

# The character map in (twisted differential) non-abelian cohomology

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## Abstract

The Chern character on K-theory has a natural extension to arbitrary generalized cohomology theories known as the Chern-Dold character. Here we further extend this to (twisted, differential) non-abelian cohomology theories, where its target is a non-abelian de Rham cohomology of twisted  $L_\infty$ -algebra valued differential forms. The construction amounts to leveraging the fundamental theorem of dg-algebraic rational homotopy theory to a twisted non-abelian generalization of the de Rham theorem. We show that the non-abelian character reproduces, besides the Chern-Dold character, also the Chern-Weil homomorphism as well as its secondary Cheeger-Simons homomorphism on (differential) non-abelian cohomology in degree 1, represented by principal bundles (with connection); and thus generalizes all these to higher (twisted, differential) non-abelian cohomology, represented by higher bundles/higher gerbes (with higher connections). As a fundamental example, we discuss the twisted non-abelian character map on twistorial Cohomotopy theory over 8-manifolds, which can be viewed as a twisted non-abelian enhancement of topological modular forms (tmf) in degree 4. This turns out to exhibit a list of subtle topological relations that in high energy physics are thought to govern the charge quantization of fluxes in M-theory.

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# 1 Introduction

Generalized cohomology theories [Wh62][Ad75] – such as K-theory, elliptic cohomology, stable Cobordism and stable Cohomotopy – are rich. This makes them fascinating but also intricate to deal with. In algebraic topology, it has become commonplace to apply filtrations by iterative *localizations* [Bou79] (review in [EKMM97, §V][Ba14]) that allow generalized cohomology to be approximated in consecutive stages; a famous example of current interest is the chromatic filtration on complex oriented cohomology theories ([MR87], review in [Ra86][Lu10]).

**The Chern-Dold character.** The primary approximation stage of generalized cohomology theories is their *rationalization* (e.g., [Hil71][Ba14]) to ordinary cohomology (e.g., singular cohomology) with rational coefficients or real coefficients (see Remark 3.51). This goes back to [Do65]; since on topological K-theory (Example 4.10) it reduces to the Chern character map [Hil71, Thm. 5.8], it has been called the *Chern-Dold character* [Bu70]:

$$\begin{array}{ccc}
 \begin{array}{l} \text{Chern-Dold} \\ \text{character} \end{array} & \begin{array}{l} \text{generalized} \\ \text{cohomology} \end{array} & \begin{array}{l} \text{rationalization} \\ \longrightarrow \\ E_{\mathbb{R}}^n(X) \end{array} & \begin{array}{l} \text{Dold's equivalence} \\ \xrightarrow{\simeq} \\ \bigoplus_k H^{n+k}(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}) \end{array} & (1) \\
 \text{ch}_E^n : E^n(X) & & & & \\
 & & \begin{array}{l} \text{dR} \circ \text{ch}_E \\ \searrow \\ \text{differential-geometric} \\ \text{Chern-Dold character} \end{array} & & \\
 & & & \begin{array}{l} \text{de Rham theorem} \\ \simeq \downarrow \\ \text{de Rham cohomology} \end{array} & \\
 & & & \text{Hom}\left(E^\bullet(*) \otimes_{\mathbb{Z}} \mathbb{R}, H_{\text{dR}}^{n+\bullet}(X)\right) & 
 \end{array}$$

That the first map in (1) is indeed the rationalization approximation on coefficient spectra is left somewhat implicit in [Bu70] (rationalization was fully formulated only in [BK72]); a fully explicit statement is in [LSW16, §2.1]. The equivalence on the right of (1) serves to make explicit how the result of that rationalization operation indeed lands in ordinary cohomology, and this was Dold’s original observation [Do65, Cor. 4] (re-derived as Prop. 4.5).

**At the heart of differential cohomology.** While rationalization is the coarsest of the localization approximations, it stands out in that it connects, via the de Rham theorem, to *differential geometric* data – when the base space  $X$  has the structure of a smooth manifold, and the coefficients are taken to be  $\mathbb{R}$  instead of  $\mathbb{Q}$ . Indeed, this “differential-geometric Chern-Dold character” shown on the bottom of (1), underlies (often without attribution to Dold or Buchstaber) the pullback-construction of differential generalized cohomology theories [HS05, §4.8] (see [BN14, p. 17][GS17b, Def. 7][GS18b, Def. 17][GS19a, Def. 1], and see Def. 4.33, Example 4.34 below).

**At the heart of non-perturbative field theory.** It is in this differential-geometric form that the Chern-Dold character plays a pivotal role in high energy physics. Here closed differential forms encode *flux densities*  $F_p \in \Omega_{\text{dR}}^p(X)$  of generalized electromagnetic fields on spacetime manifolds  $X$ ; and the condition that these lift through (i.e., are in the image of) the Chern-Dold character (1) for  $E$ -cohomology theory encodes a *charge quantization* condition in  $E$ -theory (see [Fr00][Sa10][GS19c]), generalizing Dirac’s charge quantization of the ordinary electromagnetic field in ordinary cohomology [Di31] (see [Fra97, 16.4e]):

$$\begin{array}{ccc}
 E^n(X) & \xrightarrow{\begin{array}{l} \text{differential-geometric} \\ \text{Chern-Dold character} \end{array}} & \text{Hom}\left(E^\bullet(*) \otimes_{\mathbb{Z}} \mathbb{R}, H_{\text{dR}}^{n+\bullet}(X)\right) & (2) \\
 \begin{array}{l} [c] \longmapsto \\ \text{class in} \\ E\text{-cohomology} \end{array} & & \begin{array}{l} \{ \{ F_{pi} \} \}_{i=1, \dots, \dim(E^\bullet(*) \otimes_{\mathbb{Z}} \mathbb{R})} \\ \text{charge-quantized} \\ \text{flux densities} \end{array} \\
 \text{dR} \circ \text{ch}_E^n & & & 
 \end{array}$$

This idea of charge quantization in a generalized cohomology theory turned out to be fruitful for capturing much of the expected nature of the Ramond-Ramond (RR) fields in type II/I string theory, as being charge-quantized in topological K-theory:  $E = \text{KU}, \text{KO}$  (see [FH00][Fr00][Ev06][GS19c][GS18b]).

However, various further topological conditions [FSS19b, Table 1][FSS19c, p. 2][SS20a, Table 3][FSS20, p. 2] in non-perturbative type IIA string theory (“M-theory”) are not captured by charge-quantization (2) in K-theory, or in any generalized cohomology theory, since they involve quadratic functions (301) in the fluxes. This motivates:

**Non-abelian cohomology.** Despite their established name, generalized cohomology theories in the traditional sense of [Wh62][Ad75] are not general enough for many purposes. Already the time-honored non-abelian cohomology that classifies principal bundles (Example 2.3 below), being the domain of the Chern-Weil homomorphism

[Ch50] (recalled as Def. 4.21, Prop. 4.23 below), falls outside the scope of “generalized” cohomology, as does the higher non-abelian cohomology classifying gerbes [Gi71] (Example 2.6 below). But these are just the first two stages within a truly general concept of *higher non-abelian cohomology* (Def. 2.1 below), that classifies higher bundles/higher gerbes (Example 2.7 below) and which fully subsumes Whitehead’s traditional generalized cohomology as its abelian sector (Example 2.13 below).

In higher non-abelian cohomology the very conceptualization of cohomology finds a beautiful culmination, as it is reduced to the pristine concept of homotopy types of mapping spaces (7), or rather, if geometric (differential, equivariant,...) structures are incorporated, of higher mapping stacks (Remark 2.27 below).

In particular, the concept of *twisted* non-abelian cohomology is most natural from this perspective (Def. 2.29 below) and naturally subsumes the traditional concept of twisted generalized cohomology theories (Prop. 2.38 below).

$$H^\tau(X; A) = \left\{ \begin{array}{ccc} X & \overset{\text{cocycle } c}{\dashrightarrow} & A // G \\ \swarrow \text{twist } \tau & \overset{\simeq}{\rightleftarrows} & \searrow \rho \\ & \text{BG} & \text{local coefficients} \end{array} \right\} / \text{homotopy relative BG}$$

**State of the literature.** It is fair to say that this transparent fundamental nature of higher non-abelian cohomology is not easily recognized in much of the traditional literature on the topic, which is rife with unwieldy variants of cocycle conditions presented in combinatorial  $n$ -category-theoretic language. As a consequence, the development of non-abelian cohomology theory has seen little and slow progress, certainly as compared to the flourishing of generalized cohomology theory. In particular, the concepts of higher and of twisted non-abelian cohomology had remained mysterious (see [Si97, p. 1]). It is the more recently established homotopy-theoretic formulation of  $\infty$ -category theory (e.g., via model category theory, see appendix A) in its guise as  $\infty$ -topos theory ( $\infty$ -stacks, recalled around Def. A.45 below) that provides the backdrop on which twisted higher non-abelian cohomology finds its true and elegant nature [Si97][Si99][To02][SSS12][NSS12a][NSS12b][Sch13][FSS19b][SS20b]; see §2 for details.

**The non-abelian character map.** From this homotopy-theoretic perspective, we observe in §4 and §5 that the generalization of the Chern-Dold character (1) to twisted non-abelian cohomology naturally exists (Def. 4.2), and that the non-abelian analogue of Dold’s equivalence in (1) may neatly be understood as being, up to mild re-conceptualization, the fundamental theorem of dg-algebraic rational homotopy theory (recalled as Prop. 3.60 below). We highlight that this classical theorem is fruitfully recast as constituting a *non-abelian de Rham theorem* (Theorem 3.87 below) and, more generally, a *twisted non-abelian de Rham theorem* (Theorem 3.104 below). With this in hand, the notion of the (twisted) non-abelian character map appears naturally (Def. 4.2 and Def. 5.4 below):

$$\text{ch}_\rho : \begin{array}{c} \text{twisted non-abelian cohomology} \\ \text{(Def. 2.29)} \end{array} H^\tau(X; A) \xrightarrow[\text{rationalization (Def. 3.55, Prop. 3.60)}]{(\eta_\rho^{\mathbb{R}})_*} H^{L_{\mathbb{R}}\tau}(X; L_{\mathbb{R}}A) \xrightarrow[\text{twisted non-abelian de Rham theorem (Thm. 3.104)}]{\simeq} \begin{array}{c} \text{twisted non-abelian de Rham cohomology} \\ \text{(Def. 3.98)} \end{array} H_{\text{dR}}^{\tau}(X; \mathbb{L}A). \quad (3)$$

**Twisted differential non-abelian cohomology.** Moreover, with the (twisted) non-abelian character in hand, the notion of (twisted) differential non-abelian cohomology appears naturally (Def. 4.33, Def. 5.13) together with the expected natural diagrams of twisted differential non-abelian cohomology operations:

$$\begin{array}{ccc} \begin{array}{c} \text{differential non-abelian cohomology} \\ \text{(Def. 4.33)} \end{array} \widehat{H}(X; A) & \xrightarrow[\text{curvature (234)}]{\text{flat } L_{\infty}\text{-algebra valued differential forms (Def. 3.77)}} & \Omega_{\text{dR}}(X; \mathbb{L}A) \\ \downarrow \text{characteristic class (233)} & \searrow \text{differential non-abelian character (235)} & \downarrow \\ \begin{array}{c} \text{non-abelian cohomology} \\ \text{(Def. 2.1)} \end{array} H(X; A) & \xrightarrow[\text{non-abelian character (Def. 4.2)}]{\text{ch}_A} & \begin{array}{c} \text{non-abelian de Rham cohomology} \\ \text{(Def. 3.84)} \end{array} H_{\text{dR}}(X; \mathbb{L}A) \\ \\ \begin{array}{c} \text{twisted differential non-abelian cohomology} \\ \text{(Def. 5.13)} \end{array} \widehat{H}^{\tau, \text{diff}}(X; A) & \xrightarrow[\text{twisted curvature (280)}]{\text{twisted flat } L_{\infty}\text{-algebra valued differential forms (Def. 3.92)}} & \Omega_{\text{dR}}^{\tau}(X; \mathbb{L}A) \\ \downarrow \text{twisted characteristic class (279)} & \searrow \text{twisted differential non-abelian character (281)} & \downarrow \\ \begin{array}{c} \text{twisted non-abelian cohomology} \\ \text{(Def. 2.29)} \end{array} H^{\tau}(X; A) & \xrightarrow[\text{twisted non-abelian character (Def. 5.4)}]{\text{ch}_A^{\tau}} & \begin{array}{c} \text{twisted non-abelian de Rham cohomology} \\ \text{(Def. 3.98)} \end{array} H_{\text{dR}}^{\tau}(X; \mathbb{L}A) \end{array} \quad (4)$$

**Unifying Chern-Dold, Chern-Weil and Cheeger-Simons.** In order to show that this generalization of (twisted) character maps and (twisted) differential cohomology to higher non-abelian cohomology is sound, we proceed to prove that the non-abelian character map (Def. 4.2) specializes to:

the Chern-Dold character on generalized cohomology	(Theorem 4.8),
the Chern-Weil homomorphism on degree-1 non-abelian cohomology	(Theorem 4.26),
the Cheeger-Simons homomorphism on degree-1 differential non-abelian cohomology	(Theorem 4.46).

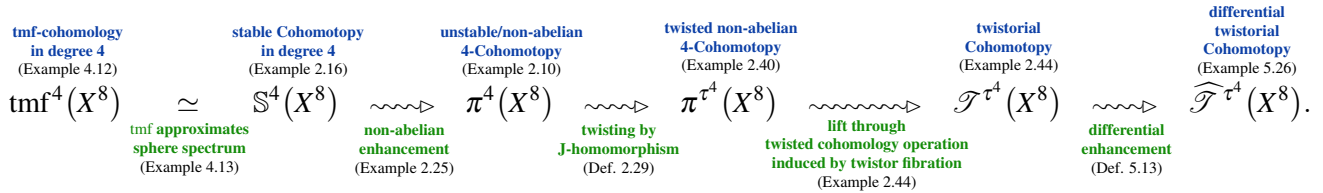
All these classical invariants are thus seen as different low-degree aspects of the higher non-abelian character map.

**Examples of twisted higher character maps.** To illustrate the mechanism, we make explicit several examples of the (twisted) non-abelian character map on cohomology theories of relevance in high energy physics:

the Chern character on complex differential K-theory	(Example 4.10, 4.36),
the Pontrjagin character on real K-theory	(Example 4.11),
the Chern character on twisted differential K-theory	(Example 5.5, 5.22),
the MMS-character on cohomotopy-twisted K-theory	(Example 5.7),
the LSW-character on twisted higher K-theory	(Example 5.10),
the character on integral Morava K-theory	(Example 4.15),
the character on topological modular forms, tmf	(Example 4.12).

Once incarnated this way within the more general context of non-abelian cohomology theory, we may ask for non-abelian enhancements (Example 2.24) of these abelian character maps:

**Non-abelian enhancement of the tmf-character – the cohomotopical character.** Our culminating example, in §5.3, is the character map on twistorial Cohomotopy theory [FSS19b][FSS20], over 8-manifolds  $X^8$  equipped with tangential  $\mathrm{Sp}(2)$ -structure  $\tau$  (59). This may be understood (Remark 4.14) as an enhancement of the tmf-character (Example 4.12) from traditional generalized cohomology to twisted differential non-abelian cohomology:



The non-abelian character map on twistorial Cohomotopy has the striking property (Prop. 5.24, the proof of which is the content of the companion physics article [FSS20, Prop. 3.9]) that the corresponding non-abelian version of Dirac’s charge quantization (2) implies Hořava-Witten’s Green-Schwarz mechanism in heterotic M-theory for heterotic line bundles  $F_2$  (see [FSS20, §1]) and other subtle effects expected in non-perturbative high energy physics; these are discussed in Remark 5.29 below.

**Quadratic character functions from Whitehead brackets in non-abelian coefficient spaces.** The crucial appearance of *quadratic functions* in the Cohomotopical character map (301) is brought about by the non-abelian nature of (twisted) Cohomotopy theory. These non-linearities originate in non-trivial Whitehead brackets (Remark 3.66) on the non-abelian coefficient spaces  $S^4$  (Example 3.68) and on  $\mathbb{C}P^3$  (Example 3.96). Generally, the non-abelian character map (3) involves also higher monomial terms of any order (cubic, quartic, ...), originating in higher order Whitehead brackets on the non-abelian coefficient space (Remark 3.66).

Note that the desire to conceptually grasp character-like but quadratic functions appearing in M-theory had been the original motivation for developing differential generalized cohomology, in [HS05]. Here, in differential non-abelian cohomology, they appear intrinsically.

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## 2 Non-abelian cohomology

We make explicit the concept of general non-abelian cohomology (Def. 2.1 below) and of twisted non-abelian cohomology (Def. 2.29 below), following [Si97][Si99][To02][SSS12][NSS12a][NSS12b][FSS19b][SS20b]; and we survey how this concept subsumes essentially every notion of cohomology known.

In the following, we make free use of the basic language of category theory and homotopy theory (for joint introduction see [Rie14][Ri20]). For  $\mathcal{C}$  a category and  $X, A \in \mathcal{C}$  a pair of its objects, we write

$$\mathcal{C}(X, A) := \text{Hom}_{\mathcal{C}}(X, A) \in \text{Sets} \quad (5)$$

for the set of morphisms from  $X$  to  $A$ . These are, of course, contravariantly and covariantly functorial in their first and second argument, respectively:

$$\mathcal{C} \xrightarrow{\mathcal{C}(X, -)} \text{Sets}, \quad \mathcal{C}^{\text{op}} \xrightarrow{\mathcal{C}(-, A)} \text{Sets}. \quad (6)$$

Basic as this is, contravariant hom-functors are of paramount interest in the case where  $\mathcal{C}$  is the *homotopy category*  $\text{Ho}(\mathbf{C})$  (Def. A.15) of a model category (Def. A.3), such as the classical homotopy category of topological spaces or, equivalently, of simplicial sets (Example A.34).

### 2.1 Non-abelian cohomology theories

**Definition 2.1** (Non-abelian cohomology). For  $X, A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Example A.34) we say that their hom-set (5) is the *non-abelian cohomology* of  $X$  with *coefficients* in  $A$ , or the *non-abelian  $A$ -cohomology* of  $X$ , to be denoted:

$$\text{non-abelian cohomology } H(X; A) := \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, A) = \left\{ \begin{array}{c} \text{map = cocycle} \\ \begin{array}{ccc} & \xrightarrow{\quad c \quad} & \\ & \parallel & \\ X & \xrightarrow{\quad \text{homotopy = coboundary} \quad} & A \\ & \downarrow & \\ & \xrightarrow{\quad c' \quad} & \\ & \text{map = cocycle} & \end{array} \\ \end{array} \right\} /_{\text{homotopy}} \quad (7)$$

We also call the contravariant hom-functor (6)

$$H(-; A) : \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \longrightarrow \text{Sets} \quad (8)$$

the *non-abelian  $A$ -cohomology theory*.

**Example 2.2** (Ordinary cohomology). For  $n \in \mathbb{N}$  and  $A$  a discrete abelian group, the ordinary cohomology (e.g. singular cohomology) in degree  $n$  with coefficients in  $A$  is equivalently ([Ei40, p. 243][EML54b, p. 520-521], review in [St72, §19][May99, §22][AGP02, §7.1, Cor. 12.1.20]) non-abelian cohomology in the sense of Def. 2.1

$$\text{ordinary cohomology } H^n(-; A) \simeq H(-; K(A, n)) \quad (9)$$

with coefficients in an *Eilenberg-MacLane space* [EML53][EML54a]:

$$K(A, n) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \quad \text{such that} \quad \pi_k(K(A, n)) = \begin{cases} A & | \quad k = n \\ 0 & | \quad k \neq n. \end{cases} \quad (10)$$

**Example 2.3** (Traditional non-abelian cohomology). For  $G$  a well-behaved topological group, the traditional non-abelian cohomology  $H^1(-; G)$  classifying  $G$ -principal bundles, is equivalently ([St51, §19.3][RS12, Thm 1.], review in [Add07, §5]) non-abelian cohomology in the general sense of Def. 2.1

$$\text{classification of principal bundles } H^1(-; G) \simeq H(-; BG) \quad (11)$$

with coefficients in the *classifying space*  $BG$  ([Mi56][Se68][St68][St70], review in [Ko96, §1.3][May99, §23.1][AGP02, §8.3][NSS12b, §3.7.1]). The latter may be given as the homotopy colimit (in the classical model structure of  $\text{TopologicalSpaces}_{\text{Qu}}$ , Example A.7) over the nerve of the topological group  $G$  (e.g. [NSS12a, Rem. 2.23]):

$$BG \simeq \underset{\rightarrow}{\text{holim}} \left( \begin{array}{c} \cdots \quad \text{.....} \\ \text{.....} \\ \text{.....} \\ \text{.....} \\ \text{.....} \\ \text{.....} \\ \text{.....} \\ \text{.....} \end{array} G \times G \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} G \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} * \right). \quad (12)$$

**Example 2.4** (Group cohomology and Characteristic classes). Conversely, the ordinary cohomology (Example 2.2) of a classifying space  $BG$  (12) is, equivalently,

- (i) the group cohomology of  $G$ ;
- (ii) the universal characteristic classes of  $G$ -principal bundles:

$$H(BG; K(A, n)) \simeq H^n(BG; A) \simeq H_{\text{Grp}}^n(G; A).$$

**Example 2.5** (Non-abelian cohomology in degree 2). For a well-behaved topological 2-group, such as the *string 2-group*  $\text{String}(G)$  (of a connected, simply connected semi-simple Lie group  $G$ ) [BCSS07][He08, Thm. 4.8][NSW11], the non-abelian cohomology  $H^1(-; \text{String}(G))$  classifying principal 2-bundles [NW11] with structure 2-group  $\text{String}(G)$  is, equivalently [BS09],

$$H^1(-; \text{String}(G)) \simeq H(-; B\text{String}(G))$$

non-abelian cohomology in the general sense of Def. 2.1 with coefficients in the classifying space  $B\text{String}(G)$ .

**Example 2.6** (Non-abelian gerbes). For  $G$  a well-behaved topological group, a non-abelian  $G$ -gerbe [Gi71][Br09] is, equivalently [NSS12a, §4.4], a fiber 2-bundle with typical 2-fiber of homotopy type of the classifying space  $BG$  (12), associated to principal 2-bundles with structure 2-group  $\text{Aut}(\mathbf{BG})$ . Hence, as in Example 2.5,  $G$ -gerbes are classified by non-abelian cohomology with coefficients in  $B\text{Aut}(\mathbf{BG})$  [NSS12a, Cor 4.51]:

$$G\text{Gerbes}(X)_{/\sim} \simeq H^1(X; \text{Aut}(\mathbf{BG})) \simeq H(X; B\text{Aut}(\mathbf{BG})).$$

**Example 2.7** (Non-abelian cohomology in unbounded degree). For any  $\infty$ -group  $\mathcal{G}$  (see [NSS12a, §2.2][NSS12b, §3.5]), the non-abelian cohomology  $H^1(-; \mathcal{G})$  classifying principal  $\infty$ -bundles [Gl82][JL06][NSS12a][NSS12b] with structure  $\infty$ -group  $\mathcal{G}$  is, equivalently [We10][RS12],

$$H^1(-; \mathcal{G}) \simeq H(-; B\mathcal{G})$$

non-abelian cohomology in the general sense of Def. 2.1 with coefficients in the classifying space  $B\mathcal{G}$  (see also [Stv12]).

Example 2.7 is, in fact, universal:

**Proposition 2.8** (Connected homotopy types are higher non-abelian classifying spaces [NSS12a, Thm. 2.19][NSS12b, Thm. 3.30, Cor. 3.34]). *Every connected homotopy type  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (339) is the classifying space of a topological group, namely of its loop group<sup>1</sup>  $\Omega A$*

$$A \simeq B(\Omega A) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}).$$

<sup>1</sup> A priori, the loop group is an  $A_\infty$ -group, for which classifying spaces are defined as in [NSS12a, Rem. 2.23], but each such is weakly equivalent to an actual topological group, see [NSS12b, Prop. 3.35].

This allows for making precise the core nature of non-abelian cohomology:

**Remark 2.9** (Non-abelian and abelian  $\infty$ -groups). For  $A \simeq BG$  (15), the  $\infty$ -group structure on  $G$  is reflected by its weak homotopy equivalence  $G \simeq \Omega BG$  with a based loop space.

- There is no commutativity of loops in a generic loop space, and hence this exhibits  $G$  as a *non-abelian*  $\infty$ -group.
- But it may happen that  $A$  itself is already equivalent to a loop space, which by (15) means that  $A \simeq B(BG) =: B^2G$  is a *double delooping*. In this case  $G \simeq \Omega(\Omega A) =: \Omega^2A$  is an *iterated loop space* [May72], specifically *double loop space*; hence a *braided*  $\infty$ -group. By the Eckmann-Hilton argument, this implies some level of commutativity of the group operation in  $G$ . Indeed, in the special case that such  $G$  is also 0-truncated (341), it implies that  $G$  is an ordinary abelian group.
- Next, it may happen that  $A \simeq B^3G$  is a 3-fold delooping, hence that  $G \simeq \Omega^3A$  is a 3-fold loop space, hence a *syllaptic*  $\infty$ -group. This is one step “more abelian” than a braided  $\infty$ -group.
- In the limiting case that  $G$  is an  $n$ -fold loop space for any  $n \in \mathbb{N}$ , hence an *infinite loop space* [May77][Ad78], it is as abelian as possible for an  $\infty$ -group. Such *abelian*  $\infty$ -groups are the coefficients of abelian cohomology theories, namely of generalized cohomology theories in the sense of Whitehead (Example 2.13)
- The fewer deloopings an  $\infty$ -group  $G$  admits, the “more non-abelian” is the cohomology theory represented by  $BG$ .

Coefficients		$H(X; BG)$	Examples
$\infty$ -group	$G \simeq \Omega B G$	non-abelian cohomology	$\pi^n(-)$ (Cohomotopy, Example 2.10) $\pi^3(-)$
braided $\infty$ -group	$G \simeq \Omega^2 B^2 G$		
syllaptic $\infty$ -group	$G \simeq \Omega^3 B^3 G$		
$\vdots$	$G \simeq \Omega^n B^n G$		
abelian $\infty$ -group	$G \simeq \Omega^\infty B^\infty G$	abelian cohomology	$E^n(-)$ (generalized cohomology, Example 2.13)

The most fundamental connected homotopy types are the  $n$ -spheres (all other are obtained by gluing  $n$ -spheres to each other):

**Example 2.10** (Cohomotopy theory). The non-abelian cohomology theory (Def. 2.1) with coefficients in the homotopy types of  $n$ -spheres is (unstable) *Cohomotopy* theory [Bo36][Sp49][Pe56][Ta09][KMT12]:

$$\overset{\text{Cohomotopy}}{\pi^n(-)} = H(-; S^n) \simeq H^1(-; \Omega S^n) \quad \text{for } n \in \mathbb{N}_+.$$

(i) By Prop. 2.8, Cohomotopy theory classifies principal  $\infty$ -bundles (Example 2.7) with structure  $\infty$ -group of the homotopy type of the  $\infty$ -group  $\Omega S^n$ .

(ii) By Remark 2.9, Cohomotopy theory is a *maximally non-abelian* cohomology theory, in that  $S^n$  does not admit deloopings, for general  $n$  (it admits a single delooping for  $n = 3$  and arbitrary deloopings for  $n = 0, 1$ ).

**Example 2.11** (Bundle gerbes). The classifying space (12) of the circle group  $U(1)$  is an Eilenberg-MacLane space (10)

$$BU(1) \simeq K(\mathbb{Z}, 2) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}).$$

Since  $U(1)$  is abelian, this space carries itself the structure of (the homotopy type of) a 2-group, and hence has a higher classifying space

$$B^2U(1) := B(BU(1)) \simeq K(\mathbb{Z}, 3) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$$

in the sense of Example 2.5, which is an Eilenberg-MacLane space in one degree higher. The higher principal 2-bundles with structure 2-group  $\mathbf{BU}(1)$  are equivalently [NSS12a, Rem. 4.36] known as *bundle gerbes* [Mu96][SWa07]. Therefore, Example 2.7 combined with Example 2.2 gives the classification of bundle gerbes by ordinary integral cohomology in degree 3:

$$\overset{\text{classification of bundle gerbes}}{H^1(-; \mathbf{BU}(1))} \simeq H(-; B^2U(1)) \simeq H^3(-; \mathbb{Z}).$$

**Example 2.12** (Higher bundle gerbes). In fact, Prop. 2.8 implies that, for all  $n \in \mathbb{N}$ ,

$$B^{n+1}\mathbf{U}(1) := B(B^n\mathbf{U}(1)) \simeq K(\mathbb{Z}, n+2) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \quad (16)$$

in the sense of Example 2.7. The higher principal bundles with structure  $(n+1)$ -group  $\mathbf{B}^n\mathbf{U}(1)$  [Ga97][FSS10, §3.2.3][FSS12b, §2.6] are also known as *higher bundle gerbes* (for  $n=2$  see [CMW97][Stv01]). On these coefficients, Example 2.7 reduces to the classification of higher bundle gerbes by ordinary integral cohomology in higher degree:

$$H^1(-; \mathbf{B}^n\mathbf{U}(1)) \simeq H(-; B^n\mathbf{U}(1)) \simeq H^{n+2}(-; \mathbb{Z}).$$

More generally, the special case of Example 2.7 where the coefficient  $\infty$ -group happens to be abelian is “generalized cohomology” in the standard sense of algebraic topology (including cohomology theories such as K-theory, elliptic cohomology, stable Cobordism theory, stable Cohomotopy theory, etc.):

**Example 2.13** (Generalized cohomology). For  $E$  a generalized cohomology theory [Wh62] (see [Ad75][Ad78]), Brown’s representability theorem ([Ad75, §III.6][Ko96, §3.4]) says that there is a *spectrum* (“ $\Omega$ -spectrum”, Example A.41) of pointed homotopy types

$$\left\{ E_n \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^*/), E_n \xrightarrow[\simeq]{\tilde{\sigma}_n} \Omega E_{n+1} \right\}_{n \in \mathbb{N}} \quad (17)$$

such that the generalized  $E$ -cohomology in degree  $n$  is equivalently non-abelian cohomology theory in the sense of Def. 2.1 with coefficients in  $E_n$ :

$$E^n(-) \simeq H(-; E_n). \quad (18)$$

**Example 2.14** (Topological K-theory). The classifying space (17) representing complex K-cohomology theory  $\text{KU}$  [AH59, §2] (review in [At67]) in degree 0 is [AH61, §1.3]:

$$\text{KU}_0 \simeq \mathbb{Z} \times \text{BU}, \quad (19)$$

where

$$\text{BU} := \lim_n \text{BU}(n) \quad (20)$$

is the classifying space (12) for the infinite unitary group (e.g. [EU14]). Hence for the case of complex K-theory, Example 2.13 says that:

$$\text{KU}^0(-) \simeq H(-; \mathbb{Z} \times \text{BU}).$$

**Example 2.15** (Iterated K-theory). Given a spectrum (17) with suitable ring structure, one can form its *algebraic K-theory spectrum*  $K(R)$  [EKMM97, §VI][BGT10, §9.5][Lu14] and hence the corresponding generalized cohomology theory (Example 2.13). Much like complex topological K-theory (Example 2.14) is the K-theory of topological  $\mathbb{C}$ -module bundles,  $K(R)$ -cohomology theory is the K-theory of suitable  $R$ -module  $\infty$ -bundles [Li13]. Specifically, for  $R = \text{ku}$  the connective spectrum for topological K-theory, its algebraic K-theory  $K(\text{ku})$  [Au09][AR02][AR07] has been argued to be the K-theory of certain categorified complex vector bundles [BDR03][BDRR09]. Moreover,  $K(R)$  is itself a suitable ring spectrum, so that the construction may be iterated to yield *iterated algebraic K-theories* [Ro14]  $K^{\circ 2}(R) := K(K(R))$ ,  $K^{\circ 3}(R) := K(K(K(R)))$ , et cetera. For  $R = \text{ku}$ , this generalizes the above forms of elliptic cohomology,  $K(\text{ku})$ , to higher degrees [LSW16]. By Example 2.13, we will regard these (connective) iterated algebraic K-theories  $K^{\circ n}(\text{ku})$  of the complex topological K-theory spectrum as examples of non-abelian cohomology theories:

$$K^{\circ n}(\text{ku})^0(-) \simeq H(-; K^{\circ n}(\text{ku})_0).$$

**Example 2.16** (Stable Cohomotopy). The generalized cohomology theory (Example 2.13) represented by the suspension spectra (Example A.42) of  $n$ -spheres is called *stable Cohomotopy theory* (e.g. [Str81][No03]) or *stable framed Cobordism theory*:

$$\mathbb{S}^n(-) = H(-; (\Sigma^\infty \mathbb{S}^n)_0). \quad (21)$$



## Non-abelian cohomology operations.

**Definition 2.17** (Non-abelian cohomology operation). For  $A_1, A_2 \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Example A.34), we say that a natural transformation in non-abelian cohomology (Def. 2.1) from  $A_1$ -cohomology theory to  $A_2$ -cohomology theory (8) is a (non-abelian) *cohomology operation*

$$\phi_* : H(-; A_1) \longrightarrow H(-; A_2) . \quad (22)$$

By the Yoneda lemma, these are in bijective correspondence to morphisms of coefficients

$$A_1 \xrightarrow{\phi} A_2 \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \quad (23)$$

via the covariant functoriality of the hom-sets (6):

$$\phi_* = H(-; \phi) := \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(-, \phi) . \quad (24)$$

**Example 2.18** (Cohomology of coefficient spaces parametrizes cohomology operations). By the Yoneda lemma (24) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Example A.34), the set of all cohomology operations (Def. 2.17) from  $A_1$ -cohomology theory to  $A_2$ -cohomology theory (22) coincides with the the non-abelian  $A_2$ -cohomology (Def. 2.1) of the coefficients  $A_1$ :

$$\begin{array}{c} \text{non-abelian } A_2\text{-cohomology of } A_1 \\ \text{acting as cohomology operations} \end{array} H(A_1; A_2) \times H(-; A_1) \xrightarrow{(-) \circ (-)} H(-; A_2) \quad (25)$$

acting by composition composition in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$ .

**Example 2.19** (Cohomology operations in ordinary cohomology). In specialization to Example 2.2 the non-abelian cohomology operations according to Def. 2.17 reduce to the classical cohomology operations in ordinary cohomology [St72][MT08] (review in [May99, §22.5]), such as Steenrod operations [St47][SE62] (review in [Ko96, §2.5]). These operations admit refinements, involving rational/real form data, to differential cohomology operations [GS18a].

**Example 2.20** (Cohomology operations in generalized cohomology). In specialization to Example 2.13, the non-abelian cohomology operations according to Def. 2.17 on a Whitehead-generalized cohomology theory  $E^\bullet(-)$  regarded as a system of non-abelian cohomology theories  $\{E^n(-)\}$  reduce to the traditional notion of *unstable* cohomology operations on generalized cohomology theories [BJW95], such as the Adams operations in K-theory [Ad62] (review in [AGP02, §10]) or the Quillen operations in stable Cobordism theory (review in [Ko96, §4.5]). For differential refinements see [GS18b].

**Example 2.21** (Characteristic classes of principal  $\infty$ -bundles). For  $G$  a topological group, the ordinary group cohomology of  $G$  (Example 2.4) parametrizes, via Example 2.18, the cohomology operations from non-abelian cohomology classifying  $G$ -principal bundles (Examples 2.3, 2.5, 2.7) to ordinary cohomology of the base space (Example 2.2):

$$\begin{array}{ccc} \begin{array}{c} \text{group} \\ \text{cohomology} \end{array} & \begin{array}{c} G\text{-principal} \\ \text{bundles} \end{array} & \begin{array}{c} \text{characteristic} \\ \text{classes} \end{array} & \begin{array}{c} \text{ordinary} \\ \text{cohomology} \end{array} \\ H_{\text{Grp}}^n(BG; A) \times H^1(-; G) & \xrightarrow{(25)} & H^n(-; A) . \end{array} \quad (26)$$

This is the assignment of *characteristic classes* to principal bundles (principal  $\infty$ -bundles). In the case when  $A = \mathbb{R}$ , this is equivalently the *Chern-Weil homomorphism*, by Chern's fundamental theorem (see Remark 4.16 and Theorem 4.26 below).

**Example 2.22** (Rationalization cohomology operation). For fairly general non-abelian coefficients  $A$  (see Def. 3.55, Def. 4.1 for details), their *rationalization*<sup>2</sup>  $A \xrightarrow{\eta_A^{\mathbb{R}}} L_{\mathbb{R}}A$  (Def. 3.55 below) induces a cohomology operation (Def. 2.17) from non-abelian  $A$ -cohomology theory (Def. 2.1) to *non-abelian real cohomology* (Def. 3.72 below):

$$\begin{array}{ccc} \begin{array}{c} \text{non-abelian} \\ \text{cohomology} \end{array} & \begin{array}{c} \text{rationalization} \\ (\eta_A^{\mathbb{R}})_* \end{array} & \begin{array}{c} \text{non-abelian} \\ \text{real cohomology} \end{array} \\ H(-; A) & \xrightarrow{\text{rationalization}} & H(-; L_{\mathbb{R}}A) . \end{array} \quad (27)$$

<sup>2</sup> For definiteness, we consider rationalization over the real numbers; see Remark 3.51 below.

**Remark 2.23** (Rationalization as character map). Up to composition with an equivalence provided by the non-abelian de Rham theorem (Theorem 3.87 below), which serves to bring the right hand side of (27) into neat minimal form, this rationalization cohomology operation is the *character map in non-abelian cohomology* (Def. 4.2 below).

**Example 2.24** (Stabilization cohomology operation). For  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$ , the non-abelian cohomology operation (Def. 2.17) induced (24) by the unit of the derived stabilization adjunction (Example A.42) goes from non-abelian  $A$ -cohomology theory (Def. 2.1) to (abelian) generalized cohomology theory (Example 2.13) represented by the 0th component space of the suspension spectrum of  $A$ :

$$\begin{array}{ccc} \begin{array}{c} \text{non-abelian} \\ A\text{-cohomology} \end{array} & & \begin{array}{c} \text{generalized} \\ \Sigma^\infty A\text{-cohomology} \end{array} \\ H(-; A) & \xrightarrow{\text{stabilization}} & H(-; (\mathbb{L}\Sigma^\infty A)_0) \end{array}$$

Hence a lift through this operation is an enhancement of generalized cohomology to non-abelian cohomology.

**Example 2.25** (Non-abelian enhancement of stable Cohomotopy). The canonical non-abelian enhancement (in the sense of Example 2.24) of stable Cohomotopy (Example 2.16) is actual Cohomotopy theory (Example 2.10):

$$\begin{array}{ccc} \text{Cohomotopy} & & \text{stable} \\ \pi^n(-) & \xrightarrow{\text{stabilization}} & \mathbb{S}^n(-) \end{array}$$

**Example 2.26** (Hurewicz homomorphism and Hopf degree theorem). By definition of Eilenberg-MacLane spaces (10) there is, for  $n \in \mathbb{N}$ , a canonical map

$$\mathbb{S}^n \xrightarrow{e^{(n)}} K(\mathbb{Z}, n) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}),$$

which represents the element  $1 \in \mathbb{Z} \simeq \pi_n(K(\mathbb{Z}, n))$ . The non-abelian cohomology operation (Def. 2.17) induced by this, from degree  $n$  Cohomotopy (Example 2.10) to degree  $n$  ordinary cohomology (Example 2.2)

$$\pi^n(-) \xrightarrow{e_*^{(n)}} H^n(-; \mathbb{Z})$$

is the cohomological version of the Hurewicz homomorphism. The *Hopf degree theorem* (e.g. [Kos93, §IX (5.8)]) is the statement that the non-abelian cohomology operation  $e_*^{(n)}$  becomes an isomorphism on connected, orientable closed manifolds of dimension  $n$ . These maps, together with their differential refinements, are analyzed in more details via Postnikov towers in [GS20].

### Structured non-abelian cohomology.

**Remark 2.27** (Structured non-abelian cohomology). More generally, it makes sense to consider the analog of Def. 2.1 for the homotopy category  $\text{Ho}(\mathbf{H})$  of a model category which is a *homotopy topos* [TV05][Lu09][Re10].

(i) This yields *structured* non-abelian cohomology [Si97][Si99][To02][SSS12][NSS12a][NSS12b][Sch13][FSS19b][SS20b]:

$$\begin{array}{ccc} \begin{array}{c} \text{structured} \\ \text{non-abelian cohomology} \end{array} & & \begin{array}{c} \text{homotopy topos} \\ \infty\text{-stacks} \end{array} \\ H(\mathcal{X}; \mathbf{A}) & := & \text{Ho}(\mathbf{H})(\mathcal{X}, \mathbf{A}), \end{array}$$

including the stacky non-abelian cohomology originally considered in [Gi71][Br90] (“gerbes”, see [NSS12a, §4.4]), and, more generally, *differential-, étale-, and equivariant-* nonabelian cohomology theories (see [SS20b, p. 6]) based on  $\infty$ -stacks.

(ii) In good cases (cohesive homotopy toposes [Sch13][SS20b, §3.1]), the homotopy topos  $\text{Ho}(\mathbf{H})$  comes equipped with a *shape* operation down to the classical homotopy category (Example A.34):

$$\begin{array}{ccc} \begin{array}{c} \text{homotopy topos} \\ \text{structured} \\ \text{non-abelian cohomology} \end{array} & \xrightarrow{\text{Shp}} & \begin{array}{c} \text{classical homotopy category} \\ \text{plain} \\ \text{non-abelian cohomology} \end{array} \\ \text{Ho}(\mathbf{H}) & & \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \\ H(\mathcal{X}; \mathbf{A}) & \dashv \longrightarrow & H(\text{Shp}(\mathcal{X}); \text{Shp}(\mathbf{A})) \end{array} \quad (28)$$

which takes, for well-behaved group  $\infty$ -stacks  $G$ , the *classifying stacks*  $\mathbf{B}G$  of  $G$ -principal bundles to the traditional classifying spaces  $BG \simeq \mathrm{Shp}(\mathbf{B}G)$  of underlying topological groups (12). This gives a forgetful functor from structured non-abelian cohomology to plain non-abelian cohomology in the sense of Def. 2.1. A classical example is the map from non-abelian Čech cohomology with coefficients in a well-behaved group  $G$  to homotopy classes of maps to the classifying space of  $G$ , in which case this comparison map is a bijection (Example 2.3).

All constructions on non-abelian cohomology have their structured analogues, for instance non-abelian cohomology operations (Def. 2.17) in structured cohomology

$$H(\mathcal{X}; \mathbf{A}_1) \xrightarrow{\phi_*} H(\mathcal{X}; \mathbf{A}_2) \quad (29)$$

are induced by postcomposition with morphisms  $\mathbf{A}_1 \xrightarrow{\phi} \mathbf{A}_2$  of coefficient stacks.

Ultimately, one is interested in working with structured non-abelian cohomology on the left of (28). However, since this is rich and intricate, it behooves us to study its projection into plain non-abelian cohomology on the right of (28). This is what we are mainly concerned with here. But we provide in §4.3 a brief discussion of non-abelian differential cohomology on smooth  $\infty$ -stacks,

## 2.2 Twisted non-abelian cohomology.

For  $\mathcal{C}$  any category and  $B \in \mathcal{C}$  any object, there is the *slice category*  $\mathcal{C}^{/X}$ , whose objects are morphisms in  $\mathcal{C}$  to  $X$  and whose morphisms are commuting triangles over  $X$  in  $\mathcal{C}$ . Basic as this is, hom-sets in the *homotopy category*  $\mathrm{Ho}(\mathbf{C}^{/B})$  (Def. A.15) of a slice model category  $\mathbf{C}^{/B}$  (Example A.10) are of paramount interest:

The slicing imposes *twisting* on the corresponding non-abelian cohomology (Def. 2.1), in that the slicing of the domain space serves as a twist, the slicing of the coefficient space as a local coefficient bundle, and the slice morphisms as twisted cocycles.

**Proposition 2.28** ( $\infty$ -Actions on homotopy types [DDK80][Pr10, §5][NSS12a, §4][Sh15][SS20b, §2.2]). *For any  $A \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})$  (Example A.34) and  $G$  a topological group, homotopy-coherent actions of  $G$  on  $A$  are equivalent to fibrations  $\rho$  with homotopy fiber  $A$  (Def. A.23) over the classifying space  $BG$  (12)*

$$\begin{array}{ccc} A & \longrightarrow & A // G \\ & & \downarrow \rho \\ & & BG. \end{array} \quad (30)$$

Here

$$A // G \simeq (A \times EG)_{/\mathrm{diag}G}$$

is the *homotopy quotient* (Borel construction) of the action.

