{\mathfrak{g}} - e_B \cdot c_A)\big(p_{A*}\alpha + e_B \cdot \alpha_{\mathrm{bas}}\big) \\ &= (\mathrm{d}\, p_{A*}\alpha + c_B \cdot \alpha_{\mathrm{bas}} - H_{\mathfrak{g}} \cdot p_{A*}\alpha) \\ &\quad + e_B \cdot (-\mathrm{d}\alpha_{\mathrm{bas}} + H_{\mathfrak{g}} \cdot \alpha_{\mathrm{bas}} - c_A \cdot p_{A*}\alpha)\end{aligned}$$

where we used the explicit swapping action of T_{FM} and the explicit form of the twist h_B from Lem. 3.52.

Computing similarly,

$$\begin{aligned}T_{\mathrm{FM}} \circ (\mathrm{d}_{\widehat{\mathfrak{g}}_A} - H_A)(\alpha) &= T_{\mathrm{FM}}\big(\mathrm{d}\alpha_{\mathrm{bas}} + c_A \cdot p_{A*}\alpha - H_{\mathfrak{g}} \cdot \alpha_{\mathrm{bas}} + e_A \cdot (-\mathrm{d}\, p_{A*}\alpha + H_{\mathfrak{g}} \cdot p_{A*}\alpha - c_B \cdot \alpha_{\mathrm{bas}})\big) \\ &= -(\mathrm{d}\, p_{A*}\alpha + c_B \cdot \alpha_{\mathrm{bas}} - H_{\mathfrak{g}} \cdot p_{A*}\alpha) \\ &\quad - e_B \cdot (-\mathrm{d}\alpha_{\mathrm{bas}} + H_{\mathfrak{g}} \cdot \alpha_{\mathrm{bas}} - c_A \cdot p_{A*}\alpha)\,,\end{aligned}$$

and hence

$$(\mathrm{d}_{\widehat{\mathfrak{g}}_B} - H_B) \circ T_{\mathrm{FM}} \;\;=\;\; - T_{\mathrm{FM}} \circ (\mathrm{d}_{\widehat{\mathfrak{g}}_A} - H_A)\,. \qquad \square$$

Corollary 3.28 (**Pull-push via automorphism of cyclified twisted K-theory spectra**). *Under the identification of twisted K-theory cocycles with* 3*-twisted cocycles from Eq.* (29)*, the action of the T-duality isomorphism* (152) *by Fourier-Mukai transform coincides with that of the composite operation of (1.) reduction, (2.) automorphism of cyclified twisted K-theory spectra (3.) reoxidation from Lem. 2.30, up to a (conventional) sign*

$$\begin{array}{ccc} \left\{ \begin{array}{ccc} \mathfrak{g}_A & \longrightarrow & \mathfrak{l}\big(\Sigma^0 \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \\ & \underset{H_A}{\searrow} \; b^2\mathbb{R} \; \swarrow & \end{array} \right\} & \xleftrightarrow{\text{Lem. } 2.30} & \left\{ \begin{array}{ccc} \mathfrak{g}_B & \longrightarrow & \mathfrak{l}\big(\Sigma^1 \mathrm{KU} /\!\!/ B\mathrm{U}(1)\big) \\ & \underset{H_B}{\searrow} \; b^2\mathbb{R} \; \swarrow & \end{array} \right\}. \\ \big\downarrow & & \big\downarrow \\ \mathrm{CE}^{0+H_A}(\mathfrak{g}_A) & \xleftrightarrow{\;\;-T_{\mathrm{FM}}\;\;} & \mathrm{CE}^{-1+H_B}(\mathfrak{g}_B) \end{array}$$

Remark 3.29 (**Strict isomorphism vs quasi-isomorphism**). The original article [FSS18a] focuses on the induced isomorphism on twisted L_∞-cohomology. Nevertheless, we stress that the map actually defines a strict isomorphism even at the level of twisted cochain complexes, and hence crucially at the level of cocycles (and further on cohomology). From a physical perspective, this means that the isomorphism is realized at the level of flux densities (prior to flux-quantization) and not only at the level of the corresponding gauge equivalency classes.

This concludes our analysis of T-duality of type II super-flux densities on super-tangent spaces. With this in hand, we now turn in §3.3 attention to possible M-theory lifts of the situation.

3.3 Lift to the M-algebra

We show that the super Lie algebra of the F-theory super-spacetime (Def. 3.30), but for T-duality reduction on a (1,9 | 32)-dimensional super-torus all the way down to the point, is further extended by the "M-algebra", with the "double" copy of the (full 10-dimensional) super-spacetime now constituted entirely by membrane wrapping modes, and with the Poincaré super 2-form (146) lifted to an M-theoretic Poincaré super 3-form. In the companion article [GSS24d] we use this to explain the M-algebra as the super-space version of the exceptional-geometric tangent space for 11d supergravity reduced all the way to the point.

F-Theory super-spacetime. Given that the derivation began in (127) on 11D super-spacetime, going through its reduction to 10D IIA super-spacetime (3.2), to arrive at its "doubled" version (145), it is natural to ask for the doubling of the T-dualized fiber to take place already in 11D, hence for extending 9D super-spacetime by *all three* extra dimensions:

(i) the IIA fiber,

(ii) the IIB fiber,

(iii) the M fiber.

At the level of super-Lie algebraic local model spaces, this request is immediate to satisfy:

Definition 3.30 (**F-theory super-spacetime** [FSS18a, Def. 8.1, Prop. 8.3])**.** Write $\mathbb{R}^{1,9+(1+1)\,|\,32}$ for the super-Lie algebra given by

$$
\mathrm{CE}\big(\mathbb{R}^{1,9+(1+1)\,|\,32}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a_A)_{a=1}^{9} \\ e^{10}, e^9_A, e^9_B \end{array}\right] \Big/ \left(\begin{array}{lll} \mathrm{d}\,\psi^\alpha &= 0 & \\ \mathrm{d}\,e^a &= \big(\overline{\psi}\,\Gamma^a\,\psi\big) & \\ \mathrm{d}\,e^9_A &= \big(\overline{\psi}\,\Gamma^9\,\psi\big) &= \big(\overline{\psi}\,\sigma_1\,\psi\big) \\ \mathrm{d}\,e^{10} &= \big(\overline{\psi}\,\Gamma^{10}\,\psi\big) &= \big(\overline{\psi}\,\sigma_2\,\psi\big) \\ \mathrm{d}\,e^9_B &= \big(\overline{\psi}\,\Gamma^9\Gamma^{10}\,\psi\big) &= \big(\overline{\psi}\,\sigma_3\,\psi\big) \end{array}\right) \tag{153}
$$

(using the notation (124) on the right) which is equivalently the homotopy-fiber product (59) of the 11D super-spacetime with the doubled super-spacetime (145):

$$
\begin{tikzcd}
 & \underset{\text{F super-spacetime}}{\mathbb{R}^{1,9+(1+1)\,|\,32}} \arrow[dl] \arrow[dr] & & \\
\underset{\text{11D super-spacetime}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \arrow[dr, two heads, "p_M"'] & & \underset{\text{doubled super- spacetime}}{\mathbb{R}^{1,8+(1+1)\,|\,32}} \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] & \\
 & \underset{\text{IIA super-spacetime}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[dr, two heads] & & \underset{\text{IIB super-spacetime}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}} \arrow[dl, two heads] \\
 & & \underset{\text{9D super-spacetime}}{\mathbb{R}^{1,8|\mathbf{16}\oplus\mathbf{16}}} &
\end{tikzcd}
\tag{154}
$$

By inspection, one sees that (cf. also [Sak00]):

Proposition 3.31 (**Superspace S-duality on F-theory super-spacetime** [FSS18a, Prop. 8.6])**.** *The group* $\mathrm{Pin}(2)$ *of Prop. 3.8 acts by super-Lie automorphisms on the F-theory super-spacetime* (153) *under which (the pullback of) flux densities* H_3^B *and* F_3 (from Prop. 3.15: the F1- and the D1-string couplings) *span the 2-dimensional vector representation*

$$
\begin{array}{ccc}
\mathbb{R}^{1,9+(1+1)\,|\,32} & \xrightarrow[\sim]{\;\exp\left(\frac{t}{2}\sigma_3\right)\;} & \mathbb{R}^{1,9+(1+1)\,|\,32} \\
e^{\frac{t}{2}\Gamma_9\Gamma_{10}}\psi & \longleftarrow\!\shortmid & \psi \\
\cos(t)\,e^9 + \sin(t)\,e^{10} & \longleftarrow\!\shortmid & e^9 \\
\cos(t)\,e^{10} - \sin(t)\,e^9 & \longleftarrow\!\shortmid & e^{10} \\
\cos(t)\,F_3 + \sin(t)\,H_3^A & \longleftarrow\!\shortmid & F_3 \\
\cos(t)\,H_3^A - \sin(t)\,F_3 & \longleftarrow\!\shortmid & H_3^A
\end{array}
$$

Noting that $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$ is the maximal compact subgroup of the S-duality group of IIB supergravity, hence the respective *local* U-duality group, this justifies the "F-theory" terminology (as concerned with the lift of T-duality on a single fiber in 10d to M-theory on a torus fiber, cf. [Jo97]); leaving open, however, the question if this local model space (145) supports a global super-field theory the way that its projection to $\mathbb{R}^{1,10|\mathbf{32}}$ (2) supports 11d SuGra. In this vein, we now ask for yet further extension to bring out more of the structure expected in M-theory.

First, we turn to T-duality not just along a 1-dimensional fiber, but along *all* spacetime directions.

10-Toroidal T-duality on super-fluxes. By combining the discussion of super-space T-duality in §3.2 with that of higher-dimensional torus reductions in §2.3 it is now immediate to T-dualize super-fluxes on super n-tori for higher n. In particular, since the type IIA super-spacetime is a $(1+9|32)$-dimensional torus extension of the super-point $\mathbb{R}^{0\,|\,32}$ (Ex. 2.34) we may consider, in immediate higher dimensional analogy to (138), the composite operation of:

(i) toroidally reducing the type IIA super-fluxes to the super-point, via Prop. 2.41,

(ii) transforming the result along the T^{10}-automorphism of Prop. 2.56,

(iii) toroidally re-oxidizing the result, but now with respect to the 10 resulting T-dual Chern classes $\overset{k}{p}_* H_3^A$

$$
\begin{tikzcd}
 & \widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "\mathrm{oxd}_{(\overline{\psi}\Gamma^\bullet\psi)}\, T^{10}\left(\mathrm{rdc}_{(\overline{\psi}\Gamma^\bullet\psi)}\left(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}}\right)\right)", "=:\,\left(H_3^{\widetilde{A}},\,(\widetilde{F}_{2k})_{k\in\mathbb{Z}}\right)"'] \arrow[ddd, two heads, "\tilde{p}"] & & \mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "\left(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}}\right)"] \arrow[ddr, two heads, "p"'] & & \mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \\
 & & & \mathrm{tor}^{10}\,\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
 & \mathbb{R}^{0\,|\,32} \arrow[urr, "T^{10}\left(\mathrm{rdc}_{(\overline{\psi}\Gamma^\bullet\psi)}\left(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}}\right)\right)"] \arrow[dr, "\mathrm{rdc}_{(\overline{\psi}\Gamma^\bullet\psi)}\left(H_3^A,\,(F_{2k})_{k\in\mathbb{Z}}\right)"'] & & \\
 & & \mathrm{tor}^{10}\,\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \arrow[uur, "T^{10}", "\sim"'] &
\end{tikzcd}
\tag{155}
$$

Here the duality operation T^{10} (89) is seen to act on the NS- and D0 fluxes as (89)

$$
\begin{array}{ccccccc}
 & \mathbb{R}^{0\,|\,32} & \xrightarrow{\mathrm{rdc}_{(\overline{\psi}\,\Gamma^\bullet\psi)}\left(H_3^A,\,(F_{2\bullet})\right)} & \mathrm{tor}^{10}\,\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) & \xrightarrow[\sim]{T^{10}} & \mathrm{tor}^{10}\,\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big) \\
+\overset{a}{c}_1 = & \big(\overline{\psi}\,\Gamma^a\,\psi\big) & \longleftarrow\!\shortmid & \overset{a}{\omega}_2 & & -\overset{a}{s}h_3 \\
 & 0 & \longleftarrow\!\shortmid & h_3 & \longleftarrow\!\shortmid & h_3 \\
-\overset{a}{\tilde{c}}_1 = & -\big(\overline{\psi}\,\Gamma^a\Gamma_{10}\,\psi\big) & \longleftarrow\!\shortmid & \overset{a}{s}h_3 & \longleftarrow\!\shortmid & -\overset{a}{\omega}_2
\end{array}
\tag{156}
$$

This means that the re-oxidized dual structure is given by:

Definition 3.32 (**The fully T-dual super-spacetime**)**.** The fully T-dual super-spacetime in (155) is $\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ given by

$$
\mathrm{CE}\big(\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}(\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{9}\end{array}\right] \Big/ \left(\begin{array}{rcl}\mathrm{d}\,\psi &=& 0 \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)\end{array}\right)
\tag{157}
$$

carrying the fully dual super-flux density

$$
H_3^{\widetilde{A}} \;=\; \sum_{a=0}^{9} e^a\,\overset{a}{c}_1 \;=\; e^a\big(\overline{\psi}\,\Gamma_a\,\psi\big).
\tag{158}
$$

