The character map in (twisted differential) non-abelian cohomology

Domenico Fiorenza, Hisham Sati, Urs Schreiber

September 23, 2020

Abstract

The Chern character on K-theory has a natural extension to arbitrary generalized cohomology theories known as the Chern-Dold character. Here we further extend this to a character map on (twisted, differential) non-abelian cohomology theories, where its target is a non-abelian de Rham cohomology of twisted L_∞-algebra valued differential forms. The construction amounts to leveraging the fundamental theorem of dg-algebraic rational homotopy theory to a twisted non-abelian generalization of the de Rham theorem. We show that the non-abelian character reproduces, besides the Chern-Dold character, also the Chern-Weil homomorphism as well as its secondary Cheeger-Simons homomorphism on (differential) non-abelian cohomology in degree 1, represented by principal bundles (with connection); and thus generalizes all these to higher (twisted, differential) non-abelian cohomology, represented by higher bundles/higher gerbes (with higher connections). As a fundamental example, we discuss the twisted non-abelian character map on twistorial Cohomotopy theory over 8-manifolds, which can be viewed as a twisted non-abelian enhancement of topological modular forms (tmf) in degree 4. This turns out to exhibit a list of subtle topological relations that in high energy physics are thought to govern the charge quantization of fluxes in M-theory.

Contents

1 Introduction

Generalized cohomology theories [Wh62][Ad75] – such as K-theory, elliptic cohomology, stable Cobordism and stable Cohomotopy – are rich. This makes them fascinating but also intricate to deal with. In algebraic topology it has become commonplace to apply filtrations by iterative *localizations* [Bou79] (review in [EKMM97, §V][Ba14]) that allow generalized cohomology to be approximated in consecutive stages; a famous example of current interest is the chromatic filtration on complex oriented cohomology theories ([MR87] review in [Ra86][Lu10]).

The Chern-Dold character. The primary approximation stage of generalized cohomology theories is their *rationalization* (e.g. [Ba14, Ex. 1.7 (4)]) to ordinary cohomology (e.g. singular cohomology) with rational coefficients or real coefficients (see Remark 3.49). This goes back to [Do65]; and since on topological K-theory (Example 4.10) it reduces to the Chern character map [Hil71, Thm. 5.8], has been called the *Chern-Dold character* [Bu70]:

$$
\begin{tikzcd}
\overset{\text{Chern-Dold character}}{\mathrm{ch}^n_E} \;:\; \overset{\text{generalized cohomology}}{E^n(X)} \arrow[r, "\text{rationalization}"] \arrow[drr, bend right, "\mathrm{dR}\circ\mathrm{ch}_E", "\text{differential-geometric Chern-Dold character}"'] & E^n_{\mathbb{R}}(X) \arrow[r, "\text{Dold's equivalence}", "\simeq"'] & \overset{\text{ordinary cohomology}}{\bigoplus_k H^{n+k}}\big(X;\, \overset{\text{rationalized homotopy groups}}{\pi_k(E)\otimes_{\mathbb{Z}}\mathbb{R}}\big) \arrow[d, "\simeq"', "\text{de Rham theorem}"] \\
 & & \mathrm{Hom}\Big(E^\bullet(*)\otimes_{\mathbb{Z}}\mathbb{R}\,,\, \underset{\text{de Rham cohomology}}{H^{n+\bullet}_{\mathrm{dR}}(X)}\Big)
\end{tikzcd}
\tag{1}
$$

That the first map in (1) is indeed the rationalization approximation on coefficient spectra is left somewhat implicit in [Bu70] (rationalization was fully formulated only in [BK72]); a fully explicit statement is in [LSW16, §2.1]. The equivalence on the right of (1) serves to make explicit how the result of that rationalization operation indeed lands in ordinary cohomology, and this was Dold's original observation [Do65, Cor. 4].

At the heart of differential cohomology. While rationalization is the coarsest of the localization approximations, it stands out in that it connects, via the de Rham theorem, to *differential geometric* data – when the base space X has the structure of a smooth manifold, and the coefficients are taken to be $\mathbb{R}$ instead of $\mathbb{Q}$. Indeed, this "differential-geometric Chern-Dold character" shown on the bottom of (1), underlies (often without attribution to Dold or Buchstaber) the pullback-construction of differential generalized cohomology theories [HS05, §4.8] (see [BN14, p. 17][GS17b, Def. 7][GS18b, Def. 17][GS19a, Def. 1], and see Def. 4.33, Example 4.34 below).

At the heart of non-perturbative field theory. It is in this differential-geometric form that the Chern-Dold character plays a pivotal role in high energy physics. Here closed differential forms encodes *flux densities* $F_p \in \Omega^p_{\mathrm{dR}}(X)$ of generalized electromagnetic fields on spacetime manifolds X; and the condition that these lift through (i.e., are in the image of) the Chern-Dold character (1) for E-cohomology theory encodes a *charge quantization* condition in E theory [Fr00][Sa10][GS19c], generalizing Dirac's charge quantization of the ordinary electromagnetic field in ordinary cohomology [Di31] (see [Fra97, 16.4e]):

$$
\begin{tikzcd}
E^n(X) \arrow[rr, "\text{differential-geometric Chern-Dold character}", "\mathrm{dR}\circ\mathrm{ch}^n_E"'] & & \mathrm{Hom}\Big(E^\bullet(*)\otimes_{\mathbb{Z}}\mathbb{R}\,,\, H^{n+\bullet}_{\mathrm{dR}}(X)\Big) \\
\underset{\text{class in } E\text{-cohomology}}{[c]} \arrow[rr, mapsto] & & \underset{\text{charge-quantized flux densities}}{\big\{[F_{p_i}]\big\}_{i=1,\cdots,\dim\left(E^\bullet(*)\otimes_{\mathbb{Z}}\mathbb{R}\right)}}
\end{tikzcd}
\tag{2}
$$

This idea of charge quantization in a generalized cohomology theory turned out to be fruitful for capturing much of the expected nature of the RR-field in type II/I string theory, as being charge-quantized in topological K-theory: $E = \mathrm{KU}, \mathrm{KO}$ [FH00][Fr00][Ev06][GS19c][GS18b].

However, various further topological conditions [FSS19b, Table 1][FSS19c, p. 2][SS20a, Table 3][FSS20, p. 2] in non-perturbative type IIA string theory ("M-theory") are not captured by charge-quantization (2) in K-theory, or in any generalized cohomology theory, since they involve quadratic functions (6) in the fluxes. This motivates:

Non-abelian cohomology. Despite their established name, generalized cohomology theories in the traditional sense of [Wh62][Ad75] are not general enough for many purposes. Already the time-honored non-abelian cohomology that classifies principal bundles (Example 2.3 below), being the domain of the Chern-Weil homomorphism

[Ch50] (recalled as Def. 4.21, Prop. 4.23 below), falls outside the scope of "generalized" cohomology, as does the higher non-abelian cohomology classifying gerbes [Gi71] (Example 2.6 below). But these are just the first two stages within a truly general concept of *higher non-abelian cohomology* (Def. 2.1 below), that classifies higher bundles/higher gerbes (Example 2.7 below) and which fully subsumes Whitehead's traditional generalized cohomology as its abelian sector (Example 2.13 below).

In higher non-abelian cohomology the very conceptualization of cohomology finds a beautiful culmination, as it is reduced to the pristine concept of homotopy types of mapping spaces (11), or rather, if geometric (differential, equivariant,...) structures are incorporated, of higher mapping stacks (Remark 2.27 below).

In particular, the concept of *twisted* non-abelian cohomology is most natural from this perspective (Def. 2.29 below) and naturally subsumes the traditional concept of twisted generalized cohomology theories (Prop. 2.37 below).

$$
\underset{\text{cohomology}}{\overset{\text{twisted}}{\text{non-abelian}}}\; H^{\tau}(X;A) \;=\; \left\{
\begin{tikzcd}
X \arrow[rr, dashed, "c"{above}, "\text{cocycle}"] \arrow[dr, "\tau"', "\text{twist}"'] & {} \arrow[d, phantom, "\Leftarrow \simeq"] & A /\!\!/ G \arrow[dl, "\rho", "\text{local coefficients}"'] \\
& \mathbf{B}G &
\end{tikzcd}
\right\}_{\text{coefficient } \infty\text{-stack } A/\!\!/G} \Big/ {}_{\text{homotopy relative } \mathbf{B}G}
$$

State of the literature. It is fair to say that this transparent fundamental nature of higher non-abelian cohomology is not easily recognized in much of the traditional literature on the topic, which is rife with unwieldy variants of cocycle conditions presented in combinatorial n-category-theoretic language. As a consequence, the development of non-abelian cohomology theory has seen little and slow progress, certainly as compared to the flourishing of generalized cohomology theory. In particular, the concepts of higher and of twisted non-abelian cohomology had remained mysterious (see [Si97, p. 1]). It is the more recently established homotopy-theoretic formulation of ∞-category theory (e.g. via model category theory, see appendix A) in its guise as ∞-topos theory (∞-*stacks*, recalled around Def. A.44 below) that provides the backdrop on which twisted higher non-abelian cohomology finds its true and elegant nature [Si97][Si99][To02][SSS12][NSS12a][NSS12b][Sc13][FSS19b][SS20b]; see §2.

The non-abelian character map. From this homotopy-theoretic perspective, we observe in §4, §5 that the generalization of the Chern-Dold character (1) to twisted non-abelian cohomology naturally exists (Def. 4.2), and that the non-abelian analogue of Dold's equivalence in (1) may neatly be understood as being, up to mild reconceptualization, the fundamental theorem of dg-algebraic rational homotopy theory (recalled as Prop. 3.58 below): We highlight that this classical theorem is fruitfully recast as constituting a *non-abelian de Rham theorem* (Theorem 3.85 below) and, more generally, a *twisted non-abelian de Rham theorem* (Theorem 3.102 below). With this in hand, the notion of the (twisted) non-abelian character map appears naturally (Def. 4.2 and Def. 5.4 below):

$$
\underset{\text{(Def. 5.4)}}{\overset{\text{twisted non-abelian}}{\text{character map}}}\quad \mathrm{ch}_{\rho} \;:\; \underset{\substack{\text{twisted non-abelian cohomology}\\ \text{(Def. 2.29)}}}{H^{\tau}(X;A)} \xrightarrow[\substack{\text{rationalization}\\ \text{(Def. 3.53, Prop. 3.58)}}]{(\eta^{\mathbb{R}}_{\rho})_{*}} \underset{\substack{\text{twisted non-abelian real cohomology}\\ \text{(Def. 3.74)}}}{H^{L_{\mathbb{R}}\tau}(X; L_{\mathbb{R}}A)} \xrightarrow[\substack{\text{twisted non-abelian}\\ \text{de Rham theorem}\\ \text{(Thm. 3.102)}}]{\simeq} \underset{\substack{\text{twisted non-abelian}\\ \text{de Rham cohomology}\\ \text{(Def. 3.96)}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X; \mathfrak{l}A)} \tag{3}
$$

Twisted differential non-abelian cohomology. Moreover, with the (twisted) non-abelian character in hand, the notion of (twisted) differential non-abelian cohomology appears naturally (Def. 4.33, Def. 5.11) together with the expected natural diagrams of twisted differential non-abelian cohomology operations:

$$
\begin{tikzcd}
\underset{\substack{\text{differential non-abelian cohomology}\\ \text{(Def. 4.33)}}}{\widehat{H}(X;A)} \arrow[rr, "\substack{\text{curvature}\\ (231)}"] \arrow[d, "\substack{\text{characteristic class}\\ (230)}"'] \arrow[drr, "\substack{\text{differential}\\ \text{non-abelian character}\\ (232)}"] & & \underset{\substack{\text{flat } L_\infty\text{-algebra valued}\\ \text{differential forms (Def. 3.75)}}}{\Omega_{\mathrm{dR}}(X; \mathfrak{l}A)} \arrow[d] \\
\underset{\substack{\text{non-abelian cohomology}\\ \text{(Def. 2.1)}}}{H(X;A)} \arrow[rr, "\mathrm{ch}_A", "\substack{\text{non-abelian character}\\ \text{(Def. 4.2)}}"'] & & \underset{\substack{\text{non-abelian}\\ \text{de Rham cohomology}\\ \text{(Def. 3.82)}}}{H_{\mathrm{dR}}(X; \mathfrak{l}A)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underset{\substack{\text{twisted differential}\\ \text{non-abelian cohomology}\\ \text{(Def. 5.11)}}}{\widehat{H}^{\tau_{\mathrm{diff}}}(X;A)} \arrow[rr, "\substack{\text{twisted}\\ \text{curvature}\\ (274)}"] \arrow[d, "\substack{\text{twisted}\\ \text{characteristic class}\\ (273)}"'] \arrow[drr, "\substack{\text{twisted differential}\\ \text{non-abelian character}\\ (275)}"] & & \underset{\substack{\text{twisted flat}\\ L_\infty\text{-algebra valued}\\ \text{differential forms (Def. 3.90)}}}{\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X; \mathfrak{l}A)} \arrow[d] \\
\underset{\substack{\text{twisted}\\ \text{non-abelian cohomology}\\ \text{(Def. 2.29)}}}{H^{\tau}(X;A)} \arrow[rr, "\mathrm{ch}^{\tau}_A", "\substack{\text{twisted}\\ \text{non-abelian character}\\ \text{(Def. 5.4)}}"'] & & \underset{\substack{\text{twisted non-abelian}\\ \text{de Rham cohomology}\\ \text{(Def. 3.96)}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X; \mathfrak{l}A)}
\end{tikzcd}
\tag{4}
$$

Unifying Chern-Dold, Chern-Weil and Cheeger-Simons. In order to show that this generalization of (twisted) character maps and (twisted) differential cohomology to higher non-abelian cohomology is sound, we proceed to prove that the non-abelian character map (Def. 4.2) specializes to

the Chern-Dold character on generalized cohomology	(Theorem 4.8),
the Chern-Weil homomorphism on degree-1 non-abelian cohomology	(Theorem 4.26)
the Cheeger-Simons homomorphism on degree-1 differential non-abelian cohomology	(Theorem 4.46)

All these classical invariants are thus seen as different low-degree aspects of the higher non-abelian character map.

Examples of twisted higher character maps. To illustrate the mechanism, we make explicit a few examples of the twisted non-abelian character map on higher K-theories of relevance in high energy physics:

the Chern character on complex differential K-theory	(Example 4.10, 4.36),
the Pontrjagin character on real K-theory	(Example 4.11),
the Chern character on twisted differential K-theory	(Example 5.5, 5.20),
the LSW-character on twisted iterated K-theory	(Example 5.8),
the character on integral Morava K-theory	(Example 4.15),
the character on topological modular forms, tmf	(Example 4.12).

Once incarnated this way within the more general context of non-abelian cohomology theory, we may ask for non-abelian enhancements (Example 2.24) of these abelian characters:

Non-abelian enhancement of the tmf**-character – the cohomotopical character.** Our culminating example, in §5.3, is the character map on twistorial Cohomotopy theory [FSS19b][FSS20], over 8-manifolds X^8 equipped with tangential $\mathrm{Sp}(2)$-structure τ (58). This may be understood (Remark 4.14) as an enhancement of the tmf-character (Example 4.12) from traditional generalized cohomology to twisted differential non-abelian cohomology:

$$
\underset{\substack{\text{tmf-cohomology}\\ \text{in degree 4}\\ \text{(Example 4.12)}}}{\mathrm{tmf}^4(X^8)}
\underset{\substack{\text{tmf approximates}\\ \text{sphere spectrum}\\ \text{(Example 4.13)}}}{\simeq}
\underset{\substack{\text{stable Cohomotopy}\\ \text{in degree 4}\\ \text{(Example 2.16)}}}{\mathbb{S}^4(X^8)}
\underset{\substack{\text{non-abelian}\\ \text{enhancement}\\ \text{(Example 2.25)}}}{\rightsquigarrow}
\underset{\substack{\text{unstable/non-abelian}\\ \text{4-Cohomotopy}\\ \text{(Example 2.10)}}}{\pi^4(X^8)}
\underset{\substack{\text{twisting by}\\ \text{J-homomorphism}\\ \text{(Def. 2.29)}}}{\rightsquigarrow}
\underset{\substack{\text{twisted non-abelian}\\ \text{4-Cohomotopy}\\ \text{(Example 2.39)}}}{\pi^{\tau^4}(X^8)}
\underset{\substack{\text{lift through}\\ \text{twisted cohomology operation}\\ \text{induced by twistor fibration}\\ \text{(Example 2.42)}}}{\rightsquigarrow}
\underset{\substack{\text{twistorial}\\ \text{Cohomotopy}\\ \text{(Example 2.42)}}}{\mathscr{T}^{\tau^4}(X^8)}
\underset{\substack{\text{differential}\\ \text{enhancement}\\ \text{(Def. 5.11)}}}{\rightsquigarrow}
\underset{\substack{\text{differential}\\ \text{twistorial}\\ \text{Cohomotopy}\\ \text{(Example 5.23)}}}{\widehat{\mathscr{T}}^{\tau^4}(X^8)}
$$

The non-abelian character map on twistorial Cohomotopy has the striking property (Prop. 5.22, the proof of which is the content of the companion article [FSS20, Prop. 3.9]) that the corresponding non-abelian version of Dirac's charge quantization (2) implies Hořava-Witten's Green-Schwarz mechanism in heterotic M-theory for heterotic line bundles F_2 (see [FSS20, §1]; here ω denotes any compatible $\mathrm{Sp}(2)$-connection on TX^8):

$$
\begin{array}{ccc}
\underset{\substack{\text{differential}\\ \text{twistorial Cohomotopy}}}{\widehat{\mathscr{T}}^{\tau_{\mathrm{diff}}}(X)} & \xrightarrow[\text{(non-abelian character form representative)}]{\text{curvature}} & \underset{\substack{\text{flat twistorial}\\ \text{differential forms}}}{\Omega_{\mathrm{dR}}^{\tau_{\mathrm{dR}}}\big(X; \mathfrak{l}\mathbb{C}P^3\big)_{\mathrm{flat}}} \\
\widehat{C}_3 & \longmapsto & \left(
\begin{array}{c|l}
H_3, & dH_3 = G_4 - \tfrac{1}{4}p_1(\omega) - F_2 \wedge F_2, \\
 & \qquad \text{Hořava-Witten Green-Schwarz mechanism} \\
 & \qquad \big[G_4 - \tfrac{1}{4}p_1(\omega)\big] = \big[F_2 \wedge F_2\big] \in H^4(X;\mathbb{Z}) \\
G_4, F_2, & d\ G_4 = 0, \;\; dF_2 = 0, \\
2G_7 & d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) - 24\, I_8(\omega) \\
 & \qquad \text{C-field tadpole cancellation \& M5 Hopf WZ term level quantization}
\end{array}
\right)
\end{array}
\tag{5}
$$

In fact, it also implies C-field tadpole cancellation [FSS19b, §3.8][SS19a], residual M5-brane anomaly cancellation [FSS19c][SS20a] and further topological conditions expected in M-theory [FSS19b, Table 1]. This suggests the *Hypothesis H* [Sa13][FSS19b][FSS19c][SS19a][SS19b][SS20a][FSS20] that the elusive cohomology theory which controls M-theory in analogy to how K-theory controls string theory is: twisted non-abelian Cohomotopy theory.

Quadratic functions from Whitehead brackets in non-abelian coefficient spaces. The crucial appearance of *quadratic functions* in the Cohomotopical character map (5)

$$
\begin{array}{lll}
\textbf{(a)} & G_4 \mapsto \big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) & \text{(integral Hopf Wess-Zumino term [FSS19c])} \\
& \quad + 24 I_8(\omega) & \\
\textbf{(b)} & F_2 \mapsto F_2 \wedge F_2 & \text{(2nd Chern class of } S\big(\mathrm{U}(1)^2\big) \subset E_8 \text{ bundle [FSS20, (7)]),}
\end{array}
\tag{6}
$$

is brought about by the non-abelian nature of (twisted) Cohomotopy theory: These non-linearities originate in non-trivial Whitehead brackets (Remark 3.64) on the non-abelian coefficient spaces S^4 (Example 3.66) and on $\mathbb{C}P^3$ (Example 3.94). Generally, the non-abelian character map (3) involves also higher monomial terms of any order (cubic, quartic, ...), originating in higher order Whitehead brackets on the non-abelian coefficient space (Remark 3.64).

Note that the desire to conceptually grasp character-like but quadratic functions appearing in M-theory had been the original motivation for developing differential generalized cohomology, in [HS05]. Here, in differential non-abelian cohomology, they appear intrinsically.

Non-abelian Hurewicz/Boardman homomorphism. These quadratic functions in the non-abelian character map (3) disappear (by Example 3.67) under the forgetful cohomology operations from non-abelian cohomology to traditional (abelian) generalized cohomology theory (Example 2.24). Specifically, there is a secondary non-abelian cohomology operation (Def. 4.42) from non-abelian differential 4-Cohomotopy (Example 4.38) to differential K-theory (Example 4.36), the *secondary non-abelian Hurewicz/Boardman homomorphism* (Example 4.43)

$$
\begin{tikzcd}
\underset{\text{differential Cohomotopy}}{\widehat{\pi}^4(\mathscr{X})} \arrow[rr, "\beta^4_{\mathrm{diff}}", "\text{secondary non-abelian Boardman homomorphism}"'] \arrow[d, "(F_{S^4})_*"'] & & \underset{\text{differential K-theory}}{\widehat{\mathrm{KU}}^0(\mathscr{X})} \arrow[d, "(F_{\mathrm{KU}_0})_*"] \\
\left\{ \begin{array}{c} 2G_7, \\ G_4 \end{array} \middle| \begin{array}{l} d\,2G_7 = -G_4 \wedge G_4 \\ d\;\;G_4 = 0 \end{array} \right\} \arrow[rr, "{\substack{G_4 \mapsto F_4 \\ G_7 \mapsto 0}}"'] \arrow[rr, phantom, shift left=4ex, "\text{curvatures/flux densities}"] & & \Big\{ (F_{2k}) \mid d F_{2k} = 0 \Big\},
\end{tikzcd}
\tag{7}
$$

which on curvature forms/flux densities forgets the quadratic function (6) in the C-field's G_4-flux (shown in (7) for trivial J-twist) and identifies what remains with the RR-field F_4-flux density. This is the identification of M/IIA fluxes envisioned in [DMW03]. The other RR-flux components also appear in the cohomotopical character after cohomological double dimensional reduction: this is discussed in detail in [BMSS19].

Secondary non-abelian charge quantization *on* K-theory. Accordingly, one may regard the non-abelian Boardman homomorphism (7) as a non-abelian but K-theory valued character, lifting the target of the plain non-abelian character (5) from rational cohomology to K-theory. As such, it imposes *secondary charge quantization conditions on* K-theory, analogous to (2) but invisible even in generalized cohomology, instead now coming from non-abelian cohomology theory (specifically from non-abelian Cohomotopy, compare [BSS19, Fig. 1]):

$$
\begin{tikzcd}
\underset{\text{differential Cohomotopy}}{\widehat{\pi}^4(\mathscr{X})} \arrow[r, "\beta^4_{\mathrm{diff}}", "{\substack{\text{differential non-abelian} \\ \text{Boardman homomorphism} \\ \text{(Example 4.43)}}}"'] & \underset{\text{differential K-theory}}{\widehat{\mathrm{KU}}^0(\mathscr{X})} \arrow[r, "\mathrm{ch}_{\mathrm{diff}}", "{\substack{\text{differential} \\ \text{Chern character} \\ \text{(Example 5.20))}}}"'] & \underset{\text{differential rational cohomology}}{\widehat{H_{\mathrm{per}}\mathbb{Q}}^0(\mathscr{X})}
\end{tikzcd}
\tag{8}
$$

$$
\xleftarrow{\rule{0pt}{0pt}} \text{charge-quantization in M-theory} \qquad \xleftarrow{\rule{0pt}{0pt}} \text{charge-quantization in string theory}
$$

Equivariant enhancement. This is particularly interesting after lifting further to *equivariant* non-abelian cohomology theory, where charge-quantizazing/lifting of RR-fields in equivariant K-theory through the Boardman homomorphism on the left of (8) encodes pertinent "tadpole cancellation" conditions [SS19a][BSS19].

The character theory presented here lifts to the required *equivariant* differential non-abelian cohomology on orbi-orientifolds by combining it with the techniques developed in [HSS18][SS20b]. We discuss the resulting *character map in equivariant (twisted differential) non-abelian cohomology* in a followup article.

Acknowledgements. We thank John Lind, Carlos Simpson, and Danny Stevenson for comments on an earlier version of this note.

2 Non-abelian cohomology

We make explicit the concept of general non-abelian cohomology (Def. 2.1 below) and of twisted non-abelian cohomology (Def. 2.29 below), following [Si97][Si99][To02][SSS12][NSS12a][NSS12b][FSS19b][SS20b]; and we survey how this concept subsumes essentially every notion of cohomology known.

In the following, we make free use of the basic language of category theory and homotopy theory (for joint introduction see [Rie14][Ri20]). For $\mathscr{C}$ a category and $X, A \in \mathscr{C}$ a pair of its objects, we write

$$\mathscr{C}(X\,,A) \;:=\; \mathrm{Hom}_{\mathscr{C}}(X\,,A) \;\in\; \mathrm{Sets} \tag{9}$$

for the set of morphisms from X to A. These are, of course, contravariantly and covariantly functorial in their first and second argument, respectively:

$$\mathscr{C} \xrightarrow{\;\mathscr{C}(X\,,-)\;} \mathrm{Sets}\;, \qquad \mathscr{C}^{\mathrm{op}} \xrightarrow{\;\mathscr{C}(-\,,A)\;} \mathrm{Sets}\,. \tag{10}$$

Basic as this is, contravariant hom-functors are of paramount interest in the case where $\mathscr{C}$ is the *homotopy category* $\mathrm{Ho}(\mathbf{C})$ (Def. A.14) of a model category (Def. A.3), such as the classical homotopy category of topological spaces or, equivalently, of simplicial sets (Example A.33).

2.1 Non-abelian cohomology theories

Definition 2.1 (Non-abelian cohomology)**.** For $X, A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33) we say that their hom-set (9) is the *non-abelian cohomology* of X *with coefficients* in A, or the *non-abelian A-cohomology* of X, to be denoted:

$$\overset{\substack{\text{non-abelian}\\ \text{cohomology}}}{H(X;A)} \;:=\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)(X\,,A) \;=\; \left\{ \begin{tikzcd} X \arrow[rr, bend left, "\overset{\text{map = cocycle}}{c}"] \arrow[rr, bend right, "\underset{\text{map = cocycle}}{c'}"'] \arrow[rr, phantom, "\Downarrow\, \substack{\text{homotopy =}\\ \text{coboundary}}"] & & A \end{tikzcd} \right\} \Big/ {\scriptstyle\text{homotopy}} \tag{11}$$

We also call the contravariant hom-functor (10)

$$H(-;A)\;:\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \longrightarrow \mathrm{Sets} \tag{12}$$

the non-abelian *A-cohomology theory*.

Example 2.2 (Ordinary cohomology)**.** For $n \in \mathbb{N}$ and A a discrete abelian group, the ordinary cohomology (e.g. singular cohomology) in degree n with coefficients in A is equivalently ([Ei40, p. 243][EML54b, p. 520-521], review in [St72, §19][May99, §22][AGP02, §7.1, Cor. 12.1.20]) non-abelian cohomology in the sense of Def. 2.1

$$\overset{\substack{\text{ordinary}\\ \text{cohomology}}}{H^n(-;A)} \;\simeq\; H\big(-;K(A,n)\big) \tag{13}$$

with coefficients in an *Eilenberg-MacLane space* [EML53][EML54a]:

$$K(A,n) \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \quad \text{such that} \quad \pi_k\big(K(A,n)\big) \;=\; \left\{ \begin{array}{c|c} A & k = n \\ 0 & k \neq n\,. \end{array} \right. \tag{14}$$

Example 2.3 (Traditional non-abelian cohomology)**.** For G a well-behaved topological group, the traditional non-abelian cohomology $H^1(-;G)$ classifying G-principal bundles, is equivalently ([St51, §19.3][RS12, Thm 1.], review in [Add07, §5]) non-abelian cohomology in the general sense of Def. 2.1

$$\overset{\substack{\text{classification of}\\ \text{principal bundles}}}{H^1(-;G)} \;\simeq\; H(-;BG) \tag{15}$$

with coefficients in the *classifying space* BG ([Mi56][Se68][St68][St70], review in [Ko96, §1.3][May99, §23.1][AGP02, §8.3][NSS12b, §3.7.1]). The latter may be given as the homotopy colimit (in the classical model structure of $\mathrm{TopologicalSpaces}_{\mathrm{Qu}}$, Example A.7) over the nerve of the topological group G (e.g. [NSS12a, Rem. 2.23]):

$$BG \;\simeq\; \underset{\longrightarrow}{\mathrm{holim}}\left(\cdots \; G \times G \;\underset{\substack{\longleftarrow\\ \longrightarrow}}{\overset{\substack{\longrightarrow\\ \longleftarrow}}{\xrightarrow{(-)\cdot(-)}}}\; G \;\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;\mathrm{e}\;}}}\; * \right). \tag{16}$$

Example 2.4 (Group cohomology and Characteristic classes)**.** Conversely, the ordinary cohomology (Example 2.2) *of* a classifying space BG (16) is, equivalently,

(i) the group cohomology of G;

(ii) the universal characteristic classes of G-principal bundles:

$$\overset{\substack{\text{group}\\ \text{cohomology}}}{H\big(BG;\, K(A,n)\big)} \;\simeq\; H^n(BG;A) \;\simeq\; H^n_{\mathrm{Grp}}(G;A)\,.$$

Example 2.5 (Non-abelian cohomology in degree 2)**.** For a well-behaved topological 2-group, such as the *string 2-group* $\mathrm{String}(G)$ (of a connected, simply connected semi-simple Lie group G) [BCSS07][He08, Thm. 4.8][NSW11], the non-abelian cohomology $H^1(-;\mathrm{String}(G))$ classifying principal 2-bundles [NW11] with structure 2-group $\mathrm{String}(G)$ is, equivalently [BS09],

$$\overset{\substack{\text{classification of}\\ \text{String-bundles}}}{H^1\big(-;\mathrm{String}(G)\big)} \;\simeq\; H\big(-;B\mathrm{String}(G)\big) \tag{17}$$

non-abelian cohomology in the general sense of Def. 2.1 with coefficients in the classifying space $B\mathrm{String}(G)$.

Example 2.6 (Non-abelian gerbes)**.** For G a well-behaved topological group, a non-abelian *G-gerbe* [Gi71][Br09] is, equivalently [NSS12a, §4.4], a fiber 2-bundle with typical 2-fiber of homotopy type of the classifying space BG (16), associated to principal 2-bundles with structure 2-group $\mathrm{Aut}(\mathbf{B}G)$. Hence, as in Example 2.5, G-gerbes are classified by non-abelian cohomology with coefficients in $B\mathrm{Aut}(\mathbf{B}G)$ [NSS12a, Cor 4.51]:

$$\overset{\substack{\text{classification of}\\ \text{non-abelian gerbes}}}{G\mathrm{Gerbes}(X)_{/\sim}} \;\simeq\; H^1\big(X;\, \mathrm{Aut}(\mathbf{B}G)\big) \;\simeq\; H\big(X;\, B\mathrm{Aut}(\mathbf{B}G)\big)\,.$$

Example 2.7 (Non-abelian cohomology in unbounded degree)**.** For any ∞-group $\mathscr{G}$ (see [NSS12a, §2.2][NSS12b, §3.5]), the non-abelian cohomology $H^1\big(-;\mathscr{G}\big)$ classifying principal ∞-bundles [Gl82][JL06][NSS12a][NSS12b] with structure ∞-group $\mathscr{G}$ is, equivalently [We10][RS12],

$$\overset{\substack{\text{classification of}\\ \text{non-abelian }\infty\text{-gerbes}}}{H^1(-;\mathscr{G})} \;\simeq\; H(-;B\mathscr{G}) \tag{18}$$

non-abelian cohomology in the general sense of Def. 2.1 with coefficients in the classifying space $B\mathscr{G}$ (see also [St12]).

Example 2.7 is, in fact, universal:

Proposition 2.8 (Connected homotopy types are higher non-abelian classifying spaces [NSS12a, Thm. 2.19][NSS12b, Thm. 3.30, Cor. 3.34])**.** *Every connected homotopy type* $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (324) *is the classifying space of a topological group, namely of its loop group*[1] ΩA

$$A \;\simeq\; B(\Omega A) \qquad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,. \tag{19}$$

[1] A priori, the loop group is an A_∞-group, for which classifying spaces are defined as in [NSS12a, Rem. 2.23], but each such is weakly equivalent to an actual topological group, see [NSS12b, Prop. 3.35].

This allows for making precise the core nature of non-abelian cohomology:

Remark 2.9 (Non-abelian and abelian ∞-groups)**.** For $A \simeq BG$ (19), the ∞-group structure on G is reflected by its weak homotopy equivalence $G \simeq \Omega BG$ with a based loop space.

- There is no commutativity of loops in a generic loop space, and hence this exhibits G as a *non-abelian* ∞-group.
- But it may happen that A itself is already equivalent to a loop space, which by (19) means that $A \simeq B\big(BG\big) =: B^2G$ is a *double delooping*. In this case $G \simeq \Omega\big(\Omega A\big) =: \Omega^2 A$ is an *iterated loop space* [May72], specifically *double loop space*; hence a *braided ∞-group*. By the Eckmann-Hilton argument, this implies some level of commutativity of the group operation in G. Indeed, in the special case that such G is also 0-truncated (326), it implies that G is an ordinary abelian group.
- Next, it may happen that $A \simeq B^3G$ is a 3-fold deloopig, hence that $G \simeq \Omega^3 A$ is a 3-fold loop space, hence a *sylleptic ∞-group*. This is one step "more abelian" than a braided ∞-group.
- In the limiting case that G is an n-fold loop space for any $n \in \mathbb{N}$, hence an *infinite loop space* [May77][Ad78], it is as abelian as possible for an ∞-group. Such *abelian ∞-groups* are the coefficients of abelian cohomology theories, namely of generalized cohomology theories in the sense of Whitehead (Example 2.13)
- The fewer deloopings an ∞-group G admits, the "more non-abelian" is the cohomology theory represented by BG.

Coefficients		$H(X;BG)$	**Examples**
∞-group	$G \simeq \Omega\ B\ G$	non-abelian cohomology	$\pi^n(-)$ (Cohomotopy, Example 2.10)
braided ∞-group	$G \simeq \Omega^2 B^2 G$		$\pi^3(-)$
sylleptic ∞-group	$G \simeq \Omega^3 B^3 G$		
$\vdots$	$G \simeq \Omega^n B^n G$		
abelian ∞-group	$G \simeq \Omega^\infty B^\infty G$	abelian cohomology	$E^n(-)$ (generalized cohomology, Example 2.13)

The most fundamental connected homotopy types are the n-spheres (all other are obtained by gluing n-spheres to each other):

Example 2.10 (Cohomotopy theory)**.** The non-abelian cohomology theory (Def. 2.1) with coefficients in the homotopy types of n-spheres is (unstable) *Cohomotopy* theory [Bo36][Sp49][Pe56][Ta09][KMT12]:

$$\overset{\text{Cohomotopy}}{\pi^n(-)} \;=\; H(-;S^n) \;\simeq\; H^1(-;\Omega S^n) \qquad \text{for } n \in \mathbb{N}_+ \,.$$

(i) By Prop. 2.8, Cohomotopy theory classifies principal ∞-bundles (Example 2.7) with structure ∞-group of the homotopy type of the ∞-group ΩS^n.
(ii) By Remark 2.9, Cohomotopy theory is a *maximally non-abelian* cohomology theory, in that S^n does not admit deloopings, for general n (it admits a single delooping for $n = 3$ and arbitrary deloopings for $n = 0,1$).

Example 2.11 (Bundle gerbes)**.** The classifying space (16) of the circle group $\mathrm{U}(1)$ is an Eilenberg-MacLane space (14)

$$B\mathrm{U}(1) \;\simeq\; K(\mathbb{Z},2) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,.$$

Since $\mathrm{U}(1)$ is abelian, this space carries itself the structure of (the homotopy type of) a 2-group, and hence has a higher classifying space

$$B^2\mathrm{U}(1) := B(B\mathrm{U}(1)) \;\simeq\; K(\mathbb{Z},3) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$$

in the sense of Example 2.5, which is an Eilenberg-MacLane space in one degree higher. The higher principal 2-bundles with structure 2-group $\mathbf{B}\mathrm{U}(1)$ are equivalently [NSS12a, Rem. 4.36] known as *bundle gerbes* [Mu96][SW07]. Therefore, Example 2.7 combined with Example 2.2 gives the classification of bundle gerbes by ordinary integral cohomology in degree 3:

$$\overset{\substack{\text{classification of} \\ \text{bundle gerbes}}}{H^1\big(-;\mathbf{B}\mathrm{U}(1)\big)} \;\simeq\; H\big(-;B^2\mathrm{U}(1)\big) \;\simeq\; H^3(-;\mathbb{Z})\,.$$

Example 2.12 (Higher bundle gerbes)**.** In fact, Prop. 2.8 implies that, for all $n \in \mathbb{N}$,

$$B^{n+1}\mathrm{U}(1) \;:=\; B\big(B^n\mathrm{U}(1)\big) \;\simeq\; K(\mathbb{Z}, n+2) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \tag{20}$$

in the sense of Example 2.7. The higher principal bundles with structure $(n+1)$-group $\mathbf{B}^n\mathrm{U}(1)$ [Ga97][FSSt10, §3.2.3][FSS12b, §2.6] are also known as *higher bundle gerbes* (for $n = 2$ see [CMW97][St01]). On these coefficients, Example 2.7 reduces to the classification of higher bundle gerbes by ordinary integral cohomology in higher degree:

$$\overset{\text{classification of higher bundle gerbes}}{H^1\big(-;\mathbf{B}^n\mathrm{U}(1)\big)} \;\simeq\; H\big(-;B^{n+1}\mathrm{U}(1)\big) \;\simeq\; H^{n+2}(-;\mathbb{Z})\,.$$

More generally, the special case of Example 2.7 where the coefficient ∞-group happens to be abelian is "generalized cohomology" in the standard sense of algebraic topology (including cohomology theories such as K-theory, elliptic cohomology, stable Cobordism theory, stable Cohomotopy theory, etc.):

Example 2.13 (Generalized cohomology)**.** For E a generalized cohomology theory [Wh62] (see [Ad75][Ad78]), Brown's representability theorem ([Ad75, §III.6][Ko96, §3.4]) says that there is a *spectrum* ("Ω-spectrum", Example A.40) of pointed homotopy types

$$\Big\{ E_n \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}}\big)\,,\; E_n \xrightarrow[\simeq]{\widetilde{\sigma}_n} \Omega E_{n+1} \Big\}_{n\in\mathbb{N}} \tag{21}$$

such that the generalized E-cohomology in degree n is equivalently non-abelian cohomology theory in the sense of Def. 2.1 with coefficients in E_n:

$$\overset{\text{generalized cohomology}}{E^n(-)} \;\simeq\; H(-;E_n)\,. \tag{22}$$

Example 2.14 (Topological K-theory)**.** The classifying space (21) representing complex K-cohomology theory KU [AH59, §2] (review in [At67]) in degree 0 is [AH61, §1.3]:

$$\mathrm{KU}_0 \;\simeq\; \mathbb{Z}\times B\mathrm{U}\,, \tag{23}$$

where

$$B\mathrm{U} \;:=\; \varinjlim_{n} B\mathrm{U}(n) \tag{24}$$

is the classifying space (16) for the infinite unitary group (e.g. [EU14]). Hence for the case of complex K-theory, Example 2.13 says that:

$$\overset{\text{topological K-theory}}{\mathrm{KU}^0(-)} \;\simeq\; H(-;\mathbb{Z}\times B\mathrm{U})\,.$$

Example 2.15 (Iterated K-theory)**.** Given a spectrum (21) with suitable ring structure, one can form its *algebraic K-theory spectrum* $K(R)$ [EKMM97, §VI][BGT10, §9.5][Lu14] and hence the corresponding generalized cohomology theory (Example 2.13). Much like complex topological K-theory (Example 2.14) is the K-theory of topological $\mathbb{C}$-module bundles, $K(R)$-cohomology theory is the K-theory of suitable R-module ∞-bundles [Li13]. Specifically, for $R = \mathrm{ku}$ the connective spectrum for topological K-theory, its algebraic K-theory $K(\mathrm{ku})$ [Au09][AR02][AR07] has been argued to be the K-theory of certain categorified complex vector bundles [BDR03][BDRR09]. Moreover, $K(R)$ is itself a suitable ring spectrum, so that the construction may be iterated to yield *iterated algebraic K-theories* [Ro14] $K^{\circ 2}(R) := K(K(R))$, $K^{\circ 3}(R) := K(K(K(R)))$, et cetera. For $R = \mathrm{ku}$, this generalizes the above forms of elliptic cohomology, $K(\mathrm{ku})$, to higher degrees [LSW16]. By Example 2.13, we will regard these (connective) iterated algebraic K-theories $K^{\circ n}(\mathrm{ku})$ of the complex topological K-theory spectrum as examples of non-abelian cohomology theories:

$$\overset{\text{iterated K-theory}}{K^{\circ n}(\mathrm{ku})^0(-)} \;\simeq\; H\big(-;K^{\circ n}(\mathrm{ku})_0\big)\,.$$

Example 2.16 (Stable Cohomotopy)**.** The generalized cohomology theory (Example 2.13) represented by the suspension spectra (Example A.41) of n-spheres is called *stable Cohomotopy theory* (e.g. [Str81][No03]) or *stable framed Cobordism theory*:

$$\mathbb{S}^n(-) \;=\; H\big(-;(\Sigma^\infty S^n)_0\big)\,. \tag{25}$$

Non-abelian cohomology operations.

Definition 2.17 (Non-abelian cohomology operation)**.** For $A_1, A_2 \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33), we say that a natural transformation in non-abelian cohomology (Def. 2.1) from A_1-cohomology theory to A_2-cohomology theory (12) is a (non-abelian) *cohomology operation*

$$\phi_* \;:\; H(-;A_1) \longrightarrow H(-;A_2)\,. \tag{26}$$

By the Yoneda lemma, these are in bijective correspondence to morphisms of coefficients

$$A_1 \xrightarrow{\;\phi\;} A_2 \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \tag{27}$$

via the covariant functoriality of the hom-sets (10):

$$\phi_* \;=\; H(-;\phi) \;:=\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)(-;\phi)\,. \tag{28}$$

Example 2.18 (Cohomology of coefficient spaces parametrizes cohomology operations)**.** By the Yoneda lemma (28) in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33), the set of all cohomology operations (Def. 2.17) from A_1-cohomology theory to A_2-cohomology theory (26) coincides with the the non-abelian A_2-cohomology (Def. 2.1) *of* the coefficients A_1:

$$\underset{\substack{\text{non-abelian } A_2\text{-cohomology of } A_1 \\ \text{acting as cohomology operations}}}{H(A_1;A_2)} \times H(-;A_1) \xrightarrow{\;(-)\circ(-)\;} H(-;A_2) \tag{29}$$

acting by composition composition in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$.

Example 2.19 (Cohomology operations in ordinary cohomology)**.** In specialization to Example 2.2 the non-abelian cohomology operations according to Def. 2.17 reduce to the classical cohomology operations in ordinary cohomology [St72][MT08] (review in [May99, §22.5]), such as Steenrod operations [St47][SE62] (review in [Ko96, §2.5]). These operations admit refinements, involving rational/real form data, to differential cohomology operations [GS18a].

Example 2.20 (Cohomology operations in generalized cohomology)**.** In specialization to Example 2.13, the non-abelian cohomology operations according to Def. 2.17 reduce to the traditional cohomology operations on generalized cohomology theories, such as the Adams operations in K-theory [Ad62] (review in [AGP02, §10]) or the Quillen operations in stable Cobordism theory (review in [Ko96, §4,5]). For differential refinements see [GS18b].

Example 2.21 (Characteristic classes of principal ∞-bundles)**.** For G a topological group, the ordinary group cohomology of G (Example 2.4) parametrizes, via Example 2.18, the cohomology operations from non-abelian cohomology classifying G-principal bundles (Examples 2.3, 2.5, 2.7) to ordinary cohomology of the base space (Example 2.2):

$$\underset{\substack{\text{group}\\ \text{cohomology}}}{H^n_{\mathrm{Grp}}\big(BG;A\big)} \;\times\; \underset{\substack{G\text{-principal}\\ \text{bundles}}}{H^1(-;G)} \xrightarrow[\;(29)\;]{\substack{\text{characteristic}\\ \text{classes}}} \underset{\substack{\text{ordinary}\\ \text{cohomology}}}{H^n(-;A)}\,. \tag{30}$$

This is the assignment of *characteristic classes* to principal bundles (principal ∞-bundles). In the case when $A = \mathbb{R}$, this is equivalently the *Chern-Weil homomorphism*, by Chern's fundamental theorem (see Remark 4.16 and Theorem 4.26 below).

Example 2.22 (Rationalization cohomology operation)**.** For fairly general non-abelian coefficients A (see Def. 3.53, Def. 4.1 for details), their *rationalization*[2] $A \xrightarrow{\eta^{\mathbb{R}}_A} L_{\mathbb{R}}A$ (Def. 3.53 below) induces a cohomology operation (Def. 2.17) from non-abelian A-cohomology theory (Def. 2.1) to *non-abelian real cohomology* (Def. 3.70 below):

$$\underset{\substack{\text{non-abelian}\\ \text{cohomology}}}{H(-;A)} \xrightarrow[\text{rationalization}]{(\eta^{\mathbb{R}}_A)_*} \underset{\substack{\text{non-abelian}\\ \text{real cohomology}}}{H\big(-;L_{\mathbb{R}}A\big)}\,. \tag{31}$$

[2] For definiteness, we consider rationalization over the real numbers; see Remark 3.49 below.

Remark 2.23 (Rationalization as character map)**.** Up to composition with an equivalence provided by the non-abelian de Rham theorem (Theorem 3.85 below), which serves to bring the right hand side of (31) into neat minimal form, this rationalization cohomology operation is the *character map in non-abelian cohomology* (Def. 4.2 below).

Example 2.24 (Stabilization cohomology operation)**.** For $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$, the non-abelian cohomology operation (Def. 2.17) induced (28) by the unit of the derived stabilization adjunction (Example A.41) goes from non-abelian A-cohomology theory (Def. 2.1) to (abelian) generalized cohomology theory (Example 2.13) represented by the 0th component space of the suspension spectrum of A:

$$\underset{\substack{\text{non-abelian}\\ A\text{-cohomology}}}{H\big(-;A\big)} \xrightarrow[\text{stabilization}]{} \underset{\substack{\text{generalized}\\ \Sigma^\infty A\text{-cohomology}}}{H\big(-;(\mathbb{L}\Sigma^\infty A)_0\big)}\,.$$

Hence a lift through this operation is an enhancement of generalized cohomology to non-abelian cohomology.

Example 2.25 (Non-abelian enhancement of stable Cohomotopy)**.** The canonical non-abelian enhancement (in the sense of Example 2.24) of stable Cohomotopy (Example 2.16) is actual Cohomotopy theory (Example 2.10):

$$\underset{\text{Cohomotopy}}{\pi^n(-)} \xrightarrow[\text{stabilization}]{} \underset{\substack{\text{stable}\\ \text{Cohomotopy}}}{\mathbb{S}^n(-)}\,.$$

Example 2.26 (Hurewicz homomorphism and Hopf degree theorem)**.** By definition of Eilenberg-MacLane spaces (14) there is, for $n \in \mathbb{N}$, a canonical map

$$S^n \xrightarrow{e^{(n)}} K(\mathbb{Z},n) \qquad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,,$$

which represents the element $1 \in \mathbb{Z} \simeq \pi_n\big(K(\mathbb{Z},n)\big)$. The non-abelian cohomology operation (Def. 2.17) induced by this, from degree n Cohomotopy (Example 2.10) to degree n ordinary cohomology (Example 2.2)

$$\pi^n(-) \xrightarrow{e^{(n)}_*} H^n(-;\mathbb{Z})$$

is the cohomological version of the Hurewicz homomorphism. The *Hopf degree theorem* (e.g. [Ko93, §IX (5.8)]) is the statement that the non-abelian cohomology operation $e^{(n)}_*$ becomes an isomorphism on connected, orientable closed manifolds of dimension n. These maps, together with their differential refinements, are analyzed in more details via Postnikov towers in [GS20].

Structured non-abelian cohomology.

Remark 2.27 (Structured non-abelian cohomology)**.** More generally, it makes sense to consider the analog of Def. 2.1 for the homotopy category $\mathrm{Ho}(\mathbf{H})$ of a model category which is a *homotopy topos* [TV05][Lu09][Re10].

(i) This yields *structured* non-abelian cohomology [Si97][Si99][To02][SSS12][NSS12a][NSS12b][Sc13][FSS19b][SS20b]:

$$\underset{\substack{\text{structured}\\ \text{non-abelian cohomology}}}{H\big(\mathscr{X};\mathbf{A}\big)} \;:=\; \overset{\text{homotopy topos}}{\mathrm{Ho}(\mathbf{H})}\big(\underset{\infty\text{-stacks}}{\mathscr{X}\,,\mathbf{A}}\big)\,,$$

including the stacky non-abelian cohomology originally considered in [Gi71][Br90] ("gerbes", see [NSS12a, §4.4]), and, more generally, *differential-, étale-, and equivariant-* nonabelian cohomology theories (see [SS20b, p. 6]) based on ∞-stacks.

(ii) In good cases (cohesive homotopy toposes [Sc13][SS20b, §3.1]), the homotopy topos $\mathrm{Ho}(\mathbf{H})$ comes equipped with a *shape* operation down to the classical homotopy category (Example A.33):

$$\begin{array}{ccc}
\overset{\text{homotopy topos}}{\mathrm{Ho}(\mathbf{H})} & \xrightarrow{\ \mathrm{Shp}\ } & \overset{\text{classical homotopy category}}{\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)} \\
\underset{\substack{\text{structured}\\ \text{non-abelian cohomology}}}{H\big(\mathscr{X};\mathbf{A}\big)} & \longmapsto & \underset{\substack{\text{plain}\\ \text{non-abelian cohomology}}}{H\big(\mathrm{Shp}(\mathscr{X});\mathrm{Shp}(\mathbf{A})\big)}
\end{array} \tag{32}$$

which takes, for well-behaved group ∞-stacks G, the *classifying stacks* $\mathbf{B}G$ of G-principal bundles to the traditional classifying spaces $BG \simeq \mathrm{Shp}(\mathbf{B}G)$ of underlying topological groups (16). This gives a forgetful functor from structured non-abelian cohomology to plain non-abelian cohomology in the sense of Def. 2.1. A classical example is the map from non-abelian Čech cohomology with coefficients in a well-behaved group G to homotopy classes of maps to the classifying space of G, in which case this comparison map is a bijection (Example 2.3).

All constructions on non-abelian have their structured analogues, for instance non-abelian cohomology operations (Def. 2.17) in structured cohomology

$$H\big(\mathscr{X};\mathbf{A}_1\big) \xrightarrow{\ \phi_*\ } H\big(\mathscr{X};\mathbf{A}_2\big) \tag{33}$$

are induced by postcomposition with morphisms $\mathbf{A}_1 \xrightarrow{\phi} \mathbf{A}_2$ of coefficient stacks.

Ultimately, one is interested in working with structured non-abelian cohomology on the left of (32). However, since this is rich and intricate, it behooves us to study its projection into plain non-abelian cohomology on the right of (32). This is what we are mainly concerned with here. But we provide in §4.3 a brief discussion of non-abelian differential cohomology on smooth ∞-stacks,

2.2 Twisted non-abelian cohomology.

For $\mathscr{C}$ any category and $B \in \mathscr{C}$ any object, there is the *slice category* $\mathscr{C}^{/X}$, whose objects are morphisms in $\mathscr{C}$ to X and whose morphisms are commuting triangles over X in $\mathscr{C}$. Basic as this is, hom-sets in the *homotopy category* $\mathrm{Ho}(\mathbf{C}^{/B})$ (Def. A.14) of a slice model category $\mathbf{C}^{/B}$ (Example A.10) are of paramount interest:

The slicing imposes *twisting* on the corresponding non-abelian cohomology (Def. 2.1), in that the slicing of the domain space serves as a twist, the slicing of the coefficient space as a local coefficient bundle, and the slice morphisms as twisted cocycles.

Proposition 2.28 (∞-Actions on homotopy types [DDK80][Pr10, §5][NSS12a, §4][Sh15][SS20b, §2.2]). *For any $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33) and G a topological group, homotopy-coherent actions of G on A are equivalent to fibrations ρ with homotopy fiber A (Def. A.22) over the classifying space BG* (16)

$$\begin{tikzcd} A \arrow[r] & A /\!\!/ G \arrow[d, "\rho"] \\ & BG . \end{tikzcd} \tag{34}$$

Here

$$A/\!\!/G \;\simeq\; \big(A \times EG\big)_{/\mathrm{diag}G}$$

is the homotopy quotient (Borel construction) of the action.

Definition 2.29 (Twisted non-abelian cohomology [NSS12a, §4][FSS19b, (10)][SS20b, Rem. 2.94]).
For $X, A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Def. A.33) we say:
(i) A *local coefficient bundle* for twisted A-cohomology is an A-fibration ρ over a classifying space BG (16) as in Prop. 2.28:

$$\begin{tikzcd} A \arrow[r] & A /\!\!/ G \arrow[d, "\rho", "\text{local coefficient bundle}"'] \\ & BG . \end{tikzcd} \tag{35}$$

(ii) A *twist* for non-abelian A-cohomology theory on X with local coefficient bundle ρ over BG is a map

$$X \xrightarrow{\ \tau\ } BG \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,. \tag{36}$$

(iii) The *non-abelian τ-twisted A-cohomology* of X with local coefficients ρ is the hom-set from τ (36) to ρ (34)

$$\underbrace{H^{\tau}(X;A)}^{\text{twisted non-abelian cohomology}} := \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{/BG}\Big)\big(\tau,\rho\big) = \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\overset{\text{cocycle}}{c}"] \arrow[dr, "\text{twist}\ \tau"'] & \arrow[d, phantom, "\Leftarrow\simeq"] & A/\!\!/G \arrow[dl, "\rho"] \arrow[dl, "\text{local coefficients}"', phantom] \\ & BG & \end{tikzcd} \right\}_{\Big/\ \text{homotopy relative } BG} \tag{37}$$

in the homotopy category (Def. A.14) of the slice model category over BG (Example A.10) of the classical model category on topological spaces (Example A.7).

Definition 2.30 (Associated coefficient bundle [NSS12a, §4.1][SS20b, Prop. 2.92])**.** Given a local coefficient A-fiber bundle ρ (35) and a twist τ (36) on a domain space X, the corresponding *associated A-fiber bundle* over X is the homotopy pullback (Def. A.23) of ρ along τ, sitting in a homotopy pullback square (315) of this form:

$$\begin{tikzcd}
\underset{\text{A-fiber bundle}}{\overset{\text{associated}}{}} E \arrow[r] \arrow[d, "\mathbb{R}\tau^*\rho"'] \arrow[dr, phantom, "\underset{\text{homotopy pullback}}{\text{(hpb)}}"] & A/\!\!/G \underset{\text{coefficient bundle}}{\overset{\text{local}}{}} \arrow[d, "\rho"] \\
X \arrow[r, "\tau"', "\text{twist}"] & BG
\end{tikzcd} \tag{38}$$

We write

$$\overset{\substack{\text{sections of} \\ \text{associated bundle}}}{\Gamma_X(E)_{/\sim}} \;:=\; \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{/X}_{\mathrm{Qu}}\Big)\big(\mathrm{id}_X\,,\,\mathbb{R}\tau^*\rho\big) \;=\; \left\{ \begin{tikzcd} & E \;\; \substack{\text{associated} \\ \text{bundle}} \arrow[d] \\ X \arrow[ur, dashed, "\sigma", "\text{section}"'] \arrow[r, equal] & X \end{tikzcd} \right\}_{\Big/ \substack{\text{vertical} \\ \text{homotopy}}} \tag{39}$$

for the set of vertical homotopy classes of section of the associated bundle, hence for the hom-set, from the identity on X to the associated bundle projection, in the homotopy category (Def. A.14) of the slice model category over X (Example A.10) of the classical model category on topological spaces (Example A.7).

Proposition 2.31 (Twisted non-abelian cohomology is sections of associated coefficient bundle [NSS12a, Prop. 4.17])**.** *Given a local coefficient bundle ρ* (35) *and a twist τ* (36)*, the τ-twisted non-abelian cohomology (Def. 2.29) with local coefficient in ρ is equivalent to the vertical homotopy classes of sections* (39) *of the associated coefficient bundle E (Def. 2.30):*

$$\overset{\substack{\text{twisted non-abelian} \\ \text{cohomology}}}{H^\tau(X;A)} \;\simeq\; \overset{\substack{\text{sections of} \\ \text{associated bundle}}}{\Gamma_X(E)_{/\sim}}\,. \tag{40}$$

Proof. Consider the following sequence of bijections:

$$\begin{aligned}
H^\tau(X;A) &= \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{/BG}_{\mathrm{Qu}}\Big)\big(\tau\,,\,\rho\big) \\
&\simeq \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{/BG}_{\mathrm{Qu}}\Big)\big(\mathbb{L}\tau_*\mathrm{id}_X\,,\,\rho\big) \\
&\simeq \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{/X}_{\mathrm{Qu}}\Big)\big(\mathrm{id}_X\,,\,\mathbb{R}\tau^*\rho\big) \\
&= \Gamma_X(E)_{/\sim}\,.
\end{aligned}$$

Here the first line is the definition (37). Then the first step is the observation that every slice object is the derived left base change (Example A.18, Prop. A.20) along itself of the identity on its domain, by (306). With this, the second step is the hom-isomorphism (290) of the derived base change adjunction $\mathbb{L}\tau_! \dashv \mathbb{R}\tau^*$. The last line is (39). □

In twisted generalization of Example 2.2 we have:

Example 2.32 (Twisted ordinary cohomology)**.** Let $n \in \mathbb{N}$, let $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Ex. A.33) be connected and consider a traditional *system of local coefficients* [St43, §3] (see also [MQRT77][ABG10][GS18c])

$$\Pi_1(X) \xrightarrow{\;t\;} \mathrm{AbelianGroups}\,,$$

namely, a functor from the fundamental groupoid of X to the category of abelian groups. Since the construction $A \mapsto K(A,n)$ of Eilenberg-MacLane spaces (14) is itself functorial and using the assumption that X is connected, this induces (see [BFGM03, Def. 3.1]) a local coefficient bundle (35) of the form

$$\begin{tikzcd}
K(A,n) \arrow[r] & K(A,n)/\!\!/\pi_1(X)\,. \arrow[d, "\rho_t"] \\
& B\pi_1(X)
\end{tikzcd} \tag{41}$$

Finally, write $X \xrightarrow{\tau} B\pi_1(X)$ for the classifying map (via Example 2.3) of the universal connected cover of X (equivalently: for the 1-truncation projection of X). Then the τ-twisted non-abelian cohomology (Def. 2.29) of X with local coefficients in ρ_t (41) is equivalently the traditional t-twisted ordinary cohomology of X in degree n:

$$\overset{\substack{\text{twisted} \\ \text{ordinary cohomology}}}{H^{n+t}(X;A)} \;\simeq\; H^{\tau}\big(X; K(A,n)\big).$$

This is manifest from comparing Def. 2.29 with [GJ99, p. 332][BFGM03, Lemma 4.2].

As a special case of Example 2.32 and in twisted generalization of Examples 2.11, 2.12 we have:

Example 2.33 (Orientifold gerbes)**.** Consider the action $\sigma_{\mathrm{U}(1)}$ of $\mathbb{Z}_2$ on the circle group $\mathrm{U}(1) \subset \mathbb{C}^{\times}$ given by complex conjugation. This deloops (see [FSS15a, §4.4]) to an action $\sigma_{B^n\mathrm{U}(1)}$ of $\mathbb{Z}_2$ on the classifying spaces $B^n\mathrm{U}(1)$ (16). By Prop. 2.28 there is a corresponding local coefficient bundle

$$\begin{tikzcd} B^n\mathrm{U}(1) \arrow[r] & B^n\mathrm{U}(1) /\!\!/ \mathbb{Z}_2 \arrow[d, "\sigma_{B^n\mathrm{U}(1)}"] \\ & \mathbb{Z}_2 \end{tikzcd} \tag{42}$$

Moreover, consider a smooth manifold X, with orientation bundle classified by $X \xrightarrow{\mathrm{or}} B\mathbb{Z}_2$. Then the or-twisted cohomology (Def. 2.29) of X...

(i) ...with local coefficients in $\sigma_{B^2\mathrm{U}(1)}$ classifies what is equivalently known as Jandl gerbes [SSW07][GSW11] or real gerbes [HMSV19] or orientifold B-fields;

(ii) ...with local coefficients in $\sigma_{B^3\mathrm{U}(1)}$ classifies what is equivalently known as topological sectors of orientifold C-fields [FSS15a, §4.4].

More generally, one can consider twisted Deligne cohomology [GS18c] as well as higher-twisted periodic integral- and Deligne-cohomology [GS19b] (see also §4.3).

Remark 2.34 (The Whitehead principle of non-abelian cohomology)**.** Let $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ be connected, so that $A \simeq BG$ (Prop. 2.8).

(i) If A is also n-truncated (327), then its Postnikov tower (Prop. A.38) says that A is the total space of a local coefficient bundle (2.29) of the form

$$\begin{tikzcd} K(\pi_n(A), n) \arrow[r, "\mathrm{hfib}(p_n)"] & A \arrow[d, "p_n^A"] \\ & A(n-1) \simeq B\big(G(n-2)\big) \end{tikzcd}$$

with homotopy fiber an Eilenberg-MacLane space (14).

(ii) Accordingly, non-abelian cohomology with coefficients in A (Def. 2.1) is equivalently the disjoint union, over the space of twists τ_n (36) in non-abelian cohohomology with coefficients in $A(n-1)$, of τ-twisted non-abelian cohomology (Def. 2.29) with coefficients in $K(\pi_n(A), n)$:

$$\overset{\substack{\text{non-abelian cohomology} \\ \text{in higher degree}}}{H(X;A)} \;\simeq\; \underset{\substack{\tau_n \in H(X; A(n-1)) \\ \text{twist in} \\ \text{non-abelian cohomology} \\ \text{of lower degree}}}{\bigsqcup} \overset{\substack{\text{higher twisted} \\ \text{ordinary cohomology}}}{H^{\tau_n}\big(X; K(\pi_n(A), n)\big)}\,. \tag{43}$$

(iii) Iterating this unravelling yields

$$H(X;A) \;\simeq\; \underset{\tau_n \in \underset{\tau_{n-1} \in H(X; A(n-2))}{\bigsqcup} H^{\tau_{n-1}}\big(X; K(\pi_{n-1}(A), n-1)\big)}{\bigsqcup} H^{\tau_n}\big(X; K(\pi_n(A), n)\big)\,. \tag{44}$$

and then

$$H(X;A) \;\simeq \coprod_{\tau_n \in \coprod\limits_{\tau_{n-1} \in \coprod\limits_{\tau_{n-2} \in H\left(X;A(n-3)\right)} H^{\tau_{n-2}}\left(X;K(\pi_{n-2}(A),n-2)\right)} H^{\tau_{n-1}}\left(X;K(\pi_{n-1}(A),n-1)\right)} H^{\tau_n}\big(X;K(\pi_n(A),n)\big)\,. \tag{45}$$

and then

$$H\big(X;A\big) \;\simeq \coprod_{\tau_n \in \coprod\limits_{\tau_{n-1} \in \coprod\limits_{\tau_{n-2} \in \coprod\limits_{\tau_{n-3} \in H\left(X;A(n-4)\right)} H^{\tau_{n-3}}\left(X;K(\pi_{n-3}(A),n-3)\right)} H^{\tau_{n-2}}\left(X;K(\pi_{n-2}(A),n-2)\right)} H^{\tau_{n-1}}\left(X;K(\pi_{n-1}(A),n-1)\right)} H^{\tau_n}\big(X;K(\pi_n(A),n)\big)\,. \tag{46}$$

and so on.

(iv) Thus non-abelian cohomology in higher degrees (Example 2.7) decomposes as a tower of consecutively higher twisted but otherwise ordinary cohomology theories, starting with a twist in non-abelian cohomology in degree 1. This phenomenon has been called the *Whitehead principle of non-abelian cohomology* [To02, p. 8] and has been interpreted as saying that "nonabelian cohomology occurs essentially only in degree 1" [Si96, p. 1].

(v) But the above formulas (43), (44), (45) make manifest that this phenomenon has two perspectives. On the one hand: non-abelian cohomology in higher degrees may be *computed* by brute force as a sequence of consecutively higher twisted abelian cohomologies, with lowest twist starting in degree-1 non-abelian cohomology. On the other hand, conversely: intricate such systems of consecutively twisted abelian cohomology theories are neatly *understood* as unified by non-abelian cohomology.

(vi) Similarly, even though Postnikov towers do exist (Prop. A.38) in the classical homotopy category (Example A.33), the latter is far from being equivalent to the stable homotopy category (336) "up to twists in degree 1".

In twisted generalization of Example 2.14, we have:

Example 2.35 (Twisted topological K-theory)**.** The classifying space $\mathrm{KU}_0 \simeq \mathbb{Z} \times B\mathrm{U}$ (23) for complex topological K-theory (Example 2.14) is the fiber of a local coefficient bundle (35) over $K(\mathbb{Z},3) \simeq B^3\mathrm{U}(1)$ (20):

$$\begin{tikzcd} \mathrm{KU}_0 \arrow[r] & \mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \arrow[d] \\ & B^2\mathrm{U}(1) \end{tikzcd} \tag{47}$$

For $X \xrightarrow{\tau} B^2\mathrm{U}(1)$ a corresponding twist (36) (hence equivalently a bundle gerbe, by Example 2.11), the corresponding twisted non-abelian cohomology (Def. 2.29) is twisted complex topological K-theory [Ka68][DK70]:

$$\overset{\substack{\text{twisted} \\ \text{topological K-theory}}}{\mathrm{KU}^\tau(-)} \;\simeq\; H^\tau\big(-;\mathbb{Z} \times B\mathrm{U}\big)\,. \tag{48}$$

This is manifest from comparing (37) with [FrHT08, (2.6)]. Alternatively, under Prop. 2.31, this is manifest from comparing the equivalent right hand side of (40) with [Ro89, Prop. 2.1] (using [NSS12a, Cor. 4.18]) or, more directly, with [AS04, §3][ABG10, §2.1].