**Definition 2.29** (Twisted non-abelian cohomology [NSS12a, §4][FSS19b, (10)][SS20b, Rem. 2.94]).

For  $X, A \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})$  (Def. A.34) we say:

(i) A *local coefficient bundle* for twisted  $A$ -cohomology is an  $A$ -fibration  $\rho$  over a classifying space  $BG$  (12) as in Prop. 2.28:

$$\begin{array}{ccc} A & \longrightarrow & A // G \\ \text{local coefficient} & & \downarrow \rho \\ \text{bundle} & & BG. \end{array} \quad (31)$$

(ii) A *twist* for non-abelian  $A$ -cohomology theory on  $X$  with local coefficient bundle  $\rho$  over  $BG$  is a map

$$X \xrightarrow{\tau} BG \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}). \quad (32)$$

(iii) The *non-abelian  $\tau$ -twisted  $A$ -cohomology* of  $X$  with local coefficients  $\rho$  is the hom-set from  $\tau$  (32) to  $\rho$  (30)

$$H^\tau(X; A) := \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{/BG})(\tau, \rho) = \left\{ \begin{array}{ccc} X & \overset{\text{cocycle } c}{\dashrightarrow} & A // G \\ \text{twist } \tau \swarrow & \overset{\simeq}{\longleftarrow} & \searrow \rho \\ & & BG \end{array} \right\} \Big/ \text{homotopy relative } BG \quad (33)$$

in the homotopy category (Def. A.15) of the slice model category over  $BG$  (Example A.10) of the classical model category on topological spaces (Example A.7).

**Definition 2.30** (Associated coefficient bundle [NSS12a, §4.1][SS20b, Prop. 2.92]). Given a local coefficient  $A$ -fiber bundle  $\rho$  (31) and a twist  $\tau$  (32) on a domain space  $X$ , the corresponding *associated  $A$ -fiber bundle* over  $X$  is the homotopy pullback (Def. A.24) of  $\rho$  along  $\tau$ , sitting in a homotopy pullback square (330) of this form:

$$\begin{array}{ccc}
 \begin{array}{c} \text{associated} \\ A\text{-fiber bundle} \end{array} E & \xrightarrow{\quad} & A // G \begin{array}{c} \text{local} \\ \text{coefficient bundle} \end{array} \\
 \mathbb{R}\tau^* \rho \downarrow & \text{(hpb)} & \downarrow \rho \\
 \text{homotopy pullback} & & \\
 X & \xrightarrow[\text{twist}]{\tau} & BG
 \end{array} \tag{34}$$

We write

$$\Gamma_X(E)_{/\sim} := \text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}^X\right)(\text{id}_X, \mathbb{R}\tau^* \rho) = \left\{ \begin{array}{c} \text{section} \nearrow E \text{ associated bundle} \\ X \xrightarrow{\quad} X \downarrow \\ \text{vertical homotopy} \end{array} \right\} \tag{35}$$

for the set of vertical homotopy classes of section of the associated bundle, hence for the hom-set, from the identity on  $X$  to the associated bundle projection, in the homotopy category (Def. A.15) of the slice model category over  $X$  (Example A.10) of the classical model category on topological spaces (Example A.7).

**Proposition 2.31** (Twisted non-abelian cohomology is sections of associated coefficient bundle [NSS12a, Prop. 4.17]). *Given a local coefficient bundle  $\rho$  (31) and a twist  $\tau$  (32), the  $\tau$ -twisted non-abelian cohomology (Def. 2.29) with local coefficient in  $\rho$  is equivalent to the vertical homotopy classes of sections (35) of the associated coefficient bundle  $E$  (Def. 2.30):*

$$\begin{array}{ccc}
 \text{twisted non-abelian} & & \text{sections of} \\
 \text{cohomology} & & \text{associated bundle} \\
 H^\tau(X; A) & \simeq & \Gamma_X(E)_{/\sim}.
 \end{array} \tag{36}$$

*Proof.* Consider the following sequence of bijections:

$$\begin{aligned}
 H^\tau(X; A) &= \text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}^{BG}\right)(\tau, \rho) \\
 &\simeq \text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}^{BG}\right)(\mathbb{L}\tau_* \text{id}_X, \rho) \\
 &\simeq \text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}^X\right)(\text{id}_X, \mathbb{R}\tau^* \rho) \\
 &= \Gamma_X(E)_{/\sim}.
 \end{aligned}$$

Here the first line is the definition (33). Then the first step is the observation that every slice object is the derived left base change (Example A.19, Prop. A.21) along itself of the identity on its domain, by (321). With this, the second step is the hom-isomorphism (304) of the derived base change adjunction  $\mathbb{L}\tau_! \dashv \mathbb{R}\tau^*$ . The last line is (35).  $\square$

In twisted generalization of Example 2.2 we have:

**Example 2.32** (Twisted ordinary cohomology). Let  $n \in \mathbb{N}$ , let  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Ex. A.34) be connected and consider a traditional *system of local coefficients* [St43, §3] (see also [MQRT77][ABG10][GS18c])

$$\Pi_1(X) \xrightarrow{t} \text{AbelianGroups},$$

namely, a functor from the fundamental groupoid of  $X$  to the category of abelian groups. Since the construction  $A \mapsto K(A, n)$  of Eilenberg-MacLane spaces (10) is itself functorial and using the assumption that  $X$  is connected, this induces (see [BFGM03, Def. 3.1]) a local coefficient bundle (31) of the form

$$\begin{array}{ccc}
 K(A, n) & \longrightarrow & K(A, n) // \pi_1(X) \\
 & & \downarrow \rho_t \\
 & & B\pi_1(X)
 \end{array} \tag{37}$$

Finally, write  $X \xrightarrow{\tau} B\pi_1(X)$  for the classifying map (via Example 2.3) of the universal connected cover of  $X$  (equivalently: for the 1-truncation projection of  $X$ ). Then the  $\tau$ -twisted non-abelian cohomology (Def. 2.29) of  $X$  with local coefficients in  $\rho_t$  (37) is equivalently the traditional  $t$ -twisted ordinary cohomology of  $X$  in degree  $n$ :

$$H^{n+t}(X; A) \simeq H^t(X; K(A, n)).$$

twisted  
ordinary cohomology

This is manifest from comparing Def. 2.29 with the characterization of twisted ordinary cohomology found in [Hir79, Cor. 1.3][GJ99, p. 332][BFGM03, Lemma 4.2].

**Example 2.33** (Classification of tangential structure). Let  $X$  be a smooth manifold of dimension  $n$ . Its *frame bundle* is an  $O(n)$ -principal bundle  $\text{Fr}(X) \rightarrow X$ , whose class (a diffeomorphism invariant of  $X$ )

$$\begin{array}{ccc} O(n)\text{Bundles}(X)_{/\sim} & \xrightarrow{\simeq} & H(X; BO(n)) \\ [\text{Fr}(X)] & \leftrightarrow & [\tau_{\text{fr}}] \end{array} \quad (38)$$

gives, by Example 2.3, the class of a twist  $\tau_{\text{fr}}$  (32) in the non-abelian  $O(n)$ -cohomology of  $X$ .

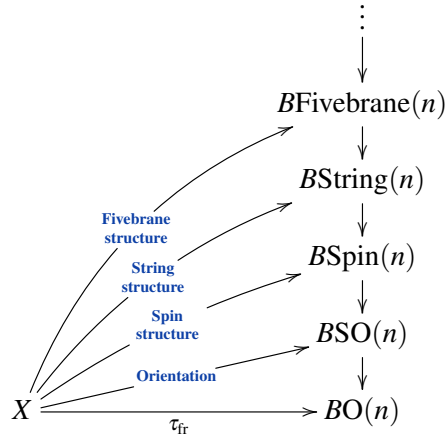
Now for  $BG$  any connected homotopy type (Prop. 2.8) and for  $BG \xrightarrow{\rho} BO(n)$  any map (equivalently the delooping of a morphism of  $\infty$ -groups  $G \rightarrow O(n)$ ), we get a local coefficient bundle (31) with (homotopy-)coset space fiber [FSS19b, Lemma 2.7]:

$$\begin{array}{ccc} O(n) // G & \xrightarrow{\text{hofib}(\rho)} & BG \\ & & \downarrow \rho \\ & & BO(n). \end{array} \quad (39)$$

The relative homotopy class of a homotopy lift of the frame bundle classifier  $\tau_{\text{fr}}$  (2.33) through this map  $\rho$

$$\left[ \begin{array}{ccc} X & \xrightarrow{\text{tangential structure}} & BG \\ & \swarrow g & \searrow \rho \\ & \tau_{\text{fr}} & BO(n) \end{array} \right] \in \text{GTangentialStructures}(X) \quad (40)$$

is known a *topological  $G$ -structure* or *tangential  $\rho$ -structure* on  $X$  (e.g. [Ko96, §1.4][GMTW09, §5][SS20b, Def. 4.48]). For instance, for  $\rho$  a stage in the Whitehead tower of  $O(n)$ , this is, in turn, *Orientation*, *Spin structure*, *String structure*, *Fivebrane structure* [SSS12], etc.:



By comparison of (40) with (33) we see that tangential  $G$ -structures on  $X$  are classified by twisted non-abelian cohomology (Def. 2.29) with coefficients in (homotopy-)coset spaces  $O(n) // G$  (39) and twisted by the class  $\tau_{\text{fr}}$  of the frame bundle (38):

$$\text{GTangentialStructures}(X) \simeq H^{\tau_{\text{fr}}}(X; O(n) // G). \quad (41)$$

According to Prop. 2.8, this example is actually universal for  $\tau_{\text{fr}}$ -twisted non-abelian cohomology.

As a special case of Example 2.32 and in twisted generalization of Examples 2.11, 2.12 we have:

**Example 2.34** (Orientifold gerbes). Consider the action  $\sigma_{U(1)}$  of  $\mathbb{Z}_2$  on the circle group  $U(1) \subset \mathbb{C}^\times$  given by complex conjugation. This deloops (see [FSS15a, §4.4]) to an action  $\sigma_{B^n U(1)}$  of  $\mathbb{Z}_2$  on the classifying spaces  $B^n U(1)$  (12). By Prop. 2.28 there is a corresponding local coefficient bundle

$$\begin{array}{ccc} B^n U(1) & \longrightarrow & B^n U(1) // \mathbb{Z}_2 \\ & & \downarrow \sigma_{B^n U(1)} \\ & & \mathbb{Z}_2 \end{array} \quad (42)$$

Moreover, consider a smooth manifold  $X$ , with orientation bundle classified by  $X \xrightarrow{\text{or}} B\mathbb{Z}_2$ . Then the or-twisted cohomology (Def. 2.29) of  $X$ ...

(i) ...with local coefficients in  $\sigma_{B^2 U(1)}$  classifies what is equivalently known as Jandl gerbes [SSW07][GSW11] or real gerbes [HMSV19] or orientifold B-fields;

(ii) ...with local coefficients in  $\sigma_{B^3 U(1)}$  classifies what is equivalently known as topological sectors of orientifold C-fields [FSS15a, §4.4].

More generally, one can consider twisted Deligne cohomology [GS18c] as well as higher-twisted periodic integral- and Deligne-cohomology [GS19b] (see also §4.3).

**Remark 2.35** (The Whitehead principle of non-abelian cohomology). Let  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  be connected, so that  $A \simeq BG$  (Prop. 2.8).

(i) If  $A$  is also  $n$ -truncated (342), then its Postnikov tower (Prop. A.39) says that  $A$  is the total space of a local coefficient bundle (2.29) of the form

$$\begin{array}{ccc} K(\pi_n(A), n) & \xrightarrow{\text{hfib}(p_n)} & A \\ & & \downarrow p_n^A \\ & & A(n-1) \simeq B(G(n-2)) \end{array}$$

with homotopy fiber an Eilenberg-MacLane space (10).

(ii) Accordingly, non-abelian cohomology with coefficients in  $A$  (Def. 2.1) is equivalently the disjoint union, over the space of twists  $\tau_n$  (32) in non-abelian cohomology with coefficients in  $A(n-1)$ , of  $\tau$ -twisted non-abelian cohomology (Def. 2.29) with coefficients in  $K(\pi_n(A), n)$ :

$$\begin{array}{ccc} \text{non-abelian cohomology} & & \text{higher twisted} \\ \text{in higher degree} & \simeq & \text{ordinary cohomology} \\ H(X; A) & \simeq & \bigsqcup_{\substack{\tau_n \in H(X; A(n-1)) \\ \text{twist in} \\ \text{non-abelian cohomology} \\ \text{of lower degree}}} H^{\tau_n}(X; K(\pi_n(A), n)). \end{array} \quad (43)$$

(iii) Iterating this unravelling yields

$$\begin{array}{ccc} H(X; A) & \simeq & \bigsqcup_{\tau_n \in \bigsqcup_{\tau_{n-1} \in H(X; A(n-2))}} H^{\tau_n}(X; K(\pi_n(A), n)). \end{array} \quad (44)$$

and then

$$\begin{array}{ccc} H(X; A) & \simeq & \bigsqcup_{\tau_n \in \bigsqcup_{\tau_{n-1} \in \bigsqcup_{\tau_{n-2} \in H(X; A(n-3))}} H^{\tau_{n-1}}(X; K(\pi_{n-1}(A), n-1))} H^{\tau_n}(X; K(\pi_n(A), n)). \end{array} \quad (45)$$

and then

$$\begin{array}{ccc} H(X; A) & \simeq & \bigsqcup_{\tau_n \in \bigsqcup_{\tau_{n-1} \in \bigsqcup_{\tau_{n-2} \in \bigsqcup_{\tau_{n-3} \in H(X; A(n-4))}} H^{\tau_{n-2}}(X; K(\pi_{n-2}(A), n-2))} H^{\tau_{n-1}}(X; K(\pi_{n-1}(A), n-1))} H^{\tau_n}(X; K(\pi_n(A), n)). \end{array} \quad (46)$$

and so on.

(iv) Thus non-abelian cohomology in higher degrees (Example 2.7) decomposes as a tower of consecutively higher twisted but otherwise ordinary cohomology theories, starting with a twist in non-abelian cohomology in degree 1. This phenomenon has been called the *Whitehead principle of non-abelian cohomology* [To02, p. 8] and has been interpreted as saying that “nonabelian cohomology occurs essentially only in degree 1” [Si96, p. 1].

(v) But the above formulas (43), (44), (45) make manifest that this phenomenon has two perspectives. On the one hand: non-abelian cohomology in higher degrees may be *computed* by brute force as a sequence of consecutively higher twisted abelian cohomologies, with lowest twist starting in degree-1 non-abelian cohomology. On the other hand, conversely: intricate such systems of consecutively twisted abelian cohomology theories are neatly *understood* as unified by non-abelian cohomology.

(vi) Similarly, even though Postnikov towers do exist (Prop. A.39) in the classical homotopy category (Example A.34), the latter is far from being equivalent to the stable homotopy category (351) “up to twists in degree 1”.

In twisted generalization of Example 2.14, we have:

**Example 2.36** (Twisted topological K-theory). The classifying space  $KU_0 \simeq \mathbb{Z} \times BU$  (19) for complex topological K-theory (Example 2.14) is the fiber of a local coefficient bundle (31) over  $K(\mathbb{Z}, 3) \simeq B^3U(1)$  (16):

$$\begin{array}{ccc} KU_0 & \longrightarrow & KU_0 // BU(1) \\ & & \downarrow \\ & & B^2U(1) \end{array} \quad (47)$$

For  $X \xrightarrow{\tau} B^2U(1)$  a corresponding twist (32) (hence equivalently a bundle gerbe, by Example 2.11), the corresponding twisted non-abelian cohomology (Def. 2.29) is twisted complex topological K-theory [Ka68][DK70]:

$$\text{KU}^{\tau}(-) \simeq H^{\tau}(-; \mathbb{Z} \times BU). \quad (48)$$

This is manifest from comparing (33) with [FrHT08, (2.6)]. Alternatively, under Prop. 2.31, this is manifest from comparing the equivalent right hand side of (36) with [Ros89, Prop. 2.1] (using [NSS12a, Cor. 4.18]) or, more directly, with [AS04, §3][ABG10, §2.1].

Generally, in twisted generalization of Example 2.13, we have:

**Example 2.37** (Local coefficient bundle for twisted generalized cohomology). Let  $R$  be a suitable ring spectrum and write  $GL_R(1)$  for its  $\infty$ -group (as in Example 2.7) of units [Schl04, §2.3][MaSi04, §22.2][ABGHR08, §3][ABGHR14a, §2]. Its canonical action on the component space  $R_0 = \mathbb{R}\Omega^{\infty}R$  (352) is given, via Prop. 2.28, by a local coefficient bundle (31) of the form

$$\begin{array}{ccc} R_0 & \longrightarrow & (R_0) // GL_R(1) \\ & & \downarrow \rho_R \\ & & BGL_R(1). \end{array} \quad (49)$$

**Proposition 2.38** (Twisted non-abelian cohomology subsumes twisted generalized cohomology). *For  $R$  a suitable ring spectrum, the twisted non-abelian cohomology (Def. 2.29) with local coefficient bundle  $\rho_R$  from Example 2.37 is, equivalently, twisted generalized  $R$ -cohomology in the traditional sense (e.g. [MaSi04, §22.1]):*

$$\text{R}^{\tau}(-) \simeq H^{\tau}(-; \rho_R). \quad (50)$$

*Proof.* Given any twist  $X \xrightarrow{\tau} BGL_R(1)$  (2.29), write  $P \rightarrow X$  for the homotopy pullback (Def. A.24) along  $\tau$  of the essentially unique point inclusion:

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & \text{(hpb)} & \downarrow \\ X & \xrightarrow{\tau} & BGL_R(1) \end{array}, \quad \begin{array}{ccc} (P \times R_0) //_{\text{diag}} GL_R(1) \simeq E & \longrightarrow & R_0 // GL_R(1) \\ \mathbb{R}\tau^* \rho_R \downarrow & \text{(hpb)} & \downarrow \rho_R \\ X & \xrightarrow{\tau} & BGL_R(1) \end{array} \quad (51)$$

This  $P$  is the  $\mathrm{GL}_R(1)$ -principal  $\infty$ -bundle which is classified by  $\tau$ , [NSS12a, Thm. 3.17], to which the coefficient bundle  $E$  (34) is  $\mathrm{GL}_R(1)$ -associated [NSS12a, Prop. 4.6], as shown on the right of (51). Consider then the following sequence of natural bijections:

$$\begin{aligned} H^\tau(X; R_0) &\simeq \Gamma_X(E) \\ &\simeq \mathrm{Ho}(\mathrm{GL}_R(1)\mathrm{Actions})(P; R_0) \\ &\simeq \mathrm{Ho}(R\mathrm{Modules})(M\tau; R) \\ &\simeq R^\tau(X). \end{aligned} \tag{52}$$

Here the first step is Prop. 2.31, while the second step is [NSS12a, Cor. 4.18]. The third step is [ABGHR08, (2.15)][ABGHR14a, (3.15)], with  $M\tau$  denoting the  $R$ -Thom spectrum of  $\tau$  [ABGHR08, Def. 2.6][ABGHR14a, Def. 3.13]. The last step is [ABGHR08, §2.5] [ABGHR14a, §1.4][ABGHR14b, §2.7]. The composite of these natural bijections is the desired (50).  $\square$

In twisted generalization of Example 2.15, we have:

**Example 2.39** (Twisted iterated K-theory). Let  $r \in \mathbb{N}$ ,  $r \geq 1$ . By [LSW16, Prop. 1.5, Def. 1.7] and using Prop. 2.38, there is a local coefficient bundle (31) of the form

$$\begin{aligned} (K^{2r-2}(\mathrm{ku}))_0 &\longrightarrow \left( (K^{2r-2}(\mathrm{ku}))_0 \right) // B^{2r-1}\mathrm{U}(1) \\ &\quad \downarrow \rho_{\mathrm{lsw}2r-1} \\ &\quad B^{2r}\mathrm{U}(1), \end{aligned} \tag{53}$$

where  $K^{2r-2}(\mathrm{ku})_0$  is the 0th space in the spectrum (17) representing iterated K-theory (Example 2.15) and  $B^{2r}\mathrm{U}(1) \simeq K(\mathbb{Z}, 2r+1)$  is the classifying space for bundle  $(2r-1)$ -gerbes (Example 2.12), such that for  $X \xrightarrow{\tau} B^{2r}\mathrm{U}(1)$  a classifying map for such a higher gerbe, the  $\tau$  twisted non-abelian cohomology (Def. 2.29) with local coefficients in (53) is equivalently integrally twisted iterated K-theory according to [LSW16]:

$$\left( K^{\circ_{2r-1}}(\mathrm{ku}) \right)^{\tau}(-) \simeq H^\tau(-; K^{\circ_{2r-2}}(\mathrm{ku})_0).$$

In twisted generalization of Example 2.10, we have:

**Example 2.40** (J-Twisted Cohomotopy theory [FSS19b, §2.1]). For  $n \in \mathbb{N}$ , consider the canonical action of the orthogonal group  $\mathrm{O}(n+1)$  on the homotopy type of the  $n$ -sphere, via the defining action on the unit sphere in  $\mathbb{R}^{n+1}$ . By Prop. 2.28 this corresponds to a local coefficient bundle (31) for twisting Cohomotopy theory (Example 2.10):

$$\begin{aligned} S^n &\longrightarrow S^n // \mathrm{O}(n) \\ &\quad \downarrow \rho_J \\ &\quad \mathrm{BO}(n). \end{aligned} \tag{54}$$

The classifying map  $\mathrm{BO}(n) \xrightarrow{J} \mathrm{Aut}(S^n)$  of this fibration is the unstable  $J$ -homomorphism. For  $X$  a smooth manifold of dimension  $d \geq k+1$ , and equipped with tangential  $\mathrm{O}(k+1)$ -structure (e.g. [SS20b, Def. 4.48])

$$\begin{array}{ccc} X & \xrightarrow{\tau} & \mathrm{BO}(k+1) \\ & \swarrow TX & \searrow Bi \\ & & \mathrm{BO}(d) \end{array}$$

the  $\tau$ -twisted non-abelian Cohomology (Def. 2.29) with local coefficients in (54) is the  $J$ -twisted Cohomotopy theory of [FSS19b][FSS19c][SS20a]:

$$\pi^\tau(-) := H^\tau(-; S^n).$$

J-twisted Cohomotopy in degree four encodes, in particular, the shifted flux quantization condition of the C-field [FSS19b, Prop. 3.13] and the vanishing of the residual M5-brane anomaly [SS20a]; while J-twisted Cohomotopy in degree four encodes, in particular, level quantization of the Hopf-Wess-Zumino term on the M5-brane [FSS19c].



**Twisted non-abelian cohomology operations.** In generalization of Def. 2.17, we set:

**Definition 2.41** (Twisted non-abelian cohomology operation). Given a transformation of local coefficient bundles (31) presented (under localization (318) to homotopy types (339)) as a strictly commuting diagram

$$\begin{array}{ccc} A_1 // G_1 & \xrightarrow{\phi_r} & A_2 // G_2 \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ BG_1 & \xrightarrow{\phi_b} & BG_2 \end{array} \in \text{TopologicalSpaces}_{\text{Qu}}, \quad (55)$$

*pasting composition* induces<sup>3</sup> for each twist  $X \xrightarrow{\tau} BG_1$  (32) a map

$$\phi_* : H^\tau(X; A_1) \xrightarrow{(\phi_r \circ (-)) \circ (\rho_1)_*} H^{\phi_b \circ \tau}(X; A_2) \quad (56)$$

of twisted non-abelian cohomology sets (Def. 2.29). We call these *twisted non-abelian cohomology operations*.

**Example 2.42** (Total non-abelian class of twisted cocycles). For any coefficient bundle  $\rho$  (31) there is the tautological transformation (55) to its total space regarded as fibered over the point:

$$\begin{array}{ccc} A // G & \xlongequal{\quad} & A // G \\ \rho \downarrow & & \downarrow \\ BG & \xrightarrow{\quad} & * \end{array}$$

The induced twisted non-abelian cohomology operation (56) goes from twisted cohomology to non-twisted cohomology with coefficient in the total space:

$$H^\tau(X; A) \xrightarrow{\rho_*} H(X; A // G) \quad (57)$$

**Example 2.43** (Hopf cohomology operation in J-twisted Cohomotopy [FSS19b, §2.3]). The quaternionic Hopf fibration  $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$  is equivariant under the symplectic unitary group  $\text{Sp}(2) \simeq \text{Spin}(5)$ , so that after passage to classifying spaces it induces a morphism of local coefficient bundles (55) for J-twisted Cohomotopy (54) in degrees 4 and 7:

$$\begin{array}{ccc} S^7 // \text{Sp}(2) & \xrightarrow[\text{Borel-equivariantized quaternionic Hopf fibration}]{h_{\mathbb{H}} // \text{Sp}(2)} & S^4 // \text{Sp}(2) \\ J_7 \downarrow & & \downarrow J_4 \\ B\text{Sp}(2) & \xlongequal{\quad} & B\text{Sp}(2) \end{array} \quad (58)$$

Via (56) this induces for each Spin 8-manifold  $X$  equipped with tangential  $\text{Sp}(2)$ -structure (Example 2.33)

$$\begin{array}{ccc} X & \xrightarrow{\tau} & B\text{Sp}(2) \\ & \searrow TX & \swarrow Bi \\ & & BO(8) \end{array} \quad (59)$$

a twisted non-abelian cohomology operation (Def. 2.41)

$$\pi^{\tau_7}(X) \xrightarrow{(h_{\mathbb{H}} // \text{Sp}(2))_*} \pi^{\tau_4}(X) \quad (60)$$

in J-twisted non-abelian Cohomotopy theory (Example 2.40).

Lifting through the twisted non-abelian cohomology transformation (60) encodes vanishing of C-field flux up to C-field background charge [FSS19b, Prop. 3.14].

<sup>3</sup> We postpone discussing the details of forming pasting composites to §5, where they are provided by Lemma 5.1 with Def. 5.2.

**Example 2.44** (Twistorial Cohomotopy [FSS20, §3.2]). The equivariantized Hopf morphism (58) of coefficient bundles factors through Borel-equivariantizations of the complex Hopf fibration  $h_{\mathbb{C}}$  followed by that of the *twistor fibration*  $t_{\mathbb{H}}$

$$\begin{array}{ccccc}
S^7 // \mathrm{Sp}(2) & \xrightarrow[\text{\scriptsize } h_{\mathbb{C}} // \mathrm{Sp}(2)]{\text{\scriptsize Borel-equivariantized complex Hopf fibration}} & \mathbb{C}P^3 // \mathrm{Sp}(2) & \xrightarrow[\text{\scriptsize } t_{\mathbb{H}} // \mathrm{Sp}(2)]{\text{\scriptsize Borel-equivariantized twistor fibration}} & S^4 // \mathrm{Sp}(2) \\
\downarrow J_{S^7} & & \downarrow J_{\mathbb{C}P^3} & & \downarrow J_{S^4} \\
B\mathrm{Sp}(2) & \xlongequal{\hspace{10em}} & B\mathrm{Sp}(2) & \xlongequal{\hspace{10em}} & B\mathrm{Sp}(2)
\end{array} \tag{61}$$

The twisted non-abelian cohomology theory (Def. 2.29) with local coefficients in the bundle appearing in this factorization is the *Twistorial Cohomotopy* of [FSS20]

$$\mathcal{T}^{\tau}(-) := H^{\tau}(-; \mathbb{C}P^3).$$

Via (56) the morphisms (61) induce, for each spin 8-manifold  $X$  equipped with tangential  $\mathrm{Sp}(2)$ -structure (59), twisted non-abelian cohomology operations (Def. 2.41)

$$\begin{array}{ccccc}
\text{\scriptsize J-twisted} & & \text{\scriptsize Twistorial} & & \text{\scriptsize J-twisted} \\
\text{\scriptsize 7-Cohomotopy} & & \text{\scriptsize Cohomotopy} & & \text{\scriptsize 4-Cohomotopy} \\
\pi^{\tau^7}(X) & \xrightarrow[\text{\scriptsize } (h_{\mathbb{C}} // \mathrm{Sp}(2))_*]{} & \mathcal{T}^{\tau}(X)_s & \xrightarrow[\text{\scriptsize } (t_{\mathbb{H}} // \mathrm{Sp}(2))_*]{} & \pi^{\tau^4}(X)
\end{array} \tag{62}$$

between J-twisted non-abelian Cohomotopy theory (Example 2.40) and Twistorial Cohomotopy.

We turn to the differential refinement of this statement in §5.3 below.

### 3 Non-abelian de Rham cohomology

We formulate (twisted) non-abelian de Rham cohomology (Def. 3.84, Def. 3.98) of differential forms with values in  $L_\infty$ -algebras (Example 3.25) and prove the (twisted) non-abelian de Rham theorem (Theorem 3.87, Theorem 3.104), as a consequence of the fundamental theorem of dg-algebraic rational homotopy theory, which we recall (Prop. 3.60).

#### 3.1 Dgc-Algebras and $L_\infty$ -algebras

Here we fix notation and conventions for the following system of categories and functors:

$$\begin{array}{ccccc}
 \begin{array}{c} \text{Def. 3.34} \\ (L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}})^{\text{op}} \end{array} & \xrightarrow[\simeq]{(104) \text{ CE}} & \begin{array}{c} \text{Def. 3.31} \\ \text{SullivanModels}_{\mathbb{R}}^{\geq 1} \end{array} & & \\
 \downarrow \text{Def. 3.25} & & \downarrow \text{Def. 3.17} & & \\
 \begin{array}{c} (L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0})^{\text{op}} \end{array} & \xrightarrow{(88) \text{ CE}} & \begin{array}{c} \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \\ \downarrow \text{Def. 3.18} \\ \text{GrddCmmttvAlgbr} \\ \downarrow \text{Def. 3.8} \\ \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \end{array} & \begin{array}{c} \xleftarrow[\perp]{\text{Def. 3.19} \text{ Sym}} \\ \xleftarrow[\perp]{\text{Def. 3.10} \text{ Sym}} \end{array} & \begin{array}{c} \text{Def. 3.14} \\ \text{CochainComplexes}_{\mathbb{R}}^{\geq 0} \\ \downarrow \text{Def. 3.15} \\ \text{GrddVctrSpc} \\ \downarrow \text{Def. 3.2} \\ \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \end{array} \\
 & & & & \\
 & & & & (63)
 \end{array}$$

**Remark 3.1** (Homotopical grading). Our grading conventions, to be detailed in the following, are strictly *homotopy theoretic*:

- (i) Any graded-algebraic object discussed here, corresponds, under the equivalences of rational homotopy theory laid out in §3.2 below, to a rational space, such that algebraic generators in degree  $n$  correspond to homotopy groups in the same degree  $n$ . Since homotopy groups of spaces are in non-negative degree  $n \in \mathbb{N}$ , all dg-algebraic objects discussed here are concentrated in non-negative degree, hence are *connective*.
- (ii) In particular, our  $L_\infty$ -algebras are in non-negative degree, naturally accommodating (as in [LM95][BFM06, §2.9]) the rationalized Whitehead homotopy Lie algebras  $\pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{R}$  of connected spaces  $X$ , with their natural non-negative grading induced from that of the homotopy groups of  $\Omega X$ . See Prop. 3.63 and Prop. 3.65 below.

#### Graded vector spaces.

**Definition 3.2** (Connective graded vector spaces). (i) We write

$$\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \in \text{Categories} \quad (64)$$

for the category whose objects are  $\mathbb{N}$ -graded (i.e. non-negatively  $\mathbb{Z}$ -graded) vector spaces over the real numbers; and we write

$$\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \text{fin}} \hookrightarrow \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \in \text{Categories} \quad (65)$$

for its full subcategory on those objects which are of *finite type*, namely degree-wise finite-dimensional.

(ii) For  $V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$  and  $k \in \mathbb{N}$  we write

$$V^k \in \text{VectorSpaces}_{\mathbb{R}}$$

for the component vector space in degree  $k$ .

**Example 3.3** (The zero-object in graded vector spaces). We write

$$0 \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \quad (66)$$

for the graded vector space which is the zero vector space in each degree. This is both the initial as well as the terminal object (hence the zero object) in  $\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$ .

**Example 3.4** (Graded linear basis). For  $n_1, n_2, \dots, n_k \in \mathbb{N}$  a finite sequence of non-negative integers, we write

$$\langle \alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_k} \rangle \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \text{fin}}$$

for the graded vector space (Def. 3.2) spanned by elements  $\alpha_{n_i}$  in degree  $n_i$ , respectively.

**Definition 3.5** (Tensor product of graded vector spaces). The category of  $\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$  (Def. 3.2) becomes a symmetric monoidal category under the graded tensor product given by

$$(V \otimes W)^k := \bigoplus_{n_1+n_2=k} V^{n_1} \otimes W^{n_2}.$$

and the symmetric braiding isomorphism given by

$$\begin{array}{ccc} V \otimes W & \xrightarrow[\simeq]{\sigma^{V,W}} & W \otimes V \\ \uparrow \text{J} & & \uparrow \text{J} \\ V^{n_1} \otimes W^{n_2} & \xrightarrow[\simeq]{\sigma_{n_1, n_2}^{V,W}} & W^{n_2} \otimes V^{n_1} \\ \cup & & \cup \\ (v, w) & \mapsto & (-1)^{n_1 n_2} \cdot (w, v) \end{array} \quad (67)$$

We denote this by

$$(\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma) \in \text{SymmetricMonoidalCategories}. \quad (68)$$

**Definition 3.6** (Degreewise linear dual). For  $V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \text{fin}}$  (Def. 3.2) we write

$$V^\vee \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \text{fin}}$$

for its degree-wise linear dual:<sup>4</sup>

$$(V^\vee)^k := (V^k)^*. \quad (69)$$

**Definition 3.7** (Degree shift). For  $V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$  (Def. 3.2) we write

$$\mathfrak{b}V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\leq 0} \quad (70)$$

for the result of shifting degrees up by 1:

$$(\mathfrak{b}V)^k := \begin{cases} V^{k-1} & | \quad k \geq 1, \\ 0 & | \quad k = 0. \end{cases}$$

### Graded-commutative algebras.

**Definition 3.8** (Graded-commutative algebras). We write

$$\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} := \text{CommMonoids}(\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma) \in \text{Categories} \quad (71)$$

for the category whose objects are non-negatively  $\mathbb{Z}$ -graded, graded-commutative unital algebras over the real numbers (hence commutative unital monoids with respect to the braided tensor product of Def. 3.5); and we write

$$\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \text{fin}} \hookrightarrow \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \in \text{Categories} \quad (72)$$

for its full sub-category in those objects which are of *finite type*, namely degree-wise finite dimensional.

**Definition 3.9** (Underlying graded vector space). We write

$$\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \xrightarrow{\text{GrddVctrSpc}} \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \quad (73)$$

for the functor on graded algebras (Def. 3.8) that forgets the algebra structure and remembers only the underlying graded vector space (Def. 3.2).

**Example 3.10** (Free graded-commutative algebras). For  $V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$  (Def. 3.2), we write

$$\text{Sym}(V) \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (74)$$

for the graded-commutative algebra (Def. 3.8) freely generated by  $V$ , hence that whose underlying graded vector space (73) is

$$\text{GrddVctrSpc}(\text{Sym}(V)) = \mathbb{R} \oplus V \oplus (V \otimes V)_{/\text{Sym}(2)} \oplus (V \otimes V \otimes V)_{/\text{Sym}(3)} \oplus \dots,$$

where the symmetric groups  $\text{Sym}(n)$  act via the braiding (67).

<sup>4</sup> This is in contrast to the intrinsic duality  $(-)^*$  in the monoidal category of graded vector spaces in *unbounded* degree (not considered here), which instead goes along with inversion of the degree:  $(V^*)^k = (V^{-k})^*$ .

**Example 3.11** (Graded Grassmann algebra). For  $V \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$  (Def. 3.2), we write

$$\wedge^{\bullet} V := \text{Sym}(\mathfrak{b}V) \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the free graded-commutative algebra (Def. 3.10) on  $V$  shifted up in degree (Def. 3.7); and we call this the *graded Grassmann-algebra* on  $V$ .

**Example 3.12** (Graded polynomial algebra). For  $n_1, n_2, \dots, n_k \in \mathbb{N}$  a finite sequence of non-negative integers, we write

$$\mathbb{R}[\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_k}] := \text{Sym}(\langle \alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_k} \rangle) \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \text{fin}}$$

for the free graded-commutative algebras (Def. 3.10) the graded vector space spanned by the  $\alpha_{n_i}$  (Def. 3.4).

**Remark 3.13** (Incarnations of Grassmann algebras). With these notation conventions from Examples 3.10, 3.11, 3.12, an ordinary Grassmann algebra on  $k$  generators is equivalently:

$$\wedge^{\bullet}(\mathbb{R}^k) = \text{Sym}(\mathfrak{b}\mathbb{R}^k) = \mathbb{R}[\theta_1^{(1)}, \theta_1^{(2)}, \dots, \theta_1^{(k)}].$$

### Cochain complexes.

**Definition 3.14** (Connective cochain complexes). We write

$$\text{CochainComplexes}_{\mathbb{R}}^{\geq 0} \in \text{Categories}$$

for the category of cochain complexes (i.e. with differential of degree +1) of real vector spaces in non-negative degree.

**Definition 3.15** (Underlying graded vector space). We write

$$\text{CochainComplexes}_{\mathbb{R}}^{\geq 0} \xrightarrow{\text{GrddVctrSpc}} \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} \quad (75)$$

for the forgetful functor on connective cochain complex (Def. 3.14) which forgets the differential and remembers only the underlying connective graded vector space (Def. 3.2).

**Definition 3.16** (Tensor product on cochain complexes). The tensor product and braiding of graded vector spaces from Def. 3.5 lifts, through (75), to a tensor product and braiding on  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.14), making it a symmetric monoidal category:

$$(\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma) \in \text{SymmetricMonoidalCategories}. \quad (76)$$

### Differential graded commutative algebras.

**Definition 3.17** (Connective differential graded commutative algebras [GM96, V.3.1]). We write

$$\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} := \text{CommMonoids}(\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}, \otimes, \sigma) \in \text{Categories}$$

for the category whose objects are differential-graded, graded-commutative, unital algebras over the real numbers concentrated in non-negative degrees (hence commutative unital monoids in the symmetric monoidal category of Def. 3.16).

**Definition 3.18** (Underlying graded-commutative algebra). We write

$$\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \xrightarrow{\text{GrddCmmstvAlgbr}} \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (77)$$

for the functor on dgc-algebras (Def. 3.17) that forgets the differential and remembers only the underlying graded-commutative algebra (Def. 3.8).

**Definition 3.19** (Free differential graded algebras). For  $V^{\bullet}$  in  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.14) we write

$$\text{Sym}(V^{\bullet}) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the free differential graded-commutative algebra on  $V^{\bullet}$ , (Def. 3.17), hence whose underlying graded-commutative algebra algebra (77) is as in Example 3.10.

**Example 3.20** (Initial algebra). The real algebra of real numbers, regarded as concentrated in degree-0

$$\mathbb{R} \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \hookrightarrow \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

is the *initial* object: For any other  $A \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 71) or  $\in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) there is a unique morphism

$$\mathbb{R} \xrightarrow{i_{\mathbb{R}}} A$$

(because our algebras are unital and homomorphisms need to preserve the unit element).

**Example 3.21** (The terminal algebra). We write

$$0 \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \hookrightarrow \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (78)$$

for the unique graded-commutative algebra (Def. 3.8) or dgc-algebra (Def. 3.17) whose underlying graded vector space (Def. 3.9) is the zero-vector space<sup>5</sup> (66). This is the terminal object<sup>6</sup> in  $\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ : For every  $A \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ , there is a unique morphism

$$A \xrightarrow{\exists!} 0 .$$

**Example 3.22** (Product and co-product algebras). In the categories  $\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.8) and  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17):

(i) the coproduct is given by the tensor product (Def. 3.5),

(ii) the product is given by the direct sum

on underlying graded vector spaces (Def. 3.9).

(The first follows by [Joh02, p. 478, Cor. 1.1.9], while the second holds since (73) is a right adjoint.)

**Example 3.23** (Smooth de Rham complex (e.g. [BT82])). For  $X$  be a smooth manifold, its de Rham algebra of smooth differential forms is a dgc-algebra in the sense of Def. 3.17, to be denoted here:

$$\Omega_{\text{dR}}^{\bullet}(X) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} .$$

**Example 3.24** (Chevalley-Eilenberg algebras of Lie algebras). For  $(\mathfrak{g}, [-, -])$  a finite-dimensional real Lie algebra, its Chevalley-Eilenberg algebra is a dgc-algebra (Def. 3.17):

$$\text{CE}(\mathfrak{g}) := (\wedge^{\bullet} \mathfrak{g}^*, d|_{\wedge^1 \mathfrak{g}^*} = [-, -]^*) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

with underlying graded-commutative algebra (Def. 3.8) the Grassmann algebra on the linear dual space  $\mathfrak{g}^*$  (Def. 3.11, Remark 3.13), and with differential given on  $\wedge^1 \mathfrak{g}^*$  by the linear dual of the Lie bracket. More explicitly, for  $\{v_a\}_{a=1}^{\dim_{\mathbb{R}}(\mathfrak{g})}$  a linear basis for the underlying vector space of the Lie algebra

$$\mathfrak{g} \simeq \langle v_1, v_2, \dots, v_{\dim(\mathfrak{g})} \rangle, \quad (79)$$

with Lie brackets

$$[v_a, v_b] = f_{ab}^c v_c, \quad \text{for structure constants } f_{ab}^c \in s\mathbb{R} \quad (80)$$

we have

$$\text{CE}(\mathfrak{g}) \simeq \mathbb{R}[\theta_1^{(1)}, \theta_1^{(2)}, \dots, \theta_1^{(\dim(\mathfrak{g}))}] / (d\theta_1^{(c)} = f_{ab}^c \theta_1^{(b)} \wedge \theta_1^{(a)}). \quad (81)$$

One observes that the Jacobi identity on  $[-, -]$  is equivalent to the condition that the differential  $d := [-, -]^*$  squares to zero, so that (81) being a dgc-algebra is actually equivalent to  $(\mathfrak{g}, [-, -])$  being a Lie algebra.

This construction is evidently contravariantly functorial and constitutes a full subcategory inclusion

$$\text{LieAlgebras}_{\mathbb{R}, \text{fin}} \xrightarrow{\text{CE}} (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}}, \quad (82)$$

meaning that, in addition, homomorphisms of Lie algebras are in natural bijection to dgc-algebra morphisms between their CE-algebras.

<sup>5</sup> Notice that the algebra 0 (78) is indeed a unital algebra (71).

<sup>6</sup> Beware that the corresponding statement in [GM96, p. 335] is incorrect.

## $L_\infty$ -algebras.

**Definition 3.25** (Chevalley-Eilenberg algebras of  $L_\infty$ -algebras [LM95, Thm . 2.3][SSS09a, Def. 13][BFM06, §2]). In direct generalization of (82), consider those  $A \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) whose underlying graded-commutative algebra (77) is free (Example 3.10, Remark 3.13) on the degreewise dual  $\mathfrak{b}\mathfrak{g}^\vee$  (Def. 3.6) of the degree shift  $\mathfrak{b}\mathfrak{g}$  (Def. 3.7) of some connective finite-type graded vector space (Def. 3.2)

$$\mathfrak{g} \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0, \text{fin}} \quad (83)$$

in that

$$A := (\wedge^\bullet \mathfrak{g}^\vee, d) := (\text{Sym}(\mathfrak{b}\mathfrak{g}^\vee), d) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}. \quad (84)$$

In this case the differential  $d$  restricted to  $\wedge^1 \mathfrak{g}^\vee$  defines, under linear dualization, a sequence of  $n$ -ary graded-symmetric multilinear maps  $\{-, \dots, -\}$  on  $\mathfrak{g}$ :

$$\begin{aligned} d|_{\wedge^1 \mathfrak{g}^\vee}(-) &= \{-\}^* + \{-, -\}^* + \{-, -, -\}^* + \dots \\ \wedge^1 \mathfrak{g}^\vee &\xrightarrow{d} \wedge^1 \mathfrak{g}^\vee \oplus \wedge^2 \mathfrak{g}^\vee \oplus \wedge^3 \mathfrak{g}^\vee \oplus \dots = \wedge^\bullet \mathfrak{g}^\vee = \text{Sym}(\mathfrak{b}\mathfrak{g}^\vee), \end{aligned} \quad (85)$$

and the condition  $d \circ d = 0$  imposes a sequence of compatibility conditions on these brackets, generalizing the Jacobi identity in Example 3.24. The corresponding graded skew-symmetric  $n$ -ary brackets ([LS93, (3)])

$$[a_1, \dots, a_n] := (-1)^{n + \sum_{i \leq n/2} \deg(a_i)} \{a_1, \dots, a_n\} \quad (86)$$

subject to these conditions give  $\mathfrak{g}$  the structure of an  $L_\infty$ -algebra (or *strong homotopy Lie algebra*):

$$\left( \mathfrak{g}, [-], [-, -], [-, -, -], \dots \right) \in L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}, \quad (87)$$

which makes  $A$  in (84) its Chevalley-Eilenberg algebra:

$$\begin{aligned} \text{CE}(\mathfrak{g}) &:= (\wedge^\bullet \mathfrak{g}^\vee, d = \{-\}^* + \{-, -\}^* + \{-, -, -\}^* + \dots) \\ &= (\text{Sym}(\mathfrak{b}\mathfrak{g}^\vee), d_{\text{CE}}). \end{aligned} \quad (88)$$

This construction constitutes a full subcategory inclusion

$$L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0} \xrightarrow{\text{CE}} (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}}. \quad (89)$$

of the category of connective finite-type  $L_\infty$ -algebras into that of connective dgc-algebras.