Remark 3.33 (**Nature of the fully T-dual IIA super-spacetime.**)**.**
(i) The fully T-dual super-Lie algebra $\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ (157) is not actually isomorphic to the ordinary IIA super-spacetime $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ as real super-Lie algebras. It behaves like the IIA super-spacetime but with the signature convention of the metric, or equivalently of its Clifford anti-commutator, swapped:
(ii) While the IIA super-spacetime is controlled by the original Clifford generators

$$
(\Gamma_a)_{a=0}^{9}\,, \qquad \Gamma_a\,\Gamma_b + \Gamma_b\Gamma_a \;=\; 2\eta_{ab}\,,
$$

its fully T-dualized version is controlled by the 10D variant of (119):

$$
\big(\widetilde{\Gamma}_a := \Gamma_a\Gamma_{10}\big)_{a=0}^{9}\,, \qquad \widetilde{\Gamma}_a\,\widetilde{\Gamma}_b + \widetilde{\Gamma}_b\,\widetilde{\Gamma}_a \;=\; -2\eta_{ab}\,.
$$

In variation of Def. 3.20, we now have:

Definition 3.34 (**The fully doubled super-spacetime**)**.** The fully doubled super-spacetime is the homotopy-fiber product (59) of the type IIA spacetime (Ex. 3.2) with its full T-dual (Def. 3.32) over the super-point (57):

$$
\begin{tikzcd}
 & \mathfrak{Dbl} \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_{\widetilde{A}}"] & \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dr, "p_A"'] & & \widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dl, "\widetilde{p}_A"] \\
 & \mathbb{R}^{0\,|\,32} &
\end{tikzcd}
\tag{159}
$$

given by

$$
\mathrm{CE}\big(\mathfrak{Dbl}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}(\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{9} \\ (\tilde{e}^a)_{a=0}^{9}\end{array}\right] \Big/ \left(\begin{array}{l}\mathrm{d}\,\psi = 0 \\ \mathrm{d}\,e^a = \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,\tilde{e}^a = \big(\overline{\psi}\,\Gamma^a\Gamma_{10}\,\psi\big)\end{array}\right).
\tag{160}
$$

A simple but important consequence for us is now the following variant of Prop. 3.21:

Proposition 3.35 (**Full Poincaré super 2-form**). *On the fully doubled super-spacetime* (160) *the 2-form*

$$P_2^{10} \;:=\; \tilde{e}_a\, e^a \;\in\; \mathrm{CE}\big(\mathfrak{Dbl}\big) \tag{161}$$

is a coboundary of the difference between the NS 3-flux H_3^A (133) *in IIA and its full T-dual $H_3^{\tilde{A}}$* (158)

$$\begin{aligned} \mathrm{d}\, P_2^{10} &= \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a - \tilde{e}_a\big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ &= \pi_A^\ast\, H_3^A - \pi_{\tilde{A}}^\ast\, H_3^{\tilde{A}}\,. \end{aligned} \tag{162}$$

The fully extended IIA super-algebra. Now we observe that the *fully doubled* super-spacetime (3.34) is further extended by what is known as the *fully extended* IIA super-spacetime (in the sense of [vHvP82]):

Definition 3.36 (**The fully extended type IIA algebra**). The translational type IIA fully extended supersymmetry algebra $\mathrm{II}\mathfrak{A}$ is (e.g. [CdAIP00, (2.16)] [14]) given by [15]

$$\mathrm{CE}\big(\mathrm{II}\mathfrak{A}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=1}^{9} \\ (\tilde{e}^a)_{a=1}^{9} \\ (e^{a_1a_2} = e^{[a_1a_2]})_{a_i=0}^{9} \\ (e^{a_1\cdots a_4} = e^{[a_1\cdots a_4]})_{a_i=0}^{9} \\ (e^{a_1\cdots a_5} = e^{[a_1\cdots a_5]})_{a_i=0}^{9} \end{array}\right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\psi &= 0 \\ \mathrm{d}\,e^a &= +\big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,\tilde{e}^a &= +\big(\overline{\psi}\,\Gamma^a\Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1a_2} &= -\big(\overline{\psi}\,\Gamma^{a_1a_2}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_4} &= +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_4}\Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_5} &= +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big) \end{array}\right). \tag{163}$$

Remark 3.37 (**Extended IIA-algebra and brane charges**).
(i) The bosonic body of the fully extended type IIA algebra (163) is

$$\begin{aligned} \big(\mathrm{II}\mathfrak{A}\big)_{\mathrm{bos}} &\simeq_{\mathbb{R}} \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^\ast \oplus \qquad \wedge^2(\mathbb{R}^{1,9})^\ast \qquad \oplus \qquad \wedge^4(\mathbb{R}^{1,9})^\ast \qquad \oplus \wedge^5(\mathbb{R}^{1,9})^\ast \\ &\simeq_{\mathbb{R}} \underbrace{\mathbb{R}^{1,9}}_{\text{space-time}} \oplus \underbrace{(\mathbb{R}^{1,9})^\ast}_{\substack{\text{T-dual} \\ \text{space-time /} \\ \text{string charges}}} \oplus \underbrace{\wedge^2(\mathbb{R}^9)^\ast}_{\substack{\text{D2-brane} \\ \text{charges}}} \oplus \underbrace{\wedge^8(\mathbb{R}^9)}_{\substack{\text{D8-brane} \\ \text{charges}}} \oplus \underbrace{\wedge^4(\mathbb{R}^9)^\ast}_{\substack{\text{D4-brane} \\ \text{charges}}} \oplus \underbrace{\wedge^6(\mathbb{R}^9)}_{\substack{\text{D6-brane} \\ \text{charges}}} \oplus \underbrace{\wedge^5(\mathbb{R}^{1,9})^\ast}_{\substack{\text{NS5-brane} \\ \text{charges}}} \end{aligned} \tag{164}$$

where in the second line we Hodge-dualized all temporal components (following [Hull98, (2.12)]) by the rule

$$\wedge^p(\mathbb{R}^{1,d})^\ast \;\simeq_{\mathbb{R}}\; \underbrace{\wedge^p(\mathbb{R}^d)^\ast}_{\text{spatial}} \oplus \underbrace{\wedge^{1+d-p}(\mathbb{R}^d)}_{\substack{\text{dualized} \\ \text{temporal}}}\,.$$

(ii) Note how the string charges in (164) play a special role as compared to the (other) brane charges: They appear with their temporal component included and, as such, may equivalently be understood as, in fact, being the T-dual *spacetime dimensions*. Thus, the fully extended IIA algebra (3.32) is a toroidal extension (Def. 2.32) of the fully doubled super-spacetime (159) by the D-brane and NS5-brane charges:

$$\begin{array}{ccccc} \mathrm{II}\mathfrak{A} & \twoheadrightarrow & \mathfrak{Dbl} & \xrightarrow{\text{brane charges}} & b\mathbb{R}^{507} \\ \psi & \longleftarrow\!\shortmid & \psi & & \\ e^a & \longleftarrow\!\shortmid & e^a & & \\ \tilde{e}_a & \longleftarrow\!\shortmid & \tilde{e}_a\,. & & \end{array} \tag{165}$$

This relation between extended IIA super-symmetry and T-duality correspondence super-space may not have previously been appreciated as such. To make it fully manifest:

[14] In [CdAIP00, (2.16)] also the D0-brane charge with differential $(\overline{\psi}\,\Gamma_{10}\,\psi)$ – is included in the extended IIA-algebra (163). But condensing D0-brane charge of course means opening up the 11th dimension, and hence here we regard this term instead as providing the further extension to the M-algebra, see Ex. 3.40.

[15] The signs in (163) are a convention that is natural in view of the further extension by the M-algebra (168), where these signs align with the Fierz identity (228), and makes the exceptional brane rotating symmetry in Prop. 3.44 come out naturally.

Remark 3.38 (Fully extended IIA as fiber product of doubled super-space with brane charges). To make manifest the double role that the string charges $\tilde{e}_a$ play in (165) — on the one hand as a doubled copy of spacetime and on the other as part of the general brane charges — consider the following super-Lie algebra of *pure brane charges* $\mathfrak{Brn}$, given by

$$\mathrm{CE}\big(\mathfrak{Brn}\big) \;\simeq\; \left(\begin{array}{rl} \mathrm{d}\,\psi & = 0 \\ \mathrm{d}\,e^a & = +\big(\overline{\psi}\,\Gamma^a\Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1a_2} & = -\big(\overline{\psi}\,\Gamma^{a_1a_2}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_4} & = +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_4}\Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_5} & = +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big) \end{array}\right), \tag{166}$$

hence the extension of the fully dual super-spacetime (157) (which may be regarded as consisting entirely of string charges) by the remaining D/NS-brane charges (164).

Then the fully extended IIA algebra $\mathrm{II}\mathfrak{A}$ (163) is the fiber product (59) over the fully T-dual super-spacetime (157) of this pure brane charge algebra (166) with the fully doubled super-spacetime (160):

$$\begin{tikzcd}[column sep=small] & & \mathrm{II}\mathfrak{A} \arrow[dl, ->>] \arrow[dr, ->>] & & \\ & \mathfrak{Dbl} \arrow[dl, ->>] \arrow[dr, ->>] & & \mathfrak{Brn} \arrow[dl, ->>] & \\ \underset{\text{IIA spacetime}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} & & \underset{\text{IIA string charges}}{\widetilde{\mathbb{R}}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} & \underset{\text{IIA brane charges}}{} & \end{tikzcd} \tag{167}$$

The M-algebra. Similar to the fully extended IIA super-symmetry algebra, we have the full extension of the 11D supersymmetry algebra, which may be understood ([To95, (13)][To98, (1)]) as incorporating charges $Z^{a_1a_2}$ of M2-branes and $Z^{a_1\cdots a_5}$ of M5-branes (the terminology *M-algebra* follows [Se97][BDPV05][Ba17, (3.1)] [16]):

Definition 3.39 (Basic M-algebra). The *basic M-algebra* is the super-Lie algebra $\mathfrak{M}$ given by [17]

$$\mathrm{CE}\big(\mathfrak{M}\big) \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{10} \\ \big(e^{a_1a_2} = e^{[a_1a_2]}\big)_{a_i=0}^{10} \\ \big(e^{a_1\cdots a_5} = e^{[a_1\cdots a_5]}\big)_{a_i=0}^{10} \end{array}\right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\psi & = 0 \\ \mathrm{d}\,e^a & = +\big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,e^{a_1a_2} & = -\big(\overline{\psi}\,\Gamma^{a_1a_2}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_5} & = +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big) \end{array}\right). \tag{168}$$

Example 3.40 (Basic M-Algebra as extension of fully extended type IIA super-spacetime.). The basic M-algebra (168) is a central extension (Def. 2.21) of the fully extended type IIA algebra (163) by (the pullback of) the same 2-cocycle (114) that classifies the M/IIA extension:

$$\begin{array}{rcccl} & \mathfrak{M} & \longrightarrow\!\!\!\!\rightarrow & \mathrm{II}\mathfrak{A} & \xrightarrow{\;(\overline{\psi}\Gamma^{10}\psi)\;} b\mathbb{R} \\ & \psi & \longleftarrow\!\!\!\shortmid & \psi & \\ & e^a & \longleftarrow\!\!\!\shortmid & e^a & \\ \text{wrapped M2-brane charges} & e^{10\,a} & \longleftarrow\!\!\!\shortmid & \tilde{e}^a & \text{string charges / doubled spacetime} \\ & e^{a_1a_2} & \longleftarrow\!\!\!\shortmid & e^{a_1a_2} & \\ & e^{10\,a_1\cdots a_4} & \longleftarrow\!\!\!\shortmid & e^{a_1\cdots a_4} & \\ & e^{a_1\cdots a_5} & \longleftarrow\!\!\!\shortmid & e^{a_1\cdots a_5}\,. & \end{array} \tag{169}$$

which means that the M-algebra is equivalently the fiber product (59) of the fully extended IIA spacetime with the 11D super-space over the 10D IIA spacetime:

$$\begin{tikzcd}[column sep=small] & \mathfrak{M} \arrow[dl, ->>] \arrow[dr, ->>] & \\ \mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[dr, ->>] & & \mathrm{II}\mathfrak{A} \arrow[dl, ->>] \\ & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} & \end{tikzcd}$$

[16] [Se97] uses the term "M-algebra" for a large further extension of (168) which includes the "hidden algebra" of [DF82][AD24]; whereas other authors like [BDPV05] say "M-algebra" for just (168). Here we disambiguate this situation by speaking of the "basic" M-algebra. More discussion of this point is in the companion article [GSS24d].

[17] The sign convention in (168) is natural in view of the Fierz identity (228), and makes the exceptional brane rotating symmetry in Prop. 3.44 come out naturally.