Generally, in twisted generalization of Example 2.13, we have:

Example 2.36 (Local coefficient bundle for twisted generalized cohomology)**.** Let R be a suitable ring spectrum and write $\mathrm{GL}_R(1)$ for its ∞-group (as in Example 2.7) of units [Schl04, §2.3][MaS04, §22.2][ABGHR08, §3][ABGHR14a, §2]. Its canonical action on the component space $R_0 = \mathbb{R}\Omega^\infty R$ (337) is given, via Prop. 2.28, by a local coefficient bundle (35) of the form

$$\begin{tikzcd} R_0 \arrow[r] & (R_0) /\!\!/ \mathrm{GL}_R(1) \arrow[d, "\rho_R"] \\ & B\mathrm{GL}_R(1)\,. \end{tikzcd} \tag{49}$$

Proposition 2.37 (Twisted non-abelian cohomology subsumes twisted generalized cohomology)**.** *For R a suitable ring spectrum, the twisted non-abelian cohomology (Def. 2.29) with local coefficient bundle ρ_R from Example 2.36 is, equivalently, twisted generalized R-cohomology in the traditional sense (e.g. [MaS04, §22.1]):*

$$\underset{\text{generalized cohomology}}{\overset{\text{twisted}}{R^\tau(-)}} \;\simeq\; H^\tau(-;\rho_R)\,. \tag{50}$$

Proof. Given any twist $X \xrightarrow{\tau} B\mathrm{GL}_R(1)$ (2.29), write $P \to X$ for the homotopy pullback (Def. A.23) along τ of the essentially unique point inclusion:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & * \arrow[d] \\ X \arrow[r, "\tau"'] & B\mathrm{GL}_R(1) \end{tikzcd} \;,\qquad \begin{tikzcd} \big(P\times R_0\big)/\!\!/_{\mathrm{diag}}\mathrm{GL}_R(1)\simeq E \arrow[r] \arrow[d, "\mathbb{R}\tau^*\rho_R"'] \arrow[dr, phantom, "\text{(hpb)}"] & R_0/\!\!/\mathrm{GL}_R(1) \arrow[d, "\rho_R"] \\ X \arrow[r, "\tau"'] & B\mathrm{GL}_R(1) \end{tikzcd} \tag{51}$$

This P is the $\mathrm{GL}_R(1)$-principal ∞-bundle which is classified by τ, [NSS12a, Thm. 3.17], to which the coefficient bundle E (38) is $\mathrm{GL}_R(1)$-associated [NSS12a, Prop. 4.6], as shown on the right of (51). Consider then the following sequence of natural bijections:

$$\begin{aligned} H^\tau\big(X;R_0\big) &\simeq \Gamma_X(E) \\ &\simeq \mathrm{Ho}\big(\mathrm{GL}_R(1)\mathrm{Actions}\big)(P;R_0) \\ &\simeq \mathrm{Ho}\big(R\mathrm{Modules}\big)(M\tau;R) \\ &\simeq R^\tau(X)\,. \end{aligned} \tag{52}$$

Here the first step is Prop. 2.31, while the second step is [NSS12a, Cor. 4.18]. The third step is [ABGHR08, (2.15)][ABGHR14a, (3.15)], with $M\tau$ denoting the R-Thom spectrum of τ [ABGHR08, Def. 2.6][ABGHR14a, Def. 3.13]. The last step is [ABGHR08, §2.5] [ABGHR14a, §1.4][ABGHR14b, §2.7]. The composite of these natural bijections is the desired (50). ☐

In twisted generalization of Example 2.15, we have:

Example 2.38 (Twisted iterated K-theory)**.** Let $r\in\mathbb{N}$, $r\geq 1$. By [LSW16, Prop. 1.5, Def. 1.7] and using Prop. 2.37, there is a local coefficient bundle (35) of the form

$$\begin{tikzcd} \big(K^{2r-2}(\mathrm{ku})\big)_0 \arrow[r] & \Big(\big(K^{2r-2}(\mathrm{ku})\big)_0\Big)/\!\!/B^{2r-1}\mathrm{U}(1) \arrow[d, "\rho_{\mathrm{lsw}_{2r-1}}"] \\ & B^{2r}\mathrm{U}(1)\,, \end{tikzcd} \tag{53}$$

where $K^{2r-2}(\mathrm{ku})_0$ is the 0th space in the spectrum (21) representing iterated K-theory (Example 2.15) and $B^{2r}\mathrm{U}(1)\simeq K(\mathbb{Z},2r+1)$ is the classifying space for bundle $(2r-1)$-gerbes (Example 2.12), such that for $X\xrightarrow{\tau}B^{2r}\mathrm{U}(1)$ a classifying map for such a higher gerbe, the τ twisted non-abelian cohomology (Def. 2.29) with local coefficients in (53) is equivalently integrally twisted iterated K-theory according to [LSW16]:

$$\underset{\text{iterated K-theory}}{\overset{\text{twisted}}{\big(K^{\circ_{2r-1}}(\mathrm{ku})\big)^\tau(-)}} \;\simeq\; H^\tau\Big(-;K^{\circ_{2r-2}}(\mathrm{ku})_0\Big)\,.$$

In twisted generalization of Example 2.10, we have:

Example 2.39 (J-Twisted Cohomotopy theory [FSS19b, §2.1])**.** For $n\in\mathbb{N}$, consider the canonical action of the orthogonal group $\mathrm{O}(n+1)$ on the homotopy type of the n-sphere, via the defining action on the unit sphere in $\mathbb{R}^{n+1}$. By Prop. 2.28 this corresponds to a local coefficient bundle (35) for twisting Cohomotopy theory (Example 2.10):

$$\begin{tikzcd} S^n \arrow[r] & S^n/\!\!/\mathrm{O}(n) \arrow[d, "\rho_J"] \\ & B\mathrm{O}(n)\,. \end{tikzcd} \tag{54}$$

The classifying map $BO(n) \xrightarrow{J} \mathrm{Aut}(S^n)$ of this fibration is the unstable *J-homomorphism*. For X a smooth manifold of dimension $d \geq k+1$, and equipped with tangential $\mathrm{O}(k+1)$-structure (e.g. [SS20b, Def. 4.48])

$$
\begin{tikzcd}
X \arrow[rr, "\tau"] \arrow[dr, "TX"'] & \arrow[d, phantom, "\Leftarrow\,\simeq"] & B\mathrm{O}(k+1) \arrow[dl, "Bi"] \\
& B\mathrm{O}(d) &
\end{tikzcd}
$$

the τ-twisted non-abelian Cohomology (Def. 2.29) with local coefficients in (54) is the *J-twisted Cohomotopy theory* of [FSS19b][FSS19c][SS20a]:

$$
\overset{\substack{\text{J-twisted} \\ \text{Cohomotopy}}}{\pi^\tau(-)} \;:=\; H^\tau\big(-; S^n\big).
$$

J-twisted Cohomotopy in degree four encodes, in particular, the shifted flux quantization condition of the C-field [FSS19b, Prop. 3.13] and the vanishing of the residual M5-brane anomaly [SS20a]; while J-twisted Cohomotopy in degree four encodes, in particular, level quantization of the Hopf-Wess-Zumino term on the M5-brane [FSS19c].

Twisted non-abelian cohomology operations. In generalization of Def. 2.17, we set:

Definition 2.40 (Twisted non-abelian cohomology operation)**.** Given a transformation of local coefficient bundles (35) presented (under localization (303) to homotopy types (324)) as a strictly commuting diagram

$$
\begin{tikzcd}
A_1 /\!\!/ G_1 \arrow[r, "\phi_t"] \arrow[d, "\rho_1"'] & A_2 /\!\!/ G_2 \arrow[d, "\rho_2"] \\
BG_1 \arrow[r, "\phi_b"] & BG_2
\end{tikzcd}
\quad \in \mathrm{TopologicalSpaces}_{\mathrm{Qu}}, \tag{55}
$$

postcomposition induces [3] for each twist $X \xrightarrow{\tau} BG_1$ (36) a map

$$
\phi_* \;:\; H^\tau(X; A_1) \longrightarrow H^{\phi_b \circ \tau}(X; A_2) \tag{56}
$$

of twisted non-abelian cohomology sets (Def. 2.29). We call these *twisted non-abelian cohomology operations*.

Example 2.41 (Hopf cohomology operation in J-twisted Cohomotopy [FSS19b, §2.3])**.** The quaternionic Hopf fibration $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$ is equivariant under the symplectic unitary group $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$, so that after passage to classifying spaces it induces a morphism of local coefficient bundles (55) for J-twisted Cohomotopy (54) in degrees 4 and 7:

$$
\begin{tikzcd}
S^7 /\!\!/ \mathrm{Sp}(2) \arrow[rr, "\substack{\text{Borel-equivariantized} \\ \text{quaternionic Hopf fibration}}", "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"'] \arrow[d, "J_7"'] & & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[d, "J_4"] \\
B\mathrm{Sp}(2) \arrow[rr, equal] & & B\mathrm{Sp}(2)
\end{tikzcd}
\tag{57}
$$

Via (56) this induces for each Spin 8-manifold X equipped with tangential $\mathrm{Sp}(2)$-structure

$$
\begin{tikzcd}
X \arrow[rr, "\tau"] \arrow[dr, "TX"'] & \arrow[d, phantom, "\Leftarrow\,\simeq"] & B\mathrm{Sp}(2) \arrow[dl, "Bi"] \\
& B\mathrm{O}(8) &
\end{tikzcd}
\tag{58}
$$

a twisted non-abelian cohomology operation (Def. 2.40)

$$
\pi^{\tau_7}(X) \xrightarrow{\;(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_*\;} \pi^{\tau^4}(X) \tag{59}
$$

in J-twisted non-abelian Cohomotopy theory (Example 2.39).

Lifting through the twisted non-abelian cohomology transformation (59) encodes vanishing of C-field flux up to C-field background charge [FSS19b, Prop. 3.14].

[3] Some care is needed in making this precise; we postpone the details to §5, where they are provided by Lemma 5.1 with Def. 5.2.

Example 2.42 (Twistorial Cohomotopy [FSS20, §3.2])**.** The equivariantized Hopf morphism (57) of coefficient bundles factors through Borel-equivariantizations of the complex Hopf fibration $h_{\mathbb{C}}$ followed by that of the *twistor fibration* $t_{\mathbb{H}}$

$$
\begin{tikzcd}
S^7 /\!/ \mathrm{Sp}(2) \arrow[rr, "\substack{\text{Borel-equivariantized} \\ \text{complex Hopf fibration}}", "h_{\mathbb{C}} /\!/ \mathrm{Sp}(2)"'] \arrow[d, "J_{S^7}"'] & & \mathbb{C}P^3 /\!/ \mathrm{Sp}(2) \arrow[rr, "\substack{\text{Borel-equivariantized} \\ \text{twistor fibration}}", "t_{\mathbb{H}} /\!/ \mathrm{Sp}(2)"'] \arrow[d, "J_{\mathbb{C}P^3}"'] & & S^4 /\!/ \mathrm{Sp}(2) \arrow[d, "J_{S^4}"] \\
B\mathrm{Sp}(2) \arrow[rr, Rightarrow] & & B\mathrm{Sp}(2) \arrow[rr, Rightarrow] & & B\mathrm{Sp}(2)
\end{tikzcd}
\tag{60}
$$

The twisted non-abelian cohomology theory (Def. 2.29) with local coefficients in the bundle appearing in this factorization is the *Twistorial Cohomotopy* of [FSS20]

$$
\overset{\substack{\text{Twistorial} \\ \text{Cohomotopy}}}{\mathscr{T}^{\tau}(-)} \;:=\; H^{\tau}\big(-;\mathbb{C}P^3\big)\,.
$$

Via (56) the morphisms (60) induce, for each spin 8-manifold X equipped with tangential $\mathrm{Sp}(2)$-structure (58), twisted non-abelian cohomology operations (Def. 2.40)

$$
\overset{\substack{\text{J-twisted} \\ \text{7-Cohomotopy}}}{\pi^{\tau^7}(X)} \xrightarrow{\;(h_{\mathbb{C}} /\!/ \mathrm{Sp}(2))_*\;} \overset{\substack{\text{Twistorial} \\ \text{Cohomotopy}}}{\mathscr{T}^{\tau}(X)s} \xrightarrow{\;(t_{\mathbb{H}} /\!/ \mathrm{Sp}(2))_*\;} \overset{\substack{\text{J-twisted} \\ \text{4-Cohomotopy}}}{\pi^{\tau^4}(X)}
\tag{61}
$$

between J-twisted non-abelian Cohomotopy theory (Example 2.39) and Twistorial Cohomotopy.

We turn to the differential refinement of this statement in §5.3 below.

3 Non-abelian de Rham cohomology

We formulate (twisted) non-abelian de Rham cohomology (Def. 3.82, Def. 3.96) of differential forms with values in L_∞-algebras (Example 3.25) and prove the (twisted) non-abelian de Rham theorem (Theorem 3.85, Theorem 3.102), as a consequence of the fundamental theorem of dg-algebraic rational homotopy theory, which we recall (Prop. 3.58).

3.1 Dgc-Algebras and L_∞-algebras

Here we fix notation and conventions for the following system of categories and functors:

$$
\begin{tikzcd}
\overset{\text{Def. 3.34}}{\big(L_\infty\mathrm{Algebras}^{\geq 0,\,\mathrm{nil}}_{\mathbb{R},\,\mathrm{fin}}\big)^{\mathrm{op}}} \arrow[r, "\overset{(102)}{\mathrm{CE}}"', "\simeq"] \arrow[d, hook] & \overset{\text{Def. 3.31}}{\mathrm{SullivanModels}^{\geq 1}_{\mathbb{R}}} \arrow[d, hook] & \\
\overset{\text{Def. 3.25}}{\big(L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\,\mathrm{fin}}\big)^{\mathrm{op}}} \arrow[r, hook, "\overset{(86)}{\mathrm{CE}}"] & \overset{\text{Def. 3.17}}{\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}} \arrow[d, "\overset{\text{Def. 3.18}}{\mathrm{GrddCmmttvAlgbr}}"'] \arrow[r, phantom, "\perp"] & \overset{\text{Def 3.14}}{\mathrm{CochainComplexes}^{\geq 0}_{\mathbb{R}}} \arrow[l, shift right, "\overset{\text{Def. 3.19}}{\mathrm{Sym}}"'] \arrow[l, shift left, leftarrow] \arrow[d, "\overset{\text{Def. 3.15}}{\mathrm{GrddVctrSpc}}"] \\
 & \underset{\text{Def. 3.8}}{\mathrm{GradedCommAlgebras}^{\geq 0}_{\mathbb{R}}} \arrow[r, phantom, "\perp"] & \underset{\text{Def. 3.2}}{\mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}}} \arrow[l, shift right, "\overset{\text{Def. 3.10}}{\mathrm{Sym}}"'] \arrow[l, shift left, leftarrow]
\end{tikzcd}
\tag{62}
$$

Remark 3.1 (Homotopical grading)**.** Our grading conventions, to be detailed in the following, are strictly *homotopy theoretic*:
(i) Any graded-algebraic object discussed here, corresponds, under the equivalences of rational homotopy theory laid out in §3.2 below, to a rational space, such that algebraic generators in degree n correspond to homotopy groups in the same degree n. Since homotopy groups of spaces are in non-negative degree $n \in \mathbb{N}$, all dg-algebraic objects discussed here are concentrated in non-negative degree, hence are *connective*.
(ii) In particular, our L_∞-algebras are in non-negative degree, naturally accommodating (as in [LM95][BFM06, §2.9]) the rationalized Whitehead homotopy Lie algebras $\pi_\bullet(\Omega X) \otimes_{\mathbb{Z}} \mathbb{R}$ of connected spaces X, with their natural non-negative grading induced from that of the homotopy groups of ΩX. See Prop. 3.61 and Prop. 3.63 below.

Graded vector spaces.

Definition 3.2 (Connective graded vector spaces)**. (i)** We write

$$\mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}} \;\in\; \mathrm{Categories} \tag{63}$$

for the category whose objects are $\mathbb{N}$-graded (i.e. non-negatively $\mathbb{Z}$-graded) vector spaces over the real numbers; and we write

$$\mathrm{GradedVectorSpaces}^{\geq 0,\,\mathrm{fin}}_{\mathbb{R}} \hookrightarrow \mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}} \;\in\; \mathrm{Categories} \tag{64}$$

for its full subcategory on those objects which are of *finite type*, namely degree-wise finite-dimensional.
(ii) For $V \in \mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}}$ and $k \in \mathbb{N}$ we write

$$V^k \;\in\; \mathrm{VectorSpaces}_{\mathbb{R}}$$

for the component vector space in degree k.

Example 3.3 (The zero-object in graded vector spaces)**.** We write

$$0 \;\in\; \mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}} \tag{65}$$

for the graded vector space which is the zero vector space in each degree. This is both the initial as well as the terminal object (hence the zero object) in $\mathrm{GradedVectorSpaces}^{\geq 0}_{\mathbb{R}}$.

Example 3.4 (Graded linear basis)**.** For $n_1, n_2, \cdots, n_k \in \mathbb{N}$ a finite sequence of non-negative integers, we write

$$\big\langle \alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_k} \big\rangle \;\in\; \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$$

for the graded vector space (Def. 3.2) spanned by elements α_{n_i} in degree n_i, respectively.

Definition 3.5 (Tensor product of graded vector spaces)**.** The category of $\mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$ (Def. 3.2) becomes a symmetric monoidal category under the graded tensor product given by

$$(V \otimes W)^k \;:=\; \bigoplus_{n_1 + n_2 = k} V^{n_1} \otimes W^{n_2} \,.$$

and the symmetric braiding isomorphism given by

$$\begin{tikzcd}
V \otimes W \arrow[rr, "\sigma^{V,W}", "\simeq"'] && W \otimes V \\
V^{n_1} \otimes W^{n_2} \arrow[u, hook] \arrow[rr, "\sigma^{V,W}_{n_1,n_2}", "\simeq"'] \arrow[d, phantom, sloped, "\in"] && W^{n_2} \otimes V^{n_1} \arrow[u, hook] \arrow[d, phantom, sloped, "\in"] \\
(v, w) \arrow[rr, mapsto] && (-1)^{n_1 n_2} \cdot (w, v)
\end{tikzcd} \tag{66}$$

We denote this by

$$\big(\mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma\big) \;\in\; \mathrm{SymmetricMonoidalCategories}\,. \tag{67}$$

Definition 3.6 (Degreewise linear dual)**.** For $V \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$ (Def. 3.2) we write

$$V^{\vee} \;\in\; \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$$

for its degree-wise linear dual:[4]

$$(V^{\vee})^k \;:=\; (V^k)^* \,. \tag{68}$$

Definition 3.7 (Degree shift)**.** For $V \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$ (Def. 3.2) we write

$$\mathfrak{b}V \;\in\; \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \tag{69}$$

for the result of shifting degrees up by 1:

$$(\mathfrak{b}V)^k \;:=\; \left\{ \begin{array}{lcl} V^{k-1} & | & k \geq 1, \\ 0 & | & k = 0. \end{array} \right.$$

Graded-commutative algebras.

Definition 3.8 (Graded-commutative algebras)**.** We write

$$\mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \;:=\; \mathrm{CommMonoids}\big(\mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma\big) \;\in\; \mathrm{Categories} \tag{70}$$

for the category whose objects are non-negatively $\mathbb{Z}$-graded, graded-commutative unital algebras over the real numbers (hence commutative unital monoids with respect to the braided tensor product of Def. 3.5); and we write

$$\mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \mathrm{fin}} \hookrightarrow \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \;\in\; \mathrm{Categories} \tag{71}$$

for its full sub-category in those objects which are of *finite type*, namely degree-wise finite dimensional.

Definition 3.9 (Underlying graded vector space)**.** We write

$$\mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \xrightarrow{\;\mathrm{GrddVctrSpc}\;} \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \tag{72}$$

for the functor on graded algebras (Def. 3.8) that forgets the algebra structure and remembers only the underlying graded vector space (Def. 3.2).

Example 3.10 (Free graded-commutative algebras)**.** For $V \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$ (Def. 3.2), we write

$$\mathrm{Sym}(V) \;\in\; \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \tag{73}$$

for the graded-commutative algebra (Def. 3.8) freely generated by V, hence that whose underlying graded vector space (72) is

$$\mathrm{GrddVctrSpc}\big(\mathrm{Sym}(V)\big) \;=\; \mathbb{R} \,\oplus\, V \,\oplus\, \big(V \otimes V\big)_{/\mathrm{Sym}(2)} \,\oplus\, \big(V \otimes V \otimes V\big)_{/\mathrm{Sym}(3)} \,\oplus\, \cdots\,,$$

where the symmetric groups $\mathrm{Sym}(n)$ act via the braiding (66).

[4] This is in contrast to the intrinsic duality $(-)^*$ in the monoidal category of graded vector spaces in *unbounded* degree (not considered here), which instead goes along with inversion of the degree: $(V^*)^k \,=\, (V^{-k})^*$.

Example 3.11 (Graded Grassmann algebra)**.** For $V \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$ (Def. 3.2), we write

$$\wedge^{\bullet} V \;:=\; \mathrm{Sym}\big(\mathfrak{b} V\big) \;\in\; \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the free graded-commutative algebra (Def. 3.10) on V shifted up in degree (Def. 3.7); and we call this the *graded Grassmann-algebra* on V.

Example 3.12 (Graded polynomial algebra)**.** For $n_1, n_2, \cdots, n_k \in \mathbb{N}$ a finite sequence of non-negative integers, we write

$$\mathbb{R}\big[\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_k}\big] \;:=\; \mathrm{Sym}\Big(\big\langle \alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_k} \big\rangle\Big) \;\in\; \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$$

for the free graded-commutative algebras (Def. 3.10) the graded vector space spanned by the α_{n_i} (Def. 3.4).

Remark 3.13 (Incarnations of Grassmann algebras)**.** With these notation conventions from Examples 3.10, 3.11, 3.12, an ordinary Grassmann algebra on k generators is equivalently:

$$\wedge^{\bullet}\big(\mathbb{R}^k\big) \;=\; \mathrm{Sym}\big(\mathfrak{b}\mathbb{R}^k\big) \;=\; \mathbb{R}\big[\theta_1^{(1)}, \theta_1^{(2)}, \cdots, \theta_1^{(k)}\big]\,.$$

Cochain complexes.

Definition 3.14 (Connective cochain complexes)**.** We write

$$\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0} \;\in\; \mathrm{Categories}$$

for the category of cochain complexes (i.e. with differential of degree $+1$) of real vector spaces in non-negative degree.

Definition 3.15 (Underlying graded vector space)**.** We write

$$\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0} \xrightarrow{\;\mathrm{GrddVctrSpc}\;} \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \tag{74}$$

for the forgetful functor on connective cochain complexex (Def. 3.14) which forgets the differential and remembers only the underlying connective graded vector space (Def. 3.2).

Definition 3.16 (Tensor product on cochain complexes)**.** The tensor product and braiding of graded vector spaces from Def. 3.5 lifts, through (74), to a tensor product and braiding on $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.14), making it a symmetric monoidal category:

$$\big(\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma\big) \;\in\; \mathrm{SymmetricMonoidalCategories}\,. \tag{75}$$

Differential graded commutative algebras.

Definition 3.17 (Connective differential graded commutative algebras [GM96, V.3.1])**.** We write

$$\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \;:=\; \mathrm{CommMonoids}\big(\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma\big) \;\in\; \mathrm{Categories}$$

for the category whose objects are differential-graded, graded-commutative, unital algebras over the real numbers concentrated in non-negative degrees (hence commutative unital monoids in the symmetric monoidal category of Def. 3.16).

Definition 3.18 (Underlying graded-commutative algebra)**.** We write

$$\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \xrightarrow{\;\mathrm{GrddCmmttvAlgbr}\;} \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \tag{76}$$

for the functor on dgc-algebras (Def. 3.17) that forgets the differential and remembers only the underlying graded-commutative algebra (Def. 3.8).

Definition 3.19 (Free differential graded algebras)**.** For $V^{\bullet}$ in $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.14) we write

$$\mathrm{Sym}(V^{\bullet}) \;\in\; \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the free differential graded-commutative algebra on $V^{\bullet}$, (Def. 3.17), hence whose underlying graded-commutative algebra algebra (76) is as in Example 3.10.

Example 3.20 (Initial algebra)**.** The real algebra of real numbers, regarded as concentrated in degree-0

$$\mathbb{R} \in \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \hookrightarrow \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

is the *initial* object: For any other $A \in \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 70) or $\in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.17) there is a unique morphism

$$\mathbb{R} \xhookrightarrow{i_{\mathbb{R}}} A$$

(because our algebras are unital and homomorphims need to preserve the unit element).

Example 3.21 (The terminal algebra)**.** We write

$$0 \in \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \hookrightarrow \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \tag{77}$$

for the unique graded-commutative algebra (Def. 3.8) or dgc-algebra (Def. 3.17) whose underlying graded vector space (Def. 3.9) is the zero-vector space[5] (65). This is the terminal object[6] in $\mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$: For every $A \in \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$, there is a unique morphism

$$A \xrightarrow{\exists !} 0 \ .$$

Example 3.22 (Product and co-product algebras)**.** In the categories $\mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.8) and $\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.17):

(i) the coproduct is given by the tensor product (Def. 3.5),

(ii) the product is given by the direct sum

on underlying graded vector spaces (Def. 3.9).
(The first follows by [Joh02, p. 478, Cor. 1.1.9], while the second holds since (72) is a right adjoint.)

Example 3.23 (Smooth de Rham complex (e.g. [BT82]))**.** For X be a smooth manifold, its de Rham algebra of smooth differential forms is a dgc-algebra in the sense of Def. 3.17, to be denoted here:

$$\Omega^\bullet_{\mathrm{dR}}(X) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}.$$

Example 3.24 (Chevalley-Eilenberg algebras of Lie algebras)**.** For $(\mathfrak{g}, [-,-])$ a finite-dimensional real Lie algebra, its Chevalley-Eilenberg algebra is a dgc-algebra (Def. 3.17):

$$\mathrm{CE}(\mathfrak{g}) := \big(\wedge^\bullet \mathfrak{g}^*, \, d|_{\wedge^1 \mathfrak{g}^*} = [-,-]^*\big) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

with underlying graded-commutative algebra (Def. 3.8) the Grassmann algebra on the linear dual space $\mathfrak{g}^*$ (Def. 3.11, Remark 3.13), and with differential given on $\wedge^1 \mathfrak{g}^*$ by the linear dual of the Lie bracket. More explicitly, for $\{v_a\}_{a=1}^{\dim_{\mathbb{R}}(\mathfrak{g})}$ a linear basis for the underlying vector space of the Lie algebra

$$\mathfrak{g} \simeq \langle v_1, v_2, \cdots, v_{\dim(\mathfrak{g})} \rangle, \tag{78}$$

with Lie brackets

$$[v_a, v_b] = f^c_{ab} v_c, \quad \text{for structure constants } f^c_{ab} \in s\mathbb{R} \tag{79}$$

we have

$$\mathrm{CE}(\mathfrak{g}) \simeq \mathbb{R}\big[\theta_1^{(1)}, \theta_1^{(2)}, \cdots \theta_1^{(\dim(\mathfrak{g}))}\big] / \big(d\,\theta_1^{(c)} = f_{ab}{}^c\, \theta_1^{(b)} \wedge \theta_1^{(a)}\big). \tag{80}$$

One observes that the Jacobi identity on $[-,-]$ is equivalent to the condition that the differential $d := [-,-]^*$ squares to zero, so that (80) being a dgc-algebra is actually equivalent to $(\mathfrak{g}, [-,-])$ being a Lie algebra.

This construction is evidently contravariantly functorial and constitutes a full subcategory inclusion

$$\mathrm{LieAlgebras}_{\mathbb{R},\mathrm{fin}} \xhookrightarrow{\mathrm{CE}} \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}} , \tag{81}$$

meaning that, in addition, homomorphisms of Lie algebras are in natural bijection to dgc-algebra morphisms between their CE-algebras.

[5] Notice that the algebra 0 (77) is indeed a unital algebra (70).
[6] Beware that the corresponding statement in [GM96, p. 335] is incorrect.

L_∞-algebras.

Definition 3.25 (Chevalley-Eilenberg algebras of L_∞-algebras [LM95, Thm . 2.3][SSS09a, Def. 13][BFM06, §2])**.** In direct generalization of (81), consider those $A \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.17) whose underlying graded-commutative algebra (76) is free (Example 3.10, Remark 3.13) on the degreewise dual $\mathfrak{b}\mathfrak{g}^\vee$ (Def. 3.6) of the degree shift $\mathfrak{b}\mathfrak{g}$ (Def. 3.7) of some connective finite-type graded vector space (Def. 3.2)

$$\mathfrak{g} \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \mathrm{fin}} \tag{82}$$

in that

$$A := \big(\wedge^\bullet \mathfrak{g}^\vee, d\big) := \big(\mathrm{Sym}(\mathfrak{b}\mathfrak{g}^\vee), d\big) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}. \tag{83}$$

In this case the differential d restricted to $\wedge^1 \mathfrak{g}^\vee$ defines, under linear dualization, a sequence of n-ary graded-symmetric multilinear maps $\{-, \cdots, -\}$ on $\mathfrak{g}$:

$$\begin{array}{ccccccccc} d|_{\wedge^1 \mathfrak{g}^\vee}(-) & = & \{-\}^* & + & \{-,-\}^* & + & \{-,-,-\}^* & + & \cdots \\ \wedge^1 \mathfrak{g}^\vee \xrightarrow{\;d\;} & & \wedge^1 \mathfrak{g}^\vee & \oplus & \wedge^2 \mathfrak{g}^\vee & \oplus & \wedge^3 \mathfrak{g}^\vee & \oplus & \cdots \; = \wedge^\bullet \mathfrak{g}^\vee = \mathrm{Sym}\big(\mathfrak{b}\mathfrak{g}^\vee\big), \end{array} \tag{84}$$

and the condition $d \circ d = 0$ imposes a sequence of compatibility conditions on these brackets, generalizing the Jacobi identity in Example 3.24. The corresponding graded skew-symmetric n-ary brackets ([LS93, (3)])

$$[a_1, \cdots, a_n] := (-1)^{n + \sum_{i \leq n/2} \deg(a_i)} \{a_1, \cdots, a_n\}$$

subject to these conditions give $\mathfrak{g}$ the structure of an *L_∞-algebra* (or *strong homotopy Lie algebra*):

$$\Big(\mathfrak{g}, [-], [-,-], [-,-,-], \cdots\Big) \in L_\infty\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0}, \tag{85}$$

which makes A in (83) its Chevalley-Eilenberg algebra:

$$\begin{aligned} \mathrm{CE}(\mathfrak{g}) &:= \big(\wedge^\bullet \mathfrak{g}^\vee, d = \{-\}^* + \{-,-\}^* + \{-,-,-\}^* + \cdots\big) \\ &= \big(\mathrm{Sym}(\mathfrak{b}\mathfrak{g}^\vee), d_{\mathrm{CE}}\big). \end{aligned} \tag{86}$$

This construction constitutes a full subcategory inclusion

$$L_\infty\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0} \xhookrightarrow{\;\mathrm{CE}\;} \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}}. \tag{87}$$

of the category of connective finite-type L_∞-algebras into that of connective dgc-algebras.

Example 3.26 (Differential graded Lie algebras)**.** A differential graded Lie algebra is an L_∞-algebra (85) whose only possibly non-vanishing brackets are the unary bracket $\partial := [-]$ (its differential) and the binary bracket $[-,-,]$ (its graded Lie bracket). In further specialization, a plain Lie algebra (Example 3.24) is an L_∞-algebra/dg-Lie algebra concentrated in degree 0:

$$\mathrm{LieAlgebras}_{\mathbb{R}, \mathrm{fin}} \hookrightarrow \mathrm{DiffGradedLieAlgebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0} \hookrightarrow L_\infty\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0}. \tag{88}$$

Example 3.27 (Line Lie n-algebra)**.** For $n \in \mathbb{N}$ we say that the *line Lie $(n+1)$-algebra* is the L_∞-algebra (Def. 3.25)

$$\mathfrak{b}^n\mathbb{R} \in L_\infty\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0} \tag{89}$$

whose Chevalley-Eilenberg algebra (86) is the polynomial dgc-algebra (Example 3.29) on a single closed generator in degree $n+1$:

$$\mathrm{CE}\big(\mathfrak{b}^n\mathbb{R}\big) := \mathbb{R}[c_{n+1}]/(d\, c_{n+1} = 0). \tag{90}$$

Example 3.28 (String Lie 2-algebra)**.** Let $\mathfrak{g} \in \mathrm{LieAlgebras}_{\mathbb{R}, \mathrm{fin}}$ be semisimple (such as $\mathfrak{g} = \mathfrak{su}(n+1), \mathfrak{so}(n+3)$, for $n \in \mathbb{N}$), hence equipped with a non-degenerate, symmetric, $\mathfrak{g}$-invariant bilinear form ("Killing form")

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\langle -,- \rangle} \mathbb{R}. \tag{91}$$

Then the element

$$\mu := \big\langle -, [-,-] \big\rangle \in \mathrm{CE}(\mathfrak{g})$$

in the Chevalley-Eilenberg (3.24) is closed (is a Lie algebra cocycle)

$$d\mu = 0.$$

In terms of a linear basis $\{v_a\}$ (78) with structure constants $\{f^c_{ab}\}$ (79) and inner product $k_{ab} := \langle v_a, v_b \rangle$ we have, in terms of (80):

$$\mu := f_{ab}{}^{c'} k_{c'c}\, \theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)}.$$

Hence we get an L_∞-algebra (Def. 3.25)

$$\mathfrak{string}_{\mathfrak{g}} \;\in\; L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}} \tag{92}$$

with the following Chevalley-Eilenberg algebra (86):

$$\mathrm{CE}\big(\mathfrak{string}_{\mathfrak{g}}\big) \;:=\; \mathbb{R}\left[\begin{matrix}\{\theta_1^a\}, \\ b_2\end{matrix}\right] \Big/ \left(\begin{aligned} d\,\theta_1^{(c)} &= f^c_{ab}\,\theta_1^{(b)} \wedge \theta_1^{(a)} \\ d\,b_2 &= \underbrace{f^{c'}_{ab} k_{c'c}\,\theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)}}_{=\mu} \end{aligned}\right). \tag{93}$$

This is known as the *string Lie 2-algebra*.

Sullivan models and nilpotent L_∞-algebras.

Example 3.29 (Polynomial dgc-algebras)**.** For $A \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (Def. 3.17), and

$$\mu \;\in\; A^{n+1} \;\subset\; A\,, \qquad d\,\mu \;=\; 0 \tag{94}$$

a closed element of homogeneous degree $n+1$, we write

$$A\big[\alpha_n\big]\big/\big(d\,\alpha_n = \mu\big) \;\in\; \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \tag{95}$$

for the dgc-algebra obtained by adjoining a generator α_n of degree n to the underlying graded-commutative algebra (76) of A and extending the differential from A to $A\big[\alpha_n\big]$ by taking its value on the new generator to be μ. The polynomial dgc-algebras (95) receives a canonical algebra inclusion of A (the unique A-algebra homomorphism):

$$A \xhookrightarrow{\;i_A\;} A[\alpha_n]/(d\,\alpha_n = \mu)\;. \tag{96}$$

Example 3.30 (Multivariate polynomial dgc-algebras)**.** Let $A \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (Def. 3.17), $\mu^{(1)} \in A^{n_1+1}$, $d\mu^{(1)} = 0$, with corresponding polynomial dgc-algebra (95) as in Example 3.29. Then, for

$$\mu^{(2)} \;\in\; A\big[\alpha^{(1)}_{n_1}\big]\big/\big(d\,\alpha^{(1)}_{n_1} \;=\; \mu^{(1)}\big)\,, \quad d\,\mu^{(2)} \;=\; 0$$

another closed element of some homogeneous degree n_2+1, in the new algebra (95) we may iterate the construction of Example 3.29 to obtain the bivariate polynomial dgc-algebra over A, to be denoted:

$$A\left[\begin{matrix}\alpha^{(2)}_{n_2+1} \\ \alpha^{(1)}_{n_1+1},\end{matrix}\right] \Big/ \left(\begin{matrix} d\,\alpha^{(2)}_{n_1} = \mu^{(2)}_{n_2+1}, \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)}_{n_1+1}\end{matrix}\right) \;:=\; \Big(A\big[\mu^{(1)}_{n_1+1}\big]\big/\big(d\,\alpha^{(1)}_{n_1+1} \;=\; \mu^{(1)}\big)\Big)\big[\alpha^{(2)}\big]\big/\big(d\,\alpha^{(2)}_{n_2+1} \;=\; \mu^{(2)}\big)\,.$$

Iterating further, we have multivariate polynomial dgc-algebras over A, to be denoted as follows:

$$A\left[\begin{matrix}\alpha^{(k)}_{n_k+1}, \\ \vdots \\ \alpha^{(2)}_{n_2+1} \\ \alpha^{(1)}_{n_1+1},\end{matrix}\right] \Big/ \left(\begin{matrix} d\,\alpha^{(k)}_{n_k}, = \mu^{(k)} \\ \vdots \\ d\,\alpha^{(2)}_{n_1} = \mu^{(2)}, \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)}\end{matrix}\right) \;\in\; \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \tag{97}$$

with

$$\mu^r \;\in\; A\left[\begin{matrix}\alpha^{(r-1)}_{n_{r-1}+1}, \\ \vdots \\ \alpha^{(1)}_{n_1+1},\end{matrix}\right], \qquad \text{for } 1 \leq r \leq k.$$

These multivariate polynomial algebras (97) receive the canonical inclusion (96) of A:

$$A \xhookrightarrow{\qquad i_A \qquad} A\left[\begin{matrix}\alpha^{(k)}_{n_k+1}, \\ \vdots \\ \alpha^{(2)}_{n_2+1} \\ \alpha^{(1)}_{n_1+1},\end{matrix}\right] \Big/ \left(\begin{matrix} d\,\alpha^{(k)}_{n_k} = \mu^{(k)}, \\ \vdots \\ d\,\alpha^{(2)}_{n_1} = \mu^{(2)}, \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)}\end{matrix}\right), \tag{98}$$

these being the composites of the stage-wise inclusions (96).

Definition 3.31 (Semifree dgc-Algebras/Sullivan models/FDAs)**.** The multivariate polynomial dgc-algebras of Example 3.30 are sometimes called **(i)** *semi-free dgc-algebras* over A (since their underlying graded-commutative algebra (76) is free, as in Example 3.10), but they are traditionally known **(ii)** in rational homotopy theory as *relative Sullivan models* (due to [Su77], review in [FHT00, II][Me13][FH17]), or, **(iii)** in supergravity theory (following [vN82][D'AF82]), as *FDAs*[7] [CDF91], (for translation see [FSS13b][FSS16a][FSS16b][HSS18][BMSS19]). Here we write:

$$\mathrm{SullivanModels}_{\mathbb{R}}^{\geq 1} \hookrightarrow \mathrm{SullivanModels}_{\mathbb{R}} \hookrightarrow \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \tag{99}$$

for, from right to left, **(a)** the full subcategory of connective dgc-algebras (Def. 3.17) on those which are isomorphic to a multivariate polynomial dgc-algebra over $\mathbb{R}$, as in Example 3.30 (i.e., the ordering of the generators in (97) is not part of the data of a Sullivan model, only the resulting dgc-algebra); and **(b)** for the further full subcategory on those Sullivan model that are generated in positive degree ≥ 1.

Example 3.32 (Polynomial dgc-algebras as pushouts)**.** For $A \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.17) the polynomial dgc-algebras over A (Def. 3.29) are pushouts in $\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ of the following form:

$$\begin{tikzcd}
A\big[\alpha_n\big]\big/\big(d\,\alpha_n = \mu\big) \arrow[dr, phantom, "\text{(po)}"] & & \mathbb{R}\left[\begin{matrix}\alpha_n\,, \\ c_{n+1}\end{matrix}\right]\Big/\left(\begin{matrix} d\,\alpha_n & = c_{n+1} \\ d\,c_{n+1} & = 0\end{matrix}\right) \arrow[ll, "{\substack{\alpha_n \mapsfrom \alpha_n \\ \mu \mapsfrom c_{n+1}}}"'] \\
A \arrow[u, hook, "i_A"] & & \mathbb{R}\big[c_{n+1}\big]\big/\big(d\,c_{n+1} = 0\big) \arrow[ll, "\mu \mapsfrom c_{n+1}"] \arrow[u, hook, "{\substack{c_{n+1} \\ \rotatebox{90}{$\mapsto$} \\ c_{n+1}}}"']
\end{tikzcd} \tag{100}$$

Here on the right we have multivariate polynomial dgc-algebras (Example 3.30) over $\mathbb{R}$ (Example 3.20) as shown. The horizontal morphisms encode the choice of $\mu \in A$ (94) and the left vertical morphism is the canonical inclusion (96).

Example 3.33 (Chevalley-Eilenberg algebras of nilpotent Lie algebras)**.** Beware that not every Lie algebra $\mathfrak{g}$ has Chevalley-Eilenberg algebra (Example 3.24) which satisfies the stratification in the Definition 3.30 of multivariate polynomial dg-algebras.

(i) For instance, the Lie algebra $\mathfrak{su}(2)$ has

$$\mathrm{CE}\big(\mathfrak{su}(2)\big) \;=\; \mathbb{R}\big[\theta_1, \theta_2, \theta_3\big]\Big/\Big(d\,\theta_i \;=\; \sum_{j,k} \varepsilon_{ijk}\theta_j \wedge \theta_k\Big)$$

and no ordering of $\{1,2,3\}$ brings this into the iterative form required in (97).

(ii) Instead, those Lie algebras whose CE-algebra is of the form (97) are precisely the nilpotent Lie algebras.

In generalization of Example 3.33, we say:

Definition 3.34 (Nilpotent L_∞-algebras)**.** An L_∞-algebra (85) is *nilpotent* if its CE-algebra (Def. 3.25) is a multivariate polynomial dgc-algebra (Example 3.30), hence is in the sub-category of $\mathrm{SullivanModels}_{\mathbb{R}}$ (99):

$$\begin{tikzcd}
L_\infty\mathrm{Algebras}_{\mathbb{R},\mathrm{fin}}^{\geq 0,\mathrm{nil}} \arrow[r, hook, "\mathrm{CE}"] \arrow[d, hook] \arrow[dr, phantom, "\text{(pb)}"] & \big(\mathrm{SulivanModels}\big)^{\mathrm{op}} \arrow[d, hook] \\
L_\infty\mathrm{Algebras}_{\mathbb{R},\mathrm{fin}}^{\geq 0} \arrow[r, hook, "\mathrm{CE}"] & \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{op}}
\end{tikzcd} \tag{101}$$

In fact, from (83) it is clear that every connected Sullivan model, hence with generators in degrees ≥ 1, is the Chevalley-Eilenberg algebra of a unique nilpotent L_∞-algebra, so that the defining inclusion at the top of (101) further restricts to an equivalence of homotopy categories:

$$L_\infty\mathrm{Algebras}_{\mathbb{R},\mathrm{fin}}^{\geq 0,\mathrm{nil}} \xrightarrow[\simeq]{\mathrm{CE}} \big(\mathrm{SullivanModels}_{\mathbb{R}}^{\geq 1}\big)^{\mathrm{op}}. \tag{102}$$

[7] Beware that "FDA" in the supergravity literature is meant to be short-hand for "free differential algebra", which is misleading, because what is really meant are not free dgc-algebras as in Example 3.19 (in general) but just "semi-free" dcg-algebras, only whose underlying graded-commutative algebras (76) is required to be free (Example 3.10).

Homotopy theory of connective dgc-Algebras. We recall the homotopy theory of connective differential graded-commutative algebras. We make free use of the language of model categories [Qu67]; for review see [Ho99][Lu09, A.2] and appendix A.

Definition 3.35 (Homotopical structure on connective dgc-algebras [BG76, §4.2][GM96, V.3.4])**.** Consider the following sub-classes of morphisms in the category of $\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (Def. 3.17):

(i) W – *weak equivalences* are the quasi-isomorphisms;

(ii) Fib – *fibrations* are the degreewise surjections;

We call this the *projective* homotopical structure on $\mathrm{dgcAlgebras}_{\mathbb{R}}^{\geq 0}$.

Proposition 3.36 (Projective model structure connective on dgc-algebras [BG76, §4.3][GM96, V.3.4])**.** *Equipped with the projective homotopical structure from Def. 3.35, the category of* $\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ *(Def. 3.17) becomes a model category (Def. A.3). We denote this as:*

$$\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}} \in \mathrm{ModelCategories}. \tag{103}$$

Remark 3.37 (All dgc-algebras are projectively fibrant)**.** Every object $A \in \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}}$ (103) is fibrant: By Example 3.21 the terminal morphism is to the 0-algebra, and this is clearly surjective, hence is a fibration, by Def. 3.35:

$$A \xrightarrow{\in \mathrm{Fib}} 0 .$$

Cofibrant dgc-algebras. In order to identify cofibrant dgc-algebras, it is useful to first consider the following:

Definition 3.38 (Homotopical structure on connective cochain complexes)**.** Consider the following sub-classes of morphisms in the category $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ (Def. 3.14):

(i) W – *weak equivalences* are the quasi-isomorphisms;

(ii) Fib – *fibrations* are the degreewise surjections;

(iii) Cof – *cofibrations* are the injections in positive degrees.

We call this the *injective* homotopical structure on $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$.

Proposition 3.39 (Injective model structure on connective cochain complexes [He07, p. 6])**.** *Equipped with the injective homotopical structure of Def. 3.38 the category of* $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ *(Def. 3.14) becomes a model category (Def. A.3). We denote this:*

$$\big(\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{inj}} \in \mathrm{ModelCategories}.$$

Proof. This is formally dual to the proof of the projective model structure on connective chain complexes [Qu67, II.4][GoS06, Thm. 1.5]; see, for instance, [Dun10, Thm. 2.4.5]. □

Remark 3.40 (Other model categories of chain complexes)**.** Prop. 3.39 is usually stated in the generality of cochain complexes of abelian groups, in which case the fibrations are only those degreewise surjections that have degreewise injective kernel, a condition that becomes trivial for abelian groups that are vector spaces.

Proposition 3.41 (Quillen adjunction between dgc-algebras and cochain complexes)**.** *The adjunction* (62) *between* $\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ *(Def. 3.17) and* $\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ *(Def. 3.14) is a Quillen adjunction (Def. A.17) with respect to the model category structures from Def. 3.39 and that from Def. 3.36.*

$$\begin{tikzcd} \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, bend right, shift right] & \big(\mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{inj}} \arrow[l, bend right, shift right, "\mathrm{Sym}"'] \end{tikzcd} .$$

Proof. It is immediate from the definitions that the forgetful right adjoint preserves the classes W and Fib. □

Lemma 3.42 (Generating cofibrations)**.** *The following inclusions of multivariate polynomial dgc-algebras (Example 3.30) are cofibrations in* $\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}}$ *(Def. 3.36)*

$$\mathbb{R}[c_{n+1}]\big/(d\, c_{n+1} = 0) \xhookrightarrow[\in \mathrm{Cof}]{c_{n+1} \mapsto c_{n-1}} \mathbb{R}\begin{bmatrix} \alpha_n\,, \\ c_{n+1} \end{bmatrix}\Big/\begin{pmatrix} d\ \alpha_n = c_{n+1}\,, \\ d\, c_{n+1} = 0 \end{pmatrix} \qquad \textit{for } n \in \mathbb{N}. \tag{104}$$

Proof. Consider the following morphisms of cochain complexes, for $n \in \mathbb{N}$:

$$
\left[
\begin{array}{c}
\vdots \\
0 \\
\uparrow{\scriptstyle d} \\
0 \\
\uparrow{\scriptstyle d} \\
\langle c_{n+1} \rangle \\
\uparrow{\scriptstyle d} \\
0 \\
\uparrow{\scriptstyle d} \\
0 \\
\uparrow{\scriptstyle d} \\
\vdots \\
\uparrow{\scriptstyle d} \\
0
\end{array}
\right]
\xhookrightarrow{\;\; i_n \;\;}
\left[
\begin{array}{c}
\vdots \\
0 \\
\uparrow{\scriptstyle d} \\
0 \\
\uparrow{\scriptstyle d} \\
\langle c_{n+1} \rangle \\
\uparrow{\scriptstyle d} \\
\langle \alpha_n \rangle \\
\uparrow{\scriptstyle d} \\
0 \\
\uparrow{\scriptstyle d} \\
\vdots \\
\uparrow{\scriptstyle d} \\
0
\end{array}
\right]
\qquad \text{with } d\alpha_n = c_{n+1}. \tag{105}
$$

Since these are injections, they are cofibrations in $\left(\mathrm{CochainComplexes}^{\geq 0}_{\mathbb{R}}\right)_{\mathrm{inj}}$ (Prop. 3.39), by Def. 3.38. Thus also their images under Sym (Def. 3.19) are cofibrations in $\left(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\right)_{\mathrm{proj}}$ (Prop. 3.36) because Sym is a left Quillen functor, by Prop. 3.41. But $\mathrm{Sym}(i_n)$ manifestly equals (104), and so the claim follows. ☐

Proposition 3.43 (Relative Sullivan algebras are cofibrations)**.** *For a multivariate polynomial dgc-algebra from Example 3.30, the canonical inclusion* (106) *of the base algebra is a cofibration in* $\left(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\right)_{\mathrm{proj}}$ *(Prop. 3.36):*

$$
A \xhookrightarrow[\;\; i_A \;\;]{\in \mathrm{Cof}} A\left[\begin{array}{c} \alpha^{(k)}_{n_k+1}, \\ \vdots \\ \alpha^{(1)}_{n_1+1}, \end{array}\right] \Big/ \left(\begin{array}{c} d\,\alpha^{(k)}_{n_k} = \mu^{(k)}, \\ \vdots \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)} \end{array}\right). \tag{106}
$$

In particular, since $\mathbb{R} \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ *is the initial object (Example 3.20), all multivariate polynomial dgc-algebras over* $\mathbb{R}$ *(the Sullivan models, Def. 3.31) are cofibrant objects in* $\left(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\right)_{\mathrm{proj}}$.

Proof. By Lemma 3.42, the right vertical morphisms in the pushout diagram (100) are cofibrations. Since the class of cofibrations is preserved under pushout, so are hence the left vertical morphisms in (100), which are the base algebra inclusions (96) of polynomial dgc-algebras. The base algebra inclusions into general multivariate polynomial dgc-algebras are composites of these, and since the class of cofibrations is presered under composition, the claim follows. ☐

Lemma 3.44 (Pushout along relative Sullivan algebras preserves quasi-isomorphisms [FHT00, Prop. 6.7 (ii), Lemma 14.2])**.** *The operation of pushout* (292) *along the canonical inclusion* (106) *of a base dgc-algebra into a multivariate polynomial dgc-algebra (Example 3.30) preserves quasi-isomorphisms. In fact, it sends quasi-isomorphism between base algebras to quasi-isomrophisms of multivariate polynomial dgc-algebras:*

$$
\begin{tikzcd}
A \arrow[r, hook, "\in \mathrm{Cof}", "i_A"'] \arrow[d, "f"', "\in \mathrm{W}"] \arrow[dr, phantom, "\text{(po)}"] & A\left[\begin{array}{c} \alpha^{(k)}_{n_k+1}, \\ \vdots \\ \alpha^{(1)}_{n_1+1}, \end{array}\right] \Big/ \left(\begin{array}{c} d\,\alpha^{(k)}_{n_k} = \mu^{(k)}, \\ \vdots \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)} \end{array}\right) \arrow[d, "(i_A)_* f"] \\
A' \arrow[r, hook, "\in \mathrm{Cof}", "i_A"'] & A'\left[\begin{array}{c} \alpha^{(k)}_{n_k+1}, \\ \vdots \\ \alpha^{(1)}_{n_1+1}, \end{array}\right] \Big/ \left(\begin{array}{c} d\,\alpha^{(k)}_{n_k} = \mu^{(k)}, \\ \vdots \\ d\,\alpha^{(1)}_{n_1} = \mu^{(1)} \end{array}\right)
\end{tikzcd}
\qquad \Rightarrow \qquad (i_a)_* f \in \mathrm{W}. \tag{107}
$$

Minimal Sullivan models

Definition 3.45 (Minimal Sullivan models [BG76, Def. 7.2][He07, Def. 1.10]). A connected (relative) Sullivan model dgc-algebra $A \in \mathrm{SullivanModels}^{\geq 1}_{\mathbb{R}}$ (Def. 3.31) is called *minimal* if it is given by a multivariate polynomial dgc-algebras as in (97) the degrees n_i of whose generators $\alpha^{(i)}_{n_i}$ are monotonically increasing

$$i < j \quad \Rightarrow \quad n_j \leq n_j\ .$$

Example 3.46 (Minimal models of simply connected dgc-algebras [BG76, Prop. 7.4]). If $A \in \mathrm{SullivanModels}^{\geq 1}_{\mathbb{R}}$ (Def. 3.31) is trivial in degree 1, then it is minimal (Def. 3.45) precisely if the unary bracket $[-]$ (84) of the corresponding L_∞-algebra (102) vanishes:

$$A^1 = 0 \quad \Rightarrow \quad \big(A \text{ is minimal} \quad \Leftrightarrow \quad [-] = 0\big).$$

Proposition 3.47 (Existence of minimal Sullivan models [BG76, Prop. 7.7, 7.8]). *If* $A \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ *is cohomologically connected, in that* $H^0(A) = \mathbb{R}$*, then:*
(i) *There exists a minimal Sullivan model* A_{min} *(Def. 3.45) with weak equivalence in* $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (103) *to* A

$$A_{\mathrm{min}} \xrightarrow{p^{\mathrm{min}}_A \in \mathrm{W}} A\,. \tag{108}$$

(ii) *This* A_{min} *is unique up to isomorphism of* $\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$.

More generally:

Proposition 3.48 (Existence of minimal relative Sullivan models [FHT00, Thm. 14.12]). *Let* $B \xrightarrow{\phi} A$ *be a morphism in* $\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ *(Def. 3.17) such that*

(a) *A and B are cohomologically connected, in that* $H^0(A) = \mathbb{R}$ *and* $H^0(B) = \mathbb{R}$,

(b) $H^1(\phi) : H^1(B) \longrightarrow H^1(A)$ *is an injection.*

Then:
(i) *There exists a minimal relative Sullivan model* $B \hookrightarrow A_{\mathrm{min}_B}$ *(Def. 3.45) equipped with a weak equivalence to* ϕ *in* $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ *(Def. 103):*

$$\begin{tikzcd} A_{\mathrm{min}_B} \arrow[rr, "\in \mathrm{W}"] & & A \\ & B \arrow[ul, hook] \arrow[ur, "\phi"'] & \end{tikzcd}$$

(ii) *This* A_{min_B} *is unique up to isomorphism in* $\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$.

3.2 Rational homotopy theory

We recall fundamental facts of dg-algebraic rational homotopy theory [Su77][BG76][GM13] (review in [FHT00] [He07][FOT08][FH17]), streamlined towards the application to non-abelian de Rham theory below in §3.3.

Remark 3.49 (Rational homotopy theory over the real numbers). Throughout, we consider rational homotopy theory over the real numbers $\mathbb{R}$ (as in [GM13]), instead of over the rational numbers $\mathbb{Q}$. This is the version in which rational homotopy theory connects to differential geometry (e.g. [FOT08]), since the *smooth* de Rham complex is not defined over $\mathbb{Q}$ but over $\mathbb{R}$ (see Lemma 3.88). The original account [BG76] of rational homotopy theory is, for the most part, formulated over an arbitrary field k of characteristic zero; and [BG76, Lem. 11.7] makes explicit that the choice of this base field does not change the form of the classical theorems. For example, the "real-ified" homotopy groups of a space X

$$\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R} \;\simeq\; \big(\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathbb{R}$$

form a real vector space with real dimension equal to the rational dimension of the corresponding rationalized homotopy groups

$$\dim_{\mathbb{Q}}\big(\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}\big) \;=\; \dim_{\mathbb{R}}\big(\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{R}\big)\,,$$

and hence the rational Whitehead L_∞-algebras (Prop. 3.61 below) have the same collection of generators and their CE-alberas/minimal Sullivan models (Prop. 3.47 below) have the same differential relations, irrespective of whether they come as algebras over $\mathbb{Q}$ or over $\mathbb{R}$. [8]

[8]While in homotopy theory $\mathbb{Q}$ and $\mathbb{R}$ coefficients behave similarly yet seem a priori not directly comparable, differential refinements might provide such comparison (with coefficients $\mathbb{R}/\mathbb{Q}$ naturally arising; see, e.g., [GS19a][GS19c]). We leave this for a future discussion.

For technical reasons, we focus on the following class of homotopy types (with little to no restriction in practice):

Definition 3.50 (Connected nilpotent spaces of finite rational type [BG76, 9.2])**.** Write

$$\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}} \hookrightarrow \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$$

for the full subcategory of homotopy types of topological spaces X (324) on those which are:

(i) *connected*: $\pi_0(X) \simeq *$;

(ii) *nilpotent*: $\pi_1(X) \in \mathrm{NilpotentGroups}$, and $\pi_{n\geq 2}(X)$ are nilpotent $\pi_1(X)$-modules (e.g. [Hil82]);

(iii) *finite rational type*: $\dim_{\mathbb{R}}\big(H^n(X;\mathbb{R})\big) < \infty$, for all $n \in \mathbb{N}$.

Remark 3.51 (Technical assumptions)**.** The connectedness assumption in Def. 3.50 is a pure convenience; for non-connected spaces all of the following applies just by iterating over connected components. On the other hand, the nilpotency and $\mathbb{R}$-finiteness condition in Def. 3.50 are strictly necessary for the plain dg-algebraic formulation of rational homotopy theory (due to [BG76][Su77]) to satisfy the fundamental theorem (Theorem 3.58 below). The generalizations required to drop these assumptions are known, but considerably more unwieldy:

(i) To drop the nilpotency assumption, all dgc-algebra models need to be equipped with the action of the fundamental group (see [FHT15]).

(ii) To drop the finite-type assumption one needs dgc-coalgebras in place of dgc-algebras, as in the original [Qu69].

Therefore, we expect that the construction of the (twisted) non-abelian character map, below in sections §4 and §5, works also without imposing these technical assumptions, but a discussion in that generality is beyond the scope of the present article.

Example 3.52 (Examples of nilpotent spaces [Hil82, §3][MP12, §3.1])**.** Such examples (Def. 3.50) include:

(i) every simply connected space X, $\pi_1(X) = 1$;

(ii) every simple space X, i.e. with abelian fundamental group acting trivially, such as tori;

(iii) hence every connected H-space;

(iv) hence every loop space $X \simeq \Omega Y$, and hence every ∞-group (Prop. 2.8);

(v) hence every infinite-loop space, i.e., every component space E_n of a spectrum E (21);

(vi) the classifying spaces BG (16) of nilpotent Lie groups G;

(vii) the mapping spaces $\mathrm{Maps}(X,A)$ out of manifolds X into nilpotent spaces A.

Rational homotopy theory is concerned with understanding the following notion:

Definition 3.53 (Rationalization [BK72, p. 133][BG76, §11.21][He07, §1.4, §1.7])**.**

(i) A connected nilpotent homotopy type $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathbb{Q}}\big)_{\geq 1,\mathrm{nil}}$ (Def. 3.50) is called *rational* if all its homotopy groups admit the structure of real vector spaces.

(ii) A *rationalization* of X is a map

$$X \xrightarrow{\eta^{\mathbb{R}}_X} L_{\mathbb{R}}(X) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)_{\geq 1,\mathrm{nil}} \tag{109}$$

such that:

(a) $L_{\mathbb{R}}$ is rational: $\pi_n(X) \in \mathrm{VectorSpaces}_{\mathbb{R}} \to \mathrm{Groups}$ for $n \geq 1$,

(b) the map $\eta^{\mathbb{R}}_X$ induces an isomorphism on rational cohomology groups:

$$H^\bullet(X;\mathbb{R}) \xrightarrow[\simeq]{H^\bullet(\eta^{\mathbb{R}}_X;\mathbb{R})} H^\bullet\big(L_{\mathbb{R}}X;,\mathbb{R}\big) .$$

Rationalization exists essentially uniquely, and defines a reflective subcategory inclusion

$$\underset{\substack{\text{connected, nilpotent, } \mathbb{R}\text{-finite,} \\ \text{rational homotopy types}}}{\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathbb{Q}}_{\geq 1,\mathrm{nil}}} \begin{array}{c} \xleftarrow{\;L_{\mathbb{R}}\;} \\[-4pt] \perp \\[-4pt] \hookrightarrow \end{array} \underset{\substack{\text{connected, nilpotent, } \mathbb{R}\text{-finite,} \\ \text{homotopy types}}}{\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)_{\geq 1,\mathrm{nil}}} \tag{110}$$

whose adjunction unit is (109).

PL de Rham theory. At the heart of dg-algebraic rational homotopy theory is the observation that a variant of the de Rham dg-algebra of a smooth manifold (Example 3.23) applies to general topological spaces: the *PL de Rham complex*[9] (Def. 3.54). This satisfies an appropriate *PL de Rham theorem* (Prop. 3.55) and makes dg-algebras of PL differential forms detect rational homotopy type (Prop. 3.58). At the same time, over a smooth manifold the PL de Rham complex is suitably equivalent to the smooth de Rham complex (Lemma 3.88).

Definition 3.54 (PL de Rham complex and PL de Rham cohomology [BG76, pp. 1-7][GM13, §9.1])**.** Write

$$\Omega^\bullet_{\mathrm{pdR}}(\Delta^\bullet) \;:\; \Delta^{\mathrm{op}} \longrightarrow \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}} \tag{111}$$

for the simplicial dgc-algebras of polynomial differential forms on the standard simplices.
(i) For $S \in \mathrm{SimplicialSets}$, its *PL de Rham complex* is the hom-object of simplicial objects from S to $\Omega^\bullet(\Delta^\bullet)$, hence is the following end in $\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$:

$$\Omega^\bullet_{\mathrm{PLdR}}(S) \;:=\; \int_{[k]\in\Delta} \underset{S_k}{\oplus}\, \Omega^\bullet_{\mathrm{pdR}}(\Delta^k)\,, \tag{112}$$

hence an element in $\omega \in \Omega^\bullet_{\mathrm{PLdR}}(S)$ is a polynomial differential form $\omega^{(n)}_\sigma \in \Omega^\bullet_{\mathrm{pdR}}(\Delta^n)$ on each n-simplex $\sigma \in S_n$ for all $n \in \mathbb{N}$, such that these are compatible under pullback along all simplex face inclusions δ_i and along all degenerate simplex projections σ_i:

$$\Omega^\bullet_{\mathrm{PLdR}}(S) \;=\; \left\{
\begin{tikzcd}[column sep=huge]
\vdots & \vdots \\
S_2 \arrow[r, dashed, "\omega^{(2)}"] \arrow[d, shift right=8, "\delta_0^*"'] \arrow[d, shift right=4] \arrow[d, "\delta_1^*"] \arrow[d, shift left=4] \arrow[d, shift left=8, "\delta_2^*"] & \Omega^\bullet_{\mathrm{pdR}}(\Delta^2) \arrow[d, shift right=8, "\delta_0^*"'] \arrow[d, "\delta_1^*"] \arrow[d, shift left=8, "\delta_2^*"] \\
S_1 \arrow[u, shift right=4, "\sigma_0^*"] \arrow[u, shift left=4, "\sigma_1^*"'] \arrow[r, dashed, "\omega^{(1)}"] \arrow[d, shift right=4, "\delta_0^*"'] \arrow[d, shift left=4, "\delta_1^*"] & \Omega^\bullet_{\mathrm{pdR}}(\Delta^1) \arrow[u, shift right=4, "\sigma_0^*"] \arrow[u, shift left=4, "\sigma_1^*"'] \arrow[d, shift right=4, "\delta_0^*"'] \arrow[d, shift left=4, "\delta_1^*"] \\
S_0 \arrow[u, "\sigma_0^*"] \arrow[r, dashed, "\omega^{(0)}"] & \Omega^\bullet_{\mathrm{pdR}}(\Delta^0) \arrow[u, "\sigma_0^*"]
\end{tikzcd}
\right\}$$

(ii) For $X \in \mathrm{TopologicalSpaces}$, its *PL de Rham complex* is that of its simgular simplicial set, according to (112):

$$\Omega^\bullet_{\mathrm{PLdR}}(X) \;:=\; \Omega^\bullet_{\mathrm{PLdR}}\big(\mathrm{Sing}(X)\big)\,. \tag{113}$$

By pullback of differential forms, this extends to a functor

$$\mathrm{SimplicialSets} \xrightarrow{\;\Omega^\bullet_{\mathrm{PLdR}}\;} \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}\,. \tag{114}$$

(iii) We write

$$H^\bullet_{\mathrm{PLdR}}(-) \;:=\; H\Omega^\bullet_{\mathrm{PLdR}}(-) \tag{115}$$

for *PL de Rham cohomology*, the cochain cohomology of the PL de Rham complex.