**Example 3.26** (Differential graded Lie algebras). A differential graded Lie algebra is an  $L_\infty$ -algebra (87) whose only possibly non-vanishing brackets are the unary bracket  $\partial := [-]$  (its differential) and the binary bracket  $[-, -]$  (its graded Lie bracket). In further specialization, a plain Lie algebra (Example 3.24) is an  $L_\infty$ -algebra/dg-Lie algebra concentrated in degree 0:

$$\text{LieAlgebras}_{\mathbb{R}, \text{fin}} \hookrightarrow \text{DiffGradedLieAlgebras}_{\mathbb{R}, \text{fin}}^{\geq 0} \hookrightarrow L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}. \quad (90)$$

**Example 3.27** (Line Lie  $n$ -algebra). For  $n \in \mathbb{N}$  we say that the *line Lie  $(n+1)$ -algebra* is the  $L_\infty$ -algebra (Def. 3.25)

$$\mathfrak{b}^n \mathbb{R} \in L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0} \quad (91)$$

whose Chevalley-Eilenberg algebra (88) is the polynomial dgc-algebra (Example 3.29) on a single closed generator in degree  $n+1$ :

$$\text{CE}(\mathfrak{b}^n \mathbb{R}) := \mathbb{R}[c_{n+1}] / (d c_{n+1} = 0). \quad (92)$$

**Example 3.28** (String Lie 2-algebra [BCSS07, §5][He08, §1.2][FSS12a, App.]). Let  $\mathfrak{g} \in \text{LieAlgebras}_{\mathbb{R}, \text{fin}}$  be semisimple (such as  $\mathfrak{g} = \mathfrak{su}(n+1), \mathfrak{so}(n+3)$ , for  $n \in \mathbb{N}$ ), hence equipped with a non-degenerate, symmetric,  $\mathfrak{g}$ -invariant bilinear form (“Killing form”)

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\langle -, - \rangle} \mathbb{R}. \quad (93)$$

Then the element

$$\mu := \langle -, [-, -] \rangle \in \text{CE}(\mathfrak{g})$$

in the Chevalley-Eilenberg (3.24) is closed (is a Lie algebra cocycle)

$$d\mu = 0.$$

In terms of a linear basis  $\{v_a\}$  (79) with structure constants  $\{f_{ab}^c\}$  (80) and inner product  $k_{ab} := \langle v_a, v_b \rangle$  we have, in terms of (81):

$$\mu := f_{ab}^c k_{c'c} \theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)}.$$

Hence we get an  $L_\infty$ -algebra (Def. 3.25)

$$\text{string}_{\mathfrak{g}} \in L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0} \quad (94)$$

with the following Chevalley-Eilenberg algebra (88):

$$\text{CE}(\text{string}_{\mathfrak{g}}) := \mathbb{R} \left[ \begin{array}{c} \{\theta_1^a\}, \\ b_2 \end{array} \right] / \left( \begin{array}{c} d\theta_1^{(c)} = f_{ab}^c \theta_1^{(b)} \wedge \theta_1^{(a)} \\ db_2 = \underbrace{f_{ab}^c k_{c'c} \theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)}}_{=\mu} \end{array} \right). \quad (95)$$

This is known as the *string Lie 2-algebra*.

### Sullivan models and nilpotent $L_\infty$ -algebras.

**Example 3.29** (Polynomial dgc-algebras). For  $A \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17), and

$$\mu \in A^{n+1} \subset A, \quad d\mu = 0 \quad (96)$$

a closed element of homogeneous degree  $n+1$ , we write

$$A[\alpha_n] / (d\alpha_n = \mu) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (97)$$

for the dgc-algebra obtained by adjoining a generator  $\alpha_n$  of degree  $n$  to the underlying graded-commutative algebra (77) of  $A$  and extending the differential from  $A$  to  $A[\alpha_n]$  by taking its value on the new generator to be  $\mu$ . The polynomial dgc-algebra (97) receives a canonical algebra inclusion of  $A$  (the unique  $A$ -algebra homomorphism):

$$A \xrightarrow{i_A} A[\alpha_n] / (d\alpha_n = \mu). \quad (98)$$

**Example 3.30** (Multivariate polynomial dgc-algebras). Let  $A \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17),  $\mu^{(1)} \in A^{n_1+1}$ ,  $d\mu^{(1)} = 0$ , with corresponding polynomial dgc-algebra (97) as in Example 3.29. Then, for

$$\mu^{(2)} \in A[\alpha_{n_1}^{(1)}] / (d\alpha_{n_1}^{(1)} = \mu^{(1)}), \quad d\mu^{(2)} = 0$$

another closed element of some homogeneous degree  $n_2+1$ , in the new algebra (97) we may iterate the construction of Example 3.29 to obtain the bivariate polynomial dgc-algebra over  $A$ , to be denoted:

$$A \left[ \begin{array}{c} \alpha_{n_2+1}^{(2)} \\ \alpha_{n_1+1}^{(1)} \end{array} \right] / \left( \begin{array}{c} d\alpha_{n_1}^{(2)} = \mu_{n_2+1}^{(2)} \\ d\alpha_{n_1}^{(1)} = \mu_{n_1+1}^{(1)} \end{array} \right) := \left( A[\mu_{n_1+1}^{(1)}] / (d\alpha_{n_1+1}^{(1)} = \mu^{(1)}) \right) [\alpha^{(2)}] / (d\alpha_{n_2+1}^{(2)} = \mu^{(2)}).$$

Iterating further, we have multivariate polynomial dgc-algebras over  $A$ , to be denoted as follows:

$$A \left[ \begin{array}{c} \alpha_{n_k+1}^{(k)} \\ \vdots \\ \alpha_{n_2+1}^{(2)} \\ \alpha_{n_1+1}^{(1)} \end{array} \right] / \left( \begin{array}{c} d\alpha_{n_k}^{(k)} = \mu^{(k)} \\ \vdots \\ d\alpha_{n_1}^{(2)} = \mu^{(2)} \\ d\alpha_{n_1}^{(1)} = \mu^{(1)} \end{array} \right) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (99)$$



with

$$\mu^r \in A \begin{bmatrix} \alpha_{n_{r-1}+1}^{(r-1)} \\ \vdots \\ \alpha_{n_1+1}^{(1)} \end{bmatrix}, \quad \text{for } 1 \leq r \leq k.$$

These multivariate polynomial algebras (99) receive the canonical inclusion (98) of  $A$ :

$$A \xrightarrow{i_A} A \begin{bmatrix} \alpha_{n_k+1}^{(k)} \\ \vdots \\ \alpha_{n_2+1}^{(2)} \\ \alpha_{n_1+1}^{(1)} \end{bmatrix} / \begin{pmatrix} d \alpha_{n_k}^{(k)} = \mu^{(k)} \\ \vdots \\ d \alpha_{n_1}^{(1)} = \mu^{(1)} \end{pmatrix}, \quad (100)$$

these being the composites of the stage-wise inclusions (98).

**Definition 3.31** (Semifree dgc-Algebras/Sullivan models/FDAs). The multivariate polynomial dgc-algebras of Example 3.30 are sometimes called **(i)** *semi-free dgc-algebras* over  $A$  (since their underlying graded-commutative algebra (77) is free, as in Example 3.10), but they are traditionally known **(ii)** in rational homotopy theory as *relative Sullivan models* (due to [Su77], review in [FHT00, II][Me13][FH17]), or, **(iii)** in supergravity theory (following [vN82][D'AF82]), as *FDAs*<sup>7</sup> [CDF91], (for translation see [FSS13b][FSS16a][FSS16b][HSS18][BMSS19][FSS19a]). Here we write:

$$\text{SullivanModels}_{\mathbb{R}}^{\geq 1} \hookrightarrow \text{SullivanModels}_{\mathbb{R}} \hookrightarrow \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (101)$$

for, from right to left, **(a)** the full subcategory of connective dgc-algebras (Def. 3.17) on those which are isomorphic to a multivariate polynomial dgc-algebra over  $\mathbb{R}$ , as in Example 3.30 (i.e., the ordering of the generators in (99) is not part of the data of a Sullivan model, only the resulting dgc-algebra); and **(b)** for the further full subcategory on those Sullivan model that are generated in positive degree  $\geq 1$ .

**Example 3.32** (Polynomial dgc-algebras as pushouts). For  $A \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) the polynomial dgc-algebras over  $A$  (Def. 3.29) are pushouts in  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  of the following form:

$$\begin{array}{ccc} A[\alpha_n]/(d\alpha_n = \mu) & \xleftarrow[\mu \leftarrow c_{n+1}]{\alpha_n \leftarrow \alpha_n} & \mathbb{R}[\alpha_n, c_{n+1}]/\left(\begin{array}{l} d\alpha_n = c_{n+1} \\ d c_{n+1} = 0 \end{array}\right) \\ \uparrow i_A & & \uparrow c_{n+1} \\ & \text{(po)} & \uparrow c_{n+1} \\ A & \xleftarrow[\mu \leftarrow c_{n+1}]{} & \mathbb{R}[c_{n+1}]/(d c_{n+1} = 0) \end{array} \quad (102)$$

Here on the right we have multivariate polynomial dgc-algebras (Example 3.30) over  $\mathbb{R}$  (Example 3.20) as shown. The horizontal morphisms encode the choice of  $\mu \in A$  (96) and the left vertical morphism is the canonical inclusion (98).

**Example 3.33** (Chevalley-Eilenberg algebras of nilpotent Lie algebras). Beware that not every Lie algebra  $\mathfrak{g}$  has Chevalley-Eilenberg algebra (Example 3.24) which satisfies the stratification in the Definition 3.30 of multivariate polynomial dg-algebras.

**(i)** For instance, the Lie algebra  $\mathfrak{su}(2)$  has

$$\text{CE}(\mathfrak{su}(2)) = \mathbb{R}[\theta_1, \theta_2, \theta_3] / \left( d\theta_i = \sum_{j,k} \varepsilon_{ijk} \theta_j \wedge \theta_k \right)$$

and no ordering of  $\{1, 2, 3\}$  brings this into the iterative form required in (99).

**(ii)** Instead, those Lie algebras whose CE-algebra is of the form (99) are precisely the nilpotent Lie algebras.

<sup>7</sup> Beware that ‘‘FDA’’ in the supergravity literature is meant to be short-hand for ‘‘free differential algebra’’, which is misleading, because what is really meant are not free dgc-algebras as in Example 3.19 (in general) but just ‘‘semi-free’’ dgc-algebras, only whose underlying graded-commutative algebras (77) is required to be free (Example 3.10).

In generalization of Example 3.33, we say (see also [Ber15, Theorem 2.3]):

**Definition 3.34** (Nilpotent  $L_\infty$ -algebras). An  $L_\infty$ -algebra (87) is *nilpotent* if its CE-algebra (Def. 3.25) is a multivariate polynomial dgc-algebra (Example 3.30), hence is in the sub-category of  $\text{SullivanModels}_{\mathbb{R}}$  (101):

$$\begin{array}{ccc} L_\infty\text{Algebras}_{\mathbb{R},\text{fin}}^{\geq 0,\text{nil}} & \xrightarrow{\text{CE}} & (\text{SullivanModels})^{\text{op}} \\ \downarrow & \text{(pb)} & \downarrow \\ L_\infty\text{Algebras}_{\mathbb{R},\text{fin}}^{\geq 0} & \xrightarrow{\text{CE}} & (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}} \end{array} \quad (103)$$

In fact, from (84) it is clear that every connected Sullivan model, hence with generators in degrees  $\geq 1$ , is the Chevalley-Eilenberg algebra of a unique nilpotent  $L_\infty$ -algebra, so that the defining inclusion at the top of (103) further restricts to an equivalence of homotopy categories:

$$L_\infty\text{Algebras}_{\mathbb{R},\text{fin}}^{\geq 0,\text{nil}} \xrightarrow[\simeq]{\text{CE}} (\text{SullivanModels}_{\mathbb{R}}^{\geq 1})^{\text{op}}. \quad (104)$$

**Homotopy theory of connective dgc-Algebras.** We recall the homotopy theory of connective differential graded-commutative algebras. We make free use of the language of model categories [Qu67]; for review see [Ho99][Lu09, A.2] and appendix A.

**Definition 3.35** (Homotopical structure on connective dgc-algebras [BG76, §4.2][GM96, §V.3.4]). Consider the following sub-classes of morphisms in the category of  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17):

- (i) *W – weak equivalences* are the quasi-isomorphisms;
- (ii) *Fib – fibrations* are the degreewise surjections;

We call this the *projective* homotopical structure on  $\text{dgcAlgebras}_{\mathbb{R}}^{\geq 0}$ .

**Proposition 3.36** (Projective model structure connective on dgc-algebras [BG76, §4.3][GM96, V.3.4]). *Equipped with the projective homotopical structure from Def. 3.35, the category of  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) becomes a model category (Def. A.3). We denote this as:*

$$(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}} \in \text{ModelCategories}. \quad (105)$$

**Remark 3.37** (All dgc-algebras are projectively fibrant). Every object  $A \in (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (105) is fibrant: By Example 3.21 the terminal morphism is to the 0-algebra, and this is clearly surjective, hence is a fibration, by Def. 3.35:

$$A \xrightarrow{\in \text{Fib}} 0.$$

**Cofibrant dgc-algebras.** In order to identify cofibrant dgc-algebras, it is useful to first consider the following:

**Definition 3.38** (Homotopical structure on connective cochain complexes). Consider the following sub-classes of morphisms in the category  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.14):

- (i) *W – weak equivalences* are the quasi-isomorphisms;
- (ii) *Fib – fibrations* are the degreewise surjections;
- (iii) *Cof – cofibrations* are the injections in positive degrees.

We call this the *injective* homotopical structure on  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$ .

**Proposition 3.39** (Injective model structure on connective cochain complexes [He07, p. 6]). *Equipped with the injective homotopical structure of Def. 3.38 the category of  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.14) becomes a model category (Def. A.3). We denote this:*

$$(\text{CochainComplexes}_{\mathbb{R}}^{\geq 0})_{\text{inj}} \in \text{ModelCategories}.$$

*Proof.* This is formally dual to the proof of the projective model structure on connective chain complexes [Qu67, II.4][GoS06, Thm. 1.5]; see, for instance, [Dun10, Thm. 2.4.5].  $\square$

**Remark 3.40** (Other model categories of chain complexes). Prop. 3.39 is usually stated in the generality of cochain complexes of abelian groups, in which case the fibrations are only those degreewise surjections that have degreewise injective kernel, a condition that becomes trivial for abelian groups that are vector spaces.

**Proposition 3.41** (Quillen adjunction between dgc-algebras and cochain complexes). *The adjunction (63) between  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) and  $\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.14) is a Quillen adjunction (Def. A.18) with respect to the model category structures from Def. 3.39 and that from Def. 3.36.*

$$\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}} \begin{array}{c} \xleftarrow{\text{Sym}} \\ \xrightarrow{\perp_{\text{Qu}}} \end{array} \left(\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}\right)_{\text{inj}}.$$

*Proof.* It is immediate from the definitions that the forgetful right adjoint preserves the classes W and Fib.  $\square$

**Lemma 3.42** (Generating cofibrations). *The following inclusions of multivariate polynomial dgc-algebras (Example 3.30) are cofibrations in  $\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}$  (Def. 3.36)*

$$\mathbb{R}[c_{n+1}]/(dc_{n+1} = 0) \xrightarrow[\in \text{Cof}]{c_{n+1} \mapsto c_{n-1}} \mathbb{R} \left[ \begin{array}{c} \alpha_n, \\ c_{n+1} \end{array} \right] / \left( \begin{array}{c} d\alpha_n = c_{n+1}, \\ dc_{n+1} = 0 \end{array} \right) \quad \text{for } n \in \mathbb{N}. \quad (106)$$

*Proof.* Consider the following morphisms of cochain complexes, for  $n \in \mathbb{N}$ :

$$\begin{array}{c} \left[ \begin{array}{c} \vdots \\ 0 \\ \uparrow d \\ 0 \\ \uparrow d \\ \langle c_{n+1} \rangle \\ \uparrow d \\ 0 \\ \uparrow d \\ 0 \\ \uparrow d \\ \vdots \\ \uparrow d \\ 0 \end{array} \right] \xrightarrow{i_n} \left[ \begin{array}{c} \vdots \\ 0 \\ \uparrow d \\ 0 \\ \uparrow d \\ \langle c_{n+1} \rangle \\ \uparrow d \\ \langle \alpha_n \rangle \\ \uparrow d \\ 0 \\ \uparrow d \\ \vdots \\ \uparrow d \\ 0 \end{array} \right] \quad \text{with } d\alpha_n = c_{n+1}. \quad (107)$$

Since these are injections, they are cofibrations in  $\left(\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}\right)_{\text{inj}}$  (Prop. 3.39), by Def. 3.38. Thus also their images under Sym (Def. 3.19) are cofibrations in  $\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}$  (Prop. 3.36) because Sym is a left Quillen functor, by Prop. 3.41. But  $\text{Sym}(i_n)$  manifestly equals (106), and so the claim follows.  $\square$

**Proposition 3.43** (Relative Sullivan algebras are cofibrations). *For a multivariate polynomial dgc-algebra from Example 3.30, the canonical inclusion (108) of the base algebra is a cofibration in  $\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}$  (Prop. 3.36):*

$$A \xrightarrow[\text{ } i_A]{\in \text{Cof}} A \left[ \begin{array}{c} \alpha_{n_k+1}^{(k)}, \\ \vdots \\ \alpha_{n_1+1}^{(1)} \end{array} \right] / \left( \begin{array}{c} d\alpha_{n_k}^{(k)} = \mu^{(k)}, \\ \vdots \\ d\alpha_{n_1}^{(1)} = \mu^{(1)} \end{array} \right). \quad (108)$$

*In particular, since  $\mathbb{R} \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  is the initial object (Example 3.20), all multivariate polynomial dgc-algebras over  $\mathbb{R}$  (the Sullivan models, Def. 3.31) are cofibrant objects in  $\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}$ .*

*Proof.* By Lemma 3.42, the right vertical morphisms in the pushout diagram (102) are cofibrations. Since the class of cofibrations is preserved under pushout, so are hence the left vertical morphisms in (102), which are the base algebra inclusions (98) of polynomial dgc-algebras. The base algebra inclusions into general multivariate polynomial dgc-algebras are composites of these, and since the class of cofibrations is preserved under composition, the claim follows.  $\square$

**Lemma 3.44** (Pushout along relative Sullivan algebras preserves quasi-isomorphisms [FHT00, Prop. 6.7 (ii), Lemma 14.2]). *The operation of pushout (306) along the canonical inclusion (108) of a base dgc-algebra into a multivariate polynomial dgc-algebra (Example 3.30) preserves quasi-isomorphisms. In fact, it sends quasi-isomorphism between base algebras to quasi-isomorphisms of multivariate polynomial dgc-algebras:*

$$\begin{array}{ccc}
A \subset & \xrightarrow[\substack{\in \text{Cof} \\ i_A}]{} & A \left[ \begin{array}{c} \alpha_{n_{k+1}}^{(k)} \\ \vdots \\ \alpha_{n_1}^{(1)} \end{array} \right] / \left( \begin{array}{c} d \alpha_{n_k}^{(k)} = \mu^{(k)} \\ \vdots \\ d \alpha_{n_1}^{(1)} = \mu^{(1)} \end{array} \right) \\
\downarrow \substack{f \in \mathbf{W}} \\
A' \subset & \xrightarrow[\substack{\in \text{Cof} \\ i_{A'}]}{} & A' \left[ \begin{array}{c} \alpha_{n_{k+1}}^{(k)} \\ \vdots \\ \alpha_{n_1}^{(1)} \end{array} \right] / \left( \begin{array}{c} d \alpha_{n_k}^{(k)} = \mu^{(k)} \\ \vdots \\ d \alpha_{n_1}^{(1)} = \mu^{(1)} \end{array} \right)
\end{array}
\quad \begin{array}{c} \text{(po)} \\ \downarrow (i_A)_* f \\ \Rightarrow (i_{A'})_* f \in \mathbf{W}. \end{array} \quad (109)$$

**Lemma 3.45** (Weak equivalences of nilpotent  $L_\infty$ -algebras [FHT00, Prop. 14.13]). *A morphism between Chevalley-Eilenberg algebras (Def. 3.25) of nilpotent  $L_\infty$ -algebras (Def. 3.34), is a quasi-isomorphism of dgc-algebras (hence a weak equivalence according to Def. 3.35) precisely if the corresponding morphism (82) of  $L_\infty$ -algebras is a quasi-isomorphism between the chain complexes given by the unary bracket operation  $\partial := [-]$  (86):*

$$\text{CE}(\mathfrak{g}) \xleftarrow[\substack{\in \mathbf{W}}]{\text{CE}(\phi)} \text{CE}(\mathfrak{h}) \quad \Leftrightarrow \quad (\mathfrak{g}, [-]_{\mathfrak{g}}) \xrightarrow[\substack{\in \mathbf{W}}]{\phi} (\mathfrak{h}, [-]_{\mathfrak{h}}).$$

**Remark 3.46** (Homotopy theory of nilpotent  $L_\infty$ -algebras inside all  $L_\infty$ -algebras).

(i) Prop. 3.43, with Remark 3.37 and Def. 3.25, allows to identify the homotopy category of finite-type nilpotent connective  $L_\infty$ -algebras (Def. 3.34), with a full subcategory of the homotopy category (Def. A.15) of the opposite (Example A.9) of dgc-algebras (Prop. 3.36):

$$\text{Ho}(L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}}) \xhookrightarrow{\text{CE}} \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{\text{op}}). \quad (110)$$

(ii) There is also the homotopy theory of more general  $L_\infty$ -algebras [Hi01][Pr10][Va14][Ro20], whose weak equivalences are the quasi-isomorphisms on chain complexes formed by the unary bracket  $[-]$  (86). Lemma 3.45 says that the homotopy theory (110) of finite-type, nilpotent connective  $L_\infty$ -algebras that we are concerned with here is fully faithfully embedded into this more general  $L_\infty$  homotopy theory:

$$\text{Ho}(L_\infty \text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}}) \xhookrightarrow{\quad} \text{Ho}(L_\infty \text{Algebras}_{\mathbb{R}}).$$

## Minimal Sullivan models

**Definition 3.47** (Minimal Sullivan models [BG76, Def. 7.2][He07, Def. 1.10]). A connected (relative) Sullivan model dgc-algebra  $A \in \text{SullivanModels}_{\mathbb{R}}^{\geq 1}$  (Def. 3.31) is called *minimal* if it is given by a multivariate polynomial dgc-algebras as in (99) the degrees  $n_i$  of whose generators  $\alpha_{n_i}^{(i)}$  are monotonically increasing

$$i < j \quad \Rightarrow \quad n_j \leq n_i.$$

**Example 3.48** (Minimal models of simply connected dgc-algebras [BG76, Prop. 7.4]). If  $A \in \text{SullivanModels}_{\mathbb{R}}^{\geq 1}$  (Def. 3.31) is trivial in degree 1, then it is minimal (Def. 3.47) precisely if the unary bracket  $[-]$  (85) of the corresponding  $L_\infty$ -algebra (104) vanishes:

$$A^1 = 0 \quad \Rightarrow \quad (A \text{ is minimal} \quad \Leftrightarrow \quad [-] = 0).$$

**Proposition 3.49** (Existence of minimal Sullivan models [BG76, Prop. 7.7, 7.8]). *If  $A \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  is cohomologically connected, in that  $H^0(A) = \mathbb{R}$ , then:*

(i) *There exists a minimal Sullivan model  $A_{\min}$  (Def. 3.47) with weak equivalence in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (105) to  $A$*

$$A_{\min} \xrightarrow{p_A^{\min} \in W} A. \quad (111)$$

(ii) *This  $A_{\min}$  is unique up to isomorphism of  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ .*

More generally:

**Proposition 3.50** (Existence of minimal relative Sullivan models [FHT00, Thm. 14.12]). *Let  $B \xrightarrow{\phi} A$  be a morphism in  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.17) such that*

(a)  *$A$  and  $B$  are cohomologically connected, in that  $H^0(A) = \mathbb{R}$  and  $H^0(B) = \mathbb{R}$ ,*

(b)  *$H^1(\phi) : H^1(B) \rightarrow H^1(A)$  is an injection.*

Then:

(i) *There exists a minimal relative Sullivan model  $B \hookrightarrow A_{\min_B}$  (Def. 3.47) equipped with a weak equivalence to  $\phi$  in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Def. 105):*

$$\begin{array}{ccc} A_{\min_B} & \xrightarrow{\in W} & A \\ & \searrow & \nearrow \phi \\ & B & \end{array}$$

(ii) *This  $A_{\min_B}$  is unique up to isomorphism in  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ .*

## 3.2 Rational homotopy theory

We recall fundamental facts of dg-algebraic rational homotopy theory [Su77][BG76][GM13] (review in [FHT00][He07][FOT08][FH17]), streamlined towards the application to non-abelian de Rham theory below in §3.3.

**Remark 3.51** (Rational homotopy theory over the real numbers). Throughout, we consider rational homotopy theory over the real numbers  $\mathbb{R}$  (as in [GM13]), instead of over the rational numbers  $\mathbb{Q}$ . This is the version in which rational homotopy theory connects to differential geometry (e.g. [FOT08]), since the *smooth* de Rham complex is not defined over  $\mathbb{Q}$  but over  $\mathbb{R}$  (see Lemma 3.90). The original account [BG76] of rational homotopy theory is, for the most part, formulated over an arbitrary field  $k$  of characteristic zero; and [BG76, Lem. 11.7] makes explicit that the choice of this base field does not change the form of the classical theorems. For example, the “real-ified” homotopy groups of a space  $X$

$$\pi_{\bullet}(X) \otimes_{\mathbb{Z}} \mathbb{R} \simeq (\pi_{\bullet}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$$

form a real vector space with real dimension equal to the rational dimension of the corresponding rationalized homotopy groups

$$\dim_{\mathbb{Q}}(\pi_{\bullet}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{R}}(\pi_{\bullet}(X) \otimes_{\mathbb{Z}} \mathbb{R}),$$

and hence the rational Whitehead  $L_{\infty}$ -algebras (Prop. 3.63 below) have the same set of generators and their Chevalley-Eilenberg algebras (Def. 3.25) have the same structure constants, irrespective of whether they come as algebras over  $\mathbb{Q}$  or over  $\mathbb{R}$ .<sup>8</sup>

<sup>8</sup> While rational homotopy theory has the same form over all ground fields of characteristic zero, there is, of course, a difference between rational homotopy equivalences over different ground fields: Two minimal Sullivan models (Prop. 3.49) over the real numbers may be isomorphic as real dgc-algebras but not as rational dgc-algebras. This happens when the isomorphism is given by an irrational linear transformation between the generators. For example, for any  $b \in \mathbb{R}$ ,  $b \geq 0$  there is a dgc-algebra isomorphism over the real numbers

$$\mathbb{R} \left[ \begin{array}{c} \omega_3 \\ \alpha_2, \beta_2 \end{array} \right] / \left( \begin{array}{l} d\omega_3 = \alpha_2 \wedge \alpha_2 + \beta_2 \wedge \beta_2 \\ d\alpha_2 = 0, d\beta_2 = 0 \end{array} \right) \xrightarrow[\simeq]{\begin{array}{l} \omega_3 \mapsto \omega_3 \\ \alpha_2 \mapsto \alpha_2 \\ \beta_2 \mapsto \sqrt{b}\beta_2 \end{array}} \mathbb{R} \left[ \begin{array}{c} \omega_3 \\ \alpha_2, \beta_2 \end{array} \right] / \left( \begin{array}{l} d\omega_3 = \alpha_2 \wedge \alpha_2 + b\beta_2 \wedge \beta_2 \\ d\alpha_2 = 0, d\beta_2 = 0 \end{array} \right).$$

But over the rational numbers this exists only when the square root of  $b$  is rational. Via the fundamental theorem (Prop. 3.60 below) the same follows for the corresponding homotopy types.

Notice that the comparison between the homotopy types over, in this order, the integers, the rational numbers and then the real numbers is provided by the character map (Def. 4.32 below); and the theory which embodies the distinction between working over these coefficients is that of homotopy fiber products of the character map, which is the theory of non-abelian differential cohomology (Def. 4.33 below), where for instance the homotopy fiber  $\mathbb{R}/\mathbb{Q} \rightarrow B\mathbb{Q} \rightarrow B\mathbb{R}$  is being detected (e.g. [GS19a][GS19c]).

For technical reasons, we focus on the following class of homotopy types (with little to no restriction in practice):

**Definition 3.52** (Connected nilpotent spaces of finite rational type [BG76, 9.2]). Write

$$\mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})_{\geq 1, \mathrm{nil}}^{\mathrm{fin}_{\mathbb{R}}} \hookrightarrow \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})$$

for the full subcategory of homotopy types of topological spaces  $X$  (339) on those which are:

- (i) *connected*:  $\pi_0(X) \simeq *$ ;
- (ii) *nilpotent*:  $\pi_1(X) \in \mathrm{NilpotentGroups}$ , and  $\pi_{n \geq 2}(X)$  are nilpotent  $\pi_1(X)$ -modules (e.g. [Hil82]);
- (iii) *finite rational type*:  $\dim_{\mathbb{R}}(H^n(X; \mathbb{R})) < \infty$ , for all  $n \in \mathbb{N}$ .

**Remark 3.53** (Technical assumptions). The connectedness assumption in Def. 3.52 is a pure convenience; for non-connected spaces all of the following applies just by iterating over connected components. On the other hand, the nilpotency and  $\mathbb{R}$ -finiteness condition in Def. 3.52 are strictly necessary for the plain dg-algebraic formulation of rational homotopy theory (due to [BG76][Su77]) to satisfy the fundamental theorem (Theorem 3.60 below). The generalizations required to drop these assumptions are known, but considerably more unwieldy:

- (i) To drop the nilpotency assumption, all dgc-algebra models need to be equipped with the action of the fundamental group (see [FHT15]).
- (ii) To drop the finite-type assumption one needs dgc-coalgebras in place of dgc-algebras, as in the original [Qu69].

Therefore, we expect that the construction of the (twisted) non-abelian character map, below in sections §4 and §5, works also without imposing these technical assumptions, but a discussion in that generality is beyond the scope of the present article.

**Example 3.54** (Examples of nilpotent spaces [Hil82, §3][MP12, §3.1]). Such examples (Def. 3.52) include:

- (i) every simply connected space  $X$ ,  $\pi_1(X) = 1$ ;
- (ii) every simple space  $X$ , i.e. with abelian fundamental group acting trivially, such as tori;
- (iii) hence every connected H-space;
- (iv) hence every loop space  $X \simeq \Omega Y$ , and hence every  $\infty$ -group (Prop. 2.8);
- (v) hence every infinite-loop space, i.e., every component space  $E_n$  of a spectrum  $E$  (17);
- (vi) the classifying spaces  $BG$  (12) of nilpotent Lie groups  $G$ ;
- (vii) the mapping spaces  $\mathrm{Maps}(X, A)$  out of manifolds  $X$  into nilpotent spaces  $A$ .

Rational homotopy theory is concerned with understanding the following notion:

**Definition 3.55** (Rationalization [BK72, p. 133][BG76, §11.1][He07, §1.4, §1.7]).

- (i) A connected nilpotent homotopy type  $X \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})_{\geq 1, \mathrm{nil}}$  (Def. 3.52) is called *rational* over the real numbers (Remark 3.51) if all its homotopy groups admit the structure of real vector spaces.
- (ii) A *rationalization* of  $X$  is a map

$$X \xrightarrow{\eta_X^{\mathbb{R}}} L_{\mathbb{R}}(X) \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})_{\geq 1, \mathrm{nil}} \quad (112)$$

such that:

- (a)  $L_{\mathbb{R}}$  is rational:  $\pi_n(X) \in \mathrm{VectorSpaces}_{\mathbb{R}} \rightarrow \mathrm{Groups}$  for  $n \geq 1$ ,
- (b) the map  $\eta_X^{\mathbb{R}}$  induces an isomorphism on rational cohomology groups:

$$H^{\bullet}(L_{\mathbb{R}}X; \mathbb{R}) \xrightarrow[\simeq]{H^{\bullet}(\eta_X^{\mathbb{R}}; \mathbb{R})} H^{\bullet}(X; \mathbb{R}).$$

Rationalization exists essentially uniquely, and defines a reflective subcategory inclusion

$$\mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})_{\geq 1, \mathrm{nil}}^{\substack{\text{connected, nilpotent, } \mathbb{R}\text{-finite,} \\ \text{rational homotopy types}}} \xleftarrow[L_{\mathbb{R}}]{\perp} \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathbb{Q}\mathrm{u}})_{\geq 1, \mathrm{nil}}^{\substack{\text{connected, nilpotent, } \mathbb{R}\text{-finite,} \\ \text{homotopy types}}} \quad (113)$$

whose adjunction unit is (112).

**PL de Rham theory.** At the heart of dg-algebraic rational homotopy theory is the observation that a variant of the de Rham dg-algebra of a smooth manifold (Example 3.23) applies to general topological spaces: the *PL de Rham complex*<sup>9</sup> (Def. 3.56). This satisfies an appropriate *PL de Rham theorem* (Prop. 3.57) and makes dg-algebras of PL differential forms detect rational homotopy type (Prop. 3.60). At the same time, over a smooth manifold the PL de Rham complex is suitably equivalent to the smooth de Rham complex (Lemma 3.90).

**Definition 3.56** (PL de Rham complex and PL de Rham cohomology [BG76, pp. 1-7][GM13, §9.1]). Write

$$\Omega_{\text{pdR}}^{\bullet}(\Delta^{\bullet}) : \Delta^{\text{op}} \longrightarrow (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}} \quad (114)$$

for the simplicial dgc-algebras of polynomial differential forms on the standard simplices.

(i) For  $S \in \text{SimplicialSets}$ , its *PL de Rham complex* is the hom-object of simplicial objects from  $S$  to  $\Omega^{\bullet}(\Delta^{\bullet})$ , hence is the following end in  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ :

$$\Omega_{\text{PLdR}}^{\bullet}(S) := \int_{[k] \in \Delta} \bigoplus_{S_k} \Omega_{\text{pdR}}^{\bullet}(\Delta^k), \quad (115)$$

hence an element in  $\omega \in \Omega_{\text{PLdR}}^{\bullet}(S)$  is a polynomial differential form  $\omega_{\sigma}^{(n)} \in \Omega_{\text{pdR}}^{\bullet}(\Delta^n)$  on each  $n$ -simplex  $\sigma \in S_n$  for all  $n \in \mathbb{N}$ , such that these are compatible under pullback along all simplex face inclusions  $\delta_i$  and along all degenerate simplex projections  $\sigma_i$ :

$$\Omega_{\text{PLdR}}^{\bullet}(S) = \left( \begin{array}{ccc} \vdots & & \vdots \\ \vdots & & \vdots \\ S_2 & \dashrightarrow \omega^{(2)} & \dashrightarrow \Omega_{\text{pdR}}^{\bullet}(\Delta^2) \\ \uparrow \delta_0^* \uparrow \delta_1^* \uparrow \delta_2^* & & \uparrow \delta_0^* \uparrow \delta_1^* \uparrow \delta_2^* \\ \downarrow \sigma_0^* \downarrow \sigma_1^* \downarrow \sigma_2^* & & \downarrow \sigma_0^* \downarrow \sigma_1^* \downarrow \sigma_2^* \\ S_1 & \dashrightarrow \omega^{(1)} & \dashrightarrow \Omega_{\text{pdR}}^{\bullet}(\Delta^1) \\ \uparrow \delta_0^* \uparrow \delta_1^* & & \uparrow \delta_0^* \uparrow \delta_1^* \\ \downarrow \sigma_0^* \downarrow \sigma_1^* & & \downarrow \sigma_0^* \downarrow \sigma_1^* \\ S_0 & \dashrightarrow \omega^{(0)} & \dashrightarrow \Omega_{\text{pdR}}^{\bullet}(\Delta^0) \end{array} \right)$$

(ii) For  $X \in \text{TopologicalSpaces}$ , its *PL de Rham complex* is that of its singular simplicial set, according to (115):

$$\Omega_{\text{PLdR}}^{\bullet}(X) := \Omega_{\text{PLdR}}^{\bullet}(\text{Sing}(X)). \quad (116)$$

By pullback of differential forms, this extends to a functor

$$\text{SimplicialSets} \xrightarrow{\Omega_{\text{PLdR}}^{\bullet}} (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}}. \quad (117)$$

(iii) We write

$$H_{\text{PLdR}}^{\bullet}(-) := H\Omega_{\text{PLdR}}^{\bullet}(-) \quad (118)$$

for *PL de Rham cohomology*, the cochain cohomology of the PL de Rham complex.

**Proposition 3.57** (PL de Rham theorem [BG76, Thm. 2.2][GM13, Thm. 9.1]). *The evident operation of integrating differential forms over simplices induces a quasi-isomorphism*

$$\Omega_{\text{PLdR}}^{\bullet}(-) \xrightarrow{\in \text{qIso}} C^{\bullet}(-; \mathbb{R})$$

from the PL de Rham complex (Def. 3.56) to the cochain complex of ordinary singular cohomology with coefficients in  $\mathbb{R}$ . Hence on cochain cohomology this induces an isomorphism

$$H_{\text{PLdR}}^{\bullet}(-) \xrightarrow{\cong} H^{\bullet}(-; \mathbb{R})$$

between PL de Rham cohomology (118) and ordinary real cohomology.

<sup>9</sup> The terminology ‘‘PL’’ or ‘‘P.L.’’ for this construction seems to have been silently introduced in [BG76], as shorthand for ‘‘piecewise linear’’, and has become widely adopted (e.g. [GM13, §9]). But beware that this refers to the piecewise-linear structure that a choice of triangulation induces on a topological space, while the actual differential forms in the PL de Rham complex are piecewise *polynomial* with respect to this piecewise linear structure.

But, in fact, before passing to cochain cohomology, the PL de Rham complex captures full rational homotopy type:

**Lemma 3.58** (Extension lemma for polynomial differential forms [GM13, Lemma 9.4]). *For  $n \in \mathbb{N}$ , the operation of pullback of piecewise polynomial differential forms (Def. 3.90) along the boundary inclusion of the  $n$ -simplex*

$\partial \Delta^n \xrightarrow{i_n} \Delta^n$  *is an epimorphism:*

$$\Omega_{\text{PLdR}}^\bullet(\Delta^n) \xrightarrow{i_n^*} \Omega_{\text{PLdR}}^\bullet(\partial \Delta^n).$$

**Proposition 3.59** (PL de Rham Quillen adjunction [BG76, §8]). *The PL de Rham complex functor (Def. 3.56) is the left adjoint in a Quillen adjunction (Def. A.18)*

$$\left( \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}}^{\text{op}} \begin{array}{c} \xleftarrow{\Omega_{\text{PLdR}}^\bullet} \\ \perp_{\text{Qu}} \\ \xrightarrow{\text{exp}_{\text{PS}}} \end{array} \text{SimplicialSets}_{\text{Qu}} \quad (119)$$

between the opposite (Def. A.9) of the model category of dgc-algebras (Prop. 3.36) and the classical model structure on simplicial sets (Prop. A.8); where the right adjoint sends a dgc-algebra  $A$  to

$$\text{exp}_{\text{PS}}(A) = \Delta[k] \longmapsto \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\Omega_{\text{PLdR}}^\bullet(\Delta^k), A) \in \text{SimplicialSets}. \quad (120)$$

*Proof.* That the right adjoint exists and is give as in (120) follows by general nerve/realization theory [Kan58], or else by direct inspection.

For the left adjoint to preserve cofibrations means to take injections of simplicial sets to degreewise surjections of dgc-algebras. This follows from the extension lemma (Lemma 3.58). Moreover, the left adjoint preserves in fact all weak equivalences, by the PL de Rham theorem (Prop. 3.57).  $\square$

**Proposition 3.60** (Fundamental theorem of dgc-algebraic rational homotopy theory). *The derived adjunction (Prop. A.21)*

$$\text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{op}}\right) \begin{array}{c} \xleftarrow{\mathbb{L}\Omega_{\text{PLdR}}^\bullet} \\ \perp \\ \xrightarrow{\mathbb{R}\text{exp}} \end{array} \text{Ho}\left(\text{SimplicialSets}_{\text{Qu}}\right) \quad (121)$$

of the Quillen adjunction (119) from Prop. 3.59 is such that:

(i) on connected, nilpotent,  $\mathbb{R}$ -finite homotopy types (Def. 3.52) the derived PLdR-adjunction unit (326) is the unit (112) of rationalization (Def. 3.55):

$$\begin{array}{ccc} X & \xrightarrow{\substack{\text{derived unit of} \\ \text{PL de Rham adjunction}}} & \mathbb{R}\text{exp} \circ \Omega_{\text{PLdR}}^\bullet(X) \\ \parallel & \Downarrow \eta_X^{\text{PLdR}} & \wr \\ X & \xrightarrow{\substack{\eta_X^{\mathbb{R}} \\ \text{rationalization unit}}} & L_{\mathbb{R}}X \end{array} \in \text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}\right)_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}. \quad (122)$$

(ii) For  $X, A$  nilpotent, connected,  $\mathbb{R}$ -finite homotopy types (Def. 3.52), the PL de Rham space functor (117) from Def. 3.56 induces natural bijections

$$\text{Ho}\left(\text{TopologicalSpaces}_{\text{Qu}}\right)(X, L_{\mathbb{R}}A) \xrightarrow[\Omega_{\text{PLdR}}^\bullet]{\simeq} \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\right)\left(\Omega_{\text{PLdR}}^\bullet(A), \Omega_{\text{PLdR}}^\bullet(X)\right). \quad (123)$$

*Proof.* (i) This is [BG76, Thm 11.2].

(ii) This follows via [BG76, Thm 9.4(i)], which says that the derived adjunction (121) restricts on connected, nilpotent,  $\mathbb{R}$ -finite (Def. 3.52) rational homotopy types (Def. 3.55) to an equivalence of categories:

$$\text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{op}}\right)_{\geq 1}^{\text{fin}} \begin{array}{c} \xleftarrow{\mathbb{L}\Omega_{\text{PLdR}}^\bullet} \\ \simeq \\ \xrightarrow{\mathbb{R}\text{exp}} \end{array} \text{Ho}\left(\text{SimplicialSets}_{\text{Qu}}\right)_{\geq 1, \text{nil}}^{\mathbb{R}, \text{fin}_{\mathbb{R}}}. \quad (124)$$

In detail, consider the following sequence of natural isomorphisms:



$$\begin{aligned}
& \text{Ho}(\text{TopologicalSpaces}_{\mathbb{Q}\text{u}})(X, L_{\mathbb{R}}A) \\
& \simeq \text{Ho}(\text{SimplicialSets}_{\mathbb{Q}\text{u}})(\text{Sing}(X), L_{\mathbb{R}}\text{Sing}(A)) \\
& \simeq \text{Ho}(\text{SimplicialSets}_{\mathbb{Q}\text{u}})(L_{\mathbb{R}}\text{Sing}(X), L_{\mathbb{R}}\text{Sing}(A)) \\
& \simeq \text{Ho}(\text{SimplicialSets}_{\mathbb{Q}\text{u}})(\mathbb{R}\exp \circ \Omega_{\text{PLdR}}^{\bullet}(X), \mathbb{R}\exp \circ \Omega_{\text{PLdR}}^{\bullet}(A)) \\
& \simeq \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{op}}\right)\left(\Omega_{\text{PLdR}}^{\bullet} \circ \mathbb{R}\exp \circ \Omega_{\text{PLdR}}^{\bullet}(X), \Omega_{\text{PLdR}}^{\bullet} \circ \mathbb{R}\exp \circ \Omega_{\text{PLdR}}^{\bullet}(A)\right) \\
& \simeq \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{op}}\right)\left(\Omega_{\text{PLdR}}^{\bullet}(X), \Omega_{\text{PLdR}}^{\bullet}(A)\right) \\
& \simeq \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\right)\left(\Omega_{\text{PLdR}}^{\bullet}(A), \Omega_{\text{PLdR}}^{\bullet}(X)\right).
\end{aligned} \tag{125}$$

Here the first step is (339); the second step uses that rationalization is a reflection (113); the third step uses (122); the fourth is the equivalence (124) along  $\mathbb{L}\Omega_{\text{PLdR}}^{\bullet}$  (using, with Example A.22, that every simplicial set is already cofibrant, Example A.8); the fifth step is the statement from (124) that  $\mathbb{R}\exp$  is the inverse equivalence. The last step is just the definition of the opposite of a category. The composite of the bijections (125) is the desired bijection (123).  $\square$

**PS de Rham theory.** The point of using piecewise *polynomial* differential forms in the PL de Rham complex (Def. 3.56) is that these, but not the piecewise smooth differential forms, can be defined over the field  $\mathbb{Q}$  of rational numbers. But since we may and do use the real numbers as the rational ground field (Remark 3.51), it is expedient to also consider piecewise smooth de Rham complexes:

**Definition 3.61** (PS de Rham complex). For  $n \in \mathbb{N}$ , we write

$$\Omega_{\text{dR}}^{\bullet}(\mathbb{R}^n \times \Delta^{\bullet}) : \Delta^{\text{op}} \longrightarrow \left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)^{\text{op}}$$

for the simplicial dgc-algebra of smooth differential forms on the product manifold of  $n$ -dimensional Cartesian space with the standard simplices, i.e., of smooth differential forms on an ambient Cartesian space (Example 3.23), restricted to the simplex. As in Def. 3.90, this induces for each  $S \in \text{SimplicialSets}$  the corresponding *piecewise smooth de Rham complexes*

$$\Omega_{\text{PSdR}}^{\bullet}(\mathbb{R}^n \times S) := \int_{[k] \in \Delta} \bigoplus_{S_n} \Omega_{\text{dR}}^{\bullet}(\mathbb{R}^n \times \Delta^n) \tag{126}$$

and by pullback of differential forms these extend to functors

$$\text{SimplicialSets} \xrightarrow{\Omega_{\text{PSdR}}^{\bullet}(\mathbb{R}^n \times (-))} \left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)^{\text{op}}. \tag{127}$$

**Proposition 3.62** (Fundamental theorem for piecewise smooth de Rham complexes). *For all  $n \in \mathbb{N}$  the piecewise smooth de Rham complex functors (Def. 3.61) participate in a Quillen adjunction analogous to the PL de Rham adjunction (Prop. 3.59)*

$$\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{op}} \begin{array}{c} \xleftarrow{\Omega_{\text{PSdR}}^{\bullet}(\mathbb{R}^n \times (-))} \\ \xrightarrow[\text{exp}_{\text{PS},n}]{\perp_{\mathbb{Q}\text{u}}} \end{array} \text{SimplicialSets}_{\mathbb{Q}\text{u}} \tag{128}$$

with right adjoint given as in (120):

$$\text{exp}_{\text{PS},n}(A) = \Delta[k] \longmapsto \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\Omega_{\text{PLdR}}^{\bullet}(\mathbb{R}^n \times \Delta^k), A) \in \text{SimplicialSets} \tag{129}$$

whose derived functors (Prop. A.21) are naturally equivalent to those of the PL de Rham adjunction (121):

$$\mathbb{L}\Omega_{\text{PSdR}}^{\bullet}(\mathbb{R}^n \times (-)) \simeq \mathbb{L}\Omega_{\text{PSdR}}^{\bullet} \simeq \mathbb{L}\Omega_{\text{PLdR}}^{\bullet}, \tag{130}$$

$$\mathbb{R}\text{exp}_{\text{PS},n} \simeq \mathbb{R}\text{exp}_{\text{PS}} \simeq \mathbb{R}\text{exp}_{\text{PL}}. \tag{131}$$

*Proof.* (i) The proofs of the PL de Rham theorem (Prop. 3.57) as well as of the extension Lemma (Lemma 3.58) apply essentially verbatim also to piecewise-smooth differential forms ([GM13, Prop. 9.8]) and hence so does the proof of the PL de Rham Quillen adjunction in the form given in Prop. 3.59.