The assignment (169) reflects the isomorphism

$$
\begin{aligned}
\mathfrak{M}_{\mathrm{bos}} &\simeq_{\mathbb{R}} \quad \mathbb{R}^{1,10} \quad \oplus \quad \wedge^2(\mathbb{R}^{1,10})^* \quad \oplus \quad \wedge^5(\mathbb{R}^{1,10})^* \\
&\simeq_{\mathbb{R}} \mathbb{R} \oplus \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^* \oplus \wedge^2(\mathbb{R}^{1,9})^* \oplus \wedge^4(\mathbb{R}^{1,9})^* \oplus \wedge^5(\mathbb{R}^{1,9})^* \\
&\simeq_{\mathbb{R}} \mathbb{R} \oplus \big(\widehat{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}}\big)_{\mathrm{bos}}\,,
\end{aligned}
$$

where in the second line we have decomposed into components that are parallel resp. orthogonal to the 10-coordinate axis, by the rule

$$
\wedge^p\big(\mathbb{R}^{1,d}\big)^* \;\simeq_{\mathbb{R}}\; \wedge^{p-1}\big(\mathbb{R}^{1,d-1}\big)^* \oplus \wedge^p\big(\mathbb{R}^{1,d-1}\big)^*\,.
$$

Of particular importance for the picture of (190) is the boxed assignment in (169), which identifies the string charges (alternatively: doubled spacetime directions) in the fully extended IIA algebra with the charges of M2-branes wrapping the 10-axis. The assignments below the box in (169) are the remaining D- and NS5-brane charges.

In order to make this distinction manifest and in view of Rem. 3.38, consider:

Remark 3.41 (M-algebra as extended doubled super-space). With the $\mathfrak{M}$-algebra being a spacetime-extension (114) of the $\mathrm{II}\mathfrak{A}$-algebra, it must, by (167) also be an extension by brane charges of a spacetime extension of the fully doubled superspacetime (160). To make this manifest — recalling from (154) that the analogous spacetime extension of the partially doubled super-spacetime is the F-theory super-spacetime — we shall write $\mathfrak{F}$ for the super-Lie algebra given by

$$
\mathrm{CE}(\mathfrak{F}) \;=\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\psi^\alpha)_{\alpha=1}^{32}\\ \big(e^a\big)_{a=0}^{10}\\ \big(\tilde e_a\big)_{a=0}^{9}\end{array}\right]\Big/\left(\begin{array}{l}\mathrm{d}\,\psi = 0\\ \mathrm{d}\,e^a = \big(\overline{\psi}\,\Gamma^a\,\psi\big)\\ \mathrm{d}\,\tilde e_a = \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)\end{array}\right) \tag{170}
$$

and thus extending the fully doubled super-spacetime in the same way that 11D super-spacetime extends 10D type IIA:

$$
\begin{array}{ccccc}
\mathfrak{F} & \twoheadrightarrow & \mathfrak{Dbl} & \xrightarrow{(\overline{\psi}\Gamma_{10}\psi)} & b\mathbb{R} \\
e^a & \longleftarrow\!\shortmid & e^a & & \\
\tilde e_a & \longleftarrow\!\shortmid & \tilde e_a\,. & &
\end{array} \tag{171}
$$

Then the M-algebra is the fiber product (59) over the fully doubled super-spacetime of this full $\mathfrak{F}$-spacetime with the fully extended $\mathrm{II}\mathfrak{A}$-spacetime:

$$
\begin{tikzcd}
 & \mathfrak{M} \arrow[dl, two heads, "p^{\mathrm{Brn}}"'] \arrow[dr, two heads, "p^M"] & \\
\mathfrak{F} \arrow[dr, two heads, "p^M"'] & & \mathrm{II}\mathfrak{A} \arrow[dl, two heads, "p^{\mathrm{Brn}}"] \\
 & \mathfrak{Dbl} &
\end{tikzcd} \tag{172}
$$

This perspective reveals a relation of the M-algebra to T-duality, cf. Prop. 174.

Remark 3.42 (Lift of the Poincaré super 2-form to the M-algebra). On the basic M-algebra (168), consider the element

$$
P_3 \;=\; \tfrac{1}{2} e^{a_1} e_{a_1 a_2} e^{a_3} \quad \in\; \mathrm{CE}(\mathfrak{M}) \tag{173}
$$

which we shall call the *Poincaré super 3-form*, since it is an M-theoretic lift of the Poincaré super 2-form from (146):

It is immediadate that the dimensional reduction the Poincaré 3-form (173) on the M-algebra (by fiber integration along e^{10}) reproduces the full Poincaré 2-form (173) on the doubled super-space:

$$
p^{\mathrm{Brn}}_{\mathrm{bas}}\; p^M_*\, P_3 \;=\; P_2\,. \tag{174}
$$

Given that the Poincaré 2-form P_2 in $\mathfrak{Dbl}$ entirely controls rational-topological T-duality in 10D, this exhibits P_3 on $\mathfrak{M}$ as analogously reflecting the T-duality phenomenon in M-theory.

We summarize the resulting picture in §4, see (190).

Remark 3.43 (M-algebra as fiber product of spacetime with brane charges). Another, equivalent, perspective is that the M-algebra is the fiber product of 11D superspace with the M-brane charges over the super-point. To make this manifest, observe that the pure brane algebra (166), which we introduced from the point of view of IIA branes, has the following isomorphic CE-algebra

$$
\begin{tikzcd}
& \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=0}^{9} \\ (e^{a_1 a_2} = e^{[a_1 a_2]})_{a_i=0}^{9} \\ (e^{a_1\cdots a_4} = e^{[a_1 a_2]})_{a_i=0}^{9} \\ (e^{a_1\cdots a_4} = e^{[a_1 a_2]})_{a_i=0}^{9} \end{array}\right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\psi & = 0 \\ \mathrm{d}\,e^a & = +\big(\overline{\psi}\,\Gamma^a \Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1 a_2} & = -\big(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_4} & = +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_4}\Gamma_{10}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_5} & = \big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big) \end{array}\right) \arrow[dd, "\wr"] & \begin{array}{ccccc} \psi & e^a & e^{a_1 a_2} & e^{a_1\cdots a_4} & e^{a_1\cdots a_5} \end{array} \\
\mathrm{CE}\big(\mathfrak{Brn}\big) \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & \\
& \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^{a_1 a_2} = e^{[a_1 a_2]})_{a_i=0}^{10} \\ (e^{a_1\cdots a_5} = e^{[a_1\cdots a_5]})_{a_i=0}^{10} \end{array}\right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\psi & = 0 \\ \mathrm{d}\,e^{a_1 a_2} & = -\big(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi\big) \\ \mathrm{d}\,e^{a_1\cdots a_5} & = +\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big) \end{array}\right) & \begin{array}{ccccc} \psi & e^{10\,a} & e^{a_1 a_2} & e^{10\,a_1\cdots a_4} & e^{a_1\cdots a_5} \\ & & (a_i \leq 9) & & \end{array} \arrow[uu, mapsto]
\end{tikzcd}
$$

witnessing the duality between IIA-branes and M-branes. But this makes it manifest that the M-algebra is also the fiber product (59) over the super-point of 11D super-spacetime with the brane charges:

$$
\begin{tikzcd}
& \mathfrak{M} \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[dr, two heads] & & \mathfrak{Brn} \arrow[dl, two heads] \\
& \mathbb{R}^{0\,|\,32} &
\end{tikzcd}
\tag{175}
$$

U-duality realized on the M-algebra. With the M-algebra thus emerging as an M-theoretic extension of the fully doubled super-spacetime on which T-duality becomes manifest, we just note that it carries a canonical action of the expected *local* hidden U-duality symmetry of M-theory, namely of the "maximal compact subalgebra" of $\mathfrak{e}_{11}$:

Proposition 3.44 (Manifestly $\mathrm{GL}(32;\mathbb{R})$-equivariant incarnation of basic M-algebra [We03, §4][BW00, §5]). *Unifying all the even generators of the M-algebra (169) into a symmetric bispinorial form like this*

$$
e^{\alpha\beta} \;:=\; \tfrac{1}{32}\big(e^a\,\Gamma_a^{\alpha\beta} + \tfrac{1}{2}e^{a_1 a_2}\,\Gamma_{a_1 a_2}^{\alpha\beta} + \tfrac{1}{5!}e^{a_1\cdots a_5}\,\Gamma_{a_1\cdots a_5}^{\alpha\beta}\big)
\tag{176}
$$

the CE-differential acquires equivalently the compact form

$$
\begin{array}{rcl}
\mathrm{d}\,\psi^\alpha & = & 0 \\
\mathrm{d}\,e^{\alpha\beta} & = & \psi^\alpha\,\psi^\beta,
\end{array}
\tag{177}
$$

which makes manifest that any $g \in \mathrm{GL}(32,\mathbb{R})$ acts via super-Lie algebra automorphisms of the M-algebra

$$
\begin{array}{rccc}
g\;: & \mathrm{CE}\big(\mathfrak{M}\big) & \longrightarrow & \mathrm{CE}\big(\mathfrak{M}\big) \\
& \psi^\alpha & \longmapsto & g^\alpha_{\alpha'}\,\psi^{\alpha'} \\
& e^{\alpha\beta} & \longmapsto & g^\alpha_{\alpha'}\,g^\beta_{\beta'}\,e^{\alpha'\beta'}.
\end{array}
\tag{178}
$$

Proof. First, to see that the transformation (176) is invertible, the trace-property (217) allows to recover:

$$
\begin{array}{rcl}
e^a & = & \Gamma^a_{\alpha\beta}\,e^{\alpha\beta} \\
e^{a_1 a_2} & = & -\Gamma^{a_1 a_2}_{\alpha\beta}\,e^{\alpha\beta} \\
e^{a_1\cdots a_5} & = & \Gamma^{a_1\cdots a_5}_{\alpha\beta}\,e^{\alpha\beta}.
\end{array}
\tag{179}
$$

The differential is as claimed due to the Fierz expansion formula (228):

$$
\begin{array}{rcll}
\mathrm{d}\,e^{\alpha\beta} & = & \tfrac{1}{32}\Big(\Gamma_a^{\alpha\beta}\,\big(\overline{\psi}\,\Gamma^a\,\psi\big) - \tfrac{1}{2}\Gamma_{a_1 a_2}^{\alpha\beta}\big(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi\big) + \tfrac{1}{5!}\Gamma_{a_1\cdots a_5}^{\alpha\beta}\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big)\Big) & \text{by (176) \& (168)} \\
& = & \psi^\alpha\,\psi^\beta & \text{by (228).}
\end{array}
$$

$\square$

Example 3.45 (Brane rotating symmetry). On the original bosonic generators (168) – the spacetime momentum e^a, the M2-brane charges $e^{a_1 a_2}$ and the M5-brane charges $e^{a_1 \cdots a_5}$ — the $\mathrm{GL}(32;\mathbb{R})$ symmetry of (177) acts by mixing them all among each other, e.g.

$$
\begin{aligned}
e^a \;&=\; \Gamma^a_{\alpha\beta} e^{\alpha\beta} && \text{by (179)} \\
&\overset{g}{\mapsto}\; \Gamma^a_{\alpha\beta} g^\alpha_{\alpha'} g^\beta_{\beta'} e^{\alpha'\beta'} && \text{by (178)} \\
&=\; \big(\tfrac{1}{32}\Gamma^a_{\alpha\beta} g^\alpha_{\alpha'} g^\beta_{\beta'} \Gamma^{\alpha'\beta'}_b\big) e^b \;+\; \big(\tfrac{1}{64}\Gamma^a_{\alpha\beta} g^\alpha_{\alpha'} g^\beta_{\beta'} \Gamma^{\alpha'\beta'}_{b_1 b_2}\big) e^{b_1 b_2} \;+\; \big(\tfrac{1}{5!\cdot 32}\Gamma^a_{\alpha\beta} g^\alpha_{\alpha'} g^\beta_{\beta'} \Gamma^{\alpha'\beta'}_{b_1\cdots b_5}\big) e^{b_1\cdots b_5} && \text{by (176),}
\end{aligned}
$$

as befits a U-duality symmetry. For this reason, the authors [BW00] speak of a "brane rotating symmetry".

Remark 3.46 (Relation to $\mathfrak{e}_{11}$-duality). This enhanced equivariance (178) of the M-algebra, which makes the basic super Lie bracket a morphism of $\mathfrak{sl}_{32}$-representations $\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \;\simeq\; \mathbf{526}$, will have to be understood as the effective part of the action of the "maximal compact" subalgebra of $\mathfrak{e}_{11}$, according to [BKS19, p. 42].

3.4 Higher T-duality

The L_∞-algebraic formulation of T-duality from §2 and §3 makes immediate a much larger generality of L_∞-algebraic (hence: rational-topological) "higher T-duality" [FSS20a] in the sense of "higher structures" and "categorified symmetries". Here, the NS-field twist H_3 in ordinary T-duality (typically thought of as the curvature 3-form of a "bundle gerbe") is allowed to have higher degrees (as befits higher bundle gerbes and yet richer higher fiber bundles) – see [Sa09] for the appearance of degree 7 (bosonic) twists in string theory. At the rational level, this story amounts to applying the constructions related to higher (odd) central extensions along the lines of §2.4.

This "higher T-duality" was identified in [FSS20a, Thm. 3.17], where it was encoded in terms of an isomorphism between the $(4t-1)$-twisted periodic Chevalley–Eilenberg cohomologies (Def. 2.14)

$$
T \;:\; H^{\bullet + h_A}_{\mathrm{CE}}(\widehat{\mathfrak{g}}_A) \xrightarrow{\;\sim\;} H^{(\bullet - 2t+1)+h_B}_{\mathrm{CE}}(\widehat{\mathfrak{g}}_B)
$$

of certain "higher T-dual pairs" of centrally extended super L_∞-algebras $\widehat{\mathfrak{g}}_A$, $\widehat{\mathfrak{g}}_B$ via $2t$-cocycles $\omega^A, \omega^B \in \mathrm{CE}(\mathfrak{g})$, for any $t \in \mathbb{N}$, over a common base $\mathfrak{g}$, supplied with suitably related $(4t-1)$-cocycle twists $h_{A/B} \in \mathrm{CE}(\widehat{\mathfrak{g}}_{A/B})$.