Proposition 3.55 (PL de Rham theorem [BG76, Thm. 2.2][GM13, Thm. 9.1])**.** *The evident operation of integrating differential forms over simplices induces a quasi-isomorphism*

$$\Omega^\bullet_{\mathrm{PLdR}}(-) \xrightarrow{\;\in \mathrm{qIso}\;} C^\bullet(-;\mathbb{R})$$

from the PL de Rham complex (Def. 3.54) to the cochain complex of ordinary singular cohomology with coefficients in $\mathbb{R}$. Hence on cochain cohomology this induces an isomorphism

$$H^\bullet_{\mathrm{PLdR}}(-) \xrightarrow{\;\simeq\;} H^\bullet(-;\mathbb{R})$$

between PL de Rham cohomology (115) *and ordinary real cohomology.*

[9] The terminology "PL" or "P.L." for this construction seems to have been silently introduced in [BG76], as shorthand for "piecewise linear", and has become widely adopted (e.g. [GM13, §9]). But beware that this refers to the piecewise-linear structure that a choice of triangulation induces on a topological space, while the actual differential forms in the PL de Rham complex are piecewise *polynomial* with respect to this piecewise linear structure.

But in fact, before passing to cochain cohomology, the PL de Rham complex captures full rational homotopy type:

Lemma 3.56 (Extension lemma for polynomial differential forms [GM13, Lemma 9.4]). *For $n \in \mathbb{N}$, the operation of pullback of piecewise polynomial differential forms (Def. 3.88) along the boundary inclusion of the n-simplex $\partial\Delta^n \xrightarrow{i_n} \Delta^n$ is an epimorphism:*

$$\Omega^\bullet_{\mathrm{PLdR}}(\Delta^n) \xrightarrow{\;\;i_n^*\;\;} \!\!\!\!\rightarrow \Omega^\bullet_{\mathrm{PLdR}}(\partial\Delta^n)\,.$$

Proposition 3.57 (PL de Rham Quillen adjunction [BG76, 8]). *The PL de Rham complex functor (Def. 3.54) is the left adjoint in a Quillen adjunction (Def. A.17)*

$$\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}} \underset{\exp_{\mathrm{PS}}}{\overset{\Omega^\bullet_{\mathrm{PLdR}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot_{\mathrm{Qu}}}}}} \mathrm{SimplicialSets}_{\mathrm{Qu}} \tag{116}$$

between the opposite (Def. A.9) of the model category of dgc-algebras (Prop. 3.36) and the classical model structure on simplicial sets (Prop. A.8); where the right adjoint sends a dgc-algebra A to

$$\exp_{\mathrm{PS}}(A) \;=\; \Delta[k] \longmapsto \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big(\Omega^\bullet_{\mathrm{PLdR}}(\Delta^k), A\big) \quad \in \mathrm{SimplicialSets}\,. \tag{117}$$

Proof. That the right adjoint exists and is give as in (117) follows by general nerve/realization theory [Ka58], or else by direct inspection.

For the left adjoint to preserve cofibrations means to take injections of simplicial sets to degreewise surjections of dgc-algebras. This follows from the extension lemma (Lemma 3.56). Moreover, the left adjoint preserves in fact all weak equivalences, by the PL de Rham theorem (Prop. 3.55). ☐

Proposition 3.58 (Fundamental theorem of dgc-algebraic rational homotopy theory). *The derived adjunction (Prop. A.20)*

$$\mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}\Big) \underset{\mathbb{R}\exp}{\overset{\mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big) \tag{118}$$

of the Quillen adjunction (116) *from Prop. 3.57 is such that:*

(i) *on connected, nilpotent, $\mathbb{R}$-finite homotopy types (Def. 3.50) the derived PLdR-adjunction unit* (311) *is the unit* (109) *of rationalization (Def. 3.53):*

$$\begin{tikzcd} X \arrow[r, "\substack{\text{derived unit of}\\ \text{PL de Rham adjunction}}", "\mathbb{D}\eta^{\mathrm{PLdR}}_X"'] \arrow[d, equal] & \mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(X) \arrow[d, phantom, "\wr"] \\ X \arrow[r, "\eta^{\mathbb{R}}_X", "\text{rationalization unit}"'] & L_{\mathbb{R}}X \end{tikzcd} \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}\,. \tag{119}$$

(ii) *For X, A nilpotent, connected, $\mathbb{R}$-finite homotopy types (Def. 3.50), the PL de Rham space functor* (114) *from Def. 3.54 induces natural bijections*

$$\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)(X, L_{\mathbb{R}}A) \xrightarrow[\Omega^\bullet_{\mathrm{PLdR}}]{\simeq} \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\Big(\Omega^\bullet_{\mathrm{PLdR}}(A), \Omega^\bullet_{\mathrm{PLdR}}(X)\Big). \tag{120}$$

Proof. **(i)** This is [BG76, Thm 11.2].

(ii) This follows via [BG76, Thm 9.4(i)], which says that the derived adjunction (118) restricts on connected, nilpotent, $\mathbb{R}$-finite (Def. 3.50) rational homotopy types (Def. 3.53) to an equivalence of categories:

$$\mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}\Big)^{\geq 1}_{\mathrm{fin}} \underset{\mathbb{R}\exp}{\overset{\mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)^{\mathbb{R},\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}\,. \tag{121}$$

In detail, consider the following sequence of natural isomorphisms:

$$
\begin{aligned}
& \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,\big(X\,,\,L_{\mathbb{R}}A\big) \\
& \simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\,\big(\mathrm{Sing}(X)\,,\,L_{\mathbb{R}}\mathrm{Sing}(A)\big) \\
& \simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\,\big(L_{\mathbb{R}}\mathrm{Sing}(X)\,,\,L_{\mathbb{R}}\mathrm{Sing}(A)\big) \\
& \simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\,\big(\mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(X)\,,\,\mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(A)\big) \\
& \simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}\Big)\,\big(\Omega^\bullet_{\mathrm{PLdR}}\circ\mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(X)\,,\,\Omega^\bullet_{\mathrm{PLdR}}\circ\mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(A)\big) \\
& \simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}\Big)\,\big(\Omega^\bullet_{\mathrm{PLdR}}(X)\,,\,\Omega^\bullet_{\mathrm{PLdR}}(A)\big) \\
& \simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\,\Big(\Omega^\bullet_{\mathrm{PLdR}}(A)\,,\,\Omega^\bullet_{\mathrm{PLdR}}(X)\Big).
\end{aligned}
\tag{122}
$$

Here the first step is (324); the second step uses that rationalization is a reflection (110); the third step uses (119); the fourth is the equivalence (121) along $\mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}$ (using, with Example A.21, that every simplicial set is already cofibrant, Example A.8); the fifth step is the statement from (121) that $\mathbb{R}\exp$ is the inverse equivalence. The last step is just the definition of the opposite of a category. The composite of the bijections (122) is the desired bijection (120). □

PS de Rham theory. The point of using piecewise *polynomial* differential forms in the PL de Rham complex (Def. 3.54) is that these, but not the piecewise smooth differential forms, can be defined over the field $\mathbb{Q}$ of rational numbers. But since we may and do use the real numbers as the rational ground field (Remark 3.49), it is expedient to also consider piecewise smooth de Rham complexes:

Definition 3.59 (PS de Rham complex)**.** For $n \in \mathbb{N}$, we write

$$
\Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^n \times \Delta^\bullet) \;:\; \Delta^{\mathrm{op}} \longrightarrow \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}
$$

for the simplicial dgc-algebra of smooth differential forms on the product manifold of n-dimensional Cartesian space with the standard simplices, i.e., of smooth differential forms on an ambient Cartesian space (Example 3.23), restricted to the simplex. As in Def. 3.88, this induces for each $S \in \mathrm{SimplicialSets}$ the corresponding *piecewise smooth de Rham complexes*

$$
\Omega^\bullet_{\mathrm{PSdR}}(\mathbb{R}^n \times S) \;:=\; \int_{[k]\in\Delta} \underset{S_n}{\oplus}\, \Omega^\bullet_{\mathrm{dR}}(\mathbb{R}^n \times \Delta^n) \tag{123}
$$

and by pullback of differential forms these extend to functors

$$
\mathrm{SimplicialSets} \xrightarrow{\;\Omega^\bullet_{\mathrm{PSdR}}(\mathbb{R}^n\times(-))\;} \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}\,. \tag{124}
$$

Proposition 3.60 (Fundamental theorem for piecewise smooth de Rham complexes)**.** *For all $n \in \mathbb{N}$ the piecewise smooth de Rham complex functors (Def. 3.59) participate in a Quillen adjunction analogous to the PL de Rham adjunction (Prop. 3.57)*

$$
\begin{tikzcd}
\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}} \arrow[r, shift right, "\exp_{\mathrm{PS},n}"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{SimplicialSets}_{\mathrm{Qu}} \arrow[l, shift right, "\Omega^\bullet_{\mathrm{PSdR}}(\mathbb{R}^n\times(-))"']
\end{tikzcd}
\tag{125}
$$

with right adjoint given as in (117)*:*

$$
\exp_{\mathrm{PS},n}(A) \;=\; \Delta[k] \;\longmapsto\; \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big(\Omega^\bullet_{\mathrm{PLdR}}(\mathbb{R}^n \times \Delta^k)\,,\,A\big) \quad \in \mathrm{SimplicialSets} \tag{126}
$$

whose derived functors (Prop. A.20) are naturally equivalent to those of the PL de Rham adjunction (118)*:*

$$
\mathbb{L}\Omega^\bullet_{\mathrm{PSdR}}(\mathbb{R}^n \times (-)) \;\simeq\; \mathbb{L}\Omega^\bullet_{\mathrm{PSdR}} \;\simeq\; \mathbb{L}\Omega^\bullet_{\mathrm{PLdR}}\,, \tag{127}
$$

$$
\mathbb{R}\exp_{\mathrm{PS},n} \;\simeq\; \mathbb{R}\exp_{\mathrm{PS}} \;\simeq\; \mathbb{R}\exp_{\mathrm{PL}}\,. \tag{128}
$$

Proof. **(i)** The proofs of the PL de Rham theorem (Prop. 3.55) as well as of the extension Lemma (Lemma 3.56) apply essentially verbatim also to piecewise-smooth differential forms ([GM13, Prop. 9.8]) and hence so does the proof of the PL de Rham Quillen adjunction in the form given in Prop. 3.57.
(ii) We have evident natural transformations

$$\Omega^\bullet_{\mathrm{PLdR}}(S) \xrightarrow{\in \mathrm{W}} \Omega^\bullet_{\mathrm{PSdR}}(S) \xrightarrow{\in \mathrm{W}} \Omega^\bullet_{\mathrm{PSdR}}(\mathbb{R}^n \times S)$$

given by inclusion of polynomial differential forms into smooth differential forms, and by pullback of differential forms along the projections $\mathbb{R}^n \times \Delta^k \longrightarrow \Delta^k$. The corresponding component morphisms are quasi-isomorphisms, hence are weak equivalences in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ ([GM13, Cor. 9.9]). Under passage to homotopy categories (Def. A.14) and derived functors (Example A.21), these natural weak equivalences become the natural isomorphisms (127) (by Prop. A.15). By essential uniqueness of adjoint functors, this implies the natural isomorphisms (128). ☐

Whitehead L_∞-algebras.

Proposition 3.61 (Rational Whitehead L_∞-algebras)**.** *For $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50), there exists a nilpotent L_∞-algebra (Def. 3.34)*

$$\mathfrak{l}X \;\in\; L_\infty\mathrm{Algebras}^{\geq 0,\mathrm{nil}}_{\mathbb{R},\mathrm{fin}}, \tag{129}$$

unique up to isomorphism, whose Chevalley-Eilenberg algebra (Def. 3.25) is the minimal model (Def. 3.45) of the PL de Rham complex of X (Def. 3.54):

$$\mathrm{CE}(\mathfrak{l}X) \;:=\; \big(\Omega^\bullet_{\mathrm{PLdR}}(X)\big)_{\mathrm{min}} \xrightarrow[\in \mathrm{W}]{p^{\mathrm{min}}_X} \Omega^\bullet_{\mathrm{PLdR}}(X)\;. \tag{130}$$

Proof. By the PL de Rham theorem (Prop. 3.55) and the assumption that X is connected, it follows that we have $H\Omega^0_{\mathrm{PLdR}}(X) \;=\; \mathbb{R}$. Therefore Prop. 3.47 applies and says that $\big(\Omega^\bullet_{\mathrm{PLdR}}(X)\big)_{\mathrm{min}} \in \mathrm{SullivanModels}^{\geq 1}_{\mathbb{R}}$ exists, and is unique up to isomorphism. With this, the equivalence (102) says that $\mathfrak{l}X$ exists and is unique up to isomorphism. ☐

Notice the immediate corollary:

Proposition 3.62 (Rational Whitehead L_∞-algebra encodes rational homotopy type)**.** *The rational Whitehead L_∞-algebra $\mathfrak{l}X$ in Prop. 3.61 encodes the rationalized homotopy type (Def. 3.53) of $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$, in that:*

$$L_{\mathbb{R}}X \;\simeq\; \exp \circ \mathrm{CE}\big(\mathfrak{l}X\big) \quad \in\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}} \tag{131}$$

with exp *from* (116)*.*

Proof. This is the composite of the following sequence of isomorphisms in the homotopy category:

$$\begin{aligned} L_{\mathbb{R}}X &\simeq \mathbb{R}\exp \circ \Omega^\bullet_{\mathrm{PLdR}}(X) \\ &\simeq \mathbb{R}\exp \circ \big(\Omega^\bullet_{\mathrm{PLdR}}(X)\big)_{\mathrm{min}} \\ &\simeq \mathbb{R}\exp \circ \mathrm{CE}(\mathfrak{l}X) \\ &\simeq \exp \circ \mathrm{CE}(\mathfrak{l}X)\;. \end{aligned}$$

Here the first step is the fundamental theorem (Prop. 3.58), the second step is the existence of the minimal model (Prop. 3.47) for the PL de Rham complex (using that it is cohomologically connected, by the PL de Rham theorem, Prop. 3.55), the third step is the existence of the rational Whitehead L_∞-algebra (Prop. 3.61), and the last step uses that Sullivan models are cofibrant (Prop. 3.43), hence fibrant in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}$, so that on these objects the right derived functor $\mathbb{R}\exp$ is given by the plain functor exp. ☐

Proposition 3.63 (Rational homotopy groups in the rational Whitehead L_∞ algebra)**.**
Let $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50).*
(i) *If X is simply connected, $\pi_1(X) = 1$ (Example 3.52), then there is an isomorphism of graded vector spaces (Def. 3.2) between the graded vector space underlying* (82) *the Whitehead L_∞-algebra $\mathfrak{l}X$ (Prop. 3.61) and the rationalized homotopy groups of the based loop space ΩX:*

$$\underset{\text{Whitehead}\atop L_\infty\text{-algebra}}{\mathfrak{l}X} \;\simeq\; \underset{\text{rationalized}\atop\text{homotopy groups}}{\pi_\bullet(\Omega X)\otimes_{\mathbb{Z}}\mathbb{R}} \quad \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}.$$

(ii) *If $\pi_1(X)$ is not necessarily trivial but abelian, then this statement still holds with $\mathfrak{l}X$ replaced by its homology with respect to the unary differential $[-]$* (84)*.*
(iii) *If $\pi_1(X)$ is not abelian, then (ii) still holds in degrees ≥ 2.*

Proof. Under translation through Prop. 3.61 and Def. 3.25, and using $\pi_\bullet(\Omega X) \simeq \pi_{\bullet+1}(X)$, claim **(i)** is equivalent to the existence of a dual isomorphism:

$$\mathrm{CE}(\mathfrak{l}X)\big/_{\mathrm{CE}(\mathfrak{l}X)^2} \;\simeq\; \mathrm{Hom}_{\mathbb{Z}}\big(\pi_\bullet(X), \mathbb{R}\big) \quad \in \mathrm{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \tag{132}$$

where the left hand side denotes the graded vector space of indecomposable elements in the Chevalley-Eilenberg algebra (the $\alpha^{(i)}_{n_i}$ in (97)). In this form, this is the statement of [BG76, Theorem 11.3 with Def. 6.12], in the special case when, with $\pi_1(X) = 1$, the unary differential $[-]$ in $\mathfrak{l}X$ vanishes (Example 3.46). The generalizations follow analogously. □

Remark 3.64 (Equivalent L_∞-structures on Whitehead products)**.** The original discussion of the Whitehead algebra structure on the homotopy groups of a space is in terms of differential-graded Lie algebras ([Hil55, Theorem B]), as are the Quillen models of rational homotopy theory [Qu69].
(i) Notice that dg-Lie algebras (Example 3.26) and L_∞-algebras with minimal CE-algebra (Def. 3.45) are two opposite classes of L_∞-algebras: The former has k-ary brackets (84) only for $k \leq 2$, the latter only for $k \geq 2$ (in the simply connected case, by Example 3.46). Yet, quasi-isomorphisms connect algebras in one class to those in the other: The transmutation of dg-Lie- into minimal L_∞-algebras is described in [BBMM16, Theorem 2.1]; that from L_∞- to dg-Lie-algebras in [FRS13, §1.0.2].
(ii) The minimal L_∞-algebra structure on $\mathfrak{l}X$ that we obtained in Prop. 3.61, 3.63, has the property that its k-ary brackets are, up to possibly a sign, equal to the order-k higher Whitehead products on X [BBMM16, Prop. 3.1].

Example 3.65 (Rationalization of Eilenberg-MacLane spaces)**.** Since the homotopy types of Eilenberg-MacLane-spaces $K(A,n) = B^{n+1}A$ (see (14)) are fully characterized by their homotopy groups (for discrete abelian groups A, e.g. [AGP02, §6]))

$$\pi_k\big(B^{n+1}A\big) \;\simeq\; \left\{ \begin{array}{ccl} A & | & k = n+1 \\ 0 & | & k \neq n+1 \end{array} \right.$$

we have, for $n \in \mathbb{N}$:
(i) The rationalization (Def. 3.53) of the integral EM-space is the real EM-space

$$L_{\mathbb{R}}\big(B^{n+1}\mathbb{Z}\big) \;\simeq\; B^{n+1}\mathbb{R}. \tag{133}$$

(ii) Their Whitehead L_∞-algebra (Prop. 3.61) is the line Lie n-algebra $\mathfrak{b}^n\mathbb{R}$ (Def. 3.27), by Prop. 3.63:

$$\mathfrak{l}B^{n+1}\mathbb{Z} \;\simeq\; \mathfrak{b}^n\mathbb{R}\,. \tag{134}$$

(iii) Hence, by Prop. 3.62,

$$B^{n+1}\mathbb{R} \;\simeq\; \exp\circ\mathrm{CE}\big(\mathfrak{b}^n\mathbb{R}\big) \quad \in \; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big). \tag{135}$$

Example 3.66 (Rationalization of n-spheres)**.** The Serre finiteness theorem (see [Ra86, Thm. 1.1.8]) says that the homotopy groups of n-spheres for $n \geq 1$ are of the form

$$\pi_{n+k}\big(S^n\big) \;\simeq\; \left\{ \begin{array}{lcl} \mathbb{Z} & | & k = 0 \\ \mathbb{Z}\oplus\mathrm{fin} & | & k = 2m \;\text{ and }\; n = 2m-1 \\ \mathrm{fin} & | & \text{otherwise} \end{array} \right.$$

where “fin” stands for some finite group. Since finite groups are pure torsion, hence have trivial rationalization, this means that the rational homotopy groups of spheres are:

$$\pi_{n+k}\big(S^n\big)\otimes_{\mathbb{Z}}\mathbb{R} \;\simeq\; \left\{ \begin{array}{ccl} \mathbb{R} & | & k=0 \\ \mathbb{R} & | & k=2m \text{ and } n=2m-1 \\ 0 & | & \text{otherwise}. \end{array} \right.$$

Moreover, the fact that ordinary cohomology is represented by Eilenberg-MacLane spaces (Example 2.2) implies that

$$H^k\big(S^n;\mathbb{R}\big) \;\sim\; \left\{ \begin{array}{ccl} \mathbb{R} & | & k=n \\ 0 & | & \text{otherwise}. \end{array} \right.$$

With this, Prop. 3.63 together with Prop. 3.55 implies that the Whitehead L_∞-algebras of spheres (Prop. 3.61) are as follows:

$$\mathrm{CE}\big(\mathfrak{l}S^n\big) \;\simeq\; \mathbb{R}\big[\omega_n\big]\big/\big(d\,\omega_n \,=\, 0.\big) \qquad\qquad \text{if } n \text{ is odd} \tag{136}$$

and

$$\mathrm{CE}\big(\mathfrak{l}S^n\big) \;\simeq\; \mathbb{R}\left[\begin{array}{c}\omega_{2n-1}\\ \omega_n\end{array}\right]\Big/\left(\begin{array}{ll} d\,\omega_{2n-1} & = -\omega_n\wedge\omega_n \\ d\,\omega_n & = 0 \end{array}\right) \qquad \text{if } n>1 \text{ is even} \tag{137}$$

Example 3.67 (Rationalization of loop spaces)**.** The minimal Sullivan model (Def. 3.45) of a loop space $A\simeq\Omega A'$ has vanishing differential (e.g. [FHT00, p. 143]). Therefore, Prop. 3.63 implies that the rational Whitehead L_∞-algebra $\mathfrak{l}A$ (Prop. 3.61) of A is the direct sum of line Lie n-algebras $\mathfrak{b}^n\mathbb{R}$ (Example 3.27):

$$\mathfrak{l}A \;\simeq\; \bigoplus_{n\in\mathbb{N}}\mathfrak{b}^n\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big) \quad\in\; L_\infty\mathrm{Algebras}^{\geq 0,\mathrm{nil}}_{\mathbb{R},\mathrm{fin}}\,.$$

Accordingly, its Chevalley-Eilenberg algebra (Def. 3.24) is the tensor product of those of the summands:

$$\mathrm{CE}\big(\mathfrak{l}A\big) \;\simeq\; \bigotimes_{n\in\mathbb{N}}\mathrm{CE}\Big(\mathfrak{b}^n\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\Big) \quad\in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\,.$$

Relative rational Whitehead L_∞-algebras. In generalization of Prop. 3.61 we have:

Proposition 3.68 (Relative rational Whitehead L_∞-algebras)**.** *For $A,B,F\in\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) and p a Serre fibration (Example A.7) from A to B with fiber F*

$$\begin{tikzcd} F \arrow[r, "\mathrm{fib}(p)"] & A \arrow[d, "p"', "\in\mathrm{Fib}"] \\ & B \end{tikzcd}$$

there exists a nilpotent L_∞-algebra (Def. 3.34)

$$\mathfrak{l}_B A \;\in\; L_\infty\mathrm{Algebras}^{\geq 0,\mathrm{nil}}_{\mathbb{R},\mathrm{fin}}\,, \tag{138}$$

unique up to isomorphism, whose Chevalley-Eilenberg algebra (Def. 3.25) is the relative minimal model (Def. 3.45, Prop. 3.48) of the PL de Rham complex of p (Def. 3.54), relative to $\mathrm{CE}(\mathfrak{l}B)$ (from Prop. 3.61):

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{l}_B A) := \big(\Omega^\bullet_{\mathrm{PLdR}}(A)\big)_{\mathrm{min}_{\mathrm{CE}(\mathfrak{l}B)}} \arrow[rr, "p_A^{\mathrm{min}_B}", "\in\mathrm{W}"'] & & \Omega^\bullet_{\mathrm{PLdR}}(A) \\ & \mathrm{CE}(\mathfrak{l}B) \arrow[ul, hook', "\text{relative minimal model } \mathrm{CE}(\mathfrak{l}p)"] \arrow[r, "p_B^{\mathrm{min}}", "\in\mathrm{W}"'] & \Omega^\bullet_{\mathrm{PLdR}}(B)\,. \arrow[u, "\Omega^\bullet_{\mathrm{PLdR}}(p)"'] \end{tikzcd} \tag{139}$$

Proof. By the PL de Rham theorem (Prop. 3.55) and the assumption that A and B are connected, it follows that we have $H\Omega^0_{\mathrm{PLdR}}(A) \,=\, \mathbb{R}$ and $H\Omega^0_{\mathrm{PLdR}}(B) \,=\, \mathbb{R}$. Moreover, by the assumption that p is a Serre fibration with connected fiber, it follows that $H^1(\Omega^\bullet_{\mathrm{PLdR}}(p))$ is injective (e.g. [FHT00, p. 196]).

Therefore Prop. 3.48 applies and says that $\big(\Omega^\bullet_{\mathrm{PLdR}}(A)\big)_{\mathrm{min}_B}\in\mathrm{SullivanModels}^{\geq 1}_{\mathbb{R}}$ exists, and is unique up to isomorphism. With this, the equivalence (102) says that $\mathfrak{l}_B A$ exists and is unique up to isomorphism. ☐

Lemma 3.69 (Minimal relative Sullivan models preserve homotopy fibers [FHT00, §15 (a)][FHT15, Thm. 5.1])**.** *Consider* $F, A, B \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50) and let p be a Serre fibration from A to B (Example A.7) such that the homology groups* $H_\bullet(F,\mathbb{R})$ *of the fiber are nilpotent as* $\pi_1(B)$*-modules (for instance in that B is simply-connected or that the fibration is principal). Then the cofiber of the minimal relative Sullivan model for p* (139) *is the minimal Sullivan model* (130) *for the homotopy fiber F (Def. A.22):*

$$
\begin{tikzcd}
F \arrow[r, "\mathrm{fib}(p)"] & A \arrow[d, "p"', "\in \mathrm{Fib}"] \\
 & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{CE}(\mathfrak{l}F) & \mathrm{CE}\big(\mathfrak{l}_{B}A\big) \arrow[l, "\mathrm{cofib}(\mathrm{CE}(\mathfrak{l}p))"'] \\
 & \mathrm{CE}(\mathfrak{l}B) \arrow[u, hook, "\mathrm{CE}(\mathfrak{l}p)"']
\end{tikzcd}
\tag{140}
$$

See Prop. 3.73 below for the key application of Lemma 3.69.

Non-abelian real cohomology.

Definition 3.70 (Non-abelian real cohomology)**.** Let $X, A \in \mathrm{TopologicalSpaces}$ Then the *non-abelian real cohomology* of X with coefficients in A is the non-abelian cohomology (Def. 2.1) of X with coefficients in the rationalization $L_{\mathbb{R}}A$ (Def. 3.53)

$$
H(X; L_{\mathbb{R}}A) \;:=\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)(X\,,\,L_{\mathbb{R}}A)\,. \tag{141}
$$

Remark 3.71 (Non-abelian real cohomology subsumes ordinary real cohomology)**.** For $n \in \mathbb{N}$, non-abelian real cohomology (Def. 3.70) with coefficients in the rationalized classifying space (Example 3.65)

$$
L_{\mathbb{R}}\big(B^{n+1}\mathbb{Z}\big) \;\simeq\; B^n\mathbb{R}
$$

is naturally equivalent, by Example 2.2, to ordinary real cohomology in degree n:

$$
H\big(X; B^{n+1}\mathbb{R}\big) \;\simeq\; H^{m+1}(X;\mathbb{R})\,.
$$

More generally:

Proposition 3.72 (Non-abelian real cohomology with coefficients in loop spaces)**.**
Let $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50) such that it admits loop space structure, hence such that there exists* A' *with*

$$
A \;\simeq\; \Omega A' \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,.
$$

Then the non-abelian real cohomology (Def. 3.70) with coefficients in $L_{\mathbb{R}}A$ *is naturally equivalent to ordinary real cohomology with coefficients in the rationalized homotopy groups of A:*

$$
H\big(X; L_{\mathbb{R}}A\big) \;\simeq\; \bigoplus_{n\in\mathbb{N}} H^{n+1}\big(X; \pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big). \tag{142}
$$

Proof. By Example 3.67 the we have

$$
\mathrm{CE}\big(\mathfrak{l}A\big) \;\simeq\; \bigotimes_{n\in\mathbb{N}} \mathrm{CE}\Big(\mathfrak{b}^n\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\Big)
$$

Noticing that the tensor product of dgc-algebras is the coproduct in the category of $\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (Example 3.22), and hence the Cartesian product in the opposite category, the right adjoint functor exp (116) preserves this, so that

$$
\exp\circ\mathrm{CE}\big(\mathfrak{l}A\big) \;\simeq\; \prod_{n\in\mathbb{N}}\Big(\exp\circ\mathrm{CE}\big(\mathfrak{b}^n(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R})\big)\Big) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,.
$$

But, by Prop. 3.62 and by (135) in Example 3.65, this says that:

$$
L_{\mathbb{R}}A \;\simeq\; \prod_{n\in\mathbb{N}}\Big(B^{n+1}\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\Big) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\,.
$$

Using this, and that cohomology preserves products of coefficients, we get the following sequence of natural bijections:

$$
\begin{aligned}
H\big(X; L_{\mathbb{R}}A\big) &\simeq H\Big(X; \prod_{n\in\mathbb{N}} B^{n+1}\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\Big) \\
&\simeq \prod_{n\in\mathbb{N}} H\Big(X; B^{n+1}\big(\pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\Big) \\
&= \prod_{n\in\mathbb{N}} H^{n+1}\big(X : \pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big) \\
&= \bigoplus_{n\in\mathbb{N}} H^{n+1}\big(X : \pi_{n+1}(A)\otimes_{\mathbb{Z}}\mathbb{R}\big)\,.
\end{aligned}
$$

The composite of these is the desired (142). $\square$

Twisted non-abelian real cohomology.

Proposition 3.73 (Rationalization of local coefficients – "fiber lemma" [BK72, §II])**.** *Let*

$$
\begin{tikzcd}
A \arrow[r] & A /\!/ G \arrow[d, "\rho"] \\
 & BG
\end{tikzcd}
$$

be a local coefficient bundle (Def. 2.29) such that all spaces are connected, nilpotent and of $\mathbb{R}$-finite tupe, $A, BG, A/\!/G \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) and such that the action of $\pi_1(BG)$ on $H_\bullet(A,\mathbb{R})$ is nilpotent (for instance in that BG is simply connected). Then:

(i) *Component-wise rationalization (Def. 3.53) yields a natural transformation to as rational local coefficient bundle as shown in the middle here:*

$$
\begin{tikzcd}
\text{minimal Sullivan model} & & & & \mathrm{CE}(\mathfrak{l}A) & \mathrm{CE}\big(\mathfrak{l}_{BG}(A/\!/G)\big) \arrow[l, "\mathrm{cofib}(\mathrm{CE}(\mathfrak{l}\rho))"'] \\
\text{rationalization} & & L_{\mathbb{R}}A \arrow[rr, "\mathrm{hofib}(L_{\mathbb{R}}\rho)"] & & (L_{\mathbb{R}}A)/\!/(L_{\mathbb{R}}G) \arrow[dd, "L_{\mathbb{R}}\rho"] & \\
\text{local coefficient bundle} & A \arrow[ur, "\eta^{\mathbb{R}}_A"] \arrow[rr, "\mathrm{hofib}(\rho)"] & & A/\!/G \arrow[ur, "\eta^{\mathbb{R}}_{A/\!/G}"'] \arrow[dd, "\rho"] & & \mathrm{CE}\big(\mathfrak{l}BG\big) \arrow[uu, hook, "\mathrm{CE}(\mathfrak{l}\rho)"'] \\
 & & & & B\big(L_{\mathbb{R}}G\big) & \\
 & & & BG \arrow[ur, "\eta^{\mathbb{R}}_{BG}"'] & &
\end{tikzcd}
\tag{143}
$$

(ii) *with minimal (relative) Sullivan model (Def. 3.45) as shown on the right.*

Proof. First, since forming classifying spaces shifts homotopy groups up in degree, it follows that $BG \xrightarrow{B\eta^{\mathbb{R}}_G} B(L_{\mathbb{R}}G)$ induces an isomorphism on rationalized homotopy groups and hence is the rationalization map (Def. 3.53) on BG.

Moreover, Lemma 3.69 says that the homotopy fiber (Def. A.22) of the rationalized fibration has Sullivan model $\mathrm{CA}(\mathfrak{l}A)$, this being the cofiber of a relative Sullivan model for the rationalized fibrations, as shown on the right of (143). Since relative Sullivan models are cofibrations in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (Prop. 3.43), hence fibrations in the opposite model structure (Example A.9), this means, with the fundamental theorem (121) that $\mathrm{CE}(\mathfrak{l}A)$ is in fact a Sullivan model for the homotopy fiber (Def. A.22) of the rationalized fibration. Hence the homotopy fiber of the rationalized fibration is the rationalization $L_{\mathbb{R}}A$ of the homotopy fiber of the original fibration, as shown in (143).

Together these say that the rationalized fibration is an $L_{\mathbb{R}}A$-fibration over $B\big(L_{\mathbb{R}}G\big)$. With this, Prop. 2.28 implies that the total space of the rationalized fibration is a homotopy quotient $\big(L_{\mathbb{R}}A\big)/\!/\big(L_{\mathbb{R}}G\big)$, which is hence the rationalization of $A/\!/G$, as shown in the middle of (143). $\square$

Due to Prop. 3.73 it makes sense to say, in generalization of Def. 3.70:

Definition 3.74 (Twisted non-abelian real cohomology)**.** Let $X \in \mathrm{TopologicalSpaces}$ and let $A/\!/G \xrightarrow{\rho} BG$ be a local coefficient bundle (Prop. 2.28, Def. 2.29) in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50). Then the *twisted non-abelian real cohomology* of X with local coefficients ρ is the twisted non-abelian $L_{\mathbb{R}}A$-cohomology (Def. 2.29) of X with coefficients in the rationalized local coefficient bundle $L_{\mathbb{R}}\rho$ from Prop. 3.73:

$$
H^{\tau}\big(X; L_{\mathbb{R}}A\big) \;:=\; \mathrm{Ho}\Big(\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{/L_{\mathbb{R}}BG}\big)\Big)\big(\tau, L_{\mathbb{R}}\rho\big)\,.
$$

3.3 Non-abelian de Rham theorem

We establish non-abelian de Rham theory for differential forms with values in (nilpotent) L_∞-algebras, following [SSS09a][FSSt10]. The main result is the non-abelian de Rham theorem, Theorem 3.85, and its generalization to the twisted non-abelian de Rham theorem, Theorem 3.102.

L_∞-Algebra valued differential forms.

Definition 3.75 (Flat L_∞-algebra valued differential forms [SSS09a, §6.5][FSSt10, §4.1])**.**
(i) For $X \in \mathrm{SmoothManifold}$ and $\mathfrak{g} \in L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.25), a *flat $\mathfrak{g}$-valued differential form on X* is a morphism of dgc-algebras (Def. 3.17)

$$\Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\;A\;} \mathrm{CE}(\mathfrak{g}) \quad \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}} \tag{144}$$

to the smooth de Rham dgc-algebra of X (Example 3.23) from the Chevalley-Eilenberg dgc-algebra of $\mathfrak{g}$ (Def. 3.25).
(ii) We write

$$\Omega_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}} := \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big(\mathrm{CE}(\mathfrak{g}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big) \tag{145}$$

for the set of all flat $\mathfrak{g}$-valued forms on X.

Example 3.76 (Flat Lie algebra valued differential forms)**.** Let $\mathfrak{g} \in \mathrm{LieAlgebras}_{\mathbb{R},\mathrm{fin}}$ be a Lie algebra (88) with Lie bracket $[-,-]$. Then a flat $\mathfrak{g}$-valued differential form in the sense of Def. 3.75 is the traditional concept: a $\mathfrak{g}$-valued 1-form satisfying the Maurer-Cartan equation:

$$\Omega^\bullet_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}} \simeq \Big\{ A \in \Omega^1_{\mathrm{dR}}(X)\otimes\mathfrak{g} \,\Big|\, dA + [A\wedge A] = 0 \Big\}. \tag{146}$$

One way to see this is to appeal to the classical fact that the Chevalley-Eilenberg algebra of a finite-dimensional Lie algebra (Example 3.24) is isomorphic to the dgc-algebra of left invariant differential forms on the corresponding Lie group G, which is generated from the Maurer-Cartan form $\theta \in \Omega^1_{\mathrm{dR}}(G)\otimes\mathfrak{g}$ satisfying $\theta_{|T_e G} = \mathrm{id}_{\mathfrak{g}}$ and $d\theta = [\theta\wedge\theta]$. More explicitly, for $\{v_a\}$ a linear basis for $\mathfrak{g}$ (78) with structure constants $\{f^c_{ab}\}$ (79), we see from (80) that a dgc-algebra homomorphims (144) has the following components (second line) and constraints (third line):

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathbb{R}\big[\{\theta_1^{(a)}\}\big]\Big/\Big(d\,\theta_1^{(c)} = f^c_{ab}\,\theta_1^{(b)}\wedge\theta_1^{(a)}\Big) \simeq \mathrm{CE}(\mathfrak{g})\,. \arrow[ll, "\text{flat Lie algebra valued differential form}"', "A"] \\
A^{(c)} \arrow[d, mapsto, "d"'] & & \theta_1^{(c)} \arrow[ll, mapsto, "\text{components}"'] \arrow[d, mapsto, "d"] \\
dA^{(c)} \arrow[r, equal, "\text{constraints}"] & f^c_{ab}\,A^{(b)}\wedge A^{(a)} & f^c_{ab}\,\theta_1^{(a)}\wedge\theta_1^{(b)} \arrow[l, mapsto]
\end{tikzcd} \tag{147}$$

Example 3.77 (Ordinary closed forms are flat line L_∞-algebra valued forms)**.** For $n\in\mathbb{N}$, consider $\mathfrak{g} = \mathfrak{b}^n\mathbb{R}$ the line Lie $(n+1)$-algebra (Example 3.27). Then the corresponding flat $\mathfrak{g}$-valued differential forms (Def. 3.75) are in natural bijection to ordinary closed $(n+1)$-forms:

$$\Omega_{\mathrm{dR}}(X;\mathfrak{b}^n\mathbb{R})_{\mathrm{flat}} \simeq \Omega^{n+1}_{\mathrm{dR}}(X)_{\mathrm{closed}}\,. \tag{148}$$

That is, by (90), we see that the elements on the left of (148) have the following component (second line) subject to the follows constraint (third line):

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) \arrow[rr, "\substack{\text{flat}\\ \text{line Lie } (n+1)\text{-algebra-valued}\\ \text{differential form}}"] & & \mathbb{R}[c_{n+1}]\big/(d\,c_{n+1} = 0) \simeq \mathrm{CE}(\mathfrak{b}^n\mathbb{R})\,. \\
C_{n+1} \arrow[d, mapsto, "d"'] & & c_{n+1} \arrow[ll, mapsto, "\text{component}"'] \arrow[d, mapsto, "d"] \\
dC_{n+1} \arrow[r, equal, "\text{constraint}"] & 0 & 0 \arrow[l, mapsto]
\end{tikzcd} \tag{149}$$

Example 3.78 (Flat String Lie 2-algebra valued differential forms)**.** Flat L_∞-algebras valued forms (Def. 3.75) with values in a String Lie 2-algebra $\mathfrak{string}_{\mathfrak{g}}$ (Example 92) are pairs consisting of a flat $\mathfrak{g}$-valued 1-form A_1 (Example 3.76) and a coboundary 2-form B_2 for its Chern-Simons form $\mathrm{CS}(A) := c\langle A \wedge [A \wedge A]\rangle$:

$$\Omega_{\mathrm{dR}}\big(X; \mathfrak{string}_{\mathfrak{g}}\big)_{\mathrm{flat}} \simeq \left\{ \begin{array}{l} B_2, \\ A_1 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \begin{array}{l} d\,B_2 = \tfrac{1}{c}\mathrm{CS}(A)\,, \\ d\,A_1 = -[A_1 \wedge A_1] \end{array} \right\}.$$

Namely, from (93) we see that in degree 1 the components of and constraints on such a differential form datum are exactly as in (147), while in degree 2 they are as follows:

$$\begin{tikzcd}[column sep=large]
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathbb{R}\left[\begin{array}{c} b_2, \\ \{\theta_1^{(a)}\} \end{array}\right] \Big/ \left(\begin{array}{ll} d\,b_2 & = \mu_{abc}\,\theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)} \\ d\theta_1^{(c)} & = f^c_{ab}\,\theta_1^{(b)} \wedge \theta_1^{(a)} \end{array} \right) \arrow[ll, "\text{flat String Lie 2-algebra valued form}"'] \simeq \mathrm{CE}\big(\mathfrak{string}_{\mathfrak{g}}\big).
\end{tikzcd} \tag{150}$$

$$\begin{tikzcd}
B_2 \arrow[d, "d"'] & & b_2 \arrow[ll, mapsto, "\text{component in degree 2}"'] \arrow[d, "d"] \\
dB_2 \arrow[r, equal, "\text{constraint}"] & \mu_{abc}\,A_1^{(c)} \wedge A_1^{(b)} \wedge A_1^{(a)} & \mu_{abc}\,\theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)} \arrow[l, mapsto]
\end{tikzcd}$$

Example 3.79 (Flat sphere-valued differential forms)**.** Flat L_∞-algebras valued forms (Def. 3.75) with values in the rational Whitehead L_∞-algebra (Prop. 3.61) of a sphere (Example 3.66) of positive even dimension $2k$ are pairs consisting of a closed differential $2k$-form and a $(4k-1)$-form whose differential equals minus the wedge square of the $2k$-form:

$$\Omega_{\mathrm{dR}}\big(-; \mathfrak{l}S^{2k}\big) \simeq \left\{ \begin{array}{l} G_{4k-1}, \\ G_{2k} \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \begin{array}{ll} d\,G_{4k-1} & = -G_{2k} \wedge G_{2k}, \\ d\,G_{2k} & = 0 \end{array} \right\}.$$

Namely, from (137) one sees that the components of and the constraints on an $\mathfrak{l}S^{2k}$-valued form are as follows:

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathbb{R}\left[\begin{array}{c} \omega_{4k-1}, \\ \omega_{2k} \end{array}\right] \Big/ \left(\begin{array}{ll} d\,\omega_{4k-1} & = -\omega_{2k} \wedge \omega_{2k}, \\ d\,\omega_{2k} & = 0 \end{array} \right) \arrow[ll, "\text{flat } \mathfrak{l}S^{2k}\text{-valued form}"'] = \mathrm{CE}\big(\mathfrak{l}S^{2k}\big)
\end{tikzcd} \tag{151}$$

$$\begin{tikzcd}
G_{2k} \arrow[d, "d"'] & & \omega_{2k} \arrow[ll, mapsto, "\text{component in degree } 2k"'] \arrow[d, "d"] \\
d\,G_{2k} \arrow[r, equal, "\text{constraint}"] & 0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
G_{4k-1} \arrow[d, "d"'] & & \omega_{4k-1} \arrow[ll, mapsto, "\text{component in degree } 4k-1"'] \arrow[d] \\
d\,G_{4k-1} \arrow[r, equal, "\text{constraint}"] & -G_{2k} \wedge G_{2k} & -\omega_{4k} \wedge \omega_{4k} \arrow[l, mapsto]
\end{tikzcd}$$

For $2k = 4$ this is the structure of the equations of motion of the C-field in 11-dimensional supergravity (modulo the Hodge self-duality constraint $G_7 = \star G_4$) [Sa13, §2.5].

Example 3.80 (PL de Rham right adjoint via L_∞-algebra valued forms)**.** For $n \in \mathbb{N}$, the right adjoint functor in the PS de Rham adjunction (125) sends the Chevalley-Eilenberg algebra (Def. 3.25) of any $\mathfrak{g} \in L_\infty\mathrm{Algebras}^{\geq 0,\mathrm{nil}}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.34) to a simplicial set of flat $\mathfrak{g}$-valued differential forms (Def. 3.75):

$$\flat\exp(\mathfrak{g})(\mathbb{R}^n) := \exp_{\mathrm{PS},n}\big(\mathrm{CE}(\mathfrak{g})\big) \;:\; [k] \longmapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^k; \mathfrak{g}\big)_{\mathrm{flat}} \quad \in \mathrm{SimplicialSets}$$

(by direct comparison of (126) with (145)). Regarded as a simplicial presheaf over CartesianSpaces (Def. 339), this construction is the moduli ∞-stack of flat L_∞-algebra valued differential forms (see §4.3 below).

Non-abelian de Rham cohomology.

Definition 3.81 (Coboundaries between flat L_∞-algebra valued forms)**.** Let $X \in \mathrm{SmoothManifolds}$ and (from Def. 3.25) $\mathfrak{g} \in L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$. For

$$A^{(0)},\, A^{(1)} \;\in\; \Omega_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}}$$

a pair of flat $\mathfrak{g}$-valued differential forms on X (Def. 3.75), we say that a *coboundary* between them is a flat $\mathfrak{g}$-valued differential form on the cylinder manifold over X (its Cartesian product manifold with the real line):

$$\widetilde{A} \;\in\; \Omega(X \times \mathbb{R};\mathfrak{g})_{\mathrm{flat}} \tag{152}$$

such that its restrictions along

$$X \simeq X \times \{0\} \xhookrightarrow{\;i_0^X\;} X \times \mathbb{R} \xhookleftarrow{\;i_1^X\;} X \times \{1\} \simeq X$$

are equal to $A^{(0)}$ and to $A^{(1)}$, respectively:

$$(i_0^X)^*\widetilde{A} \;=\; A^{(0)} \qquad \text{and} \qquad (i_1^X)^*\widetilde{A} \;=\; A^{(1)}\,. \tag{153}$$

If such a coboundary exists, we say that $A^{(0)}$ and $A^{(1)}$ are *cohomologous*, to be denoted

$$A^{(0)} \;\sim\; A^{(1)}\,.$$

Definition 3.82 (Non-abelian de Rham cohomology)**.** Let $X \in \mathrm{SmoothManifolds}$ and $\mathfrak{g} \in L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.25). Then the *non-abelian de Rham cohomology* of X with coefficients in $\mathfrak{g}$ is the set

$$H_{\mathrm{dR}}(X;\mathfrak{g}) \;:=\; \big(\Omega_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}}\big)_{/\sim} \tag{154}$$

of equivalence classes with respect to the coboundary relation from Def. 3.81 on the set of flat $\mathfrak{g}$-valued differential forms on X (Def. 3.75).

We recall the following basic facts (e.g. [GT00, Rem 3.1]):

Lemma 3.83 (Fiberwise Stokes theorem and Projection formula)**.** *Let X be a smooth manifold and let F be a compact smooth manifold with corners, e.g. $F = \Delta^k$ a standard k-simplex, which for $k=1$ is the interval $F=[0,1]$.*

Then fiberwise integration over F of differential forms on the Cartesian product manifold $X \times F$

$$\Omega^\bullet_{\mathrm{dR}}(X \times F) \xrightarrow{\;\int_F\;} \Omega^{\bullet-\dim(F)}_{\mathrm{dR}}(X) \qquad \textit{e.g.} \qquad \Omega^\bullet_{\mathrm{dR}}(X \times \mathbb{R}) \xrightarrow{\;\int_{[0,1]}\;} \Omega^{\bullet-1}_{\mathrm{dR}}(X)$$

satisfies, for all $\alpha \in \Omega^\bullet_{\mathrm{dR}}(X \times F)$ and $\beta \in \Omega^\bullet_{\mathrm{dR}}(X)$:

(i) *The fiberwise Stokes formula:*

$$\int_F d\alpha \;=\; (-1)^{\dim(F)}\, d\int_F \alpha + \int_{\partial F} \alpha \qquad \text{e.g.} \qquad d\int_{[0,1]} \alpha \;=\; (i_1^X)^*\alpha \,-\, (i_0^X)^*\alpha \,-\, \int_{[0,1]} d\alpha\,, \tag{155}$$

where

$$X \;\simeq\; X \times \{0\} \xhookrightarrow{\;i_0^X\;} X \times \mathbb{R} \xhookleftarrow{\;i_1^X\;} X \times \{1\} \;\simeq\; X$$

are the boundary inclusions.

(ii) *The projection formula*

$$\int_F \big(\mathrm{pr}_X^*\beta\big) \wedge \alpha \;=\; (-1)^{\dim(F)\deg(\beta)} \beta \wedge \int_F \alpha\,, \quad \textit{e.g.} \quad \int_{[0,1]} \big(\mathrm{pr}_X^*\beta\big) \wedge \alpha \;=\; (-1)^{\deg(\beta)} \beta \wedge \int_{[0,1]} \alpha\,, \tag{156}$$

where

$$X \times F \xrightarrow{\;\mathrm{pr}_X\;} X$$

is projection on the first factor.

Proposition 3.84 (Non-abelian de Rham cohomology subsumes ordinary de Rham cohomology)**.** *For any $n \in \mathbb{N}$, let $\mathfrak{g} = \mathfrak{b}^n\mathbb{R}$ be the line Lie $(n+1)$-algebra (Example 3.27). Then the non-abelian de Rham cohomology with coefficients in $\mathfrak{g}$ (Def. 3.82) is naturally equivalent to ordinary de Rham cohomology in degree $n+1$:*

$$H_{\mathrm{dR}}(-;\mathfrak{b}^n\mathbb{R}) \;\simeq\; H^{n+1}_{\mathrm{dR}}(-)\,. \tag{157}$$

Proof. From Example 3.77, we know that the canonical cocycle sets are in natural bijection

$$\Omega_{\mathrm{dR}}(X;\, \flat^n\mathbb{R})_{\mathrm{flat}} \;\simeq\; \Omega^{n+1}_{\mathrm{dR}}(X)_{\mathrm{closed}}\,.$$

Therefore, it just remains to see that the coboundary relations in both cases coincide. By the explicit nature (149) of the above natural bijection and by the Definition 3.81 of non-abelian coboundaries, we hence need to see that a pair of closed forms

$$C^{(0)}_{n+1},\, C^{(1)}_{n+1} \quad \in \Omega^{n+1}_{\mathrm{dR}}(X)_{\mathrm{closed}}$$

has a de Rham coboundary, i.e.,

$$\exists\, h_n \,\in\, \Omega^n_{\mathrm{dR}}(X)\,, \quad \text{such that } \; C^0_{n+1} + dh_n \;=\; C^{(1)}_{n+1}\,, \tag{158}$$

precisely if the pair extends to a closed $(n+1)$-form on the cylinder over X, as in (152) (153):

$$\exists\, \widetilde{C}_{n+1} \,\in\, \Omega^{n+1}_{\mathrm{dR}}(X\times\mathbb{R})_{\mathrm{closed}}\,, \quad \text{such that} \quad \big(i^X_0\big)^*\widetilde{C}_{n+1} \;=\; C^{(0)}_{n+1} \;\; \text{and} \;\; \big(i^X_1\big)^*\widetilde{C}_{n+1} \;=\; C^{(1)}_{n+1}\,. \tag{159}$$

That (158) $\Leftrightarrow$ (159) is a standard argument: Let t denote the canonical coordinate function on $\mathbb{R}$. In one direction, given h_n as in (158), the choice

$$\widetilde{C}_{n+1} \;:=\; (1-t)\,\mathrm{pr}^*_X\big(C^{(0)}_{n+1}\big) + t\,\mathrm{pr}^*_X\big(C^{(1)}_{n+1}\big) + dt\wedge \mathrm{pr}^*_X\big(h_n\big)$$

clearly satisfies (159). In the other direction, given $\widetilde{C}_{n+1}$ as in (159), the choice

$$h_n \;:=\; \int_{[0,1]} \widetilde{C}_{n+1}$$

satisfies (158), by the fiberwise Stokes theorem (Lemma 3.83). □

The non-abelian de Rham theorem.

Theorem 3.85 (Non-abelian de Rham theorem)**.** *Let $X, A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50), and let X admit the structure of a smooth manifold. Then the non-abelian de Rham cohomology (Def. 3.82) of X with coefficients in the real Whitehead L_∞-algebra $\mathfrak{l}A$ (Prop. 3.61) is in natural bijection with the non-abelian real cohomology (Def. 3.70) of X with coefficients in $L_{\mathbb{R}}A$ (Def. 3.53):*

$$H\big(X;\, L_{\mathbb{R}}A\big) \;\simeq\; H_{\mathrm{dR}}(X;\, \mathfrak{l}A)\,. \tag{160}$$

Proof. Consider the following sequence of natural bijections:

$$\begin{aligned} H\big(X;L_{\mathbb{R}}A\big) &= \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,\, L_{\mathbb{R}}A\big) \\ &\simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\big(\Omega^\bullet_{\mathrm{PLdR}}(A)\,,\, \Omega^\bullet_{\mathrm{PLdR}}(X)\big) \\ &\simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\big(\mathrm{CE}(\mathfrak{l}A)\,,\, \Omega^\bullet_{\mathrm{PL}}(X)\big) \\ &\simeq H_{\mathrm{dR}}(X;\,\mathfrak{l}A)\,. \end{aligned} \tag{161}$$

Here the first line is the definition Def. 3.70. Then the first step is the fundamental theorem of rational homotopy theory (Prop. 3.58). The second step uses the following isomorphisms: $\mathrm{CE}(\mathfrak{l}A) \simeq \Omega^\bullet_{\mathrm{PLdR}}(A)$ (Prop. 3.61) and $\Omega^\bullet_{\mathrm{PLdR}}(X) \simeq \Omega^\bullet_{\mathrm{dR}}(X)$ (Lemma 3.88) in the homotopy category. The last step is Lemma 3.87. The composite of these natural bijections gives the desired bijection (160). □

We now prove the three lemmas used in the proof of Theorem 3.85:

Lemma 3.86 (De Rham complex over cylinder of manifold is path space object)**.** *For $X \in$ SmoothManifolds, consider the following morphisms of dgc-algebras (Def. 3.17)*

$$\Omega^\bullet_{\mathrm{dR}}(X) \xrightarrow{\;(\mathrm{pr}_X)^*\;} \Omega^\bullet_{\mathrm{dR}}\big(X\times\mathbb{R}\big) \xrightarrow{\;(i_0^*,\, i_1^*)\;} \Omega^\bullet_{\mathrm{dR}}(X)\oplus\Omega^\bullet_{\mathrm{dR}}(X) \tag{162}$$

(from the de Rham complex of X (Example 3.23) to that of its cylinder manifold $X\times\mathbb{R}$, to its Cartesian product with itself, by Example 3.22), given by pullback of differential forms along these smooth functions:

$$X \xleftarrow{\;\mathrm{pr}_X\;} X\times\mathbb{R} \xleftarrow{\;(i_0,i_1)\;} \big(X\times\{0\}\big)\sqcup\big(X\times\{1\}\big) \;\simeq\; X\sqcup X\,.$$

This is a path space object (Def. A.11) for $\Omega^\bullet_{\mathrm{dR}}(X)$ in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (Prop. 3.36).

Proof. **(i)** It is clear by construction that the composite morphism is the diagonal.
(ii) That $(\mathrm{pr}_X)^*$ is a weak equivalence, hence a quasi-isomorphism, follows from the de Rham theorem, using that ordinary cohomology is homotopy invariant: $H^\bullet(X \times \mathbb{R}; \mathbb{R}) \simeq H^\bullet(X; \mathbb{R})$.
(iii) That (i_0^*, i_1^*) is a fibration, namely degreewise surjective, is seen from the fact that any pair of forms on the boundaries $X \times \{0\}$, $X \times \{1\}$ may be smoothly interpolated to zero along any small enough positive parameter length, and then glued to a form on $X \times \mathbb{R}$. □

Lemma 3.87 (Non-abelian de Rham cohomology via the dgc-homotopy category)**.** *Let* $X \in \mathrm{SmoothManifolds}$ *and* $\mathfrak{g} \in L_\infty\mathrm{Algebras}^{\geq 0, \mathrm{nil}}_{\mathbb{R}, \mathrm{fin}}$ *(Def. 3.34). Then the non-abelian de Rham cohomology of X with coefficients in* $\mathfrak{g}$ *(Def. 3.82) is in natural bijection with the hom-set in the homotopy category of* $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ *(Prop. 3.36) from* $\mathrm{CE}(\mathfrak{g})$ *(Def. 3.25) to* $\Omega^\bullet_{\mathrm{dR}}(X)$ *(Example 3.23):*

$$H_{\mathrm{dR}}\big(X; \mathfrak{g}\big) \;\simeq\; \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\big(\mathrm{CE}(\mathfrak{g})\,,\, \Omega^\bullet_{\mathrm{dR}}(X)\big). \tag{163}$$

Proof. Consider a pair of dgc-algebra homomorphisms

$$A^{(0)},\, A^{(1)} \;\in\; \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big(\mathrm{CE}(\mathfrak{g})\,,\, \Omega^\bullet_{\mathrm{dR}}(X)\big) \tag{164}$$

hence of flat $\mathfrak{g}$-valued differential forms, according to Def. 3.75. Observe that:

- **(i)** $\mathrm{CE}(\mathfrak{g})$ is cofibrant in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (103). (by Prop. 3.43, and since $\mathfrak{g}$ is assumed to be nilpotent (101));
- **(ii)** $\Omega^\bullet_{\mathrm{dR}}(X)$ is fibrant in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (103). (by Remark 3.37);
- **(iii)** A right homotopy (Def. A.12) between the pair (164) of morphisms, with respect to the path space object $\Omega^\bullet_{\mathrm{dR}}(X \times \mathbb{R})$ from Lemma 3.86, namely a morphism $\widetilde{A}$ making the following diagram commute

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{dR}}(X) & \\ \Omega^\bullet_{\mathrm{dR}}(X \times \mathbb{R}) \arrow[u, "i_0^*"] \arrow[d, "i_1^*"'] & \mathrm{CE}(\mathfrak{g}) \arrow[l, "\widetilde{A}"'] \arrow[ul, "A^{(0)}"'] \arrow[dl, "A^{(1)}"] \\ \Omega^\bullet_{\mathrm{dR}}(X) & \end{tikzcd} \tag{165}$$

  is manifestly the same as a coboundary $\widetilde{A}$ between the corresponding flat $\mathfrak{g}$-valued forms according to Def. 3.81.

Therefore, Prop. A.16 says that the quotient set (154) defining the non-abelian de Rham cohomology is in natural bijection to the hom-set in the homotopy category. □

Lemma 3.88 (PL de Rham complex on smooth manifold is equivalent to smooth de Rham complex)**.** *Let X be a smooth manifold. Then*
(i) *There exists a zig-zag of weak equivalences (Def. 3.35) in* $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (103) *between the smooth de Rham complex of X (Example 3.23) and the PL de Rham complex of its underlying topological space (Def. 3.54).*
(ii) *In particular, both are isomorphic in the homotopy category:*

$$X \text{ smooth manifold} \quad \Rightarrow \quad \Omega^\bullet_{\mathrm{dR}}(X) \;\simeq\; \Omega^\bullet_{\mathrm{PLdR}}(X) \quad \in \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big).$$

Proof. Let $\Omega^\bullet_{\mathrm{PSdR}}(-)$ (for "piecewise smooth") be defined as the PL de Rham complex in Def. 3.54, but with smooth differential forms on each simplex. Notice that this comes with the canonical natural inclusion

$$\Omega^\bullet_{\mathrm{PLdR}}(-) \xhookrightarrow{\;i_{\mathrm{poly}}\;} \Omega^\bullet_{\mathrm{PSdR}}(-)\;.$$

Let then $\mathrm{Tr}(X) \in \mathrm{SimplicialSets}$ be any smooth triangulation of X. This means that we have a homeomorphism out of its geometric realization to X

$$|\mathrm{Tr}(X)| \xrightarrow[\text{homeo}]{\;p\;} X\,, \tag{166}$$

which restricts on the interior of each simplex to a diffeomorphism onto its image; and that we have an inclusion

$$\mathrm{Tr}(X) \underset{\in \mathrm{W}}{\overset{\eta_{\mathrm{Tr}(X)}}{\hookrightarrow}} \mathrm{Sing}\big(|\mathrm{Tr}(X)|\big) \underset{\in \mathrm{Iso}}{\xrightarrow{\mathrm{Sing}(p)}} \mathrm{Sing}(X)\,, \tag{167}$$

which is a weak equivalence (by Example A.35). In summary, this gives us the following zig-zag of dgc-algebra homomorphisms:

$$\begin{tikzcd}
 & \Omega^\bullet_{\mathrm{PLdR}}\big(\mathrm{Tr}(X)\big) \arrow[dr, "i_{\mathrm{poly}}"] & & \Omega^\bullet_{\mathrm{dR}}(X) \arrow[dl, "p^*"'] \\
\Omega^\bullet_{\mathrm{PLdR}}(X) = \Omega^\bullet_{\mathrm{PLdR}}\big(\mathrm{Sing}(X)\big) \arrow[ur, "(\eta_S)^*"] & & \Omega^\bullet_{\mathrm{PSdR}}\big(\mathrm{Tr}(X)\big) &
\end{tikzcd}$$

Here the two morphisms on the right are quasi-isomorphisms by [GM13, Cor. 9.9] (as in Prop. 3.60). The morphism on the left is a quasi-isomorphism because i is a weak homotopy equivalence (331) and since $\Omega^\bullet_{\mathrm{PLdR}}$ preserves weak equivalences, by Ken Brown's Lemma (Lemma A.19), since it is a Quillen left adjoint, by Prop. 3.57, and since every simplicial set is cofibrant (Example A.8). □

Flat twisted L_∞-algebra valued differential forms. We generalize the above discussion to include twistings.

Definition 3.89 (Local L_∞-algebraic coefficients)**.** We say that a *local L_∞-algebraic coefficient bundle* is a fibration

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] & \widehat{\mathfrak{b}} \arrow[d, "\mathfrak{p}"] \\
 & \mathfrak{b}
\end{tikzcd} \tag{168}$$

in $L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ (Def. 3.25), hence a morphism such that under passage to Chevalley-Eilenberg algebras (87) we have a cofibration

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) & \mathrm{CE}\big(\widehat{\mathfrak{b}}\big) \arrow[l, "\mathrm{cofib}(\mathrm{CE}(\mathfrak{p}))"'] \\
 & \mathrm{CE}(\mathfrak{b}) \arrow[u, "\mathrm{CE}(\mathfrak{p})", "\in \mathrm{Cof}"']
\end{tikzcd} \tag{169}$$

in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ (Prop. 3.36).

In generalization of Def. 3.75, we say:

Definition 3.90 (Flat twisted L_∞-algebra valued differential forms)**.**
(i) Let $X \in \mathrm{SmoothManifolds}$ and $\widehat{\mathfrak{b}}$ (168) a local L_∞-algebraic coefficient bundle (Def. 3.89). For

$$\tau_{\mathrm{dR}} \;\in\; \Omega_{\mathrm{dR}}(X;\mathfrak{b})_{\mathrm{flat}} \tag{170}$$

a flat $\mathfrak{b}$-valued differential form on X (Def. 3.75), we say that a *flat τ-twisted $\mathfrak{g}$-valued differential form* on X is a morphism of dgc-algebras (Def. 3.17) in the slice over $\mathrm{CE}(\mathfrak{b})$

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathrm{CE}\big(\widehat{\mathfrak{b}}\big) \arrow[ll, "\substack{\text{flat } \tau_{\mathrm{dR}}\text{-twisted} \\ \mathfrak{g}\text{-valued differential form}}"', "A"] \\
 & \mathrm{CE}(\mathfrak{b}) \arrow[ul, "\tau_{\mathrm{dR}}"', "\text{twist}"] \arrow[ur, hook, "\mathrm{CE}(\mathfrak{p})", "\substack{\text{local} \\ L_\infty\text{-algebraic} \\ \text{coefficients}}"'] &
\end{tikzcd} \tag{171}$$

(ii) We write

$$\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}} \;:=\; \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{/\mathrm{CE}(\mathfrak{b})}\big(\tau_{\mathrm{dR}}\,,\,\mathfrak{p}\big)$$

for the set of all flat τ_{dR}-twisted $\mathfrak{g}$-valued differential forms on X.

Remark 3.91 (Underlying flat forms of flat twisted forms)**.** Let $X \in \mathrm{SmoothManifolds}$, let $\mathfrak{g} \to \widehat{\mathfrak{b}} \xrightarrow{\mathfrak{p}} \mathfrak{b}$ be a local L_∞-algebraic coefficient bundle (Def. 3.89), and let $\tau_{\mathrm{dR}} \in \Omega_{\mathrm{dR}}\big(X;\mathfrak{b}\big)$. Then there is a canonical forgetful natural transformation

$$\Omega^{\tau_{\mathrm{dR}}}(X;\mathfrak{g})_{\mathrm{flat}} \longrightarrow \Omega\big(X;\widehat{\mathfrak{b}}\,\big)_{\mathrm{flat}} \tag{172}$$

from flat τ_{dR}-twisted $\mathfrak{g}$-valued differential forms (Def. 3.90) to flat $\widehat{\mathfrak{b}}$-valued differential forms (Def. 3.75), given by remembering only the top morphism in (171).

Example 3.92 (L_∞-coefficient bundle for H_3-twisted differential forms [FSS16a, §4][FSS16b, §4][BMSS19, Lem. 2.31])**.** Consider the local L_∞-algebraic coefficient bundle (Def. 3.89) given by the following multivariate polynomial dgc-algebras (Def. 3.30):

$$
\begin{array}{ccc}
\mathrm{CE}\big(\mathfrak{l}\mathrm{ku}_1\big) \;=\mathbb{R}\left[\begin{array}{c}\vdots\\ f_5,\\ f_3,\\ f_1,\end{array}\right]\Big/\left(\begin{array}{c}\vdots\\ d\,f_5=0\\ d\,f_3=0\\ d\,f_1=0\end{array}\right) & \xleftarrow{\;\omega_{2k+1}\leftarrow\!\shortmid\,\omega_{2k+1}\;} & \mathbb{R}\left[\begin{array}{c}\vdots\\ f_5,\\ f_3,\\ f_1,\\ h_3\end{array}\right]\Big/\left(\begin{array}{l}\qquad\vdots\\ d\,f_5=h_3\wedge f_3,\\ d\,f_3=h_3\wedge f_1,\\ d\,f_1=0,\\ d\,h_3=0\end{array}\right) = \mathrm{CE}\Big(\mathfrak{l}\big(\mathrm{ku}_1/\!\!/B\mathrm{U}(1)\big)\Big)\\
 & & \Big\uparrow {\scriptstyle h_3 \,\leftarrow\!\shortmid\, h_3}\\
 & & \mathbb{R}\big[h_3\big]\big(d\,h_3\;=0\big) =\; \mathrm{CE}\big(\mathfrak{b}^2\mathbb{R}\big)
\end{array}
$$

Here the rational model of the classifying space ku_1 for complex topological K-theory in degree 1 and for its twisted version is as in [FSS16a, §4][FSS16b, §4][BMSS19, Lem. 2.31]. In this case:
(i) A twist (170) is equivalently an ordinary closed 3-form form (by Example 3.77):

$$
H_3 \;\in\; \Omega_{\mathrm{dR}}\big(X;\mathfrak{b}^2\mathbb{R}\big)_{\mathrm{flat}} \;\simeq\; \Omega^3_{\mathrm{dR}}(X)_{\mathrm{closed}}\,. \tag{173}
$$

(ii) The flat $\tau_{\mathrm{dR}}\sim H_3$-twisted $\mathfrak{l}\mathrm{ku}_1$-valued differential forms according to Def. 3.90 are equivalently sequences of odd-degree differential forms $F_{2k+1}\in\Omega^{2k+1}_{\mathrm{dR}}(X)$ satisfying the H_3-twisted de Rham closure condition (see [RW86, (23)][GS19c]):

$$
\Omega^{\tau_{\mathrm{dR}}}\big(X;\mathfrak{l}\mathrm{ku}_1\big)_{\mathrm{flat}} \;\simeq\; \Big\{F_{2\bullet+1}\in\Omega^{2\bullet+1}_{\mathrm{dR}}\;\Big|\; d\sum_k F_{2k+1} \;=\; H_3\wedge\sum_k F_{2k-1}\Big\} \tag{174}
$$

(where we set $F_{2k-1}:=0$ if $2k-1<0$, for convenience of notation).