(ii) We have evident natural transformations

$$\Omega_{\text{PLdR}}^{\bullet}(S) \xrightarrow{\in W} \Omega_{\text{PSdR}}^{\bullet}(S) \xrightarrow{\in W} \Omega_{\text{PSdR}}^{\bullet}(\mathbb{R}^n \times S)$$

given by inclusion of polynomial differential forms into smooth differential forms, and by pullback of differential forms along the projections  $\mathbb{R}^n \times \Delta^k \rightarrow \Delta^k$ . The corresponding component morphisms are quasi-isomorphisms, hence are weak equivalences in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  ([GM13, Cor. 9.9]). Under passage to homotopy categories (Def. A.15) and derived functors (Example A.22), these natural weak equivalences become the natural isomorphisms (130) (by Prop. A.16). By essential uniqueness of adjoint functors, this implies the natural isomorphisms (131).  $\square$

### Whitehead $L_{\infty}$ -algebras.

**Proposition 3.63** (Rational Whitehead  $L_{\infty}$ -algebras). *For  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52), there exists a nilpotent  $L_{\infty}$ -algebra (Def. 3.34)*

$$\mathfrak{I}X \in L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}}, \quad (132)$$

*unique up to isomorphism, whose Chevalley-Eilenberg algebra (Def. 3.25) is the minimal model (Def. 3.47) of the PL de Rham complex of  $X$  (Def. 3.56):*

$$\text{CE}(\mathfrak{I}X) := (\Omega_{\text{PLdR}}^{\bullet}(X))_{\min} \xrightarrow[\in W]{p_X^{\min}} \Omega_{\text{PLdR}}^{\bullet}(X). \quad (133)$$

*Proof.* By the PL de Rham theorem (Prop. 3.57) and the assumption that  $X$  is connected, it follows that we have  $H\Omega_{\text{PLdR}}^0(X) = \mathbb{R}$ . Therefore Prop. 3.49 applies and says that  $(\Omega_{\text{PLdR}}^{\bullet}(X))_{\min} \in \text{SullivanModels}_{\mathbb{R}}^{\geq 1}$  exists, and is unique up to isomorphism. With this, the equivalence (104) says that  $\mathfrak{I}X$  exists and is unique up to isomorphism.  $\square$

Notice the immediate corollary:

**Proposition 3.64** (Rational Whitehead  $L_{\infty}$ -algebra encodes rational homotopy type). *The rational Whitehead  $L_{\infty}$ -algebra  $\mathfrak{I}X$  in Prop. 3.63 encodes the rationalized homotopy type (Def. 3.55) of  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  in that:*

$$L_{\mathbb{R}}X \simeq \exp \circ \text{CE}(\mathfrak{I}X) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}} \quad (134)$$

*with  $\exp$  from (119).*

*Proof.* This is the composite of the following sequence of isomorphisms in the homotopy category:

$$\begin{aligned} L_{\mathbb{R}}X &\simeq \mathbb{R} \exp \circ \Omega_{\text{PLdR}}^{\bullet}(X) \\ &\simeq \mathbb{R} \exp \circ (\Omega_{\text{PLdR}}^{\bullet}(X))_{\min} \\ &\simeq \mathbb{R} \exp \circ \text{CE}(\mathfrak{I}X) \\ &\simeq \exp \circ \text{CE}(\mathfrak{I}X). \end{aligned}$$

Here the first step is the fundamental theorem (Prop. 3.60), the second step is the existence of the minimal model (Prop. 3.49) for the PL de Rham complex (using that it is cohomologically connected, by the PL de Rham theorem, Prop. 3.57), the third step is the existence of the rational Whitehead  $L_{\infty}$ -algebra (Prop. 3.63), and the last step uses that Sullivan models are cofibrant (Prop. 3.43), hence fibrant in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{\text{op}}$ , so that on these objects the right derived functor  $\mathbb{R} \exp$  is given by the plain functor  $\exp$ .  $\square$

**Proposition 3.65** (Rational homotopy groups in the rational Whitehead  $L_\infty$  algebra).

Let  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\text{fin}_{\mathbb{R}}})_{\geq 1, \text{nil}}$  (Def. 3.52).

(i) If  $X$  is simply connected,  $\pi_1(X) = 1$  (Example 3.54), then there is an isomorphism of graded vector spaces (Def. 3.2) between the graded vector space underlying (83) the Whitehead  $L_\infty$ -algebra  $\mathfrak{I}X$  (Prop. 3.63) and the rationalized homotopy groups of the based loop space  $\Omega X$ :

$$\overset{\text{Whitehead}}{\underset{L_\infty\text{-algebra}}{\mathfrak{I}X}} \simeq \overset{\text{rationalized}}{\underset{\text{homotopy groups}}{\pi_\bullet(\Omega X)}} \otimes_{\mathbb{Z}} \mathbb{R} \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}.$$

(ii) If  $\pi_1(X)$  is not necessarily trivial but abelian, then this statement still holds with  $\mathfrak{I}X$  replaced by its homology with respect to the unary differential  $[-]$  (85).

(iii) If  $\pi_1(X)$  is not abelian, then (ii) still holds in degrees  $\geq 2$ .

*Proof.* Under translation through Prop. 3.63 and Def. 3.25, and using  $\pi_\bullet(\Omega X) \simeq \pi_{\bullet+1}(X)$ , claim (i) is equivalent to the existence of a dual isomorphism:

$$\text{CE}(\mathfrak{I}X) /_{\text{CE}(\mathfrak{I}X)^2} \simeq \text{Hom}_{\mathbb{Z}}(\pi_\bullet(X), \mathbb{R}) \in \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}, \quad (135)$$

where the left hand side denotes the graded vector space of indecomposable elements in the Chevalley-Eilenberg algebra (the  $\alpha_{n_i}^{(i)}$  in (99)). In this form, this is the statement of [BG76, Theorem 11.3 with Def. 6.12], in the special case when, with  $\pi_1(X) = 1$ , the unary differential  $[-]$  in  $\mathfrak{I}X$  vanishes (Example 3.48). The generalizations follow analogously.  $\square$

**Remark 3.66** (Equivalent  $L_\infty$ -structures on Whitehead products). The original discussion of the Whitehead algebra structure on the homotopy groups of a space is in terms of differential-graded Lie algebras ([Hil55, Theorem B]), as are the Quillen models of rational homotopy theory [Qu69].

(i) Notice that dg-Lie algebras (Example 3.26) and  $L_\infty$ -algebras with minimal CE-algebra (Def. 3.47) are two opposite classes of  $L_\infty$ -algebras: The former has  $k$ -ary brackets (85) only for  $k \leq 2$ , the latter only for  $k \geq 2$  (in the simply connected case, by Example 3.48). Yet, quasi-isomorphisms connect algebras in one class to those in the other: The transmutation of dg-Lie- into minimal  $L_\infty$ -algebras is described in [BBMM16, Theorem 2.1]; that from  $L_\infty$ - to dg-Lie-algebras in [FRS13, §1.0.2].

(ii) The minimal  $L_\infty$ -algebra structure on  $\mathfrak{I}X$  that we obtained in Prop. 3.63, 3.65, has the property that its  $k$ -ary brackets are, up to possibly a sign, equal to the order- $k$  higher Whitehead products on  $X$  [BBMM16, Prop. 3.1].

**Example 3.67** (Rationalization of Eilenberg-MacLane spaces). Since the homotopy types of Eilenberg-MacLane-spaces  $K(A, n) = B^{n+1}A$  (see (10)) are fully characterized by their homotopy groups (for discrete abelian groups  $A$ , e.g. [AGP02, §6])

$$\pi_k(B^{n+1}A) \simeq \begin{cases} A & | \quad k = n + 1 \\ 0 & | \quad k \neq n + 1 \end{cases}$$

we have, for  $n \in \mathbb{N}$ :

(i) The rationalization (Def. 3.55) of the integral EM-space is the real EM-space

$$L_{\mathbb{R}}(B^{n+1}\mathbb{Z}) \simeq B^{n+1}\mathbb{R}. \quad (136)$$

(ii) Their Whitehead  $L_\infty$ -algebra (Prop. 3.63) is the line Lie  $n$ -algebra  $\mathfrak{b}^n\mathbb{R}$  (Def. 3.27), by Prop. 3.65:

$$\mathfrak{I}B^{n+1}\mathbb{Z} \simeq \mathfrak{b}^n\mathbb{R}. \quad (137)$$

(iii) Hence, by Prop. 3.64,

$$B^{n+1}\mathbb{R} \simeq \exp \circ \text{CE}(\mathfrak{b}^n\mathbb{R}) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}). \quad (138)$$

**Example 3.68** (Rationalization of  $n$ -spheres). The Serre finiteness theorem (see [Ra86, Thm. 1.1.8]) says that the homotopy groups of  $n$ -spheres for  $n \geq 1$  are of the form

$$\pi_{n+k}(S^n) \simeq \begin{cases} \mathbb{Z} & | \quad k = 0 \\ \mathbb{Z} \oplus \text{fin} & | \quad k = 2m \text{ and } n = 2m - 1 \\ \text{fin} & | \quad \text{otherwise} \end{cases}$$

where “fin” stands for some finite group. Since finite groups are pure torsion, hence have trivial rationalization, this means that the rational homotopy groups of spheres are:

$$\pi_{n+k}(S^n) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \begin{cases} \mathbb{R} & | & k = 0 \\ \mathbb{R} & | & k = 2m \text{ and } n = 2m - 1 \\ 0 & | & \text{otherwise.} \end{cases}$$

Moreover, the fact that ordinary cohomology is represented by Eilenberg-MacLane spaces (Example 2.2) implies that

$$H^k(S^n; \mathbb{R}) \sim \begin{cases} \mathbb{R} & | & k = n \\ 0 & | & \text{otherwise.} \end{cases}$$

With this, Prop. 3.65 together with Prop. 3.57 implies that the Whitehead  $L_\infty$ -algebras of spheres (Prop. 3.63) are as follows:

$$\mathrm{CE}(\mathfrak{l}S^n) \simeq \mathbb{R}[\omega_n]/(d\omega_n = 0.) \quad \text{if } n \text{ is odd} \quad (139)$$

and

$$\mathrm{CE}(\mathfrak{l}S^n) \simeq \mathbb{R} \left[ \begin{array}{c} \omega_{2n-1} \\ \omega_n \end{array} \right] / \left( \begin{array}{l} d\omega_{2n-1} = -\omega_n \wedge \omega_n \\ d\omega_n = 0 \end{array} \right) \quad \text{if } n > 1 \text{ is even} \quad (140)$$

**Example 3.69** (Rationalization of loop spaces). The minimal Sullivan model (Def. 3.47) of a loop space  $A \simeq \Omega A'$  has vanishing differential (e.g. [FHT00, p. 143]). Therefore, Prop. 3.65 implies that the rational Whitehead  $L_\infty$ -algebra  $\mathfrak{l}A$  (Prop. 3.63) of  $A$  is the direct sum of line Lie  $n$ -algebras  $\mathfrak{b}^n \mathbb{R}$  (Example 3.27):

$$\mathfrak{l}A \simeq \bigoplus_{n \in \mathbb{N}} \mathfrak{b}^n(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R}) \in L_\infty \mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0, \mathrm{nil}}.$$

Accordingly, its Chevalley-Eilenberg algebra (Def. 3.24) is the tensor product of those of the summands:

$$\mathrm{CE}(\mathfrak{l}A) \simeq \bigotimes_{n \in \mathbb{N}} \mathrm{CE}(\mathfrak{b}^n(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R})) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}.$$

**Relative rational Whitehead  $L_\infty$ -algebras.** In generalization of Prop. 3.63 we have:

**Proposition 3.70** (Relative rational Whitehead  $L_\infty$ -algebras). *For  $A, B, F \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{\mathrm{fin}_{\mathbb{R}}})_{\geq 1, \mathrm{nil}}$  (Def. 3.52) and  $p$  a Serre fibration (Example A.7) from  $A$  to  $B$  with fiber  $F$*

$$\begin{array}{ccc} F & \xrightarrow{\mathrm{fib}(p)} & A \\ & & p \downarrow \in \mathrm{Fib} \\ & & B \end{array}$$

*there exists a nilpotent  $L_\infty$ -algebra (Def. 3.34)*

$$\mathfrak{l}_B A \in L_\infty \mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0, \mathrm{nil}}, \quad (141)$$

*unique up to isomorphism, whose Chevalley-Eilenberg algebra (Def. 3.25) is the relative minimal model (Def. 3.47, Prop. 3.50) of the PL de Rham complex of  $p$  (Def. 3.56), relative to  $\mathrm{CE}(\mathfrak{l}B)$  (from Prop. 3.63):*

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{l}_B A) := (\Omega_{\mathrm{PLdR}}^\bullet(A))_{\min_{\mathrm{CE}(\mathfrak{l}B)}} & \xrightarrow[\in \mathbb{W}]{P_A^{\min_B}} & \Omega_{\mathrm{PLdR}}^\bullet(A) \\ & \swarrow \text{relative minimal model } \mathrm{CE}(\mathfrak{l}p) & \uparrow \Omega_{\mathrm{PLdR}}^\bullet(p) \\ & \mathrm{CE}(\mathfrak{l}B) & \xrightarrow[\in \mathbb{W}]{P_B^{\min}} \Omega_{\mathrm{PLdR}}^\bullet(B). \end{array} \quad (142)$$

*Proof.* By the PL de Rham theorem (Prop. 3.57) and the assumption that  $A$  and  $B$  are connected, it follows that we have  $H\Omega_{\mathrm{PLdR}}^0(A) = \mathbb{R}$  and  $H\Omega_{\mathrm{PLdR}}^0(B) = \mathbb{R}$ . Moreover, by the assumption that  $p$  is a Serre fibration with connected fiber, it follows that  $H^1(\Omega_{\mathrm{PLdR}}^\bullet(p))$  is injective (e.g. [FHT00, p. 196]).

Therefore Prop. 3.50 applies and says that  $(\Omega_{\mathrm{PLdR}}^\bullet(A))_{\min_B} \in \mathrm{SullivanModels}_{\mathbb{R}}^{\geq 1}$  exists, and is unique up to isomorphism. With this, the equivalence (104) says that  $\mathfrak{l}_B A$  exists and is unique up to isomorphism.  $\square$

**Lemma 3.71** (Minimal relative Sullivan models preserve homotopy fibers [FHT00, §15 (a)][FHT15, Thm. 5.1]). Consider  $F, A, B \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) and let  $p$  be a Serre fibration from  $A$  to  $B$  (Example A.7) such that the homology groups  $H_{\bullet}(F, \mathbb{R})$  of the fiber are nilpotent as  $\pi_1(B)$ -modules (for instance in that  $B$  is simply-connected or that the fibration is principal). Then the cofiber of the minimal relative Sullivan model for  $p$  (142) is the minimal Sullivan model (133) for the homotopy fiber  $F$  (Def. A.23):

$$\begin{array}{ccc}
 F & \xrightarrow{\text{fib}(p)} & A & & \text{CE}(\mathbb{I}F) & \xleftarrow{\text{cofib}(\text{CE}(p))} & \text{CE}(\mathbb{I}_B A) & & \\
 & & \downarrow p \in \text{Fib} & & & & \uparrow \text{CE}(p) & & \\
 & & B & & & & \text{CE}(\mathbb{I}B) & & 
 \end{array} \tag{143}$$

See Prop. 3.75 below for the key application of Lemma 3.71.

### Non-abelian real cohomology.

**Definition 3.72** (Non-abelian real cohomology). Let  $X, A \in \text{TopologicalSpaces}$  Then the *non-abelian real cohomology* of  $X$  with coefficients in  $A$  is the non-abelian cohomology (Def. 2.1) of  $X$  with coefficients in the rationalization  $L_{\mathbb{R}}A$  (Def. 3.55)

$$H(X; L_{\mathbb{R}}A) := \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, L_{\mathbb{R}}A). \tag{144}$$

**Remark 3.73** (Non-abelian real cohomology subsumes ordinary real cohomology). For  $n \in \mathbb{N}$ , non-abelian real cohomology (Def. 3.72) with coefficients in the rationalized classifying space (Example 3.67)

$$L_{\mathbb{R}}(B^{n+1}\mathbb{Z}) \simeq B^n\mathbb{R}$$

is naturally equivalent, by Example 2.2, to ordinary real cohomology in degree  $n$ :

$$H(X; B^{n+1}\mathbb{R}) \simeq H^{n+1}(X; \mathbb{R}).$$

More generally:

**Proposition 3.74** (Non-abelian real cohomology with coefficients in loop spaces).

Let  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) such that it admits loop space structure, hence such that there exists  $A'$  with

$$A \simeq \Omega A' \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}).$$

Then the non-abelian real cohomology (Def. 3.72) with coefficients in  $L_{\mathbb{R}}A$  is naturally equivalent to ordinary real cohomology with coefficients in the rationalized homotopy groups of  $A$ :

$$H(X; L_{\mathbb{R}}A) \simeq \bigoplus_{n \in \mathbb{N}} H^{n+1}(X; \pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R}). \tag{145}$$

*Proof.* By Example 3.69 the we have

$$\text{CE}(\mathbb{I}A) \simeq \bigotimes_{n \in \mathbb{N}} \text{CE}\left(\mathfrak{b}^n(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R})\right)$$

Noticing that the tensor product of dgc-algebras is the coproduct in the category of  $\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Example 3.22), and hence the Cartesian product in the opposite category, the right adjoint functor  $\exp$  (119) preserves this, so that

$$\exp \circ \text{CE}(\mathbb{I}A) \simeq \prod_{n \in \mathbb{N}} \left( \exp \circ \text{CE}\left(\mathfrak{b}^n(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R})\right) \right) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}).$$

But, by Prop. 3.64 and by (138) in Example 3.67, this says that:

$$L_{\mathbb{R}}A \simeq \prod_{n \in \mathbb{N}} \left( B^{n+1}(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R}) \right) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}).$$

Using this, and that cohomology preserves products of coefficients, we get the following sequence of natural bijections:

$$\begin{aligned}
H(X; L_{\mathbb{R}}A) &\simeq H\left(X; \prod_{n \in \mathbb{N}} B^{n+1}(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R})\right) \\
&\simeq \prod_{n \in \mathbb{N}} H\left(X; B^{n+1}(\pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R})\right) \\
&= \prod_{n \in \mathbb{N}} H^{n+1}(X; \pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R}) \\
&= \bigoplus_{n \in \mathbb{N}} H^{n+1}(X; \pi_{n+1}(A) \otimes_{\mathbb{Z}} \mathbb{R}).
\end{aligned}$$

The composite of these is the desired (145). □

### Twisted non-abelian real cohomology.

**Proposition 3.75** (Rationalization of local coefficients – “fiber lemma” [BK72, §II]). *Let*

$$\begin{array}{c}
A \longrightarrow A // G \\
\downarrow \rho \\
BG
\end{array}$$

be a local coefficient bundle (Def. 2.29) such that all spaces are connected, nilpotent and of  $\mathbb{R}$ -finite tupe,  $A, BG, A // G \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) and such that the action of  $\pi_1(BG)$  on  $H_{\bullet}(A, \mathbb{R})$  is nilpotent (for instance in that  $BG$  is simply connected). Then:

(i) Component-wise rationalization (Def. 3.55) yields a natural transformation to as rational local coefficient bundle as shown in the middle here:

$$\begin{array}{ccccc}
& & \text{minimal Sullivan model} & \text{CE}(\mathcal{I}A) & \xleftarrow{\text{cofib}(\text{CE}(\iota_p))} & \text{CE}(\iota_{BG}(A // G)) \\
& & & & & \uparrow \text{CE}(\iota_p) \\
& & & & & \text{CE}(\iota_{BG}) \\
\text{rationalization } \eta_A^{\mathbb{R}} & \nearrow & L_{\mathbb{R}}A & \xrightarrow{\text{hofib}(L_{\mathbb{R}}\rho)} & (L_{\mathbb{R}}A) // (L_{\mathbb{R}}G) & \\
\text{local coefficient bundle } A & \xrightarrow{\text{hofib}(\rho)} & A // G & \xrightarrow{\eta_{A//G}^{\mathbb{R}}} & & \\
& & \downarrow \rho & & \downarrow L_{\mathbb{R}}\rho & \\
& & BG & \xrightarrow{\eta_{BG}^{\mathbb{R}}} & B(L_{\mathbb{R}}G) & 
\end{array} \tag{146}$$

(ii) with minimal (relative) Sullivan model (Def. 3.47) as shown on the right.

*Proof.* First, since forming classifying spaces shifts homotopy groups up in degree, it follows that  $BG \xrightarrow{B\eta_G^{\mathbb{R}}} B(L_{\mathbb{R}}G)$  induces an isomorphism on rationalized homotopy groups and hence is the rationalization map (Def. 3.55) on  $BG$ .

Moreover, Lemma 3.71 says that the homotopy fiber (Def. A.23) of the rationalized fibration has Sullivan model  $\text{CA}(\mathcal{I}A)$ , this being the cofiber of a relative Sullivan model for the rationalized fibrations, as shown on the right of (146). Since relative Sullivan models are cofibrations in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Prop. 3.43), hence fibrations in the opposite model structure (Example A.9), this means, with the fundamental theorem (124) that  $\text{CE}(\mathcal{I}A)$  is in fact a Sullivan model for the homotopy fiber (Def. A.23) of the rationalized fibration. Hence the homotopy fiber of the rationalized fibration is the rationalization  $L_{\mathbb{R}}A$  of the homotopy fiber of the original fibration, as shown in (146).

Together these say that the rationalized fibration is an  $L_{\mathbb{R}}A$ -fibration over  $B(L_{\mathbb{R}}G)$ . With this, Prop. 2.28 implies that the total space of the rationalized fibration is a homotopy quotient  $(L_{\mathbb{R}}A) // (L_{\mathbb{R}}G)$ , which is hence the rationalization of  $A // G$ , as shown in the middle of (146). □

Due to Prop. 3.75 it makes sense to say, in generalization of Def. 3.72:

**Definition 3.76** (Twisted non-abelian real cohomology). Let  $X \in \text{TopologicalSpaces}$  and let  $A // G \xrightarrow{\rho} BG$  be a local coefficient bundle (Prop. 2.28, Def. 2.29) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\geq 1, \text{nil}})^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52). Then the *twisted non-abelian real cohomology* of  $X$  with local coefficients  $\rho$  is the twisted non-abelian  $L_{\mathbb{R}}A$ -cohomology (Def. 2.29) of  $X$  with coefficients in the rationalized local coefficient bundle  $L_{\mathbb{R}}\rho$  from Prop. 3.75:

$$H^{\tau}(X; L_{\mathbb{R}}A) := \text{Ho}\left(\left(\text{TopologicalSpaces}_{\text{Qu}}^{L_{\mathbb{R}}BG}\right)\right)(\tau, L_{\mathbb{R}}\rho).$$

### 3.3 Non-abelian de Rham theorem

We establish non-abelian de Rham theory for differential forms with values in (nilpotent)  $L_{\infty}$ -algebras, following [SSS09a][FSS10]. The main result is the non-abelian de Rham theorem, Theorem 3.87, and its generalization to the twisted non-abelian de Rham theorem, Theorem 3.104.

#### $L_{\infty}$ -Algebra valued differential forms.

**Definition 3.77** (Flat  $L_{\infty}$ -algebra valued differential forms [SSS09a, §6.5][FSS10, §4.1]).

(i) For  $X \in \text{SmoothManifold}$  and  $\mathfrak{g} \in L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.25), a *flat  $\mathfrak{g}$ -valued differential form on  $X$*  is a morphism of dgc-algebras (Def. 3.17)

$$\Omega_{\text{dR}}^{\bullet}(X) \xleftarrow{A} \text{CE}(\mathfrak{g}) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (147)$$

to the smooth de Rham dgc-algebra of  $X$  (Example 3.23) from the Chevalley-Eilenberg dgc-algebra of  $\mathfrak{g}$  (Def. 3.25).

(ii) We write

$$\Omega_{\text{dR}}(X; \mathfrak{g})_{\text{flat}} := \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{CE}(\mathfrak{g}), \Omega_{\text{dR}}^{\bullet}(X)) \quad (148)$$

for the set of all flat  $\mathfrak{g}$ -valued forms on  $X$ .

**Example 3.78** (Flat Lie algebra valued differential forms). Let  $\mathfrak{g} \in \text{LieAlgebras}_{\mathbb{R}, \text{fin}}$  be a Lie algebra (90) with Lie bracket  $[-, -]$ . Then a flat  $\mathfrak{g}$ -valued differential form in the sense of Def. 3.77 is the traditional concept: a  $\mathfrak{g}$ -valued 1-form satisfying the Maurer-Cartan equation:

$$\Omega_{\text{dR}}^{\bullet}(X; \mathfrak{g})_{\text{flat}} \simeq \left\{ A \in \Omega_{\text{dR}}^1(X) \otimes \mathfrak{g} \mid dA + [A \wedge A] = 0 \right\}. \quad (149)$$

One way to see this is to appeal to the classical fact that the Chevalley-Eilenberg algebra of a finite-dimensional Lie algebra (Example 3.24) is isomorphic to the dgc-algebra of left invariant differential forms on the corresponding Lie group  $G$ , which is generated from the Maurer-Cartan form  $\theta \in \Omega_{\text{dR}}^1(G) \otimes \mathfrak{g}$  satisfying  $\theta|_{T_e G} = \text{id}_{\mathfrak{g}}$  and  $d\theta = [\theta \wedge \theta]$ . More explicitly, for  $\{v_a\}$  a linear basis for  $\mathfrak{g}$  (79) with structure constants  $\{f_{ab}^c\}$  (80), we see from (81) that a dgc-algebra homomorphisms (147) has the following components (second line) and constraints (third line):

$$\begin{array}{ccc} \Omega_{\text{dR}}^{\bullet}(X) & \xleftarrow[\text{flat Lie algebra valued differential form}]{A} & \mathbb{R}[\{\theta_1^{(a)}\}] / \left( d\theta_1^{(c)} = f_{ab}^c \theta_1^{(b)} \wedge \theta_1^{(a)} \right) \simeq \text{CE}(\mathfrak{g}). & (150) \\ & & & \\ & \begin{array}{ccc} A^{(c)} & \xleftarrow[\text{components}]{} & \theta_1^{(c)} \\ d \downarrow & & \downarrow d \\ dA^{(c)} & \xleftarrow[\text{constraints}]{} & f_{ab}^c \theta_1^{(a)} \wedge \theta_1^{(b)} \end{array} & \end{array}$$

**Example 3.79** (Ordinary closed forms are flat line  $L_{\infty}$ -algebra valued forms). For  $n \in \mathbb{N}$ , consider  $\mathfrak{g} = \mathfrak{b}^n \mathbb{R}$  the line Lie  $(n+1)$ -algebra (Example 3.27). Then the corresponding flat  $\mathfrak{g}$ -valued differential forms (Def. 3.77) are in natural bijection to ordinary closed  $(n+1)$ -forms:

$$\Omega_{\text{dR}}(X; \mathfrak{b}^n \mathbb{R})_{\text{flat}} \simeq \Omega_{\text{dR}}^{n+1}(X)_{\text{closed}}. \quad (151)$$

That is, by (92), we see that the elements on the left of (151) have the following component (second line) subject to the follows constraint (third line):

$$\begin{array}{ccc}
\Omega_{\text{dR}}^\bullet(X) & \xrightarrow{\text{flat line Lie } (n+1)\text{-algebra-valued differential form}} & \mathbb{R}[c_{n+1}]/(dc_{n+1} = 0) \simeq \text{CE}(\mathfrak{b}^n \mathbb{R}). \\
C_{n+1} & \xleftarrow{\text{component}} & c_{n+1} \\
d \downarrow & & \downarrow d \\
dC_{n+1} & \xleftarrow{\text{constraint}} & 0
\end{array} \tag{152}$$

**Example 3.80** (Flat String Lie 2-algebra valued differential forms). Flat  $L_\infty$ -algebras valued forms (Def. 3.77) with values in a String Lie 2-algebra  $\text{string}_{\mathfrak{g}}$  (Example 3.28) are pairs consisting of a flat  $\mathfrak{g}$ -valued 1-form  $A_1$  (Example 3.78) and a coboundary 2-form  $B_2$  for its Chern-Simons form  $\text{CS}(A) := c\langle A \wedge [A \wedge A] \rangle$ :

$$\Omega_{\text{dR}}(X; \text{string}_{\mathfrak{g}})_{\text{flat}} \simeq \left\{ \begin{array}{l} B_2, \\ A_1 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \mid \begin{array}{l} dB_2 = \frac{1}{c} \text{CS}(A), \\ dA_1 = -[A_1 \wedge A_1] \end{array} \right\}.$$

Namely, from (95) we see that in degree 1 the components of and constraints on such a differential form datum are exactly as in (150), while in degree 2 they are as follows:

$$\begin{array}{ccc}
\Omega_{\text{dR}}^\bullet(X) & \xleftarrow{\text{flat String Lie 2-algebra valued form}} & \mathbb{R} \left[ \begin{array}{l} b_2, \\ \{\theta_1^{(a)}\} \end{array} \right] / \left( \begin{array}{l} db_2 = \mu_{abc} \theta_1^{(c)} \wedge \theta_1^{(b)} \wedge \theta_1^{(a)} \\ d\theta_1^{(c)} = f_{ab}^c \theta_1^{(b)} \wedge \theta_1^{(a)} \end{array} \right) \simeq \text{CE}(\text{string}_{\mathfrak{g}}). \\
B_2 & \xleftarrow{\text{component in degree 2}} & b_2 \\
d \downarrow & & \downarrow d \\
dB_2 & \xleftarrow{\text{constraint}} & \mu_{abc} A_1^{(c)} \wedge A_1^{(b)} \wedge A_1^{(a)}
\end{array} \tag{153}$$

**Example 3.81** (Flat sphere-valued differential forms). Flat  $L_\infty$ -algebras valued forms (Def. 3.77) with values in the rational Whitehead  $L_\infty$ -algebra (Prop. 3.63) of a sphere (Example 3.68) of positive even dimension  $2k$  are pairs consisting of a closed differential  $2k$ -form and a  $(4k-1)$ -form whose differential equals minus the wedge square of the  $2k$ -form:

$$\Omega_{\text{dR}}(-; \mathfrak{S}^{2k}) \simeq \left\{ \begin{array}{l} G_{4k-1}, \\ G_{2k} \end{array} \in \Omega_{\text{dR}}^\bullet(X) \mid \begin{array}{l} dG_{4k-1} = -G_{2k} \wedge G_{2k}, \\ dG_{2k} = 0 \end{array} \right\}.$$

Namely, from (140) one sees that the components of and the constraints on an  $\mathfrak{S}^{2k}$ -valued form are as follows:

$$\begin{array}{ccc}
\Omega_{\text{dR}}^\bullet(X) & \xleftarrow{\text{flat } \mathfrak{S}^{2k}\text{-valued form}} & \mathbb{R} \left[ \begin{array}{l} \omega_{4k-1}, \\ \omega_{2k} \end{array} \right] / \left( \begin{array}{l} d\omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k}, \\ d\omega_{2k} = 0 \end{array} \right) = \text{CE}(\mathfrak{S}^{2k}) \\
G_{2k} & \xleftarrow{\text{component in degree } 2k} & \omega_{2k} \\
d \downarrow & & \downarrow d \\
dG_{2k} & \xleftarrow{\text{constraint}} & 0 \\
G_{4k-1} & \xleftarrow{\text{component in degree } 4k-1} & \omega_{4k-1} \\
d \downarrow & & \downarrow \\
dG_{4k-1} & \xleftarrow{\text{constraint}} & -\omega_{4k} \wedge \omega_{4k}
\end{array} \tag{154}$$

For  $2k = 4$  this is the structure of the equations of motion of the C-field in 11-dimensional supergravity (modulo the Hodge self-duality constraint  $G_7 = \star G_4$ ) [Sa13, §2.5].



**Example 3.82** (PL de Rham right adjoint via  $L_\infty$ -algebra valued forms). For  $n \in \mathbb{N}$ , the right adjoint functor in the PS de Rham adjunction (128) sends the Chevalley-Eilenberg algebra (Def. 3.25) of any  $\mathfrak{g} \in L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}}$  (Def. 3.34) to a simplicial set of flat  $\mathfrak{g}$ -valued differential forms (Def. 3.77):

$$\flat \exp(\mathfrak{g})(\mathbb{R}^n) := \exp_{\text{PS}, n}(\text{CE}(\mathfrak{g})) : [k] \longmapsto \Omega_{\text{dR}}(\mathbb{R}^n \times \Delta^k; \mathfrak{g})_{\text{flat}} \in \text{SimplicialSets}$$

(by direct comparison of (129) with (148)). Regarded as a simplicial presheaf over CartesianSpaces (Def. 354), this construction is the moduli  $\infty$ -stack of flat  $L_\infty$ -algebra valued differential forms (see §4.3 below).

### Non-abelian de Rham cohomology.

**Definition 3.83** (Coboundaries between flat  $L_\infty$ -algebra valued forms). Let  $X \in \text{SmoothManifolds}$  and (from Def. 3.25)  $\mathfrak{g} \in L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$ . For

$$A^{(0)}, A^{(1)} \in \Omega_{\text{dR}}(X; \mathfrak{g})_{\text{flat}}$$

a pair of flat  $\mathfrak{g}$ -valued differential forms on  $X$  (Def. 3.77), we say that a *coboundary* between them is a flat  $\mathfrak{g}$ -valued differential form on the cylinder manifold over  $X$  (its Cartesian product manifold with the real line):

$$\tilde{A} \in \Omega(X \times \mathbb{R}; \mathfrak{g})_{\text{flat}} \quad (155)$$

such that its restrictions along

$$X \simeq X \times \{0\} \xleftarrow{i_0^X} X \times \mathbb{R} \xrightarrow{i_1^X} X \times \{1\} \simeq X$$

are equal to  $A^{(0)}$  and to  $A^{(1)}$ , respectively:

$$(i_0^X)^* \tilde{A} = A^{(0)} \quad \text{and} \quad (i_1^X)^* \tilde{A} = A^{(1)}. \quad (156)$$

If such a coboundary exists, we say that  $A^{(0)}$  and  $A^{(1)}$  are *cohomologous*, to be denoted

$$A^{(0)} \sim A^{(1)}.$$

**Definition 3.84** (Non-abelian de Rham cohomology). Let  $X \in \text{SmoothManifolds}$  and  $\mathfrak{g} \in L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.25). Then the *non-abelian de Rham cohomology* of  $X$  with coefficients in  $\mathfrak{g}$  is the set

$$H_{\text{dR}}(X; \mathfrak{g}) := (\Omega_{\text{dR}}(X; \mathfrak{g})_{\text{flat}})_{/\sim} \quad (157)$$

of equivalence classes with respect to the coboundary relation from Def. 3.83 on the set of flat  $\mathfrak{g}$ -valued differential forms on  $X$  (Def. 3.77).

We recall the following basic facts (e.g. [GT00, Rem 3.1]):

**Lemma 3.85** (Fiberwise Stokes theorem and Projection formula). *Let  $X$  be a smooth manifold and let  $F$  be a compact smooth manifold with corners, e.g.  $F = \Delta^k$  a standard  $k$ -simplex, which for  $k = 1$  is the interval  $F = [0, 1]$ .*

*Then fiberwise integration over  $F$  of differential forms on the Cartesian product manifold  $X \times F$*

$$\Omega_{\text{dR}}^\bullet(X \times F) \xrightarrow{\int_F} \Omega_{\text{dR}}^{\bullet - \dim(F)}(X) \quad \text{e.g.} \quad \Omega_{\text{dR}}^\bullet(X \times \mathbb{R}) \xrightarrow{\int_{[0,1]}} \Omega_{\text{dR}}^{\bullet-1}(X)$$

*satisfies, for all  $\alpha \in \Omega_{\text{dR}}^\bullet(X \times F)$  and  $\beta \in \Omega_{\text{dR}}^\bullet(X)$ :*

(i) *The fiberwise Stokes formula:*

$$\int_F d\alpha = (-1)^{\dim(F)} d \int_F \alpha + \int_{\partial F} \alpha \quad \text{e.g.} \quad d \int_{[0,1]} \alpha = (i_1^X)^* \alpha - (i_0^X)^* \alpha - \int_{[0,1]} d\alpha, \quad (158)$$

where

$$X \simeq X \times \{0\} \xleftarrow{i_0^X} X \times \mathbb{R} \xrightarrow{i_1^X} X \times \{1\} \simeq X$$

are the boundary inclusions.

(ii) *The projection formula*

$$\int_F (\text{pr}_X^* \beta) \wedge \alpha = (-1)^{\dim(F) \deg(\beta)} \beta \wedge \int_F \alpha, \quad \text{e.g.} \quad \int_{[0,1]} (\text{pr}_X^* \beta) \wedge \alpha = (-1)^{\deg(\beta)} \beta \wedge \int_{[0,1]} \alpha, \quad (159)$$

where

$$X \times F \xrightarrow{\text{pr}_X} X$$

is projection on the first factor.

**Proposition 3.86** (Non-abelian de Rham cohomology subsumes ordinary de Rham cohomology). *For any  $n \in \mathbb{N}$ , let  $\mathfrak{g} = \mathfrak{b}^n \mathbb{R}$  be the line Lie  $(n+1)$ -algebra (Example 3.27). Then the non-abelian de Rham cohomology with coefficients in  $\mathfrak{g}$  (Def. 3.84) is naturally equivalent to ordinary de Rham cohomology in degree  $n+1$ :*

$$H_{\text{dR}}(-; \mathfrak{b}^n \mathbb{R}) \simeq H_{\text{dR}}^{n+1}(-). \quad (160)$$

*Proof.* From Example 3.79, we know that the canonical cocycle sets are in natural bijection

$$\Omega_{\text{dR}}(X; \mathfrak{b}^n \mathbb{R})_{\text{flat}} \simeq \Omega_{\text{dR}}^{n+1}(X)_{\text{closed}}.$$

Therefore, it just remains to see that the coboundary relations in both cases coincide. By the explicit nature (152) of the above natural bijection and by the Definition 3.83 of non-abelian coboundaries, we hence need to see that a pair of closed forms

$$C_{n+1}^{(0)}, C_{n+1}^{(1)} \in \Omega_{\text{dR}}^{n+1}(X)_{\text{closed}}$$

has a de Rham coboundary, i.e.,

$$\exists h_n \in \Omega_{\text{dR}}^n(X), \quad \text{such that } C_{n+1}^{(0)} + dh_n = C_{n+1}^{(1)}, \quad (161)$$

precisely if the pair extends to a closed  $(n+1)$ -form on the cylinder over  $X$ , as in (155) (156):

$$\exists \tilde{C}_{n+1} \in \Omega_{\text{dR}}^{n+1}(X \times \mathbb{R})_{\text{closed}}, \quad \text{such that } (i_0^X)^* \tilde{C}_{n+1} = C_{n+1}^{(0)} \quad \text{and} \quad (i_1^X)^* \tilde{C}_{n+1} = C_{n+1}^{(1)}. \quad (162)$$

That (161)  $\Leftrightarrow$  (162) is a standard argument: Let  $t$  denote the canonical coordinate function on  $\mathbb{R}$ . In one direction, given  $h_n$  as in (161), the choice

$$\tilde{C}_{n+1} := (1-t) \text{pr}_X^*(C_{n+1}^{(0)}) + t \text{pr}_X^*(C_{n+1}^{(1)}) + dt \wedge \text{pr}_X^*(h_n)$$

clearly satisfies (162). In the other direction, given  $\tilde{C}_{n+1}$  as in (162), the choice

$$h_n := \int_{[0,1]} \tilde{C}_{n+1}$$

satisfies (161), by the fiberwise Stokes theorem (Lemma 3.85).  $\square$

### The non-abelian de Rham theorem.

**Theorem 3.87** (Non-abelian de Rham theorem). *Let  $X, A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}\mathbb{R}}$  (Def. 3.52), and let  $X$  admit the structure of a smooth manifold. Then the non-abelian de Rham cohomology (Def. 3.84) of  $X$  with coefficients in the real Whitehead  $L_\infty$ -algebra  $\mathfrak{A}$  (Prop. 3.63) is in natural bijection with the non-abelian real cohomology (Def. 3.72) of  $X$  with coefficients in  $L_{\mathbb{R}}\mathfrak{A}$  (Def. 3.55):*

$$H(X; L_{\mathbb{R}}\mathfrak{A}) \simeq H_{\text{dR}}(X; \mathfrak{A}). \quad (163)$$

*Proof.* Consider the following sequence of natural bijections:

$$\begin{aligned} H(X; L_{\mathbb{R}}\mathfrak{A}) &= \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, L_{\mathbb{R}}\mathfrak{A}) \\ &\simeq \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}})(\Omega_{\text{PLdR}}^\bullet(A), \Omega_{\text{PLdR}}^\bullet(X)) \\ &\simeq \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}})(\text{CE}(\mathfrak{A}), \Omega_{\text{PL}}^\bullet(X)) \\ &\simeq H_{\text{dR}}(X; \mathfrak{A}). \end{aligned} \quad (164)$$

Here the first line is the definition Def. 3.72. Then the first step is the fundamental theorem of rational homotopy theory (Prop. 3.60). The second step uses the following isomorphisms:  $\text{CE}(\mathfrak{A}) \simeq \Omega_{\text{PLdR}}^\bullet(A)$  (Prop. 3.63) and  $\Omega_{\text{PLdR}}^\bullet(X) \simeq \Omega_{\text{dR}}^\bullet(X)$  (Lemma 3.90) in the homotopy category. The last step is Lemma 3.89. The composite of these natural bijections gives the desired bijection (163).  $\square$

We now prove the three lemmas used in the proof of Theorem 3.87:

**Lemma 3.88** (De Rham complex over cylinder of manifold is path space object). *For  $X \in \text{SmoothManifolds}$ , consider the following morphisms of dgc-algebras (Def. 3.17)*

$$\Omega_{\text{dR}}^{\bullet}(X) \xrightarrow{(\text{pr}_X)^*} \Omega_{\text{dR}}^{\bullet}(X \times \mathbb{R}) \xrightarrow{(i_0^*, i_1^*)} \Omega_{\text{dR}}^{\bullet}(X) \oplus \Omega_{\text{dR}}^{\bullet}(X) \quad (165)$$

(from the de Rham complex of  $X$  (Example 3.23) to that of its cylinder manifold  $X \times \mathbb{R}$ , to its Cartesian product with itself, by Example 3.22), given by pullback of differential forms along these smooth functions:

$$X \xleftarrow{\text{pr}_X} X \times \mathbb{R} \xleftarrow{(i_0, i_1)} (X \times \{0\}) \sqcup (X \times \{1\}) \simeq X \sqcup X.$$

This is a path space object (Def. A.11) for  $\Omega_{\text{dR}}^{\bullet}(X)$  in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Prop. 3.36).