Here we provide an equivalent description via the automorphism of the higher cyclification of the corresponding twisted periodic cocycle classifying L_∞-algebras from Ex. 2.68 and Lem. 2.70. This may be viewed as a justification for the existence of the indicated isomorphism of twisted cohomologies. We motivate this new description by first further elaborating the description of the higher-self duality on the $\mathfrak{m2brane}$ from [FSS20a, §4.3].

The $\mathfrak{m2brane}$ higher self T-duality. Recall the higher extension of $\mathbb{R}^{1,10|\mathbf{32}}$ via the 4-cocycle G_4 from Ex. 2.61

$$
\mathfrak{m2brane} \xrightarrow{\;p := \mathrm{hofib}\;} \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;G_4\;} b^3\mathbb{R}\,,
$$

and consider the "Page charge" 7-cocycle

$$
\widetilde{G}_7 \;:=\; 2\,G_7 - c_3 \cdot G_4 \qquad \in \qquad \mathrm{CE}(\mathfrak{m2brane})\,. \tag{180}
$$

Note that this is indeed closed, i.e., constitutes a homomorphism

$$
\widetilde{G}_7 \;:\; \mathfrak{m2brane} \longrightarrow b^6\mathbb{R}\,,
$$

since

$$
\begin{aligned}
\mathrm{d}\,\widetilde{G}_7 &= 2\,\mathrm{d}G_7 - G_4 G_4 \\
&= G_4 G_4 - G_4 G_4 \\
&= 0\,,
\end{aligned}
$$

by the fact that (G_4, G_7) forms a $\mathfrak{l}S^4$-cocycle (Ex. 2.8). As such, it may be thought of as a 7-twisting cocycle analogous to the 3-twisting IIA/IIB NS-fluxes $H_3^{A/B} \;=\; H_3 + e^9_A \cdot c_1^{B/A}$. Apart from its higher degree, the crucial property of the 7-cocycle $\widetilde{G}_7$ is that its fiber integration down to 11d super-spacetime, via $p : \mathfrak{m2brane} \to \mathbb{R}^{1,10|\mathbf{32}}$, yields the original extending 4-cocycle G_4 up to a sign prefactor

$$
p_* \widetilde{G}_7 \;=\; -\,G_4\,.
$$

This suggests that the corresponding higher T-duality should be a *self-duality* acting on 7-twisted periodic cocycles on $\mathfrak{m2brane}$, in an appropriate sense.

More precisely, we may consider the higher central extension of $\mathbb{R}^{1,10|\mathbf{32}}$ by the opposite 4-cocycle $p_*\widetilde{G}_7 = -\,G_4$ instead, yielding an isomorphic copy of the $\mathfrak{m2brane}$-algebra

$$
\mathfrak{m2brane}^{-} \xrightarrow{\;p_- := \mathrm{hofib}\;} \mathbb{R}^{1,10\,|\,\mathbf{32}} \xrightarrow{\;-G_4\;} b^3\mathbb{R}\,,
$$

given by

$$\mathrm{CE}\big(\mathfrak{m2brane}^{-}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a)_{a=1}^{10} \\ c_3^- \end{array}\right] \Big/ \left(\begin{array}{ll} \mathrm{d}\,\psi &= 0 \\ \mathrm{d}\,e^a &= \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,c_3^- &= \underbrace{-\tfrac{1}{2}\big(\overline{\psi}\Gamma_{ab}\psi\big)e^a e^b}_{-G_4} \end{array}\right).$$

Lemma 3.47 (**Reflection symmetry**)**.** *The canonical isomorphism between the two versions of the 4-cocycle extended* $\mathbb{R}^{1,10|\mathbf{32}}$ *is given by 'reflecting' the extending generators*

$$\begin{array}{ccc} \mathfrak{m2brane} & \xleftrightarrow{\;\sim\;} & \mathfrak{m2brane}^- \\ -c_3 & \longleftarrow\!\!\shortmid & c_3^-\,, \end{array} \tag{181}$$

under which the original twisting 7-cocycle maps to the "dual" twisting cocycle

$$\widetilde{G}_7^- \;:=\; 2\,G_7 + c_3^-\cdot G_4 \qquad \in \qquad \mathrm{CE}\big(\mathfrak{m2brane}^-\big)\,. \tag{182}$$

Evidently, the fiber integration of the dual 7-cocycle under $p_-:\mathfrak{m2brane}^-\to\mathbb{R}^{1,10|\mathbf{32}}$ *recovers the 4-cocycle classifying the original* $\mathfrak{m2brane}$

$$p_{-*}\widetilde{G}_7^- \;=\; G_4\,.$$

Remark 3.48 (**Higher $\mathfrak{m2brane}$ twisting cocycles under higher T-duality**)**.**
(i) The two $b^6\mathbb{R}$-cocycles $\widetilde{G}_7$ and $\widetilde{G}_7^-$ are related in a manner analogous to that of the IIA and IIB NS-fluxes from Rem. 3.17. That is, it follows immediately that $\widetilde{G}_7^-$ is related to $\widetilde{G}_7$ via the higher T-duality operation on higher twists (Lem. 2.70), namely : the composite of (1.) reduction (100) along $\omega_4 = G_4$ followed by (2.) automorphism (103) of $b\mathcal{T}_2 \cong \mathrm{cyc}_3(b^6\mathbb{R})$ and then (3.) oxidation (100) along $\omega_4^{'} = -G_4$

$$\begin{tikzcd}[column sep=small] \widetilde{G}_7 = \overbrace{2G_7}^{\omega_7} - c_3\,\overbrace{G4}^{s_3\omega_7} \arrow[rr, mapsto, "\text{superspace higher T-duality}"] \arrow[dr, mapsto, "\mathrm{rdc}_{G_4}", "\text{reduction along extension}"'] & & \widetilde{G}_7^- = 2G_7 + c_3^- G_4\,. \\ & \begin{array}{rl} 2G_7 &= \tfrac{2}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \\ G_4 &= \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2} \\ -(-G_4) &= \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2} \end{array} \arrow[ur, mapsto, "\mathrm{oxd}_{-G_4}", "\text{oxidation along reflected extension}"'] & \end{tikzcd} \tag{183}$$

(ii) The cobounding condition on the base $\mathbb{R}^{1,10|\mathbf{32}}$, analogous to (140), is now

$$\mathrm{d}(2G_7) \;=\; -\,G_4\cdot(-G_4)\,, \tag{184}$$

being satisfied automatically by the $\mathfrak{l}S^4$-cocycle condition (or equivalently, the closure of either $\widetilde{G}_7$ or $\widetilde{G}_7^-$).

(iii) On the doubly higher extended space

$$\mathfrak{m2brane} \times_{\mathbb{R}^{1,10|\mathbf{32}}} \mathfrak{m2brane}^- \tag{185}$$

the analogous *higher Poincare form* (cf. (146))

$$P_6 \;:=\; c_3^-\cdot c_3 \tag{186}$$

is a coboundary for the difference of the (pullbacks of) the twisting cocycles

$$\mathrm{d}P_6 \;=\; \pi^*\widetilde{G}_7 - \pi_-^*\widetilde{G}_7^-\,,$$

as can be seen immediately since $\pi^*\widetilde{G}_7 - \pi_-^*\widetilde{G}_7^- = -G_4c_3 - c_3^-G_4 = \mathrm{d}(c_3^-\cdot c_3)\,.$

Proposition 3.49 (**Higher $\mathfrak{m2brane}$ twisted cocycles under higher T-duality**)**.** *The higher T-duality operation* (183) *between* $\widetilde{G}_7$ *and* $\widetilde{G}_7^-$ *extends to a bijection of the corresponding 7-twisted cocycles. In particular, it maps any* $\widetilde{G}_7$*-twisted cocycle of degree* (m mod 6, evn) *on* $\mathfrak{m2brane}$

$$\begin{array}{ccc} \mathfrak{m2brane} & \longrightarrow & \mathfrak{l}\big(\Sigma^m\mathrm{K}^3U /\!\!/ B^5\mathrm{U}(1)\big) \\ \widetilde{G}_7 & \longleftarrow\!\!\shortmid & h_7 \\ (F_{6k+m})_{k\in\mathbb{Z}} & \longleftarrow\!\!\shortmid & (f_{6k+m})_{k\in\mathbb{Z}} \end{array}$$

to the $\widetilde{G}_7^-$*-twisted cocycle of degree* $(m-3 \bmod 6, \mathrm{evn})$ *on* $\mathfrak{m2brane}^-$

$$\begin{array}{ccc} \mathfrak{m2brane}^- & \xrightarrow{\qquad\qquad\qquad} & \mathfrak{l}\big(\Sigma^{m-3}\mathrm{K}^3 U /\!\!/ B^5\mathrm{U}(1)\big) \\ \widetilde{G}_7^- & \longleftarrow & h_7 \\ \big(-p_* F_{6k+m} - c_3^- \cdot (F_{6(k-1)+m})_{\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow & (f_{6k-3+m})_{k\in\mathbb{Z}} \end{array}$$

and vice-versa.

Proof. This follows as an application of Lem. 2.70. □

Corollary 3.50 (As a self-duality on $\mathfrak{m2brane}$). *Applying a further pullback operation via the reflection isomorphism (181) on the result of Prop. 3.49, this yields an actual self-duality on* $\mathfrak{m2brane}$ *as a non-trivial isomorphism of* $\widetilde{G}_7$*-twisted cocycles. Explicitly, this self-duality maps any* $\widetilde{G}_7$*-twisted cocycle of degree* $(m \bmod 6, \mathrm{evn})$ *on* $\mathfrak{m2brane}$

$$\begin{array}{ccc} \mathfrak{m2brane} & \longrightarrow & \mathfrak{l}\big(\Sigma^{m}\mathrm{K}^3 U /\!\!/ B^5\mathrm{U}(1)\big) \\ \widetilde{G}_7 & \longleftarrow & h_7 \\ (F_{6k+m})_{k\in\mathbb{Z}} & \longleftarrow & (f_{6k+m})_{k\in\mathbb{Z}} \end{array}$$

to the $\widetilde{G}_7$*-twisted cocycle of degree* $(m-3 \bmod 6, \mathrm{evn})$ *on* $\mathfrak{m2brane}$

$$\begin{array}{ccc} \mathfrak{m2brane} & \xrightarrow{\qquad\qquad\qquad} & \mathfrak{l}\big(\Sigma^{m-3}\mathrm{K}^3 U /\!\!/ B^5\mathrm{U}(1)\big) \\ \widetilde{G}_7 & \longleftarrow & h_7 \\ \big(-p_* F_{6k+m} + c_3 \cdot (F_{6(k-1)+m})_{\mathrm{bas}}\big)_{k\in\mathbb{Z}} & \longleftarrow & (f_{6k-3+m})_{k\in\mathbb{Z}}\,. \end{array}$$

The analogous self-duality statement for $\widetilde{G}_7^-$*-twisted cocycles on* $\mathfrak{m2brane}^-$ *follows verbatim.*

Show equivalent to reduction - Γ^{10}-automorphism on base + auto classifying - oxidify to $\mathfrak{m2}$ *directly.*

Higher T-duality as a Fourier–Mukai transform In complete analogy to the case of standard superspace T-duality, Rem. 3.48 and Prop. 3.49 suggest that there should be an equivalent description in terms of the doubly extended correspondence space (cf. Def. 3.24) and a pull-push isomorphism (cf. Thm. 3.27). This is indeed the case, and in fact directly generalizes the definitions and results of the 3-twisted case with degree 1 central extensions not only to the 7-twisted case with degree 3 extensions, but to the cases of all odd $(4t-1)$-twisted cases with odd rational extensions of degree $2t-1$. We now spell out how this works in its full generality.

Definition 3.51 (Higher T-duality correspondence). Pairs $(\widehat{\mathfrak{g}}_A, H_A)$ and $(\widehat{\mathfrak{g}}_B, H_B)$ of higher centrally extended super L_∞-algebras over $\mathfrak{g}$ via even $2t$-cocycles $\omega^A_{2t}, \omega^B_{2t} \in \mathrm{CE}(\mathfrak{g})$, supplied with $(4t-1)$-cocycle twists, respectively, are said to be in *higher T-duality correspondence* if:

(i) The respective fiber integration of the twists $H_{A/B}$ yields the opposite extension cocycles $c_{B/A}$ (cf. Eq. (183) of Rem. 3.48)

$$(p_{A/B})_* H_{A/B} \;=\; \omega^{B/A}_{2t} \qquad \in \qquad \mathrm{CE}(\mathfrak{g})\,.$$

(ii) On the doubly extended space $\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B$ (cf. Eq. (185))

$$\begin{tikzcd} & \widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] \arrow[dd, phantom, "\text{(pb)}"] & \\ \widehat{\mathfrak{g}}_A \arrow[dr, two heads, "p_A"'] & & \widehat{\mathfrak{g}}_B\,, \arrow[dl, two heads, "p_B"] \\ & \mathfrak{g} & \end{tikzcd}$$

defined dually via

$$\mathrm{CE}(\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B) \;=\; \mathrm{CE}(\mathfrak{g})\big[b_A, b_B\big]\big/\big(\mathrm{d}b_{A/B} = \omega^{A/B}_{2t}\big)\,,$$

the *higher Poincaré form* (cf. Eq. (186))

$$P \;:=\; b_B \cdot b_A \qquad \in \qquad \mathrm{CE}(\widehat{\mathfrak{g}}_A \times_{\mathfrak{g}} \widehat{\mathfrak{g}}_B)$$

is a coboundary of the difference between the pullbacks of the two twisting cocycles (cf. Prop. 3.21)

$$\mathrm{d}P \;=\; \pi_A^* H_A - \pi_B^* H_B\,.$$

The conditions of a higher T-duality correspondence (Def. 3.51) may be equivalently – and concisely – expressed via data over the original base super-L_∞ algebra $\mathfrak{g}$ [FSS20a, Prop. 3.13].