In direct generalization of Example 3.92, we have:

Example 3.93 (L_∞-coefficient bundle for higher twisted differential forms [FSS18, Def. 2.14])**.** For $r\in\mathbb{N}$, $r\geq 1$, consider the local L_∞-algebraic coefficient bundle (Def. 3.89) given by the following multivariate polynomial dgc-algebras (Def. 3.30):

$$
\begin{array}{ccc}
\mathrm{CE}\Big(\underset{k\in\mathbb{N}}{\oplus}\mathfrak{b}^{2rk}\mathbb{R}\Big) & & \mathrm{CE}\Big(\Big(\underset{k\in\mathbb{N}}{\oplus}\mathfrak{b}^{2rk}\mathbb{R}\Big)/\!\!/B^{2r-1}\mathrm{U}(1)\Big)\\
\| & & \|\\
\mathbb{R}\left[\begin{array}{c}\vdots\\ f_{4r+1},\\ f_{2r+1},\\ f_1,\end{array}\right]\Big/\left(\begin{array}{r}\vdots\\ d\,f_{4r+1}=0\\ d\,f_{2r+1}=0\\ d\,f_1\;\;=0\end{array}\right) & \xleftarrow{\;f_{2rk+1}\leftarrow\!\shortmid\, f_{2rk+1}\;} & \mathbb{R}\left[\begin{array}{c}\vdots\\ f_{4r+1},\\ f_{2r+1},\\ f_1,\\ h_{2r+1}\end{array}\right]\Big/\left(\begin{array}{l}\qquad\vdots\\ d\,f_{4r+1}=h_{2r+1}\wedge f_{2r+1},\\ d\,f_{2r+1}=h_{2r+1}\wedge f_1,\\ \quad d\,f_1\;\;=0,\\ d\,h_{2r+1}=0\end{array}\right)\\
 & & \Big\uparrow {\scriptstyle h_{2r+1}\,\leftarrow\!\shortmid\, h_{2r+1}}\\
 & & \mathbb{R}\big[h_{2r+1}\big]\big(d\,h_{2r+1}\;=0\big)\\
 & & \|\\
 & & \mathrm{CE}\big(\mathfrak{b}^{2r}\mathbb{R}\big)
\end{array} \tag{175}
$$

In this case:
(i) A twist (170) is equivalently an ordinary closed $(2r+1)$-form form (by Example 3.77):

$$
H_{2r+1} \;\in\; \Omega_{\mathrm{dR}}\big(X;\mathfrak{b}^{2r}\mathbb{R}\big)_{\mathrm{flat}} \;\simeq\; \Omega^{2r+1}_{\mathrm{dR}}(X)_{\mathrm{closed}}\,. \tag{176}
$$

(ii) The flat $\tau_{\mathrm{dR}} \sim H_{2r+1}$-twisted $\underset{k\in\mathbb{N}}{\oplus} \mathfrak{b}^{2rk}\mathbb{R}$-valued differential forms according to Def. 3.90 are equivalently sequences of differential forms $F_{2r\bullet+1} \in \Omega^{2k\bullet+1}_{\mathrm{dR}}(X)$ satisfying the $H_{(2r+1)}$-twisted de Rham closure condition (186):

$$\Omega^{\tau_{\mathrm{dR}}}\big(X; \underset{k\in\mathbb{N}}{\oplus} \mathfrak{b}^{2rk}\mathbb{R}\big)_{\mathrm{flat}} \;\simeq\; \left\{ F_{2r\bullet+1} \in \Omega^{2r\bullet+1}_{\mathrm{dR}} \,\middle|\, d\sum_k F_{2rk+1} \;=\; H_{2r+1} \wedge \sum_k F_{2rk-1} \right\} \tag{177}$$

(where we set $F_{2rk-1} := 0$ if $2rk-1 < 0$, for convenience of notation).

In twisted generalization of Example 3.79, we have the following:

Example 3.94 (Flat twisted differential forms with values in Whitehead L_∞-algebras of spheres and twistor space)**.** The L_∞-algebraic local coefficient bundles (Def. 3.89) given as the relative Whitehead L_∞-algebras (Prop. 3.68) of the local coefficient bundles (60) for twisted and twistorial Cohomotopy (Example 2.42) are as shown on the right of the following diagram [FSS19b, Lemma 3.19][FSS20, Thm. 2.14]:

$$\begin{tikzcd}
 & \mathrm{CE}\big(\mathfrak{l}_{(B\mathrm{Sp}(2))}(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2))\big) \arrow[dl, dashed, "{(G_4, G_7, F_2, H_3)}"'] & = & \mathrm{CE}(\mathfrak{l}B\mathrm{Sp}(2)) \begin{bmatrix} h_3, \\ f_2, \\ \omega_7, \\ \omega_4 \end{bmatrix} \Big/ \begin{pmatrix} d\,h_3 = \omega_4 - \tfrac{1}{4}p_1 - f_2 \wedge f_2 \\ d\,f_2 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + (\tfrac{1}{2}p_1)^2 - \chi_8 \\ d\,\omega_4 = 0 \end{pmatrix} \\
\Omega^\bullet_{\mathrm{dR}}(X) & \mathrm{CE}\big(\mathfrak{l}_{(B\mathrm{Sp}(2))}(S^4 /\!\!/ \mathrm{Sp}(2))\big) \arrow[l, dashed, "{(G_4, 2G_7)}"'] \arrow[u, "(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))^*"'] & = & \mathrm{CE}(\mathfrak{l}B\mathrm{Sp}(2)) \begin{bmatrix} \omega_7, \\ \omega_4 \end{bmatrix} \Big/ \begin{pmatrix} d\,\omega_7 = -\omega_4 \wedge \omega_4 + (\tfrac{1}{4}p_1)^2 - \chi_8 \\ d\,\omega_4 = 0 \end{pmatrix} \arrow[u, hook] \\
 & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[ul, "\tau_{\mathrm{dR}}"] \arrow[u] & = & \mathbb{R}\begin{bmatrix} \chi_8, \\ \tfrac{1}{2}p_1 \end{bmatrix} \Big/ \begin{pmatrix} d\;\; \chi_8 = 0 \\ d\,\tfrac{1}{2}p_1 = 0 \end{pmatrix} \arrow[u, hook]
\end{tikzcd}$$

Therefore, given a smooth 8-dimensional spin-manifold X equipped with tangential $\mathrm{Sp}(2)$-structure τ (58), the flat τ_{dR}-twisted $\mathfrak{l}S^4$- and $\mathfrak{l}\mathbb{C}P^3$-valued differential forms (Def. 3.90) are of the following form [FSS19b, Prop. 3.20][FSS20, Prop. 3.9]:

$$\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}S^4\big) = \left\{ \begin{matrix} 2G_7, \\ G_4 \end{matrix} \in \Omega^\bullet_{\mathrm{dR}}(X) \,\middle|\, \begin{array}{l} d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \chi_8(\nabla), \\ d\;\; G_4 = 0 \end{array} \right\}$$

$$\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}\mathbb{C}P^3\big) = \left\{ \begin{matrix} H_3, \\ F_2, \\ 2G_7, \\ G_4 \end{matrix} \in \Omega^\bullet_{\mathrm{dR}}(X) \,\middle|\, \begin{array}{l} d\,H_3 = G_4 - \tfrac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ d\,F_2 = 0, \\ d\,2G_7 = -\big(G_4 - \tfrac{1}{4}\big) \wedge \big(G_4 + \tfrac{1}{4}\big) - \chi_8(\nabla), \\ d\;\; G_4 = 0, \end{array} \right\} \tag{178}$$

Here we are using (Example 4.27) that the de Rham image τ_{dR} of the rationalization $L_{\mathbb{R}}\tau$ of the twist τ is given by evaluating characteristic forms (Def. 4.19) on any $\mathrm{Sp}(2)$-connection ∇.

Twisted non-abelian de Rham cohomology. In generalization of Def. 3.81, we set:

Definition 3.95 (Coboundaries between flat twisted $L\infty$-algebraic forms)**.** Let $X \in \mathrm{SmoothManifolds}$, let $\mathfrak{g} \to \widehat{\mathfrak{b}} \xrightarrow{\mathfrak{p}} \mathfrak{b}$ be a local L_∞-algebraic coefficient bundle (Def. 3.89), and let $\tau_{\mathrm{dR}} \in \Omega_{\mathrm{dR}}(X;\mathfrak{b})$. Then for

$$A^{(0)}, A^{(1)} \;\in\; \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{g})$$

a pair of flat τ_{dR}-twisted $\mathfrak{g}$-valued differential forms on X (Def. 3.90) a *coboundary* between them is a coboundary

$$\widetilde{A} \;\in\; \Omega_{\mathrm{dR}}\big(X\times\mathbb{R};\widehat{\mathfrak{b}}\big) \tag{179}$$

in the sense of Def. 3.81 between the underlying flat $\widehat{\mathfrak{b}}$-valued forms (via Remark 3.91), such that the underling $\mathfrak{b}$-valued form of H equals the pullback of the twist τ_{dR} along $X\times\mathbb{R}\xrightarrow{\mathrm{pr}_X}X$

$$\mathfrak{p}_*(H) \;=\; \mathrm{pr}_X^*(\tau_{\mathrm{dR}})\,. \tag{180}$$

If such a coboundary exists, we say that $A^{(0)}$ and $A^{(1)}$ are *cohomologous*, to be denoted

$$A^{(0)} \;\sim\; A^{(1)}\,.$$

In generalization of Def. 3.85, we set:

Definition 3.96 (Twisted non-abelian de Rham cohomology)**.** Let $X \in \mathrm{SmoothManifolds}$, let $\mathfrak{g} \to \widehat{\mathfrak{b}} \xrightarrow{\mathfrak{p}} \mathfrak{b}$ be a local L_∞-algebraic coefficient bundle (Def. 3.89), and let $\tau_{\mathrm{dR}} \in \Omega_{\mathrm{dR}}\big(X;\mathfrak{b}\big)$. Then the τ_{dR} *-twisted non-abelian de Rham cohomology* of X with coefficients in $\mathfrak{g}$ is the set

$$H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{g}) \;:=\; \big(\Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{g})_{\mathrm{flat}}\big)_{/\sim} \tag{181}$$

of equivalence classes with respect to the coboundary relation from Def. 3.95 on the set of flat τ_{dR}-twisted $\mathfrak{g}$-valued differential forms on X (Def. 3.90).

Twisted de Rham cohomology is traditionally familiar in the form of degree-3 twisted cohomology of even/odd degree differential forms [RW86, §III, Appendix][BCMMS02, §9.3][MaS03, §3][FrHT08, §2][Te04, Prop. 3.7][Cav05, §I.4][Sa10][MW11][GS19b] (which is the target of the twisted Chern character in degree-3 twisted K-theory, see Prop. 5.5):

Definition 3.97 (Degree-3 twisted abelian de Rham cohomology)**.** For $X \in \mathrm{SmoothManifolds}$, and $H_3 \in \Omega^3_{\mathrm{dR}}(X)_{\mathrm{closed}}$ a closed differential 3-form, the H_3*-twisted de Rham cohomology* of X is the cochain cohomology [10]

$$H^{\bullet+H_3}_{\mathrm{dR}}(X) \;:=\; \frac{\ker^\bullet\big(d - H_3\wedge(-)\big)}{\mathrm{im}^\bullet\big(d-H_3\wedge(-)\big)} \tag{182}$$

of the following 2-periodic cochain complex:

$$\cdots \longrightarrow \bigoplus_k \Omega^{(n-1)+2k}_{\mathrm{dR}}(X) \xrightarrow{(d-H_3\wedge(-))} \bigoplus_k \Omega^{n+2k}_{\mathrm{dR}}(X) \xrightarrow{(d-H_3\wedge(-))} \bigoplus_k \Omega^{(n+1)+2k}_{\mathrm{dR}}(X) \longrightarrow \cdots\,.$$

We show that this is a special case of twisted non-abelian de Rham cohomology according to Def. 3.96:

Proposition 3.98 (Twisted non-abelian de Rham cohomology subsumes H_3-twisted abelian de Rham cohomology)**.** *Given a twisting 3-form as in* (173)

$$\begin{array}{ccc} \tau_{\mathrm{dR}} & \longleftrightarrow & H_3 \\ \pitchfork & & \pitchfork \\ \Omega\big(X;\mathfrak{b}^2\mathbb{R}\big)_{\mathrm{flat}} & \simeq & \Omega^3(X)_{\mathrm{closed}} \end{array}$$

the τ_{dR}-twisted non-abelian de Rham cohomology (Def. 3.96) of flat twisted $\mathfrak{l}\mathrm{ku}_1$-valued differential forms (Example 3.92) is naturally equivalent to H_3-twisted abelian de Rham cohomology (Def. 3.97) in odd degree[11]

$$\underbrace{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{l}\mathrm{ku}_1)}_{\substack{\mathfrak{b}^2\mathbb{R}\text{-twisted }\mathfrak{l}\mathrm{ku}_1\text{-valued} \\ \text{non-abelian de Rham cohomology}}} \;\simeq\; \underbrace{H^{1+H_3}_{\mathrm{dR}}(X)}_{\substack{\text{traditional }H_3\text{-twisted} \\ \text{de Rham cohomology}}}$$

[10] The notation "H_3" for the twist (and of "H_{2r+1}" for the higher twists later) originates in the physics literature and has made it as a convention in differential geometry as well. Of course, not to be confused with homology.

[11] The discussion for even degrees is directly analogous and we omit it for brevity.

Proof. By (174) in Example 3.92 the cocycle sets on both sides are in natural bijection. Hence it is sufficient to see that the coboundary relations on the cocycle sets coincide, under this identification. In one direction, consider a coboundary in the sense of twisted non-abelian de Rham cohomology (Def. 3.95) with coefficients as in Example 3.92:

$$\widetilde{F}_{2\bullet+1} \;\in\; \Omega_{\mathrm{dR}}\big(X \times \mathbb{R}; \mathfrak{l}\mathrm{ku}_1\big).$$

We claim that

$$h_{2\bullet} \;:=\; \int\limits_{[0,1]} \widetilde{F}_{2\bullet+1} \tag{183}$$

satisfies the coboundary condition (182):

$$\big(d - H_3 \wedge\big)\sum_k h_{2k} \;=\; \sum_k \big(F^{(1)}_{2k+1} - F^{(0)}_{2k+1}\big)\,. \tag{184}$$

To see this, we may compute as follows:

$$\begin{aligned}
d\sum_k h_{2k} &= \sum_k \left(F^{(1)}_{2k+1} - F^{(0)}_{2k+1} - \int\limits_{[0,1]} d\widetilde{F}_{2k+1}\right)\\
&= \sum_k \left(F^{(1)}_{2k+1} - F^{(0)}_{2k+1} - \int\limits_{[0,1]} \big(\mathrm{pr}_X^* H_3\big) \wedge \widetilde{F}_{2k-1}\right)\\
&= \sum_k \left(F^{(1)}_{2k+1} - F^{(0)}_{2k+1} + H_3 \wedge \int\limits_{[0,1]} \widetilde{F}_{2k-1}\right)\\
&= \sum_k \left(F^{(1)}_{2k+1} - F^{(0)}_{2k+1} + H_3 \wedge h_{2k-2}\right),
\end{aligned}$$

where the first step is the fiberwise Stokes formula (155) together with the defining restrictions (153) of $\widetilde{F}_{2\bullet+1}$; the second step is the cocycle condition (174) on $\widetilde{F}_{2\bullet+1}$ using the constraint (180); the third step is the projection formula (156); and the last step uses again the definition (183).

Conversely, given $h_{2\bullet}$ satisfying (184), we claim that

$$\widetilde{F}_{2\bullet+1} \;:=\; (1-t)\,\mathrm{pr}_1^*\big(F^{(0)}_{2\bullet+1}\big) + t\,\mathrm{pr}_1^*\big(F^{(1)}_{2\bullet+1}\big) + dt \wedge \mathrm{pr}_X^*(h_{2\bullet}) \tag{185}$$

is a coboundary of twisted non-abelian cocycles, in the sense of Def. 3.95: It is immediate that (185) has the required restrictions (153). We check by direct computation that it satisfies the required differential equation:

$$\begin{aligned}
d\sum_k \widetilde{F}_{2k+1} = \sum_k \Big(& - dt \wedge \mathrm{pr}_X^*\big(F^{(0)}_{2k+1}\big) + (1-t)\,\mathrm{pr}_X^*\big(H_3\big) \wedge \mathrm{pr}_X^*\big(F^{(0)}_{2k-1}\big)\\
& + dt \wedge \mathrm{pr}_X^*\big(F^{(1)}_{2k+1}\big) + t\,\mathrm{pr}_X^*\big(H_3\big) \wedge \mathrm{pr}_X^*\big(F^{(1)}_{2k-1}\big)\\
& - dt \wedge \mathrm{pr}_X^*\big(\underbrace{d\,h_{2k}}_{=F^{(1)}_{2k+1} - F^{(0)}_{2k+1} + H_3 \wedge h_{2k}}\big)\ \Big)\\
= \sum_k \Big(& \mathrm{pr}_X^*(H_3) \wedge \widetilde{F}_{2k-1}\Big)\,.
\end{aligned}$$

□

In generalization of Def. 3.97, there are twisted abelian Rham complexes with twist any odd-degree closed form [Te04][Sa09][MW11][Sa10][GS19b] (these serve as the targets[12] of the LSW-character on twisted iterated K-theories [LSW16, §2.1]; see Prop. 5.8 below).

[12] It has been been argued in [MMS20] that higher twisted de Rham cohomology is also useful for analyzing higher twists on ordinary K-theory (e.g. [Te04][Go08][DP13]).

Definition 3.99 (Higher twisted abelian de Rham cohomology)**.** For $X \in \mathrm{SmoothManifolds}$, $r \in \mathbb{N}$, $r \geq 1$, and $H_{2r+1} \in \Omega^{2r+1}_{\mathrm{dR}}(X)_{\mathrm{closed}}$ a closed differential $(2r+1)$-form, the H_{2r+1}*-twisted de Rham cohomology* of X is the cochain cohomology

$$\Omega^{\bullet + H_{2r+1}}_{\mathrm{dR}}(X) \;:=\; \frac{\ker^\bullet\big(d - H_{2r+1} \wedge (-)\big)}{\mathrm{im}^\bullet\big(d - H_{2r+1} \wedge (-)\big)} \tag{186}$$

of the following $2r$-periodic cochain complex:

$$\cdots \longrightarrow \bigoplus_k \Omega^{(n-1)+2rk}_{\mathrm{dR}}(X) \xrightarrow{(d - H_{2r+1} \wedge (-))} \bigoplus_k \Omega^{n+2rk}_{\mathrm{dR}}(X) \xrightarrow{(d - H_{2r+1} \wedge (-))} \bigoplus_k \Omega^{(n+1)+2rk}_{\mathrm{dR}}(X) \longrightarrow \cdots .$$

In direct generalization of Prop. 3.98, we find:

Proposition 3.100 (Twisted non-abelian de Rham cohomology subsumes higher twisted abelian de Rham cohomology)**.** *For $r \in \mathbb{N}$, $r \geq 1$, consider a twisting $(2r+1)$-form as in* (176)

$$\begin{array}{ccc} \tau_{\mathrm{dR}} & \longleftrightarrow & H_{2r+1} \\ \pitchfork & & \pitchfork \\ \Omega\big(X; \mathfrak{b}^{2r}\mathbb{R}\big)_{\mathrm{flat}} & \simeq & \Omega^{2r+1}\big(X\big)_{\mathrm{closed}} \end{array}$$

The τ_{dR}-twisted non-abelian de Rham cohomology (Def. 3.96) of flat twisted $\mathfrak{l}K^{2r-2}(\mathrm{ku})_1$-valued differential forms (Example 3.93) is naturally equivalent to H_{2r+1}-twisted abelian de Rham cohomology (Def. 3.99) in degree[13] *$1 \bmod 2r$.*

$$\underset{\text{non-abelian de Rham cohomology}}{\overset{\text{twisted}}{}} H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(X; \underset{k \in \mathbb{N}}{\oplus} \mathfrak{b}^{2rk}\mathbb{R}\Big) \;\simeq\; \underset{\text{de Rham cohomology}}{\overset{\text{higher } H_{2r+1}\text{-twisted}}{}} H^{1+H_{2r+1}}_{\mathrm{dR}}(X).$$

Proof. By Example 3.93, the cocycle sets on both sides are in natural bijection. Hence it remains to see that the coboundary relations correspond to each other, under this identification. This proceeds verbatim, up to degree shifts, as in the proof of Prop. 3.98 (which is the special case of $r = 1$). $\square$

Example 3.101 (Cohomology operation in (higher-) twisted de Rham cohomology)**.** Degree-3 twisted de Rham cohomology (Def. 3.97) supports the following twisted cohomology operations (Def. 2.40):

(i) *wedge product with* H_3:

$$\begin{array}{ccc} H^{\bullet + H_3}_{\mathrm{dR}}(X) & \longrightarrow & H^{\bullet + 3 + H_3}_{\mathrm{dR}}(X) \\ \sum\limits_k F_k & \longmapsto & \sum\limits_k F_k \wedge H_3 \end{array}$$

(ii) *wedge square*:

$$\begin{array}{ccc} \bigoplus\limits_r H^{2r + H_3}_{\mathrm{dR}}(X) & \longrightarrow & \bigoplus\limits_r H^{2r + 2H_3}_{\mathrm{dR}}(X) \\ \sum\limits_k F_k & \longmapsto & \Big(\sum\limits_k F_k\Big) \wedge \Big(\sum\limits_k F_k\Big) \end{array}$$

(iii) *compositions of these*:

$$\begin{array}{ccc} \bigoplus\limits_r H^{2r + H_3}_{\mathrm{dR}}(X) & \longrightarrow & \bigoplus\limits_r H^{2r + 1 + 2H_3}_{\mathrm{dR}}(X) \\ \sum\limits_k F_k & \longmapsto & \Big(\sum\limits_k F_k\Big) \wedge \Big(\sum\limits_k F_k\Big) \wedge H_3 \end{array}$$

In type IIA string theory, terms of the form **(iii)** arise, together with terms of the form $I_8 \cup [H_3]$ with I_8 a polynomial in the Pontrjagin classes (cf. Example 4.27). See [GS19c] for extensive discussions.

This evidently generalizes to higher twisted de Rham cohomology (Def. 3.99) and higher twisted real cohomology in the sense of [GS19b], with H_3 replaced by H_{2r+1} for $r \in \mathbb{N}$.

[13] The discussion for other degrees is directly analogous, and we omit it for brevity.

The twisted non-abelian de Rham theorem.

Theorem 3.102 (Twisted non-abelian de Rham theorem)**.** *Let* $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50), equipped with the structure of a smooth manifold, and let*

$$\begin{tikzcd} A \arrow[r] & A /\!\!/ G \arrow[d, "\rho", "\text{local coefficient bundle}"'] \\ & BG \end{tikzcd} \tag{187}$$

be a local coefficient bundle (35) *in* $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50) such that the action of* $\pi_1(BG) = \pi_0(G)$ *on the real homology groups of* A *is nilpotent. Consider, via Prop. 3.73, the rationalized coefficient bundle* $L_{\mathbb{R}}\rho$ *with corresponding* L_∞*-algebraic coefficient bundle* $\mathfrak{l}\rho$ *(Def. 3.89) of the relative real Whitehead* L_∞*-algebra (Prop. 3.68):*

$$\begin{tikzcd} L_{\mathbb{R}}A \arrow[r] & (L_{\mathbb{R}}A)/\!\!/(L_{\mathbb{R}}G) \arrow[d, "L_{\mathbb{R}}\rho", "\substack{\text{rationalized}\\ \text{local coefficient bundle}}"'] \\ & L_{\mathbb{R}}BG \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{l}A \arrow[r] & \mathfrak{l}_{BG}(A/\!\!/G) \arrow[d, "\mathfrak{l}\rho", "\substack{L_\infty\text{-algebraic coefficient bundle}\\ \text{of Whitehead } L_\infty\text{-algebras}}"'] \\ & \mathfrak{l}BG \end{tikzcd} .$$

Moreover, let

$$X \xrightarrow{\ \tau\ } L_{\mathbb{R}}BG \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$$

be such that $H^1(\tau;\mathbb{R})$ *is injective (for instance in that* BG *is simply connected).*

Then the τ*-twisted non-abelian real cohomology (Def. 3.74) of* X *with local coefficients in* $L_{\mathbb{R}}\rho$ *(Prop. 3.73) is in natural bijection with the twisted non-abelian de Rham cohomology (Def. 3.96) of* X *with local coefficients in* $\mathfrak{l}\rho$*,*

$$\underset{\substack{\tau\text{-twisted non-abelian}\\ \text{real cohomology}}}{H^{\tau}\big(X; L_{\mathbb{R}}A\big)} \;\simeq\; \underset{\substack{\tau_{\mathrm{dR}}\text{-twisted non-abelian}\\ \text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}A\big)}\,, \tag{188}$$

where the twists are related by the plain non-abelian de Rham theorem (Theorem 3.85):

$$\begin{array}{ccc} [\tau] & \longleftrightarrow & [\tau_{\mathrm{dR}}] \\ \cap & & \cap \\ H\big(X; L_{\mathbb{R}}BG\big) & \simeq & H_{\mathrm{dR}}\big(X; \mathfrak{l}BG\big) \end{array}$$

Proof. Consider the following sequence of natural bijections

$$\begin{aligned} H^{\tau}\big(X; L_{\mathbb{R}}A\big) &= \mathrm{Ho}\Big(\big(\mathrm{TopologicalSpaces}^{/L_{\mathbb{R}}BG}_{\mathrm{Qu}}\big)\Big)\big(\tau\,, L_{\mathbb{R}}\rho\big) \\ &\simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\Omega^\bullet_{\mathrm{PLdR}}(BG)/}_{\mathrm{proj}}\Big)\big(\Omega^\bullet_{\mathrm{PLdR}}(\rho)\,, \Omega^\bullet_{\mathrm{PLdR}}(\tau)\big) \\ &\simeq \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}\Big)\big(\mathrm{CE}(\mathfrak{l}\rho)\,, \tau^*_{\mathrm{dR}}\big) \\ &\simeq H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}A\big)\,. \end{aligned}$$

Here the first line is the definition (Def. 3.74). Then the first step is the fundamental theorem (Prop. 3.58) in the co-slice category. The substitutions in the second step are:

(a) Lemma 3.106 in the first argument (this is where the H^1-injectivity is needed);

(b) Lemma 3.88 with Theorem 3.85 in the second argument (as in the second step of (161)).

The last step is Lemma 3.104. The composite of these equivalences is the desired (188). ☐

We now establish the remaining four lemmas which enter the proof of Theorem 3.102.

Lemma 3.103 (Pullback to de Rham complex over cylinder of manifold is relative path space object)**.** *Let* $X \in \mathrm{SmoothManifolds}$*, let* $\mathfrak{b} \in L_\infty\mathrm{Algebras}^{\geq 0}_{\mathbb{R},\mathrm{fin}}$ *(Example 3.24) with Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{b}) \in \mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$ (86)*, and let* $\big\{\Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\ \tau^*_{\mathrm{dR}}\ } \mathrm{CE}(\mathfrak{b})\big\} \in \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}$ *be a morphism of dgc-algebras to the de Rham complex of* X *(Example 3.23), regarded as an object in the coslice model category (Example A.10) of* $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ *(Prop. 3.36) under* $\mathrm{CE}(\mathfrak{b})$*. Then a path space object (Def. A.11) for* τ^*_{dR} *is given by this diagram:*

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) \arrow[rr, "\mathrm{pr}_X^* \in \mathrm{W}"] & & \Omega^\bullet_{\mathrm{dR}}(X \times \mathbb{R}) \arrow[rr, "(i_0^*, i_1^*) \in \mathrm{Fib}"] & & \Omega^\bullet_{\mathrm{dR}}(X) \oplus \Omega^\bullet_{\mathrm{dR}}(X) \\
& & \mathrm{CE}(\mathfrak{b})\,, \arrow[ull, "\tau^*_{\mathrm{dR}}"] \arrow[u, "\mathrm{pr}_X^* \circ \tau^*_{\mathrm{dR}}"] \arrow[urr, "(\tau^*_{\mathrm{dR}}, \tau^*_{\mathrm{dR}})"']
\end{tikzcd}
$$

where the top morphisms are from (162).

Proof. It is clear that the diagram commutes, by construction. Moreover, the top morphisms are a weak equivalence followed by a fibration in $\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}}$, by Lemma 3.86. Therefore, by the nature of the coslice model structure (Example A.10) the total diagram constitutes a factorization of the diagonal on τ^*_{dR} through a weak equivalence followed by a fibration, as required (300). (To see that the composite really is still the diagonal morphism in the coslice, observe that Cartesian products in any coslice category are reflected in the underlying category.) It only remains to observe that τ^*_{dR} is actually a fibrant object in the coslice model category. But the terminal object in the coslice is clearly the unique morphism from $\mathrm{CE}(\mathfrak{b})$ to the zero-algebra (Example 3.21), so that in fact every object in the coslice is still fibrant

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) \arrow[rr, "\in \mathrm{Fib}"] & & 0 \\
& \mathrm{CE}(\mathfrak{b}) \arrow[ul, "\tau^*_{\mathrm{dR}}"] \arrow[ur]
\end{tikzcd}
\tag{189}
$$

as in Remark 3.37. ☐

Lemma 3.104 (Twisted non-abelian de Rham cohomology via the coslice dgc-homotopy category)**.** *Consider* $X \in \mathrm{SmoothManifolds}$*, let*

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r] & \widehat{\mathfrak{b}} \arrow[d, "\mathfrak{p}"] \\
& \mathfrak{b}
\end{tikzcd}
\quad \in L_\infty\mathrm{Algebras}^{\geq 0,\mathrm{nil}}_{\mathbb{R},\mathrm{fin}}
$$

be an L_∞-algebraic local coefficient bundle (Def. 3.89) of nilpotent L_∞-algebras (Def. 3.34) with Chevalley-Eilenberg algebra $\mathrm{CE}(\widehat{\mathfrak{b}})$, $\mathrm{CE}(\mathfrak{b}) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ (86)*, and let*

$$
\Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\;\tau^*_{\mathrm{dR}}\;} \mathrm{CE}(\mathfrak{b}) \quad \in \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}
\tag{190}
$$

be a morphism of dgc-algebras to the de Rham complex of X (Example 3.23), hence a flat $\mathfrak{b}$-valued differential form (Def. 3.75)

$$
\tau_{\mathrm{dR}} \in \Omega_{\mathrm{dR}}(X;\mathfrak{b})\,,
$$

equivalently regarded as an object in the coslice model category (Example A.10) of $\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)_{\mathrm{proj}}$ *(Prop. 3.36) under* $\mathrm{CE}(\mathfrak{b})$*. Then the τ_{dR}-twisted non-abelian de Rham cohomology of X with coefficients in $\mathfrak{g}$ (Def. 3.96) is in natural bijection with the hom-set in the homotopy category (Def. A.14) of the coslice model category* $\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{CE}(\mathfrak{b})}_{\mathrm{proj}}$ *(Example A.10) of the projective model structure on dgc-algebras (Prop. 3.36) from* $\mathrm{CE}(\mathfrak{p})$ (169) *to* τ^*_{dR} (190)*:*

$$
H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{g}) \;\simeq\; \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}\Big)\big(\mathrm{CE}(\mathfrak{p})\,,\,\tau^*_{\mathrm{dR}}\big).
\tag{191}
$$

Proof. Consider a pair of dgc-algebra homomorphisms in the coslice

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \mathrm{CE}(\widetilde{\mathfrak{b}}) \arrow[ll, bend right, "A^{(0)}"'] \arrow[ll, bend left, "A^{(1)}"] \\
& \mathrm{CE}(\mathfrak{b}) \arrow[ul, "\tau^*_{\mathrm{dR}}"] \arrow[ur, "\mathrm{CE}(\mathfrak{p})"']
\end{tikzcd}
\quad \in \big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}\big(\mathrm{CE}(\mathfrak{p})\,,\,\tau^*_{\mathrm{dR}}\big)\,,
\tag{192}
$$

hence of flat τ_{dR}-twisted $\mathfrak{g}$-valued differential forms, according to Def. 3.90. Observe that:

(i) $\mathrm{CE}(\mathfrak{p})$ is cofibrant in $\big(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big)^{\mathrm{CE}(\mathfrak{p})/}_{\mathrm{proj}}$, since:

- **(a)** the initial object in the coslice is $\mathrm{CE}(\mathfrak{b}) \xleftarrow{\mathrm{id}} \mathrm{CE}(\mathfrak{b})$,
- **(b)** the unique morphism from this object to $\mathrm{CE}(\mathfrak{p})$ is

$$
\begin{tikzcd}
\mathrm{CE}(\mathfrak{b}) \arrow[rr, "\mathrm{CE}(\mathfrak{p}) \in \mathrm{Cof}"] & & \mathrm{CE}\big(\widehat{\mathfrak{b}}\big) \\
& \mathrm{CE}(\mathfrak{b}) \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{CE}(\mathfrak{p})"'] &
\end{tikzcd}
\tag{193}
$$

(c) $\mathrm{CE}(\mathfrak{p})$ is a cofibration in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$, by (169), so that the diagram (193) is a cofibration in the coslice model category, by Example A.10.

(ii) $\mathrm{pr}_X^* \circ \tau_{\mathrm{dR}}^*$ is fibrant in $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\mathfrak{b})/}_{\mathrm{proj}}$, by (189);

(iii) A right homotopy (Def. A.12) between the pair (192) of coslice morphisms, with respect to the path space object from Lemma 3.103, namely a $\widetilde{A}$ that makes the following diagram commute

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dR}}(X) & & \\
\Omega^\bullet_{\mathrm{dR}}(X \times \mathbb{R}) \arrow[u, "i_0^*"] \arrow[d, "i_1^*"'] & & \mathrm{CE}\big(\widehat{\mathfrak{b}}\big) \arrow[ll, "\widetilde{A}"] \arrow[ull, "A^{(0)}"'] \arrow[dll, "A^{(1)}"] \\
\Omega^\bullet_{\mathrm{dR}}(X) & \mathrm{CE}\big(\widehat{\mathfrak{b}}\big) \arrow[ul, "\mathrm{pr}_X^*,\, \tau_{\mathrm{dR}}^*"] \arrow[ur, "\mathrm{CE}(\mathfrak{p})"'] &
\end{tikzcd}
\tag{194}
$$

is manifestly the same as a coboundary $\widetilde{A}$ between the corresponding flat twisted $\mathfrak{g}$-valued forms according to Def. 3.95:

(a) The top part of (194) is, just as in (165), the flat twisted $\widehat{\mathfrak{g}}$-valued form on the cylinder over X that is required by (179);

(b) the bottom part of (194) is the condition (180) on the extension of the twist to the cylinder over X.

Therefore, Prop. A.16 says that the quotient set (181) defining the twisted non-abelian de Rham cohomology is in natural bijection to the hom-set in the coslice homotopy category. □

Lemma 3.105 (Derived cobase change along quasi-isomorphism is equivalence on H^1-injectives)**.** *Let*

$$
B_1 \xrightarrow{\ \phi \in \mathrm{W}\ } B_2 \qquad \in \big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}
$$

be a quasi-isomorphism of dgc-algebras (Def. 3.17), hence a weak equivalence in the projective model structure (Prop. 3.36). Assume that either, hence both, dgc-algebras are cohomologically connected ($H^0(B_1) = \mathbb{R}$, $H^0(B_2) = \mathbb{R}$). Then the derived adjunction (Prop. A.20) of the base change Quillen adjunction (Example A.18) between the corresponding co-slice model categories (Example A.10) of the opposite model category of dgc-algebras (Example A.9) restricts to an equivalence on the full subcategories of the homotopy categories (Def. A.14) on those co-slice objects which are connected in $H^0(-)$ and injective on $H^1(-)$:

$$
\mathrm{Ho}\bigg(\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{B_2/}_{\mathrm{proj}}\Big)^{\mathrm{op}}\bigg)_{\substack{H^0-\mathrm{conn} \\ H^1-\mathrm{inj}}}
\underset{\mathbb{R}(\phi^{\mathrm{op}})^*}{\overset{\mathbb{L}(\phi^{\mathrm{op}})_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}}
\mathrm{Ho}\bigg(\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{B_1/}_{\mathrm{proj}}\Big)^{\mathrm{op}}\bigg)_{\substack{H^0-\mathrm{conn} \\ H^1-\mathrm{inj}}}
$$

Proof. Notice that if $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ were a left proper model category (Def. A.5), so that $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{op}}_{\mathrm{proj}}$ were right proper, the statement would directly follow as a special case of Prop. A.31, without any restriction to subcategories.

While $\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}$ is (apparently) not left proper, it comes close: Lemma 3.44 says that quasi-isomorphisms are preserved by pushout along at least those cofibrations that are relative Sullivan algebras (i.e. the relative cell complexes, but possibly not their retracts). Hence we adapt the logic underlying Prop. A.31 to this case. Namely, Prop. 3.47 says that those co-slice objects that are H^1-injective between H^0-connected algebras do have a cofibrant replacement by a relative Sullivan algebra:

$$
\begin{tikzcd}
A_{\mathrm{min}_B} \arrow[rr, "\in \mathrm{W}"] & & A \\
& B \arrow[ul, hook, "{\in \mathrm{RelSulAlg} \atop \subset \mathrm{Cof}}"] \arrow[ur, "\text{coslice object}"'] &
\end{tikzcd}
\tag{195}
$$

Now, first to see that the derived adjunction restricts to the given subcategories: In one direction, it is clear that $\mathbb{L}(\phi^{\mathrm{op}})_!$ preserves H^0 and H^1, as this functor is given by precomposition with the quasi-isomorphism ϕ. In the other direction: $\mathbb{R}(\phi^{\mathrm{op}})^*$ is given by pushout along ϕ of a cofibrant representative of the given coslice object, and by (195) we may take that cofibrant representative to be a relative Sullivan algebra. But then Prop. 3.44 implies that the pushout has the same cohomology.

Finally, to see that this restriction of the derived adjunction is an equivalence of categories, hence that the derived unit (311) and derived counit (312) are isomorphisms on these subcategories. This follows just as in the alternative proof (322) of Prop. A.31, using for the fibrant objects ρ there the opposites of the good fibrations given by (195), for which Prop. 3.44 guarantees the required properness condition. □

Lemma 3.106 (Pasting composition with relative Sullivan model of local coefficient bundle)**.** *Let*

$$
\begin{tikzcd}
A \arrow[r] & A /\!\!/ G \arrow[d, "\rho"] \\
\text{local coefficient bundle} & BG
\end{tikzcd}
\tag{196}
$$

be a local coefficient bundle (35) *in* $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50), and let*

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{PLdR}}\big(A /\!\!/ G\big) & & \mathrm{CE}\big(\mathfrak{l}(A /\!\!/ G)\big) \arrow[ll, "p^{\mathrm{min}_{BG}}_{A/\!\!/G}"', "\in \mathrm{W}"] \\
\Omega^\bullet_{\mathrm{PLdR}}\big(BG\big) \arrow[u, "\Omega^\bullet_{\mathrm{PLdR}}(\rho)"] & & \mathrm{CE}\big(\mathfrak{l}(BG)\big) \arrow[ll, "p^{\mathrm{min}}_{BG}"', "\in \mathrm{W}"] \arrow[u, "\mathrm{CE}(\mathfrak{l}\rho)"']
\end{tikzcd}
\tag{197}
$$

be its minimal relative Sullivan model (139)*, which exists by Prop. 3.68. Then the pasting precomposition with the square* (197) *is a natural isomorphism of hom-functors on the homotopy categories from Lemma 3.105:*

$$
\begin{tikzcd}
\mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\Omega^\bullet_{\mathrm{PLdR}}(BG)/}_{\mathrm{proj}}\Big)_{ {H^0-\mathrm{conn} \atop H^1-\mathrm{inj}} }\big(\Omega^\bullet_{\mathrm{PLdR}}(\rho)\,,\,-\big) \arrow[d, "\big(\mathbb{L}\big((p^{\mathrm{min}}_{BG})^{\mathrm{op}}\big)_!\big)^{\mathrm{op}}"', "\simeq"] \\
\mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\mathfrak{l}BG)/}_{\mathrm{proj}}\Big)_{ {H^0-\mathrm{conn} \atop H^1-\mathrm{inj}} }\big(\Omega^\bullet_{\mathrm{PLdR}}(\rho) \circ p^{\mathrm{min}}_{BG}\,,\,-\big) \arrow[d, "(-)\circ p^{\mathrm{min}_{BG}}_{A/\!\!/G}"', "\simeq"] \\
\mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)^{\mathrm{CE}(\mathfrak{l}BG)/}_{\mathrm{proj}}\Big)_{ {H^0-\mathrm{conn} \atop H^1-\mathrm{inj}} }\big(\mathrm{CE}(\mathfrak{l}\rho)\,,\,-\big)
\end{tikzcd}
\tag{198}
$$

Here the first step is the derived left co-base change along ϕ (Example A.18), while the second is composition with the diagram 197 regarded as a morphism in the co-slice under $\mathrm{CE}(\mathfrak{l}BG)$.

Proof. First notice that $\Omega^\bullet_{\mathrm{PLdR}}(\rho)$ is indeed an injection on H^1, by the assumption that the fiber A is connected (as in the proof of Prop. 3.68). With that, the first step is an isomorphism by Lemma 3.105, while the second step is evidently an isomorphism, since the weak equivalence $\phi_{A/\!\!/G}$ becomes an isomorphism in the homotopy category (and still so in the coslice homotopy category, by Example A.10). □

4 The (differential) non-abelian character map

We introduce the character map in non-abelian cohomology (Def. 4.2) and then discuss how it specializes to:
§4.1 – the Chern-Dold character on generalized cohomology;
§4.2 – the Chern-Weil homomorphism on degree-1 non-abelian cohomology;
§4.3 – the Cheeger-Simons differential characters on degree-1 non-abelian cohomology.

Definition 4.1 (Rationalization in non-abelian cohomology)**.** For $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) we write

$$(\eta^{\mathbb{R}}_A)_* \;:\; \overset{\substack{\text{non-abelian}\\ \text{cohomology}}}{H(-;A)} \xrightarrow[\text{rationalization}]{H(-;\eta^{\mathbb{R}}_A)} \overset{\substack{\text{non-abelian}\\ \text{real cohomology}}}{H\big(-;L_{\mathbb{R}}A\big)} \tag{199}$$

for the cohomology operation (Def. 2.17) from non-abelian A-cohomology (Def. 2.1) to non-abelian real cohomology (Def. 3.70), which is induced (28) by the rationalization map $\eta^{\mathbb{R}}_A$ (Def. 3.53).

Definition 4.2 (Non-abelian character map)**.** Let $X, A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) such that X admits the structure of a smooth manifold. Then we say that the *non-abelian character map* in non-abelian A-cohomology (Def. 2.1) is the cohomology operation (Def. 4.1)

$$\underset{\substack{\text{non-abelian}\\ \text{character map}}}{} \quad \mathrm{ch}_A \;:\; \overset{\substack{\text{non-abelian}\\ \text{cohomology}}}{H(X;A)} \xrightarrow[\text{rationalization}]{(\eta^{\mathbb{R}}_A)_*} \overset{\substack{\text{non-abelian}\\ \text{real cohomology}}}{H\big(X;L_{\mathbb{R}}A\big)} \xrightarrow[\substack{\text{non-abelian}\\ \text{de Rham theorem}}]{\simeq} \overset{\substack{\text{non-abelian}\\ \text{de Rham cohomology}}}{H_{\mathrm{dR}}(X;\mathfrak{l}A)} \tag{200}$$

from non-abelian A-cohomology (Def. 2.1) to non-abelian de Rham cohomology (Def. 3.82) with coefficients in the rational Whitehead L_∞-algebra $\mathfrak{l}A$ of A (Prop 3.61), which is the composite of
(i) the operation (199) of rationalization of coefficients (Def. 4.1),
(ii) the equivalence (160) of the non-abelian de Rham theorem (Theorem 3.85).

4.1 Chern-Dold character

We prove (Theorem 4.8) that the non-abelian character map reproduces the Chern-Dold character on generalized cohomology theories (recalled as Def. 4.6) and in particular the Chern character on topological K-theory (Example 4.10).

Proposition 4.3 (Dold's equivalence [Do65, Cor. 4][Hil71, Thm. 3.18][Ru98, §II.3.17])**.** *Let E be a generalized cohomology theory (Example 2.13). Then its rationalization $E_{\mathbb{R}}$ is equivalent to ordinary cohomology with coefficients in the rationalized stable homotopy groups of E:*

$$E^n_{\mathbb{R}}(X) \xrightarrow[\simeq]{\mathrm{do}_E} \bigoplus_{k\in\mathbb{Z}} H^{n+k}\big(X;\pi_k(E)\otimes_{\mathbb{R}}\mathbb{R}\big).$$

Remark 4.4 (Rational stable homotopy theory)**.** In modern stable homotopy theory, Dold's equivalence (Prop. 4.3) is a direct consequence of the fundamental theorem [SSh01, Thm. 5.1.6] that rational spectra are direct sums of Eilenberg-MacLane spectra with coefficients in the rationalized stable homotopy groups [BMSS19, Prop. 2.17].

But we may explicitly re-derive Dold's equivalence using the unstable rational homotopy theory from §3:

Proposition 4.5 (Dold's equivalence via non-abelian real cohomology)**.** *Let E be a generalized cohomology theory (Example 2.13) and let $n \in \mathbb{N}$ such that the nth coefficient space* (21) *is of $\mathbb{R}$-finite homotopy type (Def. 3.50)*

$$E_n \;\in\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}.$$

Then there is a natural equivalence between the non-abelian real cohomology (Def. 3.70) with coefficients in E_n and ordinary cohomology with coefficients in the rationalized homotopy groups of E:

$$H\big(-;L_{\mathbb{R}}E_n\big) \;\simeq\; \bigoplus_{k\in\mathbb{N}} H^{n+k}\big(-;\pi_k(E)\otimes_{\mathbb{Z}}\mathbb{R}\big). \tag{201}$$

Proof. Since E_n is an infinite-loop space, it is necessarily nilpotent (Example 3.52). We may assume without restriction that it is also connected, for otherwise we apply the following argument to each connected component (Remark 3.51). Hence $E_n \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) and the discussion in §3.2 applies:

Again since E_n is a loop space (21), Prop. 3.72 gives $H(-; L_{\mathbb{R}} E_n) \simeq \underset{k\in\mathbb{N}}{\oplus} H^k\big(-; \pi_k(E_n) \otimes_{\mathbb{Z}} \mathbb{R}\big)$. The claim follows from the definition of stable homotopy groups as $\pi_{k-n}(E) = \pi_k(E_n)$ for $k, n \geq 0$. □

Definition 4.6 (Chern-Dold character [Bu70][Hil71, p. 50])**.** Let E be a generalized cohomology theory (Example 2.13). The *Chern-Dold character* in E-cohomology theory is the cohomology operation to ordinary cohomology which is the composite of rationalization in E-cohomology with Dold's equivalence (Prop. 4.3):

$$
\begin{tikzcd}
\overset{\text{Chern-Dold character}}{\mathrm{ch}_E} \; : & E^\bullet(-) \arrow[rr, "\text{rationalization in } E\text{-cohomoloy}"] \arrow[d, "\simeq", "(22)"'] & & E^\bullet_{\mathbb{R}}(-) \arrow[r, "\mathrm{do}_E", "\simeq"', "\text{Dold's equivalence}"] \arrow[d, "(22)", "\simeq"'] & \bigoplus\limits_k H^{\bullet+k}\big(-; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}\big) \; . \\
& H(-; E_\bullet) \arrow[rr, "(\eta^{\mathbb{R}}_{E_\bullet})_*", "(199)"'] & & H(-; L_{\mathbb{R}} E_\bullet) \arrow[ur, "\simeq", "(201)"'] &
\end{tikzcd}
\tag{202}
$$

Here the bottom part in (202) serves to make the nature of the top maps fully explicit, using Example 2.13, Def. 4.1 and Prop. 4.5.

Remark 4.7 (Rationalization in the Chern-Dold character)**.** That the first map in the Dold-Chern character (202) is the rationalization localization is stated somewhat indirectly in the original definition [Bu70] (the concept of rationalization was fully formulated later in [BK72]). The role of rationalization in the Chern-Dold character is made fully explicit in [LSW16, §2.1]. The same rationalization construction of the generalized Chern character, but without attribution to [Bu70] or [Do65], is considered in [HS05, §4.8] (see also [BN14, p. 17]).

We now come to the main result in this section:

Theorem 4.8 (Non-abelian character subsumes Chern-Dold character)**.** *Let E be a generalized cohomology theory (Example 2.13) and let $n \in \mathbb{N}$ such that the nth coefficient space* (21) *is of $\mathbb{R}$-finite homotopy type (Def. 3.50). Let moreover X be a smooth manifold of connected, nilpotent, $\mathbb{R}$-finite homotopy type (Def. 3.50).*
Then the non-abelian character (Def. 4.2) coincides with the Chern-Dold character (Def. 4.6) on E-cohomology in degree n, in that the following diagram commutes:

$$
\begin{tikzcd}
H\big(X; E_n\big) \arrow[r, "\mathrm{ch}_{E_n}"'] & H_{\mathrm{dR}}\big(\mathfrak{l} E_n\big) \arrow[d, "\simeq"', "(160)\,(201)"] \\
E^n(X) \arrow[u, "(22)", "\simeq"'] \arrow[r, "\mathrm{ch}_{E_n}"'] & \bigoplus\limits_k H^{n+k}\big(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}\big).
\end{tikzcd}
\tag{203}
$$

Here the equivalence on the left is from Example 2.13, while the equivalence on the right is the inverse non-abelian de Rham theorem (Theorem 3.85) composed with that from Prop. 4.5.

Proof. Since E_n is an infinite-loop space, it is necessarily nilpotent (Example 3.52). We may assume without restriction that it is also connected, for otherwise we apply the following argument to each connected component (Remark 3.51). Hence $E_n \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) and the discussion in §3.2 and §3.3 applies:

The non-abelian de Rham isomorphism (160) in the definition (200) of the non-abelian character cancels against its inverse on the right of (203). Commutativity of the remaining diagram

$$
\begin{tikzcd}
H\big(X; E_n\big) \arrow[r, "(\eta^{\mathbb{R}}_{E_n})_*"'] & H\big(X; L_{\mathbb{R}} E_n\big) \arrow[d, "\simeq"', "(201)"] \\
E^n(X) \arrow[u, "(22)", "\simeq"'] \arrow[r, "\mathrm{ch}_{E_n}"'] & \bigoplus\limits_k H^{n+k}\big(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}\big)
\end{tikzcd}
$$

is the very definition of the Chern-Dold character (Def. 4.6). □

Example 4.9 (de Rham homomorphism in ordinary cohomology)**.** On ordinary integral cohomology (Example 2.2), the non-abelian character (Def. 4.2) reduces to extension of scalars from the integers to the real numbers, followed by the de Rham isomorphism, in that the following diagram commutes:

$$
\begin{tikzcd}
H\big(-; B^{n+1}\mathbb{Z}\big) \arrow[rr, "\substack{\text{non-abelian character}\\ \text{on ordinary cohomology}}", "\mathrm{ch}_{B^{n+1}\mathbb{Z}}"'] & & H_{\mathrm{dR}}\big(-; \mathfrak{l}B^{n+1}\mathbb{Z}\big) \arrow[d, "\simeq"', "(134)\,(157)"] \\
H^{n+1}(-;\mathbb{Z}) \arrow[u, "(13)"', "\simeq"] \arrow[r, "\substack{\text{extension}\\ \text{of scalars}}"'] & H^{n+1}(-;\mathbb{R}) \arrow[r, "\simeq", "\substack{\text{ordinary}\\ \text{de Rham isomorphism}}"'] & H^{n+1}_{\mathrm{dR}}(-)
\end{tikzcd}
$$

Example 4.10 (Chern character on complex K-theory)**.** The spectrum (21) representing complex K-theory has 0th component space $\mathrm{KU}_0 \simeq \mathbb{Z} \times B\mathrm{U}$ (23). Here the connected components $B\mathrm{U}$, the classifying space of the infinite unitary group (24), are clearly of finite $\mathbb{R}$-type (since their real cohomology is the ring of universal Chern classes, e.g. [Ko96, Thm. 2.3.1]). Therefore, Theorem 4.8 applies and says that the non-abelian character map (Def. 4.2) for coefficients in $\mathbb{Z} \times B\mathrm{U}$ reduces to the Chern-Dold character on complex K-theory. This, in turn, is equivalent (by [Hil71, Thm. 5.8]) to the original Chern character ch on complex K-theory [Hi56, §12.1][BH58, §9.1][AH61, §1.10] (review in [Hil71, §V]):

$$
\underset{\substack{\text{Chern character on}\\ \text{complex K-theory}}}{\mathrm{ch}} \;\;\simeq\;\; \mathrm{ch}_{\mathbb{Z}\times B\mathrm{U}}\,.
$$

Example 4.11 (Pontrjagin character on real K-theory)**.** The *Pontrjagin character* ph on real topological K-theory (see [GHV73, §9.4][IK99][Ig08][GS18b, §2.1]) is defined to be the composite

$$
\begin{tikzcd}
\mathrm{KSpin}^\bullet(-) \arrow[r] & \mathrm{KSO}^\bullet(-) \arrow[r] \arrow[rrr, bend right, "\widetilde{\mathrm{ph}}^\bullet"'] & \mathrm{KO}^\bullet(-) \arrow[r, "\mathrm{cplx}"] \arrow[rr, bend right, "\mathrm{ph}^\bullet"'] & \mathrm{KU}^\bullet(-) \arrow[r, "\mathrm{ch}^\bullet"] & \bigoplus_k H^{4\bullet}(-;\mathbb{R})
\end{tikzcd}
$$

of the complexification map (on representing virtual vector bundles) with the Chern character on complex K-theory (Example 4.10).

(i) By naturality of the complexification map and since the complex Chern character is a Chern-Dold character (by [Hil71, Thm. 5.8]), so is the Pontrjagin character, as well as its restriction $\widetilde{\mathrm{ph}}$ to oriented real K-theory KSO and further to ph on KO-theory and to Spin K-theory, etc.

(ii) The connected components $B\mathrm{O}$ of the classifying space KO_0 for real topological K-theory are of finite $\mathbb{R}$-type (since the real cohomology is the ring of universal Pontrjagin classes). Therefore, Theorem 4.8 applies and says that the non-abelian Chern character (Def. 4.2) for coefficients in $\mathbb{Z} \times B\mathrm{SO}$ coincides with the Pontrjagin character $\widetilde{\mathrm{ph}}$ in KSO-theory:

$$
\underset{\substack{\text{Pontrjagin character}\\ \text{on oriented real K-theory}}}{\widetilde{\mathrm{ph}}} \;\;\simeq\;\; \mathrm{ch}_{\mathbb{Z}\times B\mathrm{SO}}\,.
$$

(iii) By Remark 3.51, the construction extends to the Pontrjagin character ph on KO-theory.

(iv) The same applies to the further restriction of the Pontrjagin character to KSpin; see [LD91][Th62] for some subtleties involved and [Sa08, §7] for interpretation and applications.

Example 4.12 (Chern-Dold character on Topological Modular Forms)**.** The connective ring spectrum tmf of *topological modular forms* [Ho94, §9][Ho02, §4] (see [DFHH14]) is, essentially by design, such that under rationalization it yields the graded ring of rational modular forms (e.g [DH11, p. 2]):

$$
\underset{\substack{\text{topological}\\ \text{modular forms}}}{\pi_\bullet(\mathrm{tmf})} \xrightarrow{\;(-)\otimes_{\mathbb{Z}}\mathbb{R}\;} \underset{\substack{\text{rational}\\ \text{modular forms}}}{\mathrm{mf}^{\mathbb{R}}_\bullet} \simeq \mathbb{R}\big[\overbrace{c_4}^{\deg=8}, \overbrace{c_6}^{\deg=12}\big]. \tag{204}
$$

It follows that the Chern-Dold character (Def. 4.6) on tmf takes values in real cohomology with coefficients in modular forms

$$
\mathrm{tmf}^\bullet(-) \xrightarrow[\mathrm{ch}^\bullet_{\mathrm{tmf}}]{\substack{\text{Chern-Dold character}\\ \text{on topological modular forms}}} H^\bullet\big(-;\mathrm{mf}^{\mathbb{R}}_\bullet\big). \tag{205}
$$

(This is often considered over the rational numbers, sometimes over the complex numbers [BE13, Fig. 1]; we may just as well stay over the real numbers, by Remark 3.49, to retain contact to the de Rham theorem.)

By Theorem 4.8 this is an instance of the non-abelian character map:

$$\underset{\text{topological nodular forms}}{\overset{\text{Chern-Dold character on}}{\mathrm{ch}^{\bullet}_{\mathrm{tmf}}}} \quad \simeq \quad \mathrm{ch}_{\mathrm{tmf}_\bullet}\,.$$

Example 4.13 (The Hurewicz/Boardman homomorphism on topological modular forms)**.** The spectrum tmf (Example 4.12) carries the structure of a suitable (E_∞) ring spectrum and hence receives an essentially unique homomorphism of ring spectra from the sphere spectrum:

$$\Sigma^\infty S^0 \;=\; \mathbb{S} \xrightarrow{\;\;e_{\mathrm{tmf}}\;\;} \mathrm{tmf}\,.$$

This is also known as the *Hurewicz homomorphism* or rather the *Boardman homomorphism* (e.g. [Ad75, §II.7][Ko96, §4.3]) for tmf. The Boardman homomorphism on tmf happens to be a stable weak equivalence in degrees ≤ 6, in that it is an isomorphism on stable homotopy groups in these degrees [Ho02, Prop. 4.6][DFHH14, §13]:

$$\pi^s_{\bullet\leq 6} \;=\; \pi_{\bullet\leq 6}(\mathbb{S}) \xrightarrow[\;\simeq\;]{\;\pi_{\bullet\leq 6}(e_{\mathrm{tmf}})\;} \pi_{\bullet\leq 6}(\mathrm{tmf})\,.$$

Hence, in particular, when X^9 is a manifold of dimension $\dim(X) \leq 9$, the Boardman homomorphism identifies the stable Cohomotopy (Example 2.16) of X^9 in degree 4 with $\mathrm{tmf}^4\big(X^9\big)$ (by Prop. A.37):

$$\begin{tikzcd}
\underset{\text{stable}\atop\text{4-Cohomotopy}}{\pi^4_s\big(X^9\big)} = \mathbb{S}^4(X^9) \arrow[rr, "\underset{\text{Boardman homomorphism}}{e^4_{\mathrm{tmf}}}", "\simeq"'] \arrow[dr, "\mathrm{ch}_{\mathbb{S}^4}"'] & & \underset{\text{tmf-cohomology}\atop\text{in degree 4}}{\mathrm{tmf}^4(X^9)}\,. \arrow[dl, "\mathrm{ch}_{\mathrm{tmf}^4}"] \\
& H^4_{\mathrm{dR}}(X^9) &
\end{tikzcd} \tag{206}$$

In this situation, the character map from Example 4.12 extracts exactly the datum of a real 4-class.

Remark 4.14 (Clarifying the role of tmf in string theory)**.** Since the famous computation of [Wi87] showed that the partition function of the heterotic string lands in modular forms, and since the theorem of [AHS01][AHR10] showed that, mathematically, this statement lifts through (what we call above) the tmf-Chern-Dold character (205), there have been proposals about a possible role of tmf-cohomology theory in controlling elusive aspects of string theory (see [KS05][Sa10][DH11][ST11][Sa14][GJF18][GPPV18][Sa19]). While good progress has been made, it might be fair to say that the situation has remained inconclusive. But with the non-abelian generalization (Def. 4.2) of the Chern-Dold character in hand, we may ask for a non-abelian enhancement (Example 2.24) of tmf-theory on string background spacetimes. By Example 4.13, this is, in degree 4, equivalent to asking for a non-abelian enhancement of stable Cohomotopy theory (Example 2.25). This exists canonically: given by actual Cohomotopy theory (Example 2.10). We consider the non-abelian character map on twisted 4-Cohomotopy in Example 5.21 below. The concluding Prop. 5.22 shows that this does capture core aspects of non-perturbative string theory.

Example 4.15 (Chern-Dold character on integral Morava K-theory)**.** We highlight that a particularly interesting example of the Chern-Dold character, which is not widely known, is that on integral Morava K-theory, whose codomain in real cohomology has a rich coefficient system. Morava K-theories $K(n)$ [JW75] (reviewed in [Wu89][Ru98, §IX.7]) form a sequence of spectra labeled by chromatic level $n \in \mathbb{N}$ and by a prime p (notationally left implicit). Their coefficient ring is pure torsion, and hence vanishes upon rationalization. However, there is an integral version $\widetilde{K}(n)$, highlighted in [KS03][Sa10][Buh11][SW15][GS17b], which has an integral p-adic coefficient ring:

$$\widetilde{K}(n)_* \;=\; \mathbb{Z}_p[v_n, v_n^{-1}]\,, \qquad\qquad \text{with } \deg(v_n) = 2(p^n - 1). \tag{207}$$

This theory more closely resembles complex K-theory than is the case for $K(n)$; in fact, for $n = 1$, it coincides with the p-completion of complex K-theory.

Therefore, the Chern-Dold character (Def. 4.6) on integral Morava K-theory [GS17b, p. 53] is of the form

$$\mathrm{ch}_{\mathrm{Mor}} \;:\; \widetilde{K}(n)(-) \xrightarrow{\qquad\quad} H^*\big(-; \mathbb{Q}_p[v_n, v_n^{-1}] \otimes_{\mathbb{Q}} \mathbb{R}\big)\,, \tag{208}$$

where we used (207) in (202) together with the fact that the rationalization of the p-adic integers is the rational (here: real, by Remark 3.49) p-adic numbers[14] $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{R}$.

Now $\mathbb{Q}_p$ is not finite-dimensional over $\mathbb{Q}$, whence $\mathbb{Q}_p \otimes \mathbb{R}$ is not finite-dimensional over $\mathbb{R}$, so that the classifying space for integral Morava K-theory is not of $\mathbb{R}$-finite type (Def. 3.50). Therefore, our *proof* of the non-abelian de Rham theorem (Theorem 3.85), being based on the fundamental theorem of dgc-algebraic rational homotopy theory (Prop. 3.58), does not immediately apply to integral Morava K-theory coefficients; and hence the non-abelian character on integral Morava K-theory with de Rham codomain, in the form defined in Def. 4.2, is not established here. While this is a purely technical issue, as discussed in Remark 3.51, further discussion is beyond the scope of the present article.

4.2 Chern-Weil homomorphism

We prove (Theorem 4.26) that the non-abelian character subsumes the Chern-Weil homomorphism (recalled as Prop. 4.21, review in [Ch51, §III][KN63, §XII][CS74, §2][MS74, §C][FSSt10, §2.1]) in degree-1 non-abelian cohomology.

Chern-Weil theory. For definiteness, we recall the statements of Chern-Weil theory that we need to prove Theorem 4.26 below.

Remark 4.16 (Attributions in Chern-Weil theory)**. (i)** What came to be known as the *Chern-Weil homomorphism* (recalled as Def. 4.21 below) seems to be first publicly described by H. Cartan (in May 1950), in his prominent *Séminaire* [Ca50, §7], published as [Ca51]. Later that year at the ICM (in Aug.-Sep. 1950), Chern discusses this construction in a talk [Ch50, (10)], including a brief reference to unpublished work by Weil (which remained unpublished until appearance in Weil's collected works [We49]) for the proof that the construction is independent of the choice of connection (which is stated with an announcement of a proof in [Ca50, §7]).
(ii) The new result of Chern's talk was the observation [Ch50, (15)] – later called the *fundamental theorem* in [Ch51, §III.6], recalled as Prop. 4.23 below – that this differential-geometric construction coincides with the topological construction of real characteristic classes (Example 2.21). This crucially uses the identification [Ch50, (11)] of the real cohomology of classifying space BG with invariant polynomials, later expanded on by Bott [Bo73, p. 239]. (Various subsequent authors, e.g. [Fr02, (1.14)], suggest to prove Chern's equation (15) by making sense of a connection on the universal G-bundle (which is possible though notoriously subtle, e.g. [Mo79]); but the proof in [Ch50] simply observes that for any given domain manifold the classifying space for G-bundles may be truncated to a finite cell complex (Prop. A.37), thus carrying a finite dimensional smooth G-bundle with ordinary connection. This argument was later worked out in [NR61][NR63][Sc80]).
(iii) It is this fundamental theorem [Ch50, (15)][Ch51, §III.6] which allows to identify the Chern-Weil homomorphism as an instance of the non-abelian character, in Theorem 4.26 below.