*Proof.* (i) It is clear by construction that the composite morphism is the diagonal.

(ii) That  $(\text{pr}_X)^*$  is a weak equivalence, hence a quasi-isomorphism, follows from the de Rham theorem, using that ordinary cohomology is homotopy invariant:  $H^{\bullet}(X \times \mathbb{R}; \mathbb{R}) \simeq H^{\bullet}(X; \mathbb{R})$ .

(iii) That  $(i_0^*, i_1^*)$  is a fibration, namely degreewise surjective, is seen from the fact that any pair of forms on the boundaries  $X \times \{0\}$ ,  $X \times \{1\}$  may be smoothly interpolated to zero along any small enough positive parameter length, and then glued to a form on  $X \times \mathbb{R}$ .  $\square$

**Lemma 3.89** (Non-abelian de Rham cohomology via the dgc-homotopy category). *Let  $X \in \text{SmoothManifolds}$  and  $\mathfrak{g} \in L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0, \text{nil}}$  (Def. 3.34). Then the non-abelian de Rham cohomology of  $X$  with coefficients in  $\mathfrak{g}$  (Def. 3.84) is in natural bijection with the hom-set in the homotopy category of  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Prop. 3.36) from  $\text{CE}(\mathfrak{g})$  (Def. 3.25) to  $\Omega_{\text{dR}}^{\bullet}(X)$  (Example 3.23):*

$$H_{\text{dR}}(X; \mathfrak{g}) \simeq \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}})(\text{CE}(\mathfrak{g}), \Omega_{\text{dR}}^{\bullet}(X)). \quad (166)$$

*Proof.* Consider a pair of dgc-algebra homomorphisms

$$A^{(0)}, A^{(1)} \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{CE}(\mathfrak{g}), \Omega_{\text{dR}}^{\bullet}(X)) \quad (167)$$

hence of flat  $\mathfrak{g}$ -valued differential forms, according to Def. 3.77. Observe that:

(i)  $\text{CE}(\mathfrak{g})$  is cofibrant in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (105). (by Prop. 3.43, and since  $\mathfrak{g}$  is assumed to be nilpotent (103));

(ii)  $\Omega_{\text{dR}}^{\bullet}(X)$  is fibrant in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (105). (by Remark 3.37);

(iii) A right homotopy (Def. A.13) between the pair (167) of morphisms, with respect to the path space object  $\Omega_{\text{dR}}^{\bullet}(X \times \mathbb{R})$  from Lemma 3.88, namely a morphism  $\tilde{A}$  making the following diagram commute

$$\begin{array}{ccc} \Omega_{\text{dR}}^{\bullet}(X) & & \\ \uparrow i_0^* & \swarrow A^{(0)} & \\ \Omega_{\text{dR}}^{\bullet}(X \times \mathbb{R}) & \xleftarrow{\tilde{A}} & \text{CE}(\mathfrak{g}) \\ \downarrow i_1^* & \searrow A^{(1)} & \\ \Omega_{\text{dR}}^{\bullet}(X) & & \end{array} \quad (168)$$

is manifestly the same as a coboundary  $\tilde{A}$  between the corresponding flat  $\mathfrak{g}$ -valued forms according to Def. 3.83.

Therefore, Prop. A.17 says that the quotient set (157) defining the non-abelian de Rham cohomology is in natural bijection to the hom-set in the homotopy category.  $\square$

**Lemma 3.90** (PL de Rham complex on smooth manifold is equivalent to smooth de Rham complex). *Let  $X$  be a smooth manifold. Then*

(i) *There exists a zig-zag of weak equivalences (Def. 3.35) in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (105) between the smooth de Rham complex of  $X$  (Example 3.23) and the PL de Rham complex of its underlying topological space (Def. 3.56).*

(ii) *In particular, both are isomorphic in the homotopy category:*

$$X \text{ smooth manifold} \quad \Rightarrow \quad \Omega_{\text{dR}}^{\bullet}(X) \simeq \Omega_{\text{PLdR}}^{\bullet}(X) \in \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}).$$

*Proof.* Let  $\Omega_{\text{PSdR}}^\bullet(-)$  (for “piecewise smooth”) be defined as the PL de Rham complex in Def. 3.56, but with smooth differential forms on each simplex. Notice that this comes with the canonical natural inclusion

$$\Omega_{\text{PLdR}}^\bullet(-) \xrightarrow{i_{\text{poly}}} \Omega_{\text{PSdR}}^\bullet(-) .$$

Let then  $\text{Tr}(X) \in \text{SimplicialSets}$  be any smooth triangulation of  $X$ . This means that we have a homeomorphism out of its geometric realization to  $X$

$$|\text{Tr}(X)| \xrightarrow[\text{homeo}]{p} X, \quad (169)$$

which restricts on the interior of each simplex to a diffeomorphism onto its image; and that we have an inclusion

$$\text{Tr}(X) \xrightarrow[\in \mathbb{W}]{\eta_{\text{Tr}(X)}} \text{Sing}(|\text{Tr}(X)|) \xrightarrow[\in \text{Iso}]{\text{Sing}(p)} \text{Sing}(X), \quad (170)$$

which is a weak equivalence (by Example A.36). In summary, this gives us the following zig-zag of dgc-algebra homomorphisms:

$$\begin{array}{ccc} & \Omega_{\text{PLdR}}^\bullet(\text{Tr}(X)) & \Omega_{\text{dR}}^\bullet(X) \\ & \nearrow (\eta_s)^* & \searrow p^* \\ \Omega_{\text{PLdR}}^\bullet(X) = \Omega_{\text{PLdR}}^\bullet(\text{Sing}(X)) & & \Omega_{\text{PSdR}}^\bullet(\text{Tr}(X)) \\ & \searrow i_{\text{poly}} & \nearrow \end{array}$$

Here the two morphisms on the right are quasi-isomorphisms by [GM13, Cor. 9.9] (as in Prop. 3.62). The morphism on the left is a quasi-isomorphism because  $i$  is a weak homotopy equivalence (346) and since  $\Omega_{\text{PLdR}}^\bullet$  preserves weak equivalences, by Ken Brown’s Lemma (Lemma A.20), since it is a Quillen left adjoint, by Prop. 3.59, and since every simplicial set is cofibrant (Example A.8).  $\square$

**Flat twisted  $L_\infty$ -algebra valued differential forms.** We generalize the above discussion to include twistings.

**Definition 3.91** (Local  $L_\infty$ -algebraic coefficients). We say that a *local  $L_\infty$ -algebraic coefficient bundle* is a fibration

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \widehat{\mathfrak{b}} \\ & & \downarrow p \\ & & \mathfrak{b} \end{array} \quad (171)$$

in  $L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.25), hence a morphism such that under passage to Chevalley-Eilenberg algebras (89) we have a cofibration

$$\begin{array}{ccc} \text{CE}(\mathfrak{g}) & \xleftarrow{\text{cofib}(\text{CE}(p))} & \text{CE}(\widehat{\mathfrak{b}}) \\ & & \text{CE}(\mathfrak{p}) \uparrow \in \text{Cof} \\ & & \text{CE}(\mathfrak{b}) \end{array} \quad (172)$$

in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Prop. 3.36).

In generalization of Def. 3.77, we say:

**Definition 3.92** (Flat twisted  $L_\infty$ -algebra valued differential forms).

(i) Let  $X \in \text{SmoothManifolds}$  and  $\widehat{\mathfrak{b}}$  (171) a local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91). For

$$\tau_{\text{dR}} \in \Omega_{\text{dR}}(X; \mathfrak{b})_{\text{flat}} \quad (173)$$

a flat  $\mathfrak{b}$ -valued differential form on  $X$  (Def. 3.77), we say that a *flat  $\tau$ -twisted  $\mathfrak{g}$ -valued differential form* on  $X$  is a morphism of dgc-algebras (Def. 3.17) in the slice over  $\text{CE}(\mathfrak{b})$

$$\begin{array}{ccc} \Omega_{\text{dR}}^\bullet(X) & \xleftarrow[\text{flat } \tau_{\text{dR}}\text{-twisted } \mathfrak{g}\text{-valued differential form}]{A} & \text{CE}(\widehat{\mathfrak{b}}) \\ & \searrow \tau_{\text{dR}} \text{ twist} & \nearrow \text{CE}(\mathfrak{p}) \\ & & \text{CE}(\mathfrak{b}) \end{array} \quad \begin{array}{l} \text{local} \\ L_\infty\text{-algebraic} \\ \text{coefficients} \end{array} \quad (174)$$

(ii) We write

$$\Omega_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{g})_{\text{flat}} := (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{/\text{CE}(\mathfrak{b})}(\tau_{\text{dR}}, \mathfrak{p})$$

for the set of all flat  $\tau_{\text{dR}}$ -twisted  $\mathfrak{g}$ -valued differential forms on  $X$ .

**Remark 3.93** (Underlying flat forms of flat twisted forms). Let  $X \in \text{SmoothManifolds}$ , let  $\mathfrak{g} \xrightarrow{\widehat{\mathfrak{b}}} \mathfrak{b}$  be a local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91), and let  $\tau_{\text{dR}} \in \Omega_{\text{dR}}(X; \mathfrak{b})$ . Then there is a canonical forgetful natural transformation

$$\Omega^{\tau_{\text{dR}}}(X; \mathfrak{g})_{\text{flat}} \longrightarrow \Omega(X; \widehat{\mathfrak{b}})_{\text{flat}} \quad (175)$$

from flat  $\tau_{\text{dR}}$ -twisted  $\mathfrak{g}$ -valued differential forms (Def. 3.92) to flat  $\widehat{\mathfrak{b}}$ -valued differential forms (Def. 3.77), given by remembering only the top morphism in (174).

**Example 3.94** ( $L_\infty$ -coefficient bundle for  $H_3$ -twisted differential forms [FSS16a, §4][FSS16b, §4][BMSS19, Lem. 2.31]). Consider the local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91) given by the following multivariate polynomial dgc-algebras (Def. 3.30):

$$\text{CE}(\text{lku}_1) = \mathbb{R} \left[ \begin{array}{c} \vdots \\ f_5, \\ f_3, \\ f_1, \end{array} \right] / \left( \begin{array}{c} \vdots \\ df_5 = 0 \\ df_3 = 0 \\ df_1 = 0 \end{array} \right) \xleftarrow{\omega_{2k+1} \leftarrow \omega_{2k+1}} \mathbb{R} \left[ \begin{array}{c} \vdots \\ f_5, \\ f_3, \\ f_1, \\ h_3 \end{array} \right] / \left( \begin{array}{c} \vdots \\ df_5 = h_3 \wedge f_3, \\ df_3 = h_3 \wedge f_1, \\ df_1 = 0, \\ dh_3 = 0 \end{array} \right) = \text{CE}(\iota(\text{ku}_1 // BU(1)))$$

$$\begin{array}{c} \uparrow h_3 \\ \uparrow \hat{=} \\ \uparrow h_3 \\ \mathbb{R}[h_3](dh_3 = 0) = \text{CE}(\mathfrak{b}^2\mathbb{R}) \end{array}$$

Here the rational model of the classifying space  $\text{ku}_1$  for complex topological K-theory in degree 1 and for its twisted version is as in [FSS16a, §4][FSS16b, §4][BMSS19, Lem. 2.31]. In this case:

(i) A twist (173) is equivalently an ordinary closed 3-form form (by Example 3.79):

$$H_3 \in \Omega_{\text{dR}}(X; \mathfrak{b}^2\mathbb{R})_{\text{flat}} \simeq \Omega_{\text{dR}}^3(X)_{\text{closed}}. \quad (176)$$

(ii) The flat  $\tau_{\text{dR}} \sim H_3$ -twisted  $\text{lku}_1$ -valued differential forms according to Def. 3.92 are equivalently sequences of odd-degree differential forms  $F_{2k+1} \in \Omega_{\text{dR}}^{2k+1}(X)$  satisfying the  $H_3$ -twisted de Rham closure condition (see [RW86, (23)][GS19c]):

$$\Omega^{\tau_{\text{dR}}}(X; \text{lku}_1)_{\text{flat}} \simeq \left\{ F_{2\bullet+1} \in \Omega_{\text{dR}}^{2\bullet+1} \mid d \sum_k F_{2k+1} = H_3 \wedge \sum_k F_{2k-1} \right\} \quad (177)$$

(where we set  $F_{2k-1} := 0$  if  $2k-1 < 0$ , for convenience of notation).

In direct generalization of Example 3.94, we have:

**Example 3.95** ( $L_\infty$ -coefficient bundle for higher twisted differential forms [FSS18, Def. 2.14]). For  $r \in \mathbb{N}$ ,  $r \geq 1$ , consider the local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91) given by the following multivariate polynomial dgc-algebras (Def. 3.30):

$$\text{CE} \left( \bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk}\mathbb{R} \right) \quad \text{CE} \left( \left( \bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk}\mathbb{R} \right) // B^{2r-1}U(1) \right)$$

$$\parallel \quad \parallel \quad (178)$$

$$\mathbb{R} \left[ \begin{array}{c} \vdots \\ f_{4r+1}, \\ f_{2r+1}, \\ f_1, \end{array} \right] / \left( \begin{array}{c} \vdots \\ df_{4r+1} = 0 \\ df_{2r+1} = 0 \\ df_1 = 0 \end{array} \right) \xleftarrow{f_{2rk+1} \leftarrow f_{2rk+1}} \mathbb{R} \left[ \begin{array}{c} \vdots \\ f_{4r+1}, \\ f_{2r+1}, \\ f_1, \\ h_{2r+1} \end{array} \right] / \left( \begin{array}{c} \vdots \\ df_{4r+1} = h_{2r+1} \wedge f_{2r+1}, \\ df_{2r+1} = h_{2r+1} \wedge f_1, \\ df_1 = 0, \\ dh_{2r+1} = 0 \end{array} \right)$$

$$\begin{array}{c} \uparrow h_{2r+1} \\ \uparrow \hat{=} \\ \uparrow h_{2r+1} \\ \mathbb{R}[h_{2r+1}](dh_{2r+1} = 0) \\ \parallel \\ \text{CE}(\mathfrak{b}^{2r}\mathbb{R}) \end{array}$$

In this case:

(i) A twist (173) is equivalently an ordinary closed  $(2r+1)$ -form form (by Example 3.79):

$$H_{2r+1} \in \Omega_{\text{dR}}(X; \mathfrak{b}^{2r}\mathbb{R})_{\text{flat}} \simeq \Omega_{\text{dR}}^{2r+1}(X)_{\text{closed}}. \quad (179)$$

(ii) The flat  $\tau_{\text{dR}} \sim H_{2r+1}$ -twisted  $\bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk}\mathbb{R}$ -valued differential forms according to Def. 3.92 are equivalently sequences of differential forms  $F_{2r\bullet+1} \in \Omega_{\text{dR}}^{2k\bullet+1}(X)$  satisfying the  $H_{(2r+1)}$ -twisted de Rham closure condition (189):

$$\Omega^{\tau_{\text{dR}}}(X; \bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk}\mathbb{R})_{\text{flat}} \simeq \left\{ F_{2r\bullet+1} \in \Omega_{\text{dR}}^{2r\bullet+1} \mid d \sum_k F_{2rk+1} = H_{2r+1} \wedge \sum_k F_{2rk-1} \right\} \quad (180)$$

(where we set  $F_{2rk-1} := 0$  if  $2rk-1 < 0$ , for convenience of notation).

In twisted generalization of Example 3.81, we have the following:

**Example 3.96** (Flat twisted differential forms with values in Whitehead  $L_\infty$ -algebras of spheres and twistor space). The  $L_\infty$ -algebraic local coefficient bundles (Def. 3.91) given as the relative Whitehead  $L_\infty$ -algebras (Prop. 3.70) of the local coefficient bundles (61) for twisted and twistorial Cohomotopy (Example 2.44) are as shown on the right of the following diagram [FSS19b, Lemma 3.19][FSS20, Thm. 2.14]:

$$\begin{array}{ccc}
 & \text{CE}(\mathfrak{l}_{B\text{Sp}(2)})(\mathbb{C}P^3 // \text{Sp}(2)) & = \text{CE}(\mathfrak{l}B\text{Sp}(2)) \left[ \begin{array}{c} h_3, \\ f_2, \\ \omega_7, \\ \omega_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 - \chi_8 \\ d\omega_4 = 0 \end{array} \right) \\
 \swarrow (G_4, G_7, F_2, H_3) & \uparrow (t_{\mathbb{H}} // \text{Sp}(2))^* & \uparrow \\
 \Omega_{\text{dR}}^\bullet(X) \xleftarrow{(G_4, 2G_7)} \text{CE}(\mathfrak{l}_{B\text{Sp}(2)})(S^4 // \text{Sp}(2)) & = \text{CE}(\mathfrak{l}B\text{Sp}(2)) \left[ \begin{array}{c} \omega_7, \\ \omega_4 \end{array} \right] / \left( \begin{array}{l} d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 - \chi_8 \\ d\omega_4 = 0 \end{array} \right) \\
 \uparrow \tau_{\text{dR}} & \nearrow & \uparrow \\
 \text{CE}(\mathfrak{l}B\text{Sp}(2)) & = \mathbb{R} \left[ \begin{array}{c} \chi_8, \\ \frac{1}{2}p_1 \end{array} \right] / \left( \begin{array}{l} d\chi_8 = 0 \\ d\frac{1}{2}p_1 = 0 \end{array} \right) & \nearrow
 \end{array}$$

Therefore, given a smooth 8-dimensional spin manifold  $X$  equipped with tangential  $\text{Sp}(2)$ -structure  $\tau$  (59), the flat  $\tau_{\text{dR}}$ -twisted  $\mathbb{S}^4$ - and  $\mathbb{C}P^3$ -valued differential forms (Def. 3.92) are of the following form [FSS19b, Prop. 3.20][FSS20, Prop. 3.9]:

$$\begin{aligned}
 \Omega_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathbb{S}^4) &= \left\{ \begin{array}{c} 2G_7, \\ G_4 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \mid \begin{array}{l} d2G_7 = -(G_4 - \frac{1}{4}p_1(\nabla)) \wedge (G_4 + \frac{1}{4}p_1(\nabla)) - \chi_8(\nabla), \\ dG_4 = 0 \end{array} \right\} \\
 \Omega_{\text{dR}}^{\tilde{\tau}_{\text{dR}}}(X; \mathbb{C}P^3) &= \left\{ \begin{array}{c} H_3, \\ F_2, \\ 2G_7, \\ G_4 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \mid \begin{array}{l} dH_3 = G_4 - \frac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ dF_2 = 0, \\ d2G_7 = -(G_4 - \frac{1}{4}) \wedge (G_4 + \frac{1}{4}) - \chi_8(\nabla), \\ dG_4 = 0, \end{array} \right\} \quad (181)
 \end{aligned}$$

Notice:

- (a) Here we are using (Example 4.27) that the de Rham image  $\tau_{\text{dR}}$  of the rationalization  $L_{\mathbb{R}}\tau$  of the twist  $\tau$  is given by evaluating characteristic forms (Def. 4.19) on any  $\text{Sp}(2)$ -connection  $\nabla$ .
- (b) In the second equation of (181) we are using the above minimal model for  $\mathbb{C}P^3 // \text{Sp}(2)$  relative to  $\mathbb{S}^4 // \text{Sp}(2)$  (instead of relative to  $B\text{Sp}(2)$ ).

**Twisted non-abelian de Rham cohomology.** In generalization of Def. 3.83, we set:

**Definition 3.97** (Coboundaries between flat twisted  $L_\infty$ -algebraic forms). Let  $X \in \text{SmoothManifolds}$ , let  $\mathfrak{g} \rightarrow \widehat{\mathfrak{b}} \xrightarrow{\mathfrak{p}} \mathfrak{b}$  be a local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91), and let  $\tau_{\text{dR}} \in \Omega_{\text{dR}}(X; \mathfrak{b})$ . Then for

$$A^{(0)}, A^{(1)} \in \Omega_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{g})$$

a pair of flat  $\tau_{\text{dR}}$ -twisted  $\mathfrak{g}$ -valued differential forms on  $X$  (Def. 3.92) a *coboundary* between them is a coboundary

$$\widetilde{A} \in \Omega_{\text{dR}}(X \times \mathbb{R}; \widehat{\mathfrak{b}}) \quad (182)$$

in the sense of Def. 3.83 between the underlying flat  $\widehat{\mathfrak{b}}$ -valued forms (via Remark 3.93), such that the underlying  $\mathfrak{b}$ -valued form of  $H$  equals the pullback of the twist  $\tau_{\text{dR}}$  along  $X \times \mathbb{R} \xrightarrow{\text{pr}_X} X$

$$\mathfrak{p}_*(H) = \text{pr}_X^*(\tau_{\text{dR}}). \quad (183)$$

If such a coboundary exists, we say that  $A^{(0)}$  and  $A^{(1)}$  are *cohomologous*, to be denoted

$$A^{(0)} \sim A^{(1)}.$$

In generalization of Def. 3.84, we set:

**Definition 3.98** (Twisted non-abelian de Rham cohomology). Let  $X \in \text{SmoothManifolds}$ , let  $\mathfrak{g} \rightarrow \widehat{\mathfrak{b}} \xrightarrow{\mathfrak{p}} \mathfrak{b}$  be a local  $L_\infty$ -algebraic coefficient bundle (Def. 3.91), and let  $\tau_{\text{dR}} \in \Omega_{\text{dR}}(X; \mathfrak{b})$ . Then the  $\tau_{\text{dR}}$ -*twisted non-abelian de Rham cohomology* of  $X$  with coefficients in  $\mathfrak{g}$  is the set

$$H_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{g}) := (\Omega_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{g})_{\text{flat}})_{/\sim} \quad (184)$$

of equivalence classes with respect to the coboundary relation from Def. 3.97 on the set of flat  $\tau_{\text{dR}}$ -twisted  $\mathfrak{g}$ -valued differential forms on  $X$  (Def. 3.92).

Twisted de Rham cohomology is traditionally familiar in the form of degree-3 twisted cohomology of even/odd degree differential forms [RW86, §III, Appendix][BCMMS02, §9.3][MaSt03, §3][FrHT08, §2][Te04, Prop. 3.7][Cav05, §I.4][Sa10][MW11][GS19b] (which is the target of the twisted Chern character in degree-3 twisted K-theory, see Prop. 5.5):

**Definition 3.99** (Degree-3 twisted abelian de Rham cohomology). For  $X \in \text{SmoothManifolds}$ , and  $H_3 \in \Omega_{\text{dR}}^3(X)_{\text{closed}}$  a closed differential 3-form, the  $H_3$ -*twisted de Rham cohomology* of  $X$  is the cochain cohomology<sup>10</sup>

$$H_{\text{dR}}^{\bullet+H_3}(X) := \frac{\ker^\bullet(d - H_3 \wedge (-))}{\text{im}^\bullet(d - H_3 \wedge (-))} \quad (185)$$

of the following 2-periodic cochain complex:

$$\dots \longrightarrow \bigoplus_k \Omega_{\text{dR}}^{(n-1)+2k}(X) \xrightarrow{(d-H_3 \wedge (-))} \bigoplus_k \Omega_{\text{dR}}^{n+2k}(X) \xrightarrow{(d-H_3 \wedge (-))} \bigoplus_k \Omega_{\text{dR}}^{(n+1)+2k}(X) \longrightarrow \dots$$

We show that this is a special case of twisted non-abelian de Rham cohomology according to Def. 3.98:

**Proposition 3.100** (Twisted non-abelian de Rham cohomology subsumes  $H_3$ -twisted abelian de Rham cohomology). *Given a twisting 3-form as in (176)*

$$\begin{array}{ccc} \tau_{\text{dR}} & \longleftrightarrow & H_3 \\ \cap & & \cap \\ \Omega(X; \mathfrak{b}^2 \mathbb{R})_{\text{flat}} & \simeq & \Omega^3(X)_{\text{closed}} \end{array}$$

the  $\tau_{\text{dR}}$ -*twisted non-abelian de Rham cohomology* (Def. 3.98) of flat twisted  $\mathfrak{ku}_1$ -valued differential forms (Example 3.94) is naturally equivalent to  $H_3$ -*twisted abelian de Rham cohomology* (Def. 3.99) in odd degree<sup>11</sup>

$$\begin{array}{ccc} \text{\color{blue}b}^2\mathbb{R}\text{-twisted } \mathfrak{ku}_1\text{-valued} & & \text{\color{blue}traditional } H_3\text{-twisted} \\ \text{\color{blue}non-abelian de Rham cohomology} & & \text{\color{blue}de Rham cohomology} \\ H_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{ku}_1) & \simeq & H_{\text{dR}}^{1+H_3}(X) \end{array}$$

<sup>10</sup>The notation “ $H_3$ ” for the twist (and of “ $H_{2r+1}$ ” for the higher twists later) originates in the physics literature and has made it as a convention in differential geometry as well. Of course, not to be confused with homology.

<sup>11</sup>The discussion for even degrees is directly analogous and we omit it for brevity.

*Proof.* By (177) in Example 3.94 the cocycle sets on both sides are in natural bijection. Hence it is sufficient to see that the coboundary relations on the cocycle sets coincide, under this identification. In one direction, consider a coboundary in the sense of twisted non-abelian de Rham cohomology (Def. 3.97) with coefficients as in Example 3.94:

$$\tilde{F}_{2\bullet+1} \in \Omega_{\text{dR}}(X \times \mathbb{R}; \mathfrak{k}u_1).$$

We claim that

$$h_{2\bullet} := \int_{[0,1]} \tilde{F}_{2\bullet+1} \quad (186)$$

satisfies the coboundary condition (185):

$$(d - H_3 \wedge) \sum_k h_{2k} = \sum_k (F_{2k+1}^{(1)} - F_{2k+1}^{(0)}). \quad (187)$$

To see this, we may compute as follows:

$$\begin{aligned} d \sum_k h_{2k} &= \sum_k \left( F_{2k+1}^{(1)} - F_{2k+1}^{(0)} - \int_{[0,1]} d\tilde{F}_{2k+1} \right) \\ &= \sum_k \left( F_{2k+1}^{(1)} - F_{2k+1}^{(0)} - \int_{[0,1]} (\text{pr}_X^* H_3) \wedge \tilde{F}_{2k-1} \right) \\ &= \sum_k \left( F_{2k+1}^{(1)} - F_{2k+1}^{(0)} + H_3 \wedge \int_{[0,1]} \tilde{F}_{2k-1} \right) \\ &= \sum_k \left( F_{2k+1}^{(1)} - F_{2k+1}^{(0)} + H_3 \wedge h_{2k-2} \right), \end{aligned}$$

where the first step is the fiberwise Stokes formula (158) together with the defining restrictions (156) of  $\tilde{F}_{2\bullet+1}$ ; the second step is the cocycle condition (177) on  $\tilde{F}_{2\bullet+1}$  using the constraint (183); the third step is the projection formula (159); and the last step uses again the definition (186).

Conversely, given  $h_{2\bullet}$  satisfying (187), we claim that

$$\tilde{F}_{2\bullet+1} := (1-t) \text{pr}_1^*(F_{2\bullet+1}^{(0)}) + t \text{pr}_1^*(F_{2\bullet+1}^{(1)}) + dt \wedge \text{pr}_X^*(h_{2\bullet}) \quad (188)$$

is a coboundary of twisted non-abelian cocycles, in the sense of Def. 3.97: It is immediate that (188) has the required restrictions (156). We check by direct computation that it satisfies the required differential equation:

$$\begin{aligned} d \sum_k \tilde{F}_{2k+1} &= \sum_k \left( -dt \wedge \text{pr}_X^*(F_{2k+1}^{(0)}) + (1-t) \text{pr}_X^*(H_3) \wedge \text{pr}_X^*(F_{2k-1}^{(0)}) \right. \\ &\quad \left. + dt \wedge \text{pr}_X^*(F_{2k+1}^{(1)}) + t \text{pr}_X^*(H_3) \wedge \text{pr}_X^*(F_{2k-1}^{(1)}) \right. \\ &\quad \left. - dt \wedge \text{pr}_X^*(\underbrace{dh_{2k}}_{=F_{2k+1}^{(1)} - F_{2k+1}^{(0)} + H_3 \wedge h_{2k}}) \right) \\ &= \sum_k \left( \text{pr}_X^*(H_3) \wedge \tilde{F}_{2k-1} \right). \quad \square \end{aligned}$$

In generalization of Def. 3.99, there are twisted abelian Rham complexes with twist any odd-degree closed form [Te04][Sa09][MW11][Sa10][GS19b] (these serve as the targets for the Chern character [MMS20] on higher-twisted ordinary K-theory [Te04][Go08][DP13][Pen16], see Example 5.7 below; and for the LSW-character on twisted higher K-theories [LSW16, §2.1], see Prop. 5.10 below):



**Definition 3.101** (Higher twisted abelian de Rham cohomology). For  $X \in \text{SmoothManifolds}$ ,  $r \in \mathbb{N}$ ,  $r \geq 1$ , and  $H_{2r+1} \in \Omega_{\text{dR}}^{2r+1}(X)_{\text{closed}}$  a closed differential  $(2r+1)$ -form, the  $H_{2r+1}$ -twisted de Rham cohomology of  $X$  is the cochain cohomology

$$\Omega_{\text{dR}}^{\bullet+H_{2r+1}}(X) := \frac{\ker^{\bullet}(d - H_{2r+1} \wedge (-))}{\text{im}^{\bullet}(d - H_{2r+1} \wedge (-))} \quad (189)$$

of the following  $2r$ -periodic cochain complex:

$$\dots \longrightarrow \bigoplus_k \Omega_{\text{dR}}^{(n-1)+2rk}(X) \xrightarrow{(d-H_{2r+1} \wedge (-))} \bigoplus_k \Omega_{\text{dR}}^{n+2rk}(X) \xrightarrow{(d-H_{2r+1} \wedge (-))} \bigoplus_k \Omega_{\text{dR}}^{(n+1)+2rk}(X) \longrightarrow \dots$$

In direct generalization of Prop. 3.100, we find:

**Proposition 3.102** (Twisted non-abelian de Rham cohomology subsumes higher twisted abelian de Rham cohomology). For  $r \in \mathbb{N}$ ,  $r \geq 1$ , consider a twisting  $(2r+1)$ -form as in (179)

$$\begin{array}{ccc} \tau_{\text{dR}} & \longleftrightarrow & H_{2r+1} \\ \cap & & \cap \\ \Omega(X; \mathfrak{b}^{2r}\mathbb{R})_{\text{flat}} & \simeq & \Omega^{2r+1}(X)_{\text{closed}} \end{array}$$

The  $\tau_{\text{dR}}$ -twisted non-abelian de Rham cohomology (Def. 3.98) of flat twisted  $\mathbb{K}^{2r-2}(\text{ku})_1$ -valued differential forms (Example 3.95) is naturally equivalent to  $H_{2r+1}$ -twisted abelian de Rham cohomology (Def. 3.101) in degree<sup>12</sup>  $1 \bmod 2r$ .

$$H_{\text{dR}}^{\tau_{\text{dR}}}(X; \bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk}\mathbb{R}) \simeq H_{\text{dR}}^{1+H_{2r+1}}(X).$$

twisted non-abelian de Rham cohomology
higher  $H_{2r+1}$ -twisted de Rham cohomology

*Proof.* By Example 3.95, the cocycle sets on both sides are in natural bijection. Hence it remains to see that the coboundary relations correspond to each other, under this identification. This proceeds verbatim, up to degree shifts, as in the proof of Prop. 3.100 (which is the special case of  $r = 1$ ).  $\square$

**Example 3.103** (Cohomology operation in (higher-) twisted de Rham cohomology). Degree-3 twisted de Rham cohomology (Def. 3.99) supports the following twisted cohomology operations (Def. 2.41):

(i) wedge product with  $H_3$ :

$$\begin{array}{ccc} H_{\text{dR}}^{\bullet+H_3}(X) & \longrightarrow & H_{\text{dR}}^{\bullet+3+H_3}(X) \\ \sum_k F_k & \mapsto & \sum_k F_k \wedge H_3 \end{array}$$

(ii) wedge square:

$$\begin{array}{ccc} \bigoplus_r H_{\text{dR}}^{2r+H_3}(X) & \longrightarrow & \bigoplus_r H_{\text{dR}}^{2r+2H_3}(X) \\ \sum_k F_k & \mapsto & \left( \sum_k F_k \right) \wedge \left( \sum_k F_k \right) \end{array}$$

(iii) compositions of these:

$$\begin{array}{ccc} \bigoplus_r H_{\text{dR}}^{2r+H_3}(X) & \longrightarrow & \bigoplus_r H_{\text{dR}}^{2r+1+2H_3}(X) \\ \sum_k F_k & \mapsto & \left( \sum_k F_k \right) \wedge \left( \sum_k F_k \right) \wedge H_3 \end{array}$$

In type IIA string theory, terms of the form (iii) arise, together with terms of the form  $I_8 \cup [H_3]$  with  $I_8$  a polynomial in the Pontrjagin classes (cf. Example 4.27). See [GS19c] for extensive discussions.

This evidently generalizes to higher twisted de Rham cohomology (Def. 3.101) and higher twisted real cohomology in the sense of [GS19b], with  $H_3$  replaced by  $H_{2r+1}$  for  $r \in \mathbb{N}$ .

<sup>12</sup>The discussion for other degrees is directly analogous, and we omit it for brevity.

### The twisted non-abelian de Rham theorem.

**Theorem 3.104** (Twisted non-abelian de Rham theorem). *Let  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\geq 1, \text{nil}})^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52), equipped with the structure of a smooth manifold, and let*

$$\begin{array}{ccc} A & \longrightarrow & A // G \\ \text{local coefficient bundle} \downarrow \rho & & \downarrow \rho \\ & & BG \end{array} \quad (190)$$

be a local coefficient bundle (31) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\geq 1, \text{nil}})^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) such that the action of  $\pi_1(BG) = \pi_0(G)$  on the real homology groups of  $A$  is nilpotent. Consider, via Prop. 3.75, the rationalized coefficient bundle  $L_{\mathbb{R}}\rho$  with corresponding  $L_{\infty}$ -algebraic coefficient bundle  $\mathfrak{l}\rho$  (Def. 3.91) of the relative real Whitehead  $L_{\infty}$ -algebra (Prop. 3.70):

$$\begin{array}{ccc} L_{\mathbb{R}}A & \longrightarrow & (L_{\mathbb{R}}A) // (L_{\mathbb{R}}G) \\ \text{rationalized local coefficient bundle} \downarrow L_{\mathbb{R}}\rho & & \downarrow L_{\mathbb{R}}\rho \\ & & L_{\mathbb{R}}BG \end{array} \quad \begin{array}{ccc} \mathfrak{l}A & \longrightarrow & \mathfrak{l}_{BG}(A // G) \\ \text{L}_{\infty}\text{-algebraic coefficient bundle of Whitehead } L_{\infty}\text{-algebras} \downarrow \mathfrak{l}\rho & & \downarrow \mathfrak{l}\rho \\ & & \mathfrak{l}BG \end{array} .$$

Moreover, let

$$X \xrightarrow{\tau} L_{\mathbb{R}}BG \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$$

be such that  $H^1(\tau; \mathbb{R})$  is injective (for instance in that  $BG$  is simply connected).

Then the  $\tau$ -twisted non-abelian real cohomology (Def. 3.76) of  $X$  with local coefficients in  $L_{\mathbb{R}}\rho$  (Prop. 3.75) is in natural bijection with the twisted non-abelian de Rham cohomology (Def. 3.98) of  $X$  with local coefficients in  $\mathfrak{l}\rho$ ,

$$\begin{array}{ccc} \text{\scriptsize } \tau\text{-twisted non-abelian} & & \text{\scriptsize } \tau_{\text{dR}}\text{-twisted non-abelian} \\ \text{\scriptsize } \text{real cohomology} & & \text{\scriptsize } \text{de Rham cohomology} \\ H^{\tau}(X; L_{\mathbb{R}}A) & \simeq & H_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{l}A), \end{array} \quad (191)$$

where the twists are related by the plain non-abelian de Rham theorem (Theorem 3.87):

$$\begin{array}{ccc} [\tau] & \longleftarrow & [\tau_{\text{dR}}] \\ \cap & & \cap \\ H(X; L_{\mathbb{R}}BG) & \simeq & H_{\text{dR}}(X; \mathfrak{l}BG) \end{array}$$

*Proof.* Consider the following sequence of natural bijections

$$\begin{aligned} H^{\tau}(X; L_{\mathbb{R}}A) &= \text{Ho}\left(\left(\text{TopologicalSpaces}_{\text{Qu}}^{/L_{\mathbb{R}}BG}\right)\right)(\tau, L_{\mathbb{R}}\rho) \\ &\simeq \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\Omega_{\text{PLdR}}^{\bullet}(BG)/}\right)\left(\Omega_{\text{PLdR}}^{\bullet}(\rho), \Omega_{\text{PLdR}}^{\bullet}(\tau)\right) \\ &\simeq \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{CE}(\mathfrak{b})/}\right)\left(\text{CE}(\mathfrak{l}\rho), \tau_{\text{dR}}^*\right) \\ &\simeq H_{\text{dR}}^{\tau_{\text{dR}}}(X; \mathfrak{l}A). \end{aligned}$$

Here the first line is the definition (Def. 3.76). Then the first step is the fundamental theorem (Prop. 3.60) in the co-slice category. The substitutions in the second step are:

(a) Lemma 3.108 in the first argument (this is where the  $H^1$ -injectivity is needed);

(b) Lemma 3.90 with Theorem 3.87 in the second argument (as in the second step of (164)).

The last step is Lemma 3.106. The composite of these equivalences is the desired (191).  $\square$

We now establish the remaining four lemmas which enter the proof of Theorem 3.104.

**Lemma 3.105** (Pullback to de Rham complex over cylinder of manifold is relative path space object).

Let  $X \in \text{SmoothManifolds}$ , let  $\mathfrak{b} \in L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Example 3.24) with Chevalley-Eilenberg algebra  $\text{CE}(\mathfrak{b}) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (88), and let  $\{\Omega_{\text{dR}}^{\tau_{\text{dR}}}(X) \xleftarrow{\tau_{\text{dR}}^*} \text{CE}(\mathfrak{b})\} \in (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{\text{CE}(\mathfrak{b})/}$  be a morphism of dgc-algebras to the de Rham complex of  $X$  (Example 3.23), regarded as an object in the coslice model category (Example A.10) of  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  (Prop. 3.36) under  $\text{CE}(\mathfrak{b})$ . Then a path space object (Def. A.11) for  $\tau_{\text{dR}}^*$  is given by this diagram:

$$\begin{array}{ccccc}
\Omega_{\mathrm{dR}}^{\bullet}(X) & \xrightarrow{\mathrm{pr}_X^* \in \mathcal{W}} & \Omega_{\mathrm{dR}}^{\bullet}(X \times \mathbb{R}) & \xrightarrow{(i_0^*, i_1^*) \in \mathrm{Fib}} & \Omega_{\mathrm{dR}}^{\bullet}(X) \oplus \Omega_{\mathrm{dR}}^{\bullet}(X) \\
& & \uparrow \mathrm{pr}_X^* \circ \tau_{\mathrm{dR}}^* & & \nearrow (\tau_{\mathrm{dR}}^*, \tau_{\mathrm{dR}}^*) \\
& \swarrow \tau_{\mathrm{dR}}^* & \mathrm{CE}(\mathfrak{b}), & & \\
& & \downarrow \tau_{\mathrm{dR}}^* & & 
\end{array}$$

where the top morphisms are from (165).

*Proof.* It is clear that the diagram commutes, by construction. Moreover, the top morphisms are a weak equivalence followed by a fibration in  $(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}$ , by Lemma 3.88. Therefore, by the nature of the coslice model structure (Example A.10) the total diagram constitutes a factorization of the diagonal on  $\tau_{\mathrm{dR}}^*$  through a weak equivalence followed by a fibration, as required (314). (To see that the composite really is still the diagonal morphism in the coslice, observe that Cartesian products in any coslice category are reflected in the underlying category.) It only remains to observe that  $\tau_{\mathrm{dR}}^*$  is actually a fibrant object in the coslice model category. But the terminal object in the coslice is clearly the unique morphism from  $\mathrm{CE}(\mathfrak{b})$  to the zero-algebra (Example 3.21), so that in fact every object in the coslice is still fibrant

$$\begin{array}{ccc}
\Omega_{\mathrm{dR}}^{\bullet}(X) & \xrightarrow{\in \mathrm{Fib}} & 0 \\
& \swarrow \tau_{\mathrm{dR}}^* & \nearrow \\
& \mathrm{CE}(\mathfrak{b}) & 
\end{array} \tag{192}$$

as in Remark 3.37.  $\square$

**Lemma 3.106** (Twisted non-abelian de Rham cohomology via the coslice dgc-homotopy category). *Consider  $X \in \mathrm{SmoothManifolds}$ , let*

$$\mathfrak{g} \longrightarrow \begin{array}{c} \widehat{\mathfrak{b}} \\ \downarrow \mathfrak{p} \\ \mathfrak{b} \end{array} \in L_{\infty}\mathrm{Algebras}_{\mathbb{R}, \mathrm{fin}}^{\geq 0, \mathrm{nil}}$$

be an  $L_{\infty}$ -algebraic local coefficient bundle (Def. 3.91) of nilpotent  $L_{\infty}$ -algebras (Def. 3.34) with Chevalley-Eilenberg algebra  $\mathrm{CE}(\widehat{\mathfrak{b}})$ ,  $\mathrm{CE}(\mathfrak{b}) \in \mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (88), and let

$$\Omega_{\mathrm{dR}}^{\bullet}(X) \xleftarrow{\tau_{\mathrm{dR}}^*} \mathrm{CE}(\mathfrak{b}) \in (\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}^{\mathrm{CE}(\mathfrak{b})/} \tag{193}$$

be a morphism of dgc-algebras to the de Rham complex of  $X$  (Example 3.23), hence a flat  $\mathfrak{b}$ -valued differential form (Def. 3.77)

$$\tau_{\mathrm{dR}} \in \Omega_{\mathrm{dR}}(X; \mathfrak{b}),$$

equivalently regarded as an object in the coslice model category (Example A.10) of  $(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}$  (Prop. 3.36) under  $\mathrm{CE}(\mathfrak{b})$ . Then the  $\tau_{\mathrm{dR}}$ -twisted non-abelian de Rham cohomology of  $X$  with coefficients in  $\mathfrak{g}$  (Def. 3.98) is in natural bijection with the hom-set in the homotopy category (Def. A.15) of the coslice model category  $(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}^{\mathrm{CE}(\mathfrak{b})/}$  (Example A.10) of the projective model structure on dgc-algebras (Prop. 3.36) from  $\mathrm{CE}(\mathfrak{p})$  (172) to  $\tau_{\mathrm{dR}}^*$  (193):

$$H_{\mathrm{dR}}^{\tau_{\mathrm{dR}}}(X; \mathfrak{g}) \simeq \mathrm{Ho}\left((\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}^{\mathrm{CE}(\mathfrak{b})/}\right)(\mathrm{CE}(\mathfrak{p}), \tau_{\mathrm{dR}}^*). \tag{194}$$

*Proof.* Consider a pair of dgc-algebra homomorphisms in the coslice

$$\begin{array}{ccc}
\Omega_{\mathrm{dR}}^{\bullet}(X) & \xleftarrow{A^{(0)}} & \mathrm{CE}(\widetilde{\mathfrak{b}}) \\
& \xleftarrow{A^{(1)}} & \nearrow \\
& \xleftarrow{\tau_{\mathrm{dR}}^*} & \mathrm{CE}(\mathfrak{b}) \xrightarrow{\mathrm{CE}(\mathfrak{p})} \mathrm{CE}(\widetilde{\mathfrak{b}})
\end{array} \in (\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}^{\mathrm{CE}(\mathfrak{b})/}(\mathrm{CE}(\mathfrak{p}), \tau_{\mathrm{dR}}^*), \tag{195}$$

hence of flat  $\tau_{\mathrm{dR}}$ -twisted  $\mathfrak{g}$ -valued differential forms, according to Def. 3.92. Observe that:

- (i)  $\mathrm{CE}(\mathfrak{p})$  is cofibrant in  $(\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\mathrm{proj}}^{\mathrm{CE}(\mathfrak{p})/}$ , since:
  - (a) the initial object in the coslice is  $\mathrm{CE}(\mathfrak{b}) \xleftarrow{\mathrm{id}} \mathrm{CE}(\mathfrak{b})$ ,
  - (b) the unique morphism from this object to  $\mathrm{CE}(\mathfrak{p})$  is

$$\begin{array}{ccc}
\text{CE}(\mathfrak{b}) & \xrightarrow{\text{CE}(\mathfrak{p}) \in \text{Cof}} & \text{CE}(\widehat{\mathfrak{b}}) \\
& \swarrow \text{id} & \nearrow \text{CE}(\mathfrak{p}) \\
& & \text{CE}(\mathfrak{b})
\end{array} \tag{196}$$

(c)  $\text{CE}(\mathfrak{p})$  is a cofibration in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$ , by (172), so that the diagram (196) is a cofibration in the coslice model category, by Example A.10.