Lemma 3.52 (Higher T-duality conditions on the base). *Two pairs* $(\widehat{\mathfrak{g}}_A, H_A)$ *and* $(\widehat{\mathfrak{g}}_B, H_B)$ *are in higher T-duality correspondence (Def. 3.51) if and only if the twists are of the form*

$$H_{A/B} \;=\; H_{\mathfrak{g}} + b_{A/B} \cdot \omega^{B/A}_{2t}$$

for a common basic $(4t-1)$-cochain $H_{\mathfrak{g}} \in \mathrm{CE}(\mathfrak{g})$ whose differential trivializes the product of the corresponding extending $2t$-cocycles

$$\mathrm{d}H_{\mathfrak{g}} \;=\; -\,\omega_{2t}^{A} \cdot \omega_{2t}^{B}\,.$$

Proof. Follows verbatim as that of Lem. 3.25, by modifying the degrees appropriately. □

Corollary 3.53 (Higher T-duality correspondence classifying space). *It follows that the T-duality L_∞-algebra $b\mathcal{T}_t \cong \mathrm{cyc}_{2t-1}\, b^{2t}\mathbb{R}$ from Ex. 2.67 classifies the set of higher T-duality correspondences over any super-L_∞ algebra $\mathfrak{g}$, in that morphisms of super L_∞-algebras*

$$\begin{array}{ccc}
\mathfrak{g} & \longrightarrow & b\mathcal{T}_t \\
\omega_{2t}^{A} & \longleftarrow\!\shortmid & \omega_{2t} \\
\omega_{2t}^{B} & \longleftarrow\!\shortmid & -\widetilde{\omega}_{2t} \\
H_{\mathfrak{g}} & \longleftarrow\!\shortmid & h_{4t-1}\,,
\end{array}$$

are in canonical bijection with the set of higher T-duality correspondences over $\mathfrak{g}$ in terms of Lem. 3.52 (hence equivalently in terms of Def. 3.51).

Example 3.54 ($\mathfrak{m2brane}$ higher T-duality correspondence via the (G_4, G_7)-cocycle). The cobounding condition (184) translates to a higher T-duality correspondence via the map

$$\begin{array}{ccc}
\mathbb{R}^{1,10|\mathbf{32}} & \longrightarrow & b\mathcal{T}_2 \\
G_4 & \longleftarrow\!\shortmid & \omega_4 \\
-G_4 & \longleftarrow\!\shortmid & -\widetilde{\omega}_4 \\
2G_7 & \longleftarrow\!\shortmid & h_7\,,
\end{array}$$

which evidently factors through the fixed (G_4, G_7)-cocycle as

$$\mathbb{R}^{1,10|\mathbf{32}} \xrightarrow{\;(G_4,G_7)\;} \mathfrak{l}S^4 \xrightarrow{\;\;\iota_2\;\;} b\mathcal{T}_2\,, \tag{187}$$

where the latter 'embedding' morphism of the rational 4-sphere into the higher T-duality algebra is given by

$$\begin{array}{ccc}
\mathfrak{l}S^4 & \xrightarrow{\iota_2} & b\mathcal{T}_2 \\
g_4 & \longleftarrow\!\shortmid & \omega_4 \\
g_4 & \longleftarrow\!\shortmid & \widetilde{\omega}_4 \\
2g_7 & \longleftarrow\!\shortmid & h_7\,.
\end{array}$$

Remark 3.55 (Canonical higher T-duality correspondences of a basic $\mathfrak{l}S^{2t}$-cocycle).
(i) The factorization (187) implies immediately that a fixed $\mathfrak{l}S^4$-cocycle on base super-L_∞ algebra $\mathfrak{g}$, yields a class of different higher T-duality correspondences via different choices of embeddings of $\mathfrak{l}S^4$ into the higher T-duality algebra $b\mathcal{T}_t$. For example, it is immediate to see (Lem. 3.52) that post-composition of $(G_4, G_7) : \mathbb{R}^{1,10|\mathbf{32}} \longrightarrow \mathfrak{l}S^4$ with the embedding

$$\begin{array}{ccc}
\mathfrak{l}S^4 & \xrightarrow{\widehat{\iota}_2} & b\mathcal{T}_2 \\
g_4 & \longleftarrow\!\shortmid & \omega_4 \\
-g_4 & \longleftarrow\!\shortmid & \widetilde{\omega}_4 \\
-2g_7 & \longleftarrow\!\shortmid & h_7\,,
\end{array}$$

yields (directly) a self-correspondence on

$$\mathfrak{m2brane}$$

but instead with the opposite twist of (180)

$$\widehat{G}_7 \;:=\; -2G_7 + c_3 G_4 \;\equiv\; -\widetilde{G}_7\,.$$

(ii) Evidently, this observation generalizes to higher even sphere-valued cocycles on any super-L_∞ algebra. That is, for any fixed $\mathfrak{l}S^{2t}$-cocycle

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\;(G_{2t},\, G_{4t-1})\;} & \mathfrak{l}S^{2t} \\
G_{2t} & \longleftarrow\!\shortmid & g_{2t} \\
G_{4t-1} & \longleftarrow\!\shortmid & g_{4t-1}\,,
\end{array}$$

postcomposition with (any of) the embeddings

$$\mathfrak{l}S^{2t} \xrightarrow{\iota_t} b\mathcal{T}_t$$

immediately yields a higher T-duality correspondence (Lem. 3.52).

For any higher T-duality correspondence, the natural pull-push homomorphism of $(4t-1)$-twisted cochain complexes

$$T_{\mathrm{FM}} \quad := \quad \pi_{B*} \circ e^{-P} \circ \pi_A^* \equiv \pi_{B*} \circ (1 + b_A \cdot b_B) \circ \pi_A^* \tag{188}$$

is an isomorphism of degree $(-n_t \bmod 2n_t, \mathrm{evn})$, for

$$n_t \;:=\; 2t-1\,.$$

In particular, it descends to an isomorphism on cohomology [FSS20a, Thm. 3.17].

Theorem 3.56 (Higher T-duality/Fourier-Mukai isomorphism). *Let $(\widehat{\mathfrak{g}}_A, H_A)$ and $(\widehat{\mathfrak{g}}_B, H_B)$ be in higher T-duality correspondence. Then the pull-push morphism (188) is an isomorphism of $(4t-1)$-twisted cochain complexes (Def. 2.14) and acts explicitly as*

$$\begin{array}{rccc} T_{\mathrm{FM}} \;:\; & \mathrm{CE}^{\bullet + H_A}(\widehat{\mathfrak{g}}_A) & \xrightarrow{\;\sim\;} & \mathrm{CE}^{(\bullet - n_t)+H_B}(\widehat{\mathfrak{g}}_B) \\ & \alpha = \alpha_{\mathrm{bas}} + b_A \cdot p_{A*}\alpha & \longmapsto & p_{A*}\alpha + b_B \cdot \alpha_{\mathrm{bas}}\,, \end{array} \tag{189}$$

thereby swapping the "winding" and "non-winding" modes.

In particular, it descends to a "Fourier-Mukai" isomorphism of higher twisted cocycles and furthermore higher twisted cohomologies

$$T_{\mathrm{FM}} \;:\; H^{\bullet + H_A}_{\mathrm{CE}}(\widehat{\mathfrak{g}}_A) \xrightarrow{\;\sim\;} H^{(\bullet - n_t)+H_B}_{\mathrm{CE}}(\widehat{\mathfrak{g}}_B)\,.$$

Proof. Follows verbatim as that of Thm. 3.27, by modifying the degrees appropriately. □

Corollary 3.57 (Pull-push via automorphism of higher cyclified twisted cocycle classifying algebras). *Under the identification of $(2n_t+1) = (4t-1)$-twisted Chevalley-Eilenberg cocycles with maps into the corresponding classifying super-L_∞-algebras from Eq. (32), the action of the higher T-duality isomorphism (189) by Fourier-Mukai transform coincides with that of the composite operation of (1.) reduction, (2.) automorphism of higher cyclified twisted cocycle classifying algebras (3.) reoxidation from Lem. 2.70, up to a (conventional) sign*

$$\begin{tikzcd}[column sep=small]
\left\{ \begin{tikzcd} \mathfrak{g}_A \arrow[r] \arrow[dr, "H_A"'] & \mathfrak{l}\big(\Sigma^{m}\mathrm{K}^{n_t}\mathrm{U} /\!\!/ B^{2n_t-1}\mathrm{U}(1)\big) \arrow[d] \\ & b^{2n_t}\mathbb{R} \end{tikzcd} \right\} \arrow[rr, leftrightarrow, "\text{Lem. 2.30}"] \arrow[d, hook] & & \left\{ \begin{tikzcd} \mathfrak{g}_B \arrow[r] \arrow[dr, "H_B"'] & \mathfrak{l}\big(\Sigma^{m-n_t}\mathrm{K}^{n_t}\mathrm{U} /\!\!/ B^{2n_t-1}\mathrm{U}(1)\big) \arrow[d] \\ & b^{2n_t}\mathbb{R} \end{tikzcd} \right\}. \arrow[d, hook] \\
\mathrm{CE}^{m+H_A}(\mathfrak{g}_A) \arrow[rr, leftrightarrow, "-T_{\mathrm{FM}}"] & & \mathrm{CE}^{m-n_t+H_B}(\mathfrak{g}_B)
\end{tikzcd}$$

4 Conclusion & Outlook

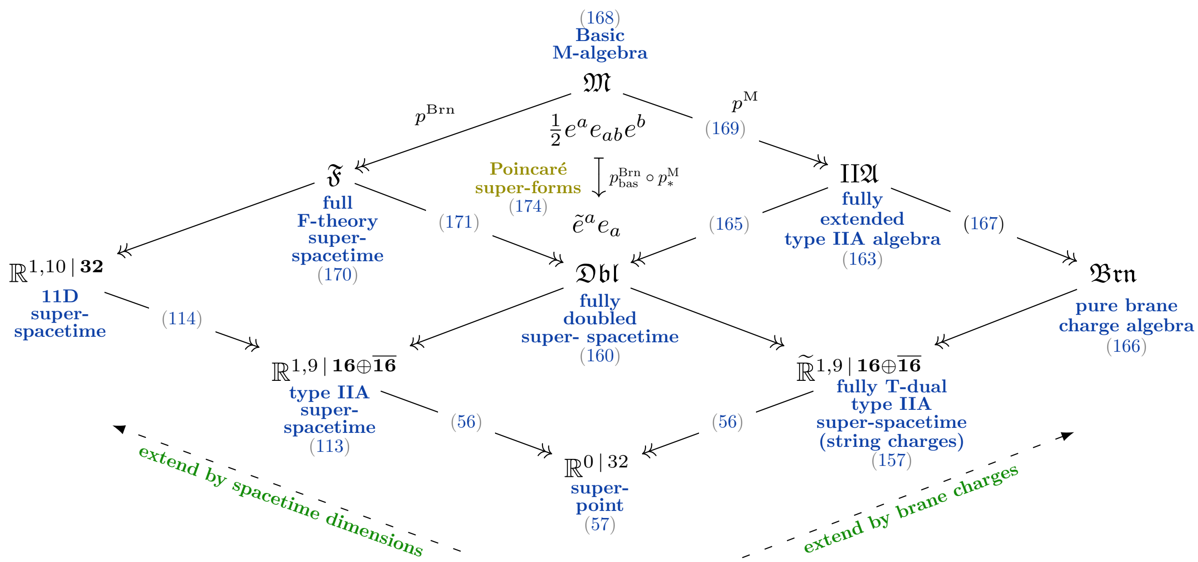
In view of the fact that on-shell 11D supergravity is entirely determined by the structure of its avatar super-flux densities on the typical super-tangent space [GSS24a, Thm. 3.1], which there are naturally understood as an $\mathfrak{l}S^4$-valued super-L_∞ algebra cocycle [FSS17], we have given a super-L_∞ algebraic re-analysis of T-duality on the analogous avatar super-flux densities of 10D type II supergravity, first for type A/B dualization along one spacetime direction (streamlining the previous result of [FSS18a]) and then for type A/$\widetilde{\mathrm{A}}$-duality along *all* spacetime directions, via reduction/oxidation all the way to/from the super-point $\mathbb{R}^{0|32}$.