Notation 4.17 (Principal bundles with connection)**.** For $G \in$ LieGroups $X \in$ SmoothManifolds, we write

$$G\mathrm{Connections}(X)_{/\sim} \twoheadrightarrow G\mathrm{Bundles}(X)_{/\sim} \tag{209}$$

for the forgetful map from the set of isomorphism classes of G-bundles equipped with connections to those of G-bundles without connection, over X.

The function (209) is surjective and admits sections, corresponding to a choice of the class of a principal connection on any class of G-principal bundles.

Definition 4.18 (Invariant polynomials [We49][Ca50, §7])**.** For $\mathfrak{g} \in \mathrm{LieAlgebras}_{\mathbb{R},\mathrm{fin}}$, we write

$$\mathrm{inv}^\bullet(\mathfrak{g}) := \mathrm{Sym}\big(\mathfrak{b}^2\mathfrak{g}^*\big)^G \in \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the graded sub-algebra (70) on those elements in the symmetric algebra (73) of the linear dual of $\mathfrak{g}$ shifted up (Def. 3.7) into degree 2, which are invariant under the adjoint action of G on $\mathfrak{g}^*$.

[14] Note, parenthetically, that the classical Chern character ch itself can be extended to cohomology theories with values in graded $\mathbb{Q}$-algebras; see, e.g., [Ma06].

Definition 4.19 (Characteristic forms [Ca50, §7][Ch50, (10)]). Let G be a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$, and let $P \xrightarrow{p} X$ be G-principal bundle with connection ∇ (Def. 4.17). Then for $\omega \in \mathrm{inv}^{2n}(\mathfrak{g})$ an invariant polynomial (Def. 4.18), its evaluation on the curvature 2-form $F_\nabla \in \Omega^2(P) \otimes \mathfrak{g}$ of the connection yields a differential form

$$\omega(F_\nabla) \;\in\; \Omega^{2n}_{\mathrm{dR}}(X) \xrightarrow{\;p^*\;} \Omega^{2n}_{\mathrm{dR}}(P)$$

which, by the second condition on an Ehresmann connection, is *basic*, namely in the image of the pullback operation along the bundle projection p, as shown. Regarded as a differential form on X, this is called the *characteristic form* corresponding to ω.

Lemma 4.20 (Characteristic de Rham classes of characteristic forms [We49][Ch50, p. 401][Ch51, §III.4]). *The class in de Rham cohomology*

$$\big[\omega(F_\nabla)\big] \;\in\; H^{2n}_{\mathrm{dR}}(X)$$

of a characteristic form in Def. 4.19 is independent of the choice of connection ∇ and depends only on the isomorphism class of the principal bundle P.

Definition 4.21 (Chern-Weil homomorphism [Ca50, §7][Ch50, (10)]). Let G be a finite-dimensional Lie group, with classifying space denoted BG. The *Chern-Weil homomorphism* is the composite map

$$\begin{array}{cccccc}
\underset{\text{homomorphism}}{\text{Chern-Weil}}\ \mathrm{cw}_G & : & G\mathrm{Bundles}(X)_{/\sim} & \longrightarrow & G\mathrm{Connections}(X)_{/\sim} & \longrightarrow \mathrm{Hom}\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, H^\bullet_{\mathrm{dR}}(X)\big) \\
 & & \underset{[P]}{\text{principal bundle}} & \longmapsto & \underset{[P,\nabla]}{\text{with connection}} & \longmapsto \Big(\underset{\omega}{\text{invariant polynomial}} \;\mapsto\; \underset{[\omega(F_\nabla)]}{\text{de Rham class of characteristic form}} \Big),
\end{array} \tag{210}$$

where the first map is any section of (209), given by choosing any connection on a given principal bundle; and the second map is the construction of characteristic forms according to Def. 4.19. (The Hom on the right is that in $\mathrm{GradedCommAlgebras}^{\geq 0}_{\mathbb{R}}$.) By Lemma 4.20 the second map is well-defined (and its composition with the first turns out to be independent of the choices made, by Prop. 4.23 below).

That this construction is useful, in that it produces interesting real characteristic classes of G-principal bundles (Example 2.21), is the following statement:

Proposition 4.22 (Abstract Chern-Weil homomorphism [Ch50, (11)][Ch51, §III.5][Bo73, p. 239]). *Let G be a finite-dimensional, compact Lie group, with Lie algebra denoted $\mathfrak{g}$. Then the real cohomology algebra of its classifying space BG is isomorphic to the algebra of invariant polynomials (Def. 4.18):*

$$\mathrm{inv}^\bullet(\mathfrak{g}) \;\simeq\; H^\bullet(BG;\mathbb{R}) \quad \in \mathrm{GradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\,. \tag{211}$$

We can also obtain the following:

Proposition 4.23 (Fundamental theorem of Chern-Weil theory [Ch50, (15)][Ch51, §III.6] (Rem. 4.16)). *Let G be a finite-dimensional compact Lie group. Then the Chern-Weil homomorphism (Def. 4.21) coincides with the operation of pullback of universal characteristic classes along the classifying maps of G-bundles (Example 2.21), in that the following diagram commutes:*

$$\begin{tikzcd}
H(X;BG) \arrow[rr, "\substack{\text{pullback of}\\ \text{universal characteristic classes}\\ \text{along classifying map (30)}}", "c \,\mapsto\, c^*(-)"'] && \mathrm{Hom}\big(H^\bullet(BG;\mathbb{R}),\, H^\bullet(X;\mathbb{R})\big) \arrow[d, "\simeq"', "(211)"] \\
G\mathrm{Bundles}(X)_{/\sim} \arrow[u, "(15)", "\simeq"'] \arrow[rr, "\mathrm{cw}_G", "\text{Chern-Weil homomorphism (210)}"'] && \mathrm{Hom}\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, H^\bullet_{\mathrm{dR}}(X)\big)
\end{tikzcd} \tag{212}$$

Here the isomorphism on the left is from Example 2.3, while that from the right is from Prop. 4.22 and using the de Rham theorem.

Chern-Weil homomorphism as a special case of the non-abelian character.

Lemma 4.24 (Sullivan model of classifying space)**.** *Let G be a finite-dimensional, compact and simply-connected Lie group, with Lie algebra denoted $\mathfrak{g}$. Then the minimal Suillvan model (Def. 3.45) of its classifying space BG is the graded algebra of invariant polynomials (Def. 4.18), regarded as a dgc-algebra with vanishing differential:*

$$\big(\mathrm{inv}(\mathfrak{g}), d=0\big) \;\simeq\; \mathrm{CE}(\mathfrak{l}BG) \quad \in \; \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\,. \tag{213}$$

Proof. According to [FOT08, Example 2.42], we have

$$\mathrm{CE}(\mathfrak{l}BG) \;\simeq\; \big(H^\bullet(BG;\mathbb{R}), d=0\big)\,. \tag{214}$$

The composition of (214) with the isomorphism (211) from Prop. 4.22 yields the desired (213). □

Lemma 4.25 (Non-abelian de Rham cohomology with coefficients in a classifying space)**.** *Let G be a finite-dimensional, compact and simply-connected Lie group, with Lie algebra denoted $\mathfrak{g}$. Then the non-abelian de Rham cohomology (Def. 3.82) with coefficients in the rational Whitehead L_∞-algebra $\mathfrak{l}BG$ (Prop. 3.61) of the classifying space is canonically identified with the codomain of the classical Chern-Weil construction* (210)*:*

$$\underset{\text{de Rham cohomology}}{\overset{\text{nonabelian}}{}} H_{\mathrm{dR}}\big(X;\mathfrak{l}BG\big) \;\simeq\; \underset{\text{Chern-Weil construction}}{\overset{\text{traditional codomain of}}{}} \mathrm{Hom}\big(\mathrm{inv}^\bullet(\mathfrak{g}), H^\bullet_{\mathrm{dR}}(X)\big)\,. \tag{215}$$

Proof. Consider the following sequence of natural bijections:

$$\begin{aligned}
H_{\mathrm{dR}}\big(X;\mathfrak{l}BG\big) &:= \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big(\mathrm{CE}\big(\mathfrak{l}BG\big)\,,\,\Omega^\bullet_{\mathrm{dR}}(X)\big)_{/\sim} \\
&\simeq \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\Big(\big(\mathrm{inv}^\bullet(\mathfrak{g}), d=0\big)\,,\,\Omega^\bullet_{\mathrm{dR}}(X)\Big)_{/\sim} \\
&\simeq \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\Big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\Omega^\bullet_{\mathrm{dR}}(X)_{\mathrm{closed}}\Big)_{/\sim} \\
&\simeq \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\Big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\big(\Omega^\bullet_{\mathrm{dR}}(X)_{\mathrm{closed}}\big)_{/\sim}\Big) \\
&\simeq \mathrm{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,H^\bullet_{\mathrm{dR}}(X)\big) \\
&=: \mathrm{Hom}\big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,H^\bullet_{\mathrm{dR}}(X)\big)\,.
\end{aligned}$$

Here the first line is the definition (Def. 3.82). After that, the first step is Lemma 4.25. The second step unwinds what it means to hom out of a dgc-algebra with vanishing differential (which is generator-wise as in Example 3.77), while the third and fourth steps unwind what this means for the coboundary relations (which is generator-wise as in Prop. 3.84). The last line just matches the result to the abbreviated notation used in (210). □

Theorem 4.26 (Non-abelian character map subsumes Chern-Weil homomorphism)**.** *Let G be a finite-dimensional compact, connected and simply-connected Lie group, with Lie algebra $\mathfrak{g}$. Let $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) be equipped with the structure of a smooth manifold. Then the non-abelian character ch_{BG} (Def 4.2) on non-abelian cohomology (Def. 2.1) of X with coefficients in BG coincides with the Chern-Weil homomorphism cw_G (Def. 4.21) with coefficients in G, in that the following diagram (of cohomology sets) commutes:*

$$\begin{tikzcd}
H(X;BG) \arrow[r, "\text{non-abelian character}", "\mathrm{ch}_{BG}"'] & H_{\mathrm{dR}}(X;\mathfrak{l}BG) \arrow[d, "\simeq"', "(215)"] \\
G\mathrm{Bundles}(X)_{/\sim} \arrow[u, "(15)", "\simeq"'] \arrow[r, "\mathrm{cw}_G", "\text{Chern-Weil homomorphism}"'] & \mathrm{Hom}\big(\mathrm{inv}^\bullet(\mathfrak{g}), H^\bullet_{\mathrm{dR}}(X)\big)
\end{tikzcd} \tag{216}$$

Here the isomorphism on the left is from Example 2.3, while that on the right is from Lemma 4.25.

Proof. First, notice that BG is simply connected (hence nilpotent), by the assumption that G is connected, and that it is of finite rational type by Prop. 4.22. Hence, with Def. 3.50,

$$BG \;\in\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}\,. \tag{217}$$

Now, by Definition 4.2, the non-abelian character map on the top of (216)

$$\mathrm{ch}_{BG} \;:\; H(X;BG) \xrightarrow{(\eta^{\mathbb{R}}_{BG})} H\big(X;L_{\mathbb{R}}BG\big) \xrightarrow{\;\simeq\;} H_{\mathrm{dR}}\big(X;L_{\mathbb{R}}BG\big)$$

sends a classifying map

$$X \xrightarrow{c} BG \;\in\; H(X;BG) \;=\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)(X\,,BG)$$

first to its composite with the rationalization map (Def. 3.53). By the fundamental theorem (Theorem 3.58 (i), using (217)), this is given by the derived adjunction unit $\mathbb{D}\eta_{BG}$ of $\mathbb{R}\exp \dashv \Omega^\bullet_{\mathrm{PLdR}}$ (118):

$$X \xrightarrow{\;c\;} BG \xrightarrow{L_{\mathbb{R}}BG \simeq \mathbb{D}\eta_{BG}} \mathbb{R}\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(BG) \qquad \in\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,L_{\mathbb{R}}BG\big) \;=\; H\big(X;\,,L_{\mathbb{R}}BG\big)\,.$$

Moreover, by part (ii) of the fundamental theorem, the adjunct of the morphism $\mathbb{D}\eta_{BG}\circ c$ under (118) is

$$\Omega^\bullet_{\mathrm{PLdR}}(X) \xleftarrow{\;c^*\;} \Omega^\bullet_{\mathrm{PLdR}}(BG) \qquad \in\; \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)$$

(using that $\Omega^\bullet_{\mathrm{PLdR}}(\mathbb{D}\eta^{\mathbb{R}})$ is an equivalence, by reflectivity of rationalization (110)). Hence it is the pullback operation of rational cocyles on BG along the classifying map c. Sending this further along the isomorphism to the bottom right in (216) (via Theorem 3.85 and Lemma 4.25) gives, by (161):

$$\mathrm{ch}_{BG} \;:\; c \;\mapsto\quad \Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\;c^*\;} \Omega^\bullet_{\mathrm{PLdR}}(BG) \xleftarrow{\;\simeq\;} \mathrm{inv}^\bullet(\mathfrak{g}) \qquad \in\; \mathrm{Ho}\Big(\big(\mathrm{DiffGradedCommAlgebras}^{\geq 0}_{\mathbb{R}}\big)_{\mathrm{proj}}\Big)\,. \tag{218}$$

In conclusion, we have found that the commutativity of (216) is equivalent to the statement that the characteristic forms obtained by the Chern-Weil construction (210) represent the pullback (218) of the universal real characteristic classes on BG along the classifying map c of the underlying principal bundle (Example 2.21). This is the case by the fundamental theorem of Chern-Weil theory, Prop. 4.23. □

Example 4.27 (de Rham representative of tangential Sp(2)-twist)**.** For X a smooth 8-dimensional spin-manifold equipped with tangential Sp(2)-structure τ (58), Theorem 4.26 says that there exists a smooth Sp(2)-principal bundle on X equipped with an Ehresmann connection ∇ such that the rationalization (Def. 3.53) of the twist τ corresponds, under the non-abelian de Rham theorem (Theorem 3.85) to a flat $\mathfrak{l}B\mathrm{Sp}(2)$-valued differential form whose components are the characteristic forms of the Sp(2)-principal connection ∇:

$$\begin{array}{ccccccc}
H\big(X;B\mathrm{Sp}(2)\big) & \overset{(\eta^{\mathbb{R}}_{BG})_*}{\longrightarrow} & H\big(X;L_{\mathbb{R}}B\mathrm{Sp}(2)\big) & \simeq & H_{\mathrm{dR}}\big(X;\mathfrak{l}B\mathrm{Sp}(2)\big) & & \\
\tau & \longmapsto & L_{\mathbb{R}}\tau & \longleftrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\;\tau_{\mathrm{dR}}\;} & \mathbb{R}\left[\begin{array}{c}\chi_8, \\ \tfrac{1}{2}p_1\end{array}\right]\Big/\left(\begin{array}{l} d\,\tfrac{1}{2}p_1 = 0 \\ d\;\;\chi_8 = 0\end{array}\right) & = \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \\
 & & & & \tfrac{1}{2}p_1(\nabla) \longleftarrow\!\!\!\shortmid & \tfrac{1}{2}p_1 & \\
 & & & & \chi_8(\nabla) \longleftarrow\!\!\!\shortmid & \chi_8 &
\end{array}$$

Here on the right we are using [CV98, Thm . 8.1], see [FSS20, Lemma 2.12] to identify generating universal characteristic classes on $B\mathrm{Sp}(2)$: $\frac{1}{2}p_1$ is the first Pontrjagin class (of degree 4) and $\chi_8 = \Big(\frac{1}{2}p_2 - \big(\frac{1}{2}p_1\big)^2\Big)$ is the Euler 8-class, which on $B\mathrm{Sp}(2)$ happens to be proportional to the I_8-polynomial (see [FSS19b, Prop. 3.7]).

4.3 Cheeger-Simons homomorphism

We show (Theorem 4.46) that the non-abelian character map induces *secondary* non-abelian cohomology operations (Def. 4.42) which subsume the Cheeger-Simons homomorphism, recalled around (255) below, with values in ordinary differential cohomology, recalled around (242) below. We follow [FSSt10] [SSS12][Sc13] where the Cheeger-Simons homomorphism, generalized to higher principal bundles, is called the ∞*-Chern-Weil homomorphism*.

Underlying this is a differential enhancement of the non-abelian character map (Def. 4.32), and an induced notion of differential non-abelian cohomology (Def. 4.33) on smooth ∞-stacks (recalled as Def. A.44).

The differential non-abelian character map. We introduce (in Def. 4.32 below) the differential refinement of the non-abelian character map; given as before by rationalization, but now followed not by a map to non-abelian de Rham cohomology, but to its refinement by the full cocycle space of flat non-abelian differential forms (Def. 4.28 below). It is this refinement of the codomain of the character map that allows it to be fibered over the smooth space of actual flat non-abelian differential forms (instead of just their non-abelian de Rham classes), thus producing differential non-abelian cohomology (Def. 4.33 below).

Definition 4.28 (Moduli ∞-stack of flat L_∞-algebra valued forms [Sc13, 4.4.14.2])**.** Let $A \in$ SimplicialSets be of connected, nilpotent, $\mathbb{R}$-finite homotopy type (Def. 3.50). In view of the system of sets (Def. 3.75)

$$X \;\longmapsto\; \Omega_{\mathrm{dR}}\big(X; \mathfrak{l}A\big) \;\in\; \mathrm{Sets}$$

of flat non-abelian differential forms with coefficient in the Whitehead L_∞-algebra $\mathfrak{l}A$ of A (Prop. 3.61), which are contravariantly assigned to smooth manifolds X, we consider in $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (Def. A.44):
(i) the *smooth space of flat $\mathfrak{l}A$-valued differential forms*

$$\Omega_{\mathrm{dR}}\big(-; \mathfrak{l}A\big)_{\mathrm{flat}} \;:=\; \Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big)\,, \tag{219}$$

regarded as a simplicially constant simplicial presheaf (341);
(ii) the *smooth ∞-stack of flat $\mathfrak{l}A$-valued differential forms* (Example 3.80)

$$\flat\exp(\mathfrak{l}A) \;:=\; \Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^k; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big) \tag{220}$$

which to any Cartesian space assigns the simplicial set that in degree k is the set of flat $\mathfrak{l}A$-valued differential forms on the product manifold of the Cartesian space with the standard smooth k-simplex $\Delta^k \subset \mathbb{R}^k$;
(iii) the canonical inclusion

$$\begin{tikzcd}
\underset{\text{flat } \mathfrak{l}A\text{-valued forms}}{\overset{\text{smooth space of}}{\Omega(-;\mathfrak{l}A)_{\mathrm{flat}}}} \arrow[r, "\text{atlas}"] \arrow[d, equal] & \underset{\text{flat } \mathfrak{l}A\text{-valued forms}}{\overset{\text{smooth } \infty\text{-stack of}}{\flat\exp(\mathfrak{l}A)}} \arrow[d, equal] \\
\Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big) \arrow[r, hook] & \Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^k; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big)
\end{tikzcd} \tag{221}$$

exhibiting $\Omega(-;\mathfrak{l}A)$ (219) as the presheaf of 0-simplices in the simplicial presheaf $\flat\exp(\mathfrak{l}A)$ (220) (more abstractly: this is the canonical *atlas* of the smooth moduli ∞-stack, see [SS20b, Prop. 2.70]).

Lemma 4.29 (Moduli ∞-stack of flat forms is equivalent to discrete rational ∞-stack)**.**
For $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50), the evident inclusion (by inclusion of polynomial forms into smooth differential forms followed by pullback along pr_{Δ^k})

$$\begin{tikzcd}
\mathrm{Disc}\big(L_{\mathbb{R}}A\big) \simeq \mathrm{Disc} \circ \mathbb{R}\exp \circ \mathrm{CE}\big(\mathfrak{l}A\big) \arrow[r, "\in \mathrm{W}"] \arrow[d, equal] & \flat\exp\big(\mathfrak{l}A\big) \arrow[d, equal] \\
\Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{PLdR}}\big(\Delta^k; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big) \arrow[r, hook] & \Big(\mathbb{R}^n \mapsto \Big(\Delta[k] \mapsto \Omega_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^k; \mathfrak{l}A\big)_{\mathrm{flat}}\Big)\Big)
\end{tikzcd} \tag{222}$$

of the image under Disc (345) *of the dg-algebraic model* (119) *for the rationalization of A (Def. 3.53) given by the fundamental theorem (Prop. 3.58), into the moduli ∞-stack of flat $\mathfrak{l}A$-valued differential forms (Def. 4.28) is an equivalence in* $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ *(Def. A.44).*

Proof. By Prop. 3.60, the inclusion is for each $\mathbb{R}^n$ a weak equivalence (127) in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8), hence is a weak equivalence already in the global projective model structure on simplicial presheaves, and hence also in the local projective model structure. (Example A.43). $\square$

Lemma 4.30 (Moduli ∞-stack of closed differential forms is shifted de Rham complex)**.** *For $n \in \mathbb{N}$, we have an equivalence in* $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ *(Def. A.44) from the moduli ∞-stack $\flat\exp\big(\mathfrak{b}^n\mathbb{R}\big)$ of flat differential forms (Def. 4.28) with values in the line Lie $(n+1)$-algebra $\mathfrak{b}^n\mathbb{R}$ (Example 3.27) to the image under the Dold-Kan construction (Def. A.53) of the smooth de Rham complex $\Omega^\bullet_{\mathrm{dR}}(-)$ (Example 3.23), naturally regarded as a presheaf on* CartesianSpaces (338) *with values in connective chain complexes (Example A.48) (i.e., with de Rham differential lowering the chain degree) shifted up in degree by n and then homologically truncated in degree 0, as shown on the right.*

$$
\flat\exp\big(\mathfrak{b}^n\mathbb{R}\big) \xrightarrow{\ \simeq\ } \mathrm{DK}\left(\begin{array}{c} \vdots \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \Omega^0_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^1_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \vdots \\ \downarrow{\scriptstyle d} \\ \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \end{array}\right) \quad \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)
$$

Proof. First observe, with Example 3.77, that the simplicial presheaf

$$
\flat\exp\big(\mathfrak{b}^n\mathbb{R}\big)(-) \;=\; \Big(\Delta[k] \mapsto \Omega^{n+1}_{\mathrm{dR}}\big((-)\times\Delta^k\big)_{\mathrm{clsd}}\Big) \tag{223}
$$

naturally carries the structure of a presheaf of simplicial abelian groups, given by addition of differtial forms. Therefore, by the Dold-Kan Quillen equivalence (Prop A.52), it is sufficient to prove that we have a quasi-isomorphism of presheaves of chain complexes from the corresponding normalized chain complex (346) of (223) to the shifted and truncated de Rham complex itself:

$$
N\Big(\Delta[k] \mapsto \Omega^{n+1}_{\mathrm{dR}}\big((-)\times\Delta^k\big)_{\mathrm{clsd}}\Big) \xrightarrow[\simeq]{\int_{\Delta^\bullet}} \Big(\cdots \to 0 \to 0 \to \Omega^0_{\mathrm{dR}}(-) \xrightarrow{d} \Omega^1_{\mathrm{dR}}(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}}\Big). \tag{224}
$$

We claim that such is given by fiber integration of differential forms over the simplices Δ^k:

First, to see that fiber integration does constitute a chain map, we compute for $\omega \in \Omega^\bullet_{\mathrm{dR}}\big((-)\times\Delta^k\big)_{\mathrm{clsd}}$ on the left of (224):

$$
\int_{\Delta^k}\partial\omega \;=\; (-1)^k\int_{\partial\Delta^k}\omega \;=\; d\int_{\Delta^k}\omega\,, \tag{225}
$$

where the first step is the definition of the differential in the normalized chain complex (346) and the second step is the fiberwise Stokes formula (155).

Finally, to see that $\int_{\Delta^\bullet}$ is a quasi-isomorphism, notice that the chain homology groups on both sides are

$$
H_k(-) \;=\; \left\{\begin{array}{lcl} \mathbb{R} & | & k = n+1 \\ 0 & | & \text{otherwise} \end{array}\right.
$$

over each Cartesian space: For the left hand side this follows via the weak equivalence (127) from the fundamental theorem (Prop. 3.58) via Example 3.65, while for the right hand side this follows from the Poincaré lemma.

Hence it is sufficient to see that fiber integration over Δ^{n+1} is an isomorphism on the $(n+1)$st chain homology groups. But a generator of this group on the left is clearly given by the pullback $\mathrm{pr}^*_{\Delta^{n+1}}\omega$ of any $\omega \in \Omega^{n+1}_{\mathrm{dR}}(\Delta^{n+1})$ of unit weight and supported in the interior of the simplex. That this is sent under $\int_{\Delta^{n+1}}$ to a generator $\pm 1 \in \mathbb{R} \simeq \Omega^0_{\mathrm{dR}}(-)_{\mathrm{clsd}}$ on the right follows by the projection formula (156). $\square$

Remark 4.31 (Moduli of closed forms via stable Dold-Kan correspondence)**.** Expressed in terms of the stable Dold-Kan construction $\mathrm{DK}_{\mathrm{st}}$ (Prop. A.55) via the derived stabilization adjunction (Example A.41), Lemma 4.30 says, equivalently, that:

$$
\flat\exp\big(\mathfrak{b}^n\mathbb{R}\big) \;\simeq\; \mathbb{R}\Omega^\infty\Big(\mathrm{DK}_{\mathrm{st}}\big(\,\Omega^\bullet_{\mathrm{dR}}(-)\otimes_{\mathbb{Z}}\mathfrak{b}^{n+1}\mathbb{R}\,\big)\Big) \quad \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\,, \tag{226}
$$

where now $\Omega^\bullet_{\mathrm{dR}}(-) \in \mathrm{PSh}\big(\mathrm{CartesianSpaces}\,,\,\mathrm{ChainComplexes}_{\mathbb{Z}}\big)$ is in non-positive degrees, with $\Omega^0_{\mathrm{dR}}(-)$ in degree 0, and where $\mathfrak{b}^{n+1}\mathbb{R}$ (Def. 3.7) is concentrated on $\mathbb{R}$ in degree $n+1$.

Definition 4.32 (Differential non-abelian character map [FSS15b, §4])**.** Given $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50), the *differential non-abelian character map* in A-cohomology theory, to be denoted $\mathbf{ch}_A$, is the morphism in $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (342) from $\mathrm{Disc}(A)$ (345) to the moduli ∞-stack of flat $\mathfrak{l}A$-valued forms $\flat\exp(\mathfrak{l}A)$ (220) given by the composite

$$
\begin{tikzcd}
\overset{\substack{\text{coefficient space as}\\ \text{geometrically discrete}\\ \text{moduli } \infty\text{-stack}}}{\mathrm{Disc}(A)} \arrow[r, "\mathrm{Disc}(\eta^{\mathrm{PLdR}}_A)"] \arrow[rrr, bend left, "\overset{\text{differential non-abelian character map}}{\mathbf{ch}_A}"] \arrow[rr, bend right, "\mathrm{Disc}(\mathbb{D}\eta^{\mathrm{PLdR}}_A)"', "\substack{\text{rationalization}\\ (119)}"'] & \mathrm{Disc}\circ\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(A) \arrow[r, "\mathrm{Disc}\circ\exp(p^{\mathrm{min}})", "(130)"'] & \mathrm{Disc}\circ\mathbb{R}\exp\circ\mathrm{CE}(\mathfrak{l}A) \arrow[r, "\in \mathrm{W}"', "(222)"'] & \overset{\substack{\text{moduli } \infty\text{-stack of}\\ \text{flat } \mathfrak{l}A\text{-valued}\\ \text{differential forms}}}{\flat\exp(\mathfrak{l}A)}
\end{tikzcd}
\tag{227}
$$

of

(a) the image under Disc (345) of the derived adjunction unit $\mathbb{D}\eta^{\mathrm{PLdR}}_A$ (311) of the PS de Rham adjunction (125), specifically with (co-)fibrant replacement p^{min} being the minimal Sullivan model replacement (108); (recalling that exp is a contravariant functor),

with

(b) the weak equivalence from Lemma 4.29.

Differential non-abelian cohomology.

Definition 4.33 (Differential non-abelian cohomology [FSS15b, §4])**.** For $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) we say that:

(i) the *moduli ∞-stack of ΩA-connections* is the object $A_{\mathrm{diff}} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ in the homotopy category of smooth ∞-stacks (Def. A.44), which is given by the homotopy pullback (Def. A.23) of the smooth space of flat non-abelian differential forms $\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{flat}}$ (221) along the differential non-abelian character map $\mathbf{ch}_A$ (Def. 4.32):

$$
\begin{tikzcd}
\overset{\substack{\text{moduli } \infty\text{-stack}\\ \text{of } \Omega A\text{-connections}}}{A_{\mathrm{diff}}} \arrow[r, "\substack{\mathfrak{l}A\text{-valued}\\ \text{curvature forms}}", "F_A"'] \arrow[d, "c_A", "\substack{\text{universal characteristic class}\\ \text{in non-abelian } A\text{-cohomology}}"'] \arrow[dr, phantom, "(\mathrm{hpb})"] & \overset{\substack{\text{smooth space of}\\ \text{flat } \mathfrak{l}A\text{-valued forms}}}{\Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{flat}}} \arrow[d, "\text{atlas}"] \\
\mathrm{Disc}(A) \arrow[r, "\mathbf{ch}_A", "\substack{\text{differential non-abelian}\\ \text{character map}}"'] & \underset{\substack{\text{moduli } \infty\text{-stack of}\\ \text{flat } \mathfrak{l}A\text{-valued forms}}}{\flat\exp(\mathfrak{l}A)}
\end{tikzcd}
\quad \in \quad \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\,;
\tag{228}
$$

(ii) the *differential non-abelian cohomology* of a smooth ∞-stack $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (342) with coefficients in A is the structured non-abelian cohomology (Remark 2.27) with coefficients in the moduli ∞-stack A_{diff} of ΩA-connections (228), hence the hom-set in the homotopy category of ∞-stacks (Def. A.44) from $\mathscr{X}$ to A_{diff}

$$
\widehat{H}\big(\mathscr{X};A\big) \;:=\; H\big(\mathscr{X};A_{\mathrm{diff}}\big) \;:=\; \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\big(\mathscr{X}\,,A_{\mathrm{diff}}\big)\,.
\tag{229}
$$

(iii) We call the non-abelian cohomology operations induced from the maps in (228) as follows (see (4)):

$$
\textbf{(a)}\ \textit{characteristic class}: \quad \widehat{H}\big(\mathscr{X};A\big) \xrightarrow{(c_A)_*} H\big(\mathrm{Shp}(\mathscr{X});A\big) \qquad \text{(Def. 2.1)}
\tag{230}
$$

$$
\textbf{(b)}\ \textit{curvature}: \quad \widehat{H}\big(\mathscr{X};A\big) \xrightarrow{(F_A)_*} \Omega_{\mathrm{dR}}\big(\mathscr{X};\mathfrak{l}A\big)_{\mathrm{flat}} \qquad \text{(Def. 3.75)}
\tag{231}
$$

$$
\textbf{(c)}\ \textit{differential character}: \quad \widehat{H}\big(\mathscr{X};A\big) \xrightarrow{(\mathbf{ch}_A\circ c_A)_*} H_{\mathrm{dR}}\big(\mathscr{X};\mathfrak{l}A\big) \qquad \text{(Def. 3.82)}
\tag{232}
$$

In differential enhancement of Example 2.13, we have:

Differential generalized cohomology.

Example 4.34 (Differential generalized cohomology)**.** Let $E^\bullet$ be a generalized cohomology theory (Example 2.13) with representing spectrum E (21) which is connective and whose component spaces E_n are of finite $\mathbb{R}$-type, so that their connected components are, by Example 3.52, in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50).

(i) Then differential non-abelian cohomology, in the sense of Def. 4.33, with coefficients in the component spaces $E_\bullet$, coincides with canonical differential generalized E-cohomology in the traditional sense of [HS05, §4.1][Bun12, Def. 4.53][BG13, §2.2][BNV13, §4.4]:

$$\underset{\substack{\text{generalized}\\ \text{differential cohomology}}}{\widehat{E}^n(-)} \;\simeq\; \widehat{H}(-;E_n)\,. \tag{233}$$

(ii) Here "canonical", in the sense of [Bun12, Def. 4.46], refers to choosing the curvature differential form coefficients to be $\pi_\bullet(E)\otimes\mathbb{R}$ (instead of some chain complex quasi-isomorphic to this). By Example 3.67, this choice corresponds in our Def. 4.33 to the *minimality* (Def. 3.45) of the minimal Sullivan model $\mathrm{CE}(\mathfrak{l}E_n)$ for E_n (Prop. 3.61) that controls the flat L_∞-algebra valued differential forms $\Omega_{\mathrm{dR}}(-;\mathfrak{l}E_n)_{\mathrm{flat}}$ (Def. 3.75) in the top right of (247).

(iii) Hence for canonical/minimal curvature coefficients, we have from Example 3.67, Lemma 4.30 and Remark 226 that

$$\flat\exp\big(\mathfrak{l}E_n\big) \;\simeq\; \mathbb{R}\Omega^\infty\Big(\mathrm{DK}_{\mathrm{st}}\big(\Omega^\bullet_{\mathrm{dR}}(-)\otimes_{\mathbb{Z}}\pi_\bullet(E_n)\big)\Big) \quad\in\; \mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \tag{234}$$

and

$$\Omega_{\mathrm{dR}}\big(-;\mathfrak{l}E_n\big)_{\mathrm{flat}} \;\simeq\; \mathbb{R}\Omega^\infty\Big(\mathrm{DK}_{\mathrm{st}}\big(\Omega^\bullet_{\mathrm{dR}}(-)\otimes_{\mathbb{Z}}\pi_\bullet(E_n)\big)_{\leq 0}\Big) \quad\in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\,. \tag{235}$$

(iv) With this, the equivalence 233 follows by observing that the defining homotopy pullback diagram (228) for differential non-abelian cohomology with coefficients in $A := E_n$ (337) is the image under $\mathbb{R}\Omega^\infty$ (336) of the defining homotopy pullback diagram for canonical differential E-cohomology according to [HS05, (4.12)] [Bun12, Def. 4.51][BNV13, (24)], and using that right adjoints preserve homotopy pullbacks:

$$\underset{\substack{\text{moduli }\infty\text{-stack}\\ \text{of }\Omega E_0\text{-connections}}}{\begin{tikzcd} (E_0)_{\mathrm{diff}} \arrow[r, "F_{E_0}"] \arrow[d, "c_{E_0}"'] \arrow[dr, phantom, "\text{(hpb)}"] & \Omega_{\mathrm{dR}}(-;\mathfrak{l}E_0)_{\mathrm{flat}} \arrow[d, "\text{atlas}"] \\ \mathrm{Disc}(E_0) \arrow[r, "\mathbf{ch}_{E_0}"'] & \flat\exp(\mathfrak{l}E_0) \end{tikzcd}} \;\simeq\; \mathbb{R}\Omega^\infty \underset{\substack{\text{"differential function spectrum"}\\ \text{of differential generalized }E\text{-cohomology}}}{\left(\begin{tikzcd} \mathrm{Diff}(E,\mathrm{can}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \big(\Omega^\bullet_{\mathrm{dR}}(-)\otimes_{\mathbb{Z}}\pi_\bullet(E)\big)_{\leq 0} \arrow[d] \\ \mathrm{Disc}(E) \arrow[r, "H\mathbb{R}\wedge(-)"'] & \Omega^\bullet_{\mathrm{dR}}(-)\otimes_{\mathbb{Z}}\pi_\bullet(E) \end{tikzcd} \right)} \tag{236}$$

The same applies to $(E_n)_{\mathrm{diff}}$, by replacing E with $\mathbb{L}\Sigma^n E$ (336) on the right of (236).

Remark 4.35 (The canonical atlas for the moduli stack of connections)**.** The operation $(-)_{\leq 0}$ in (235) is the naive truncation functor on the category of chain complexes

$$\begin{array}{ccc} \mathrm{ChainComplexes}_{\mathbb{Z}} & \xrightarrow{\qquad (-)_{\leq 0} \qquad} & \mathrm{ChainComplexes}^{\leq 0}_{\mathbb{Z}} \\ \big(\cdots \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} V_{-1} \to \cdots\big) & \longmapsto & \big(V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} V_{-1} \to \cdots\big)\,. \end{array}$$

In contrast to the homological truncation involved in Ω^∞ (353), this naive truncation is not homotopy-invariant and does not have a derived functor. Instead, as seen from (235) and (221), once regarded in differential non-abelian cohomology, this operation serves to construct the canonical *atlas* [SS20b, Prop. 2.70] of the moduli ∞-stack of flat $\mathfrak{l}E_n$-valued differential forms.

Via the defining homotopy pullback (228), (236) this becomes hallmark of differential cohomology: Differential cohomology is the universal solution to lifting the values of the character map from cohomology classes to cochain representatives, namely to curvature forms.

In differential enhancement of Example 2.14 and Example 4.10 we have:

Example 4.36 (Differential complex K-theory)**.** With the coefficient space $A := \mathrm{KU}_0 = \mathbb{Z}\times B\mathrm{U}$ (23) for topological complex K-theory (Example 2.14), the corresponding differential non-abelian cohomology theory (Def. 4.33) is, by Example 4.34, differential K-theory, whose diagram (4) of cohomology operations is of this form

$$\begin{tikzcd} \widehat{H}\big(\mathscr{X};\mathrm{KU}_0\big) \simeq \widehat{\mathrm{KU}}^0(\mathscr{X}) \arrow[r, "F_{\mathrm{KU}_0}"] \arrow[d, "c_{\mathrm{KU}_0}"'] & \Big\{ \big\{F_{2k}\in\Omega^{2k}_{\mathrm{dR}}(\mathscr{X})\big\}_{k\in\mathbb{N}} \,\big|\, d\,F_{2k}=0 \Big\} \arrow[d] \\ \mathrm{KU}^0(\mathscr{X}) \arrow[r, "\mathrm{ch}"'] & \underset{k\in\mathbb{N}}{\bigoplus} H^{2k}_{\mathrm{dR}}(\mathscr{X}), \end{tikzcd} \tag{237}$$

where the bottom map is the ordinary Chern character from Example 4.10, and the curvature differential forms are identified as in Example 3.92.

Examples of differential non-abelian cohomology. In differential enhancement of Example 2.3, we have:

Proposition 4.37 (Differential cohomology of principal connections)**.** *Let G be a compact Lie group with classifying space BG* (16)*. Then there is a natural map over manifolds X, shown dashed in* (238)*, from equivalence classes of G-principal connections (Notation 4.17) to differential non-abelian cohomology with coefficients in BG (Def. 4.33) which covers the classification of G-principal bundles by plain non-abelian cohomology with coefficients in BG (Example 2.3), in that the following diagram commutes:*

$$\begin{tikzcd} G\mathrm{Connections}(X)_{/\sim} \arrow[r, dashed] \arrow[d, "\substack{\text{forget}\\ \text{connection}}"'] & \overset{\substack{\text{differential}\\ \text{non-abelian cohomology}}}{\widehat{H}(X;BG)} \arrow[d, "c_{BG}"] \\ G\mathrm{Bundles}(X)_{/\sim} \arrow[r, "\simeq"'] & \underset{\text{non-abelian cohomology}}{H(X;\,BG)} \end{tikzcd} \tag{238}$$

Proof. By Lemma 4.25, the differential form coefficient in the given case is

$$\Omega_{\mathrm{dR}}(-;\mathfrak{l}BG)_{\mathrm{flat}} \;\simeq\; \mathrm{Hom}_{\mathbb{R}}\Big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\Omega^\bullet_{\mathrm{dR}}(-)_{\mathrm{clsd}}\Big)\,.$$

Therefore, with Example 3.65, we find that

$$\Big(\Delta[k]\mapsto \mathrm{Hom}_{\mathbb{R}}\big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\Omega^\bullet_{\mathrm{dR}}(\Delta^k)_{\mathrm{clsd}}\big)\Big) \;\simeq\; \prod_k K\big(\mathrm{inv}^n(\mathfrak{g}),n\big) \quad \in \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)$$

is a product of Eilenberg-MacLane spaces (14) for real coefficient groups spanned by the invariant polynomials, and so the defining homotopy pullback (228) is here of this form:

$$\begin{tikzcd} BG_{\mathrm{diff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \mathrm{Hom}_{\mathbb{R}}\big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\Omega^\bullet_{\mathrm{dR}}(-)_{\mathrm{clsd}}\big) \arrow[d] \\ \mathrm{Disc}(BG) \arrow[r, "(c_k)_{k\in\mathbb{N}}"'] & \mathrm{Disc}\Big(\underset{k\in\mathbb{N}}{\prod} K\big(\mathrm{inv}^n(\mathfrak{g}),n\big)\Big), \end{tikzcd}$$

where the bottom map classifies the real characteristic classes of BG via Example 2.2. It follows (by Example A.26) that maps into BG_{diff} are equivalence classes of triples

$$\widehat{H}(X;BG) \;\simeq\; \left\{ \big(f,\phi,(\alpha_k)\big) \;\middle|\; \begin{tikzcd} X \arrow[r, dashed, "(\alpha_k)"] \arrow[d, dashed, "f"'] \arrow[dr, phantom, "\Swarrow \phi"] & \mathrm{Hom}_{\mathbb{R}}\big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\,\Omega^\bullet_{\mathrm{dR}}(-)_{\mathrm{clsd}}\big) \arrow[d] \\ BG \arrow[r] & \mathrm{Disc}\Big(\underset{k\in\mathbb{N}}{\prod} K\big(\mathrm{inv}^n(\mathfrak{g}),n\big)\Big) \end{tikzcd} \right\} \tag{239}$$

consisting of **(a)** a classifying map f for a G-principal bundle (Example 2.3), **(b)** a set of closed differential forms α labeled by the invariant polynomials, and **(c)** a set of coboundaries ϕ in real cohomology between these differential forms and the pullbacks $f^* c_k$.

Now, given a G-connection ∇ on a G-principal bundle f^*EG over X, we obtain such a triple by **(a)** taking f to be the classifying map of the underlying G-principal bundle, **(b)** taking $\alpha_k := \omega_k(F_\nabla)$ to be the characteristic forms (Def. 4.19) of the connection, and **(c)** taking ϕ to be given by the relative Chern-Simons forms [CS74] between the given connection and the pullback along f of the universal connection (see Remark 4.16). This construction is an invariant of the isomorphism class of the connection (see [HS05, p. 28]) and hence defines the desired map (238):

$$\begin{array}{ccc} G\mathrm{Connections}(X)_{/\sim} & \longrightarrow & \widehat{H}(X;BG) \\ \big[f^*EG, \nabla\big] & \longmapsto & \big[f,\, \big(\mathrm{cs}_k(\nabla, f^*\nabla_{\mathrm{univ}})\big),\, \big(\omega_k(F_\nabla)\big)\big] \end{array} \tag{240}$$

□

In differential enhancement of Example 2.10, we have:

Example 4.38 (Differential Cohomotopy [FSS15b])**.** The canonical differential enhancement of (unstable) Cohomotopy theory (Example 2.10) in degree n is differential non-abelian cohomology (Def. 4.33) with coefficients in S^n:

$$\underset{\text{differential Cohomotopy}}{\widehat{\pi}^n(-)} := \widehat{H}\big(-;S^n\big)\,.$$

(i) By Example 3.79, a cocycle $\widehat{C}_3 \in \widehat{\pi}^4(X)$ in differential 4-Cohomotopy has as curvature (228) a pair consisting of a differential 4-form G_4 and a differential 7-form G_7, satisfying the *cohomotopical Bianchi identity* shown here:

$$\begin{array}{ccc} \underset{}{\overset{\text{differential 4-Cohomotopy}}{\widehat{\pi}^4(X)}} & \xrightarrow[]{\substack{\text{cohomotopical curvature} \\ F_{S^4}}} & \Omega\big(X;\mathfrak{l}S^4\big)_{\mathrm{flat}} \\ \underset{\substack{\text{cohomotopically} \\ \text{charge-quantized} \\ C_3\text{-field}}}{\widehat{C}_3} & \longmapsto & \left\{ \begin{array}{c} G_7(\widehat{C}_3), \\ G_4(\widehat{C}_3) \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \begin{array}{l} d\,G_7(\widehat{C}_3) = -G_4(\widehat{C}_3)\wedge G_4(\widehat{C}_3) \\ d\,G_4(\widehat{C}_3) = 0 \end{array} \right\}. \end{array} \tag{241}$$

Such differential form data is exactly what characterizes the flux densities of the C_3-field in 11-dimensional supergravity (up to the self-duality constraint $G_7 = \star G_4$). By comparison with Dirac's charge quantization (2), we thus see that a natural candidate for charge quantization of the supergravity C_3-field is (nonabelian/unstable) 4-Cohomotopy theory π^4 [Sa13, §2.5][FSS16a, §2][BMSS19, §3] (review in [FSS19a, §7]) or rather: differential 4-Cohomotopy theory $\widehat{\pi}^4$ [FSS15b, p. 9][GS20].

(ii) The consequence of this Cohomotopical charge quantization is readily seen from the Hurewicz operation on Cohomotopy theory (Example 2.26): The de Rham class of the 4-flux density is constrained to be integral, hence to be in the image of the de Rham homomorphism (Example 4.9) and its cup square is forced to vanish

$$\big[G_4(\widehat{C}_3)\big] \in H^4\big(X;\mathbb{Z}\big) \longrightarrow H^4_{\mathrm{dR}}(X)\,, \qquad \big[G_4(\widehat{C}_3)\big] \cup \big[G_4(\widehat{C}_3)\big] = 0\,.$$

This leads to interpretation via Massey products [KS05], with corresponding differential refinement in [GS17a].

(iii) Passing from 11-dimensional supergravity to M-theory, the curvature data in (241) is expected to be subjected to more refined topological constraints, forcing the class of G_4 to be integral up to a fractional shift by the first Pontrjagin class of the tangent bundle, and deforming its cup square to a quadratic function with non-trivial "background charge" ([FSS19b, Table 1]). Furthermore, comparing with integral and differential cohomology yields subtle but natural conditions on the theory [GS20]. We see, in Prop. 5.22 below, that these more subtle M-theoretic constraints on the C_3-field flux densities are imposed by charge quantization in – hence lifting through the non-abelian character map of – the corresponding *twisted* non-abelian cohomology theory, namely: *J-twisted* 4-Cohomotopy [FSS19b][FSS20] (Example 5.21 below).

Differential ordinary cohomology. The *ordinary differential cohomology* $\widehat{H}^\bullet(X)$ [SiSu08] of a smooth manifold X combines ordinary integral cohomology classes (Example 2.2) with closed differential forms that represent the same class in real cohomology, in that it makes a diagram of the following form commute:

$$\begin{tikzcd}
\underset{\text{differential cohomology}}{\overset{\text{ordinary}}{\widehat{H}^\bullet(X)}} \arrow[r, "\text{curvature}"] \arrow[d, "\text{underlying integral class}"'] & \Omega^\bullet_{\mathrm{dR}}(X)_{\mathrm{clsd}} \arrow[d, "\text{via de Rham theorem}"] \\
H^\bullet(X;\mathbb{Z}) \arrow[r, "\text{rationalization}"] & H^\bullet(X;\mathbb{R})
\end{tikzcd} \tag{242}$$

In fact, differential cohomology is universal with this property, but not at the coarse level of cohomology sets shown above (where the universal property is shallow) but at the fine level of of complexes of sheaves of coefficients (i.e. of moduli ∞-stacks), as made precise in Prop. 4.40 below.

In degree 2, ordinary differential cohomology classifies ordinary $\mathrm{U}(1)$-principal bundles (equivalently: complex line bundles) with connection [Bry93, §II], and the curvature map in (242) assigns their traditional curvature 2-form. In degree 3 ordinary differential cohomology classifies bundle gerbes with connection [Mu96][SW07] with their curvature 3-form. In general degree it classifies higher bundle gerbes with connection [Ga97], or equivalently higher $\mathrm{U}(1)$-principal bundles with connection [FSS12b, 2.6].

One construction of ordinary differential cohomology over smooth manifolds is given in [CS85, §1], now known as *Cheeger-Simons characters*. An earlier construction over schemes, now known as *Deligne cohomology* (Example 4.39), due independently to [De71, §2.2][MM74, §3.1.7][AM77, §III.1] and brought to seminal application in [Bei85] (review in [EV88]) is readily adapted to smooth manifolds [Bry93, §I.5][Ga97]. The advantage of Deligne cohomology over Cheeger-Simons characters is that is immediately generalizes from smooth manifolds to smooth ∞-stacks, [FSSt10, §3.2.3][FSS12b, §2.5], such as to orbifolds [SS20a] and to moduli ∞-stacks of higher principal connections where it yields higher Chern-Simons functionals [SSS12][FSS12a][FSS13a][FSS15a], as well as allowing for twists in a systematic manner [GS18c][GS19b].

In differential enhancement of Example 2.12, we have:

Example 4.39 (Ordinary differential cohomology on smooth ∞-stacks [FSSt10, §3.2.3][FSS12b, §2.5])**.** Let $n \in \mathbb{N}$.
(i) The smooth *Deligne-Beilinson complex* in degree $n+1$ is the presheaf of connective chain complexes (Example A.48) over CartesianSpaces (338) given by the truncated and shifted smooth de Rham complex (Example 3.23) with a copy of the integers included in degree $n+1$ (as integer valued 0-forms, hence as smooth real-valued functions constant on an integer):

$$\mathrm{DB}^{n+1}_\bullet := \Big(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \hookrightarrow \Omega^0_{\mathrm{dR}}(-) \xrightarrow{d} \Omega^1_{\mathrm{dR}}(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathrm{dR}}(-) \Big). \tag{243}$$

(ii) The de Rham differential in degree 0 gives a morphism of presheaves of complexes

$$\mathrm{DB}^{n+1}_\bullet \xrightarrow{(0,0,\cdots,0,d)} \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \tag{244}$$

from the Deligne-Beilinson complex (243) to the presheaf of closed $(n+1)$-forms, regarded as a presheaf of chain complexes regarded in degree 0.
(iii) *Ordinary differential cohomology* is stacky non-abelian cohomology (Remark 2.27) with coefficients in the Deligne-Beilinson complex (243) regarded as a smooth ∞-stack (Def. A.44) under the ∞-stackified Dold-Kan construction from Example A.53 (hence sheaf hypercohomology with coefficients in the Deligne complex):

$$\underset{\text{differential cohomology}}{\overset{\text{ordinary}}{\widehat{H}^{n+1}\big(\mathscr{X}\big)}} := \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\Big(\mathscr{X}\;,\; \underset{\text{correspondence}}{\overset{\text{Dold-Kan}}{L^{\mathrm{loc}} \circ \mathrm{DK}}}\big(\underset{\text{complex}}{\overset{\text{Deligne-Beilinson}}{\mathrm{DB}^{n+1}_\bullet}}\big)\Big). \tag{245}$$

(iv) The *curvature map* on ordinary differential cohomology is the cohomology operation induced by (244):

$$\begin{tikzcd}
\underset{\text{differential cohomology}}{\overset{\text{ordinary}}{\widehat{H}^{n+1}(\mathscr{X})}} \arrow[r, "\text{curvature}", "F"'] \arrow[d, equal] & \Omega^{n+1}_{\mathrm{dR}}(\mathscr{X})_{\mathrm{clsd}} \arrow[d, equal] \\
\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\Big(\mathscr{X}, L_{\mathrm{loc}} \circ \mathrm{DK}\big(\mathrm{DB}^{n+1}_\bullet\big)\Big) \arrow[r, "\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)(\mathscr{X}{,}L_{\mathrm{loc}}\circ \mathrm{DK}(d))"] & \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)\Big(\mathscr{X}, L_{\mathrm{loc}} \circ \mathrm{DK}\big(\Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}}\big)\Big)
\end{tikzcd} \tag{246}$$

Proposition 4.40 (Differential non-abelian cohomology subsumes differential ordinary cohomology [FSSt10, Prop. 3.2.26])**.** *Let $n \in \mathbb{N}$ and consider $A = B^n\mathrm{U}(1) \simeq K(\mathbb{Z}, n+1)$ (Example 2.12). Then:*

(i) *Differential non-abelian A-cohomology (Def. 4.33) coincides with ordinary differential cohomology (Def. 4.39):*

$$\overset{\text{ordinary differential cohomology}}{\widehat{H}^{n+1}\big(\mathscr{X}\big)} \;\simeq\; \widehat{H}\big(\mathscr{X}\,;B^n\mathrm{U}(1)\big)\,. \tag{247}$$

(ii) *The abstract curvature map in differential A-cohomology* (228) *reproduces the ordinary curvature map* (246)*.*

Proof. First we use the Dold-Kan correspondence (Prop. A.50) to obtain a convenient presentation of the differential character:

(a) Since the Dold-Kan construction DK (Def. A.53) realizes homotopy groups from homology groups (348), and since Eilenberg-MacLane spaces are characterized by their homotopy groups (14), we have the vertical identifications on the left of the following diagram:

$$\begin{tikzcd}
\mathrm{Disc}\big(B^{n+1}\mathbb{Z}\big) \arrow[r, "\eta^{\mathbb{R}}_{B^{n+1}\mathbb{Z}}"'] \arrow[rr, bend left, "\mathbf{ch}_{B^n\mathrm{U}(1)}"] \arrow[d, equal] & \mathrm{Disc}\big(B^{n+1}\mathbb{R}\big) \arrow[r, "\simeq"'] \arrow[d, equal] & \flat\exp(\mathfrak{b}^n\mathbb{R}) \arrow[d, "\int_{\Delta^\bullet}"', "\simeq"] \\
\mathrm{DK}\left(\begin{array}{c} \mathbb{Z} \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \end{array}\right) \arrow[r, hook] & \mathrm{DK}\left(\begin{array}{c} \mathbb{R} \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \end{array}\right) \arrow[r, hook] & \mathrm{DK}\left(\begin{array}{c} \Omega^0_{\mathrm{dR}}(-) \\ \downarrow d \\ \Omega^1_{\mathrm{dR}}(-) \\ \downarrow d \\ \vdots \\ \downarrow d \\ \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \end{array}\right)
\end{tikzcd} \tag{248}$$

Under this identification, it is clear that the rationalization map $\eta^{\mathbb{R}}_{B^{n+1}\mathbb{Z}}$ (Def. 3.53) is presented by the canonical inclusion of the integers into the real numbers, as on the bottom left of (248).

Moreover, the right vertical equivalence in (248) is that from Lemma 4.30.

(b) Since the differential character (227) in the present case evidently comes from a morphism of (presheaves of) simplicial abelian groups, with group structure given by addition of ordinary differential forms (Example 3.77), we may, using the Dold-Kan correspondence (Prop. A.50), analyze the remainder of the diagram on normalized chain complexes $N(-)$ (347).

Using this, it follows by inspection of the bottom map in (227) that the bottom right square in (248) commutes, with the bottom morphism on the right being the canonical inclusion of (presheaves of) chain complexes.

Now to use this presentation for identifying the resulting homotopy fiber product:

(i) Since the DK-construction (Def. A.53), applied objectwise over CartesianSpaces, is a right Quillen functor into the global model structure from Example A.43, and since ∞-stackification preserves homotopy pullbacks (Lemma A.46), it is now sufficient to show, by definition (245), that the homotopy pullback (Def. A.23) along the bottom map in (248), formed in presheaves of chain complexes is the Deligne-Beilinson complex (243). For this, by (315) it is sufficient to find a fibration replacement of the bottom map in (248) whose ordinary fiber product with $\Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}}$ is the Deligne-Beilinson complex. This is the case for the following factorization:

$$
\begin{tikzcd}[ampersand replacement=\&]
\& \left(\begin{array}{c} \mathbb{Z} \\ \downarrow{\scriptstyle i} \\ \Omega^0_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^1_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \vdots \\ \downarrow{\scriptstyle d} \\ \Omega^{n-1}_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^n_{\mathrm{dR}}(-) \end{array}\right)
\arrow[r, "{\left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ d \end{array}\right)}"]
\arrow[d, "i_1"']
\arrow[dr, phantom, "\text{(pb)}"]
\& \left(\begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \\ \downarrow \\ \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \end{array}\right)
\arrow[d, "i"]
\\
\left(\begin{array}{c} \mathbb{Z} \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{array}\right)
\arrow[r, "n \mapsto (n,n)", "\in \mathrm{W}"']
\& \left(\begin{array}{ccc} \mathbb{Z} & \oplus & \Omega^0_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle i} & \swarrow{\scriptstyle -\mathrm{id}} & \downarrow{\scriptstyle d} \\ \Omega^0_{\mathrm{dR}}(-) & \oplus & \Omega^1_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} & \swarrow{\scriptstyle +\mathrm{id}} & \downarrow{\scriptstyle d} \\ \Omega^1_{\mathrm{dR}}(-) & \oplus & \Omega^2_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} & \swarrow{\scriptstyle -\mathrm{id}} & \downarrow{\scriptstyle d} \\ \vdots & \vdots & \vdots \\ \downarrow{\scriptstyle d} & \swarrow & \downarrow{\scriptstyle d} \\ \Omega^{n-1}_{\mathrm{dR}}(-) & \oplus & \Omega^n_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} & \swarrow & \\ \Omega^n_{\mathrm{dR}}(-) & & \end{array}\right)
\arrow[r, "{\left(\begin{array}{c} \mathrm{pr}_2 \\ \mathrm{pr}_2 \\ \vdots \\ \mathrm{pr}_2 \\ d \end{array}\right)}", "\in \mathrm{Fib}"']
\& \left(\begin{array}{c} \Omega^0_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^1_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^2_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \vdots \\ \downarrow{\scriptstyle d} \\ \Omega^n_{\mathrm{dR}}(-) \\ \downarrow{\scriptstyle d} \\ \Omega^{n+1}_{\mathrm{dR}}(-)_{\mathrm{clsd}} \end{array}\right)
\end{tikzcd}
\tag{249}
$$

Here the total bottom morphims is the total bottom morphism from (248), factored as a weak equivalence (quasi-isomorhism) followed by a fibration (positve degreewise surjection). The ordinary pullback of the fibration is shown, and represents the homotopy pullback, since all chain complexes are projectively fibrant.

(ii) Finally, the top morphism in (249), thus being the abstract curvature map (228) is seen to coincide with the curvature map (244) on the Deligne complex. □

Secondary non-abelian cohomology operations. We define secondary non-abelian cohomology operations (Def. 4.42 below) which generalize the classical notion of secondary characteristic classes (Theorem 4.46, see Remark 4.47 for the terminology) to higher non-abelian cohomology. To formulate the concept in this generality, we need a technical condition (Def. 4.41) which happens to be trivially satisfied in the classical case (Lemma 4.44 below):

Definition 4.41 (Absolute minimal model)**.** For $A_1, A_2 \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) we say that an *absolute minimal model* for a morphism $A_1 \overset{c}{\dashrightarrow} A_2$ in SimplicialSets is a morphism $\mathfrak{l}A_1 \overset{\mathfrak{c}}{\dashrightarrow} \mathfrak{l}A_2$ between the respective Whitehead L_∞-algebras (Prop. 3.61) which makes the square on the left and hence the square on the far right of the following diagram commute:

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{dRPL}}(A_1) & \mathrm{CE}(\mathfrak{l}A_1) \arrow[l, "p^{\min}_{A_1}"'] \\
\Omega^\bullet_{\mathrm{dRPL}}(A_2) \arrow[u] & \mathrm{CE}(\mathfrak{l}A_2) \arrow[l, "p^{\min}_{A_2}"] \arrow[u, dashed, "\mathfrak{c}"'] ,
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\eta^{\mathrm{PLdR}}_{A_1}"] \arrow[d, "c"'] \arrow[rr, bend left, "\mathbb{D}\eta^{\mathrm{PLdR}}_{A_1}"] & \exp \circ \Omega^\bullet_{\mathrm{PLdR}}(A_1) \arrow[r, "\exp(p^{\min}_{A_1})"] \arrow[d, "\exp \circ \Omega^\bullet_{\mathrm{PLdR}}(c)"] & \exp \circ \mathrm{CE}(\mathfrak{l}A_1) \arrow[d, dashed, "\exp \circ \mathrm{CE}(\mathfrak{c})"] \\
A_2 \arrow[r, "\eta^{\mathrm{PLdR}}_{A_2}"] \arrow[rr, bend right, "\mathbb{D}\eta^{\mathrm{PLdR}}_{A_2}"'] & \exp \circ \Omega^\bullet_{\mathrm{PLdR}}(A_2) \arrow[r, "\exp(p^{\min}_{A_2})"] & \exp \circ \mathrm{CE}(\mathfrak{l}A_2) ,
\end{tikzcd}
\tag{250}
$$

$\in$ DiffGradedCommAlgebras$^{\geq 0}_{\mathbb{R}}$ $\qquad\qquad$ $\in$ SimplicialSets

hence a morphism that yields a transformation between exactly those derived adjunction units $\mathbb{D}\eta^{\mathrm{PLdR}}$ (311) of the PL-de Rham adjunction (116) that are given by *minimal* fibrant replacement.[15] In this case, the commuting diagram (250) evidently extends to a strict transformation between the differential non-abelian characters (227) on the A_i (Def. 4.32), in that the following diagram of simplicial presheaves (Def. 339) commutes:

$$\begin{tikzcd} \mathrm{Disc}(A_1) \arrow[r, "\mathbf{ch}_{A_1}"] \arrow[d, "\mathrm{Disc}(c)"'] & \flat\exp(\mathfrak{l}A_1) \arrow[d, "\flat\exp(\mathfrak{c})"] \\ \mathrm{Disc}(A_2) \arrow[r, "\mathbf{ch}_{A_2}"] & \flat\exp(\mathfrak{l}A_2) \end{tikzcd} \in \mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets}\big). \tag{251}$$

In differential enhacement of Def. 2.17 we have:

Definition 4.42 (Secondary non-abelian cohomology operation)**.** Let $A_1 \xrightarrow{c} A_2$ in SimplicialSets, with induced cohomology operation (Def. 2.17)

$$H(-;A_1) \xrightarrow{c_*} H(-;A_2),$$

have an absolute minimal model $\mathfrak{c}$ (Def 4.41). Then the corresponding *secondary non-abelian cohomology operation* is the structured cohomology operation (Remark 2.27)

$$\widehat{H}(-;A_1) \xrightarrow[\substack{\text{secondary}\\ \text{non-abelian character}}]{(c_{\mathrm{diff}})_*} \widehat{H}(-;A_2) \tag{252}$$

on differential non-abelian cohomology (Def. 4.33) which is induced (33) by the dashed morphism c_{diff} in the following diagram, which in turn is induced from c and $\mathfrak{c}$ (251) by the universal property of the defining homotopy pullback operation (227):

$$\begin{tikzcd} \text{secondary/differential cohomology operation} & (A_1)_{\mathrm{diff}} \arrow[rrr, dashed, "c_{\mathrm{diff}}"] \arrow[dd, "c_{A_1}"'] \arrow[dr, "F_{A_1}"] & & & (A_2)_{\mathrm{diff}} \arrow[dd, "c_{A_2}"'] \arrow[dr, "F_{A_2}"] & \\ & & \Omega_{\mathrm{dR}}\big(-;\mathfrak{l}A_1\big)_{\mathrm{flat}} \arrow[rrr, "\mathfrak{c}_*"] \arrow[dd] & & & \Omega_{\mathrm{dR}}\big(-;\mathfrak{l}A_2\big)_{\mathrm{flat}} \arrow[dd] \\ \text{plain/primary cohomology operation} & \mathrm{Disc}(A_1) \arrow[rrr, "\mathrm{Disc}(c)"] \arrow[dr, "\mathbf{ch}_{A_1}"'] & & & \mathrm{Disc}(A_2) \arrow[dr, "\mathbf{ch}_{A_2}"] & \\ \text{transformation of differential characters} & & \flat\exp(\mathfrak{l}A_1) \arrow[rrr, "\mathfrak{c}_*"] & & & \flat\exp(\mathfrak{l}A_2). \end{tikzcd} \tag{253}$$

The left and right squares are the homotopy pullback squares defining differential non-abelian cohomology (Def. 4.33) while the bottom square is the transformation of differential non-abelian characters (Def. 4.32) from (251).

In differential enhancement of Examples 2.26, 4.13 we have:

Example 4.43 (Secondary non-abelian Hurewicz/Boardman homomorphism to differential K-theory)**.** Consider the map

$$S^4 \xrightarrow{\beta^4} B\mathrm{U} \qquad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$$

from the 4-sphere to the classifying space of the infinite unitary group (24) which classifies a generator in $\pi_4\big(B\mathrm{U}\big) \simeq \mathbb{Z}$. By Example 3.66 and Examples 3.67, 3.92 the corresponding Whitehead L_∞-algebras (Prop. 3.61) are as shown here:

$$\begin{tikzcd} \mathrm{CE}\big(\mathfrak{l}S^4\big) \arrow[d, equal] & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{U}\big) \simeq \underset{k\in\mathbb{N}}{\otimes}\mathrm{CE}\big(\mathfrak{l}K(\mathbb{Z},2k)\big) \arrow[ll] \arrow[d, equal] \\ \mathbb{R}\begin{bmatrix}\omega_7, \\ \omega_4\end{bmatrix} \Big/ \begin{pmatrix} d\,\omega_7 = -\omega_4\wedge\omega_4 \\ d\,\omega_4 = 0 \end{pmatrix} & & \mathbb{R}\begin{bmatrix}\vdots \\ f_4, \\ f_2,\end{bmatrix} \Big/ \begin{pmatrix} \vdots \\ d f_4 = 0 \\ d f_2 = 0 \end{pmatrix} \arrow[ll, hook', "{\left.\begin{matrix}\omega_4 \\ 0\end{matrix}\middle|\begin{matrix}2k=4 \\ \text{else}\end{matrix}\right\} \mapsfrom f_{2k}}"'] \end{tikzcd} \tag{254}$$

[15] Notice that the existence of morphisms $\mathfrak{c}$ making this diagram commute is not guaranteed; it is only the existence of the *relative* minimal morphism $\mathfrak{l}_{A_2}(c)$ from Prop. 3.68 which is guaranteed to make the square (139) commute.