(ii)  $\text{pr}_X^* \circ \tau_{\text{dR}}^*$  is fibrant in  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{\text{CE}(\mathfrak{b})/}$ , by (192);

(iii) A right homotopy (Def. A.13) between the pair (195) of coslice morphisms, with respect to the path space object from Lemma 3.105, namely a  $\widetilde{A}$  that makes the following diagram commute

$$\begin{array}{ccc}
\Omega_{\text{dR}}^{\bullet}(X) & & \\
\uparrow i_0^* & \swarrow A^{(0)} & \\
\Omega_{\text{dR}}^{\bullet}(X \times \mathbb{R}) & \xleftarrow{\widetilde{A}} & \text{CE}(\widehat{\mathfrak{b}}) \\
\downarrow i_1^* & \swarrow \text{pr}_X^*, \tau_{\text{dR}}^* & \nearrow A^{(1)} \\
\Omega_{\text{dR}}^{\bullet}(X) & & \text{CE}(\widehat{\mathfrak{b}})
\end{array} \tag{197}$$

is manifestly the same as a coboundary  $\widetilde{A}$  between the corresponding flat twisted  $\mathfrak{g}$ -valued forms according to Def. 3.97:

(a) The top part of (197) is, just as in (168), the flat twisted  $\widehat{\mathfrak{g}}$ -valued form on the cylinder over  $X$  that is required by (182);

(b) the bottom part of (197) is the condition (183) on the extension of the twist to the cylinder over  $X$ .

Therefore, Prop. A.17 says that the quotient set (184) defining the twisted non-abelian de Rham cohomology is in natural bijection to the hom-set in the coslice homotopy category.  $\square$

**Lemma 3.107** (Derived cobase change along quasi-isomorphism is equivalence on  $H^1$ -injectives). *Let*

$$B_1 \xrightarrow{\phi \in \mathbb{W}} B_2 \in (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$$

be a quasi-isomorphism of dgc-algebras (Def. 3.17), hence a weak equivalence in the projective model structure (Prop. 3.36). Assume that either, hence both, dgc-algebras are cohomologically connected ( $H^0(B_1) = \mathbb{R}$ ,  $H^0(B_2) = \mathbb{R}$ ). Then the derived adjunction (Prop. A.21) of the base change Quillen adjunction (Example A.19) between the corresponding co-slice model categories (Example A.10) of the opposite model category of dgc-algebras (Example A.9) restricts to an equivalence on the full subcategories of the homotopy categories (Def. A.15) on those co-slice objects which are connected in  $H^0(-)$  and injective on  $H^1(-)$ :

$$\text{Ho} \left( \left( (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{B_2/} \right)^{\text{op}} \right)_{H^0\text{-conn}, H^1\text{-inj}} \xrightarrow[\mathbb{R}(\phi^{\text{op}})^*]{\mathbb{L}(\phi^{\text{op}})_!} \text{Ho} \left( \left( (\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{B_1/} \right)^{\text{op}} \right)_{H^0\text{-conn}, H^1\text{-inj}}$$

*Proof.* Notice that if  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  were a left proper model category (Def. A.5), so that  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}^{\text{op}}$  were right proper, the statement would directly follow as a special case of Prop. A.32, without any restriction to subcategories.

While  $(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}$  is (apparently) not left proper, it comes close: Lemma 3.44 says that quasi-isomorphisms are preserved by pushout along at least those cofibrations that are relative Sullivan algebras (i.e. the relative cell complexes, but possibly not their retracts). Hence we adapt the logic underlying Prop. A.32 to this case. Namely, Prop. 3.49 says that those co-slice objects that are  $H^1$ -injective between  $H^0$ -connected algebras do have a cofibrant replacement by a relative Sullivan algebra:

$$\begin{array}{ccc}
A_{\min_B} & \xrightarrow{\in W} & A \\
& \searrow^{\substack{\in \text{RelSulAlg} \\ \subset \text{Cof}}} & \nearrow_{\text{coslice object}} \\
& & B
\end{array} \tag{198}$$

Now, first to see that the derived adjunction restricts to the given subcategories: In one direction, it is clear that  $\mathbb{L}(\phi^{\text{op}})_!$  preserves  $H^0$  and  $H^1$ , as this functor is given by precomposition with the quasi-isomorphism  $\phi$ . In the other direction:  $\mathbb{R}(\phi^{\text{op}})^*$  is given by pushout along  $\phi$  of a cofibrant representative of the given coslice object, and by (198) we may take that cofibrant representative to be a relative Sullivan algebra. But then Prop. 3.44 implies that the pushout has the same cohomology.

Finally, to see that this restriction of the derived adjunction is an equivalence of categories, hence that the derived unit (326) and derived counit (327) are isomorphisms on these subcategories. This follows just as in the alternative proof (337) of Prop. A.32, using for the fibrant objects  $\rho$  there the opposites of the good fibrations given by (198), for which Prop. 3.44 guarantees the required properness condition.  $\square$

**Lemma 3.108** (Pasting composition with relative Sullivan model of local coefficient bundle). *Let*

$$\begin{array}{ccc}
A & \longrightarrow & A // G \\
& & \downarrow \rho \\
& & BG
\end{array} \tag{199}$$

*local coefficient bundle*

*be a local coefficient bundle (31) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52), and let*

$$\begin{array}{ccc}
\Omega_{\text{PLdR}}^{\bullet}(A // G) & \xleftarrow[\in W]{P_{A // G}^{\min_{BG}}} & \text{CE}(I(A // G)) \\
\uparrow \Omega_{\text{PLdR}}^{\bullet}(\rho) & & \uparrow \text{CE}(I\rho) \\
\Omega_{\text{PLdR}}^{\bullet}(BG) & \xleftarrow[\in W]{P_{BG}^{\min}} & \text{CE}(I(BG))
\end{array} \tag{200}$$

*be its minimal relative Sullivan model (142), which exists by Prop. 3.70. Then the pasting precomposition with the square (200) is a natural isomorphism of hom-functors on the homotopy categories from Lemma 3.107:*

$$\begin{aligned}
& \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\Omega_{\text{PLdR}}^{\bullet}(BG)/}\right)_{H^0\text{-conn}, H^1\text{-inj}}(\Omega_{\text{PLdR}}^{\bullet}(\rho), -) \\
& \quad \left(\mathbb{L}\left((P_{BG}^{\min})^{\text{op}}\right)_!\right)^{\text{op}} \downarrow \simeq \\
& \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{CE}(IBG)/}\right)_{H^0\text{-conn}, H^1\text{-inj}}(\Omega_{\text{PLdR}}^{\bullet}(\rho) \circ P_{BG}^{\min}, -) \\
& \quad (-) \circ P_{A // G}^{\min_{BG}} \downarrow \simeq \\
& \text{Ho}\left(\left(\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}^{\text{CE}(IBG)/}\right)_{H^0\text{-conn}, H^1\text{-inj}}(\text{CE}(I\rho), -)
\end{aligned} \tag{201}$$

*Here the first step is the derived left co-base change along  $\phi$  (Example A.19), while the second is composition with the diagram 200 regarded as a morphism in the co-slice under  $\text{CE}(IBG)$ .*

*Proof.* First notice that  $\Omega_{\text{PLdR}}^{\bullet}(\rho)$  is indeed an injection on  $H^1$ , by the assumption that the fiber  $A$  is connected (as in the proof of Prop. 3.70). With that, the first step is an isomorphism by Lemma 3.107, while the second step is evidently an isomorphism, since the weak equivalence  $\phi_{A // G}$  becomes an isomorphism in the homotopy category (and still so in the coslice homotopy category, by Example A.10).  $\square$

## 4 The (differential) non-abelian character map

We introduce the character map in non-abelian cohomology (Def. 4.2) and then discuss how it specializes to:

- §4.1 – the Chern-Dold character on generalized cohomology;
- §4.2 – the Chern-Weil homomorphism on degree-1 non-abelian cohomology;
- §4.3 – the Cheeger-Simons differential characters on degree-1 non-abelian cohomology.

**Definition 4.1** (Rationalization in non-abelian cohomology). For  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) we write

$$(\eta_A^{\mathbb{R}})_* : H(-; A) \xrightarrow[\text{rationalization}]{\substack{\text{non-abelian} \\ \text{cohomology}}} H(-; \eta_A^{\mathbb{R}}) \xrightarrow[\text{non-abelian}]{\substack{\text{non-abelian} \\ \text{real cohomology}}} H(-; L_{\mathbb{R}}A) \quad (202)$$

for the cohomology operation (Def. 2.17) from non-abelian  $A$ -cohomology (Def. 2.1) to non-abelian real cohomology (Def. 3.72), which is induced (24) by the rationalization map  $\eta_A^{\mathbb{R}}$  (Def. 3.55).

**Definition 4.2** (Non-abelian character map). Let  $X, A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) such that  $X$  admits the structure of a smooth manifold. Then we say that the *non-abelian character map* in non-abelian  $A$ -cohomology (Def. 2.1) is the cohomology operation (Def. 4.1)

$$\text{ch}_A : H(X; A) \xrightarrow[\text{rationalization}]{\substack{\text{non-abelian} \\ \text{cohomology}}} (\eta_A^{\mathbb{R}})_* : H(X; L_{\mathbb{R}}A) \xrightarrow[\text{non-abelian}]{\substack{\text{non-abelian} \\ \text{de Rham theorem}}} \simeq H_{\text{dR}}(X; \mathfrak{A}) \quad (203)$$

from non-abelian  $A$ -cohomology (Def. 2.1) to non-abelian de Rham cohomology (Def. 3.84) with coefficients in the rational Whitehead  $L_{\infty}$ -algebra  $\mathfrak{A}$  of  $A$  (Prop 3.63), which is the composite of

- (i) the operation (202) of rationalization of coefficients (Def. 4.1),
- (ii) the equivalence (163) of the non-abelian de Rham theorem (Theorem 3.87).

### 4.1 Chern-Dold character

We prove (Theorem 4.8) that the non-abelian character map reproduces the Chern-Dold character on generalized cohomology theories (recalled as Def. 4.6) and in particular the Chern character on topological K-theory (Example 4.10).

**Proposition 4.3** (Dold's equivalence [Do65, Cor. 4][Hi171, Thm. 3.18][Ru98, §II.3.17]). *Let  $E$  be a generalized cohomology theory (Example 2.13). Then its rationalization  $E_{\mathbb{R}}$  is equivalent to ordinary cohomology with coefficients in the rationalized stable homotopy groups of  $E$ :*

$$E_{\mathbb{R}}^n(X) \xrightarrow[\simeq]{\text{do}_E} \bigoplus_{k \in \mathbb{Z}} H^{n+k}(X; \pi_k(E) \otimes_{\mathbb{R}} \mathbb{R}).$$

**Remark 4.4** (Rational stable homotopy theory). In modern stable homotopy theory, Dold's equivalence (Prop. 4.3) is a direct consequence of the fundamental theorem [SSH01, Thm. 5.1.6] that rational spectra are direct sums of Eilenberg-MacLane spectra with coefficients in the rationalized stable homotopy groups [BMSS19, Prop. 2.17].

But we may explicitly re-derive Dold's equivalence using the unstable rational homotopy theory from §3:

**Proposition 4.5** (Dold's equivalence via non-abelian real cohomology). *Let  $E$  be a generalized cohomology theory (Example 2.13) and let  $n \in \mathbb{N}$  such that the  $n$ th coefficient space (17) is of  $\mathbb{R}$ -finite homotopy type (Def. 3.52)*

$$E_n \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}.$$

*Then there is a natural equivalence between the non-abelian real cohomology (Def. 3.72) with coefficients in  $E_n$  and ordinary cohomology with coefficients in the rationalized homotopy groups of  $E$ :*

$$H(-; L_{\mathbb{R}}E_n) \simeq \bigoplus_{k \in \mathbb{N}} H^{n+k}(-; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}). \quad (204)$$

*Proof.* Since  $E_n$  is an infinite-loop space, it is necessarily nilpotent (Example 3.54). We may assume without restriction that it is also connected, for otherwise we apply the following argument to each connected component (Remark 3.53). Hence  $E_n \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) and the discussion in §3.2 applies:

Again since  $E_n$  is a loop space (17), Prop. 3.74 gives  $H(-; L_{\mathbb{R}}E_n) \simeq \bigoplus_{k \in \mathbb{N}} H^k(-; \pi_k(E_n) \otimes_{\mathbb{Z}} \mathbb{R})$ . The claim follows from the definition of stable homotopy groups as  $\pi_{k-n}(E) = \pi_k(E_n)$  for  $k, n \geq 0$ .  $\square$

**Definition 4.6** (Chern-Dold character [Bu70][Hil71, p. 50]). Let  $E$  be a generalized cohomology theory (Example 2.13). The *Chern-Dold character* in  $E$ -cohomology theory is the cohomology operation to ordinary cohomology which is the composite of rationalization in  $E$ -cohomology with Dold's equivalence (Prop. 4.3):

$$\begin{array}{ccccc}
 \text{Chern-Dold} & & & & \\
 \text{character} & & & & \\
 \text{ch}_E : & E^\bullet(-) & \xrightarrow{\text{rationalization}} & E_{\mathbb{R}}^\bullet(-) & \xrightarrow[\simeq]{\text{do}_E} & \bigoplus_k H^{\bullet+k}(-; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}) . \\
 & \downarrow (18) \simeq & & \downarrow (18) \simeq & \nearrow (204) \simeq \\
 & H(-; E_\bullet) & \xrightarrow[\text{(202)}]{(\eta_{E_\bullet}^{\mathbb{R}})_*} & H(-; L_{\mathbb{R}}E_\bullet) & 
 \end{array} \quad (205)$$

Here the bottom part in (205) serves to make the nature of the top maps fully explicit, using Example 2.13, Def. 4.1 and Prop. 4.5.

**Remark 4.7** (Rationalization in the Chern-Dold character). That the first map in the Dold-Chern character (205) is the rationalization localization is stated somewhat indirectly in the original definition [Bu70] (the concept of rationalization was fully formulated later in [BK72]). The role of rationalization in the Chern-Dold character is made fully explicit in [LSW16, §2.1]. The same rationalization construction of the generalized Chern character, but without attribution to [Bu70] or [Do65], is considered in [HS05, §4.8] (see also [BN14, p. 17]).

We now come to the main result in this section:

**Theorem 4.8** (Non-abelian character subsumes Chern-Dold character). *Let  $E$  be a generalized cohomology theory (Example 2.13) and let  $n \in \mathbb{N}$  such that the  $n$ th coefficient space (17) is of  $\mathbb{R}$ -finite homotopy type (Def. 3.52). Let moreover  $X$  be a smooth manifold of connected, nilpotent,  $\mathbb{R}$ -finite homotopy type (Def. 3.52).*

*Then the non-abelian character (Def. 4.2) coincides with the Chern-Dold character (Def. 4.6) on  $E$ -cohomology in degree  $n$ , in that the following diagram commutes:*

$$\begin{array}{ccc}
 H(X; E_n) & \xrightarrow{\text{ch}_{E_n}} & H_{\text{dR}}(\mathfrak{t}E_n) \\
 (18) \uparrow \simeq & & \simeq \downarrow (163) (204) \\
 E^n(X) & \xrightarrow{\text{ch}_{E_n}} & \bigoplus_k H^{n+k}(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}) .
 \end{array} \quad (206)$$

*Here the equivalence on the left is from Example 2.13, while the equivalence on the right is the inverse non-abelian de Rham theorem (Theorem 3.87) composed with that from Prop. 4.5.*

*Proof.* Since  $E_n$  is an infinite-loop space, it is necessarily nilpotent (Example 3.54). We may assume without restriction that it is also connected, for otherwise we apply the following argument to each connected component (Remark 3.53). Hence  $E_n \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) and the discussion in §3.2 and §3.3 applies:

The non-abelian de Rham isomorphism (163) in the definition (203) of the non-abelian character cancels against its inverse on the right of (206). Commutativity of the remaining diagram

$$\begin{array}{ccc}
 H(X; E_n) & \xrightarrow{(\eta_{E_n}^{\mathbb{R}})_*} & H(X; L_{\mathbb{R}}E_n) \\
 (18) \uparrow \simeq & & \simeq \downarrow (204) \\
 E^n(X) & \xrightarrow{\text{ch}_{E_n}} & \bigoplus_k H^{n+k}(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R})
 \end{array}$$

is the very definition of the Chern-Dold character (Def. 4.6).  $\square$

**Example 4.9** (de Rham homomorphism in ordinary cohomology). On ordinary integral cohomology (Example 2.2), the non-abelian character (Def. 4.2) reduces to extension of scalars from the integers to the real numbers, followed by the de Rham isomorphism, in that the following diagram commutes:

$$\begin{array}{ccc}
 H(-; B^{n+1}\mathbb{Z}) & \xrightarrow{\text{non-abelian character on ordinary cohomology}} & H_{\text{dR}}(-; B^{n+1}\mathbb{Z}) \\
 \uparrow \text{(9)} \simeq & \text{ch}_{B^{n+1}\mathbb{Z}} & \simeq \downarrow \text{(137) (160)} \\
 H^{n+1}(-; \mathbb{Z}) & \xrightarrow{\text{extension of scalars}} H^{n+1}(-; \mathbb{R}) \xrightarrow{\text{ordinary de Rham isomorphism}} & H_{\text{dR}}^{n+1}(-)
 \end{array}$$

**Example 4.10** (Chern character on complex K-theory). The spectrum (17) representing complex K-theory has 0th component space  $KU_0 \simeq \mathbb{Z} \times BU$  (19). Here the connected components  $BU$ , the classifying space of the infinite unitary group (20), are clearly of finite  $\mathbb{R}$ -type (since their real cohomology is the ring of universal Chern classes, e.g. [Ko96, Thm. 2.3.1]). Therefore, Theorem 4.8 applies and says that the non-abelian character map (Def. 4.2) for coefficients in  $\mathbb{Z} \times BU$  reduces to the Chern-Dold character on complex K-theory. This, in turn, is equivalent (by [Hil71, Thm. 5.8]) to the original Chern character  $\text{ch}$  on complex K-theory [Hi56, §12.1][BH58, §9.1][AH61, §1.10] (review in [Hil71, §V]):

$$\begin{array}{ccc}
 \text{non-abelian cohomology with } \mathbb{Z} \times BU\text{-coefficients} & \xrightarrow{\text{non-abelian character map with } \mathbb{Z} \times BU\text{-coefficients}} & \text{non-abelian de Rham homology with } l(\mathbb{Z} \times BU)\text{-coefficients} \\
 H(X; \mathbb{Z} \times BU) & \xrightarrow{\text{ch}_{\mathbb{Z} \times BU}} & H_{\text{dR}}(X; l(\mathbb{Z} \times BU)) \\
 \parallel & & \parallel \\
 KU^0(X) & \xrightarrow{\text{ch}} & \bigoplus_{k \in \mathbb{Z}} H_{\text{dR}}^{2k}(X) \\
 \nabla & \longmapsto & [\text{tr exp}(F_{\nabla})] \\
 & & \text{traditional Chern character}
 \end{array}$$

**Example 4.11** (Pontrjagin character on real K-theory). The *Pontrjagin character*  $\text{ph}$  on real topological K-theory (see [GHV73, §9.4][IK99][Ig08][GS18b, §2.1]) is defined to be the composite

$$\text{KSpin}^\bullet(-) \longrightarrow \text{KSO}^\bullet(-) \longrightarrow \text{KO}^\bullet(-) \xrightarrow{\text{cplx}} \text{KU}^\bullet(-) \xrightarrow{\text{ch}^\bullet} \bigoplus_k H^{4\bullet}(-; \mathbb{R})$$

$\searrow \text{ph}^\bullet \quad \swarrow \text{ph}^\bullet$

of the complexification map (on representing virtual vector bundles) with the Chern character on complex K-theory (Example 4.10).

(i) By naturality of the complexification map and since the complex Chern character is a Chern-Dold character (by [Hil71, Thm. 5.8]), so is the Pontrjagin character, as well as its restriction  $\tilde{\text{ph}}$  to oriented real K-theory  $\text{KSO}$  and further to  $\text{ph}$  on  $\text{KO}$ -theory and to  $\text{Spin}$  K-theory, etc.

(ii) The connected components  $BO$  of the classifying space  $\text{KO}_0$  for real topological K-theory are of finite  $\mathbb{R}$ -type (since the real cohomology is the ring of universal Pontrjagin classes). Therefore, Theorem 4.8 applies and says that the non-abelian Chern character (Def. 4.2) for coefficients in  $\mathbb{Z} \times BSO$  coincides with the Pontrjagin character  $\tilde{\text{ph}}$  in  $\text{KSO}$ -theory:

$$\begin{array}{c}
 \text{Pontrjagin character on oriented real K-theory} \\
 \tilde{\text{ph}} \simeq \text{ch}_{\mathbb{Z} \times BSO}.
 \end{array}$$

(iii) By Remark 3.53, the construction extends to the Pontrjagin character  $\text{ph}$  on  $\text{KO}$ -theory.

(iv) The same applies to the further restriction of the Pontrjagin character to  $\text{KSpin}$ ; see [LD91][Th62] for some subtleties involved and [Sa08, §7] for interpretation and applications.

**Example 4.12** (Chern-Dold character on Topological Modular Forms). The connective ring spectrum  $\text{tmf}$  of *topological modular forms* [Ho94, §9][Ho02, §4] (see [DFHH14]) is, essentially by design, such that under rationalization it yields the graded ring of rational modular forms (e.g [DH11, p. 2]):

$$\pi_\bullet(\text{tmf}) \xrightarrow{(-) \otimes_{\mathbb{Z}} \mathbb{R}} \text{mf}_\bullet^{\mathbb{R}} \simeq \mathbb{R} \left[ \overbrace{c_4}^{\text{deg} = 8}, \overbrace{c_6}^{\text{deg} = 12} \right]. \quad (207)$$



It follows that the Chern-Dold character (Def. 4.6) on  $\mathrm{tmf}$  takes values in real cohomology with coefficients in modular forms

$$\mathrm{tmf}^\bullet(-) \xrightarrow[\mathrm{ch}_{\mathrm{tmf}}^\bullet]{\text{Chern-Dold character on topological modular forms}} H^\bullet(-; \mathrm{mf}_\bullet^{\mathbb{R}}). \quad (208)$$

(This is often considered over the rational numbers, sometimes over the complex numbers [BE13, Fig. 1]; we may just as well stay over the real numbers, by Remark 3.51, to retain contact to the de Rham theorem.)

By Theorem 4.8 this is an instance of the non-abelian character map:

$$\text{Chern-Dold character on topological modular forms} \quad \mathrm{ch}_{\mathrm{tmf}}^\bullet \simeq \mathrm{ch}_{\mathrm{tmf}_\bullet}.$$

**Example 4.13** (The Hurewicz/Boardman homomorphism on topological modular forms). The spectrum  $\mathrm{tmf}$  (Example 4.12) carries the structure of a suitable  $(E_\infty)$  ring spectrum and hence receives an essentially unique homomorphism of ring spectra from the sphere spectrum:

$$\Sigma^\infty \mathcal{S}^0 = \mathbb{S} \xrightarrow{e_{\mathrm{tmf}}} \mathrm{tmf}.$$

This is also known as the *Hurewicz homomorphism* or rather the *Boardman homomorphism* (e.g. [Ad75, §II.7][Ko96, §4.3]) for  $\mathrm{tmf}$ . The Boardman homomorphism on  $\mathrm{tmf}$  happens to be a stable weak equivalence in degrees  $\leq 6$ , in that it is an isomorphism on stable homotopy groups in these degrees [Ho02, Prop. 4.6][DFHH14, §13]:

$$\pi_{\bullet \leq 6}^s = \pi_{\bullet \leq 6}(\mathbb{S}) \xrightarrow[\simeq]{\pi_{\bullet \leq 6}(e_{\mathrm{tmf}})} \pi_{\bullet \leq 6}(\mathrm{tmf}).$$

Hence, in particular, when  $X^{10}$  is a manifold of dimension  $\dim(X) \leq 6 + 4 = 10$ , the Boardman homomorphism identifies the stable Cohomotopy (Example 2.16) of  $X^{10}$  in degree 4 with  $\mathrm{tmf}^4(X^{10})$  (by Prop. A.38):

$$\begin{array}{ccc} \text{stable 4-Cohomotopy} & \text{Boardman homomorphism} & \text{tmf-cohomology in degree 4} \\ \pi_s^4(X^{10}) = \mathbb{S}^4(X^{10}) & \xrightarrow[\simeq]{e_{\mathrm{tmf}}^4} & \mathrm{tmf}^4(X^{10}) \\ & \searrow \mathrm{ch}_{\mathbb{S}^4} & \swarrow \mathrm{ch}_{\mathrm{tmf}^4} \\ & & H_{\mathrm{dR}}^4(X^{10}) \end{array} \quad (209)$$

In this situation, the character map from Example 4.12 extracts exactly the datum of a real 4-class.

**Remark 4.14** (Clarifying the role of  $\mathrm{tmf}$  in string theory). Since the famous computation of [Wi87] showed that the partition function of the heterotic string lands in modular forms, and since the theorem of [AHS01][AHR10] showed that, mathematically, this statement lifts through (what we call above) the  $\mathrm{tmf}$ -Chern-Dold character (208), there have been proposals about a possible role of  $\mathrm{tmf}$ -cohomology theory in controlling elusive aspects of string theory (see [KS05][Sa10][DH11][ST11][Sa14][GJF18][GPPV18][Sa19]). While good progress has been made, it might be fair to say that the situation has remained inconclusive. But with the non-abelian generalization (Def. 4.2) of the Chern-Dold character in hand, we may ask for a non-abelian enhancement (Example 2.24) of  $\mathrm{tmf}$ -theory on string background spacetimes. By Example 4.13, this is, in degree 4, equivalent to asking for a non-abelian enhancement of stable Cohomotopy theory (Example 2.25). This exists canonically: given by actual Cohomotopy theory (Example 2.10). We consider the non-abelian character map on twisted 4-Cohomotopy in Example 5.23 below. The concluding Prop. 5.24 shows that this does capture core aspects of non-perturbative string theory.

**Example 4.15** (Chern-Dold character on integral Morava K-theory). We highlight that a particularly interesting example of the Chern-Dold character, which is not widely known, is that on integral Morava K-theory, whose codomain in real cohomology has a rich coefficient system. Morava K-theories  $K(n)$  [JW75] (reviewed in [Wu89][Ru98, §IX.7]) form a sequence of spectra labeled by chromatic level  $n \in \mathbb{N}$  and by a prime  $p$  (notationally left implicit). Their coefficient ring is pure torsion, and hence vanishes upon rationalization. However, there is an integral version  $\tilde{K}(n)$ , highlighted in [KS03][Sa10][Buh11][SW15][GS17b], which has an integral  $p$ -adic coefficient ring:

$$\tilde{K}(n)_* = \mathbb{Z}_p[v_n, v_n^{-1}], \quad \text{with } \deg(v_n) = 2(p^n - 1). \quad (210)$$

This theory more closely resembles complex K-theory than is the case for  $K(n)$ ; in fact, for  $n = 1$ , it coincides with the  $p$ -completion of complex K-theory.

Therefore, the Chern-Dold character (Def. 4.6) on integral Morava K-theory [GS17b, p. 53] is of the form

$$\mathrm{ch}_{\mathrm{Mor}} : \tilde{K}(n)(-) \longrightarrow H^*(-; \mathbb{Q}_p[v_n, v_n^{-1}] \otimes_{\mathbb{Q}} \mathbb{R}), \quad (211)$$

where we used (210) in (205) together with the fact that the rationalization of the  $p$ -adic integers is the rational (here: real, by Remark 3.51)  $p$ -adic numbers<sup>13</sup>  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{R}$ .

Now  $\mathbb{Q}_p$  is not finite-dimensional over  $\mathbb{Q}$ , whence  $\mathbb{Q}_p \otimes \mathbb{R}$  is not finite-dimensional over  $\mathbb{R}$ , so that the classifying space for integral Morava K-theory is not of  $\mathbb{R}$ -finite type (Def. 3.52). Therefore, our *proof* of the non-abelian de Rham theorem (Theorem 3.87), being based on the fundamental theorem of dgc-algebraic rational homotopy theory (Prop. 3.60), does not immediately apply to integral Morava K-theory coefficients; and hence the non-abelian character on integral Morava K-theory with de Rham codomain, in the form defined in Def. 4.2, is not established here. While this is a purely technical issue, as discussed in Remark 3.53, further discussion is beyond the scope of the present article.

## 4.2 Chern-Weil homomorphism

We prove (Theorem 4.26) that the non-abelian character subsumes the Chern-Weil homomorphism (recalled as Prop. 4.21, review in [Ch51, §III][KN63, §XII][CS74, §2][MS74, §C][FSS10, §2.1]) in degree-1 non-abelian cohomology.

**Chern-Weil theory.** For definiteness, we recall the statements of Chern-Weil theory that we need to prove Theorem 4.26 below.

**Remark 4.16** (Attributions in Chern-Weil theory). **(i)** What came to be known as the *Chern-Weil homomorphism* (recalled as Def. 4.21 below) seems to be first publicly described by H. Cartan (in May 1950), in his prominent *Séminaire* [Ca50, §7], published as [Ca51]. Later that year at the ICM (in Aug.-Sep. 1950), Chern discusses this construction in a talk [Ch50, (10)], including a brief reference to unpublished work by Weil (which remained unpublished until appearance in Weil’s collected works [We49]) for the proof that the construction is independent of the choice of connection (which is stated with an announcement of a proof in [Ca50, §7]).

**(ii)** The new result of Chern’s talk was the observation [Ch50, (15)] – later called the *fundamental theorem* in [Ch51, §III.6], recalled as Prop. 4.23 below – that this differential-geometric construction coincides with the topological construction of real characteristic classes (Example 2.21). This crucially uses the identification [Ch50, (11)] of the real cohomology of classifying space  $BG$  with invariant polynomials, later expanded on by Bott [Bo73, p. 239]. (Various subsequent authors, e.g. [Fr02, (1.14)], suggest to prove Chern’s equation (15) by making sense of a connection on the universal  $G$ -bundle – which is possible though notoriously subtle, e.g. [Mo79] – but the proof in [Ch50] simply observes that for any fixed bound  $\leq d$  on the dimension of the domain space, the classifying space for  $G$ -principal bundles may be truncated to a  $d + 1$ -dimensional sub-complex (as follows by the cellular approximation theorem [Sp66, p. 404]), this carrying a smooth  $G$ -principal bundle with ordinary connection, which is universal for  $G$ -principal bundles over  $\leq d$ -manifolds. This argument was later worked out in [NR61][NR63][Sc80]).

**(iii)** It is this fundamental theorem [Ch50, (15)][Ch51, §III.6] which allows to identify the Chern-Weil homomorphism as an instance of the non-abelian character, in Theorem 4.26 below.

**Notation 4.17** (Principal bundles with connection). For  $G \in \mathrm{LieGroups}$   $X \in \mathrm{SmoothManifolds}$ , we write

$$G\mathrm{Connections}(X)_{/\sim} \longrightarrow G\mathrm{Bundles}(X)_{/\sim} \quad (212)$$

for the forgetful map from the set of isomorphism classes of  $G$ -bundles equipped with connections to those of  $G$ -bundles without connection, over  $X$ .

<sup>13</sup>Note, parenthetically, that the classical Chern character  $\mathrm{ch}$  itself can be extended to cohomology theories with values in graded  $\mathbb{Q}$ -algebras; see, e.g., [Ma06].

The function (212) is surjective and admits sections, corresponding to a choice of the class of a principal connection on any class of  $G$ -principal bundles.

**Definition 4.18** (Invariant polynomials [We49][Ca50, §7]). For  $\mathfrak{g} \in \text{LieAlgebras}_{\mathbb{R}, \text{fin}}$ , we write

$$\text{inv}^\bullet(\mathfrak{g}) := \text{Sym}(\mathfrak{b}^2 \mathfrak{g}^*)^G \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

for the graded sub-algebra (71) on those elements in the symmetric algebra (74) of the linear dual of  $\mathfrak{g}$  shifted up (Def. 3.7) into degree 2, which are invariant under the adjoint action of  $G$  on  $\mathfrak{g}^*$ .

**Definition 4.19** (Characteristic forms [Ca50, §7][Ch50, (10)]). Let  $G$  be a finite-dimensional Lie group with Lie algebra  $\mathfrak{g}$ , and let  $P \xrightarrow{p} X$  be  $G$ -principal bundle with connection  $\nabla$  (Def. 4.17). Then for  $\omega \in \text{inv}^{2n}(\mathfrak{g})$  an invariant polynomial (Def. 4.18), its evaluation on the curvature 2-form  $F_\nabla \in \Omega^2(P) \otimes \mathfrak{g}$  of the connection yields a differential form

$$\omega(F_\nabla) \in \Omega_{\text{dR}}^{2n}(X) \xrightarrow{p^*} \Omega_{\text{dR}}^{2n}(P)$$

which, by the second condition on an Ehresmann connection, is *basic*, namely in the image of the pullback operation along the bundle projection  $p$ , as shown. Regarded as a differential form on  $X$ , this is called the *characteristic form* corresponding to  $\omega$ .

**Lemma 4.20** (Characteristic de Rham classes of characteristic forms [We49][Ch50, p. 401][Ch51, §III.4]). *The class in de Rham cohomology*

$$[\omega(F_\nabla)] \in H_{\text{dR}}^{2n}(X)$$

*of a characteristic form in Def. 4.19 is independent of the choice of connection  $\nabla$  and depends only on the isomorphism class of the principal bundle  $P$ .*

**Definition 4.21** (Chern-Weil homomorphism [Ca50, §7][Ch50, (10)]). Let  $G$  be a finite-dimensional Lie group, with classifying space denoted  $BG$ . The *Chern-Weil homomorphism* is the composite map

$$\begin{array}{c} \text{Chern-Weil} \\ \text{homomorphism} \end{array} \text{cw}_G : \text{GBundles}(X)_{/\sim} \longrightarrow \text{GConnections}(X)_{/\sim} \longrightarrow \text{Hom}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X)) \quad (213)$$

$$\begin{array}{ccc} \text{principal bundle} & \text{with connection} & \text{invariant polynomial} \quad \text{de Rham class of characteristic form} \\ [P] \longmapsto & [P, \nabla] \longmapsto & (\omega \mapsto [\omega(F_\nabla)]) \end{array}$$

where the first map is any section of (212), given by choosing any connection on a given principal bundle; and the second map is the construction of characteristic forms according to Def. 4.19. (The Hom on the right is that in  $\text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ .) By Lemma 4.20 the second map is well-defined (and its composition with the first turns out to be independent of the choices made, by Prop. 4.23 below).

That this construction is useful, in that it produces interesting real characteristic classes of  $G$ -principal bundles (Example 2.21), is the following statement:

**Proposition 4.22** (Abstract Chern-Weil homomorphism [Ch50, (11)][Ch51, §III.5][Bo73, p. 239]). *Let  $G$  be a finite-dimensional, compact Lie group, with Lie algebra denoted  $\mathfrak{g}$ . Then the real cohomology algebra of its classifying space  $BG$  is isomorphic to the algebra of invariant polynomials (Def. 4.18):*

$$\text{inv}^\bullet(\mathfrak{g}) \simeq H^\bullet(BG; \mathbb{R}) \in \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}. \quad (214)$$

We can also obtain the following:

**Proposition 4.23** (Fundamental theorem of Chern-Weil theory [Ch50, (15)][Ch51, §III.6] (Rem. 4.16)). *Let  $G$  be a finite-dimensional compact Lie group. Then the Chern-Weil homomorphism (Def. 4.21) coincides with the operation of pullback of universal characteristic classes along the classifying maps of  $G$ -bundles (Example 2.21), in that the following diagram commutes:*

$$\begin{array}{ccc}
H(X; BG) & \xrightarrow[\substack{\text{pullback of} \\ \text{universal characteristic classes} \\ \text{along classifying map (26)}}]{c \mapsto c^*(-)} & \text{Hom}(H^\bullet(BG; \mathbb{R}), H^\bullet(X; \mathbb{R})) \\
\uparrow \simeq \text{(11)} & & \simeq \downarrow \text{(214)} \\
GBundles(X)_{/\sim} & \xrightarrow[\text{Chern-Weil homomorphism (213)}]{\text{cw}_G} & \text{Hom}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X))
\end{array} \tag{215}$$

Here the isomorphism on the left is from Example 2.3, while that from the right is from Prop. 4.22 and using the de Rham theorem.

### Chern-Weil homomorphism as a special case of the non-abelian character.

**Lemma 4.24** (Sullivan model of classifying space). *Let  $G$  be a finite-dimensional, compact and simply-connected Lie group, with Lie algebra denoted  $\mathfrak{g}$ . Then the minimal Sullivan model (Def. 3.47) of its classifying space  $BG$  is the graded algebra of invariant polynomials (Def. 4.18), regarded as a dgc-algebra with vanishing differential:*

$$(\text{inv}(\mathfrak{g}), d = 0) \simeq \text{CE}(\mathbb{1}BG) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}. \quad (216)$$

*Proof.* According to [FOT08, Example 2.42], we have

$$\text{CE}(\mathbb{1}BG) \simeq (H^\bullet(BG; \mathbb{R}), d = 0). \quad (217)$$

The composition of (217) with the isomorphism (214) from Prop. 4.22 yields the desired (216).  $\square$

**Lemma 4.25** (Non-abelian de Rham cohomology with coefficients in a classifying space). *Let  $G$  be a finite-dimensional, compact and simply-connected Lie group, with Lie algebra denoted  $\mathfrak{g}$ . Then the non-abelian de Rham cohomology (Def. 3.84) with coefficients in the rational Whitehead  $L_\infty$ -algebra  $\mathbb{1}BG$  (Prop. 3.63) of the classifying space is canonically identified with the codomain of the classical Chern-Weil construction (213):*

$$H_{\text{dR}}(X; \mathbb{1}BG) \simeq \text{Hom}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X)). \quad (218)$$

nonabelian  
de Rham cohomology
traditional codomain of  
Chern-Weil construction

*Proof.* Consider the following sequence of natural bijections:

$$\begin{aligned} H_{\text{dR}}(X; \mathbb{1}BG) &:= \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{CE}(\mathbb{1}BG), \Omega_{\text{dR}}^\bullet(X))_{/\sim} \\ &\simeq \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}((\text{inv}^\bullet(\mathfrak{g}), d = 0), \Omega_{\text{dR}}^\bullet(X))_{/\sim} \\ &\simeq \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{inv}^\bullet(\mathfrak{g}), \Omega_{\text{dR}}^\bullet(X)_{\text{closed}})_{/\sim} \\ &\simeq \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{inv}^\bullet(\mathfrak{g}), (\Omega_{\text{dR}}^\bullet(X)_{\text{closed}})_{/\sim}) \\ &\simeq \text{GradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X)) \\ &=: \text{Hom}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X)). \end{aligned}$$

Here the first line is the definition (Def. 3.84). After that, the first step is Lemma 4.25. The second step unwinds what it means to hom out of a dgc-algebra with vanishing differential (which is generator-wise as in Example 3.79), while the third and fourth steps unwind what this means for the coboundary relations (which is generator-wise as in Prop. 3.86). The last line just matches the result to the abbreviated notation used in (213).  $\square$

**Theorem 4.26** (Non-abelian character map subsumes Chern-Weil homomorphism). *Let  $G$  be a finite-dimensional compact, connected and simply-connected Lie group, with Lie algebra  $\mathfrak{g}$ . Let  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\text{fin}_{\mathbb{R}}}_{\geq 1, \text{nil}})$  (Def. 3.52) be equipped with the structure of a smooth manifold. Then the non-abelian character  $\text{ch}_{BG}$  (Def 4.2) on non-abelian cohomology (Def. 2.1) of  $X$  with coefficients in  $BG$  coincides with the Chern-Weil homomorphism  $\text{cw}_G$  (Def. 4.21) with coefficients in  $G$ , in that the following diagram (of cohomology sets) commutes:*

$$\begin{array}{ccc} H(X; BG) & \xrightarrow[\text{ch}_{BG}]{\text{non-abelian character}} & H_{\text{dR}}(X; \mathbb{1}BG) \\ \uparrow (11) \simeq & & \simeq \downarrow (218) \\ \text{GBundles}(X)_{/\sim} & \xrightarrow[\text{Chern-Weil homomorphism}]{\text{cw}_G} & \text{Hom}(\text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X)) \end{array} \quad (219)$$

Here the isomorphism on the left is from Example 2.3, while that on the right is from Lemma 4.25.

*Proof.* First, notice that  $BG$  is simply connected (hence nilpotent), by the assumption that  $G$  is connected, and that it is of finite rational type by Prop. 4.22. Hence, with Def. 3.52,

$$BG \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\text{fin}_{\mathbb{R}}}_{\geq 1, \text{nil}}). \quad (220)$$

Now, by Definition 4.2, the non-abelian character map on the top of (219)

$$\text{ch}_{BG} : H(X; BG) \xrightarrow{(\eta_{BG}^{\mathbb{R}})} H(X; L_{\mathbb{R}}BG) \xrightarrow{\simeq} H_{\text{dR}}(X; L_{\mathbb{R}}BG)$$

sends a classifying map

$$X \xrightarrow{c} BG \in H(X; BG) = \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, BG)$$

first to its composite with the rationalization map (Def. 3.55). By the fundamental theorem (Theorem 3.60 (i), using (220)), this is given by the derived adjunction unit  $\mathbb{D}\eta_{BG}$  of  $\mathbb{R}\exp \dashv \Omega_{\text{PLdR}}^{\bullet}$  (121):

$$X \xrightarrow{c} BG \xrightarrow{L_{\mathbb{R}}BG \simeq \mathbb{D}\eta_{BG}} \mathbb{R}\exp \circ \Omega_{\text{PLdR}}^{\bullet}(BG) \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, L_{\mathbb{R}}BG) = H(X; , L_{\mathbb{R}}BG).$$

Moreover, by part (ii) of the fundamental theorem, the adjunct of the morphism  $\mathbb{D}\eta_{BG} \circ c$  under (121) is

$$\Omega_{\text{PLdR}}^{\bullet}(X) \xleftarrow{c^*} \Omega_{\text{PLdR}}^{\bullet}(BG) \in \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}})$$

(using that  $\Omega_{\text{PLdR}}^{\bullet}(\mathbb{D}\eta^{\mathbb{R}})$  is an equivalence, by reflectivity of rationalization (113)). Hence it is the pullback operation of rational cocycles on  $BG$  along the classifying map  $c$ . Sending this further along the isomorphism to the bottom right in (219) (via Theorem 3.87 and Lemma 4.25) gives, by (164):

$$\text{ch}_{BG} : c \mapsto \Omega_{\text{dR}}^{\bullet}(X) \xleftarrow{c^*} \Omega_{\text{PLdR}}^{\bullet}(BG) \xleftarrow{\simeq} \text{inv}^{\bullet}(\mathfrak{g}) \in \text{Ho}((\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}}). \quad (221)$$

In conclusion, we have found that the commutativity of (219) is equivalent to the statement that the characteristic forms obtained by the Chern-Weil construction (213) represent the pullback (221) of the universal real characteristic classes on  $BG$  along the classifying map  $c$  of the underlying principal bundle (Example 2.21). This is the case by the fundamental theorem of Chern-Weil theory, Prop. 4.23.  $\square$

**Example 4.27** (de Rham representative of tangential  $\text{Sp}(2)$ -twist). For  $X$  a smooth 8-dimensional spin manifold equipped with tangential  $\text{Sp}(2)$ -structure  $\tau$  (59), Theorem 4.26 says that there exists a smooth  $\text{Sp}(2)$ -principal bundle on  $X$  equipped with an Ehresmann connection  $\nabla$  such that the rationalization (Def. 3.55) of the twist  $\tau$  corresponds, under the non-abelian de Rham theorem (Theorem 3.87) to a flat  $\mathfrak{lBSp}(2)$ -valued differential form whose components are the characteristic forms of the  $\text{Sp}(2)$ -principal connection  $\nabla$ :

$$\begin{array}{ccc} H(X; B\text{Sp}(2)) & \xrightarrow{(\eta_{BG}^{\mathbb{R}})^*} & H(X; L_{\mathbb{R}}B\text{Sp}(2)) \simeq H_{\text{dR}}(X; \mathfrak{lBSp}(2)) \\ \tau & \longmapsto & L_{\mathbb{R}}\tau \longleftrightarrow \Omega_{\text{dR}}^{\bullet}(X) \xleftarrow{\tau_{\text{dR}}} \mathbb{R} \left[ \begin{array}{c} \chi_8 \\ \frac{1}{2}p_1 \end{array} \right] / \left( \begin{array}{c} d\frac{1}{2}p_1 = 0 \\ d\chi_8 = 0 \end{array} \right) = \text{CE}(\mathfrak{lBSp}(2)) \\ & & \frac{1}{2}p_1(\nabla) \longleftarrow \frac{1}{2}p_1 \\ & & \chi_8(\nabla) \longleftarrow \chi_8 \end{array}$$

Here on the right we are using [CV98, Thm . 8.1], see [FSS20, Lemma 2.12] to identify generating universal characteristic classes on  $B\text{Sp}(2)$ :  $\frac{1}{2}p_1$  is the first Pontrjagin class (of degree 4) and  $\chi_8 = \left(\frac{1}{2}p_2 - \left(\frac{1}{2}p_1\right)^2\right)$  is the Euler 8-class, which on  $B\text{Sp}(2)$  happens to be proportional to the  $I_8$ -polynomial (see [FSS19b, Prop. 3.7]).