The result of this analysis is that:

(i) super-space T-duality is entirely controlled by a super Fourier-Mukai transform with integral kernel a Poincaré super 2-form P_2 on fully doubled 10D superspace $\mathfrak{Dbl}$ serving as a correspondence between type IIA super-spacetime and its II$\widetilde{\mathrm{A}}$ dual,

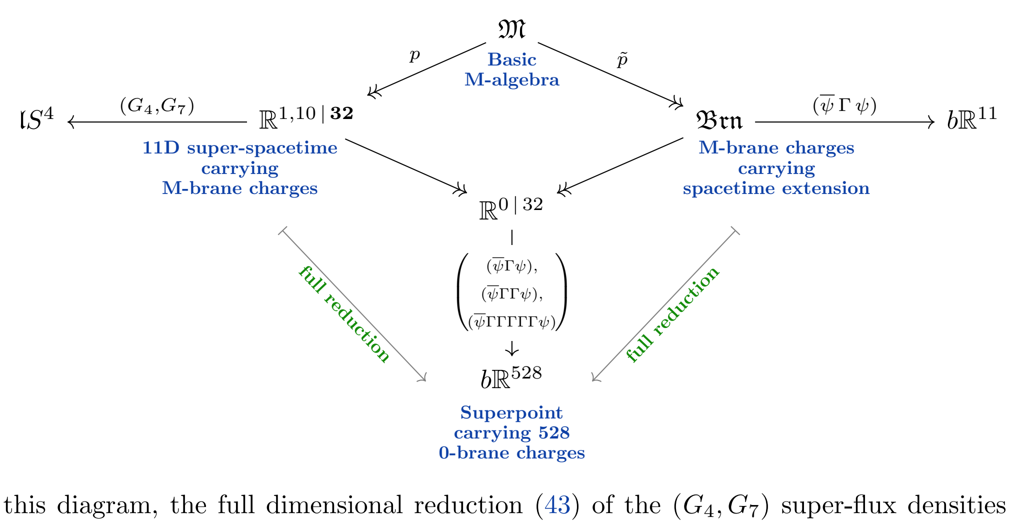
(ii) extending this super-correspondence by (i) the 11th spacetime dimension and (ii) the remaining brane charges, yields a whole network of further correspondences that bring in increasingly more M-theoretic structure into type II T-duality,

(iii) the tip of this network is formed by the basic M-algebra $\mathfrak{M}$ carrying a super-invariant 3-form P_3 whose dimensional reduction along the 11th spacetime dimension coincides with the above Poincaré 2-form P_2:



(190)

Observing here, by the pasting law (61), that every composite diamond and hence also the whole outer diagram exhibits a fiber product and hence a correspondence, this suggests that the M-algebra itself is to be regarded as a correspondence space exhibiting an M-theoretic version of T-duality whereby 11D super-spacetime inter-transmutes with all the M-brane charges:



(191)

As shown in this diagram, the full dimensional reduction (43) of the (G_4, G_7) super-flux densities to the super-point factors there through a 528-tuple of 2-cocycles, which — after swapping spacetime dimensions with brane charges as appropriate for a T-duality — re-oxidize to the pure brane charge algebra on the other side of the big correspondence, now dually carrying 0-brane charge in lieu of the corresponding spacetime "condensate" (cf. [FSS15, §4.2]).

The Poincaré super 3-form as the "decomposed" hidden M-theory 3-form. This suggests, in view of the situation in type II T-duality from Prop. 3.21,that the M-theoretic Poincaré 3-form P_3 ought to be a coboundary for the difference of the pullback to $\mathfrak{M}$ of the corresponding direct product $\mathfrak{l}S^4 \times b\mathbb{R}^{11}$-valued cocycles — compare the analogous direct product (52) in ordinary T-duality. But with $\big(\overline{\psi}\,\Gamma\,\psi\big)$ already being cobounded by the spacetime component of the coframe field, this gives the following curious requirements on candidate M-theoretic Poincaré super 3-forms:

$$
\begin{aligned}
P_3 \;&\equiv\; \tfrac{1}{2} e^{a_1} e_{a_1 a_2} e^{a_2} \;+\; \cdots \quad \in\; \mathrm{CE}(\mathfrak{M})^{\mathrm{Spin}(1,10)}\,,
\\
\mathrm{d}\, P_3 \;&=\; G_4 \;\equiv\; \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big) e^{a_1} e^{a_2}\,,
\\
p^{\mathrm{Brn}}_{\mathrm{bas}} \circ p^{M}_{*}(P_3) \;&=\; P_2 \;\equiv\; \tilde{e}_a e^a\,.
\end{aligned}
\tag{192}
$$

Here we may observe (details in [GSS24d]) that such P_3 does not exist on the *basic* M-algebra $\mathfrak{M}$, but that it does exist (known as the "decomposition" of the M-theory 3-form) on its "hidden" fermionic extension $\widehat{\mathfrak{M}}$ given [DF82][BDIPV04][AD24] by

$$
\mathrm{CE}\big(\widehat{\mathfrak{M}}\big) \;\simeq\; \mathrm{CE}(\mathfrak{M})\big[(\phi^\alpha)_{\alpha=1}^{32}\big] \big/ \big(\mathrm{d}\,\phi \;=\; 2\big((1+s)\Gamma_a \psi\, e^a \;+\; \Gamma^{a_1 a_2}\psi\, e_{a_1 a_2} \;+\; \tfrac{6+s}{6!}\Gamma^{a_1 \cdots a_5}\, e_{a_1 \cdots a_5}\big)
\tag{193}
$$

which as such exists for any $s \in \mathbb{R}$.

The first two conditions in (192) are known [DF82][BDIPV04][AD24] to be satisfied for $s \neq 0$ by

$$
\begin{aligned}
P_3 \;:=\;\; & \alpha_0\, e_{a_1 a_2}\, e^{a_1}\, e^{a_2}
\\
& + \alpha_1\, e^{a_1}{}_{a_2}\, e^{a_2}{}_{a_3}\, e^{a_3}{}_{a_1}
\\
& + \alpha_2\, e^{a_1 \cdots a_4 b_1}\, e_{b_1}{}^{b_2}\, e_{b_2 a_1 \cdots a_4}
\\
& + \alpha_3\, \epsilon_{a_1 \cdots a_5 b_1 \cdots b_5 c}\, e^{a_1 \cdots a_5} e^{b_1 \cdots b_5} e^{c}
\\
& + \alpha_4\, \epsilon_{a_1 a_2 a_3 b_1 b_2 b_3 c_1 \cdots c_5}\, e^{a_1 a_2 a_3 d_1 d_2}\, e_{d_1 d_2}{}^{b_1 b_2 b_3}\, e^{c_1 \cdots c_5}
\\
& + \beta_1\, \big(\overline{\psi}\,\Gamma_a\, \phi\big) e^a
\\
& + \beta_2\, \big(\overline{\psi}\,\Gamma_{a_1 a_2}\, \phi\big)\, e^{a_1 a_2}
\\
& + \beta_3\, \big(\overline{\psi}\,\Gamma_{a_1 \cdots a_5}\, \phi\big)\, e^{a_1 \cdots a_5}\,,
\end{aligned}
\qquad\quad \in \;\; \mathrm{CE}\big(\widehat{\mathfrak{M}}\big)\,,
\tag{194}
$$

for real prefactors α_i, β_i that are rational functions of s.

Incidentally, this means [FSS20b, Lem. 3.7] that the hidden M-algebra serves as an *atlas* (in the sense of stacks) of the M2-brane-extended super-spacetime (Ex. 2.61), in that we have a homomorphism (14) of super-L_∞-algebras

$$
\begin{array}{ccc}
\widehat{\mathfrak{M}} & \longrightarrow\!\!\!\!\rightarrow & \mathfrak{m2brane}
\\
\psi & \longleftarrow\!\shortmid & \psi
\\
e^a & \longleftarrow\!\shortmid & e^a
\\
P_3 & \longleftarrow\!\shortmid & c_3
\end{array}
\tag{195}
$$

which is surjective in degree=0 and whose domain is, by construction, an ordinary super-Lie algebra (instead of a higher super-L_∞ algebra). Under this atlas, the rational higher T-duality on $\mathfrak{m2brane}$ from §3.4 transfers to a corresponding higher duality on $\widehat{\mathfrak{M}}$ (see [FSS20a, Prop. 4.17]).

The hidden M-algebra as the local model for M-theoretic duality correspondence space. Previously, the meaning or further preferred specification of the parameter s in (193) had remained mysterious, but now we may observe (details again in [GSS24d]) that:

(i) there are exactly two values of s for which also the third condition in (192) is satisfied, in that $\alpha_0 = -\frac{1}{2}$, hence for which P_3 really qualifies as an M-theoretic lift of the Poincaré 2-form exhibiting T-duality (an aspect not previously considered), namely:

$$
s = -1 \qquad \text{and} \qquad s = 3/2\,.
$$

(ii) The case $s = -1$ is moreover special because exactly here the hidden extension (193) becomes independent of the spacetime coframe and hence induces already a fermionic extension $\widehat{\mathfrak{Brn}} \longrightarrow\!\!\!\!\rightarrow \mathfrak{Brn}$ of the pure brane algebra, given via (193) by:

$$
\mathrm{CE}\big(\widehat{\mathfrak{Brn}}\big) \;\simeq\; \mathrm{CE}(\mathfrak{Brn})\big[(\phi^\alpha)_{\alpha=1}^{32}\big]\big/\big(\mathrm{d}\,\phi \;=\; 2\Gamma^{a_1 a_2}\psi\, e_{a_1 a_2} \;+\; 288\,\Gamma^{a_1\cdots a_5}\, e_{a_1\cdots a_5}\big)\,. \tag{196}
$$

Interestingly, thereby we find a fiber product analogous to (191) but now exhibiting as correspondence space the hidden M-algebra $\widehat{\mathfrak{M}}$ on which a Poincaré 3-form, satisfying all of (192), does exist:

$$
\begin{tikzcd}[column sep=small]
 & \underset{\text{Hidden M-algebra}}{\widehat{\mathfrak{M}}} \arrow[dl, ->>] \arrow[dr, ->>] & \\
\underset{\text{11D super-spacetime}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \arrow[dr, ->>] & & \underset{\text{Hidden extension of brane charges}}{\widehat{\mathfrak{Brn}}}\,. \arrow[dl, ->>] \\
 & \underset{\text{Super-point}}{\mathbb{R}^{0|32}} &
\end{tikzcd}
\tag{197}
$$

This plausibly exhibits the hidden M-algebra $\widehat{\mathfrak{M}}$ (at $s = -1$) as the Kleinian local model space for a topological T-duality- (and possibly U-duality-)covariant completion of 11D superspace supergravity. We hope to further discuss this elsewhere.

Global duality and flux quantization. Finally, to round up all this discussion of super-flux duality on nothing but super-tangent spaces (albeit extended ones), we highlight some profound implications for global solitonic field structure in higher dimensional supergravity. Namely, the point is that all these the super-L_∞ algebraic cocycle relations discussed above may be understood as shadows (precisely: "rationalizations") of *flux quantization laws* which govern the global (solitonic) field content on curved spacetimes ([SS24c][FSS23]):

For example, the super-L_∞ cocycles for the type II NS/RR flux densities, which we saw in (3.13) to have coefficients in the real Whitehead L_∞ algebra of the 3-twisted K-theory spectrum, entail that *one* admissible choice for the global topological structure of the RR-fields is actual twisted K-theory — which is of course the statement of a famous conjecture in string theory:

$$
\begin{tikzcd}
\overset{\text{curved type IIA super-spacetime super-manifold}}{X^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[rr, dashed, "\text{flux-quantized configuration of type II NS/RR fields}"] & & \overset{\text{higher moduli stack for differential twisted K-theory}}{\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)_{\mathrm{diff}}} \arrow[d, "\text{rationalization}"] \\
\underset{\text{type IIA super-algebra}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[u, "\text{super-tangent space around any point}"] \arrow[rr, "(6)", "\text{avatar super-flux densities}"'] & & \underset{\text{Whitehead } L_\infty \text{ algebra of twisted K-theory}}{\mathfrak{l}\big(\mathrm{KU}/\!\!/B\mathrm{U}(1)\big)}
\end{tikzcd}
\tag{198}
$$

Similarly, the fact that the avatar super-flux densities of 11D SuGra have coefficients in the Whitehead L_∞-algebra of the 4-sphere means that the actual 4-sphere serves as the classifying space for *one* admissible choice for the global topological structure of the C-field (aka a "model for the C-field"):

$$
\begin{tikzcd}
\overset{\text{curved 11D super-spacetime super-manifold}}{X^{1,10\,|\,\mathbf{32}}} \arrow[rr, dashed, "\text{flux-quantized configuration of the C-field}"] & & \overset{\text{higher moduli stack for differential 4-Cohomotopy}}{\big(S^4\big)_{\mathrm{diff}}} \arrow[d, "\text{rationalization}"] \\
\underset{\text{11D super-algebra}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \arrow[u, "\text{super-tangent space around any point}"] \arrow[rr, "(5)", "\text{avatar super-flux densities}"'] & & \underset{\text{Whitehead } L_\infty \text{ algebra of 4-Cohomotopy}}{\mathfrak{l}\big(S^4\big)}
\end{tikzcd}
$$

With this understood, our observations characterize the admissible flux quantization laws for the Poincaré 3-form on M-extended superspacetime.