The morphism shown in (254) evidently restricts to the relative rational Whitehead L_∞-algebra inclusion (Prop. 3.68) on the factor $K(\mathbb{R},4) \subset L_{\mathbb{R}}B\mathrm{U}$ and is zero elsewhere, hence fits into the required diagram (250) exhibiting it as an absolute minimal model (Def. 4.41) for β^4 (by the commuting diagram in Prop. 3.48).

Therefore, Def. 4.42 induces from β^4 a secondary non-abelian cohomology operation, going from differential 4-Cohomotopy (Example 4.38) to differential K-theory (Example 4.36), which on curvature forms (231) injects the 4-form curvature G_4 in differential Cohomotopy to the 4-form component F_4 in differential K-theory, as shown in (7).

Cheeger-Simons homomorphism. Where the construction of the Chern-Weil homomorphism (Def. 4.21) invokes connections on principal bundles without actually being sensitive to this choice (by Prop. 4.23), the *Cheeger-Simons homomorphism* [CS85, §2][HS05, §3.3] (based on [CS74]) is a refinement of the Chern-Weil homomorphism, now taking values in differential ordinary cohomology (Example 4.39), that does detect connection data (hence "differential" data):

$$
\begin{tikzcd}
G\mathrm{Connections}(X)_{/\sim} \arrow[r, "\substack{\text{Cheeger-Simons}\\ \text{homomorphism}}", "\mathrm{cs}_G"'] \arrow[d, "\substack{\text{forget}\\ \text{connection}}"'] & \mathrm{Hom}_{\mathbb{Z}}\Big(H^\bullet(BG;\mathbb{Z})\,,\, \underset{\substack{\text{differential}\\ \text{cohomology}}}{\widehat{H}^\bullet(X)}\Big) \arrow[d, "\text{curvature map}"] \\
G\mathrm{Bundles}(X)_{/\sim} \arrow[r, "\mathrm{cw}_G", "\substack{\text{Chern-Weil}\\ \text{homomorphism}}"'] & \mathrm{Hom}_{\mathbb{R}}\Big(\mathrm{inv}^\bullet(\mathfrak{g})\,,\, \underset{\substack{\text{de Rham}\\ \text{cohomology}}}{H^\bullet_{\mathrm{dR}}(X)}\Big)
\end{tikzcd}
\tag{255}
$$

We discuss how the general notion of secondary non-abelian cohomology operations (Def. 4.42) specializes on ordinary principal bundles to the Cheeger-Simons homomorphism, and hence generalizes it to higher non-abelian cohomology:

Lemma 4.44 (Characteristic classes of G-principal bundles have absolute minimal models)**.** *Let G be a connected compact Lie group with classifying space BG (16). For $n \in \mathbb{N}$, let $[c] \in H^{n+1}(BG;\mathbb{Z})$ be a universal integral characteristic class (Example 2.4). Then every representative classifying map $BG \xrightarrow{c} B^{n+1}\mathbb{Z}$ has an absolute minimal model in the sense of Def. 4.41.*

Proof. By Lemma 4.24, the minimal Sullivan model for BG has vanishing differential, while the minimal Sullivan model of $B^{n+1}\mathbb{Z}$ is a tensor factor (by Example 3.65), whose inclusion is already the relative minimal Sullivan model $\mathfrak{l}_{B^{n+1}\mathbb{Z}}(c)$ (Prop. 3.68) of c. Therefore, setting

$$
\mathrm{CE}(\mathfrak{c}) := \mathrm{CE}\big(\mathfrak{l}_{B^{n+1}\mathbb{Z}}(c)\big) \;:\; \mathbb{R}[c]/(d\,c=0) \hookrightarrow \mathrm{inv}^\bullet(\mathfrak{g})
\tag{256}
$$

gives the required morphism of minimal models that makes makes the square (250) commute, by (139). □

In differential enhancement of Example 2.18 we have:

Definition 4.45 (Secondary characteristic classes of differential non-abelian G-cohomology)**.** Let G be a connected compact Lie group with classifying space BG (16). By Lemma 4.44), the construction of secondary characteristic classes (Def. 4.42, on differential non-abelian G-cohomology (Example 4.37) exists generally, and yields a $\mathbb{Z}$-linear map of the form

$$
H\big(BG; B^\bullet\mathbb{Z}\big) \xrightarrow{(-)_{\mathrm{diff}}} \widehat{H}\big(BG_{\mathrm{diff}}; B^\bullet\mathbb{Z}\big) \;=\; H\big(BG_{\mathrm{diff}}; B^\bullet\mathbb{Z}_{\mathrm{diff}}\big)\,,
$$

where on the right we have the ordinary differential non-abelian cohomology (Prop. 4.40) *of* the moduli ∞-stack BG_{diff} (228). Combined with the composition operation in $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (A.44) this gives a map

$$
\widehat{H}\big(X;BG\big)\times H\big(BG;B^\bullet\mathbb{Z}\big) \xrightarrow{\mathrm{id}\times(-)_{\mathrm{diff}}} H\big(X;BG_{\mathrm{diff}}\big)\times H\big(BG_{\mathrm{diff}};B^\bullet\mathbb{Z}_{\mathrm{diff}}\big) \xrightarrow{\circ} H\big(X;B^\bullet\mathbb{Z}_{\mathrm{diff}}\big) \;=\; \widehat{H}\big(X;B^\bullet\mathbb{Z}\big)
$$

which is $\mathbb{Z}$-linear in its second argument, and whose hom-adjunct is

$$
\widehat{H}(X;BG) \xrightarrow{\nabla\mapsto(c\mapsto c_{\mathrm{diff}}(\nabla))} \mathrm{Hom}_{\mathbb{Z}}\big(H(BG;B^\bullet\mathbb{Z})\,,\,\widehat{H}(X;B^\bullet\mathbb{Z})\big)\,.
\tag{257}
$$

Theorem 4.46 (Secondary non-abelian cohomology operations subsume Cheeger-Simons homomorphism)**.** *Let G be a connected compact Lie group, with classifying space denoted BG* (16)*. Then the canonical construction* (257) *of secondary characteristic classes on differential non-abelian G-cohomology (Def. 4.45) coincides with the Cheeger-Simons homomorphim* (255)*, in that the following diagram commutes:*

$$
\begin{tikzcd}
G\mathrm{Connections}(X) \arrow[rr, "\substack{\text{Cheeger-Simons}\\ \text{homomorphism}}", "\mathrm{cs}_G"'] \arrow[d, "(238)"'] & & \mathrm{Hom}_{\mathbb{Z}}\Big(H^\bullet(BG;\mathbb{Z}) \,, \underset{\substack{\text{differential}\\ \text{ordinary}\\ \text{cohomology}}}{\widehat{H}^\bullet(X)} \Big) \\
\underset{\substack{\text{differential non-abelian}\\ \text{cohomology}}}{\widehat{H}(X;BG)} \arrow[rr, "\nabla \mapsto (c \mapsto c_{\mathrm{diff}}(\nabla))", "\substack{\text{secondary}\\ \text{non-abelian cohomology operations}}"'] & & \mathrm{Hom}_{\mathbb{Z}}\Big(H(BG;B^\bullet\mathbb{Z}) \,, \widehat{H}(X;B^\bullet\mathbb{Z}) \Big) \arrow[u, "\simeq\ (247)"']
\end{tikzcd}
\quad , \tag{258}
$$

where on the left we have the map from G-connections to differential non-abelian G-cohomology from Prop. 4.37, and on the right the identification of ordinary differential cohomology from Prop. 4.40.

Proof. Let $c \in H\big(BG; B^\bullet\mathbb{Z}\big)$ be a characteristic class, and let (f^*EG, ∇) be a G-principal bundle equipped with a G-connection. By Prop. 4.37, its image in differential non-abelian cohomology is given by the first map in the following diagram

$$
\begin{array}{ccccccc}
G\mathrm{Connections}(X)_{/\sim} & \longrightarrow & \widehat{H}(X;BG) & \xrightarrow{(c_{\mathrm{diff}})_*} & \widehat{H}\big(X;B^{n+1}\mathbb{Z}\big) & \xrightarrow{\simeq} & \widehat{H}^{n+1}(X) \\
\big[f^*EG, \nabla\big] & \longmapsto & \big[f,\, \big(\mathrm{cs}_k(\nabla, f^*\nabla_{\mathrm{univ}})\big)\, \big(\omega_k(F_\nabla)\big)\big] & \longmapsto & \big[f^*c,\, \mathrm{cs}_c(\nabla, f^*\nabla_{\mathrm{univ}}),\, c(F_\nabla)\big] & &
\end{array}
\tag{259}
$$

Here the triple of data are the three components (Example A.26) of a map into the defining homotopy pullback of differential non-abelian cohomology (239). Therefore, the secondary operation induced by the transformation (253) of these homotopy pullbacks, which in the present case is of this form:

$$
\begin{tikzcd}
BG_{\mathrm{diff}} \arrow[rr, "\substack{\text{secondary}\\ \text{characteristic class}}", "c_{\mathrm{diff}}"'] \arrow[dr] \arrow[dd, "c_{BG}"'] & & B^{n+1}\mathbb{Z}_{\mathrm{diff}} \arrow[dr] \arrow[dd, "c_{B^{n+1}\mathbb{Z}}"' near start] & \\
& \Omega_{\mathrm{dR}}(-;\mathfrak{l}BG)_{\mathrm{flat}} \arrow[rr] \arrow[dd] & & \Omega_{\mathrm{dR}}(-;\mathfrak{l}B^{n+1}\mathbb{Z})_{\mathrm{flat}} \arrow[dd] \\
BG \arrow[rr, "\text{characteristic class}" near end, "c"' near end] \arrow[dr, "\mathbf{ch}_{BG}"'] & & B^{n+1}\mathbb{Z} \arrow[dr, "\mathbf{ch}_{B^{n+1}\mathbb{Z}}"] & \\
& \flat\exp(\mathfrak{l}BG) \arrow[rr, "\mathfrak{c}_*"'] & & \flat\exp(\mathfrak{l}B^{n+1}\mathbb{Z})
\end{tikzcd}
\quad , \tag{260}
$$

acts **(a)** on the first component in the triple by postcomposition with c, hence as

$$
f \mapsto f^*c := c \circ f
$$

and **(b)** on the other two components by composition with $\mathfrak{c}$, which by (256) corresponds to projecting out the Chern-Simons form and characteristic form corresponding to c, respectively. This is shown as the second map in (259). Hence we are reduced to showing that the total map in (259) gives the Cheeger-Simons homomorphism. This statement is the content of [HS05, §3.3]. □

Remark 4.47 (Secondary characteristic classes of G-connections)**.** The traditional reason for referring to the Cheeger-Simons homomorphism (258) as producing *secondary* invariants is that Cheeger-Simons classes $\mathrm{cs}_G(P,\nabla) \in \widehat{H}(X)$ may be non-trivial even if the underlying characteristic class $\mathrm{cw}_G(P)$ (the "primary" class) vanishes. In this case the $\mathrm{cs}_G(P,\nabla)$ are also called *Chern-Simons invariants*.

(i) This happens, in particular, when the G-connection ∇ is flat, $F(\nabla) = 0$ (by Def. 4.19). Such secondary Chern-Simons invariants exhibit some subtle phenomena ([Rzn95][Rzn96][IS07][Es09]).

(ii) In fact, the proof of Theorem 258, via the triples (239) of homotopy data, shows that, in this case, $\mathrm{cs}_G(P,\nabla)$ measures *how* (or "why") $\mathrm{cw}_G(P)$ vanishes, namely by which class of homotopies.

(iii) Here we may understand secondary classes more abstractly, and explicitly related to the non-abelian character map: Where a (primary) non-abelian cohomology operation, according to Def. 2.17, is induced by a morphism of coefficient spaces (28), a secondary non-abelian cohomology operation, according to Def. 4.42, is induced (252) by a morphism of non-abelian character maps (251) – hence by a morphism of morphisms – on these coefficient spaces.

(iv) Note that classical secondary cohomology operations themselves admit differential refinements. For instance, for the case of Massey products as secondary operations for the cup product [GS17a]. While these can also fit into our context on general grounds, we will not demonstrate that explicitly here.

5 The twisted (differential) non-abelian character map

We introduce the character map in twisted non-abelian cohomology (Def. 5.4) and then discuss how it specializes to:

§5.1 – the twisted Chern character on (higher) K-theory;
§5.3 – the twisted character on Cohomotopy theory.

Rationalization in twisted non-abelian cohomology. In generalization of Def. 4.1 we now define rationalization of local coefficient bundles (35). This operation is transparent in the language of ∞-category theory, where it simply amounts to forming the pasting composite with the homotopy-coherent naturality square of the rationalization unit $\eta^{\mathbb{R}}$:

$$
\overset{\tau\text{-twisted cocycle with local coefficients } \rho}{
\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Swarrow\simeq"] & A /\!\!/ G \arrow[dl, "\rho"] \\
& BG &
\end{tikzcd}}
\quad \overset{\text{rationalization}}{\longmapsto} \quad
\overset{\tau\text{-twisted cocycle with rationalized local coefficients } L_{\mathbb{R}}\rho}{
\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Swarrow\simeq"] & A /\!\!/ G \arrow[dl, "\rho"] \arrow[r, "\eta^{\mathbb{R}}_{A /\!\!/ G}"] \arrow[ddl, phantom, "\Swarrow\simeq"] & L_{\mathbb{R}}(A /\!\!/ G) \arrow[ddll, "L_{\mathbb{R}}\rho"] \\
& BG \arrow[d, "\eta^{\mathbb{R}}_{BG}"'] & & \\
& L_{\mathbb{R}}BG & &
\end{tikzcd}}
\tag{261}
$$

Slightly less directly but equivalently, this operation is the composite of **(a)** "base change" along $\eta^{\mathbb{R}}_{BG}$ from the slice over BG to the slice over $L_{\mathbb{R}}BG$, **(b)** followed by the composition with the naturality square, now regarded as a morphism in the slice over $L_{\mathbb{R}}BG$:

$$
\overset{\tau\text{-twisted cocycle with local coefficients } \rho}{
\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Swarrow\simeq"] & A /\!\!/ G \arrow[dl, "\rho"] \\
& BG &
\end{tikzcd}}
\overset{\text{base change}}{\longmapsto}
\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Swarrow\simeq"] & A /\!\!/ G \arrow[dl, "\rho"] \\
& BG \arrow[d, "\eta^{\mathbb{R}}_{BG}"'] & \\
& L_{\mathbb{R}}BG &
\end{tikzcd}
\overset{\substack{\text{composition} \\ \text{in slice}}}{\longmapsto}
\overset{\tau\text{-twisted cocycle with local coefficients } \rho}{
\begin{tikzcd}
X \arrow[rr, "c"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Swarrow\simeq"] & A /\!\!/ G \arrow[dl, "\rho"] \arrow[r, "\eta^{\mathbb{R}}_{A /\!\!/ G}"] \arrow[ddl, phantom, "\Swarrow\simeq"] & L_{\mathbb{R}}(A /\!\!/ G) \arrow[ddll, "L_{\mathbb{R}}\rho"] \\
& BG \arrow[d, "\eta^{\mathbb{R}}_{BG}"'] & & \\
& L_{\mathbb{R}}BG & &
\end{tikzcd}}
$$

It is in this second form that the operation lends itself to formulation in model category theory (Def. 5.2 below). For that we just need to produce a rectified (strictly commuting) incarnation of the $\eta^{\mathbb{R}}$-naturality square:

Lemma 5.1 (Rectified rationalization unit on coefficient bundle)**.** *Let*

$$
\begin{tikzcd}
A \arrow[r] & A /\!\!/ G \arrow[d, "\rho", "\text{local coefficient bundle}"'] \\
& BG
\end{tikzcd}
\tag{262}
$$

be a local coefficient bundle (35) *in* $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ *(Def. 3.50), and let*

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{PLdR}}\big(A /\!\!/ G\big) & \mathrm{CE}\big(\mathfrak{l}_{BG}(A /\!\!/ G)\big) \arrow[l, "p^{\mathrm{min}_{BG}}_{A /\!\!/ G}"', "\in \mathrm{W}"] \\
\Omega^\bullet_{\mathrm{PLdR}}\big(BG\big) \arrow[u, "\Omega^\bullet_{\mathrm{PLdR}}(\rho)"] & \mathrm{CE}\big(\mathfrak{l}(BG)\big) \arrow[l, "p^{\mathrm{min}}_{BG}"', "\in \mathrm{W}"] \arrow[u, "\mathrm{CE}(\mathfrak{l}p)"']
\end{tikzcd}
\tag{263}
$$

be its minimal relative Sullivan model (139)*, which exists by Prop. 3.68. Then the composite of the image of* (263) *under* exp *with the* $\Omega^\bullet_{\mathrm{PLdR}} \dashv \exp$*-adjunction unit (from Prop. 3.57):*

$$
\mathbb{D}\eta^{\mathrm{PLdR}}_{\rho} \;:=\;
\begin{tikzcd}
A/\!\!/G \arrow[r, "\eta^{\mathrm{PLdR}}_{A/\!\!/G}"] \arrow[d, "\rho"'] \arrow[rr, bend left, "\mathbb{D}\eta^{\mathrm{PLdR}}_{A/\!\!/G} \simeq \eta^{\mathbb{R}}_{A/\!\!/G}"] & \exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(A/\!\!/G\big) \arrow[r, "\exp(p^{\min_{BG}}_{A/\!\!/G})"] \arrow[d, "\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(\rho)"] & \exp\circ\mathrm{CE}\big(\mathfrak{l}_{BG}(A/\!\!/G)\big) \arrow[d, "\exp\circ\mathrm{CE}(\mathfrak{l}\rho)"] \\
BG \arrow[r, "\eta^{\mathrm{PLdR}}_{BG}"] \arrow[rr, bend right, "\mathbb{D}\eta^{\mathrm{PLdR}}_{BG} \simeq \eta^{\mathbb{R}}_{BG}"'] & \exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(BG\big) \arrow[r, "\exp(p^{\min}_{BG})"] & \exp\circ\mathrm{CE}\big(\mathfrak{l}(BG)\big)
\end{tikzcd}
\tag{264}
$$

is, after passage (303) *to the classical homotopy category (Example A.33), equivalent to the naturality square of the rationalization unit on* ρ (109)*:*

$$
\mathbb{D}\eta^{\mathrm{PLdR}}_{\rho} \;\simeq\; \eta^{\mathbb{R}}_{\rho}\,.
$$

Proof. By Prop. 3.43 the right part of (264) is the image under exp of a fibrant replacement morphism. By (311) this identifies the diagram as the naturality square of the derived PLdR adjunction unit, and by (119) in the fundamental theorem (Prop. 3.58) this implies the claim. □

Definition 5.2 (Rationalization in twisted non-abelian cohomology)**.** Given a local coefficient bundle ρ and its rectified rationalization unit $\mathbb{D}\eta^{\mathrm{PLdR}}_{\rho}$ as in Lemma 5.1 we say that *rationalization* in twisted non-abelian cohomology with local coefficients ρ (Def. 2.29) is the twisted non-abelian cohomology operation (Def. 2.40)

$$
\big(\eta^{\mathbb{R}}_{\rho}\big)_{*} \;:\; H^{\tau}(X;A) \xrightarrow{\;(\mathbb{D}\eta^{\mathrm{PLdR}}_{\rho}\circ(-))\,\circ\,\mathbb{L}(\eta^{\mathbb{R}}_{BG})_{!}\;} H^{L_{\mathbb{R}}\tau}\big(X;L_{\mathbb{R}}A\big)
\tag{265}
$$

given by the composite of

(a) derived left base change $\mathbb{L}(\eta^{\mathbb{R}}_{BG})_{!}$ (Example A.18) along the rationalization unit (109) on the classifying space of twists,

(b) composition with the rectified rationalization unit (264) on the coefficient bundle, regarded as a morphism in the homotopy category (303) of the slice model category (Example A.10) of $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8) over $\exp\circ\mathrm{CE}(\mathfrak{l}BG))$.

Remark 5.3 (Commutativity of rationalization over twisting)**.** The existence of the transformation (264) from a local coefficient bundle to its rationalization, inducing the cohomology operation (265) from any twisted cohomology theory to its twisted rational cohomology, may be thought of as exhibiting commutativity of rationalization over twisting. For twisted KO-theory this is discussed in [GS19d, Prop. 4].

Twisted non-abelian character map. In generalization of Def. 4.2 we set:

Definition 5.4 (Twisted non-abelian character map)**.** Let $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50) equipped with the structure of a smooth manifold, and

$$
\begin{tikzcd}
A \arrow[r] & A/\!\!/G \arrow[d, "\rho", "\text{local coefficient bundle}"'] \\
& BG
\end{tikzcd}
\tag{266}
$$

be a local coefficient bundle (35) in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50). Then the *twisted non-abelian character map* in twisted non-abelian cohomology is the twisted cohomology operation

$$
\underset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{character map}}}{\mathrm{ch}_{\rho}} \;:\; \overset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{cohomology}}}{H^{\tau}(X;A)} \xrightarrow[\text{rationalization}]{(\eta^{\mathbb{R}}_{\rho})_{*}} \overset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{real cohomology}}}{H^{L_{\mathbb{R}}\tau}\big(X;L_{\mathbb{R}}A\big)} \xrightarrow[\substack{\text{twisted}\\ \text{non-abelian}\\ \text{de Rham theorem}}]{\simeq} \overset{\substack{\text{twisted}\\ \text{non-abelian}\\ \text{de Rham cohomology}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}(X;\mathfrak{l}A)}
\tag{267}
$$

from twisted non-abelian A-cohomology (Def. 2.29) to twisted non-abelian de Rham cohomology (Def. 3.96) with local coefficients in the rational relative Whitehead L_∞-algebra $\mathfrak{l}\rho$ of ρ (Prop. 3.73) which is the composite of

(i) the operation (265) of rationalization of local coefficients (Def. 5.2),

(ii) the equivalence (188) of the twisted non-abelian de Rham theorem (Theorem 3.102).

5.1 Twisted Chern character on higher K-theory

We discuss how the twisted non-abelian character map reproduces the the twisted Chern character in twisted topological K-theory [BCMMS02, 6.3][MaS03][AS06, §7][TX06][MaS06, §6][FrHT08, §2][BGNT08][GT10, §4][Ka12, §8.3][GS19a, §3.2][GS19c] (Prop. 5.5) and in twisted iterated K-theory [LSW16, §2.2] (Prop. 5.8).

In twisted enhancement of Example 4.10, we have:

Proposition 5.5 (Twisted Chern character in twisted topological K-theory)**.** *Consider twisted complex topological K-theory* $\mathrm{KU}^{\tau}(-)$ *(Example 2.35), for degree-3 twists given (via Example 2.11) by*

$$\tau \,\in\, H\big(-; B^2\mathrm{U}(1)\big) \,\simeq\, H^3(-;,\mathbb{Z})\,,$$

and regarded, via (48)*, as twisted non-abelian cohomology with local coefficients in* $\mathbb{Z}\times B\mathrm{U} \,/\!\!/\, B^2\mathrm{U}(1)$ (47)*. Then the twisted non-abelian character map (Def. 5.4)* $\mathrm{ch}^{\tau}_{\mathbb{Z}\times B\mathrm{U}}$ *is equivalent to the traditional twisted Chern character* ch^{τ} *on twisted K-theory with values in* H_3*-twisted de Rham cohomology (Def. 3.97):*

$$\underset{\text{character map}}{\overset{\text{twisted non-abelian}}{\mathrm{ch}^{\tau}_{\mathbb{Z}\times B\mathrm{U}}}} \;\;\simeq\;\; \underset{\text{Chern character}}{\overset{\text{twisted}}{\mathrm{ch}^{\tau}}}.$$

Proof. That the codomain of the twisted non-abelian character map, in this case, is indeed H_3-twisted de Rham cohomology is the content of Prop. 3.98. With this, and due to the twisted non-abelian de Rham theorem (Theorem 3.102), it is sufficient to see that the general rationalization map of local non-abelian coefficients from Def. 5.2 reproduces the rationalization map underlying the twisted Chern character. This is manifest from comparing the rationalization operation (261), that is made formally precise by Def. 5.2, to the description of the twisted Chern character as given in [FrHT08, (2.8)-(2.9)]. ☐

Remark 5.6 (Twisted Pontrjagin character in twisted KO-theory)**.** Similarly, an analogous statement holds for the twisted Pontrjagin character (as in Example 4.11) on twisted real K-theory [GS19d, Prop. 2].

Lemma 5.7 (Higher twisted de Rham coefficients inside rational twisted iterated K-theory)**.** *There is a non-trivial twisted cohomology operation (Def. 2.40) from* **(a)** *twisted non-abelian de Rham cohomology (Def. 3.96) with coefficients in the relative rational Whitehead* L_∞*-algebra (Prop. 3.68) of the coefficient bundle* (53) *of twisted iterated K-theory (Example 2.38) to* **(b)** *higher twisted de Rham cohomology (Def. 3.99) regarded as twisted non-abelian de Rham cohomology via Prop. 3.100):*

$$H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(-;\, \mathfrak{l}K^{\circ_{2r-2}}(\mathrm{ku})_1\Big) \xrightarrow{\;\;\phi_*\;\;} H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\Big(-;\, \underset{k\in\mathbb{N}}{\bigoplus}\, \mathfrak{b}^{2rk}\mathbb{R}\Big)\,, \tag{268}$$

given, under the twisted non-abelian de Rham theorem (Theorem 3.102) by the LSW-character from [LSW16, §2.2] applied to rational coefficients.

Proposition 5.8 (Twisted Chern character in twisted iterated K-theory)**.** *For* $r\in\mathbb{N}$*,* $r\geq 1$*, consider twisted iterated K-theory* $\big(K^{\circ_{2r-2}}(\mathrm{ku})\big)^{\tau}$ *(Example 2.38), for degree-*$(2r+1)$ *twists given (via Example 2.12) by*

$$\tau \,\in\, H\big(-; B^{2r}\mathrm{U}(1)\big) \,\simeq\, H^{2r+1}(-;,\mathbb{Z})\,,$$

and regarded, via Example 2.38, as twisted non-abelian cohomology with local coefficients in $\big(K^{\circ_{2r-2}}(\mathrm{ku})\big)_0$*. Then the twisted non-abelian character map (Def. 5.4)* $\mathrm{ch}^{\tau}_{K^{\circ_{2r-2}}(\mathrm{ku})_0}$ *composed with the projection operation* (268) *onto higher twisted de Rham cohomology, (Def. 3.99) from Lemma 5.7, is equivalent to the LSW character map* ch_{2r-1} *[LSW16, Def. 2.20] restricted along the connective inclusion*

$$\underset{\text{LSW character}}{\overset{\text{twisted}}{\mathrm{ch}^{\tau}_{2r-1}}} \;\;\simeq\;\; \underset{\substack{\text{projection onto}\\ \text{higher twisted}\\ \text{de Rham cohomology}}}{\phi_*} \;\circ\; \underset{\text{character map}}{\overset{\text{twisted non-abelian}}{\mathrm{ch}^{\tau}_{K^{\circ_{2r-2}}(\mathrm{ku})_0}}}.$$

Proof. After unwinding the definitions, the statement reduces to the commutativity of the square diagram in [LSW16, p. 15]: The top morphism there is the plain rationalization map (Def. 5.2), the right vertical morphism is ϕ_* from Lemma 5.7 before passing from real to de Rham cohomology, the left morphism is restriction to the connective part and the bottom morphism is the LSW character. ☐

5.2 Twisted differential non-abelian character

We introduce twisted differential non-abelian cohomology (Def. 5.11 below) and discuss how the corresponding twisted differential non-abelian character subsumes existing constructions on twisted differential K-theory (Examples 5.17 Example 5.20 below).

Twisted differential non-abelian cohomology. From the perspective of structured non-abelian cohomology (Remark 2.27) that we have developed, it is now evident how to canonically combine

(a) twisted non-abelian cohomology (Def. 2.29) with

(b) differential non-abelian cohomology (Def. 4.33) to get

twisted differential non-abelian cohomology:

Definition 5.9 (Differential non-abelian local coefficient bundles)**.** Let

$$\begin{tikzcd} A \arrow[r] & A /\!/ G \arrow[d, "\rho", "\text{local coefficient bundle}"'] \\ & BG \end{tikzcd}$$

be a local coefficient bundle (35) in $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)^{\mathrm{fin}_{\mathbb{R}}}_{\geq 1,\mathrm{nil}}$ (Def. 3.50).

(i) By Lemma 3.69, Lemma 5.1, and using that exp preserves fibrations (Prop. 3.60), this induces a homotopy fibration (Def. A.22) in $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (Def. A.44) of differential non-abelian character maps (Def. 4.32) of this form:

$$\begin{tikzcd}[column sep=small] \mathrm{Disc}(A) \arrow[rr, "\substack{\text{differential non-abelian character map} \\ \text{with coefficients in fiber space}}", "\mathbf{ch}_A"'] \arrow[dr, "\mathrm{hofib}(\mathrm{Disc}(\rho))"'] & & \flat\exp(\mathfrak{l}A) \arrow[dr, "\mathrm{hofib}((\mathfrak{l}\rho)_*)"] & & \Omega_{\mathrm{dR}}(-;\mathfrak{l}A)_{\mathrm{flat}} \arrow[ll, "\mathrm{atlas}"'] \arrow[dr, "\mathrm{hofib}((\mathfrak{l}\rho)_*)"] & \\ & \mathrm{Disc}\big(A/\!/G\big) \arrow[rr, "\mathbf{ch}^{BG}_{A/\!/G}", "\text{twisted differential non-abelian character map}"'] \arrow[d, "\mathrm{Disc}(\rho)"] & & \flat\exp\big(\mathfrak{l}(A/\!/G)\big) \arrow[d, "(\mathfrak{l}\rho)_*"] & & \Omega_{\mathrm{dR}}\big(-;\mathfrak{l}_{BG}(A/\!/G)\big)_{\mathrm{flat}} \arrow[ll, "\mathrm{atlas}"] \arrow[d, "(\mathfrak{l}\rho)_*"] \\ & \mathrm{Disc}(BG) \arrow[rr, "\mathbf{ch}_{BG}", "\substack{\text{differential non-abelian character map} \\ \text{with coefficients in space of twists}}"'] & & \flat\exp(\mathfrak{l}BG) & & \Omega_{\mathrm{dR}}(-;\mathfrak{l}BG)_{\mathrm{flat}} \arrow[ll, "\mathrm{atlas}"] \end{tikzcd} \tag{269}$$

(ii) Here the *twisted differential non-abelian character map* $\mathbf{ch}^{BG}_{A/\!/G}$ is defined just as in Def. 4.32, but with coefficients the relative Whitehead L_∞-algebra $\mathfrak{l}_{BG}(A/\!/G)$ (Prop. 3.68), as opposed to the absolute Whitehead L_∞-algebra $\mathfrak{l}(A/\!/G)$ (Prop. 3.61).

Remark 5.10 (Differential local coefficient bundles)**.** Since homotopy limits commute over each other, passage to the homotopy fiber products (Def. A.23) formed from the horizontal stages of (269) yields a homotopy fibration of moduli ∞-stacks of ∞-connections (228) of this form:

$$\begin{tikzcd}[column sep=small] & & & \Omega_{\mathrm{dR}}\big(-;\mathfrak{l}_{BG}(A/\!/BG)\big)_{\mathrm{flat}} \arrow[dr, "\mathrm{atlas}"] \arrow[ddd, "(\mathfrak{l}\rho)_*"'] & \\ \underset{\substack{\text{moduli } \infty\text{-stack of} \\ \Omega A\text{-connections}}}{A_{\mathrm{diff}}} \arrow[r, "\mathrm{hofib}(\rho_{\mathrm{diff}})"] & \big(A/\!/G\big)_{\mathrm{diff}_{BG}} \arrow[urr, "F^{BG}_{A/\!/G}"] \arrow[dr, "c^{BG}_{A/\!/G}"] \arrow[ddd, "\rho_{\mathrm{diff}}", "\substack{\text{differential non-abelian} \\ \text{local coefficient bundle}}"'] & & & \flat\exp\big(\mathfrak{l}_{BG}(A/\!/G)\big) \arrow[ddd, "(\mathfrak{l}\rho)_*"] \\ & & \mathrm{Disc}\big(A/\!/G\big) \arrow[urr, "\mathbf{ch}^{BG}_{A/\!/G}"] \arrow[ddd, "\mathrm{Disc}(\rho)"] & & \\ & & & \Omega_{\mathrm{dR}}(-;\mathfrak{l}BG)_{\mathrm{flat}} \arrow[dr, "\mathrm{atlas}"] & \\ & \underset{\substack{\text{moduli } \infty\text{-stack of} \\ G\text{-connections}}}{BG_{\mathrm{diff}}} \arrow[urr, "F_{BG}"] \arrow[dr, "c_{BG}"'] & & & \flat\exp(\mathfrak{l}BG) \\ & & \mathrm{Disc}(BG) \arrow[urr, "\mathbf{ch}_{BG}"] & & \end{tikzcd} \tag{270}$$

Definition 5.11 (Twisted differential non-abelian cohomology)**.** Given a differential non-abelian local coefficient bundle ρ_{diff} (270) according to Def. 5.9, we say that:

(i) A *differential twist* on a $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (Def. A.44) is a cocycle τ_{diff} in differential non-abelian cohomology with coefficients in BG (Def. 4.33)

$$\big[\tau_{\mathrm{diff}}\big] \;\in\; \widehat{H}\big(\mathscr{X}; BG\big). \tag{271}$$

(ii) The τ_{diff}*-twisted differential non-abelian cohomology* with local coefficients in ρ_{diff} is the structured (Remark 2.27) τ_{diff} twisted non-abelian cohomology (Def. 2.29) with coefficients in ρ_{diff}, hence the hom-set in the homotopy category (Def. A.14) of the slice model structure (Def. A.10) of the local projective model structure $\mathrm{SmoothStacks}_\infty$ on simplicial presheaves over CartesianSpaces (Example A.43) from τ_{diff} (271) to ρ_{diff} (270):

$$\underset{\substack{\text{twisted differential}\\ \text{non-abelian cohomology}}}{\widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}; A\big)} \;:=\; \mathrm{Ho}\big(\mathrm{SmoothStacks}_\infty^{/BG_{\mathrm{diff}}}\big)\big(\tau_{\mathrm{diff}}\,,\,\rho_{\mathrm{diff}}\big) = \left\{ \begin{tikzcd} \mathscr{X} \arrow[rr, dashed, "c_{\mathrm{diff}}", "\text{differential cocycle}"'] \arrow[dr, "\tau_{\mathrm{diff}}"', "\substack{\text{differential}\\ \text{twist}}"] & \arrow[d, phantom, "\Leftarrow\ \simeq"] & (A /\!\!/ G)_{\mathrm{diff}_{BG}} \arrow[dl, "\rho_{\mathrm{diff}}", "\substack{\text{differential local}\\ \text{coefficients}}"'] \\ & BG_{\mathrm{diff}} & \end{tikzcd} \right\}_{\Big/ \substack{\text{homotopy}\\ \text{relative } BG_{\mathrm{diff}}}} \tag{272}$$

(iii) The twisted non-abelian cohomology operations induced from the maps in (270) we call (see (4)):

$$\text{(a) \textit{characteristic class}:} \qquad \widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}; A\big) \xrightarrow{\;c_A^{\tau} := \big(c^{BG}_{A/\!\!/G}\big)_*\;} H^{\tau}\big(\mathrm{Shp}(\mathscr{X}); A\big) \quad \text{(Def. 2.29)} \tag{273}$$

$$\text{(b) \textit{curvature}:} \qquad \widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}; A\big) \xrightarrow{\;F_A^{\tau_{\mathrm{dR}}} := \big(F^{BG}_{A/\!\!/G}\big)_*\;} \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(\mathscr{X}; \mathfrak{l}A\big)_{\mathrm{flat}} \quad \text{(Def. 3.90)} \tag{274}$$

$$\text{(c) \textit{differential character}:} \qquad \widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}; A\big) \xrightarrow{\;\mathrm{ch}_A^{\tau} := \big(\mathbf{ch}^{BG}_{A/\!\!/G} \circ c^{BG}_{A/\!\!/G}\big)_*\;} H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(\mathscr{X}; \mathfrak{l}A\big) \quad \text{(Def. 3.96)} \tag{275}$$

Twisted differential non-abelian cohomology as non-abelian ∞-sheaf hypercohomology. While the formulation of twisted differential non-abelian cohomology as hom-sets in a slice of $\mathrm{SmoothStacks}_\infty$ (Def. 5.11) is natural and useful, we indicate how this is equivalently incarnated as a non-abelian sheaf hypercohomology over $\mathscr{X}$. This serves to make the connection to existing literature (in Example 5.16 below), but is not otherwise needed for the development here. We shall be brief, referring to [SS20b] for some technical background that is beyond the scope of our presentation here.

Proposition 5.12 (Étale ∞-topos over ∞-stack [SS20b, Prop. 3.33, Rem. 3.34])**.** *For* $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ *(Def. A.44) let*

$$\mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big) \xhookrightarrow{\;\;\mathbb{L}i_{\mathscr{X}}\;\;} \mathrm{Ho}\big(\mathrm{SmoothStacks}_\infty^{/\mathscr{X}}\big)$$

be the full subcategory of the homotopy category (Def. A.14) of the slice model structure over $\mathscr{X}$ *(Example A.10) of the local projective model structure on simplicial presheaves (Example A.43) on those* $\mathscr{E} \to \mathscr{X}$ *which are local diffeomorphisms ([SS20b, Def. 3.26]).*

(i) *The inclusion* $\mathbb{L}i_{\mathscr{X}}$ *is a left-exact homotopy co-reflection, in that it preserves finite homotopy limits and has a derived right adjoint* $\mathbb{R}\mathrm{LcclCnstnt}$ *(sending ∞-bundles to their ∞-sheaves of ∞-sections).*

(ii) *There is a global section functor* $\mathbb{R}\Gamma_{\mathscr{X}}$ *from* $\mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big)$ *to* $\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ *(Example A.33) which also admits a left exact left adjoint:*

$$\begin{tikzcd} \underset{\text{∞-bundles over } \mathscr{X}}{\mathrm{Ho}\big(\mathrm{SmoothStacks}_\infty^{/\mathscr{X}}\big)} \arrow[r, shift right=1.5ex, "\mathbb{R}\mathrm{LcclConstnt}"', "\text{∞-sheaf of local sections}"] \arrow[r, phantom, "\perp"] & \underset{\text{∞-sheaves over } \mathscr{X}}{\mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big)} \arrow[l, hook', shift right=1.5ex, "\mathbb{L}i_{\mathscr{X}}"'] \arrow[r, shift right=1.5ex, "\mathbb{R}\Gamma_{\mathscr{X}}"', "\text{global sections}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \arrow[l, shift right=1.5ex, "\Delta_{\mathscr{X}}"'] \end{tikzcd}. \tag{276}$$

Definition 5.13 (Non-abelian ∞-sheaf hypercohomology over ∞-stacks)**.** Given $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (Def. A.44) and $\mathscr{A} \in \mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big)$ (Prop. 5.12) we say that the set of connected components of the *derived global sections* (276) of $\mathscr{A}$ over $\mathscr{X}$

$$H\big(\mathscr{X}\,,\mathscr{A}\big) \;:=\; \pi_0\big(\mathbb{R}\Gamma_{\mathscr{X}}(\mathscr{A})\big)$$

is the *non-abelian ∞-sheaf hypercohomology* of $\mathscr{X}$ with coefficients in $\mathscr{A}$.

Lemma 5.14 (Twisted differential non-abelian cohomology as non-abelian ∞-sheaf hyper-cohomology)**.** *Given a differential twist* τ_{diff} (271) *on some* $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (342) *consider the object*

$$\underline{A}_{\tau_{\mathrm{diff}}} := \mathbb{R}\mathrm{LcllCnstnt}_{\mathscr{X}}\big(\mathbb{R}\tau^*_{\mathrm{diff}}(A /\!\!/ G)_{\mathrm{diff}}\big) \quad \in \mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big) \tag{277}$$

in the étale ∞-topos over $\mathscr{X}$ Prop. 5.12. The non-abelian ∞-sheaf hypercohomology (Def. 5.13) of $\underline{A}_{\tau_{\mathrm{diff}}}$ over $\mathscr{X}$ coincides with the τ_{diff}-twisted differential non-abelian cohomology of $\mathscr{X}$ (Def. 5.11):

$$\overset{\substack{\text{non-abelian}\\ \infty\text{-sheaf hypercohomology}}}{\pi_0\,\mathbb{R}\Gamma_{\mathscr{X}}\big(\underline{A}_{\tau_{\mathrm{diff}}}\big)} \;\simeq\; \overset{\substack{\text{twisted differential}\\ \text{non-abelian cohomology}}}{\widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}, A\big)}. \tag{278}$$

Proof. As in [SS20b, Remark 3.34]. ☐

It is useful to decompose this construction of twisted differential cohomology via ∞-sheaf hypercohomology again as a homotopy pullback of corresponding ∞-sheaves representing plain twisted cohomology and plain twisted differential forms:

Remark 5.15 (Homotopy pullback of ∞-sheaves representing twisted differential cohomology)**.** Given a differential twist τ_{diff} (271) on some $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (342) with components $(\tau, \tau_{\mathrm{dR}}, L_{\mathbb{R}}\tau)$ (Example A.26),

(i) Consider the pullback stacks over $\mathscr{X}$ in the following diagram

$$\begin{tikzcd}
\mathbb{R}\tau^*\,\mathrm{Disc}(A/\!\!/G) \arrow[rrr] \arrow[dd] \arrow[dr] & & & \mathrm{Disc}(A/\!\!/G) \arrow[dd, "\rho"] \arrow[dr] & & \\
& \mathbb{R}(L_{\mathbb{R}}\tau)^*\flat\exp\big(\mathfrak{l}_{BG}(A/\!\!/G)\big) \arrow[rrr] \arrow[ddd] \arrow[ddr] & & & \flat\exp\big(\mathfrak{l}_{BG}(A/\!\!/G)\big) \arrow[ddd] \arrow[ddr] & \\
\mathscr{X} \arrow[rrr, "\tau"] \arrow[ddr, equal] & & & \mathrm{Disc}(BG) \arrow[ddr] & & \\
& & \mathbb{R}\tau^*_{\mathrm{dR}}\Omega_{\mathrm{dR}}\big(-;\mathfrak{l}_{BG}(A/\!\!/G)\big)_{\mathrm{flat}} \arrow[rrr] \arrow[dd] & & & \Omega_{\mathrm{dR}}\big(-;\mathfrak{l}_{BG}(A/\!\!/G)\big)_{\mathrm{flat}} \arrow[dd] \\
& \mathscr{X} \arrow[rrr, "L_{\mathbb{R}}\tau"] \arrow[dr, equal] & & & \flat\exp\big(\mathfrak{l}BG\big) \arrow[dr] & \\
& & \mathscr{X} \arrow[rrr, "\tau_{\mathrm{dR}}"] & & & \Omega_{\mathrm{dR}}(-;\mathfrak{l}BG)
\end{tikzcd}$$

Here the right hand side is (269) and all front-facing squares are homotopy pullbacks (Def. A.23).

(ii) By commutativity of homotopy limits over each other, these form a homotopy pullback square as on the right of the following diagram, which gives, under the derived right adjoint $\mathbb{R}\mathrm{LcllCnstnt}$ (276) a homotopy pullback diagram of ∞-sheaves of sections as shown on the left:

$$\begin{tikzcd}
\underline{A}_{\tau_{\mathrm{diff}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \underline{\Omega\big(-;\mathfrak{l}A\big)_{\mathrm{flat}}}_{\tau_{\mathrm{dR}}} \arrow[d] \\
\underline{A}_{\tau} \arrow[r] & \underline{\flat\exp\big(\mathfrak{l}A\big)}_{L_{\mathbb{R}}\tau}
\end{tikzcd}
:= \mathbb{R}\mathrm{LcllCnstst}
\left(
\begin{tikzcd}
\mathbb{R}\tau^*_{\mathrm{diff}}(A/\!\!/G)_{\mathrm{diff}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \mathbb{R}\tau^*_{\mathrm{dR}}\Omega_{\mathrm{dR}}\big(-;\mathfrak{l}_{BG}(A/\!\!/G)\big) \arrow[d] \\
\mathbb{R}\tau^*(A/\!\!/G) \arrow[r] & \mathbb{R}(L_{\mathbb{R}}\tau)^*\flat\exp\big(\mathfrak{l}(A/\!\!/G)\big)
\end{tikzcd}
\right)
\in \mathrm{Ho}\big(\mathbf{\acute{E}t}_{\mathscr{X}}\big). \tag{279}$$

Here the top left item $\underline{A}_{\tau_{\mathrm{diff}}}$ from (277) is the ∞-sheaf whose global sections give the τ_{diff}-twisted differential cohomology, by Lemma 5.14.

In differential enhancement of Prop. 2.37 and in twisted enhancement of Example 4.34, we have:

Example 5.16 (Twisted differential generalized cohomology)**.** Let $\mathscr{X} = X$ be a smooth manifold (Example A.45) and R be a suitable ring spectrum, and let

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "\mathbb{R}\tau^*\rho_R"'] \arrow[dr, phantom, "\text{(hpb)}"] & (R_0)/\!\!/\mathrm{GL}_R(1) \arrow[d, "\rho_R"] \\ X \arrow[r, "\tau"'] & B\mathrm{GL}_R(1) \end{tikzcd}$$

be a twist for twisted generalized R-cohomology over X (51), as in Lemma 2.37.
(i) Then the corresponding homotopy pullback diagram (279), which exhibits, by Lemma 5.14, twisted differential non-abelian cohomology (Def. 5.11) with coefficients in E_0 as ∞-sheaf hypercohomology (Def. 5.13), is the image under $\mathbb{R}\Omega^\infty_X$ of the homotopy pullback diagram of sheaves of spectra considered in [BN14, Def. 4.11], shown on the right below, for canonical/minimal differential refinement as in Example 4.34:

$$\begin{tikzcd} \underline{R_0}_{\tau_{\mathrm{diff}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & \underline{\Omega\big(-;\,\mathfrak{l}R_0\big)_{\mathrm{flat}}}_{\tau_{\mathrm{dR}}} \arrow[d] \\ \underline{R_0}_{\tau} \arrow[r] & \underline{\flat \exp\big(\mathfrak{l}R_0\big)}_{L_{\mathbb{R}}\tau} \end{tikzcd} \quad \simeq \quad \mathbb{R}\Omega_X{}^{\infty}\left(\begin{tikzcd} \mathrm{Diff}(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(hpb)}"] & H\mathscr{M}_{\leq 0} \arrow[d] \\ \mathrm{Disc}(E) \arrow[r] & H\mathscr{M} \end{tikzcd}\right)$$

This is the twisted/parametrized analog of the relation (236).
(ii) Accordingly, the twisted differential generalized R-cohomology according to [BN14, Def. 4.13] is subsumed by twisted differential non-abelian cohomology, via Lemma 5.14.

In differential enhancement of Prop. 5.5 and in twisted generalization of Example 4.36 we have:

Example 5.17 (Twisted Chern character in twisted differential K-theory)**.** Consider again the local coefficient bundle

$$\begin{tikzcd} \mathrm{KU}_0 \arrow[r] & \mathrm{KU}_0/\!\!/B\mathrm{U}(1) \arrow[d, "\rho"] \\ & B^2\mathrm{U}(1) \end{tikzcd}$$

for complex topological K-theory (Example 2.35). By Example 5.16, the twisted differential non-abelian cohomology theory (Def. 5.11) induced from these local coefficients is twisted differential K-theory, as discussed in [CMW09] for torsion twists (review in [BS12, §7]). By the diagram (4) of cohomology operations on twisted differential cohomology, one may regard the corresponding twisted curvature map (274)

$$\widehat{K}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}\big) \xrightarrow{\ \big(F^{\tau_{\mathrm{dR}}}_{\mathrm{KU}_0}\big)_*\ } \Omega^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(\mathscr{X};\,\mathfrak{l}\mathrm{KU}_0\big)_{\mathrm{flat}}$$

(with values in flat $\tau_{\mathrm{dR}} \simeq H_3$-twisted differential forms, by Example 3.92) as an incarnation of the Chern character map on twisted differential K-theory. This is the perspective taken in [CMW09, p. 2][Pa18] for torsion twists, and in [BN14, p. 6] for general twists.

However, in the spirit of the Cheeger-Simons homomorphism (4.3), any lift of a cohomology operation (here: rationalization) to differential cohomology should be enhanced all the way to a secondary cohomology operation (Def. 4.42, now to be generalized to a *twisted* secondary cohomology operation, Def. 5.19 below) whose codomain is itself a (twisted) differential cohomology theory. The twisted Chern character enhanced to a secondary cohomology operation this way is Example 5.20 below, following the perspective taken in [GS19a, §3.2][GS19c, §2.3].

Secondary twisted non-abelian cohomology operations. We introduce the twisted generalization of secondary non-abelian cohomology operations (Def. 5.19 below). This requires the following twisted analog of the technical condition in Def. 4.41:

Definition 5.18 (Twisted absolute minimal model)**.** For

$$
\begin{tikzcd}
A_1 /\!\!/ G_1 \arrow[r, "c_t"] \arrow[d, "\rho_1"'] & A_2 /\!\!/ G_2 \arrow[d, "\rho_2"] \\
BG_1 \arrow[r, "c_b"'] & BG_2
\end{tikzcd}
\quad \in \ \text{SimplicialSets}
$$

a transformation (55) between local coefficient bundles (35), and for $\mathfrak{c}_b$ an absolute minimal model (Def. 4.41) of the map c_b between spaces of twists, hence with induced transformation (251)

$$
\begin{tikzcd}
\mathrm{Disc}\big(BG_1\big) \arrow[rr, "\mathrm{Disc}(c_b)"] \arrow[dr, "\mathbf{ch}_{BG_1}"] & & \mathrm{Disc}\big(BG_2\big) \arrow[dr, "\mathbf{ch}_{BG_2}"] & \\
& \flat \exp\big(\mathfrak{l}BG_1\big) \arrow[rr, "(\mathfrak{c}_b)_*"'] & & \flat \exp\big(\mathfrak{l}BG_1\big)
\end{tikzcd}
$$

between the differential character maps (Def. 4.32) on the spaces of twists, we say that a corresponding *twisted absolute minimal model* is a lift of $\mathfrak{c}_b$ to a morphism

$$
\mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big) \overset{\mathfrak{c}_t}{\dashrightarrow} \mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big) \tag{280}
$$

between the relative rational Whitehead L_∞-algebras of the local coefficient bundles (Prop. 3.68) which
(i) yields a transformation

$$
\begin{tikzcd}
\mathrm{Disc}\big(A_1 /\!\!/ G_1\big) \arrow[rr, "\mathrm{Disc}(c_t)"] \arrow[dr, "\mathbf{ch}^{BG_1}_{A_1 /\!\!/ G_1}"] & & \mathrm{Disc}\big(A_2 /\!\!/ G_2\big) \arrow[dr, "\mathbf{ch}^{BG_2}_{A_2 /\!\!/ G_2}"] & \\
& \flat \exp\Big(\mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big)\Big) \arrow[rr, dashed, "(\mathfrak{c}_t)_*"'] & & \flat \exp\Big(\mathfrak{l}_{BG_2}\big(A_2 /\!\!/ G_2\big)\Big)
\end{tikzcd}
$$

of the twisted differential characters (269) (thus being an "absolute minimal model for c_t relative to c_b"),
(ii) compatible with the transformation of the differential characters on the twisting space, in that the following cube commutes:

$$
\begin{tikzcd}
\mathrm{Disc}\big(A_1 /\!\!/ G_1\big) \arrow[rrr, "\mathrm{Disc}(c_t)"] \arrow[dd, "\rho_1"'] \arrow[dr, "\mathbf{ch}^{BG_1}_{A_1 /\!\!/ G_1}"] & & & \mathrm{Disc}\big(A_2 /\!\!/ G_2\big) \arrow[dd, "\rho_2"'] \arrow[dr, "\mathbf{ch}^{BG_2}_{A_2 /\!\!/ G_2}"] & & \\
& \flat \exp\Big(\mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big)\Big) \arrow[rrr, "(\mathfrak{c}_t)_*"] \arrow[ddd, "\mathfrak{l}_{BG_1}(A_1 /\!\!/ G_1)"'] & & & \flat \exp\Big(\mathfrak{l}_{BG_1}\big(A_2 /\!\!/ G_2\big)\Big) \arrow[ddd, "(\mathfrak{l}_{BG_2} p_2)_*"'] & \\
\mathrm{Disc}\big(BG_1\big) \arrow[rrr, "\mathrm{Disc}(c_b)"] \arrow[ddr, "\mathbf{ch}_{BG_1}"'] & & & \mathrm{Disc}\big(BG_2\big) \arrow[ddr, "\mathbf{ch}_{BG_2}"] & & \\
& & \Omega\Big(-; \mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big)\Big)_{\mathrm{flat}} \arrow[uul, "\mathrm{atlas}"'] \arrow[rrr, dashed, "(\mathfrak{c}_t)_*"] \arrow[dd, "(\mathfrak{l}_{BG_1} p_1)_*"] & & & \Omega\Big(-; \mathfrak{l}_{BG_1}\big(A_1 /\!\!/ G_1\big)\Big)_{\mathrm{flat}} \arrow[uul, "\mathrm{atlas}"'] \arrow[dd, "(\mathfrak{l}_{BG_2} p_2)_*"] \\
& \flat \exp\big(\mathfrak{l}BG_1\big) \arrow[rrr, "(\mathfrak{c}_b)_*"] & & & \flat \exp\big(\mathfrak{l}BG_2\big) & \\
& & \Omega\big(-; \mathfrak{l}BG_1\big)_{\mathrm{flat}} \arrow[ul, "\mathrm{atlas}"] \arrow[rrr, "(\mathfrak{c}_b)_*"] & & & \Omega\big(-; \mathfrak{l}BG_2\big)_{\mathrm{flat}} \arrow[ul, "\mathrm{atlas}"]
\end{tikzcd}
\tag{281}
$$

At the level of dgc-algebras, the condition that $\mathfrak{c}_t$ (280) is a twisted absolute minimal model for the transformation of local coefficient bundles means equivalently that it makes the following cube commute:

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{PLdR}}\big(A_1 /\!\!/ G_1\big) & & \Omega^\bullet_{\mathrm{PLdR}}\big(A_2 /\!\!/ G_2\big) \arrow[ll, "\Omega^\bullet_{\mathrm{PLdR}}(c_t)"'] & \\
& \mathrm{CE}\big(\mathfrak{l}_{BG_1}(A_1 /\!\!/ G_1)\big) \arrow[ul, "p^{\mathrm{min} BG_1}_{A_1 /\!\!/ G_1}"'] \arrow[dd, "\mathrm{CE}(\mathfrak{l}\rho_1)"] & & \mathrm{CE}\big(\mathfrak{l}_{BG_2}(A_2 /\!\!/ G_2)\big) \arrow[ul, "p^{\mathrm{min} BG_1}_{A_1 /\!\!/ G_1}"'] \arrow[ll, dashed, "\mathrm{CE}(\mathfrak{c}_t)"'] \arrow[dd, "\mathrm{CE}(\mathfrak{l}\rho_2)"] \\
\Omega^\bullet_{\mathrm{PLdR}}(BG_1) \arrow[uu, "\Omega^\bullet_{\mathrm{PLdR}}(\rho_1)"] & & \Omega^\bullet_{\mathrm{PLdR}}(BG_2) \arrow[uu, "\Omega^\bullet_{\mathrm{PLdR}}(\rho_2)"'] \arrow[ll, "\Omega^\bullet_{\mathrm{PLdR}}(c_t)"'] & \\
& \mathrm{CE}(\mathfrak{l}BG_1) \arrow[ul, "p^{\mathrm{min}}_{BG_1}"] & & \mathrm{CE}(\mathfrak{l}BG_1) \arrow[ul, "p^{\mathrm{min}}_{BG_2}"] \arrow[ll, "\mathfrak{c}_b"']
\end{tikzcd}
\tag{282}
$$

In differential enhancement of Def. 2.40 and in twisted generalization of Def. 4.42, we set:

Definition 5.19 (Twisted secondary non-abelian cohomology operations)**.** Let

$$
\begin{tikzcd}
A_1 /\!\!/ G_1 \arrow[r, "c_t"] \arrow[d, "\rho_1"'] & A_2 /\!\!/ G_2 \arrow[d, "\rho_2"] \\
BG_1 \arrow[r, "c_b"] & BG_2
\end{tikzcd}
\quad \in \; \mathrm{SimplicialSets}
$$

be a transformation (55) between local coefficient bundles (35), together with an absolute minimal model $\mathfrak{c}_b$ (Def. 4.41) for the base map, and a compatible twisted absolute minimal model $\mathfrak{c}_t$ (Def. 5.18) for the total map. Then forming stage-wise homotopy pullbacks (Def. A.23) in the required commuting cube (281) yields a transformation of corresponding differential coefficient bundles (270):

$$
\begin{tikzcd}
(A_1 /\!\!/ G_1)_{\mathrm{diff}} \arrow[r, "(c_t)_{\mathrm{diff}}"] \arrow[d, "(\rho_1)_{\mathrm{diff}}"'] & (A_2 /\!\!/ G_2)_{\mathrm{diff}} \arrow[d, "(\rho_2)_{\mathrm{diff}}"] \\
(BG_1)_{\mathrm{diff}} \arrow[r, "(c_b)_{\mathrm{diff}}"'] & (BG_2)_{\mathrm{diff}}
\end{tikzcd}
\quad \in \mathrm{PSh}\big(\mathrm{CartesianSpaces}\,,\, \mathrm{SimplicialSets}\big)\,.
$$

This yields, in turn, a natural transformation of twisted differential non-abelian cohomology sets (Def. 5.11), hence a *twisted secondary non-abelian cohomology operation*, by pasting composition, hence by right derived base change (Example A.18) along $(\rho_1)_{\mathrm{diff}}$ followed by composition with $(c_t)_{\mathrm{diff}}$ regarded as a morphism in the slice (Example A.10) over $(BG_1)_{\mathrm{diff}}$:

$$
\widehat{H}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}\,; A_1\big) \xrightarrow{\;\; ((c_t)_{\mathrm{diff}} \circ (-)) \circ ((\rho_1)_{\mathrm{diff}})_* \;\;} \widehat{H}^{(c_b)_{\mathrm{diff}} \circ \tau_{\mathrm{diff}}}\big(\mathscr{X}\,; A_2\big).
$$

In differential enhancement of Prop. 5.5, we have:

Example 5.20 (Twisted differential character on twisted differential K-theory)**.** Consider the rationalization (Def. 3.53) over the actual *rational numbers* (see Remark 3.49) of the local coefficient bundle (47) for degree-3 twisted complex topological K-theory (Example 2.35).

(i) This is captured by the diagram

$$
\begin{tikzcd}
\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1) \arrow[r, "\eta^{\mathbb{Q}}_{\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)}"] \arrow[d, "\rho"'] & L_{\mathbb{Q}}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) \arrow[d, "L_{\mathbb{R}}\rho"] \\
B^2\mathrm{U}(1) \arrow[r, "\eta^{\mathbb{Q}}_{B^2\mathrm{U}(1)}"] & L_{\mathbb{Q}}\big(B^2\mathrm{U}(1)\big)
\end{tikzcd}
\tag{283}
$$

regarded as a transformation of local coefficient bundles from twisted K-theory to twisted even-periodic rational cohomology:

$$
L_{\mathbb{Q}}\mathrm{KU}_0 \;\simeq\; \Omega^\infty\Big(\underbrace{\bigoplus_k \Sigma^{2k} H\mathbb{Q}}_{=: H_{\mathrm{per}}\mathbb{Q}}\Big)\,.
$$

(ii) Since rationalization is idempotent (110), which here means that $L_{\mathbb{R}} \circ L_{\mathbb{Q}} \;\simeq\; L_{\mathbb{R}}$, in this situation an absolute minimal model (Def. 4.41) of the base map $c_b = \eta^{\mathbb{R}}_{B^2\mathrm{U}(1)}$ and a twisted absolute minimal model (Def. 5.18) of the total map $c_t = \eta^{\mathbb{R}}_{K_0 /\!/ B\mathrm{U}(1)}$ exist and are given, respectively, simply by the identity morphisms

$$\mathfrak{c}_b := \mathrm{id}_{\mathfrak{l}B^2\mathrm{U}(1)} \qquad \text{and} \qquad \mathfrak{c}_t := \mathrm{id}_{\mathfrak{l}_{B^2\mathrm{U}(1)}(K_0 /\!/ B\mathrm{U}(1))}.$$

(iii) Therefore, the induced twisted secondary cohomology operation Def. 5.19 exists, and is for each differential twist τ_{diff} a transformation

$$\widehat{K}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}\big) \xrightarrow{\;\mathrm{ch}^{\tau_{\mathrm{diff}}}_{\mathrm{diff}} := \left(\eta^{\mathbb{R}}_{K_0 /\!/ B\mathrm{U}(1)}\right)_{\mathrm{diff}}\;} \widehat{H_{\mathrm{per}}\mathbb{Q}}^{\,L_{\mathbb{Q}}\tau_{\mathrm{diff}}}\big(\mathscr{X}\big) \tag{284}$$

from twisted differential K-theory to twisted differential periodic rational cohomology theory.

(iv) This is the twisted differential Chern character map on twisted differential complex K-theory as conceived in [GS19a, §3.2][GS19c, Prop. 4]. The analogous statement holds for the twisted differential Pontrjagin character (as in Example 4.11) on twisted differential real K-theory [GS19d, Thm. 12].

(v) Notice that this construction is close to but more structured than the plain curvature map on twisted differential K-theory (Example 5.17): If we considered the transformation of local coefficients as in (283) but for rationalization $L_{\mathbb{R}}$ over the real numbers (Remark 3.49), then the induced twisted secondary cohomology operation would be equivalent to the twisted curvature map. Instead, (284) refines the plain curvature map to a twisted secondary operation that retains information about rational periods.

5.3 Twisted character on twisted differential Cohomotopy

We discuss here (Example 5.21 below) the twisted non-abelian character map (Def. 5.4) on J-twisted Cohomotopy (Example 2.39) in degree 4, and on Twistorial Cohomotopy (Example 2.42). We close by exhibiting the key fact that makes twisted cohomotopical characters so interesting (Prop. 5.22 below). These twisted non-abelian character maps have been introduced and analyzed in [FSS19b] and [FSS20], respectively. The general theory developed here shows how these cohomotopical characters are cousins both of abelian generalized characters such as the twisted Chern character on higher K-theory (§5.1) as well as of non-abelian characters such as the Chern-Weil homomorphism (§4.2).

Character map on twisted 4-Cohomotopy and on twistorial Cohomotopy. Recall from Example 4.13, and Remark 4.14 that 4-Cohomotopy is a non-abelian enhancement (Example 2.25) of tmf^4 on 8-manifolds, and recall un-twisted differential Cohomotopy from Example 4.38.

Example 5.21 (Character map on J-twisted Cohomotopy and on Twistorial Cohomotopy)**.** Let X be an 8-dimensional smooth spin-manifold equipped with tangential $\mathrm{Sp}(2)$-structure τ (58). Consider the twisted non-abelian character maps (Def. 5.4) on J-twisted Cohomotopy (Example 2.39) in degree 4, and on Twistorial Cohomotopy (Example 2.42):

$$\begin{tikzcd}[column sep=small]
\mathbb{C}P^3 /\!/ \mathrm{Sp}(2) \arrow[d, "{\substack{\text{Borel-equivariantized} \\ \text{twistor fibration} \\ t_{\mathbb{H}} /\!/ \mathrm{Sp}(2)}}"'] & \underset{\text{Twistorial Cohomotopy}}{\mathcal{T}^\tau(X)} := H^\tau\big(X; \mathbb{C}P^3\big) \arrow[rr, "{\substack{\text{character map on} \\ \text{Twistorial Cohomotopy} \\ \mathrm{ch}^\tau_{\mathbb{C}P^3}}}"] \arrow[d, "{\substack{\text{cohomology operation} \\ \text{along twistor fibration} \\ (t_{\mathbb{H}})_*}}"] & & H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}\mathbb{C}P^3\big) \arrow[d, "(\mathfrak{l}t_{\mathbb{H}})_*"] = \left\{ \begin{array}{c|l} H_3, & d\,H_3 = G_4 - \tfrac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ F_2, & d\,F_2 = 0, \\ 2G_7, & d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \chi_8(\nabla), \\ G_4 & d\;\;G_4 = 0 \end{array} \right\}_{/\sim} \arrow[d, "{\substack{H_3 \;\; F_2 \;\; 2G_7 \;\; G_4 \\ \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \\ 0 \;\; 0 \;\; 2G_7 \;\; G_4}}"] \\
S^4 /\!/ \mathrm{Sp}(2) & \underset{\substack{\text{J-twisted} \\ \text{4-Cohomotopy}}}{\pi^\tau\big(X\big)} := H^\tau\big(X; S^4\big) \arrow[rr, "{\substack{\mathrm{ch}^\tau_{S^4} \\ \text{character map in} \\ \text{J-twisted Cohomotopy}}}"'] & & H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(X; \mathfrak{l}S^4\big) = \left\{ \begin{array}{c|l} 2G_7, & d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \chi_8(\nabla), \\ G_4 & d\;\;G_4 = 0, \end{array} \right\}_{/\sim}
\end{tikzcd}$$

Here:

- **(i)** The twisted non-abelian de Rham cohomology targets on the right are as shown, by Example 3.94.
- **(ii)** The vertical twisted non-abelian cohomology operation (Def. 2.40) on the left is induced from the Borel-equivariantized twistor fibration (60), and that on the right from its associated morphism of rational Whitehead L_∞-algebras (Prop. 3.68).