### 4.3 Cheeger-Simons homomorphism

We show (Theorem 4.46) that the non-abelian character map induces *secondary* non-abelian cohomology operations (Def. 4.42) which subsume the Cheeger-Simons homomorphism, recalled around (258) below, with values in ordinary differential cohomology, recalled around (245) below. We follow [FSSt10] [SSS12][Sch13] where the Cheeger-Simons homomorphism, generalized to higher principal bundles, is called the  $\infty$ -Chern-Weil homomorphism.

Underlying this is a differential enhancement of the non-abelian character map (Def. 4.32), and an induced notion of differential non-abelian cohomology (Def. 4.33) on smooth  $\infty$ -stacks (recalled as Def. A.45).

**The differential non-abelian character map.** We introduce (in Def. 4.32 below) the differential refinement of the non-abelian character map; given as before by rationalization, but now followed not by a map to non-abelian de Rham cohomology, but to its refinement by the full cocycle space of flat non-abelian differential forms (Def. 4.28 below). It is this refinement of the codomain of the character map that allows it to be fibered over the smooth space of actual flat non-abelian differential forms (instead of just their non-abelian de Rham classes), thus producing differential non-abelian cohomology (Def. 4.33 below).

**Definition 4.28** (Moduli  $\infty$ -stack of flat  $L_\infty$ -algebra valued forms [Sch13, 4.4.14.2]). Let  $A \in \text{SimplicialSets}$  be of connected, nilpotent,  $\mathbb{R}$ -finite homotopy type (Def. 3.52). In view of the system of sets (Def. 3.77)

$$X \longmapsto \Omega_{\text{dR}}(X; \mathfrak{A}) \in \text{Sets}$$

of flat non-abelian differential forms with coefficient in the Whitehead  $L_\infty$ -algebra  $\mathfrak{A}$  of  $A$  (Prop. 3.63), which are contravariantly assigned to smooth manifolds  $X$ , we consider in  $\text{Ho}(\text{SmoothStacks}_\infty)$  (Def. A.45):

(i) the *smooth space of flat  $\mathfrak{A}$ -valued differential forms*

$$\Omega_{\text{dR}}(-; \mathfrak{A})_{\text{flat}} := \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{dR}}(\mathbb{R}^n; \mathfrak{A})_{\text{flat}} \right) \right), \quad (222)$$

regarded as a simplicially constant simplicial presheaf (356);

(ii) the *smooth  $\infty$ -stack of flat  $\mathfrak{A}$ -valued differential forms* (Example 3.82)

$$\flat \exp(\mathfrak{A}) := \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{dR}}(\mathbb{R}^n \times \Delta^k; \mathfrak{A})_{\text{flat}} \right) \right) \quad (223)$$

which to any Cartesian space assigns the simplicial set that in degree  $k$  is the set of flat  $\mathfrak{A}$ -valued differential forms on the product manifold of the Cartesian space with the standard smooth  $k$ -simplex  $\Delta^k \subset \mathbb{R}^k$ ;

(iii) the canonical inclusion

$$\begin{array}{ccc} \begin{array}{c} \text{smooth space of} \\ \text{flat } \mathfrak{A}\text{-valued forms} \\ \Omega(-; \mathfrak{A})_{\text{flat}} \end{array} & \xrightarrow{\text{atlas}} & \begin{array}{c} \text{smooth } \infty\text{-stack of} \\ \text{flat } \mathfrak{A}\text{-valued forms} \\ \flat \exp(\mathfrak{A}) \end{array} \\ \parallel & & \parallel \\ \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{dR}}(\mathbb{R}^n; \mathfrak{A})_{\text{flat}} \right) \right) & \hookrightarrow & \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{dR}}(\mathbb{R}^n \times \Delta^k; \mathfrak{A})_{\text{flat}} \right) \right) \end{array} \quad (224)$$

exhibiting  $\Omega(-; \mathfrak{A})$  (222) as the presheaf of 0-simplices in the simplicial presheaf  $\flat \exp(\mathfrak{A})$  (223) (more abstractly: this is the canonical *atlas* of the smooth moduli  $\infty$ -stack, see [SS20b, Prop. 2.70]).

**Lemma 4.29** (Moduli  $\infty$ -stack of flat forms is equivalent to discrete rational  $\infty$ -stack).

For  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52), the evident inclusion (by inclusion of polynomial forms into smooth differential forms followed by pullback along  $\text{pr}_{\Delta^k}$ )

$$\begin{array}{ccc} \text{Disc}(L_{\mathbb{R}}A) \simeq \text{Disc} \circ \mathbb{R} \exp \circ \text{CE}(\mathfrak{A}) & \xrightarrow{\in \mathbb{W}} & \flat \exp(\mathfrak{A}) \\ \parallel & & \parallel \\ \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{PLdR}}(\Delta^k; \mathfrak{A})_{\text{flat}} \right) \right) & \hookrightarrow & \left( \mathbb{R}^n \mapsto \left( \Delta[k] \mapsto \Omega_{\text{dR}}(\mathbb{R}^n \times \Delta^k; \mathfrak{A})_{\text{flat}} \right) \right) \end{array} \quad (225)$$

of the image under  $\text{Disc}$  (360) of the dg-algebraic model (122) for the rationalization of  $A$  (Def. 3.55) given by the fundamental theorem (Prop. 3.60), into the moduli  $\infty$ -stack of flat  $\mathfrak{A}$ -valued differential forms (Def. 4.28) is an equivalence in  $\text{Ho}(\text{SmoothStacks}_\infty)$  (Def. A.45).

*Proof.* By Prop. 3.62, the inclusion is for each  $\mathbb{R}^n$  a weak equivalence (130) in  $\text{SimplicialSets}_{\text{Qu}}$  (Example A.8), hence is a weak equivalence already in the global projective model structure on simplicial presheaves, and hence also in the local projective model structure. (Example A.44).  $\square$

**Lemma 4.30** (Moduli  $\infty$ -stack of closed differential forms is shifted de Rham complex).

For  $n \in \mathbb{N}$ , we have an equivalence in  $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$  (Def. A.45) from the moduli  $\infty$ -stack  $\mathfrak{b}\exp(\mathfrak{b}^n\mathbb{R})$  of flat differential forms (Def. 4.28) with values in the Lie  $(n+1)$ -algebra  $\mathfrak{b}^n\mathbb{R}$  (Example 3.27) to the image under the Dold-Kan construction (Def. A.54) of the smooth de Rham complex  $\Omega_{\mathrm{dR}}^\bullet(-)$  (Example 3.23), naturally regarded as a presheaf on  $\mathrm{CartesianSpaces}$  (353) with values in connective chain complexes (Example A.49) (i.e., with de Rham differential lowering the chain degree) shifted up in degree by  $n$  and then homologically truncated in degree 0, as shown on the right.

$$\mathfrak{b}\exp(\mathfrak{b}^n\mathbb{R}) \xrightarrow{\cong} \mathrm{DK} \left( \begin{array}{c} \vdots \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \Omega_{\mathrm{dR}}^0(-) \\ \downarrow d \\ \Omega_{\mathrm{dR}}^1(-) \\ \downarrow d \\ \vdots \\ \downarrow d \\ \Omega_{\mathrm{dR}}^{n+1}(-)_{\mathrm{clsd}} \end{array} \right) \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$$

*Proof.* First observe, with Example 3.79, that the simplicial presheaf

$$\mathfrak{b}\exp(\mathfrak{b}^n\mathbb{R})(-) = \left( \Delta[k] \mapsto \Omega_{\mathrm{dR}}^{n+1}((-) \times \Delta^k)_{\mathrm{clsd}} \right) \quad (226)$$

naturally carries the structure of a presheaf of simplicial abelian groups, given by addition of differential forms. Therefore, by the Dold-Kan Quillen equivalence (Prop A.53), it is sufficient to prove that we have a quasi-isomorphism of presheaves of chain complexes from the corresponding normalized chain complex (361) of (226) to the shifted and truncated de Rham complex itself:

$$N\left(\Delta[k] \mapsto \Omega_{\mathrm{dR}}^{n+1}((-) \times \Delta^k)_{\mathrm{clsd}}\right) \xrightarrow[\cong]{j_{\Delta^\bullet}} \left( \cdots \rightarrow 0 \rightarrow 0 \rightarrow \Omega_{\mathrm{dR}}^0(-) \xrightarrow{d} \Omega_{\mathrm{dR}}^1(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathrm{dR}}^{n+1}(-)_{\mathrm{clsd}} \right). \quad (227)$$

We claim that such is given by fiber integration of differential forms over the simplices  $\Delta^k$ :

First, to see that fiber integration does constitute a chain map, we compute for  $\omega \in \Omega_{\mathrm{dR}}^\bullet((-) \times \Delta^k)_{\mathrm{clsd}}$  on the left of (227):

$$\int_{\Delta^k} \partial \omega = (-1)^k \int_{\partial \Delta^k} \omega = d \int_{\Delta^k} \omega, \quad (228)$$

where the first step is the definition of the differential in the normalized chain complex (361) and the second step is the fiberwise Stokes formula (158).

Finally, to see that  $\int_{\Delta^\bullet}$  is a quasi-isomorphism, notice that the chain homology groups on both sides are

$$H_k(-) = \begin{cases} \mathbb{R} & | \quad k = n+1 \\ 0 & | \quad \text{otherwise} \end{cases}$$

over each Cartesian space: For the left hand side this follows via the weak equivalence (130) from the fundamental theorem (Prop. 3.60) via Example 3.67, while for the right hand side this follows from the Poincaré lemma.

Hence it is sufficient to see that fiber integration over  $\Delta^{n+1}$  is an isomorphism on the  $(n+1)$ st chain homology groups. But a generator of this group on the left is clearly given by the pullback  $\mathrm{pr}_{\Delta^{n+1}}^* \omega$  of any  $\omega \in \Omega_{\mathrm{dR}}^{n+1}(\Delta^{n+1})$  of unit weight and supported in the interior of the simplex. That this is sent under  $\int_{\Delta^{n+1}}$  to a generator  $\pm 1 \in \mathbb{R} \simeq \Omega_{\mathrm{dR}}^0(-)_{\mathrm{clsd}}$  on the right follows by the projection formula (159).  $\square$

**Remark 4.31** (Moduli of closed forms via stable Dold-Kan correspondence). Expressed in terms of the stable Dold-Kan construction  $\mathrm{DK}_{\mathrm{st}}$  (Prop. A.56) via the derived stabilization adjunction (Example A.42), Lemma 4.30 says, equivalently, that:

$$\mathfrak{b}\exp(\mathfrak{b}^n\mathbb{R}) \simeq \mathbb{R}\Omega^\infty \left( \mathrm{DK}_{\mathrm{st}} \left( \Omega_{\mathrm{dR}}^\bullet(-) \otimes_{\mathbb{Z}} \mathfrak{b}^{n+1}\mathbb{R} \right) \right) \in \mathrm{Ho}(\mathrm{SmoothStacks}_\infty), \quad (229)$$

where now  $\Omega_{\mathrm{dR}}^\bullet(-) \in \mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{ChainComplexes}_{\mathbb{Z}})$  is in non-positive degrees, with  $\Omega_{\mathrm{dR}}^0(-)$  in degree 0, and where  $\mathfrak{b}^{n+1}\mathbb{R}$  (Def. 3.7) is concentrated on  $\mathbb{R}$  in degree  $n+1$ .



**Definition 4.32** (Differential non-abelian character map [FSS15b, §4]). Given  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\text{fin}_{\mathbb{R}}}_{\geq 1, \text{nil}})$  (Def. 3.52), the *differential non-abelian character map* in  $A$ -cohomology theory, to be denoted  $\mathbf{ch}_A$ , is the morphism in  $\text{Ho}(\text{SmoothStacks}_{\infty})$  (357) from  $\text{Disc}(A)$  (360) to the moduli  $\infty$ -stack of flat  $\mathbb{A}^1$ -valued forms  $\flat \exp(\mathbb{A}^1)$  (223) given by the composite

$$\begin{array}{ccc}
 \text{coefficient space as} & & \text{moduli } \infty\text{-stack of} \\
 \text{geometrically discrete} & & \text{flat } \mathbb{A}^1\text{-valued} \\
 \text{moduli } \infty\text{-stack} & & \text{differential forms} \\
 \text{Disc}(A) & \xrightarrow{\text{Disc}(\eta_A^{\text{PLdR}})} & \text{Disc} \circ \exp \circ \Omega_{\text{PLdR}}^{\bullet}(A) \xrightarrow[\text{(133)}]{\text{Disc} \circ \exp(p^{\text{min}})} \text{Disc} \circ \mathbb{R} \exp \circ \text{CE}(\mathbb{A}^1) \xrightarrow[\text{(225)}]{\in \mathbb{W}} \flat \exp(\mathbb{A}^1) \quad (230) \\
 & \searrow \text{Disc}(\mathbb{D}\eta_A^{\text{PLdR}}) & \\
 & \text{rationalization} & \\
 & \text{(122)} & 
 \end{array}$$

differential non-abelian character map  $\mathbf{ch}_A$

of

(a) the image under  $\text{Disc}$  (360) of the derived adjunction unit  $\mathbb{D}\eta_A^{\text{PLdR}}$  (326) of the PS de Rham adjunction (128), specifically with (co-)fibrant replacement  $p^{\text{min}}$  being the minimal Sullivan model replacement (111); (recalling that  $\exp$  is a contravariant functor),

with

(b) the weak equivalence from Lemma 4.29.

### Differential non-abelian cohomology.

**Definition 4.33** (Differential non-abelian cohomology [FSS15b, §4]). For  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\text{fin}_{\mathbb{R}}}_{\geq 1, \text{nil}})$  (Def. 3.52) we say that:

(i) the *moduli  $\infty$ -stack of  $\Omega A$ -connections* is the object  $A_{\text{diff}} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  in the homotopy category of smooth  $\infty$ -stacks (Def. A.45), which is given by the homotopy pullback (Def. A.24) of the smooth space of flat non-abelian differential forms  $\Omega_{\text{dR}}(-; \mathbb{A}^1)_{\text{flat}}$  (224) along the differential non-abelian character map  $\mathbf{ch}_A$  (Def. 4.32):

$$\begin{array}{ccc}
 \text{moduli } \infty\text{-stack} & & \text{smooth space of} \\
 \text{of } \Omega A\text{-connections} & & \text{flat } \mathbb{A}^1\text{-valued forms} \\
 A_{\text{diff}} & \xrightarrow{F_A} & \Omega_{\text{dR}}(-; \mathbb{A}^1)_{\text{flat}} \in \text{Ho}(\text{SmoothStacks}_{\infty}); \quad (231) \\
 \downarrow c_A & \text{(hpb)} & \downarrow \text{atlas} \\
 \text{universal characteristic class} & & \\
 \text{in non-abelian } A\text{-cohomology} & & \\
 \text{Disc}(A) & \xrightarrow{\mathbf{ch}_A} & \flat \exp(\mathbb{A}^1) \\
 & \text{differential non-abelian} & \text{moduli } \infty\text{-stack of} \\
 & \text{character map} & \text{flat } \mathbb{A}^1\text{-valued forms}
 \end{array}$$

(ii) the *differential non-abelian cohomology* of a smooth  $\infty$ -stack  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (357) with coefficients in  $A$  is the structured non-abelian cohomology (Remark 2.27) with coefficients in the moduli  $\infty$ -stack  $A_{\text{diff}}$  of  $\Omega A$ -connections (231), hence the hom-set in the homotopy category of  $\infty$ -stacks (Def. A.45) from  $\mathcal{X}$  to  $A_{\text{diff}}$

$$\widehat{H}(\mathcal{X}; A) := H(\mathcal{X}; A_{\text{diff}}) := \text{Ho}(\text{SmoothStacks}_{\infty})(\mathcal{X}, A_{\text{diff}}). \quad (232)$$

(iii) We call the non-abelian cohomology operations induced from the maps in (231) as follows (see (4)):

$$\text{(a) characteristic class: } \widehat{H}(\mathcal{X}; A) \xrightarrow{(c_A)_*} H(\text{Shp}(\mathcal{X}); A) \quad (\text{Def. 2.1}) \quad (233)$$

$$\text{(b) curvature: } \widehat{H}(\mathcal{X}; A) \xrightarrow{(F_A)_*} \Omega_{\text{dR}}(\mathcal{X}; \mathbb{A}^1)_{\text{flat}} \quad (\text{Def. 3.77}) \quad (234)$$

$$\text{(c) differential character: } \widehat{H}(\mathcal{X}; A) \xrightarrow{(\mathbf{ch}_A \circ c_A)_*} H_{\text{dR}}(\mathcal{X}; \mathbb{A}^1) \quad (\text{Def. 3.84}) \quad (235)$$

In differential enhancement of Example 2.13, we have the following.

## Differential generalized cohomology.

**Example 4.34** (Differential generalized cohomology). Let  $E^\bullet$  be a generalized cohomology theory (Example 2.13) with representing spectrum  $E$  (17) which is connective and whose component spaces  $E_n$  are of finite  $\mathbb{R}$ -type, so that their connected components are, by Example 3.54, in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52).

(i) Then differential non-abelian cohomology, in the sense of Def. 4.33, with coefficients in the component spaces  $E_\bullet$ , coincides with canonical differential generalized  $E$ -cohomology in the traditional sense of [HS05, §4.1][Bun12, Def. 4.53][BG13, §2.2][BNV13, §4.4]:

$$\widehat{E}^n(-) \simeq \widehat{H}(-; E_n). \quad (236)$$

(ii) Here “canonical”, in the sense of [Bun12, Def. 4.46], refers to choosing the curvature differential form coefficients to be  $\pi_\bullet(E) \otimes \mathbb{R}$  (instead of some chain complex quasi-isomorphic to this). By Example 3.69, this choice corresponds in our Def. 4.33 to the *minimality* (Def. 3.47) of the minimal Sullivan model  $\text{CE}(\mathbb{L}E_n)$  for  $E_n$  (Prop. 3.63) that controls the flat  $L_\infty$ -algebra valued differential forms  $\Omega_{\text{dR}}(-; \mathbb{L}E_n)_{\text{flat}}$  (Def. 3.77) in the top right of (250).

(iii) Hence for canonical/minimal curvature coefficients, we have from Example 3.69, Lemma 4.30 and Remark 229 that

$$\flat \exp(\mathbb{L}E_n) \simeq \mathbb{R}\Omega^\infty \left( \text{DK}_{\text{st}}(\Omega_{\text{dR}}^\bullet(-) \otimes_{\mathbb{Z}} \pi_\bullet(E_n)) \right) \in \text{Ho}(\text{SmoothStacks}_\infty) \quad (237)$$

and

$$\Omega_{\text{dR}}(-; \mathbb{L}E_n)_{\text{flat}} \simeq \mathbb{R}\Omega^\infty \left( \text{DK}_{\text{st}}(\Omega_{\text{dR}}^\bullet(-) \otimes_{\mathbb{Z}} \pi_\bullet(E_n))_{\leq 0} \right) \in \text{Ho}(\text{SmoothStacks}_\infty). \quad (238)$$

(iv) With this, the equivalence 236 follows by observing that the defining homotopy pullback diagram (231) for differential non-abelian cohomology with coefficients in  $A := E_n$  (352) is the image under  $\mathbb{R}\Omega^\infty$  (351) of the defining homotopy pullback diagram for canonical differential  $E$ -cohomology according to [HS05, (4.12)] [Bun12, Def. 4.51][BNV13, (24)], and using that right adjoints preserve homotopy pullbacks:

$$\begin{array}{ccc} (E_0)_{\text{diff}} & \xrightarrow{F_{E_0}} & \Omega_{\text{dR}}(-; \mathbb{L}E_0)_{\text{flat}} \\ \downarrow c_{E_0} & \text{(hpb)} & \downarrow \text{atlas} \\ \text{Disc}(E_0) & \xrightarrow{\text{ch}_{E_0}} & \flat \exp(\mathbb{L}E_0) \end{array} \simeq \mathbb{R}\Omega^\infty \left( \begin{array}{ccc} \text{Diff}(E, \text{can}) & \longrightarrow & (\Omega_{\text{dR}}^\bullet(-) \otimes_{\mathbb{Z}} \pi_\bullet(E))_{\leq 0} \\ \downarrow & \text{(hpb)} & \downarrow \\ \text{Disc}(E) & \xrightarrow{H\mathbb{R}\wedge(-)} & \Omega_{\text{dR}}^\bullet(-) \otimes_{\mathbb{Z}} \pi_\bullet(E) \end{array} \right) \quad (239)$$

moduli  $\infty$ -stack of  $\Omega_{E_0}$ -connections
“differential function spectrum” of differential generalized  $E$ -cohomology

The same applies to  $(E_n)_{\text{diff}}$ , by replacing  $E$  with  $\mathbb{L}\Sigma^n E$  (351) on the right of (239).

**Remark 4.35** (The canonical atlas for the moduli stack of connections). The operation  $(-)_{\leq 0}$  in (238) is the naive truncation functor on the category of chain complexes

$$\begin{array}{ccc} \text{ChainComplexes}_{\mathbb{Z}} & \xrightarrow{(-)_{\leq 0}} & \text{ChainComplexes}_{\mathbb{Z}}^{\leq 0} \\ (\dots \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} V_{-1} \rightarrow \dots) & \mapsto & (V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} V_{-1} \rightarrow \dots). \end{array}$$

In contrast to the homological truncation involved in  $\Omega^\infty$  (368), this naive truncation is not homotopy-invariant and does not have a derived functor. Instead, as seen from (238) and (224), once regarded in differential non-abelian cohomology, this operation serves to construct the canonical *atlas* [SS20b, Prop. 2.70] of the moduli  $\infty$ -stack of flat  $\mathbb{L}E_n$ -valued differential forms.

Via the defining homotopy pullback (231), (239) this becomes hallmark of differential cohomology: Differential cohomology is the universal solution to lifting the values of the character map from cohomology classes to cochain representatives, namely to curvature forms.

In differential enhancement of Example 2.14 and Example 4.10 we have:

**Example 4.36** (Differential complex K-theory). With the coefficient space  $A := \text{KU}_0 = \mathbb{Z} \times BU$  (19) for topological complex K-theory (Example 2.14), the corresponding differential non-abelian cohomology theory (Def. 4.33) is, by Example 4.34, differential K-theory, whose diagram (4) of cohomology operations is of this form (see [HS05][BS09][BS12][GS17b])

$$\begin{array}{ccc} \widehat{H}(\mathcal{X}; \text{KU}_0) \simeq \widehat{\text{KU}}^0(\mathcal{X}) & \xrightarrow{F_{\text{KU}_0}} & \left\{ \{F_{2k} \in \Omega_{\text{dR}}^{2k}(\mathcal{X})\}_{k \in \mathbb{N}} \mid dF_{2k} = 0 \right\} \\ \text{c}_{\text{KU}_0} \downarrow & & \downarrow \\ \text{KU}^0(\mathcal{X}) & \xrightarrow{\text{ch}} & \bigoplus_{k \in \mathbb{N}} H_{\text{dR}}^{2k}(\mathcal{X}), \end{array} \quad (240)$$

where the bottom map is the ordinary Chern character from Example 4.10, and the curvature differential forms are identified as in Example 3.94.

**Examples of differential non-abelian cohomology.** In differential enhancement of Example 2.3, we have:

**Proposition 4.37** (Differential non-abelian cohomology of principal connections). *Let  $G$  be a compact Lie group with classifying space  $BG$  (12). Then there is a natural map over manifolds  $X$ , shown dashed in (241), from equivalence classes of  $G$ -principal connections (Notation 4.17) to differential non-abelian cohomology with coefficients in  $BG$  (Def. 4.33) which covers the classification of  $G$ -principal bundles by plain non-abelian cohomology with coefficients in  $BG$  (Example 2.3), in that the following diagram commutes:*

$$\begin{array}{ccc} G\text{Connections}(X)_{/\sim} & \overset{\text{differential non-abelian cohomology}}{\dashrightarrow} & \widehat{H}(X; BG) \\ \text{forget connection} \downarrow & & \downarrow c_{BG} \\ GBundles(X)_{/\sim} & \xrightarrow{\simeq} & H(X; BG) \\ & & \text{non-abelian cohomology} \end{array} \quad (241)$$

*Proof.* By Lemma 4.25, the differential form coefficient in the given case is

$$\Omega_{\text{dR}}(-; \mathfrak{L}BG)_{\text{flat}} \simeq \text{Hom}_{\mathbb{R}}(\text{inv}^\bullet(\mathfrak{g}), \Omega_{\text{dR}}^\bullet(-)_{\text{clsd}}).$$

Therefore, with Example 3.67, we find that

$$\left( \Delta[k] \mapsto \text{Hom}_{\mathbb{R}}(\text{inv}^\bullet(\mathfrak{g}), \Omega_{\text{dR}}^\bullet(\Delta^k)_{\text{clsd}}) \right) \simeq \prod_k K(\text{inv}^n(\mathfrak{g}), n) \in \text{Ho}(\text{SimplicialSets}_{\text{Qu}})$$

is a product of Eilenberg-MacLane spaces (10) for real coefficient groups spanned by the invariant polynomials, and so the defining homotopy pullback (231) is here of this form:

$$\begin{array}{ccc} BG_{\text{diff}} & \longrightarrow & \text{Hom}_{\mathbb{R}}(\text{inv}^\bullet(\mathfrak{g}), \Omega_{\text{dR}}^\bullet(-)_{\text{clsd}}) \\ \downarrow & \text{(hpb)} & \downarrow \\ \text{Disc}(BG) & \xrightarrow{(c_k)_{k \in \mathbb{N}}} & \text{Disc}\left(\prod_{k \in \mathbb{N}} K(\text{inv}^n(\mathfrak{g}), n)\right), \end{array}$$

where the bottom map classifies the real characteristic classes of  $BG$  via Example 2.2. It follows (by Example A.27) that maps into  $BG_{\text{diff}}$  are equivalence classes of triples

$$\widehat{H}(X; BG) \simeq \left\{ (f, \phi, (\alpha_k)) \left| \begin{array}{ccc} X & \overset{(\alpha_k)}{\dashrightarrow} & \text{Hom}_{\mathbb{R}}(\text{inv}^\bullet(\mathfrak{g}), \Omega_{\text{dR}}^\bullet(-)_{\text{clsd}}) \\ \downarrow f & \Leftarrow \phi & \downarrow \\ BG & \longrightarrow & \text{Disc}\left(\prod_{k \in \mathbb{N}} K(\text{inv}^n(\mathfrak{g}), n)\right) \end{array} \right. \right\} \quad (242)$$

consisting of **(a)** a classifying map  $f$  for a  $G$ -principal bundle (Example 2.3), **(b)** a set of closed differential forms  $\alpha$  labeled by the invariant polynomials, and **(c)** a set of coboundaries  $\phi$  in real cohomology between these differential forms and the pullbacks  $f^*c_k$ .

Now, given a  $G$ -connection  $\nabla$  on a  $G$ -principal bundle  $f^*EG$  over  $X$ , we obtain such a triple by **(a)** taking  $f$  to be the classifying map of the underlying  $G$ -principal bundle, **(b)** taking  $\alpha_k := \omega_k(F_\nabla)$  to be the characteristic forms (Def. 4.19) of the connection, and **(c)** taking  $\phi$  to be given by the relative Chern-Simons forms [CS74] between the given connection and the pullback along  $f$  of the universal connection (see Remark 4.16). This construction is an invariant of the isomorphism class of the connection (see [HS05, p. 28]) and hence defines the desired map (241):

$$\begin{array}{ccc} G\text{Connections}(X)_{/\sim} & \xrightarrow{\quad} & \widehat{H}(X; BG) \\ [f^*EG, \nabla] & \mapsto & [f, (cS_k(\nabla, f^*\nabla_{\text{univ}}), (\omega_k(F_\nabla)))] \end{array} \quad (243)$$

In differential enhancement of Example 2.10, we have:

**Example 4.38** (Differential Cohomotopy [FSS15b]). The canonical differential enhancement of (unstable) Cohomotopy theory (Example 2.10) in degree  $n$  is differential non-abelian cohomology (Def. 4.33) with coefficients in  $S^n$ :

$$\widehat{\pi}^n(-) := \widehat{H}(-; S^n).$$

**(i)** By Example 3.81, a cocycle  $\widehat{C}_3 \in \widehat{\pi}^4(X)$  in differential 4-Cohomotopy has as curvature (231) a pair consisting of a differential 4-form  $G_4$  and a differential 7-form  $G_7$ , satisfying the *Cohomotopical Bianchi identity* shown here:

$$\begin{array}{ccc} \begin{array}{c} \text{differential} \\ \text{4-Cohomotopy} \end{array} \widehat{\pi}^4(X) & \xrightarrow{\text{cohomotopical curvature } F_{S^4}} & \Omega(X; \mathbb{S}^4)_{\text{flat}} \\ \widehat{C}_3 & \mapsto & \left\{ \begin{array}{l} G_7(\widehat{C}_3), \\ G_4(\widehat{C}_3) \end{array} \in \Omega_{\text{dR}}^\bullet(X) \left| \begin{array}{l} dG_7(\widehat{C}_3) = -G_4(\widehat{C}_3) \wedge G_4(\widehat{C}_3) \\ dG_4(\widehat{C}_3) = 0 \end{array} \right. \right\}. \end{array} \quad (244)$$

Such differential form data is exactly what characterizes the flux densities of the  $C_3$ -field in 11-dimensional supergravity (up to the self-duality constraint  $G_7 = \star G_4$ ). By comparison with Dirac's charge quantization (2), we thus see that a natural candidate for charge quantization of the supergravity  $C_3$ -field is (nonabelian/unstable) 4-Cohomotopy theory  $\pi^4$  [Sa13, §2.5][FSS16a, §2][BMSS19, §3] (review in [FSS19a, §7]) or rather: differential 4-Cohomotopy theory  $\widehat{\pi}^4$  [FSS15b, p. 9][GS20].

**(ii)** The consequence of this Cohomotopical charge quantization is readily seen from the Hurewicz operation on Cohomotopy theory (Example 2.26): The de Rham class of the 4-flux density is constrained to be integral, hence to be in the image of the de Rham homomorphism (Example 4.9) and its cup square is forced to vanish

$$[G_4(\widehat{C}_3)] \in H^4(X; \mathbb{Z}) \longrightarrow H_{\text{dR}}^4(X), \quad [G_4(\widehat{C}_3)] \cup [G_4(\widehat{C}_3)] = 0.$$

This leads to interpretation via Massey products [KS05], with corresponding differential refinement in [GS17a].

**(iii)** Passing from 11-dimensional supergravity to M-theory, the curvature data in (244) is expected to be subjected to more refined topological constraints, forcing the class of  $G_4$  to be integral up to a fractional shift by the first Pontrjagin class of the tangent bundle, and deforming its cup square to a quadratic function with non-trivial “background charge” ([FSS19b, Table 1][GS20]). We see, in Prop. 5.24 below, that these more subtle M-theoretic constraints on the  $C_3$ -field flux densities are imposed by charge quantization in – hence lifting through the non-abelian character map of – the corresponding *twisted* non-abelian cohomology theory, namely: *J-twisted* 4-Cohomotopy [FSS19b][FSS20] (Example 5.23 below).

**Differential ordinary cohomology.** The *ordinary differential cohomology*  $\widehat{H}^\bullet(X)$  [SiSu08] of a smooth manifold  $X$  combines ordinary integral cohomology classes (Example 2.2) with closed differential forms that represent the same class in real cohomology, in that it makes a diagram of the following form commute:

$$\begin{array}{ccc} \begin{array}{c} \text{ordinary} \\ \text{differential cohomology} \end{array} \widehat{H}^\bullet(X) & \xrightarrow{\text{curvature}} & \Omega_{\text{dR}}^\bullet(X)_{\text{clsd}} \\ \downarrow \text{underlying} & & \downarrow \text{via} \\ H^\bullet(X; \mathbb{Z}) & \xrightarrow{\text{rationalization}} & H^\bullet(X; \mathbb{R}) \end{array} \quad (245)$$

In fact, differential cohomology is universal with this property, but not at the coarse level of cohomology sets shown above (where the universal property is shallow) but at the fine level of complexes of sheaves of coefficients (i.e. of moduli  $\infty$ -stacks), as made precise in Prop. 4.40 below.

In degree 2, ordinary differential cohomology classifies ordinary  $U(1)$ -principal bundles (equivalently: complex line bundles) with connection [Bry93, §II], and the curvature map in (245) assigns their traditional curvature 2-form. In degree 3 ordinary differential cohomology classifies bundle gerbes with connection [Mu96][SWa07] with their curvature 3-form. In general degree it classifies higher bundle gerbes with connection [Ga97], or equivalently higher  $U(1)$ -principal bundles with connection [FSS12b, 2.6].

One construction of ordinary differential cohomology over smooth manifolds is given in [CS85, §1], now known as *Cheeger-Simons characters*. An earlier construction over schemes, now known as *Deligne cohomology* (Example 4.39), due independently to [De71, §2.2][MM74, §3.1.7][AM77, §III.1] and brought to seminal application in [Bei85] (review in [EV88]) is readily adapted to smooth manifolds [Bry93, §I.5][Ga97]. The advantage of Deligne cohomology over Cheeger-Simons characters is that it immediately generalizes from smooth manifolds to smooth  $\infty$ -stacks, [FSS10, §3.2.3][FSS12b, §2.5], such as to orbifolds [SS20a] and to moduli  $\infty$ -stacks of higher principal connections where it yields higher Chern-Simons functionals [SSS12][FSS12a][FSS13a][FSS15a], as well as allowing for twists in a systematic manner [GS18c][GS19b].

In differential enhancement of Example 2.12, we have:

**Example 4.39** (Ordinary differential cohomology on smooth  $\infty$ -stacks [FSS10, §3.2.3][FSS12b, §2.5]). Let  $n \in \mathbb{N}$ .

(i) The smooth *Deligne-Beilinson complex* in degree  $n+1$  is the presheaf of connective chain complexes (Example A.49) over CartesianSpaces (353) given by the truncated and shifted smooth de Rham complex (Example 3.23) with a copy of the integers included in degree  $n+1$  (as integer valued 0-forms, hence as smooth real-valued functions constant on an integer):

$$DB_{\bullet}^{n+1} := \left( \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \hookrightarrow \Omega_{\mathrm{dR}}^0(-) \xrightarrow{d} \Omega_{\mathrm{dR}}^1(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathrm{dR}}^n(-) \right). \quad (246)$$

(ii) The de Rham differential in degree 0 gives a morphism of presheaves of complexes

$$DB_{\bullet}^{n+1} \xrightarrow{(0,0,\dots,0,d)} \Omega_{\mathrm{dR}}^{n+1}(-)_{\mathrm{clsd}} \quad (247)$$

from the Deligne-Beilinson complex (246) to the presheaf of closed  $(n+1)$ -forms, regarded as a presheaf of chain complexes regarded in degree 0.

(iii) *Ordinary differential cohomology* is stacky non-abelian cohomology (Remark 2.27) with coefficients in the Deligne-Beilinson complex (246) regarded as a smooth  $\infty$ -stack (Def. A.45) under the  $\infty$ -stackified Dold-Kan construction from Example A.54 (hence sheaf hypercohomology with coefficients in the Deligne complex):

$$\widehat{H}^{n+1}(\mathcal{X}) := \mathrm{Ho}(\mathrm{SmoothStacks}_{\infty}) \left( \mathcal{X}, L^{\mathrm{loc}} \circ \mathrm{DK}(DB_{\bullet}^{n+1}) \right). \quad (248)$$

ordinary differential cohomology
Dold-Kan correspondence
Deligne-Beilinson complex

(iv) The *curvature map* on ordinary differential cohomology is the cohomology operation induced by (247):

$$\begin{array}{ccc} \widehat{H}^{n+1}(\mathcal{X}) & \xrightarrow[\text{curvature } F]{} & \Omega_{\mathrm{dR}}^{n+1}(\mathcal{X})_{\mathrm{clsd}} \\ \parallel & & \parallel \\ \mathrm{Ho}(\mathrm{SmoothStacks}_{\infty}) \left( \mathcal{X}, L^{\mathrm{loc}} \circ \mathrm{DK}(DB_{\bullet}^{n+1}) \right) & \xrightarrow{\mathrm{Ho}(\mathrm{SmoothStacks}_{\infty})(\mathcal{X}, L^{\mathrm{loc}} \circ \mathrm{DK}(d))} & \mathrm{Ho}(\mathrm{SmoothStacks}_{\infty}) \left( \mathcal{X}, L^{\mathrm{loc}} \circ \mathrm{DK}(\Omega_{\mathrm{dR}}^{n+1}(-)_{\mathrm{clsd}}) \right) \end{array} \quad (249)$$

**Proposition 4.40** (Differential non-abelian cohomology subsumes differential ordinary cohomology [FSS10, Prop. 3.2.26]). Let  $n \in \mathbb{N}$  and consider  $A = B^n U(1) \simeq K(\mathbb{Z}, n+1)$  (Example 2.12). Then:

(i) *Differential non-abelian A-cohomology* (Def. 4.33) coincides with *ordinary differential cohomology* (Def. 4.39):

$$\widehat{H}^{n+1}(\mathcal{X}) \simeq \widehat{H}(\mathcal{X}; B^n U(1)). \quad (250)$$

ordinary differential cohomology

(ii) *The abstract curvature map in differential A-cohomology* (231) reproduces the *ordinary curvature map* (249).

*Proof.* First we use the Dold-Kan correspondence (Prop. A.51) to obtain a convenient presentation of the differential character:

(a) Since the Dold-Kan construction DK (Def. A.54) realizes homotopy groups from homology groups (363), and since Eilenberg-MacLane spaces are characterized by their homotopy groups (10), we have the vertical identifications on the left of the following diagram:

$$\begin{array}{ccc}
 \text{Disc}(\mathcal{B}^{n+1}\mathbb{Z}) & \xrightarrow{\eta_{\mathcal{B}^{n+1}\mathbb{Z}}^{\mathbb{R}}} & \text{Disc}(\mathcal{B}^{n+1}\mathbb{R}) \xrightarrow{\simeq} \mathfrak{b}\exp(\mathfrak{b}^n\mathbb{R}) \\
 \parallel & & \parallel \\
 \text{DK} \left( \begin{array}{c} \mathbb{Z} \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \end{array} \right) & \hookrightarrow & \text{DK} \left( \begin{array}{c} \mathbb{R} \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \end{array} \right) \hookrightarrow \text{DK} \left( \begin{array}{c} \Omega_{\text{dR}}^0(-) \\ \downarrow d \\ \Omega_{\text{dR}}^1(-) \\ \downarrow d \\ \vdots \\ \downarrow d \\ \Omega_{\text{dR}}^{n+1}(-)_{\text{clsd}} \end{array} \right)
 \end{array} \quad (251)$$

Under this identification, it is clear that the rationalization map  $\eta_{\mathcal{B}^{n+1}\mathbb{Z}}^{\mathbb{R}}$  (Def. 3.55) is presented by the canonical inclusion of the integers into the real numbers, as on the bottom left of (251).

Moreover, the right vertical equivalence in (251) is that from Lemma 4.30.

(b) Since the differential character (230) in the present case evidently comes from a morphism of (presheaves of) simplicial abelian groups, with group structure given by addition of ordinary differential forms (Example 3.79), we may, using the Dold-Kan correspondence (Prop. A.51), analyze the remainder of the diagram on normalized chain complexes  $N(-)$  (362).

Using this, it follows by inspection of the bottom map in (230) that the bottom right square in (251) commutes, with the bottom morphism on the right being the canonical inclusion of (presheaves of) chain complexes.

Now to use this presentation for identifying the resulting homotopy fiber product:

(i) Since the DK-construction (Def. A.54), applied objectwise over CartesianSpaces, is a right Quillen functor into the global model structure from Example A.44, and since  $\infty$ -stackification preserves homotopy pullbacks (Lemma A.47), it is now sufficient to show, by definition (248), that the homotopy pullback (Def. A.24) along the bottom map in (251), formed in presheaves of chain complexes is the Deligne-Beilinson complex (246). For this, by (330) it is sufficient to find a fibration replacement of the bottom map in (251) whose ordinary fiber product with  $\Omega_{\text{dR}}^{n+1}(-)_{\text{clsd}}$  is the Deligne-Beilinson complex. This is the case for the following factorization:

$$\begin{array}{ccc}
 \left( \begin{array}{c} \mathbb{Z} \\ \downarrow i \\ \Omega_{\text{dR}}^0(-) \\ \downarrow d \\ \Omega_{\text{dR}}^1(-) \\ \downarrow d \\ \vdots \\ \downarrow d \\ \Omega_{\text{dR}}^{n-1}(-) \\ \downarrow d \\ \Omega_{\text{dR}}^n(-) \end{array} \right) & \xrightarrow{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ d \end{pmatrix}} & \left( \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \\ \downarrow \\ \Omega_{\text{dR}}^{n+1}(-)_{\text{clsd}} \end{array} \right) \\
 \downarrow i_1 & \text{(pb)} & \downarrow i \\
 \left( \begin{array}{c} \mathbb{Z} \\ \downarrow \\ 0 \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{array} \right) & \xrightarrow[\in \mathbb{W}]{n \mapsto (n,n)} & \left( \begin{array}{c} \mathbb{Z} \oplus \Omega_{\text{dR}}^0(-) \\ \downarrow i \quad \swarrow -\text{id} \quad \downarrow d \\ \Omega_{\text{dR}}^0(-) \oplus \Omega_{\text{dR}}^1(-) \\ \downarrow d \quad \swarrow +\text{id} \quad \downarrow d \\ \Omega_{\text{dR}}^1(-) \oplus \Omega_{\text{dR}}^2(-) \\ \downarrow d \quad \swarrow -\text{id} \quad \downarrow d \\ \vdots \\ \downarrow d \\ \Omega_{\text{dR}}^{n-1}(-) \oplus \Omega_{\text{dR}}^n(-) \\ \downarrow d \quad \swarrow \\ \Omega_{\text{dR}}^n(-) \end{array} \right) & \xrightarrow[\in \text{Fib}]{\begin{pmatrix} \text{pr}_2 \\ \vdots \\ \text{pr}_2 \\ d \end{pmatrix}} & \left( \begin{array}{c} \Omega_{\text{dR}}^0(-) \\ \downarrow d \\ \Omega_{\text{dR}}^1(-) \\ \downarrow d \\ \Omega_{\text{dR}}^2(-) \\ \downarrow d \\ \vdots \\ \downarrow d \\ \Omega_{\text{dR}}^n(-) \\ \downarrow d \\ \Omega_{\text{dR}}^{n+1}(-)_{\text{clsd}} \end{array} \right)
 \end{array} \quad (252)$$

Here the total bottom morphisms is the total bottom morphism from (251), factored as a weak equivalence (quasi-isomorphism) followed by a fibration (positive degreewise surjection). The ordinary pullback of the fibration is shown, and represents the homotopy pullback, since all chain complexes are projectively fibrant.