To see how this works, first consider the analogous question for the Poincaré 2-form P_2 on doubled superspace super-manifolds, whose classifying L_∞-algebra is $\mathfrak{poin}_2$ (148). Hence admissible flux quantization laws for P_2 have classifying spaces Poin_2 whose Whitehead L_∞-algebra is $\mathfrak{l}\mathrm{Poin}_2 \simeq \mathfrak{poin}_2$. With any such choice, the actual *twisted* *super Poincaré line bundles* on doubled super-spacetime super-manifolds are dashed maps of this form:

$$
\begin{tikzcd}
\text{curved doubled super-spacetime super-manifold} & \text{flux-quantized configuration of Poincaré 2-flux} & \text{higher moduli stack for twisted Poincaré line bundles} \\
\mathrm{Dbl} \arrow[rr, dashed] && \big(\mathrm{Poin}_2\big)_{\mathrm{diff}} \arrow[d, "\text{rationalization}"] \\
\mathfrak{Dbl} \arrow[u, "\text{super-tangent space around any point}"] \arrow[rr, "(150)"] && \mathfrak{l}\big(\mathrm{Poin}_2\big) \\
\text{11D super-algebra} & \text{avatar Poincaré super 2-form} & L_\infty\text{-classifier for Poincaré 2-forms}
\end{tikzcd}
$$

An evident choice (among others) for the Poincaré bundle flux quantization, which is also compatible with the traditional choice in (198), is the actual homotopy fiber of spaces analogous to (148):

$$
\mathrm{Poin}_2 \xrightarrow{\ \mathrm{hofib}\ } B^2\mathrm{U}(1) \times B^2\mathrm{U}(1) \xrightarrow{\ \pi_L - \pi_R\ } \mathrm{B}^2\mathrm{U}(1)
$$

With this choice, a dashed map to Poin_2 as above modulates twisted complex line bundles whose twist is the difference of the type IIA/IIB B-field bundle gerbes pulled back to the doubled super-spacetime, hence isomorphisms between this pair of B-field bundle gerbes. This is the situation familiar from topological T-duality (e.g [BRS06, (2.4)][Wa24, Def. 4.1.2]).

Finally then the L_∞-classifier for a Poincaré 3-form as in (192) analogous to the 2-form analogue (148) is the homotopy fiber of the canonical map from $\mathfrak{l}S^4$ to $b^3\mathbb{R}$,

$$
\mathfrak{poin}_3 \xrightarrow{\ \mathrm{hofib}\ } \mathfrak{l}S^4 \xrightarrow{\ (23)\ } b^3\mathbb{R} \tag{199}
$$

in that the Bianchi identity (192) characterizes dashed maps making the following diagram commute – in analogy with (150):

$$
\begin{tikzcd}
\widehat{\mathfrak{M}} \arrow[r, dashed, "P_3"] \arrow[d, two heads] & \mathfrak{poin}_3 \arrow[d] \\
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, "(G_4, G_7)"] & \mathfrak{l}S^4\,,
\end{tikzcd}
\tag{200}
$$

and a flux quantization law for the Poincaré 3-form is specified by classifying spaces Poin_3 with $\mathfrak{l}\mathrm{Poin}_3 \simeq \mathfrak{poin}_3$. Given such as choice, a flux-quantized Poincaré 3-bundle is then modulated by a dashed map of the following kind:

$$
\begin{tikzcd}
\text{Curved M-theoretic super-spacetime super-manifold} & \text{flux-quantized configuration of Poincaré 3-flux} & \text{higher moduli stack for M-theoretic Poincaré bundles} \\
\widehat{\mathrm{M}} \arrow[rr, dashed] && \big(\mathrm{Poin}_3\big)_{\mathrm{diff}} \arrow[d, "\text{rationalization}"] \\
\widehat{\mathfrak{M}} \arrow[u, "\text{super-tangent space around any point}"] \arrow[rr, "(200)"] && \mathfrak{l}\big(\mathrm{Poin}_3\big) \\
\text{M-algebra} & \text{avatar Poincaré super 3-form} & L_\infty\text{-classifier for Poincaré 3-forms}
\end{tikzcd}
$$

Now, one admissible such flux quantization for the Poincaré 3-form is the homotopy fiber of the unit map $S^4 \to B^3\mathrm{U}(1)$. However, in view of (24) an alternative suggestive choice is the 7-sphere, $\mathrm{Poin}_3 = S^7$, sitting in the homotopy fiber sequence

$$
S^7 \longrightarrow S^4 \longrightarrow B\mathrm{SU}(2)
$$

which witnesses as S^7 as a SU(2)-principal bundle over the 4-sphere.

This choice gives the flux quantization law previously discussed in [GSS24b] (there for the H_3-flux density on the worldvolume of M5-branes), which has the curious property that it provably entails [SS24d] the kind of anyonic quantum states that motivated our discussion back in §1.

A Background

For ease of reference we briefly record some conventions, definitions and facts used in the main text.

Tensor conventions

Our tensor conventions are standard, but since superspace computations crucially depend on the corresponding prefactors, here to briefly make them explicit:

- The Einstein summation convention applies throughout: Given a product of terms indexed by some $i \in I$, with the index of one factor in superscript and the other in subscript, then a sum over I is implied: $x_i\, y^i := \sum_{i\in I} x_i\, y^i$.
- Our Minkowski metric is "mostly plus"
$$\big(\eta_{ab}\big)_{a,b=0}^{d} \;=\; \big(\eta^{ab}\big)_{a,b=0}^{d} \;:=\; \big(\mathrm{diag}(-1,+1,+1,\cdots,+1)\big)_{a,b=0}^{d}\,. \tag{201}$$
- Shifting position of frame indices always refers to contraction with the Minkowski metric (201):
$$V^a \;:=\; V_b\,\eta^{ab}\,, \quad V_a \;=\; V^b\eta_{ab}\,.$$
- Skew-symmetrization of indices is denoted by square brackets ($(-1)^{|\sigma|}$ is sign of the permutation σ):
$$V_{[a_1\cdots a_p]} \;:=\; \tfrac{1}{p!}\sum_{\sigma\in\mathrm{Sym}(n)} (-1)^{|\sigma|} V_{a_{\sigma(1)}\cdots a_{\sigma(p)}}\,.$$
- We normalize the Levi-Civita symbol to
$$\epsilon_{012\cdots} \;:=\; +1 \quad \text{hence} \quad \epsilon^{012\cdots} \;:=\; -1\,. \tag{202}$$
- We normalize the Kronecker symbol to
$$\delta^{a_1\cdots a_p}_{b_1\cdots b_p} \;:=\; \delta^{[a_1}_{[b_1}\cdots\delta^{a_p]}_{b_p]} \;=\; \delta^{a_1}_{[b_1}\cdots\delta^{a_p}_{b_p]} \;=\; \delta^{[a_1}_{b_1}\cdots\delta^{a_p]}_{b_p} \tag{203}$$
so that
$$V_{a_1\cdots a_p}\delta^{a_1\cdots a_p}_{b_1\cdots b_p} \;=\; V_{[b_1\cdots b_p]} \quad\text{and}\quad \epsilon^{c_1\cdots c_p a_1\cdots a_q}\,\epsilon_{c_1\cdots c_p b_1\cdots b_q} \;=\; -\,p!\cdot q!\,\delta^{a_1\cdots a_q}_{b_1\cdots b_q}\,. \tag{204}$$

Super-algebra

In *homological* super-algebra, where a homological degree $n \in \mathbb{Z}$ (such as of flux densities) interacts with a super-degree $\sigma \in \mathbb{Z}_2$ there are – beware – two different *sign rules* in use (cf. [DM99, p. 62]), whose relation is a little subtle. The traditional sign rule in supergravity (e.g. [CDF91, (II.2.106-9)]) that we follow here comes from $\mathbb{Z}\times\mathbb{Z}_2$-*bi-grading*. (The alternative sign rule which collapses this bi-degree to a single "parity" degree in $\mathbb{Z}_2$ is popular with authors who say the word "Q-manifold").

Sign rule. For homological super-algebra we consider bigrading in the direct product ring $\mathbb{Z}\times\mathbb{Z}_2$ — where the first factor $\mathbb{Z}$ is the homological degree and the second $\mathbb{Z}_2 \simeq \{\mathrm{evn}, \mathrm{odd}\}$ the super-degree – with sign rule
$$\mathrm{deg}_1 = (n_1,\sigma_1),\ \mathrm{deg}_2 = (n_2,\sigma_2) \in \mathbb{Z}\times\mathbb{Z}_2 \qquad \vdash \qquad \mathrm{sgn}\big(\mathrm{deg}_1,\, \mathrm{deg}_2\big) \;:=\; (-1)^{n_1\cdot n_2 + \sigma_1\cdot\sigma_2}\,. \tag{205}$$

For $(v_i)_{i\in I}$ a set of generators with bi-degrees $(\mathrm{deg}_i)_{i\in I}$ we write:

(i) $\mathbb{R}\big\langle (v_i)_{i\in I}\big\rangle$ for the graded super-vector space spanned by these elements,

(ii) $\mathbb{R}\big[(v_i)_{i\in I}\big]$ for the graded-commutative polynomial algebra generated by these elements, hence the tensor algebra on $|I|$ generators modulo the relation
$$v_1\cdot v_2 \;=\; (-1)^{\mathrm{sgn}(\mathrm{deg}_1,\mathrm{deg}_2)}\; v_2\cdot v_1\,, \tag{206}$$
hence the (graded, super) *symmetric algebra* on the above super-vector space:
$$\mathbb{R}\big[(v_i)_{i\in I}\big] \;:=\; \mathrm{Sym}\big(\mathbb{R}\big\langle (v_i)_{i\in I}\big\rangle\big).$$

(iii) $\mathbb{R}_{\mathrm{d}}\big[(v_i)_{i\in I}\big]$ for the (free) differential graded-commutative algebra (dgca) generated by these elements and their *differentials*
$$(\mathrm{d}v_i)_{i\in I}$$
treated as primitive elements with $\mathrm{deg}(\mathrm{d}e_i) = \mathrm{deg}(e_i) + (1,\mathrm{evn})$ and modulo the corresponding relation (206), with differential defined by
$$e_i \longmapsto \mathrm{d}e_i \qquad , \qquad \mathrm{d}e_i \longmapsto 0$$
and extended as a (graded) 'derivation, hence the dgca
$$\mathbb{R}_{\mathrm{d}}\big[(v_i)_{i\in I}\big] \;:=\; \Big(\mathrm{Sym}\big(\mathbb{R}\big\langle (v_i)_{i\in I},\, (\mathrm{d}v_i)_{i\in I}\big\rangle\big),\, \mathrm{d}\Big). \tag{207}$$

Spinors in 11d

We briefly record the following standard facts about the Majorana spinor representation **32** of $\mathrm{Spin}(1,10)$ (proofs and references may be found in [MiSc06, §2.5][GSS24a, §2.2.1]).

(We may and do take this to be the only spinor representation that we construct from "from scratch"; all other spin representations we extract via simple algebra from this one. For instance the **16** and $\overline{\mathbf{16}}$ of $\mathrm{Spin}(1,9)$ are conveniently identified with the images $P(\mathbf{32})$ and $\overline{P}(\mathbf{32})$ of **32** under the projector $P := \frac{1}{2}(1+\Gamma_{10})$ and its adjoint, respectively cf. (110) below.)

There exists an irreducible $\mathbb{R}$-linear representation **32** of $\mathrm{Pin}^+(1,10)$ with Clifford generators to be denoted

$$\Gamma_a \;:\; \mathbf{32} \to \mathbf{32} \tag{208}$$

and equipped with a $\mathrm{Spin}(1,10)$-equivariant skew-symmetric and non-degenerate bilinear form

$$\big(\overline{(-)}(-)\big) \;:\; \mathbf{32} \otimes \mathbf{32} \longrightarrow \mathbb{R} \tag{209}$$

satisfying all of the following properties.