Proposition 5.22 (Charge-quantization in J-twisted Cohomotopy [FSS19b, Prop. 3.13][FSS20, Cor. 3.11])**.** *Consider the twisted non-abelian character maps (Def. 5.4) in J-twisted Cohomotopy and in Twistorial Cohomotopy from Example 5.21.*

(i) *A necessary condition for a flat* $\mathrm{Sp}(2)$*-twisted* $\mathrm{l}S^4$*-valued differential form datum* (G_4, G_7) *to lift through the J-twisted cohomotopical character map (i.e. to be in its image) is that the de Rham class of* G_4*, when shifted by the fourth fraction of the Pontrjagin form, is in the image, under the de Rham homomorphism (Example 4.9), of an integral class:*

$$\big[G_4 - \tfrac{1}{4}p_1(\nabla)\big] \;\in\; H^4(X;\mathbb{Z}) \longrightarrow H^4_{\mathrm{dR}}(X)\,. \tag{285}$$

(ii) *A necessary condition for a flat* $\mathrm{Sp}(2)$*-twisted* $\mathrm{l}\mathbb{C}P^3$*-valued differential form datum* (G_4, G_7, F_2, H_3) *to lift through the character map in Twistorial Cohomotopy is that the de Rham class of* G_4 *shifted by the fourth fraction of the Pontrjagin form is in the image, under the de Rham homomorphism (Example 4.9), of an integral class, and as such equal to the* $[F_2]$ *cup-square:*

$$\big[G_4 - \tfrac{1}{4}p_1(\nabla)\big] \;=\; \big[F_2 \wedge F_2\big] \;\in\; H^4(X;\mathbb{Z}) \longrightarrow H^4_{\mathrm{dR}}(X)\,. \tag{286}$$

Differential twisted Cohomotopy theory.

Example 5.23 (Differential twistorial Cohomotopy)**.** Consider the local coefficient bundle (60) for *twistorial Cohomotopy* (Def. 2.42)

$$\begin{tikzcd}
\mathbb{C}P^3 \arrow[r] & \mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)\,. \arrow[d, "J_{\mathbb{C}P^3}"] \\
& B\mathrm{Sp}(2)
\end{tikzcd}$$

(i) This induces, via Def. 5.11, a twisted differential non-abelian cohomology theory $\widehat{\mathscr{T}}^{\tau_{\mathrm{diff}}}(-)$, to be called *differential twistorial Cohomotopy*, whose value, over any $\mathscr{X} \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (Def. A.44) equipped with twist τ_{diff} given by an $\mathrm{Sp}(2)$-connection ω on the frame bundle (via Prop. 4.37)

$$\begin{array}{ccc}
\mathrm{Sp}(2)\mathrm{Connections}(\mathscr{X})_{/\sim} & \longrightarrow & \widehat{H}\big(\mathscr{X}; B\mathrm{Sp}(2)\big) \\
{[\omega]} & \longmapsto & [\tau_{\mathrm{diff}}]
\end{array}$$

sits in a cohomology operation diagram (4) of this form:

$$\begin{tikzcd}
\overset{\substack{\text{differential}\\ \text{twistorial}\\ \text{Cohomotopy}}}{\widehat{\mathscr{T}}^{\tau_{\mathrm{diff}}}(X)} \arrow[r, "\substack{\text{twistorial}\\ \text{curvature}}", "F^{\tau_{\mathrm{dR}}}_{\mathbb{C}P^3}"'] \arrow[d] & \overset{\text{twistorial Bianchi identities (Example 3.94)}}{\left\{ \begin{array}{c} H_3, \\ F_2, \\ 2G_7, \\ G_4 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle\vert\; \begin{array}{l} dH_3 = G_4 - \tfrac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ dF_2 = 0, \\ d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \chi_8(\nabla), \\ d\;\; G_4 = 0 \end{array} \right\}} \arrow[d] \\
\underset{\substack{\text{twistorial}\\ \text{Cohomotopy}\\ \text{(Example 2.42)}}}{\mathscr{T}^{\tau}(\mathscr{X})} \arrow[r, "\mathrm{ch}^{\tau}_{\mathbb{C}P^3}", "\substack{\text{character map}\\ \text{on twistorial Cohomotopy}\\ \text{(Example 5.21)}}"'] & \underset{\substack{\text{twistorial}\\ \text{de Rham cohomology}\\ \text{(Def 3.96)}}}{H^{\tau_{\mathrm{dR}}}_{\mathrm{dR}}\big(\mathscr{X}; \mathrm{l}\mathbb{C}P^3\big)}\,.
\end{tikzcd} \tag{287}$$

(ii) The twisted non-abelian curvature map (274) lifts the values of the twistorial character map (Example 5.21) to differential form representatives (Example 3.94). These twistorial curvature forms satisfy the integrality condition (286), by Prop. 5.22. In summary, this establishes the situation announced in (5).

(iii) Similarly, we obtain the differential refinement (Def. 5.11) of J-twisted 4-Cohomotopy theory (Example 2.39):

$$\begin{tikzcd}
\overset{\substack{\text{differential}\\ \text{J-twisted}\\ \text{4-Cohomotopy}}}{\widehat{\pi}^{\tau^4_{\mathrm{diff}}}(X)} \arrow[r, "\substack{\text{J-twisted}\\ \text{cohomotopical}\\ \text{curvature}}", "F^{\tau^4_{\mathrm{dR}}}_{S^4}"'] \arrow[d] & \overset{\text{J-twisted cohomotopical Bianchi identities (Example 3.94)}}{\left\{ \begin{array}{c} 2G_7, \\ G_4 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle\vert\; \begin{array}{l} d\,2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \chi_8(\nabla), \\ d\;\; G_4 = 0 \end{array} \right\}} \arrow[d] \\
\underset{\substack{\text{J-twisted}\\ \text{4-Cohomotopy}\\ \text{(Example 2.39)}}}{\pi^{\tau^4}(\mathscr{X})} \arrow[r, "\mathrm{ch}^{\tau^4}_{S^4}", "\substack{\text{character map}\\ \text{on J-twisted Cohomotopy}\\ \text{(Example 5.21)}}"'] & \underset{\substack{\text{J-twisted}\\ \text{de Rham cohomology}\\ \text{(Def 3.96)}}}{H^{\tau^4_{\mathrm{dR}}}_{\mathrm{dR}}\big(\mathscr{X}; \mathrm{l}S^4\big)}\,.
\end{tikzcd} \tag{288}$$

Proposition 5.24 (Twisted secondary operation from twistorial to J-twisted Cohomotopy)**.** *The defining twisted non-abelian cohomology operation* (61) *from twistorial Cohomotopy (Example 2.42) to J-twisted 4-Cohomotopy (Example 2.39), induced by the* $\mathrm{Sp}(2)$*-equivariatized twistor fibration* $t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ (60) *refines to a twisted secondary cohomology operation (Def. 5.19) from differential twistorial Cohomotopy (Example 5.23) to differential J-twisted Cohomotopy* (288)*:*

$$
\begin{tikzcd}
\substack{\text{differential}\\ \text{twistorial}\\ \text{Cohomotopy}} & \widehat{\mathcal{T}}^{\tau_{\mathrm{diff}}}\big(\mathscr{X}\big) \arrow[rr, "c^{\tau}_{\mathbb{C}P^3}"] \arrow[d, "((t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_{\mathrm{diff}})_*"', "\substack{\text{along}\\ \mathrm{Sp}(2)\text{-equivariantized}\\ \text{twistor fibration}}"] & & \mathcal{T}^{\tau}\big(\mathscr{X}\big) \arrow[d, "(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))_*"'] & \substack{\text{twisted primary}\\ \text{cohomology operation}} \\
\substack{\text{twisted secondary}\\ \text{cohomology operation}} \\[-20pt]
\substack{\text{differential}\\ \text{J-twisted}\\ \text{4-Cohomotopy}} & \widehat{\pi}^{\tau^4_{\mathrm{diff}}}\big(\mathscr{X}\big) \arrow[rr, "c^{\tau^4}_{S^4}"] & & \pi^{\tau^4}\big(\mathscr{X}\big)
\end{tikzcd}
$$

Proof. By Def. 5.19, we need to show that we have a twisted absolute minimal model (Def. 5.18) for the $\mathrm{Sp}(2)$-equivariantized twistor fibration (60). By (282) this means that we can find a morphism

$$
\mathfrak{l}_{B\mathrm{Sp}(2)}\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)\big) \xrightarrow{\ \mathfrak{t}_{\mathbb{H}} /\!\!/ \mathfrak{l}\mathrm{Sp}(2)\ } \mathfrak{l}_{B\mathrm{Sp}(2)} S^4 /\!\!/ \mathrm{Sp}(2)\big) \tag{289}
$$

between the relative Whitehead L_∞-algebras (Prop. 3.68) of the two local coefficient bundles, which makes the following cube of transformations of derived PL-de Rham adjunction units commute:

$$
\begin{tikzcd}[column sep=small]
\exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)\big) \arrow[rr, "\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))"] \arrow[dd, "\exp\circ\Omega^\bullet_{\mathrm{PLdR}}(J_{\mathbb{C}P^3})"'] \arrow[dr, "\exp\left(p^{\mathrm{min}_{\mathrm{Sp}(2)}}_{\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)}\right)"] & & \exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(S^4 /\!\!/ \mathrm{Sp}(2)\big) \arrow[dd] \arrow[dr, "\exp\left(p^{\mathrm{min}_{B\mathrm{Sp}(2)}}_{S^4 /\!\!/ \mathrm{Sp}(2)}\right)"] & \\
& \exp\circ\mathrm{CE}\Big(\mathfrak{l}_{B\mathrm{Sp}(2)}\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)\big)\Big) \arrow[rr, dashed, "\exp\circ\mathrm{CE}(\mathfrak{t}_{\mathbb{H}} /\!\!/ \mathfrak{l}\mathrm{Sp}(2))"] \arrow[dd] & & \exp\circ\mathrm{CE}\Big(\mathfrak{l}\big(S^4 /\!\!/ \mathrm{Sp}(2)\big)\Big) \arrow[dd] \\
\exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(B\mathrm{Sp}(2)\big) \arrow[rr, equal] \arrow[dr, "\exp\left(p^{\mathrm{min}}_{B\mathrm{Sp}(2)}\right)"'] & & \exp\circ\Omega^\bullet_{\mathrm{PLdR}}\big(B\mathrm{Sp}(2)\big) \arrow[dr, "\exp\left(p^{\mathrm{min}}_{B\mathrm{Sp}(2)}\right)"] & \\
& \exp\circ\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[rr, equal] & & \exp\circ\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)
\end{tikzcd}
$$

But, from Example 3.94, we see that the total object of the relative Whitehead L_∞-algebra of $\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)$ relative to $\mathfrak{l}B\mathrm{Sp}(2)$ coincides with that relative to $\mathfrak{l}_{B\mathrm{Sp}(2)} S^4 /\!\!/ \mathrm{Sp}(2)$. Therefore, we may take the twisted absolute minimal model (289) to be equal to the relative Whitehead L_∞-algebra projection (the top arrow in Example 3.94). This makes the required top front square commute by the commuting triangle in Prop. 3.47. $\square$

A Model category theory

For ease of reference and to highlight some less widely used aspects needed in the main text, we record basics of homotopy theory via model category theory [Qu67] (review in [Ho99][Hir02][Lu09, A.2]) and of homotopy topos theory [Re10] via model categories of simplicial presheaves [Bro73][Ja87][Du01] (review in [Du98][Lu09, §A.3.3][Ja15]).

Topology. By

$$\mathrm{TopologicalSpaces} \in \mathrm{Categories}$$

we denote a *convenient* [St67] (in particular: cartesian closed) category of topological spaces such as compactly-generated spaces [St09] or Δ-generated spaces [Du03], equivalently numerically-generated spaces [SYH10], or D-topological spaces [SS20a, Prop. 2.4].

Categories. Let $\mathscr{C}$ be a category.

(i) For $X, A \in \mathscr{C}$ a pair of objects, we write

$$\mathscr{C}(X,A) := \mathrm{Hom}_{\mathscr{C}}(X,A)$$

for the set of *morphisms* from X to A.

(ii) For $\mathscr{C}, \mathscr{D}$ two categories, we denote a pair of *adjoint functors* between them by

$$\begin{tikzcd} \mathscr{D} \arrow[r, shift right=1ex, "R"'] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, shift right=1ex, "L"'] \end{tikzcd} \quad \Leftrightarrow \quad \mathscr{D}(L(-), -) \simeq \mathscr{C}(-, R(-)) \,. \tag{290}$$

(iii) A Cartesian square in $\mathscr{C}$ we indicate by *pullback* notation $f^*(-)$ and/or by the symbol "(pb)":

$$\begin{tikzcd} f^*A \arrow[r, "f^*p"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "p"] \\ B_1 \arrow[r, "f"'] & B_2 \,. \end{tikzcd} \tag{291}$$

(iv) Dually, a co-Cartesian square in $\mathscr{C}$ we indicate by *pushout* notation $f_*(-)$ and/or by the symbol "(po)":

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{(po)}"] & A \arrow[d, "q_* f"] \\ Y \arrow[r] & f_*A \,. \end{tikzcd} \tag{292}$$

Model categories.

Definition A.1 (Weak equivalences)**.** A *category with weak equivalences* is a category $\mathscr{C}$ equipped with a sub-class $\mathrm{W} \subset \mathrm{Mor}(\mathscr{C})$ of its morphisms, to be called the class of *weak equivalences*, such that

(i) W contains the class of isomorphisms;

(ii) W satisfies the cancellation property ("2-out-of-3"): if in any commuting triangle in $\mathscr{C}$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z \end{tikzcd} \tag{293}$$

two morphisms are in W, then so is the third.

Definition A.2 (Weak factorization system)**.** A *weak factorization system* in a category $\mathscr{C}$ is a pair of sub-classes of morphisms $\mathrm{Proj}, \mathrm{Inj} \subset \mathrm{Mor}(\mathscr{C})$ such that

(i) every morphisms $X \xrightarrow{f} Y$ in $\mathscr{C}$ may be factored through a morphism in Proj followed by one in Inj:

$$f \,:\, X \xrightarrow{\in \mathrm{Proj}} Z \xrightarrow{\in \mathrm{Inj}} Y \tag{294}$$

(ii) For every commuting square in $\mathscr{C}$ with left morphism in Proj and right morphism in Inj, there exists a lift, namely a dashed morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\in \mathrm{Proj}"'] & A \arrow[d, "\in \mathrm{Inj}"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd} \tag{295}$$

making the resulting triangles commute.

(iii) Given Inj (resp. Proj), the class Proj (resp. Inj) is the largest class for which (295) holds.

Definition A.3 (Model category [Jo08, Def. E.1.2][Rie09])**.** A *model category* is a category **C** that has all small limits and colimits, equipped with three sub-classes of its class of morphisms, to be denoted

W – *weak equivalences*

Fib – *fibrations*

Cof – *cofibrations*

such that

(i) The class W makes **C** a category with weak equivalences (Def. A.1);

(ii) The pairs $(\mathrm{Fib}\,,\,\mathrm{Cof}\cap\mathrm{W})$ and $(\mathrm{Fib}\cap\mathrm{W}\,,\,\mathrm{Cof})$ are weak factorization systems (Def. A.2).

Remark A.4 (Minimal assumptions)**.** By item (iii) in Def. A.2 a model category structure is specified already by the classes W and Fib, or alternatively by the classes W and Cof. Moreover, it follows from Def. A.3 that also the class W is stable under retracts [Jo08, Prop. E.1.3][Rie09, Lemma 2.4]: Given a commuting diagram in the model category **C** of the form on the left here

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, equal] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "\in \mathrm{W}"] & X \arrow[d, "f"] \\ A \arrow[r] \arrow[rr, bend right, equal] & B \arrow[r] & A \end{tikzcd} \qquad \Rightarrow \qquad f \;\in\; \mathrm{W} \tag{296}$$

with the middle morphism a weak equivalence, then also f is a weak equivalence.

Definition A.5 (Proper model category)**.** A model category **C** Def. A.3 is called

(i) *right proper*, if pullback along fibrations preserves weak equivalences:

$$\begin{tikzcd} X \arrow[r, "p^*f"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "p"', "\in \mathrm{Fib}"] \\ Y \arrow[r, "f \in \mathrm{W}"'] & B \end{tikzcd} \qquad \Rightarrow \qquad p^*f \;\in\; \mathrm{W} \tag{297}$$

(ii) *left proper*, if pushout along cofibrations preserves weak equivalences, hence if the opposite model category (Example A.9) is right proper.

Notation A.6 (Fibrant and cofibrant objects)**.** Let **C** be a model category (Def. A.3)

(i) We write $* \in \mathbf{C}$ for the terminal object and $\varnothing \in \mathbf{C}$ for the initial object.

(ii) An object $X \in \mathbf{C}$ is called:

(a) *fibrant* if the unique morphism to the terminal object is a fibration, $X \xrightarrow{\in \mathrm{Fib}} *$;

(b) *cofibrant* if the unique morphism from the initial object is a cofibration, $\varnothing \xrightarrow{\in \mathrm{Cof}} X$.

We write $\mathbf{C}_{\mathrm{fib}}, \mathbf{C}^{\mathrm{cof}}, \mathbf{C}^{\mathrm{cof}}_{\mathrm{fib}} \subset \mathbf{C}$ for the full subcategories on, respectively, fibrant objects, or cofibrant objects or objects that are both fibrant and cofibrant.

(iii) Given an object $X \in \mathbf{C}$

(a) A *fibrant replacement* is a factorization (294) of the terminal morphism as

$$X \xrightarrow[\in \mathrm{Cof}\cap\mathrm{W}]{j_X} PX \xrightarrow[\in \mathrm{Fib}]{q_X} * \,. \tag{298}$$

(b) A *cofibrant replacement* is a factorization (294) of the initial morphism as

$$\varnothing \xrightarrow[\in \mathrm{Cof}]{i_X} QX \xrightarrow[\in \mathrm{Fib}\cap\mathrm{W}]{p_X} X \,. \tag{299}$$

Example A.7 (Classical model structure on topological spaces [Qu67, §II.3][Hir15])**.** The category of TopologicalSpaces carries a model category structure whose

(i) W – weak equivalence are the weak homotopy equivalences;

(ii) Fib – fibrations are the Serre fibrations.

We denote this model category by

$$\mathrm{TopologicalSpaces}_{\mathrm{Qu}} \;\in\; \mathrm{ModelCategories}\,.$$

Example A.8 (Classical model structure on simplicial sets [Qu67, §II.3][GJ99])**.** The category of SimplicialSets carries a model category structure whose

(i) W – weak equivalence are those whose geometric realization is a weak homotopy equivalence;

(ii) Cof – cofibrations are the monomorphisms (degreewise injections).

We denote this model category by

$$\mathrm{SimplicialSets}_{\mathrm{Qu}} \in \mathrm{ModelCategories}\,.$$

Example A.9 (Opposite model category [Hir02, §7.1.8])**.** If $\mathbf{C}$ is a model category (Def. A.3) then the opposite underlying category becomes a model category $\mathbf{C}^{\mathrm{op}}$ with the same weak equivalences (up to reversal) and with fibrations (resp. cofibrations) the cofibrations (resp. fibrations) of $\mathbf{C}$, up to reversal.

Example A.10 (Slice model structure [Hir02, §7.6.4])**.** Let $\mathbf{C}$ be a model category (Def. A.3)

(i) For $X \in \mathbf{C}$ any object, the slice category $\mathbf{C}^{/X}$, whose objects are morphisms to X and whose morphisms are commuting triangles in $\mathbf{C}$ over X

$$\mathbf{C}^{/X}\big(a\,,b\big) \quad := \quad \left\{ \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & X & \end{tikzcd} \right\}$$

becomes itself a model category, whose weak equivalence, fibrations and cofibrations are those morphims whose underlying morphisms f are such in $\mathbf{C}$.

(ii) Dually there is the coslice model category $\mathbf{C}^{X/} := \big((\mathbf{C}^{\mathrm{op}})^{/X}\big)^{\mathrm{op}}$, being the opposite model category (Example A.9) of the slice category of the opposite of $\mathbf{C}$:

$$\mathbf{C}^{/X}(a\,,b) \quad := \quad \left\{ \begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd} \right\}$$

Homotopy categories.

Definition A.11 (Path space objects [Qu67, Def. I.4])**.** Let $\mathbf{C}$ be a model category (Def. A.3), and $A \in \mathbf{C}_{\mathrm{fib}}$ be a fibrant object (Notation A.6). Then a *path space object* for A is a factorization of the diagonal morphism Δ_A through a weak equivalence followed by a fibration:

$$\begin{tikzcd} A \arrow[r, "\in \mathrm{W}"] \arrow[rr, bend right, "\Delta_A"'] & \mathrm{Paths}(A) \arrow[r, "(p_0,p_1)\in \mathrm{Fib}"] & A \times A\,. \end{tikzcd} \tag{300}$$

Definition A.12 (Right homotopy)**.** Let $\mathbf{C}$ be a model category (Def. A.3), $X \in \mathbf{C}^{\mathrm{cof}}$ a cofibrant object, $A \in \mathbf{C}_{\mathrm{fib}}$ a fibrant object (Notation A.6) and let $\mathrm{Paths}(A)$ be a path space object for A (Def. A.11). Then a *right homotopy* between a pair of morphisms $f,g \in \mathbf{C}(X,A)$, to be denoted

$$\phi \,:\, f \Rightarrow_r g \qquad \text{or} \qquad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & A \end{tikzcd}$$

is a morphism $\phi \in \mathbf{C}\big(X, \mathrm{Paths}(A)\big)$ which makes this diagram commute:

$$\begin{tikzcd} & A \\ X \arrow[ur, "f"] \arrow[r, "\phi"] \arrow[dr, "g"'] & \mathrm{Paths}(A) \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & A \end{tikzcd}$$

Proposition A.13 (Right homotopy classes)**.** *Let $\mathbf{C}$ be a model category, $X \in \mathbf{C}^{\mathrm{cof}}$ and $A \in \mathbf{C}_{\mathrm{fib}}$ (Notation A.6). Then right homotopy (Def. A.12) is an equivalence relation $\sim_r$ on the hom-set $\mathbf{C}(X,A)$. We write*

$$\mathbf{C}(X,A)_{/\sim_r} \;\in\; \mathrm{Sets} \tag{301}$$

for the corresponding set of right homotopy classes of morphisms from X to A.

Definition A.14 (Homotopy category of a model category)**.** For $\mathbf{C}$ a model category (Def. A.3),
(i) we write

$$\mathrm{Ho}(\mathbf{C}) := \big(\mathbf{C}^{\mathrm{cof}}_{\mathrm{fib}}\big)_{/\sim_r} \in \text{Categories} \tag{302}$$

for the category whose objects are those objects of $\mathbf{C}$ that are both fibrant and cofibrant (Notation A.6), and whose morphisms are the right homotopy classes of morphisms in $\mathbf{C}$ (Def. 301):

$$X, A \in \mathbf{C}^{\mathrm{cof}}_{\mathrm{fib}} \quad \Rightarrow \quad \mathrm{Ho}(\mathbf{C})(X\,,A) := \mathbf{C}(X,A)_{/\sim_r}$$

and composition of morphisms is induced from composition of representatives in $\mathbf{C}$.
(ii) Given a choice of fibrant replacement P and of cofibrant replacement Q for each object of $\mathbf{C}$ (Notation A.6) we obtain a functor

$$\mathbf{C} \xrightarrow{\gamma_{\mathbf{C}}} \mathrm{Ho}(\mathbf{C})\,, \tag{303}$$

which **(a)** sends any object $X \in \mathbf{C}$ to PQX and sends **(b)** any morphism $X \xrightarrow{f} A$ to the right homotopy class (301) of any lift (295) PQf obtained from any lift Qf in the following diagrams:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "i_X"'] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "f \circ p_x"'] \arrow[ur, dashed, "Qf"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} QX \arrow[r, "j_{QY} \circ Qf"] \arrow[d, "j_{QX}"'] & PQY \arrow[d, "q_{QY}"] \\ PQX \arrow[r] \arrow[ur, dashed, "PQf"] & * \end{tikzcd}$$

Proposition A.15 (Homotopy category is localization)**.** *Given a model category* $\mathbf{C}$ *(Def. A.3) the functor* $\mathbf{C} \xrightarrow{\gamma_{\mathbf{C}}} \mathrm{Ho}\big(\mathbf{C}\big)$ (303) *from Def. A.14 exhibits the homotopy category as the localization of the model category at its class of weak equivalences:* $\gamma_{\mathbf{C}}$ *sends all weak equivalences in* $\mathbf{C}$ *to isomorphisms, and is the universal functor with this property.*

The restriction to fibrant-and-cofibrant objects in Def. A.14 is convenient for defining composition of morphisms, but for computing hom-sets in the homotopy category it is sufficient that the domain object is cofibrant, and the codomain fibrant:

Proposition A.16 ([Qu67, §I.1 Cor. 7])**.** *Let* $\mathscr{C}$ *be a model category (Def. A.3). For* $X \in \mathscr{C}^{\mathrm{cof}}$ *a cofibrant object and* $A \in \mathscr{C}_{\mathrm{fib}}$ *a fibrant object, any choice of fibrant replacement* PX *and cofibrant replacement* QA *(Notation A.6). induces a bijection between the set of right homotopy classes (Def. A.12) and the hom-set in the homotopy category (Def. A.14) between* X *and* A*:*

$$\mathscr{C}(X,A)_{/\sim_r} \xrightarrow[\simeq]{\mathscr{C}(j_X, p_A)} \mathrm{Ho}(\mathbf{C})(X,A)\ .$$

Quillen adjunctions.

Definition A.17 (Quillen adjunction)**.** Let $\mathbf{D}, \mathbf{C}$ be model categories (Def. A.3). Then a pair of adjoint functors $(L \dashv R)$ (290) between their underlying categories is called a *Quillen adjunction*, to be denoted

$$\begin{tikzcd} \mathbf{D} \arrow[r, shift right=1.5ex, "R"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathbf{C} \arrow[l, shift right=1.5ex, "L"'] \end{tikzcd} \tag{304}$$

if the following equivalent conditions hold:

- L preserves Cof, and R preserves Fib;
- L preserves Cof and Cof$\cap$W;
- R preserves Fib and Fib$\cap$W.

Example A.18 (Base change Quillen adjunction)**.** Let $\mathbf{C}$ be a model category (Def. A.3), $B_1, B_2 \in \mathbf{C}_{\mathrm{fib}}$ a pair of fibrant objects (Notation A.6) and

$$B_1 \xrightarrow{f} B_2 \quad \in \mathbf{C} \tag{305}$$

a morphism. Then we have a Quillen adjunction (Def. A.17)

$$\begin{tikzcd} \mathbf{C}^{/B_2} \arrow[r, shift right=1.5ex, "f^*"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathbf{C}^{/B_1} \arrow[l, shift right=1.5ex, "f_!"'] \end{tikzcd}$$

between the slice model categories (Example A.10), where:

(i) The left adjoint functor $f_!$ is given by postcomposition in $\mathbf{C}$ with f (305):

$$
f_* \;:\; \begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "\tau"'] && A \arrow[dl, "p"] \\ & B_1 \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} X \arrow[rr, "c"] \arrow[dr, "\tau"'] && A \arrow[dl, "p"] \\ & B_1 \arrow[d, "f"] \\ & B_2 \end{tikzcd} \tag{306}
$$

(ii) The right adjoint functor f^* is given by pullback (291) along f (305).

That these functors indeed form an adjunction $f_! \dashv f^*$ follows from the defining universal property of the pullback (291):

$$
\begin{array}{ccc}
\mathbf{C}^{/B_2}\big(f_*\tau,\, \rho\big) & \simeq & \mathbf{C}^{/B_1}\big(\tau,\, f^*\rho\big) \\
\begin{tikzcd} X \arrow[rr, dashed, "c"] \arrow[dr, "\tau"'] && A \arrow[dd, "\rho"] \\ & B_1 \arrow[dr, "f"'] & \\ && B_2 \end{tikzcd} & \leftrightarrow & \begin{tikzcd} X \arrow[r, dashed, "\widetilde{c}"] \arrow[dr, "\tau"'] & f^*A \arrow[r] \arrow[d, "f^*\rho"] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[dd, "\rho"] \\ & B_1 \arrow[dr, "f"'] & \\ && B_2 \end{tikzcd}
\end{array} \tag{307}
$$

That this adjunction is a Quillen adjunction (Def. A.17) follows since $f_!$ (306) evidently preserves each of W and Cof (even Fib) separately, by Example A.10.

Lemma A.19 (Ken Brown's lemma [Ho99, Lemma 1.1.12][Bro73])**.** *Given a Quillen adjunction $L \dashv R$ (Def. A.17),*
(i) *the right Quillen functor R preserves all weak equialences between fibrant objects.*
(ii) *the left Quillen functor L preserves all weak equivalences between cofibrant objects.*

Proposition A.20 (Derived functors)**.** *Given a Quillen adjunction* $(L \dashv_{\mathrm{Qu}} R)$ *(Def. A.17), there are adjoint functors* $\mathbb{L}L \dashv \mathbb{R}R$ (290) *between the homotopy categories (Def. A.14)*

$$
\begin{tikzcd} \mathrm{Ho}(\mathbf{D}) \arrow[r, bend right, "\mathbb{R}R"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathbf{C}) \arrow[l, bend right, "\mathbb{L}L"'] \end{tikzcd} \tag{308}
$$

whose composites with the localization functors (303) *make the following squares commute up to natural isomorphism:*

$$
\begin{tikzcd} \mathbf{D} \arrow[r, "R"] \arrow[d, "\gamma_{\mathbf{D}}"'] \arrow[dr, phantom, "\swArrow \simeq"] & \mathbf{C} \arrow[d, "\gamma_{\mathbf{C}}"] \\ \mathrm{Ho}(\mathbf{D}) \arrow[r, "\mathbb{R}R"'] & \mathrm{Ho}(\mathbf{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{D} \arrow[d, "\gamma_{\mathbf{D}}"'] \arrow[dr, phantom, "\simeq \seArrow"] & \mathbf{C} \arrow[l, "L"'] \arrow[d, "\gamma_{\mathbf{C}}"] \\ \mathrm{Ho}(\mathbf{D}) & \mathrm{Ho}(\mathbf{C}) \arrow[l, "\mathbb{L}L"] \end{tikzcd}.
$$

These are unique up to natural isomorphism, and are called the left and right derived functors of L and R, respectively.

Example A.21 (Derived functors via (co-)fibrant replacement)**.** It is convenient to leave the localization functors γ (303) notationally implicit, and understand objects of $\mathbf{C}$ as objects of $\mathrm{Ho}(\mathbf{C})$, via γ. Then:

(i) The value of a left derived functor $\mathbb{L}L$ (Prop. A.20) on an object $c \in \mathbf{C}$ is equivalently the value of L on a cofibrant replacement Qc (299):

$$
\mathbb{L}L(c) \simeq L(Qc) \quad \in \mathrm{Ho}(\mathbf{D})\,. \tag{309}
$$

(ii) The value of a right derived functor $\mathbb{R}R$ (Prop. A.20) on an object $d \in \mathbf{D}$ is equivalently the value of R on a cofibrant replacement Pd (298):

$$
\mathbb{R}R(d) \simeq R(Pd) \quad \in \mathrm{Ho}(\mathbf{C})\,. \tag{310}
$$

(iii) The derived unit $\mathbb{D}\eta$ of the derived adjunction (308), is, on any cofibrant object $c \in \mathbf{C}^{\mathrm{cof}}$, given by

$$
\mathbb{D}\eta_c \;:\; c \xrightarrow{\eta_c} R\big(L(c)\big) \xrightarrow{R(j_{L(c)})} R\big(PL(c)\big) \qquad \in \mathrm{Ho}(\mathbf{C}) \tag{311}
$$

where $L(c) \xrightarrow{j_{L(c)}} PL(c)$ is any fibrant replacement (298).

(iv) The derived co-unit $\mathbb{D}\varepsilon$ of the derived adjunction (308), is, on any fibrant object $d \in \mathbf{D}_{\mathrm{fib}}$, given by

$$
\mathbb{D}\varepsilon_d \;:\; L\big(QR(d)\big) \xrightarrow{L(p_{R(d)})} L\big(R(d)\big) \xrightarrow{\varepsilon_d} d \qquad \in \mathrm{Ho}(\mathbf{D}) \tag{312}
$$

where $QR(d) \xrightarrow{p_{R(d)}} R(d)$ is any cofibrant replacement (299).

Homotopy fibers and homotopy pullback.

Definition A.22 (Homotopy fiber)**.** Let $\mathbf{C}$ be a model category (Def. A.3).

(i) For $A \xrightarrow{p} B$ a morphism in $\mathbf{C}$ with $B \in \mathbf{C}_{\text{fib}} \subset \mathbf{C}$ a fibrant object (Notation A.6), and for $* \xrightarrow{b} B$ a morphism from the terminal object (a "point in B"), the *homotopy fiber* of p over b is the image in the homotopy category (303) of the ordinary fiber over b, i.e. the pullback (291) along b in $\mathbf{C}$, of any fibration $\widetilde{p}$ weakly equivalent to p:

$$\begin{tikzcd} \mathrm{hofib}_b(\rho) \arrow[r] & A \arrow[d, "\rho"] \\ & B \end{tikzcd} \quad := \quad \gamma_{\mathbf{C}} \left(\begin{tikzcd} \mathrm{fib}_b(\widetilde{p}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widetilde{A} \arrow[d, "\widetilde{\rho}"', "\in \mathrm{Fib}"] & A \arrow[l, "\in \mathrm{W}"'] \arrow[dl, "\rho"] \\ * \arrow[r, "b"'] & B & \end{tikzcd} \right) \in \mathrm{Ho}(\mathbf{C}) \,. \tag{313}$$

This is well-defined in that $\mathrm{hofib}_b(p) \in \mathrm{Ho}(\mathbf{C})$ depends on the choice of fibration replacement $\widetilde{p}$ only up to isomorphism in the homotopy category.

(ii) Dually, homotopy co-fibers are homotopy fibers in the opposite model category (Def. A.9).

More generally:

Definition A.23 (Homotopy pullback)**.** Given a model category $\mathbf{C}$ (Def. A.3) and a pair of coincident morphisms

$$\begin{tikzcd} & A \arrow[d, "\rho"] \\ X \arrow[r, "\tau"'] & B \end{tikzcd}$$

between fibrant objects, the *homotopy pullback* of ρ along τ (or *homotopy fiber product of ρ with τ*) is the image of ρ, regarded as an object in the homotopy category (Def. A.14) of the slice model category (Example A.10) under the right derived functor (Prop. A.20) of the right base change functor along τ (Example A.18):

$$\mathrm{Ho}\big(\mathbf{C}^{/B}\big) \ni \begin{tikzcd} A \arrow[d, "\rho"] \\ B \end{tikzcd} \quad \overset{\substack{\text{homotopy} \\ \text{pullback}}}{\longmapsto} \quad \begin{tikzcd} \mathbb{R}\tau^* A \arrow[d, "\mathbb{R}\tau^*\rho"] \\ X \end{tikzcd} \;:=\; \mathrm{Ho}\big(\mathbf{C}^{/X}\big)\,, \tag{314}$$

By (307) the derived adjunction counit (312) on (314) gives a commuting square in (303) the homotopy category of $\mathbf{C}$

$$\begin{tikzcd} \mathbb{R}\tau^* A \arrow[r] \arrow[d, "\mathbb{R}\tau^*\rho"'] \arrow[dr, phantom, "\text{(hpb)}"] & A \arrow[d, "\rho"] \\ X \arrow[r, "\tau"'] & B \end{tikzcd} \quad := \quad \gamma_{\mathbf{C}} \left(\begin{tikzcd} \tau^*\widetilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \widetilde{A} \arrow[d, "\widetilde{\rho}"', "\in \mathrm{Fib}"] & A \arrow[l, "\in \mathrm{W}"'] \arrow[dl, "\rho"] \\ X \arrow[r, "\tau"'] & B & \end{tikzcd} \right) \in \mathrm{Ho}(\mathbf{C}) \,. \tag{315}$$

This square in the homotopy category, together with its pre-image pullback square in the model category, is the *homotopy pullback square* of ρ along τ.

Example A.24 (Homotopy fiber is homotopy pullback to the point)**.** Homotopy fibers (Def. A.22) are the homotopy pullbacks (Def. A.23) to the terminal object, by (310).

Lemma A.25 (Factorization lemma [Bro73, p. 431])**.** *Let* $\mathbf{C}$ *be a model category (Def. A.3) and* $A \xrightarrow{\rho} B \in \mathbf{C}_{\text{fib}}$ *a morphism between fibrant objects. Then for* $\mathrm{Paths}(B)$ *a path space object for* B *(Def. A.11) the vertical composite in the following diagram*

$$\begin{tikzcd} A \arrow[r, "\in \mathrm{W}"] \arrow[ddr, "\rho"'] & p_1^* A \arrow[r, "\in \mathrm{W}"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\rho"] \\ & \mathrm{Paths}(B) \arrow[r, "p_1"'] \arrow[d, "p_0"] & B \\ & B & \end{tikzcd} \tag{316}$$

is a fibration, and in fact a fibration resolution of ρ*, in that it factors* ρ *through a weak equivalence.*

Example A.26 (Homotopy pullback via triples)**.** Given a model category $\mathbf{C}$ (Def. A.3) and a pair of coincident morphisms

$$\begin{tikzcd} & A \arrow[d, "\rho"] \\ X \arrow[r, "\tau"] & B \end{tikzcd}$$

between fibrant objects, Lemma A.25 says that the corresponding homotopy pullback (Def. A.23) is computed by the following diagram

$$\begin{tikzcd} \mathbb{R}\tau^* A \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\phi"] & & A \arrow[d, "\rho"] \\ & \mathrm{Paths}(B) \arrow[r, "p_0"'] \arrow[d, "p_1"] & B \\ X \arrow[r, "\tau"] & B & \end{tikzcd} \quad = \quad \begin{tikzcd} \tau^*\big(p_0 \circ p_1^* \rho\big) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{(pb)}"] & p_1^* A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\rho"] \\ & \mathrm{Paths}(B) \arrow[r, "p_0"'] \arrow[d, "p_1"] & B \\ X \arrow[r, "\tau"] & B & \end{tikzcd}$$

Here the right hand side exhibits the left hand side as a limit cone. This means that the homotopy pullback $\mathbb{R}\tau^* A$ is universally characterized by the fact that morphisms into it are triples (f, g, ϕ) , consisting of a pair of morphisms f, g to A, X, respectively, and a right homotopy ϕ (Def. A.12) between their composites with ρ and τ, respectively:

$$\mathbf{C}\big(-; \mathbb{R}\tau^* A\big) \;\; \simeq \;\; \left\{ (f, g, \phi) \;\middle|\; \begin{tikzcd} \cdot \arrow[r, dashed, "f"] \arrow[d, dashed, "g"'] \arrow[dr, phantom, "\Leftarrow \phi"] & A \arrow[d, "\rho"] \\ X \arrow[r, "\tau"] & B \end{tikzcd} \right\} \tag{317}$$

Quillen equivalences.

Lemma A.27 (Conditions characterizing Quillen equivalences)**.** *Given a Quillen adjunction $L \dashv_{\mathrm{Qu}} R$ (Def. A.17), the following conditions are equivalent:*

- *The left derived functor (Prop. A.20) is an equivalence of homotopy categories (Def. A.14)* $\mathrm{Ho}(\mathscr{D}) \underset{\simeq}{\xleftarrow{\mathbb{L}L}} \mathrm{Ho}(\mathscr{C})$.
- *The right derived functor (Prop. A.20) is an equivalence of homotopy categories (Def. A.14)* $\mathrm{Ho}(\mathscr{D}) \underset{\simeq}{\xrightarrow{\mathbb{R}R}} \mathrm{Ho}(\mathscr{C})$.
- *Both of the following two conditions hold:*
  - **(i)** *The derived adjunction unit* $\mathbb{D}\eta$ (311) *is a natural isomorphism, hence* (311) *is a weak equivalence in* $\mathbf{C}$*;*
  - **(ii)** *The derived adjunction counit* $\mathbb{D}\varepsilon$ (312) *is a natural isomorphism, hence* (312) *is a weak equivalence in* $\mathbf{D}$.
- *For $c \in \mathbf{C}^{\mathrm{cof}}$ and $d \in \mathbf{D}_{\mathrm{fib}}$, a morphism out of $L(c)$ is a weak equivalence precisely if its adjunct into $R(d)$ is:*

$$L(c) \underset{\in \mathrm{W}}{\xrightarrow{\;\;f\;\;}} d \qquad \Leftrightarrow \qquad c \underset{\in \mathrm{W}}{\xrightarrow{\;\;\widetilde{f}\;\;}} R(d)\,. \tag{318}$$

Definition A.28 (Quillen equivalence)**.** If the equivalent conditions from Lemma A.27 are met, a Quillen adjunction $L \dashv_{\mathrm{Qu}} R$ (Def. A.17) is called a *Quillen equivalence*, which we denote as follows:

$$\mathscr{D} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Qu}}}}}} \mathscr{C}\,.$$

Hence:

Proposition A.29 (Derived equivalence of homotopy categories)**.** *The derived adjunction (Prop. A.20) of a Quillen equivalence (Def. A.28) is an adjoint equivalence of homotopy categories (Def. A.14):*

$$\mathrm{Ho}(\mathbf{D}) \underset{\mathbb{R}R}{\overset{\mathbb{L}L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ho}(\mathbf{C})\,. \tag{319}$$

Lemma A.30 (Quillen equivalence when left adjoint creates weak equivalences [EI19, Lemma 3.3])**.** *Let $L \dashv_{\mathrm{Qu}} R$ be a Quillen adjunction (Def. A.17) such that the left adjoint functor L creates weak equivalences, in that for all morphisms f in $\mathbf{C}$ we have*

$$f \in \mathrm{W}_{\mathbf{C}} \qquad \Leftrightarrow \qquad L(f) \in \mathrm{W}_{\mathbf{D}}\,. \tag{320}$$

Then $L \dashv_{\mathrm{Qu}} R$ is a Quillen equivalence (Def. A.28) precisely if the adjunction co-unit ε_d is a weak equivalence on all fibrant objects $d \in \mathbf{C}_{\mathrm{fib}}$.

Proof. By Lemma A.27, it is sufficient to check that the **(i)** derived unit and **(ii)** derived counit of the adjunction are weak equivalences precisely if the ordinary counit is a weak equivalence.
(ii) For the derived counit (312)

$$\mathbb{D}\varepsilon_c \;:\; L\big(QR(d)\big) \xrightarrow[\in \mathrm{W}]{L(p_{R(d)})} L\big(R(d)\big) \xrightarrow{\varepsilon_d} d$$

we have that $p_{R(d)}$ is a weak equivalence (299), and since L preserves this, by assumption, so is $L\big(p_{R(d)}\big)$. Therefore $\mathbb{D}\varepsilon_d$ is a weak equivalence precisely if ε_d is, by 2-out-of-3 (293).
(i) For the derived unit (311)

$$c \xrightarrow{\eta_c} R\big(L(c)\big) \xrightarrow{R(j_{L(c)})} R\big(PL(c)\big)$$

consider the composite of its image under L with the adjunction counit, as shown in the middle row of the following diagram:

$$\begin{tikzcd} L(c) \arrow[r, "L(\eta_c)"] \arrow[rr, bend left, "L(\mathbb{D}\eta_c)"] \arrow[rrr, bend right, "j_{L(c)} \in \mathrm{W}"'] & L \circ R\big(L(c)\big) \arrow[r, "L\circ R(j_{L(c)})"] & L \circ R\big(PL(c)\big) \arrow[r, "\varepsilon_{PL(c)}"] & PL(c)\,. \end{tikzcd}$$

By the formula for adjuncts, this composite equals the adjunct of the derived adjunction unit, hence $j_{L(c)}$, as shown by the bottom morphism, which is a weak equivalence (298). Now, since L creates weak equivalences by assumption, $L(\mathbb{D}\eta_c)$ is a weak equivalence precisely if $\mathbb{D}\eta_c$ is a weak equivalence. Therefore it follows, again by 2-out-of-3 (293), that this is the case precisely if the adjunction counit ε is a weak equivalence on the fibrant object $PL(c)$. ☐

Proposition A.31 (Base change along weak equivalence in right proper model category)**.** *Let $\mathbf{C}$ be a right proper model category (Def. A.5). Then its base change Quillen adjunction (Example A.18) along any weak equivalence*

$$B_1 \xrightarrow[\in \mathrm{W}]{f} B_2 \qquad \in \mathbf{C}$$

is a Quillen equivalence (Def. A.28):

$$\begin{tikzcd} \mathbf{C}^{/B_2} \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] & \mathbf{C}^{/B_1}\,. \arrow[l, shift right=2, "f_!"'] \arrow[l, leftarrow, shift left=2, "f^*"'] \end{tikzcd}$$

Proof. Observe that $B_2 \xrightarrow{\mathrm{id}} B_2$ is the terminal object of $\mathbf{C}^{/B_2}$, so that the fibrant objects of $\mathbf{C}^{/B_2}$ correspond to the fibrations in $\mathbf{C}$ over B_2. Therefore, the condition (318) says that for $f_! \dashv f^*$ to be a Quillen equivalence it is sufficient that in (307) c is a weak equivalence precisely if $\widetilde{c}$ is, assuming that ρ is a fibration:

$$\begin{tikzcd} X \arrow[rr, dashed, "c"] \arrow[dr, "\tau"'] & & A \arrow[d, "\rho \in \mathrm{Fib}"] \\ & B_1 \arrow[r, "f \in \mathrm{W}"'] & B_2 \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} X \arrow[r, dashed, "\widetilde{c}"] \arrow[dr, "\tau"'] & f^*A \arrow[r, "\rho^* f \in \mathrm{W}"] \arrow[d, "f^*\rho"] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\rho \in \mathrm{Fib}"] \\ & B_1 \arrow[r, "f \in \mathrm{W}"'] & B_2 \end{tikzcd} \tag{321}$$

But under this assumption, right-properness implies that $\rho^* f$ is a weak equivalence (297), so that the statement follows by 2-out-of-3 (293).

Alternative Proof. The conclusion also follows with Lemma A.30: The left adjoint functor $L = f_!$ clearly creates weak equivalences (320) (by the nature of the slice model structure, Example A.10), so that Lemma A.30 asserts that we have a Quillen equivalence as soon as the ordinary adjunction counit is a weak equivalence on all fibrant objects. By (307), the adjunction counit on a fibration $\rho \in \mathrm{Fib}$ is the dashed morphism $\rho^* f$ in the following diagram on the right:

$$
\begin{tikzcd}
f^*A \arrow[r, dashed, "\mathrm{id}"] \arrow[dr, "f^*\rho"'] & f^*A \arrow[r, "\rho^* f \in \mathrm{W}"] \arrow[d, "f^*\rho"] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\rho \in \mathrm{Fib}"] \\
& B_1 \arrow[r, "f \in \mathrm{W}"'] & B_2
\end{tikzcd}
\quad \leftrightarrow \quad
\begin{tikzcd}
f^*A \arrow[rr, dashed, "\rho^* f \in \mathrm{W}"] \arrow[dr, "f^*\rho"'] & & A \arrow[d, "\rho \in \mathrm{Fib}"] \\
& B_1 \arrow[r, "f \in \mathrm{W}"'] & B_2
\end{tikzcd}
\tag{322}
$$

And hence this is a weak equivalence, again by right-properness. □

Example A.32 (Quillen equivalence between topological spaces and simplicial sets [Qu67])**.** Forming simplicial sets constitutes a Quillen equivalence (Def. A.28)

$$
\mathrm{TopologicalSpaces}_{\mathrm{Qu}} \underset{\underset{\text{singular simplicial complex}}{\mathrm{Sing}}}{\overset{\overset{\text{geometric realization}}{|-|}}{\underset{\simeq_{\mathrm{Qu}}}{\rightleftarrows}}} \mathrm{SimplicialSets}_{\mathrm{Qu}}
\tag{323}
$$

between the classical model structure on topological spaces (Example A.7) and the classical model structure on simplicial sets (Example A.8).

Example A.33 (Classical homotopy category)**.** By Prop. A.29 and the derived adjunction (Prop. A.20) of the $|-| \dashv \mathrm{Sing}$-adjunction (Example A.32) is an equivalence between the homotopy categories (Def. A.14) of the classical model category of topological spaces (Example A.7) and the classical model category of simplicial sets (Example A.8):

$$
\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big) \underset{\mathbb{R}\mathrm{Sing}}{\overset{\mathbb{L}|-|}{\underset{\simeq}{\rightleftarrows}}} \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big).
\tag{324}
$$

Either of these is the *classical homotopy category*. We refer to its objects as *homotopy types*, to be distinguished from the actual topological spaces or simplicial set that represent them.

Cell complexes.

Proposition A.34 (Skeleta and truncation [May67, §II.8][DK84, §1.2 (vi)])**.** *For each $n \in \mathbb{N}$ there is a pair of adjoint functors*

$$
\mathrm{SimplicialSets} \underset{\mathrm{cosk}}{\overset{\mathrm{sk}}{\underset{\perp}{\rightleftarrows}}} \mathrm{SimplicialSets}\,,
\tag{325}
$$

where $\mathrm{sk}_n(S)$ is the simplicial sub-set generated by the simplices in S of dimension $\leq n$ (hence including only all their degenerate higher simplices), and where

$$
\mathrm{cosk}_n(S) \;:\; [k] \;\mapsto\; \mathrm{SimplicialSets}\big(\mathrm{sk}_n(\Delta[k])\,,\,S\big)\,.
$$

Here cosk_{n+1} preserves fibrant objects of the classical model structure (Example A.8) and models n-truncation, in that:

$$
\pi_k\big|\mathrm{cosk}_{n+1}(S)\big| \;=\; 0 \qquad \text{for } k \geq n+1
$$

and there are natural fibrations $S \xrightarrow{p_n} \mathrm{cosk}_n(S)$ such that

$$
\pi_k\big|S\big| \xrightarrow[\simeq]{\pi_k|p_n|} \pi_k\big|\mathrm{cosk}_{n+1}(S)\big| \qquad \text{for } k \leq n\,.
$$

For $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ we write

$$
A(n) \;:=\; \big|\mathrm{cosk}_{n+1}\big(\mathrm{Sing}(A)\big)\big|
\tag{326}
$$

We say that A is *n-truncated* if it is equivalent to its n-truncation:

$$
A \text{ is } n\text{-truncated} \quad \Leftrightarrow \quad A \;\simeq\; A(n)\,.
\tag{327}
$$

for its n-truncation.

Example A.35 (Simplicial sets are weakly equivalent to singular simplicial sets of their realization)**.** For $S \in$ SimplicialSets, the unit of the adjunction (323) is a weak equivalence:

$$S \xrightarrow[\in \mathrm{W}]{\eta_S} \mathrm{Sing}(|S|) \,. \tag{328}$$

Notice that, a priori, the characterization of Quillen equivalences (Lemma A.27) only says, with Example A.32, that the derived adjunction unit, hence the composite

$$S \xrightarrow{\eta_S} \mathrm{Sing}(|S|) \xrightarrow{\mathrm{Sing}(|j_{|S|}|)} \mathrm{Sing}(P\,|S|)$$

is a weak equivalence, where $j_{|S|}$ is a Kan fibrant replacement for $|S|$. But since all topological spaces are fibrant (Example A.7), the above simpler condition follows.

Example A.36 (Homotopy types of manifolds via triangulations)**.** For $X \in$ TopologicalSpaces equipped with the stucture of an n-manifold, there exists a *triangulation* of X, namely an n-skeletal simplicial set (Prop. A.34)

$$\mathrm{Tr}(X) \in \mathrm{SimplicialSets}\,, \qquad \mathrm{sk}_n\big(\mathrm{Tr}(X)\big) \;=\; \mathrm{Tr}(X) \tag{329}$$

equipped with a homeomorphism to X out of its geometric realization (323)

$$|\mathrm{Tr}(X)| \xrightarrow[\text{homeo}]{p} X\,, \tag{330}$$

Since the inclusion

$$\mathrm{Tr}(X) \xhookrightarrow[\in \mathrm{W}]{\eta_{\mathrm{Tr}(X)}} \mathrm{Sing}\big(|\mathrm{Tr}(X)|\big) \xrightarrow[\in \mathrm{Iso}]{\mathrm{Sing}(p)} \mathrm{Sing}(X)\,, \tag{331}$$

is a weak equivalence (by Example A.35), the triangulation represents the homotopy type (324) of the manifold.

Proposition A.37 (Homotopy classes of maps out of n-manifolds)**.** *Let $X \in$ TopologicalSpaces admit the stucture of an n-manifold. Then for any $A \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33) the homotopy classes of maps $X \to A$ are in natural bijection to the homotopy classes into the $(n-1)$-truncation* (326) *of A:*

$$\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,A\big) \;\simeq\; \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,A(n-1)\big) \tag{332}$$

Proof. Consider the following sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,A\big) &\simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\big(\mathrm{Sing}(X)\,,\mathrm{Sing}(A)\big)\\
&\simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\big(\mathrm{Tr}(X)\,,\mathrm{Sing}(A)\big)\\
&\simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\Big(\mathrm{sk}_n\big(\mathrm{Tr}(X)\big)\,,\mathrm{Sing}(A)\Big)\\
&\simeq \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big)\Big(\mathrm{Tr}(X)\,,\mathrm{cosk}_n\big(\mathrm{Sing}(A)\big)\Big)\\
&\simeq \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\Big(\big|\mathrm{Tr}(X)\big|\,,\big|\mathrm{cosk}_n\big(\mathrm{Sing}(A)\big)\big|\Big)\\
&\simeq \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)\big(X\,,A(n-1)\big).
\end{aligned}$$

Here the first step is (A.33), using, with Example A.21, that all topological spaces are fibrant and all simplicial sets cofibrant. The second step uses (331). The third step is (329). Using that we do not need to fibrantly replace the skeleton in the domain, by Prop. A.16, the fourth step is the skeleta-adjunction (325). The fifth step is the reverse of the first step, with the same argument on (co-)fibrancy. The last step uses (330) in the first argument and (326) in the second. The composite of these isomorphisms is the desired (332). □

Proposition A.38 (Postnikov tower [GJ99, Cor. 3.7])**.** *Let $X \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}\big)$ (Example A.33). If X is connected, then its sequence of n-truncations* (326) *forms a system of fibrations with homotopy fibers (Def. A.22) the Eilenberg-MacLane spaces* (14) *of the homotopy group in the given degree:*

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
K(\pi_3(X),3) \arrow[r, "\mathrm{hfib}(p_3^X)"] & X(3) \arrow[d, "p_3^X"] \\
K(\pi_2(X),2) \arrow[r, "\mathrm{hfib}(p_2^X)"] & X(2) \arrow[d, "p_2^X"] \\
K(\pi_1(X),1) \arrow[r, "\mathrm{hfib}(p_1^X)"] & X(1) \arrow[d, "p_1^X"] \\
 & X(0)\,.
\end{tikzcd}
$$

If X is not connected then this applies to each of its connected components.

Stable model categories.

Example A.39 (Looping/suspension-adjunction)**.** On the category of pointed topological spaces, equipped with the coslice model structure under the point (Example A.10) of the classical model structure (Example A.7), the operation of forming based loop spaces $\Omega X := \mathrm{Maps}^{*/}(S^1, X)$ is the right adjoint in a Quillen adjunction (Def. A.17)

$$
\begin{tikzcd}
\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}} \arrow[r, shift right, "\Omega"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}} \arrow[l, shift right, "\Sigma"']
\end{tikzcd}
\tag{333}
$$

whose left adjoint is the reduced suspension operation $\Sigma X := S^1 \wedge X := \big(S^1 \times X\big)/\big(S^1 \times \{*_X\} \sqcup \{*_{S^1}\} \times X\big)$.

Example A.40 (Stable model category of sequential spectra [BF78][GJ99, §X.4])**.** There exists a model category (Def. A.3) $\mathrm{SequentialSpectra}_{\mathrm{BF}}$ whose objects are sequences

$$
E \;:=\; \Big\{ E_n \,\in\, \mathrm{TopologicalSpaces},\; \Sigma E_n \xrightarrow{\sigma_n} E_{n+1} \,\Big\}_{n \in \mathbb{N}}
$$

of topological spaces E_n and continuous function σ_n from their suspension ΣE_n (Example A.39) to the next space in the sequences; and whose morphisms $E \xrightarrow{f} F$ are sequences of component maps $E_n \xrightarrow{f_n} F_n$ that commute with the σs. Moreover:

- W – *weak equivalences* are the morphisms that induce isomorphisms on all *stable homotopy groups* $\pi_\bullet(X) := \underset{\underset{n}{\longrightarrow}}{\lim} \pi_{\bullet+k}(X_k)$ (where the colimit is formed using the σ's);
- Cof – *cofibrations* are those morphisms $E \xrightarrow{f} F$ such that the maps

$$
E_0 \xrightarrow[\in \mathrm{Cof}]{f_0} F_0 \quad \text{and} \quad \underset{n\in\mathbb{N}}{\forall} \;\; E_{n+1} \underset{\Sigma E_n}{\sqcup} \Sigma F_n \xrightarrow[\in \mathrm{Cof}]{(f_{n+1}, \sigma_n^F)} F_{n+1}
$$

  are cofibrations in the classical model structure on topological spaces (Example A.7).
- Fib – *Fibrant objects* are the Ω-*spectra*, namely those sequences of spaces $\{E_n\}$ for which the $\Sigma \dashv \Omega$-adjunct (333) of each σ_n is a weak equivalence:

$$
\Big\{ E_n \in \mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}}\,,\;\; E_n \xrightarrow[\in \mathrm{W}]{\tilde{\sigma}_n} \Omega E_{n+1} \,\Big\}_{n\in\mathbb{N}}
\tag{334}
$$

Example A.41 (Derived stabilization adjunction)**.** The suspension/looping Quillen adjunction on pointed spaces (Example A.39) extends to a commuting diagram of Quillen adjunctions (Def. A.17) to and on the stable model category of spectra (Example A.40)

$$
\begin{tikzcd}
\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}} \arrow[r, shift right, "\Omega"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[d, shift right, "\Sigma^\infty"'] \arrow[d, phantom, "\dashv_{\mathrm{Qu}}"] & \mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}} \arrow[l, shift right, "\Sigma"'] \arrow[d, shift right, "\Sigma^\infty"'] \arrow[d, phantom, "\dashv_{\mathrm{Qu}}"] \\
\mathrm{SequentialSpectra}_{\mathrm{BF}} \arrow[r, shift right, "\Omega"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[u, shift right, "\Omega^\infty"'] & \mathrm{SequentialSpectra}_{\mathrm{BF}}\,. \arrow[l, shift right, "\Sigma"'] \arrow[u, shift right, "\Omega^\infty"']
\end{tikzcd}
\tag{335}
$$

such that the bottom adjunction is a Quillen equivalence (Def. A.28), hence such that under passage to derived adjunctions (Prop. A.20)

$$\begin{tikzcd} \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}}\Big) \arrow[d, shift right, "\mathbb{L}\Sigma^\infty"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\Big(\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}}\Big) \arrow[l, shift right, "\mathbb{L}\Sigma"'] \arrow[d, shift right, "\mathbb{L}\Sigma^\infty"'] \\ \mathrm{Ho}\Big(\mathrm{SequentialSpectra}_{\mathrm{BF}}\Big) \arrow[u, shift right, "\mathbb{R}\Omega^\infty"'] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}\Big(\mathrm{SequentialSpectra}_{\mathrm{BF}}\Big) \arrow[l, shift right, "\mathbb{L}\Sigma"'] \arrow[u, shift right, "\mathbb{R}\Omega^\infty"'] \end{tikzcd} \tag{336}$$

the bottom adjunction is an equivalence, thus exhibiting the homotopy category of spectra as being *stable* under looping/suspension.
We say that
(i) $\mathrm{Ho}\big(\mathrm{SequentialSpectra}_{\mathrm{BF}}\big)$ is the *stable homotopy category* of spectra;
(ii) the vertical adjunction $(\mathbb{L}\Sigma^\infty \dashv \mathbb{R}\Omega^\infty)$ is the *stabilization adjunction* between homotopy types (324) and spectra.
(iii) the images of Σ^∞ are the *suspension spectra*.
(iv) For $E \in \mathrm{Ho}\big(\mathrm{SequentialSpectra}_{\mathrm{BF}}\big)$ and $n \in \mathbb{N}$ we write (for brevity and in view of (334))

$$E_n := \mathbb{R}\Omega^\infty\big((\mathbb{L}\Sigma)^n E\big) \quad \in \mathrm{Ho}\big(\mathrm{TopologicalSpaces}^{*/}_{\mathrm{Qu}}\big) \tag{337}$$

for the homotopy type of the nth component space of the spectrum.

Smooth ∞-stacks.

Definition A.42 (Simplicial presheaves over Cartesian spaces)**.** We write
(i)

$$\mathrm{CartesianSpaces} := \big\{\mathbb{R}^{n_1} \xrightarrow{\text{smooth}} \mathbb{R}^{n_2}\big\}_{n_i \in \mathbb{N}} \tag{338}$$

for the category whose objects are the Cartesian spaces $\mathbb{R}^n$, for $n \in \mathbb{N}$, and whose morphisms are the *smooth* functions between these (hence the full subcategory of SmoothManifolds on the Cartesian spaces).
(ii)

$$\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big) := \mathrm{Functors}\big(\mathrm{CartesianSpaces}^{\mathrm{op}}, \mathrm{SimplicalSets}\big) \tag{339}$$

for the category of functors from the opposite of CartesianSpaces (338) to SimplicialSets,

Example A.43 (Model structure on simplicial presheaves over Cartesian spaces [Du98][Du01][FSSt10, §A])**.** The category of simplicial presheaves over Cartesian spaces (Prop. 339) carries the following model category structures (Def. A.3):
(i) The global projective model structure

$$\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\mathrm{proj}} \tag{340}$$

whose

- W – *weak equivalences* are the morphisms which over each $\mathbb{R}^n$ are weak equivalence in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8),
- Fib – *fibrations* are the morphisms which over each $\mathbb{R}^n$ are fibrations in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8),

(ii) The local projective model structure

$$\mathrm{SmoothStack}_\infty := \mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \tag{341}$$

whose:

- W – *weak equivalences* are the morphisms whose stalk at $0 \in \mathbb{R}^n$ is a weak equivalence in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8), for all $n \in \mathbb{N}$;
- Cof – *cofibrations* are the morphisms with the left lifting property (295) against the class of morphisms which over each $\mathbb{R}^n$ are in $\mathrm{Fib} \cap \mathrm{W}$ of $\mathrm{SimplicialSets}_{\mathrm{Qu}}$.

Definition A.44 (Homotopy category of smooth ∞-stacks)**.** We write

$$\mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \;:=\; \mathrm{Ho}\Big(\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\substack{\mathrm{proj}\\ \mathrm{loc}}}\Big) \tag{342}$$

for the homotopy category (Def. A.14) of the local projective model category of simplicial presheaves over CartesianSpaces (Example A.43). We say that the objects of $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$ (342) are *smooth ∞-stacks*.

For exposition of smooth ∞-stack theory see [FSS12b, §2][FSS13a][SS20a, §1]. In particular, notice:

Example A.45 (Smooth manifolds as smooth ∞-stacks)**.** For $X \in$ SmoothManifolds it is incarnated as a smooth ∞-stack (Def. A.44) by the rule

$$\mathscr{X} \;=\; \Big(\mathbb{R}^n \mapsto \big(\Delta[k] \mapsto \mathrm{SmoothManifolds}(\mathbb{R}^n, X)\big)\Big). \tag{343}$$

This construction constitutes to a full embedding

$$\mathrm{SmoothManifolds} \hookrightarrow \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$$

of smooth manifolds into smooth ∞-stacks.

Lemma A.46 (∞-Stackification preserves finite homotopy limits)**.** *The identity functors constitute a Quillen adjunction (Def. A.17) between the local and the global projective model categories of Example A.43:*

$$\begin{tikzcd} \mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \arrow[r, shift right=1.5ex, "\mathrm{id}"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\mathrm{proj}} \arrow[l, shift right=1.5ex, "\mathrm{id}"'] \end{tikzcd}.$$

Moreover, this is such that the derived left adjoint functor (Prop. A.20)

$$L^{\mathrm{loc}} :\; \mathrm{Ho}\Big(\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\mathrm{proj}}\Big) \xrightarrow{\;\mathbb{L}\mathrm{id}\;} \mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \tag{344}$$

(the ∞-stackification functor) preserves homotopy pullbacks (Def. A.23).

Proposition A.47 (Shape Quillen adjunction [Sc13, Prop. 4.4.8][SS20a, Example 3.18])**.** *We have a Quillen adjunction (Def. A.17)*

$$\begin{tikzcd} \mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicalSets}\big)_{\substack{\mathrm{proj}\\ \mathrm{loc}}} \arrow[r, shift left=1.5ex, "\mathrm{Shp}"] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{SimplicialSets}_{\mathrm{Qu}} \arrow[l, shift left=1.5ex, "\mathrm{Disc}"] \end{tikzcd}$$

between the projective local model structure on simplicial presheaves over CartesianSpaces *(Example A.43) and the classical model structure on simplicial sets (Example A.8), hence a derived adjunction (Prop. A.20) between homotopy category of ∞-stacks (Def. A.44) and the classical homotopy category (Example A.33)*

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \arrow[r, shift left=1.5ex, "\mathbb{L}\mathrm{Shp}"] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big) \arrow[l, shift left=1.5ex, "\mathbb{R}\mathrm{Disc}"] \end{tikzcd}$$

whose (underived) right adjoint sends a simplicial set to the presheaf which is constant on that simplicial set:

$$\mathrm{Disc}(S) \;:=\; \mathrm{const}(S) \;:\; \big(\mathbb{R}^n \mapsto S\big). \tag{345}$$

Homological algebra.

Example A.48 (Projective model structure on connective chain complexes [Qu67, §II.4 (5.)])**.** The category $\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}$ of connective chain complexes of abelian groups (i.e. concentrated in non-negative degrees with differential of degree -1) carries a model category structure (Def. A.3) whose

W – weak equivalences are the quasi-isomorphisms (those inducing isomorphisms on all chain homology groups)

Fib – fibrations are the positive-degree wise surjections

Cof – cofibrations are the morphisms with degreewise injective kernels.

We write $\big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}}$ for this model category.