In stating these we use the following notation:

- We denote, as usual, the skew-symmetrized product of k Clifford generators by
$$\Gamma_{a_1 \cdots a_k} \;:=\; \tfrac{1}{k!} \sum_{\sigma \in \mathrm{Sym}(k)} \mathrm{sgn}(\sigma)\, \Gamma_{a_{\sigma(1)}} \cdot \Gamma_{a_{\sigma(2)}} \cdots \Gamma_{a_{\sigma(n)}} \;: \tag{210}$$
- The spinor pairing (209) serves as the *spinor metric* whose components – being the odd partner of the Minkowski metric (201) – we denote by $\big(\eta_{\alpha\beta}\big)_{\alpha,\beta=1}^{32}$:
$$\psi^\alpha\, \eta_{\alpha\beta}\, \phi^\beta \;\;:=\;\; \big(\overline{\psi}\,\phi\big)\,. \tag{211}$$
These are skew symmetric in their indices
$$\eta_{\alpha\beta} \;=\; -\eta_{\beta\alpha} \tag{212}$$
which together with the inverse matrix $(\eta^{\alpha\beta})$ is used to lower and raise spinor indices by contraction "from the right" (the position of the terms is irrelevant, since the components $\eta_{\alpha\beta}$ are commuting numbers, but the order of the indices matters due to the skew-symmetry):
$$\psi_\alpha \;:=\; \psi^{\alpha'}\eta_{\alpha'\alpha}\,, \qquad \psi^\alpha \;=\; \psi_{\alpha'}\eta^{\alpha'\alpha}\,, \qquad \psi_\alpha\phi^\alpha \;=\; -\psi^\alpha\phi_\alpha\,. \tag{213}$$

Now, conventions may be chosen such that all of the following holds true:

- The Clifford generators (208) square to the mostly plus Minkowski metric (201)
$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a \;=\; +2\,\eta_{ab}\,\mathrm{id}_{\mathbf{32}}\,. \tag{214}$$
- The Clifford product is given on the basis elements (210) as
$$\Gamma^{a_j \cdots a_1}\,\Gamma_{b_1 \cdots b_k} \;=\; \sum_{l=0}^{\min(j,k)} \pm l! \binom{j}{l}\binom{k}{l}\, \delta^{[a_1 \cdots a_l}_{[b_1 \cdots b_l}\,\Gamma^{a_j \cdots a_{l+1}]}{}_{b_{l+1} \cdots b_k]}\,. \tag{215}$$
- The Clifford volume form equals the Levi-Civita symbol (202):
$$\Gamma_{a_1 \cdots a_{11}} \;=\; \epsilon_{a_1 \cdots a_{11}}\mathrm{id}_{\mathbf{32}}\,. \tag{216}$$
- The trace of all positive index Clifford basis elements vanishes:
$$\mathrm{Tr}(\Gamma_{a_1 \cdots a_p}) \;=\; \left\{ \begin{array}{ccl} 32 & | & p = 0 \\ 0 & | & p > 0\,. \end{array} \right. \tag{217}$$
- The Hodge duality relation on Clifford elements is:
$$\Gamma^{a_1 \cdots a_p} \;=\; \tfrac{(-1)^{(p+1)(p-2)/2}}{(11-p)!}\,\epsilon^{a_1 \cdots a_p\, b_1 \cdots a_{11-p}}\,\Gamma_{b_1 \cdots b_{11-p}}\,. \tag{218}$$
For instance:
$$\begin{array}{ll} \Gamma^{a_1 \cdots a_{11}} = \epsilon^{a_1 \cdots a_{11}}\mathrm{Id}_{\mathbf{32}}\,, & \Gamma^{a_1 \cdots a_6} \;=\; +\tfrac{1}{5!}\,\epsilon^{a_1 \cdots a_6\, b_1 \cdots b_5}\,\Gamma_{b_1 \cdots b_5}\,, \\ \Gamma^{a_1 \cdots a_{10}} = \epsilon^{a_1 \cdots a_{10} b}\,\Gamma_b\,, & \Gamma^{a_1 \cdots a_5} \;=\; -\tfrac{1}{6!}\,\epsilon^{a_1 \cdots a_5\, b_1 \cdots b_6}\,\Gamma_{b_1 \cdots b_6}\,. \end{array} \tag{219}$$
- The Clifford generators are skew self-adjoint with respect to the pairing (209)
$$\overline{\Gamma_a} \;=\; -\Gamma_a \qquad \text{in that} \qquad \underset{\phi,\psi \in \mathbf{32}}{\forall}\; \big(\overline{(\Gamma_a\phi)}\,\psi\big) \;=\; -\big(\overline{\phi}\,(\Gamma_a\psi)\big)\,, \tag{220}$$
so that generally
$$\overline{\Gamma_{a_1 \cdots a_p}} \;=\; (-1)^{p+p(p-1)/2}\,\Gamma_{a_1 \cdots a_p}\,. \tag{221}$$

- The $\mathbb{R}$-vector space of $\mathbb{R}$-linear endomorphisms of $\mathbf{32}$ has a linear basis given by the $\leq$ 5-index Clifford elements

$$\mathrm{End}_{\mathbb{R}}(\mathbf{32}) \;=\; \big\langle 1,\, \Gamma_{a_1},\, \Gamma_{a_1 a_2},\, \Gamma_{a_1 a_2 a_3},\, \Gamma_{a_1 \cdots a_4},\, \Gamma_{a_1 \cdots a_5} \big\rangle_{a_i = 0,1,\cdots}, \tag{222}$$

- The $\mathbb{R}$-vector space of *symmetric* bilinear forms on $\mathbf{32}$ has a linear basis given by the expectation values with respect to (209) of the 1-, 2-, and 5-index Clifford basis elements:

$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32}\otimes\mathbf{32})_{\mathrm{sym}},\,\mathbb{R}\Big) \;\simeq\; \Big\langle \big(\overline{(-)}\Gamma_a(-)\big),\; \big(\overline{(-)}\Gamma_{a_1 a_2}(-)\big),\; \big(\overline{(-)}\Gamma_{a_1\cdots a_5}(-)\big)\Big\rangle_{a_i=0,1,\cdots}, \tag{223}$$

which means in components that these Clifford generators are symmetric in their lowered indices (213):

$$\Gamma^a_{\alpha\beta} \;=\; \Gamma^a_{\beta\alpha}\,, \qquad \Gamma^{a_1 a_2}_{\alpha\beta} \;=\; \Gamma^{a_1 a_2}_{\beta\alpha}\,, \qquad \Gamma^{a_1\cdots a_5}_{\alpha\beta} \;=\; \Gamma^{a_1\cdots a_5}_{\beta\alpha}\,, \tag{224}$$

while a basis for the *skew-symmetric* bilinear forms is given by

$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32}\otimes\mathbf{32})_{\mathrm{skew}},\,\mathbb{R}\Big) \;\simeq\; \Big\langle \big(\overline{(-)}(-)\big),\; \big(\overline{(-)}\Gamma_{a_1 a_2 a_3}(-)\big),\; \big(\overline{(-)}\Gamma_{a_1\cdots a_4}(-)\big)\Big\rangle_{a_i=0,1,\cdots}, \tag{225}$$

which means in components that these Clifford generators are skew-symmetric in their lowered indices (213):

$$\eta_{\alpha\beta} \;=\; -\eta_{\beta\alpha}\,, \qquad \Gamma^{a_1 a_2 a_3}_{\alpha\beta} \;=\; -\Gamma^{a_1 a_2 a_3}_{\beta\alpha}\,, \qquad \Gamma^{a_1\cdots a_5}_{\alpha\beta} \;=\; -\Gamma^{a_1\cdots a_5}_{\beta\alpha} \tag{226}$$

- Any linear endomorphism $\phi \in \mathrm{End}_{\mathbb{R}}(\mathbf{32})$ is uniquely a linear combination of Clifford elements as:

$$\phi \;=\; \tfrac{1}{32}\sum_{p=0}^{5} \tfrac{(-1)^{p(p-1)/2}}{p!}\mathrm{Tr}\big(\phi\circ\Gamma_{a_1\cdots a_p}\big)\Gamma^{a_1\cdots a_p}\,. \tag{227}$$

- which implies in particular the Fierz expansion

$$\big(\overline{\phi}_1\,\psi\big)\big(\overline{\psi}\,\phi_2\big) \;=\; \tfrac{1}{32}\Big(\big(\overline{\psi}\,\Gamma^a\,\psi\big)\big(\overline{\phi}_1\,\Gamma_a\,\phi_2\big) - \tfrac{1}{2}\big(\overline{\psi}\,\Gamma^{a_1 a_2}\,\psi\big)\big(\overline{\phi}_1\,\Gamma_{a_1 a_2}\,\phi_2\big) + \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma^{a_1\cdots a_5}\,\psi\big)\big(\overline{\phi}_1\,\Gamma_{a_1\cdots a_5}\,\phi_2\big)\Big). \tag{228}$$

Proposition A.1 (The general Fierz identities [DF82, (3.1-3) & Table 2][CDF91, (II.8.69) & Table II.8.XI]**).**
(i) *The* $\mathrm{Spin}(1,10)$*-irrep decomposition of the first few symmetric tensor powers of* $\mathbf{32}$ *is:*

$$\begin{aligned}
\big(\mathbf{32}\otimes\mathbf{32}\big)_{\mathrm{sym}} &\;\cong\; \mathbf{11}\oplus\mathbf{55}\oplus\mathbf{462}\\
\big(\mathbf{32}\otimes\mathbf{32}\otimes\mathbf{32}\big)_{\mathrm{sym}} &\;\cong\; \mathbf{32}\oplus\mathbf{320}\oplus\mathbf{1408}\oplus\mathbf{4424}\\
\big(\mathbf{32}\otimes\mathbf{32}\otimes\mathbf{32}\otimes\mathbf{32}\big)_{\mathrm{sym}} &\;\cong\; \mathbf{1}\oplus\mathbf{165}\oplus\mathbf{330}\oplus\mathbf{462}\oplus\mathbf{65}\oplus\mathbf{429}\oplus\mathbf{1144}\oplus\mathbf{17160}\oplus\mathbf{32604}\,.
\end{aligned} \tag{229}$$

(ii) *In more detail, the irreps appearing on the right are tensor-spinors spanned by basis elements*

$$\begin{aligned}
&\big\langle \Xi^{\alpha}_{a_1\cdots a_p} \;=\; \Xi^{\alpha}_{[a_1\cdots a_p]}\big\rangle_{a_i\in\{0,\cdots,10\},\alpha\in\{1,\cdots 32\}} \quad\in\quad \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,10)\big)\\
&\text{with}\;\; \Gamma^{a_1}\Xi_{a_1 a_2\cdots a_p} \;=\; 0
\end{aligned} \tag{230}$$

(jointly to be denoted $\Xi^{(N)}$ for the case of the irrep $\mathbf{N}$) *such that:*

$$\begin{aligned}
\psi\big(\overline{\psi}\,\Gamma_a\,\psi\big) &\;=\; \tfrac{1}{11}\,\Gamma_a\,\Xi^{(32)} + \Xi^{(320)}_a\,,\\
\psi\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big) &\;=\; \tfrac{1}{11}\,\Gamma_{a_1 a_2}\,\Xi^{(32)} - \tfrac{2}{9}\,\Gamma_{[a_1}\,\Xi^{(320)}_{a_2]} + \Xi^{(1408)}_{a_1 a_2}\,,\\
\psi\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big) &\;=\; -\tfrac{1}{77}\Gamma_{a_1\cdots a_5}\Xi^{(32)} + \tfrac{5}{9}\Gamma_{[a_1\cdots a_4}\Xi^{(320)}_{a_5]} + 2\,\Gamma_{[a_1 a_2 a_3}\,\Xi^{(1408)}_{a_4 a_5]} + \Xi^{(4224)}_{a_1\cdots a_5}\,.
\end{aligned} \tag{231}$$

Background formulas for 11d Supergravity. Our notation and conventions for super-geometry and for on-shell 11d supergravity on super-space follow [GSS24a, §2.2 & §3], to which we refer for further details and exhaustive referencing. We denote the local data of a super-Cartan connection on (a surjective submersion $\widetilde{X}$ of) (super-)spacetime X, representing a super-gravitational field configuration, as[18]

$$\begin{array}{llll}
\textbf{Graviton} & \big(E^a\big)_{a=0}^{D-1} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\,\mathbb{R}^{1,D-1}\big)\\
\textbf{Gravitino} & \big(\Psi^\alpha\big)_{\alpha=1}^{N} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\,\mathbf{N}_{\mathrm{odd}}\big)\\
\textbf{Spin-connection} & \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{D-1} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\,\mathfrak{so}(1,D-1)\big)
\end{array} \tag{232}$$

[18] Our use of different letters for the even and odd components of a super co-frame follows e.g. [CDF91]. Other authors write "E^α" for what we denote "Ψ^α", e.g. [BaSo23]. While it is of course part of the magic of supergravity that E^a and E^α/Ψ^α are unified into a single super-coframe field E, we find that for reading and interpreting formulas it is helpful to use different symbols for its even and odd components.

and the corresponding Cartan structural equations (cf. [GSS24a, Def. 2.78]) for the supergravity field strengths as

$$
\begin{array}{llcll}
\textbf{Super-Torsion} & \big(T^a & := & \mathrm{d}\,E^a & - \Omega^a{}_b\,E^b - \big(\,\overline{\Psi}\,\Gamma^a\,\Psi\big)\big)_{a=0}^{D-1} \\
\textbf{Gravitino field strength} & \big(\rho & := & \mathrm{d}\,\Psi & - \tfrac{1}{4}\Omega^{ab}\,\Gamma_{ab}\psi\big)_{\alpha=1}^{N} \\
\textbf{Curvature} & \big(R^{ab} & := & \mathrm{d}\,\Omega^{ab} & - \Omega^a{}_c\,\Omega^{cb}\big)_{a,b=0}^{D-1}.
\end{array}
\tag{233}
$$

Finally, we denote the corresponding components in the given local super-coframe (E, Ψ) by [GSS24a, (127-8)]:

$$
\begin{array}{lll}
T^a & \equiv & 0 \\
\rho & =: & \tfrac{1}{2}\rho_{ab}\,E^a\,E^b \,+\, H_a\Psi\,E^a \\
R^{a_1a_2} & =: & \tfrac{1}{2}R^{a_1a_2}{}_{b_1b_2}\,E^{a_1}\,E^{a_2} \,+\, \big(\overline{J}^{a_1a_2}{}_b\Psi\big)E^b \,+\, \big(\overline{\Psi}\,K^{a_1a_2}\,\Psi\big)\,,
\end{array}
\tag{234}
$$

where all components not explicitly appearing vanish identically by the superspace torsion constraints [GSS24a, (121), (137)]. In addition, in the main text we consider the situation that also $\rho_{ab} = 0$ whence also $J^{a_1a_2}{}_b = 0$.

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