More generally:

Example A.49 (Projective model structure on presheaves of connective chain complexes [Ja03, p. 7])**.** The category of presheaves of connective chain complexes over CartesianSpaces (338) carries the structure of a model category whose weak equivalences and fibrations are objectwise those of $\big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}}$ (Example A.48). We write $\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}}$ for this model category.

Proposition A.50 (Dold-Kan correspondence [Do58, Thm 1.9][Ka58][GJ99, §III.2][SSh03a, §2.1])**.** *Given* $A_\bullet \in$ SimplicialAbelianGroups*, its normalized chain complex is the connective chain complex of abelian groups (Example A.48) which in degree* $n \in \mathbb{N}$ *is the quotient of* A_n *by the degenerate cells and whose differential is the alternating sum of the face maps:*

$$N(A)_\bullet := \Big\{ N(A)_n := A_n/\sigma(A_{n+1}) \,,\; \partial_n := \sum_{i=0}^{n} (-1)^i d_i \,:\, N(A)_n \longrightarrow N(A)_{n-1} \Big\}_{n\in\mathbb{N}} \in \mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \,. \quad (346)$$

(i) *This construction constitutes an adjoint equivalence of categories*

$$\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \underset{\longrightarrow}{\overset{N}{\underset{\simeq}{\longleftarrow}}} \mathrm{SimplicialAbelianGroups} \quad (347)$$

(ii) *such that simplicial homotopy groups of* $A \in$ SimplicialAbelianGroups $\to$ SimplicialSet *are identified with chain homology groups of the normalized chain complex ([GJ99, Cor. III.2.5]):*

$$\pi_\bullet(A) \simeq H_\bullet(NA) \,. \quad (348)$$

Example A.51 (Model structure on simplicial abelian groups [Qu69, §III.2][SSh03a, §4.1])**.** The category of SimplicialAbelianGroups carries a model category structure (Def. A.3) whose

- W – weak equivalences are the morphisms which are weak equivaleces as morphisms in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8)
- Fib – fibrations are the morphisms which are fibrations as morphisms in $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ (Example A.8)

In other words, this is the transferred model structure along the free/forgetful adjunction, which thus becomes a Quillen adjunction (Def. A.17):

$$\mathrm{SimplicialAbelianGroup}_{\mathrm{proj}} \underset{\longrightarrow}{\overset{\mathbb{Z}[-]}{\underset{\perp_{\mathrm{Qu}}}{\longleftarrow}}} \mathrm{SimplicialSets}_{\mathrm{Qu}} \,. \quad (349)$$

Proposition A.52 (Dold-Kan Quillen equivalence [SSh03a, §4.1][Ja03, Lemma 1.5])**.** *With respect to the projective model structure on connective chain complexes (Example A.48) and the projective model structure on simplicial abelian groups (Example A.51) the Dold-Kan correspondence (Prop. A.50) is a Quillen equivalence (Def. A.28):*

$$\big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}} \underset{\longrightarrow}{\overset{N}{\underset{\simeq_{\mathrm{Qu}}}{\longleftarrow}}} \mathrm{SimplicialAbelianGroups}_{\mathrm{proj}} \,, \quad (350)$$

where both functors preserve all three classes of morphims, Fib*,* Cof *and* W*, separately.*

Example A.53 (Dold-Kan construction [FSSt10, §3.2.3][FSS12b, §2.4])**. i)** We write DK for the total right adjoint in the composite of the free Quillen adjunction (349) and the Dold-Kan equivalence (350):

$$\begin{tikzcd} \big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}} \arrow[r, shift left, "\simeq_{\mathrm{Qu}}"'] \arrow[rr, bend right, "\mathrm{DK}"'] & \mathrm{SimplicialAbelianGroups}_{\mathrm{proj}} \arrow[l, shift left, "N"'] \arrow[r, shift left, "\perp_{\mathrm{Qu}}"'] & \mathrm{SimplicialSets}_{\mathrm{Qu}} \arrow[l, shift left, "\mathbb{Z}[-]"'] \end{tikzcd} \,. \quad (351)$$

ii) This extends to a right Quillen functor on global projective model categories of presheaves (Example A.43, Example A.49). whose right derived functor (Prop. A.20) $\mathbb{R}$DK composed with the ∞-stackification functor (344) is thus of the form

$$\begin{tikzcd} \mathrm{Ho}\Big(\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}}\Big) \arrow[r, "\substack{\text{derived}\\ \text{Dold-Kan construction}\\ \mathbb{R}\mathrm{DK}}"] \arrow[dr, "\substack{\infty\text{-stackified}\\ \text{Dold-Kan construction}}"'] & \mathrm{Ho}\Big(\mathrm{PSh}\big(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets}\big)_{\mathrm{proj}}\Big) \arrow[d, "L^{\mathrm{loc}}\ \infty\text{-stackification}"] \\ & \mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \end{tikzcd}$$

and preserves homotopy pullbacks (by Lemma A.46).

Example A.54 (Projective model structure on unbounded chain complexes [Ho99, Thm. 2.3.11])**.** The category $\mathrm{ChainComplexes}_{\mathbb{Z}}$ of unbounded chain complexes of abelian groups carries a model category structure (Def. A.3) whose:

W – weak equivalences are the quasi-isomorphisms;
Fib – fibrations are the degreewise surjections.

We write $\big(\mathrm{ChainComplexes}_{\mathbb{Z}}\big)_{\mathrm{proj}}$ for this model category.

Proposition A.55 (Stable Dold-Kan construction)**.** *The Dold-Kan construction (Def. A.53) lifts along the stabilization adjunction (Example A.41) from connective to unbounded chain complexes (Example A.54), such as to make the following diagram commute:*

$$
\begin{tikzcd}
\mathrm{Ho}\Big(\big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}}\Big) \arrow[r, "\simeq"] \arrow[rr, bend left, "\overset{\text{Dold-Kan correspondence}}{\mathbb{R}\mathrm{DK}}"] \arrow[d, hook, shift right] & \mathrm{Ho}\big(\mathrm{SimplicialAbelianGroups}_{\mathrm{proj}}\big) \arrow[r] & \mathrm{Ho}\big(\mathrm{SimplicialSets}_{\mathrm{Qu}}\big) \arrow[d, shift right, "\mathbb{L}\Sigma^\infty"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Ho}\Big(\big(\mathrm{ChainComplexes}_{\mathbb{Z}}\big)_{\mathrm{proj}}\Big) \arrow[u, shift right, "\mathbb{R}\Omega^\infty"'] \arrow[u, phantom, "\dashv"] \arrow[r, "\simeq"] \arrow[rr, bend right, "\underset{\text{stable Dold-Kan construction}}{\mathbb{R}\mathrm{DK}_{\mathrm{st}}}"'] & \mathrm{Ho}\big((H\mathbb{Z})\mathrm{ModuleSpectra}\big) \arrow[r] & \mathrm{Ho}\big(\mathrm{SequentialSpectra}_{\mathrm{BF}}\big) . \arrow[u, shift right, "\mathbb{R}\Omega^\infty"']
\end{tikzcd}
\tag{352}
$$

Here the right adjoint on chain complexes is the homological truncation from below:

$$
\mathbb{R}\Omega^\infty\Big(\cdots \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} \cdots\Big) \;=\; \Big(\cdots \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} \ker(\partial_{-1})\Big). \tag{353}
$$

Proof. **(i)** It is clear from inspection that the assignment (353) is right adjoint to the inclusion of connective chain complexes, so that we have a pair of adjoint functors

$$
\begin{tikzcd}
\big(\mathrm{ChainComplexes}_{\mathbb{Z}}\big)_{\mathrm{proj}} \arrow[r, shift right, "\Omega^\infty"'] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \big(\mathrm{ChainComplexes}_{\mathbb{Z}}^{\geq 0}\big)_{\mathrm{proj}} . \arrow[l, hook, shift right]
\end{tikzcd}
\tag{354}
$$

Moreover, it is immediate that this is a Quillen adjunction (Def. A.17) between the projective model structure on connective chain complexes (Example A.48) and that on unbounded chain complexes (Example A.54): Ω^∞ clearly preserves fibrations (using that those between connective chain complexes need to be surjective only in positive degrees!) and clearly preserves all weak equivalences. Finally, since all chain complexes in the projective model structure are fibrant, we have that with Ω^∞ also $\mathbb{R}\Omega^\infty$ is given by (353), via Example A.21.
(ii) A Quillen adjunction of the form

$$
\begin{tikzcd}
\big(\mathrm{ChainComplexes}_{\mathbb{Z}}\big)_{\mathrm{proj}} \arrow[r, shift right, "H"'] \arrow[r, phantom, "\perp_{\simeq}"] \arrow[rr, bend right, "\mathrm{DK}_{\mathrm{st}}"'] & (H\mathbb{Z})\mathrm{ModuleSpectra} \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] & \mathrm{SequentialSpectra}_{\mathrm{BF}} \arrow[l, shift right]
\end{tikzcd}
\tag{355}
$$

is established in [SSh01, §B.1], where
(a) the first step is a Quillen equivalence (Def. A.28) between the projective model structure on unbounded chain complexes (Example A.54) and a model category of module spectra over the Eilenberg-MacLane spectrum $H\mathbb{Z}$ [SSh01, §B.1.11];
(b) the second step is a Quillen adjunction [SSh01, p. 37, item ii)] to the Bousfield-Friedlander model structure (Example A.40) whose right adjoint assigns underlying sequential spectra; such that
(c) the composite right adjoint $\mathrm{DK}_{\mathrm{st}}$ (355) further composed with Ω^∞ on spectra (335) equals the composite of Ω^∞ on chain complexes (354) with the unstable Dold-Kan construction (351):

$$
\Omega^\infty \circ \mathrm{DK}_{\mathrm{st}} \;\simeq\; \mathrm{DK} \circ \Omega^\infty
$$

(by immediate inspection of the construction in [SSh01, p. 38-39]).
(iii) By uniqueness of adjoints, this implies that the Quillen adjunction of the stable Dold-Kan construction (355) is intertwined by the Quillen adjunctions involving Ω^∞ with the Quillen adjunction of the unstable Dold-Kan construction (351), and hence the commuting diagram of derived functors (A.55) follows (Prop. A.20). ☐

References

[Ad62] J. F. Adams, *Vector fields on spheres*, Bull. Amer. Math. Soc. **68** (1962), 39-41, [euclid:bams/1183524456].

[Ad75] J. F. Adams, *Stable homotopy and generalized homology*, The University of Chicago Press, 1974, [ucp:bo21302708].

[Ad78] J. F. Adams, *Infinite loop spaces*, Annals of Mathematics Studies 90, Princeton University Press, 1978, [doi:10.1515/9781400821259].

[Add07] N. Addington, *Fiber bundles and nonabelian cohomology*, 2007, [pages.uoregon.edu/adding/notes/gstc2007.pdf]

[AGP02] M. Aguilar, S. Gitler, and C. Prieto, *Algebraic topology from a homotopical viewpoint*, Springer, 2002, [doi:10.1007/b97586].

[ABG10] M. Ando, A. Blumberg and D. Gepner, *Twists of K-theory and TMF*, in: R. Doran, G. Friedman, J. Rosenberg (eds.), *Superstrings, Geometry, Topology, and C^*-algebras*, Proc. Sympos. Pure Math. 81, Amer. Math. Soc., 2010, [doi:10.1090/pspum/081], [arXiv:1002.3004].

[ABGHR08] M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk, *Units of ring spectra and Thom spectra*, [arXiv:0810.4535].

[ABGHR14a] M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk, *Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory*, J. Topol. **7** (2014), 1077-1117, [doi:10.1112/jtopol/jtu009], [arXiv:1403.4320].

[ABGHR14b] M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology*, J. Topol. **7** (2014), 869-893, [arXiv:1403.4325].

[AHS01] M. Ando, M. Hopkins and N. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146** (2001) 595-687, [doi:10.1007/s002220100175].

[AHR10] M. Ando, M. Hopkins and C. Rezk, *Multiplicative orientations of* KO*-theory and the spectrum of topological modular forms*, 2010, [faculty.math.illinois.edu/~mando/papers/koandtmf.pdf]

[AM77] M. Artin and B. Mazur, *Formal Groups Arising from Algebraic Varieties*, Ann. Sci. École Norm. Sup. Sér. 4, **10** (1977), 87-131, [numdam:ASENS_1977_4_10_1_87_0].

[At67] M. Atiyah, *K-theory*, Harvard Lecture 1964 (notes by D. W. Anderson), Benjamin, NY, 1967, [https://www.maths.ed.ac.uk/~v1ranick/papers/atiyahk.pdf]

[AH59] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math Soc. **65** (1959), 276-281, [euclid:bams/1183523205].

[AH61] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, pp. 7-38, American Mathematical Society, 1961, [doi:10.1142/9789814401319_0008].

[AS04] M. Atiyah and G. Segal, *Twisted K-theory*, Ukrainian Math. Bull. **1** (2004), 287-330, [iamm.su/en/journals/j879/?VID=10], [arXiv:math/0407054].

[AS06] M. F. Atiyah and G. Segal, *Twisted K-theory and cohomology*, Inspired by S. S. Chern, 5-43, Nankai Tracts Math. 11, World Sci. Publ., Hackensack, NJ, 2006, [doi:10.1142/97898127726880002], [arXiv:math/0510674].

[Au09] C. Ausoni, *On the algebraic K-theory of the complex K-theory spectrum*, Invent. Math. **180** (2010), 611-668, [doi:10.1007/s00222-010-0239-x], [arXiv:0902.2334].

[AR02] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188**, (2002), 1-39, [euclid:acta/1485891473],

[AR07] C. Ausoni and J. Rognes, *Rational algebraic K-theory of topological K-theory*, Geom. Topol. **16** (2012), 2037-2065, [doi:10.2140/gt.2012.16.2037], [arXiv:0708.2160].

[BDRR09] N. Baas, B. I. Dundas, B. Richter and J. Rognes, *Stable bundles over rig categories*, J. Topol. **4** (2011), 623-640, [doi:10.1112/jtopol/jtr016], [arXiv:0909.1742].

[BDR03] N. Baas, B. I. Dundas and J. Rognes, *Two-vector bundles and forms of elliptic cohomology*, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004, [doi:10.1017/CBO9780511526398.005], [arXiv:math/0306027].

[BCSS07] J. Baez, A. Crans, U. Schreiber, and D. Stevenson, *From loop groups to 2-groups*, Homology Homotopy Appl. **9** (2007), 101-135, [doi:10.4310/HHA.2007.v9.n2.a4], [arXiv:math/0504123].

[BS09] J. Baez and D. Stevenson, *The Classifying Space of a Topological 2-Group*, In: N. Baas et al. (eds.), *Algebraic Topology*, Abel Symposia, vol 4., Springer 2009, [arXiv:0801.3843].

[Ba14] T. Bauer, *Bousfield localization and the Hasse square*, Chapter 9 in [DFHH14], [https://math.mit.edu/conferences/talbot/2007/tmfproc/Chapter09/bauer.pdf].

[Bei85] A. Beilinson, *Higher regulators and values of L-functions*, J. Math. Sci. **30** (1985), 2036-2070, [doi:10.1007/BF02105861].

[BBMM16] F. Belchí, U. Buijs, J. M. Moreno-Fernández, and A. Murillo, *Higher order Whitehead products and L_∞ structures on the homology of a DGL*, Linear Algeb. Appl. **520** (2017), 16-31, [doi:10.1016/j.laa.2017.01.008], [arXiv:1604.01478].

[BE13] D. Berwick-Evans, *Perturbative sigma models, elliptic cohomology and the Witten genus*, [arXiv:1311.6836] [math.AT].

[BGT10] A. Blumberg, D. Gepner and G. Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), 733-838, [doi:10.2140/gt.2013.17.733], [arXiv:1001.2282].

[BH58] A. Borel and F. Hirzebruch, *Characteristic Classes and Homogeneous Spaces I*, Amer. J. Math. **80** (1958), 458-538, [jstor:2372795].

[Bo36] K. Borsuk, *Sur les groupes des classes de transformations continues*, CR Acad. Sci. Paris **202** (1936), 1400-1403, [doi:10.24033/asens.603].

[Bo73] R. Bott, *On the Chern-Weil homomorphism and the continuous cohomology of Lie-groups*, Adv. Math. **11** (1973), 289-303, [doi:10.1016/0001-8708(73)90012-1].

[BT82] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982, [doi:10.1007/978-1-4757-3951-0].

[Bou79] A. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257-281, [doi:10.1016/0040-9383(79)90018-1].

[BF78] A. Bousfield and E. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Lecture Notes in Math. **658**, Springer 1978, pp. 80-130, [web.math.rochester.edu/people/faculty/doug/otherpapers/bousfield-friedlander.pdf]

[BG76] A. Bousfield and V. Gugenheim, *On PL deRham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179** (1976), [ams:memo-8-179].

[BK72] A. Bousfield and D. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics **304**, Springer, 1972, [doi:10.1007/978-3-540-38117-4].

[BCMMS02] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, Comm. Math. Phys. **228** (2002), 17-45, [doi:10.1007/s002200200646], [arXiv:hep-th/0106194].

[BMSS19] V. Braunack-Mayer, H. Sati and U. Schreiber, *Gauge enhancement of Super M-Branes via rational parameterized stable homotopy theory*, Comm. Math. Phys. **371** (2019), 197-265, [10.1007/s00220-019-03441-4], [arXiv:1806.01115].

[Br90] L. Breen, *Bitorseurs et cohomologie non-Abélienne*, in: P. Cartier et al. (eds.) *The Grothendieck Festschrift*, Vol I. 401-476, Birkhäuser, 1990, [doi:10.1007/978-0-8176-4574-8_10].

[Br09] L. Breen, *Notes on 1- and 2-gerbes*, in J. Baez and P May (eds.) *Towards Higher Categories*, The IMA Volumes in Mathematics and its Applications, vol 152, Springer, 2009, [doi:10.1007/978-1-4419-1524-5_5], [arXiv:math/0611317].

[BGNT08] P. Bressler, A. Gorokhovsky, R. Nest, and B. Tsygan, *Chern character for twisted complexes*, in: *Geometry and Dynamics of Groups and Spaces*, Progr. Math. **265** (2008), 309-324, [doi:10.1007/978-3-7643-8608-5_5], [https://arxiv.org/abs/0710.0643].

[Bro73] K. S. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419-458, [jstor:1996573].

[Bry93] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and geometric Quantization*, Birkhäuser, 1993, [doi:10.1007/978-0-8176-4731-5].

[Bu70] V. M. Buchstaber, *The Chern-Dold character in cobordisms. I*, Math. Sb. **12**, AMS (1970), [doi:10.1070/SM1970v012n04ABEH000939].

[Buh11] L. Buhné, *Properties of integral Morava K-theory and the asserted application to the Diaconescu-Moore-Witten anomaly*, Diploma thesis, Hamburg University, 2011, [spire:899170].

[BFM06] U. Buijs, Y. Félix, and A. Murillo, *L_∞-rational homotopy of mapping spaces*, published as: *L_∞-models of based mapping spaces*, J. Math. Soc. Japan **63** (2011), 503-524, [doi:10.2969/jmsj/06320503], [arXiv:1209.4756].

[BFGM03] M. Bullejos, E. Faro, and M. A. García-Muñoz, *Homotopy colimits and cohomology with local coefficients*, Cah. Top. Géom. Diff. Cat. **44** (2003), 63-80, [numdam:CTGDC_2003__44_1_63_0].

[Bun12] U. Bunke, *Differential cohomology*, [arXiv:1208.3961].

[BG13] U. Bunke and D. Gepner, *Differential function spectra, the differential Becker-Gottlieb transfer, and applications to differential algebraic K-theory*, [arXiv:1306.0247]

[BN14] U. Bunke and T. Nikolaus, *Twisted differential cohomology*, Algebr. Geom. Topol. **19** (2019), 1631-1710, [doi:10.2140/agt.2019.19.1631], [arXiv:1406.3231].

[BNV13] U. Bunke, T. Nikolaus and M. Völkl, *Differential cohomology theories as sheaves of spectra*, J. Homotopy Relat. Struct. **11** (2016), 1-66, [doi:10.1007/s40062-014-0092-5], [arXiv:1311.3188].

[BS12] U. Bunke and T. Schick, *Differential K-theory: A survey*, in: C. Br et al. (eds.), *Global Differential Geometry*, Springer 2012 [doi:10.1007/978-3-642-22842-1]

[BSS19] S. Burton, H. Sati, and U. Schreiber, *Lift of fractional D-brane charge to equivariant Cohomotopy theory*, Journal of Geometry & Physics, 2020 [arXiv:1812.09679].

[CV98] M. Čadek and J. Vanžura, *Almost quaternionic structures on eight-manifolds*, Osaka J. Math. **35** (1998), 165-190, [euclid:1200787905].

[CMW09] A. L. Carey, J. Mickelsson, and B.-L. Wang, *Differential twisted K-theory and applications*, J. Geom. Phys. **59** (2009), 632-653, [doi:10.1016/j.geomphys.2009.02.002], [arXiv:0708.3114].

[CMW97] A. Carey, M. Murray, and B. L. Wang, *Higher bundle gerbes and cohomology classes in gauge theories*, J. Geom. Phys. **21** (1997), 183-197, [doi:10.1016/S0393-0440(96)00014-9], [arXiv:hep-th/9511169].

[Ca50] H. Cartan, *Cohomologie réelle d'un espace fibré principal diffrentiable. I: notions d'algbre différentielle, algèbre de Weil d'un groupe de Lie*, Séminaire Henri Cartan, vol. 2 (1949-1950), Talk no. 19, May 1950, [numdam:SHC_1949-1950__2__A18_0].

[Ca51] H. Cartan, *Notions d'algèbre différentielle; applications aux groupes de Lie et aux variétés où opère un groupe de Lie*, in: Centre Belge de Recherches Mathématiques, *Colloque de Topologie (Espaces Fibrés), Bruxelles 5-8 juin 1950*, Georges Thone, Liége; Masson et Cie., Paris, 1951. [ncatlab.org/nlab/files/CartanNotionsDAlgebreDifferentielle.pdf]

[CDF91] L. Castellani, R. D'Auria, and P. Fré, *Supergravity and Superstrings – A Geometric Perspective*, World Scientific, 1991, [doi:doi:10.1142/0224].

[Cav05] G. Cavalcanti, *New aspects of the dd^c-lemma*, PhD thesis, Oxford, 2005, [arXiv:math/0501406].

[CS85] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, in: Geometry and Topology, Lecture Notes in Mathematics **1167**, 50-80, Springer, 1985, [doi:10.1007/BFb0075212].

[Ch50] S.-S. Chern, *Differential geometry of fiber bundles*, in: *Proceedings of the International Congress of Mathematicians*, Cambridge, Mass., (Aug.-Sep. 1950), vol. 2, pp. 397-411, Amer. Math. Soc., Providence, R. I., 1952, [ncatlab.org/nlab/files/Chern-DifferentialGeometryOfFiberBundles.pdf]

[Ch51] S.-S. Chern, *Topics in Differential Geometry*, Institute for Advanced Study, Princeton, 1951, [ncatlab.org/nlab/files/Chern-IASNotes1951.pdf]

[CS74] S.-S. Chern and J. Simons, *Characteristic Forms and Geometric Invariants*, Ann. Math. **99** (1974), 48-69, [jstor:1971013].

[DP13] M. Dadarlat and U. Pennig, *A Dixmier-Douady theory for strongly self-absorbing C^*- algebras*, Algebr. Geom. Topol. **15** (2015), 137-168, [doi:10.2140/agt.2015.15.137], [arXiv:1306.2583].

[D'AF82] R. D'Auria and P. Fré, *Geometric supergravity in $D = 11$ and its hidden supergroup*, Nucl. Phys. **B 201** (1982), 101–140, [doi:10.1016/0550-3213(82)90376-5].

[De71] P. Deligne, *Théorie de Hodge II*, Publ. Math. IHÉS **40** (1971), 5-57, [numdam:PMIHES_1971__40__5_0].

[DMW03] D. Diaconescu, G. Moore and E. Witten, *E_8 Gauge Theory, and a Derivation of K-Theory from M-Theory*, Adv. Theor. Math. Phys. 6:1031-1134, 2003 [arXiv:hep-th/0005090] [arXiv:hep-th/0005091]

[Di31] P.A.M. Dirac, *Quantized Singularities in the Electromagnetic Field*, Proc. Royal Soc. **A133** (1931), 60-72, [doi:10.1098/rspa.1931.0130].

[Do58] A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. Math. **68** (1958), 54-80, [jstor:1970043].

[Do65] A. Dold, *Relations between ordinary and extraordinary homology*, Matematika **9** (1965), 8-14, [mathnet:mat350]; in: J. Adams et al. (eds.), *Algebraic Topology: A Student's Guide*, LMS Lecture Note Series, pp. 166-177, Cambridge, 1972, [doi:10.1017/CBO9780511662584.015].

[DK70] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Publ. Math. IHÉS **38** (1970), 5-25, [numdam:PMIHES_1970__38__5_0].

[DH11] C. Douglas and A. Henriques, *Topological modular forms and conformal nets*, in: H. Sati, U. Schreiber (eds.), *Mathematical Foundations of Quantum Field and Perturbative String Theory*, Proc. Sympo. Pure Math. 83, Amer. Math. Soc., 2011, [doi:10.1090/pspum/083], [arXiv:1103.4187].

[DFHH14] C. L. Douglas, J. Francis, A. G. Henriques, and M. A. Hill (eds.), *Topological Modular Forms*, Mathematical Surveys and Monographs vol. 201, Amer. Math. Soc., 2014, ISBN:978-1-4704-1884-7].

[DDK80] E. Dror, W. Dwyer and D. Kan, *Equivariant maps which are self homotopy equivalences*, Proc. Amer. Math. Soc. **80** (1980), 670-672, [jstor:2043448].

[Du98] D. Dugger, *Sheaves and homotopy theory*, 1998, [ncatlab.org/nlab/files/DuggerSheavesAndHomotopyTheory.pdf]

[Du01] D. Dugger, *Universal homotopy theories*, Adv. Math. **164** (2001), 144-176, [doi:10.1006/aima.2001.2014], [arXiv:math/0007070].

[Du03] D. Dugger, *Notes on Delta-generated spaces*, 2003, [pages.uoregon.edu/ddugger/delta.html].

[Dun10] G. Dungan, *Review of model categories*, Florida State University, 2010, [ncatlab.org/nlab/files/DunganModelCategories.pdf]

[DK84] W. Dwyer and D. Kan, *An obstruction theory for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **87** (1984), 139-146, [doi:10.1016/1385-7258(84)90015-5].

[Ei40] S. Eilenberg, *Cohomology and Continuous Mappings*, Ann. Math. **41** (1940), 231-251, [jstor:1968828].

[EML53] S. Eilenberg and S. Mac Lane, *On the Groups $H(\Pi,n)$, I*, Ann. Math. **58** (1953), 55-106, [jstor:1969820].

[EML54a] S. Eilenberg and S. MacLane, *On the Groups $H(\Pi,n)$, II: Methods of Computation*, Ann. Math. **60** (1954), 49-139, [jstor:1969702].

[EML54b] S. Eilenberg and S. Mac Lane, *On the Groups $H(\Pi,n)$, III: Operations and Obstructions*, Ann. Math. **60** (1954), 513-557, [jstor:1969849].

[EKMM97] A. Elmendorf, I. Kriz, M. Mandell, and P. May, *Rings, modules and algebras in stable homotopy theory*, Mathematical Surveys and Monographs vol. 47, Amer. Math. Soc., Providence, RI, 1997, [ISBN:978-0-8218-4303-1], [www.math.uchicago.edu/~may/BOOKS/EKMM.pdf]

[EI19] M. A. Erdal and A. G. Ílhan, *A model structure via orbit spaces for equivariant homotopy*, J. Homotopy Relat. Struc. **14** (2019), 1131-1141, [doi:10.1007/s40062-019-00241-4], [arXiv:1903.03152].

[Es09] H. Esnault, *Algebraic Differential Characters of Flat Connections with Nilpotent Residues*, In: N. Baas et. al (eds.), *Algebraic Topology*, Abel Symposia vol 4, Springer, 2009, [doi:10.1007/978-3-642-01200-6_5].

[EV88] H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, in: Rapoport et al. (eds.) *Beilinson's Conjectures on Special Values of L-Functions*, Perspectives in Math. 4, Academic Press (1988) 43-91, [ISBN:978-0-12-581120-0].

[EU14] J. Espinoza and B. Uribe, *Topological properties of the unitary group*, JP J. Geom. Topol. **16** (2014), 45-55, [doi:10.18257/raccefyn.317], [arXiv:1407.1869].

[Ev06] J. Evslin, *What Does(n't) K-theory Classify?*, Second Modave Summer School in Mathematical Physics [arXiv:hep-th/0610328]

[FRS13] D. Fiorenza, C. L. Rogers and U. Schreiber, *L_∞-algebras of local observables from higher prequantum bundles* Homology, Homotopy Appl. **16** (2014), 107-142, [doi:10.4310/HHA.2014.v16.n2.a6], [arXiv:1304.6292].

[FSS12a] D. Fiorenza, H. Sati and U. Schreiber, *Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory*, Adv. Theor. Math. Phys. **18** (2014), 229 - 321, [arXiv:1201.5277].

[FSS12b] D. Fiorenza, H. Sati and U. Schreiber, *Extended higher cup-product Chern-Simons theories*, J. Geom. Phys. **74** (2013), 130-163, [doi:10.1016/j.geomphys.2013.07.011], [arXiv:1207.5449].

[FSS13a] D. Fiorenza, H. Sati, and U. Schreiber, *A higher stacky perspective on Chern-Simons theory*, in D. Calaque et al. (eds.) *Mathematical Aspects of Quantum Field Theories*, Mathematical Physics Studies, Springer, 2014, [doi:10.1007/978-3-319-09949-1], [arXiv:1301.2580].

[FSS13b] D. Fiorenza, H. Sati, and U. Schreiber, *Super Lie n-algebra extensions, higher WZW models, and super p-branes with tensor multiplet fields*, Intern. J. Geom. Meth. Mod. Phys. **12** (2015) 1550018, [arXiv:1308.5264].

[FSS15a] D. Fiorenza, H. Sati, and U. Schreiber, *The E_8 moduli 3-stack of the C-field in M-theory*, Commun. Math. Phys. **333** (2015), 117-151, [doi:10.1007/s00220-014-2228-1], [arXiv:1202.2455] [hep-th].

[FSS15b] D. Fiorenza, H. Sati, U. Schreiber, *The WZW term of the M5-brane and differential cohomotopy*, J. Math. Phys. **56** (2015), 102301, [doi:10.1063/1.4932618], [arXiv:1506.07557].

[FSS16a] D. Fiorenza, H. Sati, and U. Schreiber, *Rational sphere valued supercocycles in M-theory and type IIA string theory*, J. Geom. Phys. **114** (2017), 91-108, [doi:10.1016/j.geomphys.2016.11.024], [arXiv:1606.03206].

[FSS16b] D. Fiorenza, H. Sati, and U. Schreiber, *T-Duality from super Lie n-algebra cocycles for super p-branes*, Adv. Theor. Math. Phys. **22** (2018), 1209–1270, [doi:10.4310/ATMP.2018.v22.n5.a3], [arXiv:1611.06536].

[FSS18] D. Fiorenza, H. Sati and U. Schreiber, *Higher T-duality of super M-branes* Adv. Theor. Math. Phys. **24** (2020), 621-708, [doi:10.4310/ATMP.2020.v24.n3.a3], [arXiv:1803.05634].

[FSS19a] D. Fiorenza, H. Sati, and U. Schreiber, *The rational higher structure of M-theory*, Proc. LMS-EPSRC Durham Symposium *Higher Structures in M-Theory*, Aug. 2018, Fortsch. Phys., 2019, [doi:10.1002/prop.201910017] [arXiv:1903.02834].

[FSS19b] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Comm. Math. Phys. **377** (2020), 1961-2025, [doi:10.1007/s00220-020-03707-2], [arXiv:1904.10207].

[FSS19c] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M5 WZ term level quantization*, [arXiv:1906.07417].

[FSS20] D. Fiorenza, H. Sati, and U. Schreiber, *Twistorial Cohomotopy Implies Green-Schwarz anomaly cancellation*, [arXiv:2008.08544].

[FSSt10] D. Fiorenza, U. Schreiber and J. Stasheff, *Čech cocycles for differential characteristic classes – An ∞-Lie theoretic construction*, Adv. Theor. Math. Phys. **16** (2012), 149-250, [doi:10.4310/ATMP.2012.v16.n1.a5], [arXiv:1011.4735].

[FH17] Y. Félix and S. Halperin, *Rational homotopy theory via Sullivan models: a survey*, Notices of the International Congress of Chinese Mathematicians vol. 5 (2017) no. 2, [doi:10.4310/ICCM.2017.v5.n2.a3], [arXiv:1708.05245].

[FHT00] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, 205, Springer-Verlag, 2000, [doi:10.1007/978-1-4613-0105-9].

[FHT15] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory II*, World Scientific, 2015, [doi:10.1142/9473].

[FOT08] Y. Félix, J. Oprea and D. Tanré, *Algebraic models in geometry*, Oxford University Press, 2008, [ISBN:9780199206520].

[Fra97] T. Frankel, *The Geometry of Physics - An introduction*, Cambridge University Press, 2012, [doi:10.1017/CBO9781139061377].

[Fr00] D. Freed, *Dirac charge quantization and generalized differential cohomology*, Surveys in Differential Geometry, Int. Press, Somerville, MA, 2000, pp. 129-194, [doi:10.4310/SDG.2002.v7.n1.a6], [arXiv:hep-th/0011220].

[Fr02] D. Freed, *Classical Chern-Simons theory, part II*, Houston J. Math. **28** (2002), 293-310, [web.ma.utexas.edu/users/dafr/cs2.pdf]

[FH00] D. Freed and M. Hopkins, *On Ramond-Ramond fields and K-theory*, JHEP 0005 (2000) 044 [arXiv:hep-th/0002027]

[FrHT08] D. S. Freed, M. J. Hopkins, and C. Teleman, *Twisted equivariant K-theory with complex coefficients*, J. Topology **1** (2008), 16-44, [doi:10.1112/jtopol/jtm001], [arXiv:math/0206257] [math.AT].

[GJF18] D. Gaiotto and T. Johnson-Freyd, *Holomorphic SCFTs with small index*, [arXiv:1811.00589].

[Ga97] P. Gajer, *Geometry of Deligne cohomology*, Invent. Math. **127** (1997), 155-207, [doi:10.1007/s002220050118], [arXiv:alg-geom/9601025].

[GSW11] K. Gawedzki, R. Suszek and K. Waldorf, *Bundle Gerbes for Orientifold Sigma Models*, Adv. Theor. Math. Phys. **15** (2011), 621-688, [doi:10.4310/ATMP.2011.v15.n3.a1], [arXiv:0809.5125].

[GM96] S. Gelfand and Y. Manin, *Methods of homological algebra*, Springer, 2003, [doi:10.1007/978-3-662-12492-5].

[Gi71] J. Giraud, *Cohomologie non abélienne*, Springer, 1971, [doi:10.1007/978-3-662-62103-5].

[Gl82] P. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure Appl. Algebra **25** (1982), 33- 105, [doi:10.1016/0022-4049(82)90094-9].

[GJ99] P. Goerss and R. F. Jardine, *Simplicial homotopy theory*, Birkhäuser, Boston, 2009, [doi:10.1007/978-3-0346-0189-4].

[GoS06] P. Goerss and K. Schemmerhorn, *Model categories and simplicial methods*, in: *Interactions between Homotopy Theory and Algebra*, Contemporary Mathematics 436, Amer. Math. Soc., 2007, [doi:10.1090/conm/436], [arXiv:math/0609537].

[Go08] J. M. Gomez Guerra, *Models of twisted K-theory*, PhD thesis, University of Michigan, 2008, [ISBN:978-0549-81502-0].

[GT00] K. Gomi and Y. Terashima, *A Fiber Integration Formula for the Smooth Deligne Cohomology*, Int. Math. Res. Notices (2000), no. 13, [doi:10.1155/S1073792800000386].

[GT10] K. Gomi and Y. Terashima, *Chern-Weil construction for twisted K-theory*, Comm. Math. Phys. **299** (2010), 225–254, [doi:10.1007/s00220-010-1080-1].

[GS17a] D. Grady and H. Sati, *Massey products in differential cohomology via stacks* J. Homotopy Relat. Struct. **13** (2017), 169-223, [doi:10.1007/s40062-017-0178-y], [arXiv:1510.06366] [math.AT].

[GS17b] D. Grady and H. Sati, *Spectral sequences in smooth generalized cohomology*, Algebr. Geom. Top. **17** (2017), 2357-2412, [doi:10.2140/agt.2017.17.2357], [arXiv:1605.03444] [math.AT].

[GS18a] D. Grady and H. Sati, *Primary operations in differential cohomology*, Adv. Math. **335** (2018), 519-562, [doi:10.1016/j.aim.2018.07.019], [arXiv:1604.05988] [math.AT].

[GS18b] D. Grady and H. Sati, *Differential KO-theory: constructions, computations, and applications*, [arXiv:1809.07059].

[GS18c] D. Grady and H. Sati, *Twisted smooth Deligne cohomology*, Ann. Glob. Anal. Geom. **53** (2018), 445-466, [doi:10.1007/s10455-017-9583-z], [arXiv:1706.02742] [math.DG].

[GS19a] D. Grady and H. Sati, *Twisted differential generalized cohomology theories and their Atiyah-Hirzebruch spectral sequence*, Algebr. Geom. Topol. **19** (2019), 2899-2960, [doi:10.2140/agt.2019.19.2899], [arXiv:1711.06650] [math.AT].

[GS19b] D. Grady and H. Sati, *Higher-twisted periodic smooth Deligne cohomology*, Homology Homotopy Appl. **21** (2019), 129-159, [doi:10.4310/HHA.2019.v21.n1.a7], [arXiv:1712.05971] [math.DG].

[GS19c] D. Grady and H. Sati, *Ramond-Ramond fields and twisted differential K-theory*, [arXiv:1903.08843].

[GS19d] D. Grady and H. Sati, *Twisted differential KO-theory*, [arXiv:1905.09085] [math.AT].

[GS20] D. Grady and H. Sati, *Differential cohomotopy versus differential cohomology for M-theory and differential lifts of Postnikov towers*, [arXiv:2001.07640] [hep-th].

[GHV73] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature, and Cohomology Vol. II: Lie groups, principal bundles and characteristic classes*, Academic Press, New York-London, 1973, [ISBN:9780123027023].

[GM13] P. Griffiths and J. Morgan, *Rational Homotopy Theory and Differential Forms*, Progress in Mathematics Volume 16, Birkhäuser (2013), [doi:10.1007/978-1-4614-8468-4].

[GPPV18] S. Gukov, D. Pei, P. Putrov, and C. Vafa, *4-manifolds and topological modular forms*, [arXiv:1811.07884] [hep-th].

[HMSV19] P. Hekmati, M. Murray, R. Szabo, and R. Vozzo, *Real bundle gerbes, orientifolds and twisted KR-homology*, Adv. Theor. Math. Phys. **23** (2019), 2093-2159, [doi:10.4310/ATMP.2019.v23.n8.a5], [arXiv:1608.06466].

[He08] A. Henriques, *Integrating L_∞-algebras*, Compos. Math. **144** (2008), 1017-1045, [doi:10.1112/S0010437X07003405], [arXiv:math/0603563].

[He07] K. Hess, *Rational homotopy theory: a brief introduction*, in: L. Avramov et al. (eds.), *Interactions between Homotopy Theory and Algebra*, Contemporary Mathematics 436, Amer. Math. Soc., 2007, [doi:10.1090/conm/436].

[Hil55] P. Hilton, *On the homotopy groups of unions of spheres*, J. London Math. Soc. **30** (1955), 154-172, [arXiv:10.1112/jlms/s1-30.2.154].

[Hil71] P. Hilton, *General cohomology theory and K-theory*, London Mathematical Society Lecture Note Series 1, Cambridge University Press, 1971, [doi:10.1017/CBO9780511662577].

[Hil82] P. Hilton, *Nilpotency in group theory and topology*, Publ. Secció Mat. **26** (1982), 47-78, [jstor:43741908].

[Hir02] P. Hirschhorn, *Model Categories and Their Localizations*, AMS Math. Survey and Monographs Vol 99, American Mathematical Society, 2002, [ISBN:978-0-8218-4917-0].

[Hir15] P. Hirschhorn, *The Quillen model category of topological spaces*, Expositiones Math. **37** (2019), 2-24, [doi:10.1016/j.exmath.2017.10.004], [arXiv:1508.01942].

[Hi56] F. Hirzebruch, *Neue topologische Methoden in der Algebraischen Geometrie*, Ergebnisse der Mathematik und Ihrer Grenzgebiete 1 Folge, Springer, 1956, [doi:10.1007/978-3-662-41083-7].

[Ho94] M. Hopkins, *Topological modular forms, the Witten Genus, and the theorem of the cube*, Proceedings of the ICM, Zürich, 1994, [doi:10.1007/978-3-0348-9078-6_49].

[Ho02] M. Hopkins, *Algebraic topology and modular forms*, Proceedings of the ICM, Beijing 2002, vol. 1, 2830-309, [arxiv:math/0212397].

[HS05] M. Hopkins and I. Singer, *Quadratic Functions in Geometry, Topology, and M-Theory*, J. Differential Geom. **70** (2005), 329-452, [arXiv:math.AT/0211216].

[Ho99] M. Hovey, *Model Categories*, Amer. Math. Soc., Provdence, RI, 1999, [ISBN:978-0-8218-4361-1].

[HSS18] J. Huerta, H. Sati, and U. Schreiber, *Real ADE-equivariant (co)homotopy of super M-branes*, Commun. Math. Phys. **371** (2019) 425, [doi:10.1007/s00220-019-03442-3], [arXiv:1805.05987].

[Ig08] K. Igusa, *Pontrjagin classes and higher torsion of sphere bundles*, in: R. Penner et al. (eds.), *Groups of Diffeomorphisms*, pp. 21-29, Adv. Stud. Pure Math. 52, 2008, [euclid:aspm/1543447476].

[IK99] M. Imaoka and K. Kuwana, *Stably extendible vector bundles over the quaternionic projective spaces*, Hiroshima Math. J. **29** (1999), 273-279, [euclid:hmj/1206125008].

[IS07] J. N. Iyer and C. Simpson, *Regulators of canonical extensions are torsion: the smooth divisor case*, [arXiv:0707.0372].

[Ja87] J. F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algeb. **47** (1987), 35-87, [core.ac.uk/download/pdf/82485559.pdf]

[Ja03] J. F. Jardine, *Presheaves of chain complexes*, K-theory **30** (2003), 365-420, [ncatlab.org/nlab/files/JardinePresheavesOfChainComplexes.pdf]

[Ja09] J. F. Jardine, *Cocycle categories*, in: N. Baas et, al. (eds.) *Algebraic Topology*, Abel Symposia, vol 4. Springer 2009, [doi:10.1007/978-3-642-01200-6_8], [arXiv:math/0605198].

[Ja15] J. F. Jardine, *Local homotopy theory*, Springer Monographs in Mathematics, 2015, [doi:10.1007/978-1-4939-2300-7].

[JL06] J. F. Jardine and Z. Luo, *Higher principal bundles*, Math. Proc. Camb. Phil. Soc. **140** (2006), 221-243, [doi:10.1017/S0305004105008911].

[JW75] D. C. Johnson and W. S. Wilson, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975) 55-75, [http://www.math.jhu.edu/~wsw/papers2/math/08-BP-Kn-J2W-1975.pdf]

[Joh02] P. Johnstone *Sketches of an Elephant – A Topos Theory Compendium*, Oxford University Press, 2002, vol. 1 [ISBN:9780198534259], vol. 2 [ISBN:9780198515982].

[Jo08] A. Joyal, *The theory of quasi-categories and its applications*, [mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf]

[Ka58] D. Kan, *Functors involving c.s.s complexes*, Trans. Amer. Math. Soc. **87** (1958), 330-346, [jstor:1993103].

[Ka68] M. Karoubi, *Algèbres de Clifford et K-théorie*, Ann. Sci. Ecole Norm. Sup. **1** (1968), 161-270, [numdam:ASENS_1968_4_1_2_161_0].

[Ka12] M. Karoubi, *Twisted bundles and twisted K-theory*, Topics in noncommutative geometry, 223-257, Clay Math. Proc. vol. **16**, Amer. Math. Soc., Providence, RI, 2012, [arXiv:1012.2512].

[KMT12] R. Kirby, P. Melvin, and P. Teichner, *Cohomotopy sets of 4-manifolds*, Geom. & Top. Monographs **18** (2012), 161-190, [arXiv:1203.1608].

[KN63] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. 1*, Wiley, 1963, [ISBN:9780471157335].

[Ko96] S. Kochman, *Bordism, Stable Homotopy and Adams Spectral Sequences*, Fields Institute Monographs, Amer. Math. Soc., 1996, [cds:2264210].

[Ko93] A. Kosinski, *Differential manifolds*, Academic Press, 1993, [ISBN:978-0-12-421850-5].

[KS03] I. Kriz and H. Sati, *M-theory, type IIA superstrings, and elliptic cohomology*, Adv. Theor. Math. Phys. **8** (2004) 345-394, [doi:10.4310/ATMP.2004.v8.n2.a3], [arXiv:hep-th/0404013].

[KS05] I. Kriz and H. Sati, *Type II string theory and modularity*, J. High Energy Phys. **08** (2005) 038, [doi:10.1088/1126-6708/2005/08/038], [arXiv:hep-th/0501060].

[LM95] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Commun. Algebra **23** (1995), 2147-2161, [doi:10.1080/00927879508825335], [arXiv:hep-th/9406095s].

[LS93] T. Lada and J. Stasheff, *Introduction to sh Lie algebras for physicists*, Int. J. Theo. Phys. **32** (1993), 1087-1103, [doi:10.1007/BF00671791], [arXiv:hep-th/9209099].

[Lee12] J. Lee, *Introduction to Smooth Manifolds*, Springer 2012 [doi:10.1007/978-1-4419-9982-5]

[LD91] B. Li and H. Duan, *Spin characteristic classes and reduced KSpin group of a low-dimensional complex*, Proc. Amer. Math. Soc. **113** (1991), 479-491, [doi:10.1090/S0002-9939-1991-1079895-1].

[Li13] J. Lind, *Bundles of spectra and algebraic K-theory*, Pacific J. Math. **285** (2016), 427-452, [doi:10.2140/pjm.2016.285.427], [arXiv:1304.5676].

[LSW16] J. Lind, H. Sati, and C. Westerland, *Twisted iterated algebraic K-theory and topological T-duality for sphere bundles*, Ann. K-Th. **5** (2020), 1-42, [doi:10.2140/akt.2020.5.1], [arXiv:1601.06285].

[Lu09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies 170, Princeton University Press, 2009, [pup:8957].

[Lu10] J. Lurie, *Chromatic Homotopy Theory*, Harvard, 2010, [people.math.harvard.edu/~lurie/252x.html].

[Lu14] J. Lurie, *Algebraic K-Theory of Ring Spectra*, Lecture 19 of: *Algebraic K-Theory and Manifold Topology*, 2014 [people.math.harvard.edu/~lurie/281notes/Lecture19-Rings.pdf]

[Ma06] H. Maakestad, *Notes on the Chern-character*, J Gen. Lie Theory Appl. **11** (2017), 243, [doi:10.4172/1736-4337.1000253], [arXiv:math/0612060] [math.AG].

[MMS20] L. Macdonald, V. Mathai, and H. Saratchandran, *On the Chern character in Higher Twisted K-theory and spherical T-duality*, [arXiv:2007.02507] [math.DG].

[MR87] M. Mahowald and D. Ravenel, *Towards a Global Understanding of the Homotopy Groups of Spheres*, pp. 57-74 in Part II of: *The Lefschetz Centennial Conference – Proceedings on Algebraic Topology*, Contemporary Mathematics 58, Amer. Math. Soc., 1987, [ISBN:978-0-8218-5063-3].

[MaS03] V. Mathai and D. Stevenson, *Chern character in twisted K-theory: equivariant and holomorphic cases*, Commun. Math. Phys. **236** (2003), 161-186, [doi:10.1007/s00220-003-0807-7], [arXiv:hep-th/0201010].

[MaS06] V. Mathai and D. Stevenson, *On a generalized Connes-Hochschild-Kostant-Rosenberg theorem*, Adv. Math. **200** (2006), 303–335, [doi:10.1016/j.aim.2004.11.006], arXiv:math/0404329].

[MW11] V. Mathai and S. Wu, *Analytic torsion for twisted de Rham complexes*, J. Differential Geom. **88** (2011), 297-332, [arXiv:0810.4204] [math.DG].

[May67] P. May, *Simplicial objects in algebraic topology*, The University of Chicago Press, 1967, [ISBN:9780226511818].

[May72] P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer, 1972, [doi:10.1007/BFb0067491].

[May77] P. May, *Infinite loop space theory*, Bull. Amer. Math. Soc. **83** (1977), 456-494, [euclid:bams/1183538891]

[MQRT77] J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, with contributions by F. Quinn, N. Ray, and J. Tornehave, Lecture Notes in Mathematics **577**, Springer-Verlag, Berlin, 1977, [ISBN:978-3-540-37409-1].

[May99] P. May, *A concise course in algebraic topology*, University of Chicago Press, 1999, [ISBN:978-0226511832].

[MP12] P. May and K. Ponto, *More Concise Algebraic Topology*, University of Chicago Press, 2012, [www.math.uchicago.edu/∼may/TEAK/KateBookFinal.pdf]

[MaS04] P. May and J. Sigurdsson, *Parametrized Homotopy Theory*, Mathematical Surveys and Monographs, vol. 132, Amer. Math. Soc., 2006, [ISBN:978-0-8218-3922-5], [arXiv:math/0411656].

[MM74] B. Mazur and W. Messing, *Universal extensions and one-dimensional crystalline cohomology*, Lecture Notes in Mathematics **370**, Springer-Verlag, Berlin, 1974, [doi:10.1007/BFb0061628].

[Me13] L. Menichi, *Rational homotopy – Sullivan models*, In: Free loop spaces in geometry and topology, 111-136, IRMA Lect. Math. Theor. Phys., 24, Eur. Math. Soc., Zürich, 2015, [arXiv:1308.6685].

[Mi56] J. Milnor, *Construction of Universal Bundles, II*, Ann. Math. **63** (1956), 430-436, [jstor:1970012].

[MS74] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton Univ. Press, 1974, [ISBN:9780691081229].

[MT08] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Dover Publications, NY, 2008, [ISBN10:0486466647].

[Mo79] M. Mostow, *The differentiable space structures of Milnor classifying spaces, simplicial complexes, and geometric realizations*, J. Differential Geom. **14** (1979), 255-293, [euclid:jdg/1214434974].

[Mu96] M. K. Murray, *Bundle gerbes*, J. London Math. Soc. **54** (1996), 403-416, [doi:10.1112/jlms/54.2.403], [arXiv:dg-ga/9407015].

[NR61] M. Narasimhan and S. Ramanan, *Existence of Universal Connections*, Amer. J. Math. **83** (1961), 563-572, [jstor:2372896].

[NR63] M. Narasimhan and S. Ramanan, *Existence of Universal Connections II*, Amer. J. Math. **85** (1963), 223-231, [jstor:2373211].

[NSW11] T. Nikolaus, C. Sachse, and C. Wockel, *A Smooth Model for the String Group*, Int. Math. Res. Not. **16** (2013), 3678-3721, [doi:10.1093/imrn/rns154], [arXiv:1104.4288].

[NSS12a] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞-bundles – General theory*, J. Homotopy Rel. Struc. **10** 4 (2015), 749–801, [doi:10.1007/s40062-014-0083-6], [arXiv:1207.0248].

[NSS12b] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞-bundles – Presentations*, J. Homotopy Rel. Struc. **10**, 3 (2015), 565-622, [doi:10.1007/s40062-014-0077-4], [arXiv:1207.0249].

[NW11] T. Nikolaus and K. Waldorf, *Four Equivalent Versions of Non-Abelian Gerbes*, Pacific J. Math. **264** (2013), 355-420, [doi:10.2140/pjm.2013.264.355], [arXiv:1103.4815].

[No03] S. Nowak, *Stable cohomotopy groups of compact spaces*, Fund. Math. **180** (2003), 99-137, [doi:10.4064/fm180-2-1].

[Pa18] B. Park, *Geometric models of twisted differential K-theory I*, J. Homotopy Relat. Struct. **13** (2018), 143-167, [doi:10.1007/s40062-017-0177-z], [arXiv:1602.02292].

[Pe56] F. P. Peterson, *Some Results on Cohomotopy Groups*, Amer. J. Math. **78** (1956), 243-258, [jstor:2372514].

[Pr10] M. Prezma, *Homotopy normal maps*, Algebr. Geom. Topol. **12** (2012), 1211-1238, [doi:10.2140/agt.2012.12.1211], [arXiv:1011.4708].

[Qu67] D. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics **43**, Springer, 1967, [doi:10.1007/BFb0097438].

[Qu69] D. Quillen, *Rational homotopy theory*, Ann. Math. **90** (1969), 205-295, [jstor:1970725].

[Ra86] D. Ravenel, *Complex cobordism and stable homotopy groups of spheres* Academic Press Orland, 1986; reprinted as: AMS Chelsea Publishing, Volume 347, 2004, [ISBN:978-0-8218-2967-7], [web.math.rochester.edu/people/faculty/doug/mu.html].

[Re10] C. Rezk, *Toposes and homotopy toposes*, 2010, [https://faculty.math.illinois.edu/~rezk/homotopy-topos-sketch.pdf]

[Rzn95] A. Reznikov, *All regulators of flat bundles are torsion*, Ann. Math. **141** (1995), 373-386, [jstor:2118525], [arXiv:dg-ga/9407006].

[Rzn96] A. Reznikov, *Rationality of secondary classes*, J. Differential Geom. **43** (1996), 674-692, [doi:10.4310/jdg/1214458328], [arXiv:dg-ga/9407007].

[Ri20] B. Richter, *From categories to homotopy theory*, Cambridge University Press, 2020, [doi:10.1017/9781108855891].

[Rie09] E. Riehl, *A concise definition of model category*, 2009, [www.math.jhu.edu/∼eriehl/modelcat.pdf]

[Rie14] E. Riehl, *Categorical Homotopy Theory* Cambridge University Press, 2014, [doi:10.1017/CBO9781107261457].

[RS12] D. Roberts and D. Stevenson, *Simplicial principal bundles in parametrized spaces*, New York J. Math. **22** (2016), 405-440, [arXiv:1203.2460].

[Ro14] J. Rognes, *Chromatic redshift*, lecture notes, MSRI, Berkeley, 2014, [arXiv:1403.4838].

[RW86] R. Rohm and E. Witten, *The antisymmetric tensor field in superstring theory*, Ann. Phys. **170** (1986), 454-489, [doi:10.1016/0003-4916(86)90099-0].

[Ro89] J. Rosenberg, *Continuous-trace algebras from the bundle theoretic point of view*, J. Austral. Math. Soc. Ser. A **47** (1989), 368-381, [doi:10.1017/S1446788700033097].

[Ru98] Y. Rudyak, *On Thom Spectra, Orientability, and Cobordism*, Springer, 1998, [doi:10.1007/978-3-540-77751-9].

[Sa08] H. Sati, *An approach to anomalies in M-theory via KSpin*, J. Geom. Phys. **58** (2008), 387-401, [doi:10.1016/j.geomphys.2007.11.010], [arXiv:0705.3484].

[Sa09] H. Sati, *A higher twist in string theory*, J. Geom. Phys. **59** (2009), 369-373, [doi:10.1016/j.geomphys.2008.11.009], [arXiv:hep-th/0701232].

[Sa10] H. Sati, *Geometric and topological structures related to M-branes*, in: R. Doran, G. Friedman and J. Rosenberg (eds.), *Superstrings, Geometry, Topology, and C^*-algebras*, Proc. Symp. Pure Math. **81**, AMS, Providence, 2010, pp. 181-236, [doi:10.1090/pspum/081], [arXiv:1001.5020].

[Sa13] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018), 062304, [doi:10.1063/1.5007185], [arXiv:1310.1060].

[Sa14] H. Sati, *M-Theory with Framed Corners and Tertiary Index Invariants*, SIGMA **10** (2014), 024, 28 pages, [doi:10.3842/SIGMA.2014.024], [arXiv:1203.4179].

[Sa19] H. Sati, *Six-dimensional gauge theories and (twisted) generalized cohomology*, [arXiv:1908.08517].

[SS19a] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [doi:10.1016/j.geomphys.2020.103775], [arXiv:1909.12277].

[SS19b] H. Sati and U. Schreiber, *Differential Cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams*, [arXiv:1912.10425].

[SS20a] H. Sati and U. Schreiber, *Twisted Cohomotopy implies M5-brane anomaly cancellation*, [arXiv:2002.07737].

[SS20b] H. Sati and U. Schreiber, *Proper Orbifold Cohomology*, [arXiv:2008.01101].

[SSS09a] H. Sati, U. Schreiber and J. Stasheff, *L_∞-algebra connections and applications to String- and Chern-Simons n-transport* in *Quantum Field Theory*, Birkhäuser (2009), 303-424, [doi:10.1007/978-3-7643-8736-5_17], [arXiv:0801.3480].

[SSS12] H. Sati, U. Schreiber, and J. Stasheff, *Twisted differential string and fivebrane structures*, Commun. Math. Phys. **315** (2012), 169-213, [doi:10.1007/s00220-012-1510-3], [arXiv:0910.4001].

[SW15] H. Sati and C. Westerland, *Twisted Morava K-theory and E-theory*, J. Topol. **8** (2015), 887–916, [doi:10.1112/jtopol/jtv020], [arXiv:1109.3867] [math.AT].

[Sc80] R. Schlafly, *Universal connections*, Invent. Math. **59** (1980), 59-65, [doi:10.1007/BF01390314].

[Schl04] C. Schlichtkrull, *Units of ring spectra and their traces in algebraic K-theory*, Geom. Topol. **8** (2004), 645-673, [euclid:gt/1513883412], [arXiv:math/0405079].

[Sc13] U. Schreiber, *Differential cohomology in a cohesive infinity-topos*, [arXiv:1310.7930] [math-ph].

[SSW07] U. Schreiber, C. Schweigert and K. Waldorf, *Unoriented WZW models and Holonomy of Bundle Gerbes*, Commun. Math. Phys. **274** (2007), 31-64, [doi:10.1007/s00220-007-0271-x], [arXiv:hep-th/0512283].

[SSh01] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), 103-153, [doi:10.1016/S0040-9383(02)00006-X], [arXiv:math/0108143].

[SSh03a] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287-334, [euclid:agt/1513882376], [arXiv:math/0209342].

[SSh03b] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), 103-153, [doi:10.1016/S0040-9383(02)00006-X].

[SW07] C. Schweigert and K. Waldorf, *Gerbes and Lie Groups*, In: KH. Neeb, A. Pianzola (eds.) *Developments and Trends in Infinite-Dimensional Lie Theory*, Progress in Mathematics, vol 288. Birkhäuser, Boston, 2011, [doi:10.1007/978-0-8176-4741-4_10], [arXiv:0710.5467].

[Se68] G. Segal, *Classifying spaces and spectral sequences*, Publ. Math. IHÉS **34** (1968), 105-112, [numdam:PMIHES_1968__34__105_0].

[Sh15] A. Sharma, *On the homotopy theory of G-spaces*, Intern. J. Math. Stat. Inv. **7** (2019), 22-55, [arXiv:1512.03698].

[SYH10] K. Shimakawa, K. Yoshida, and T. Haraguchi, *Homology and cohomology via enriched bifunctors*, Kyushu J. Math. **72** (2018), 239-252, [arXiv:1010.3336].

[SiSu08] J. Simons and D. Sullivan, *Axiomatic characterization of ordinary differential cohomology*, J. Topology **1** (2008), 45-56, [doi:10.1112/jtopol/jtm006], [arXiv:math/0701077].

[Si96] C. Simpson, *The Hodge filtration on nonabelian cohomology*, Algebraic geometry – Santa Cruz 1995, Proc. Sympos. Pure Math. **62**, Part 2, 217-281, Amer. Math. Soc., Providence, RI, 1997, [doi:10.1090/pspum/062.2], [arXiv:alg-geom/9604005].

[Si97] C. Simpson, *Secondary Kodaira-Spencer classes and nonabelian Dolbeault cohomology*, [arXiv:alg-geom/9712020].

[Si99] C. Simpson, *Algebraic aspects of higher nonabelian Hodge theory*, in: F. Bogomolov, L. Katzarkov (eds.), *Motives, polylogarithms and Hodge theory, Part II* (Irvine, CA, 1998), Int. Press, 2002, 2016, 417-604, [ISBN:9781571462909], [arXiv:math/9902067].

[Sp49] E. Spanier, *Borsuk's Cohomotopy Groups*, Ann. Math. **50** (1949), 203-245, [jstor:1969362].

[St70] J. Stasheff, *H-spaces and classifying spaces: foundations and recent developments*, Algebraic topology, 247-272, (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, RI, 1971.

[St43] N. Steenrod, *Homology With Local Coefficients*, Ann. Math. **44** (1943), 610-627, [jstor:1969099].

[St47] N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. Math. **48** (1947), 290-320, [jstor:1969172].

[St51] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951, [jstor:j.ctt1bpm9t5].

[St67] N. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133-152, [euclid:mmj/1028999711].

[St68] N. Steenrod, *Milgram's classifying space of a topological group*, Topology **7** (1968), 349-368, [doi:10.1016/0040-9383(68)90012-8].

[St72] N. Steenrod, *Cohomology operations, and obstructions to extending continuous functions*, Adv. Math. **8** (1972), 371-416, [doi:10.1016/0001-8708(72)90004-7].

[SE62] N. Steenrod and D. Epstein, *Cohomology operations*, Annals of Mathematics Studies, Princeton University Press, 1962, [jstor:j.ctt1b7x52h].

[St01] D. Stevenson, *Bundle 2-gerbes*, Proc. London Math. Soc. **88** (2004), 405-435, [doi:10.1112/S0024611503014357], [arXiv:math/0106018].

[St12] D. Stevenson, *Classifying theory for simplicial parametrized groups*, [arXiv:1203.2461].

[ST11] S. Stolz and P. Teichner, *Supersymmetric field theories and generalized cohomology* in: H. Sati, U. Schreiber (eds.), *Mathematical Foundations of Quantum Field and Perturbative String Theory*, Proceedings of Symposia in Pure Mathematics, AMS, 2011, [doi:10.1090/pspum/083], [arXiv:1108.0189].

[Str81] C. T. Stretch, *Stable cohomotopy and cobordism of abelian groups*, Math. Proc. Camb. Phil. Soc. **90** (1981), 273-278, [doi:10.1017/S0305004100058734].

[St09] N. Strickland, *The category of CGWH spaces*, 2009, [neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf]

[Su77] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. IHÉS **47** (1977), 269-331, [numdam:PMIHES_1977__47__269_0].

[Ta09] L. Taylor, *The principal fibration sequence and the second cohomotopy set*, Geom. & Topol. Monogr. **18** (2012), 235-251, [doi:10.2140/gtm.2012.18.235], [arXiv:0910.1781].

[Te04] C. Teleman, *K-theory and the moduli space of bundles on a surface and deformations of the Verlinde algebra*, Topology, geometry and quantum field theory, 358-378, Cambridge Univ. Press, Cambridge, 2004, [arXiv:math/0306347] [math.AG].

[Th62] E. Thomas, *On the cohomology groups of the classifying space for the stable spinor groups*, Bol. Soc. Mat. Mexicana (2) **7** (1962), 57-69.

[To02] B. Toën, *Stacks and Non-abelian cohomology*, lecture at *Introductory Workshop on Algebraic Stacks, Intersection Theory, and Non-Abelian Hodge Theory*, MSRI, 2002, [perso.math.univ-toulouse.fr/btoen/files/2015/02/msri2002.pdf]

[TV05] B. Toën and G. Vezzosi, *Homotopical Algebraic Geometry I: Topos theory*, Adv. Math. **193** (2005), 257-372, [doi:10.1016/j.aim.2004.05.004], [arXiv:math/0207028].

[TX06] J.-L. Tu and P. Xu, *Chern character for twisted K-theory of orbifolds*, Adv. Math. **207** (2006), 455–483, [doi:10.1016/j.aim.2005.12.001], [arXiv:math/0505267].

[vN82] P. van Nieuwenhuizen, *Free Graded Differential Superalgebras*, Istanbul 1982, Proceedings, Group Theoretical Methods In Physics, 228-247, [spire:182644].

[We49] A. Weil, *Géométrie différentielle des espaces fibres*, originally unpublished; published as paper [1949e] in: Oeuvres Scientifiques / Collected Papers, vol. 1 (1926-1951), 422-436, Springer, Berlin, 2009, [ISBN:978-3-662-45256-1].

[We10] M. Wendt, *Classifying spaces and fibrations of simplicial sheaves*, J. Homotopy Rel. Struc **6** (2011), 1-38, [arXiv:1009.2930].

[Wh62] G. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227-283, [jstor:1993676].

[Wi87] E. Witten, *Elliptic Genera And Quantum Field Theory*, Commun. Math. Phys. **109** (1987), 525-536, [euclid:cmp/1104117076].

[Wu89] U. Würgler, *Morava K-theories: a survey*, Algebraic topology Poznań 1989, Lecture Notes in Math. **1474**, pp. 111-138, Springer, Berlin, 1991, [doi:10.1007/BFb0084741].

Domenico Fiorenza, *Dipartimento di Matematica, La Sapienza Universita di Roma, Piazzale Aldo Moro 2, 00185 Rome, Italy.*
fiorenza@mat.uniroma1.it

Hisham Sati, *Mathematics, Division of Science, New York University Abu Dhabi, UAE.*
hsati@nyu.edu

Urs Schreiber, *Mathematics, Division of Science, New York University Abu Dhabi, UAE; on leave from Czech Academy of Science, Prague.*
urs.schreiber@googlemail.com