(ii) Finally, the top morphism in (252), thus being the abstract curvature map (231) is seen to coincide with the curvature map (247) on the Deligne complex.  $\square$

**Secondary non-abelian cohomology operations.** We define secondary non-abelian cohomology operations (Def. 4.42 below) which generalize the classical notion of secondary characteristic classes (Theorem 4.46, see Remark 4.47 for the terminology) to higher non-abelian cohomology. To formulate the concept in this generality, we need a technical condition (Def. 4.41) which happens to be trivially satisfied in the classical case (Lemma 4.44 below):

**Definition 4.41** (Absolute minimal model). For  $A_1, A_2 \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) we say that an *absolute minimal model* for a morphism  $A_1 - c \rightarrow A_2$  in  $\text{SimplicialSets}$  is a morphism  $\mathfrak{I}A_1 - c \rightarrow \mathfrak{I}A_2$  between the respective Whitehead  $L_\infty$ -algebras (Prop. 3.63) which makes the square on the left and hence the square on the far right of the following diagram commute:

$$\begin{array}{ccccc}
\Omega_{\text{dRPL}}^\bullet(A_1) & \xleftarrow{p_{A_1}^{\text{min}}} & \text{CE}(\mathfrak{I}A_1) & & \\
\uparrow & & \uparrow c & & \\
\Omega_{\text{dRPL}}^\bullet(A_2) & \xleftarrow{p_{A_2}^{\text{min}}} & \text{CE}(\mathfrak{I}A_2) & & \\
\in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} & & & & \\
A_1 & \xrightarrow{\eta_{A_1}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^\bullet(A_1) & \xrightarrow{\exp(p_{A_1}^{\text{min}})} & \exp \circ \text{CE}(\mathfrak{I}A_1) \\
\downarrow c & & \downarrow \exp \circ \Omega_{\text{PLdR}}^\bullet(c) & & \downarrow \exp \circ \text{CE}(c) \\
A_2 & \xrightarrow{\eta_{A_2}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^\bullet(A_2) & \xrightarrow{\exp(p_{A_2}^{\text{min}})} & \exp \circ \text{CE}(\mathfrak{I}A_2) \\
\in \text{SimplicialSets} & & & & 
\end{array} \quad (253)$$

hence a morphism that yields a transformation between exactly those derived adjunction units  $\mathbb{D}\eta^{\text{PLdR}}$  (326) of the PL-de Rham adjunction (119) that are given by *minimal* fibrant replacement.<sup>14</sup> In this case, the commuting diagram (253) evidently extends to a strict transformation between the differential non-abelian characters (230) on the  $A_i$  (Def. 4.32), in that the following diagram of simplicial presheaves (Def. 354) commutes:

$$\begin{array}{ccc}
\text{Disc}(A_1) & \xrightarrow{\text{ch}_{A_1}} & \flat \exp(\mathfrak{I}A_1) \\
\text{Disc}(c) \downarrow & & \flat \exp(c) \downarrow \\
\text{Disc}(A_2) & \xrightarrow{\text{ch}_{A_2}} & \flat \exp(\mathfrak{I}A_2) \\
\in \text{PSh}(\text{CartesianSpaces}, \text{SimplicialSets}). & & (254)
\end{array}$$

In differential enhancement of Def. 2.17 we have:

**Definition 4.42** (Secondary non-abelian cohomology operation). Let  $A_1 \xrightarrow{c} A_2$  in  $\text{SimplicialSets}$ , with induced cohomology operation (Def. 2.17)

$$H(-; A_1) \xrightarrow{c_*} H(-; A_2),$$

have an absolute minimal model  $c$  (Def 4.41). Then the corresponding *secondary non-abelian cohomology operation* is the structured cohomology operation (Remark 2.27)

$$\widehat{H}(-; A_1) \xrightarrow[\text{secondary non-abelian character}]{(c_{\text{diff}})^*} \widehat{H}(-; A_2) \quad (255)$$

on differential non-abelian cohomology (Def. 4.33) which is induced (29) by the dashed morphism  $c_{\text{diff}}$  in the following diagram, which in turn is induced from  $c$  and  $c$  (254) by the universal property of the defining homotopy pullback operation (230):

<sup>14</sup> Notice that the existence of morphisms  $c$  making this diagram commute is not guaranteed; it is only the existence of the *relative* minimal morphism  $\mathfrak{I}A_2(c)$  from Prop. 3.70 which is guaranteed to make the square (142) commute.

$$\begin{array}{ccc}
\text{secondary/differential} & (A_1)\text{diff} \xrightarrow{\quad c_{\text{diff}} \quad} (A_2)\text{diff} & \\
\text{cohomology operation} & \searrow F_{A_1} \quad \quad \quad \searrow F_{A_2} & \\
& \Omega_{\text{dR}}(-; \mathfrak{A}_1)_{\text{flat}} \xrightarrow{\quad c_* \quad} \Omega_{\text{dR}}(-; \mathfrak{A}_2)_{\text{flat}} & \\
\text{plain/primary} & \downarrow c_{A_1} \quad \quad \quad \downarrow c_{A_2} & \\
\text{cohomology operation} & \text{Disc}(A_1) \xrightarrow{\quad \text{Disc}(c) \quad} \text{Disc}(A_2) & \\
\text{transformation of} & \searrow \text{ch}_{A_1} \quad \quad \quad \searrow \text{ch}_{A_2} & \\
\text{differential characters} & \mathfrak{bexp}(\mathfrak{A}_1) \xrightarrow{\quad c_* \quad} \mathfrak{bexp}(\mathfrak{A}_2) &
\end{array} \tag{256}$$

The left and right squares are the homotopy pullback squares defining differential non-abelian cohomology (Def. 4.33) while the bottom square is the transformation of differential non-abelian characters (Def. 4.32) from (254).

In differential enhancement of Examples 2.26, 4.13 we have:

**Example 4.43** (Secondary non-abelian Hurewicz/Boardman homomorphism to differential K-theory). Consider the map

$$S^4 \xrightarrow{\beta^4} BU \in \text{Ho}(\text{TopologicalSpaces}_{\mathbb{Q}\text{u}})$$

from the 4-sphere to the classifying space of the infinite unitary group (20) which classifies a generator in  $\pi_4(BU) \simeq \mathbb{Z}$ . By Example 3.68 and Examples 3.69, 3.94 the corresponding Whitehead  $L_\infty$ -algebras (Prop. 3.63) are as shown here:

$$\begin{array}{ccc}
\text{CE}(\mathfrak{tS}^4) & \longleftarrow & \text{CE}(\mathfrak{tBU}) \simeq \bigotimes_{k \in \mathbb{N}} \text{CE}(\mathfrak{tK}(\mathbb{Z}, 2k)) \\
\parallel & & \parallel \\
\mathbb{R} \left[ \begin{array}{c} \omega_7, \\ \omega_4 \end{array} \right] / \left( \begin{array}{l} d\omega_7 = -\omega_4 \wedge \omega_4 \\ d\omega_4 = 0 \end{array} \right) & \xleftarrow{\left. \begin{array}{l} \omega_4 \\ 0 \end{array} \right| \left. \begin{array}{l} 2k=4 \\ \text{else} \end{array} \right\} \leftarrow f_{2k}} & \mathbb{R} \left[ \begin{array}{c} \vdots \\ f_4, \\ f_2, \end{array} \right] / \left( \begin{array}{l} \vdots \\ df_4 = 0 \\ df_2 = 0 \end{array} \right)
\end{array} \tag{257}$$

The morphism shown in (257) evidently restricts to the relative rational Whitehead  $L_\infty$ -algebra inclusion (Prop. 3.70) on the factor  $K(\mathbb{R}, 4) \subset L_{\mathbb{R}}BU$  and is zero elsewhere, hence fits into the required diagram (253) exhibiting it as an absolute minimal model (Def. 4.41) for  $\beta^4$  (by the commuting diagram in Prop. 3.50).

**Cheeger-Simons homomorphism.** Where the construction of the Chern-Weil homomorphism (Def. 4.21) invokes connections on principal bundles without actually being sensitive to this choice (by Prop. 4.23), the *Cheeger-Simons homomorphism* [CS85, §2][HS05, §3.3] (based on [CS74]) is a refinement of the Chern-Weil homomorphism, now taking values in differential ordinary cohomology (Example 4.39), that does detect connection data (hence “differential” data):

$$\begin{array}{ccc}
\text{GConnections}(X)_{/\sim} & \xrightarrow[\text{Cheeger-Simons homomorphism}]{c_{\text{SG}}} & \text{Hom}_{\mathbb{Z}} \left( H^\bullet(BG; \mathbb{Z}), \widehat{H}^\bullet(X) \right) \\
\downarrow \text{forget connection} & & \downarrow \text{curvature map} \\
\text{GBundles}(X)_{/\sim} & \xrightarrow[\text{Chern-Weil homomorphism}]{c_{\text{WG}}} & \text{Hom}_{\mathbb{R}} \left( \text{inv}^\bullet(\mathfrak{g}), H_{\text{dR}}^\bullet(X) \right) \\
& & \text{de Rham cohomology}
\end{array} \tag{258}$$

We discuss how the general notion of secondary non-abelian cohomology operations (Def. 4.42) specializes on ordinary principal bundles to the Cheeger-Simons homomorphism, and hence generalizes it to higher non-abelian cohomology:



**Lemma 4.44** (Characteristic classes of  $G$ -principal bundles have absolute minimal models). *Let  $G$  be a connected compact Lie group with classifying space  $BG$  (12). For  $n \in \mathbb{N}$ , let  $[c] \in H^{n+1}(BG; \mathbb{Z})$  be a universal integral characteristic class (Example 2.4). Then every representative classifying map  $BG \xrightarrow{c} B^{n+1}\mathbb{Z}$  has an absolute minimal model in the sense of Def. 4.41.*

*Proof.* By Lemma 4.24, the minimal Sullivan model for  $BG$  has vanishing differential, while the minimal Sullivan model of  $B^{n+1}\mathbb{Z}$  is a tensor factor (by Example 3.67), whose inclusion is already the relative minimal Sullivan model  $\mathfrak{l}_{B^{n+1}\mathbb{Z}}(c)$  (Prop. 3.70) of  $c$ . Therefore, setting

$$\text{CE}(c) := \text{CE}(\mathfrak{l}_{B^{n+1}\mathbb{Z}}(c)) : \mathbb{R}[c]/(dc=0) \hookrightarrow \text{inv}^\bullet(\mathfrak{g}) \quad (259)$$

gives the required morphism of minimal models that makes the square (253) commute, by (142).  $\square$

In differential enhancement of Example 2.18 we have:

**Definition 4.45** (Secondary characteristic classes of differential non-abelian  $G$ -cohomology). Let  $G$  be a connected compact Lie group with classifying space  $BG$  (12). By Lemma 4.44, the construction of secondary characteristic classes (Def. 4.42, on differential non-abelian  $G$ -cohomology (Example 4.37) exists generally, and yields a  $\mathbb{Z}$ -linear map of the form

$$H(BG; B^\bullet\mathbb{Z}) \xrightarrow{(-)^{\text{diff}}} \widehat{H}(BG_{\text{diff}}; B^\bullet\mathbb{Z}) = H(BG_{\text{diff}}; B^\bullet\mathbb{Z}_{\text{diff}}),$$

where on the right we have the ordinary differential non-abelian cohomology (Prop. 4.40) of the moduli  $\infty$ -stack  $BG_{\text{diff}}$  (231). Combined with the composition operation in  $\text{Ho}(\text{SmoothStacks}_\infty)$  (A.45) this gives a map

$$\widehat{H}(X; BG) \times H(BG; B^\bullet\mathbb{Z}) \xrightarrow{\text{id} \times (-)^{\text{diff}}} H(X; BG_{\text{diff}}) \times H(BG_{\text{diff}}; B^\bullet\mathbb{Z}_{\text{diff}}) \xrightarrow{\circ} H(X; B^\bullet\mathbb{Z}_{\text{diff}}) = \widehat{H}(X; B^\bullet\mathbb{Z})$$

which is  $\mathbb{Z}$ -linear in its second argument, and whose hom-adjunct is

$$\widehat{H}(X; BG) \xrightarrow{\nabla \mapsto (c \mapsto c_{\text{diff}}(\nabla))} \text{Hom}_{\mathbb{Z}}(H(BG; B^\bullet\mathbb{Z}), \widehat{H}(X; B^\bullet\mathbb{Z})). \quad (260)$$

**Theorem 4.46** (Secondary non-abelian cohomology operations subsume Cheeger-Simons homomorphism). *Let  $G$  be a connected compact Lie group, with classifying space denoted  $BG$  (12). Then the canonical construction (260) of secondary characteristic classes on differential non-abelian  $G$ -cohomology (Def. 4.45) coincides with the Cheeger-Simons homomorphism (258), in that the following diagram commutes:*

$$\begin{array}{ccc} G\text{Connections}(X) & \xrightarrow[\text{CS}_G]{\text{Cheeger-Simons homomorphism}} & \text{Hom}_{\mathbb{Z}}(H^\bullet(BG; \mathbb{Z}), \widehat{H}^\bullet(X)) \\ \downarrow (241) & & \uparrow \simeq (250) \\ \widehat{H}(X; BG) & \xrightarrow[\text{secondary non-abelian cohomology operations}]{\nabla \mapsto (c \mapsto c_{\text{diff}}(\nabla))} & \text{Hom}_{\mathbb{Z}}(H(BG; B^\bullet\mathbb{Z}), \widehat{H}(X; B^\bullet\mathbb{Z})), \end{array} \quad (261)$$

differential non-abelian cohomology
differential ordinary cohomology

where on the left we have the map from  $G$ -connections to differential non-abelian  $G$ -cohomology from Prop. 4.37, and on the right the identification of ordinary differential cohomology from Prop. 4.40.

*Proof.* Let  $c \in H(BG; B^\bullet\mathbb{Z})$  be a characteristic class, and let  $(f^*EG, \nabla)$  be a  $G$ -principal bundle equipped with a  $G$ -connection. By Prop. 4.37, its image in differential non-abelian cohomology is given by the first map in the following diagram

$$\begin{array}{ccccc} G\text{Connections}(X)_{/\sim} & \longrightarrow & \widehat{H}(X; BG) & \xrightarrow{(c_{\text{diff}})^*} & \widehat{H}(X; B^{n+1}\mathbb{Z}) & \xrightarrow{\simeq} & \widehat{H}^{n+1}(X) \\ [f^*EG, \nabla] & \mapsto & [f, (cS_k(\nabla, f^*\nabla_{\text{univ}})) (\omega_k(F_\nabla))] & \mapsto & [f^*c, cS_c(\nabla, f^*\nabla_{\text{univ}}), c(F_\nabla)] & & \end{array} \quad (262)$$

Here the triple of data are the three components (Example A.27) of a map into the defining homotopy pullback of differential non-abelian cohomology (242). Therefore, the secondary operation induced by the transformation (256) of these homotopy pullbacks, which in the present case is of this form:

$$\begin{array}{ccc}
BG_{\text{diff}} & \xrightarrow[\text{secondary characteristic class}]{c_{\text{diff}}} & B^{n+1}\mathbb{Z}_{\text{diff}} \\
\downarrow c_{BG} & \searrow & \downarrow c_{B^{n+1}\mathbb{Z}} \\
& \Omega_{\text{dR}}(-; (BG)_{\text{flat}}) & \xrightarrow{\hspace{10em}} \Omega_{\text{dR}}(-; (B^{n+1}\mathbb{Z})_{\text{flat}}) \\
& \downarrow & \downarrow \\
BG & \xrightarrow[\text{characteristic class}]{c} & B^{n+1}\mathbb{Z} \\
\downarrow \text{ch}_{BG} & \searrow & \downarrow \text{ch}_{B^{n+1}\mathbb{Z}} \\
\flat \exp(LBG) & \xrightarrow[c_*]{} & \flat \exp(LB^{n+1}\mathbb{Z}),
\end{array}
\tag{263}$$

acts **(a)** on the first component in the triple by postcomposition with  $c$ , hence as

$$f \mapsto f^*c := c \circ f$$

and **(b)** on the other two components by composition with  $c$ , which by (259) corresponds to projecting out the Chern-Simons form and characteristic form corresponding to  $c$ , respectively. This is shown as the second map in (262). Hence we are reduced to showing that the total map in (262) gives the Cheeger-Simons homomorphism. This statement is the content of [HS05, §3.3].  $\square$

**Remark 4.47** (Secondary characteristic classes of  $G$ -connections). The traditional reason for referring to the Cheeger-Simons homomorphism (261) as producing *secondary* invariants is that Cheeger-Simons classes  $\text{cs}_G(P, \nabla) \in \widehat{H}(X)$  may be non-trivial even if the underlying characteristic class  $\text{cw}_G(P)$  (the “primary” class) vanishes. In this case the  $\text{cs}_G(P, \nabla)$  are also called *Chern-Simons invariants*.

**(i)** This happens, in particular, when the  $G$ -connection  $\nabla$  is flat,  $F(\nabla) = 0$  (by Def. 4.19). Such secondary Chern-Simons invariants exhibit some subtle phenomena ([Rzn95][Rzn96][IS07][Es09]).

**(ii)** In fact, the proof of Theorem 261, via the triples (242) of homotopy data, shows that, in this case,  $\text{cs}_G(P, \nabla)$  measures *how* (or “why”)  $\text{cw}_G(P)$  vanishes, namely by which class of homotopies.

**(iii)** Here we may understand secondary classes more abstractly, and explicitly related to the non-abelian character map: Where a (primary) non-abelian cohomology operation, according to Def. 2.17, is induced by a morphism of coefficient spaces (24), a secondary non-abelian cohomology operation, according to Def. 4.42, is induced (255) by a morphism of non-abelian character maps (254) – hence by a morphism of morphisms – on these coefficient spaces.

**(iv)** Note that classical secondary cohomology operations themselves admit differential refinements. For instance, for the case of Massey products as secondary operations for the cup product [GS17a]. While these can also fit into our context on general grounds, we will not demonstrate that explicitly here.

## 5 The twisted (differential) non-abelian character map

We introduce the character map in twisted non-abelian cohomology (Def. 5.4) and then discuss how it specializes to:

- §5.1 – the twisted Chern character on (higher) K-theory;
- §5.3 – the twisted character on Cohomotopy theory.

**Rationalization in twisted non-abelian cohomology.** In generalization of Def. 4.1 we now define rationalization of local coefficient bundles (31). This operation is transparent in the language of  $\infty$ -category theory, where it simply amounts to forming the pasting composite with the homotopy-coherent naturality square of the rationalization unit  $\eta^{\mathbb{R}}$ :

$$\begin{array}{ccc}
 \begin{array}{c} \tau\text{-twisted cocycle with} \\ \text{local coefficients } \rho \\ X \xrightarrow{c} A//G \\ \swarrow \tau \quad \searrow \rho \\ \simeq \\ BG \end{array} & \xrightarrow{\text{rationalization}} & \begin{array}{c} \tau\text{-twisted cocycle with rationalized local coefficients } L_{\mathbb{R}}\rho \\ X \xrightarrow{c} A//G \xrightarrow{\eta_{A//G}^{\mathbb{R}}} L_{\mathbb{R}}(A//G) \\ \swarrow \tau \quad \searrow \rho \\ \simeq \\ BG \xrightarrow{\eta_{BG}^{\mathbb{R}}} L_{\mathbb{R}}BG \\ \swarrow L_{\mathbb{R}}\rho \\ \simeq \\ L_{\mathbb{R}}BG \end{array}
 \end{array} \tag{264}$$

Slightly less directly but equivalently, this operation is the composite of (a) “base change” along  $\eta_{BG}^{\mathbb{R}}$  from the slice over  $BG$  to the slice over  $L_{\mathbb{R}}BG$ , (b) followed by the composition with the naturality square, now regarded as a morphism in the slice over  $L_{\mathbb{R}}BG$ :

$$\begin{array}{ccc}
 \begin{array}{c} \tau\text{-twisted cocycle with} \\ \text{local coefficients } \rho \\ X \xrightarrow{c} A//G \\ \swarrow \tau \quad \searrow \rho \\ \simeq \\ BG \end{array} & \xrightarrow{\text{base change}} & \begin{array}{c} X \xrightarrow{c} A//G \\ \swarrow \tau \quad \searrow \rho \\ \simeq \\ BG \\ \eta_{BG}^{\mathbb{R}} \downarrow \\ L_{\mathbb{R}}BG \end{array} & \xrightarrow{\text{composition in slice}} & \begin{array}{c} \tau\text{-twisted cocycle with} \\ \text{local coefficients } \rho \\ X \xrightarrow{c} A//G \xrightarrow{\eta_{A//G}^{\mathbb{R}}} L_{\mathbb{R}}(A//G) \\ \swarrow \tau \quad \searrow \rho \\ \simeq \\ BG \xrightarrow{\eta_{BG}^{\mathbb{R}}} L_{\mathbb{R}}BG \\ \swarrow L_{\mathbb{R}}\rho \\ \simeq \\ L_{\mathbb{R}}BG \end{array}
 \end{array}$$

It is in this second form that the operation lends itself to formulation in model category theory (Def. 5.2 below). For that we just need to produce a rectified (strictly commuting) incarnation of the  $\eta^{\mathbb{R}}$ -naturality square:

**Lemma 5.1** (Rectified rationalization unit on coefficient bundle). *Let*

$$\begin{array}{ccc}
 A & \longrightarrow & A//G \\
 \text{local coefficient bundle} & & \downarrow \rho \\
 & & BG
 \end{array} \tag{265}$$

be a local coefficient bundle (31) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52), and let

$$\begin{array}{ccc}
 \Omega_{\text{PLdR}}^{\bullet}(A//G) & \xleftarrow[p_{A//G}^{\min} \in \mathbb{W}]{} & \text{CE}(l_{BG}(A//G)) \\
 \uparrow \Omega_{\text{PLdR}}^{\bullet}(\rho) & & \uparrow \text{CE}(l_{\rho}) \\
 \Omega_{\text{PLdR}}^{\bullet}(BG) & \xleftarrow[p_{BG}^{\min} \in \mathbb{W}]{} & \text{CE}(l(BG))
 \end{array} \tag{266}$$

be its minimal relative Sullivan model (142), which exists by Prop. 3.70. Then the composite of the image of (266)

under  $\exp$  with the  $\Omega_{\text{PLdR}}^\bullet \dashv \exp$ -adjunction unit (from Prop. 3.59):

$$\mathbb{D}\eta_\rho^{\text{PLdR}} := \begin{array}{ccccc} & & \mathbb{D}\eta_{A/G}^{\text{PLdR}} \simeq \eta_{A/G}^{\mathbb{R}} & & \\ & \nearrow & & \searrow & \\ A//G & \xrightarrow{\eta_{A/G}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^\bullet(A//G) & \xrightarrow{\exp(\rho_{A//G}^{\text{min}})} & \exp \circ \text{CE}(l_{BG}(A//G)) \\ \rho \downarrow & & \exp \circ \Omega_{\text{PLdR}}^\bullet(\rho) \downarrow & & \exp \circ \text{CE}(l_\rho) \downarrow \\ BG & \xrightarrow{\eta_{BG}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^\bullet(BG) & \xrightarrow{\exp(\rho_{BG}^{\text{min}})} & \exp \circ \text{CE}(l(BG)) \\ & \searrow & & \nearrow & \\ & & \mathbb{D}\eta_{BG}^{\text{PLdR}} \simeq \eta_{BG}^{\mathbb{R}} & & \end{array} \quad (267)$$

is, after passage (318) to the classical homotopy category (Example A.34), equivalent to the naturality square of the rationalization unit on  $\rho$  (112):

$$\mathbb{D}\eta_\rho^{\text{PLdR}} \simeq \eta_\rho^{\mathbb{R}}.$$

*Proof.* By Prop. 3.43 the right part of (267) is the image under  $\exp$  of a fibrant replacement morphism. By (326) this identifies the diagram as the naturality square of the derived PLdR adjunction unit, and by (122) in the fundamental theorem (Prop. 3.60) this implies the claim.  $\square$

**Definition 5.2** (Rationalization in twisted non-abelian cohomology). Given a local coefficient bundle  $\rho$  and its rectified rationalization unit  $\mathbb{D}\eta_\rho^{\text{PLdR}}$  as in Lemma 5.1 we say that *rationalization* in twisted non-abelian cohomology with local coefficients  $\rho$  (Def. 2.29) is the twisted non-abelian cohomology operation (Def. 2.41)

$$(\eta_\rho^{\mathbb{R}})_* : H^\tau(X; A) \xrightarrow{(\mathbb{D}\eta_\rho^{\text{PLdR}} \circ (-)) \circ \mathbb{L}(\eta_{BG}^{\mathbb{R}})!} H^{L_{\mathbb{R}}}\tau(X; L_{\mathbb{R}}A) \quad (268)$$

given by the composite of

- (a) derived left base change  $\mathbb{L}(\eta_{BG}^{\mathbb{R}})!$  (Example A.19) along the rationalization unit (112) on the classifying space of twists,
- (b) composition with the rectified rationalization unit (267) on the coefficient bundle, regarded as a morphism in the homotopy category (318) of the slice model category (Example A.10) of  $\text{SimplicialSets}_{\text{Qu}}$  (Example A.8) over  $\exp \circ \text{CE}(l(BG))$ .

**Remark 5.3** (Commutativity of rationalization over twisting). The existence of the transformation (267) from a local coefficient bundle to its rationalization, inducing the cohomology operation (268) from any twisted cohomology theory to its twisted rational cohomology, may be thought of as exhibiting commutativity of rationalization over twisting. For twisted KO-theory this is discussed in [GS19d, Prop. 4].

**Twisted non-abelian character map.** In generalization of Def. 4.2 we set:

**Definition 5.4** (Twisted non-abelian character map). Let  $X \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\geq 1, \text{nil}}})^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52) equipped with the structure of a smooth manifold, and

$$\begin{array}{ccc} A & \longrightarrow & A//G \\ & \text{local coefficient bundle} \downarrow \rho & \\ & & BG \end{array} \quad (269)$$

be a local coefficient bundle (31) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}^{\geq 1, \text{nil}}})^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52). Then the *twisted non-abelian character map* in twisted non-abelian cohomology is the twisted cohomology operation

$$\begin{array}{ccccc} \text{twisted non-abelian character map} & \text{twisted non-abelian cohomology} & \text{twisted non-abelian real cohomology} & \text{twisted non-abelian de Rham cohomology} & \\ \text{ch}_\rho : & H^\tau(X; A) & \xrightarrow{(\eta_\rho^{\mathbb{R}})_*} & H^{L_{\mathbb{R}}}\tau(X; L_{\mathbb{R}}A) & \xrightarrow{\simeq} & H_{\text{dR}}^{\tau}(X; lA) \\ & \text{rationalization} & & \text{twisted non-abelian de Rham theorem} & \end{array} \quad (270)$$

from twisted non-abelian  $A$ -cohomology (Def. 2.29) to twisted non-abelian de Rham cohomology (Def. 3.98) with local coefficients in the rational relative Whitehead  $L_\infty$ -algebra  $l\rho$  of  $\rho$  (Prop. 3.75) which is the composite of

- (i) the operation (268) of rationalization of local coefficients (Def. 5.2),
- (ii) the equivalence (191) of the twisted non-abelian de Rham theorem (Theorem 3.104).

## 5.1 Twisted Chern character on higher K-theory

We discuss how the twisted non-abelian character map reproduces the twisted Chern character in twisted topological K-theory [BCMMS02, §6.3][MaSt03][AS06, §7] – see also [TX06][MaSt06, §6][FrHT08, §2][BGNT08][GT10, §4][Ka12, §8.3][GS19a, §3.2][GS19c] – (Prop. 5.5) and in twisted iterated K-theory [LSW16, §2.2] (Prop. 5.10).

**Character maps on higher-twisted ordinary K-theories.** In twisted enhancement of Example 4.10, we have:

**Proposition 5.5** (Twisted Chern character in twisted topological K-theory). *Consider twisted complex topological K-theory  $KU^\tau(-)$  (Example 2.36), for degree-3 twists given (via Example 2.11) by*

$$\tau \in H(-; B^2U(1)) \simeq H^3(-; \mathbb{Z}),$$

and regarded, via (48), as twisted non-abelian cohomology with local coefficients in  $\mathbb{Z} \times BU // B^2U(1)$  (47). Then the twisted non-abelian character map (Def. 5.4)  $\text{ch}_{\mathbb{Z} \times BU}^\tau$  is equivalent to the traditional twisted Chern character  $\text{ch}^\tau$  on twisted K-theory with values in  $H_3$ -twisted de Rham cohomology (Def. 3.99):

$$\begin{array}{ccc} \text{twisted non-abelian} & & \text{twisted} \\ \text{character map} & & \text{Chern character} \\ \text{ch}_{\mathbb{Z} \times BU}^\tau & \simeq & \text{ch}^\tau. \end{array}$$

*Proof.* That the codomain of the twisted non-abelian character map, in this case, is indeed  $H_3$ -twisted de Rham cohomology is the content of Prop. 3.100. With this, and due to the twisted non-abelian de Rham theorem (Theorem 3.104), it is sufficient to see that the general rationalization map of local non-abelian coefficients from Def. 5.2 reproduces the rationalization map underlying the twisted Chern character. This is manifest from comparing the rationalization operation (264), that is made formally precise by Def. 5.2, to the description of the twisted Chern character as given in [FrHT08, (2.8)-(2.9)].  $\square$

**Remark 5.6** (Twisted Pontrjagin character in twisted KO-theory). Similarly, an analogous statement holds for the twisted Pontrjagin character (as in Example 4.11) on twisted real K-theory [GS19d, Prop. 2].

There are also higher twists of ordinary K-theory, specifically twists of K-theory *by Cohomotopy* (Example 2.10):

**Example 5.7** (Chern character on higher cohomotopy-twisted K-theory). For any  $k \in \mathbb{N}$ , there is a non-trivial local coefficient bundle (31) for twists of topological K-theory (Example 2.14) over the  $(2k+1)$ -sphere [MMS20, §2.1]:

$$\begin{array}{ccc} KU_0 & \longrightarrow & Y_{2k+1} \\ & & \downarrow \rho_{Y_{2k+1}} \\ & & S^{2k+1}. \end{array} \quad (271)$$

(i) By Def. 2.29, this defines a twisted version of topological K-theory (Example 2.14) with the twist  $\tau = \lambda$  in Cohomotopy theory (Example 2.10):

$$\begin{array}{ccc} \text{cohomotopically-twisted topological K-theory} & & \text{twist in Cohomotopy} \\ K_\lambda^0(X) := H^\lambda(X; KU_0), & [\lambda] \in \pi^{2k+1}(X). \end{array} \quad (272)$$

Direct comparison shows that this recovers the higher twisted K-theory of [MMS20, Def. 2.5].

(ii) Moreover, by Def. 5.4, we obtain the character map on cohomotopically twisted topological K-theory (272), and by Theorems 3.87, 3.98 and Examples 3.67, 3.68 we see that it lands in  $\lambda_{\text{dR}} = H_{2k+1}$ -twisted de Rham cohomology (Example 3.95, Prop. 3.102):

$$K_\lambda^0(X) := H^\lambda(X; KU_0) \xrightarrow{\text{ch}_{KU_0}^\lambda} H_{\text{dR}}^{\lambda_{\text{dR}}}(X; iKU_0) \simeq H^{\bullet+H_{2k+1}}(X). \quad (273)$$

(iii) A map of this form has been defined in [MMS20, §2.2] by direct construction on form representatives. However, [MMS20, Thm. 4.19] implies that this component construction coincides with the rationalization map (Def. 5.2) on the local coefficient bundle (271), up to application of the de Rham theorem. Therefore, the twisted character map (273) obtained as a special case of Def. 5.4, reproduces the MMS-Character on higher (cohomotopically) twisted K-theory, from [MMS20, §2.2].

**Remark 5.8** (Charge quantization of spherical T-duality in M-theory). For  $k = 3$ , the character map (273) on 7-Cohomotopy-twisted K-theory is a candidate for charge quantization (2) of the super-rational M-theory fields participating in 3-spherical T-duality over 11-dimensional super-spacetime, as derived in [FSS18, Prop. 4.17, Rem. 4.18] (review in [SS18, (8), (19)]). However, the 7-Cohomotopy-twisted K-theory character has some spurious fields of 2-periodic degree in its image, which are not seen in the physics application, where the field degrees are 6-periodic [Sa09, §3][FSS18, (65)]. Another candidate for charge-quantization of the super-rational M-theory fields participating in 3-spherical T-duality, possibly more accurately reflecting the physics, is the character map on twisted higher K-theory [LSW16], which we turn to next (Lemma 5.10).

### Character map on twisted higher K-theory.

**Lemma 5.9** (Higher twisted de Rham coefficients inside rational twisted iterated K-theory). *There is a non-trivial twisted cohomology operation (Def. 2.41) from (a) twisted non-abelian de Rham cohomology (Def. 3.98) with coefficients in the relative rational Whitehead  $L_\infty$ -algebra (Prop. 3.70) of the coefficient bundle (53) of twisted iterated K-theory (Example 2.39) to (b) higher twisted de Rham cohomology (Def. 3.101) regarded as twisted non-abelian de Rham cohomology via Prop. 3.102):*

$$H_{\text{dR}}^{\tau} \left( -; \mathfrak{K}^{\circ 2r-2}(\mathbf{ku})_1 \right) \xrightarrow{\phi_*} H_{\text{dR}}^{\tau} \left( -; \bigoplus_{k \in \mathbb{N}} \mathfrak{b}^{2rk} \mathbb{R} \right), \quad (274)$$

given, under the twisted non-abelian de Rham theorem (Theorem 3.104) by the LSW-character from [LSW16, §2.2] applied to rational coefficients.

**Proposition 5.10** (Twisted Chern character in twisted iterated K-theory). *For  $r \in \mathbb{N}$ ,  $r \geq 1$ , consider twisted iterated K-theory  $(\mathfrak{K}^{\circ 2r-2}(\mathbf{ku}))^\tau$  (Example 2.39), for degree- $(2r+1)$  twists given (via Example 2.12) by*

$$\tau \in H(-; \mathfrak{B}^{2r} \mathbf{U}(1)) \simeq H^{2r+1}(-; \mathbb{Z}),$$

and regarded, via Example 2.39, as twisted non-abelian cohomology with local coefficients in  $(\mathfrak{K}^{\circ 2r-2}(\mathbf{ku}))_0$ . Then the twisted non-abelian character map (Def. 5.4)  $\text{ch}_{\mathfrak{K}^{\circ 2r-2}(\mathbf{ku})_0}^\tau$  composed with the projection operation (274) onto higher twisted de Rham cohomology, (Def. 3.101) from Lemma 5.9, is equivalent to the LSW character map  $\text{ch}_{2r-1}$  [LSW16, Def. 2.20] restricted along the connective inclusion

$$\begin{array}{ccc} \text{twisted LSW character} & & \text{twisted non-abelian character map} \\ \text{ch}_{2r-1}^\tau \simeq & \phi_* \circ & \text{ch}_{\mathfrak{K}^{\circ 2r-2}(\mathbf{ku})_0}^\tau \\ & \text{projection onto higher twisted de Rham cohomology} & \end{array}$$

*Proof.* After unwinding the definitions, the statement reduces to the commutativity of the square diagram in [LSW16, p. 15]: The top morphism there is the plain rationalization map (Def. 5.2), the right vertical morphism is  $\phi_*$  from Lemma 5.9 before passing from real to de Rham cohomology, the left morphism is restriction to the connective part and the bottom morphism is the LSW character.  $\square$

## 5.2 Twisted differential non-abelian character

We introduce twisted differential non-abelian cohomology (Def. 5.13 below) and discuss how the corresponding twisted differential non-abelian character subsumes existing constructions on twisted differential K-theory (Examples 5.19 Example 5.22 below).

**Twisted differential non-abelian cohomology.** From the perspective of structured non-abelian cohomology (Remark 2.27) that we have developed, it is now evident how to canonically combine

(a) twisted non-abelian cohomology (Def. 2.29) with

(b) differential non-abelian cohomology (Def. 4.33) to get

*twisted differential non-abelian cohomology:*

**Definition 5.11** (Differential non-abelian local coefficient bundles). Let

$$\begin{array}{ccc} A & \longrightarrow & A // G \\ & \text{local coefficient bundle} & \downarrow \rho \\ & & BG \end{array}$$

be a local coefficient bundle (31) in  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})_{\geq 1, \text{nil}}^{\text{fin}_{\mathbb{R}}}$  (Def. 3.52).

(i) By Lemma 3.71, Lemma 5.1, and using that  $\text{exp}$  preserves fibrations (Prop. 3.62), this induces a homotopy fibration (Def. A.23) in  $\text{Ho}(\text{SmoothStacks}_{\infty})$  (Def. A.45) of differential non-abelian character maps (Def. 4.32) of this form:

$$\begin{array}{ccccc} \text{Disc}(A) & \xrightarrow{\text{ch}_A} & \mathfrak{b}\text{exp}(lA) & \xleftarrow{\text{atlas}} & \Omega_{\text{dR}}(-; lA)_{\text{flat}} \\ & \searrow \text{hofib}(\text{Disc}(\rho)) & \downarrow \text{hofib}((l\rho)_*) & & \downarrow \text{hofib}((l\rho)_*) \\ & & \text{Disc}(A // G) & \xrightarrow{\text{ch}_{A // G}^{BG}} & \mathfrak{b}\text{exp}(l(A // G)) & \xleftarrow{\text{atlas}} & \Omega_{\text{dR}}(-; l_{BG}(A // G))_{\text{flat}} \\ & & \downarrow \text{Disc}(\rho) & & \downarrow (l\rho)_* & & \downarrow (l\rho)_* \\ & & \text{Disc}(BG) & \xrightarrow{\text{ch}_{BG}} & \mathfrak{b}\text{exp}(lBG) & \xleftarrow{\text{atlas}} & \Omega_{\text{dR}}(-; lBG)_{\text{flat}} \end{array} \quad (275)$$

differential non-abelian character map with coefficients in fiber space  
twisted differential non-abelian character map  
differential non-abelian character map with coefficients in space of twists

(ii) Here the *twisted differential non-abelian character map*  $\text{ch}_{A // G}^{BG}$  is defined just as in Def. 4.32, but with coefficients the relative Whitehead  $L_{\infty}$ -algebra  $l_{BG}(A // G)$  (Prop. 3.70), as opposed to the absolute Whitehead  $L_{\infty}$ -algebra  $l(A // G)$  (Prop. 3.63).

**Remark 5.12** (Differential local coefficient bundles). Since homotopy limits commute over each other, passage to the homotopy fiber products (Def. A.24) formed from the horizontal stages of (275) yields a homotopy fibration of moduli  $\infty$ -stacks of  $\infty$ -connections (231) of this form:

$$\begin{array}{ccccc} & & & \xrightarrow{F_{A // G}^{BG}} & \Omega_{\text{dR}}(-; l_{BG}(A // BG))_{\text{flat}} \\ & & & \downarrow (l\rho)_* & \downarrow \text{atlas} \\ A_{\text{diff}} & \xrightarrow{\text{hofib}(\rho_{\text{diff}})} & (A // G)_{\text{diff}_{BG}} & \xrightarrow{\text{ch}_{A // G}^{BG}} & \mathfrak{b}\text{exp}(l_{BG}(A // G)) \\ & \downarrow \rho_{\text{diff}} & \downarrow \text{Disc}(\rho) & \downarrow \text{atlas} & \downarrow (l\rho)_* \\ & & \text{Disc}(A // G) & \xrightarrow{\text{ch}_{A // G}^{BG}} & \mathfrak{b}\text{exp}(l_{BG}(A // G)) \\ & & \downarrow \text{Disc}(\rho) & \downarrow \text{atlas} & \downarrow (l\rho)_* \\ & & \text{Disc}(BG) & \xrightarrow{\text{ch}_{BG}} & \mathfrak{b}\text{exp}(lBG) \end{array} \quad (276)$$

moduli  $\infty$ -stack of  $\Omega A$ -connections  
differential non-abelian local coefficient bundle  
moduli  $\infty$ -stack of  $G$ -connections

**Definition 5.13** (Twisted differential non-abelian cohomology). Given a differential non-abelian local coefficient bundle  $\rho_{\text{diff}}$  (276) according to Def. 5.11, we say that:

(i) A *differential twist* on a  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (Def. A.45) is a cocycle  $\tau_{\text{diff}}$  in differential non-abelian cohomology with coefficients in  $BG$  (Def. 4.33)

$$[\tau_{\text{diff}}] \in \widehat{H}(\mathcal{X}; BG). \quad (277)$$

(ii) The  $\tau_{\text{diff}}$ -*twisted differential non-abelian cohomology* with local coefficients in  $\rho_{\text{diff}}$  is the structured (Remark 2.27)  $\tau_{\text{diff}}$  twisted non-abelian cohomology (Def. 2.29) with coefficients in  $\rho_{\text{diff}}$ , hence the hom-set in the homotopy category (Def. A.15) of the slice model structure (Def. A.10) of the local projective model structure  $\text{SmoothStacks}_{\infty}$  on simplicial presheaves over  $\text{CartesianSpaces}$  (Example A.44) from  $\tau_{\text{diff}}$  (277) to  $\rho_{\text{diff}}$  (276):

$$\widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}; A) := \text{Ho}(\text{SmoothStacks}_{\infty}^{/BG_{\text{diff}}})(\tau_{\text{diff}}, \rho_{\text{diff}}) = \left\{ \begin{array}{ccc} \mathcal{X} & \overset{\text{differential cocycle}}{\dashrightarrow^{c_{\text{diff}}}} & (A//G)_{\text{diff}_{BG}} \\ & \swarrow \simeq & \searrow \\ & BG_{\text{diff}} & \end{array} \right\} \Big/ \text{homotopy relative } BG_{\text{diff}} \quad (278)$$

$\tau_{\text{diff}}$   
differential twist
 $\rho_{\text{diff}}$   
differential local coefficients

(iii) The twisted non-abelian cohomology operations induced from the maps in (276) we call (see (4)):

$$(a) \text{ characteristic class: } \widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}; A) \xrightarrow{c_A^{\tau} := (c_{A//G}^{BG})_*} H^{\tau}(\text{Shp}(\mathcal{X}); A) \quad (\text{Def. 2.29}) \quad (279)$$

$$(b) \text{ curvature: } \widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}; A) \xrightarrow{F_A^{\tau_{\text{dR}}} := (F_{A//G}^{BG})_*} \Omega_{\text{dR}}^{\tau_{\text{dR}}}(\mathcal{X}; \mathbb{A})_{\text{flat}} \quad (\text{Def. 3.92}) \quad (280)$$

$$(c) \text{ differential character: } \widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}; A) \xrightarrow{\text{ch}_A^{\tau} := (\text{ch}_{A//G}^{BG} \circ c_{A//G}^{BG})_*} H_{\text{dR}}^{\tau_{\text{dR}}}(\mathcal{X}; \mathbb{A}) \quad (\text{Def. 3.98}) \quad (281)$$

**Twisted differential non-abelian cohomology as non-abelian  $\infty$ -sheaf hypercohomology.** While the formulation of twisted differential non-abelian cohomology as hom-sets in a slice of  $\text{SmoothStacks}_{\infty}$  (Def. 5.13) is natural and useful, we indicate how this is equivalently incarnated as a non-abelian sheaf hypercohomology over  $\mathcal{X}$ . This serves to make the connection to existing literature (in Example 5.18 below), but is not otherwise needed for the development here. We shall be brief, referring to [SS20b] for some technical background that is beyond the scope of our presentation here.

**Proposition 5.14** (Étale  $\infty$ -topos over  $\infty$ -stack [SS20b, Prop. 3.33, Rem. 3.34]). *For  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (Def. A.45) let*

$$\text{Ho}(\mathbf{Ét}_{\mathcal{X}}) \xhookrightarrow{\text{Li}_{\mathcal{X}}} \text{Ho}(\text{SmoothStacks}_{\infty}^{/\mathcal{X}})$$

*be the full subcategory of the homotopy category (Def. A.15) of the slice model structure over  $\mathcal{X}$  (Example A.10) of the local projective model structure on simplicial presheaves (Example A.44) on those  $\mathcal{E} \rightarrow \mathcal{X}$  which are local diffeomorphisms ([SS20b, Def. 3.26]).*

(i) *The inclusion  $\text{Li}_{\mathcal{X}}$  is a left-exact homotopy co-reflection, in that it preserves finite homotopy limits and has a derived right adjoint  $\mathbb{R}\text{LclCnstnt}$  (sending  $\infty$ -bundles to their  $\infty$ -sheaves of  $\infty$ -sections).*

(ii) *There is a global section functor  $\mathbb{R}\Gamma_{\mathcal{X}}$  from  $\text{Ho}(\mathbf{Ét}_{\mathcal{X}})$  to  $\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Example A.34) which also admits a left exact left adjoint:*

$$\begin{array}{ccc} \text{Ho}(\text{SmoothStacks}_{\infty}^{/\mathcal{X}}) & \xleftarrow{\text{Li}_{\mathcal{X}}} & \text{Ho}(\mathbf{Ét}_{\mathcal{X}}) & \xleftarrow{\Delta_{\mathcal{X}}} & \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \\ \text{\scriptsize } \infty\text{-bundles over } \mathcal{X} & & \text{\scriptsize } \infty\text{-sheaves over } \mathcal{X} & & \\ \text{\scriptsize } \perp & & \text{\scriptsize } \perp & & \\ \text{\scriptsize } \mathbb{R}\text{LclCnstnt} & & \text{\scriptsize } \mathbb{R}\Gamma_{\mathcal{X}} & & \\ \text{\scriptsize } \infty\text{-sheaf of local sections} & & \text{\scriptsize } \text{global sections} & & \end{array} \quad (282)$$

**Definition 5.15** (Non-abelian  $\infty$ -sheaf hypercohomology over  $\infty$ -stacks). Given  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (Def. A.45) and  $\mathcal{A} \in \text{Ho}(\mathbf{Ét}_{\mathcal{X}})$  (Prop. 5.14) we say that the set of connected components of the *derived global sections* (282) of  $\mathcal{A}$  over  $\mathcal{X}$

$$H(\mathcal{X}, \mathcal{A}) := \pi_0(\mathbb{R}\Gamma_{\mathcal{X}}(\mathcal{A}))$$

is the *non-abelian  $\infty$ -sheaf hypercohomology* of  $\mathcal{X}$  with coefficients in  $\mathcal{A}$ .

**Lemma 5.16** (Twisted differential non-abelian cohomology as non-abelian  $\infty$ -sheaf hyper-cohomology). *Given a differential twist  $\tau_{\text{diff}}$  (277) on some  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (357) consider the object*

$$\underline{A}_{\tau_{\text{diff}}} := \mathbb{R}\text{LclCnstnt}_{\mathcal{X}}(\mathbb{R}\tau_{\text{diff}}^*(A//G)_{\text{diff}}) \in \text{Ho}(\mathbf{Ét}_{\mathcal{X}}) \quad (283)$$

*in the étale  $\infty$ -topos over  $\mathcal{X}$  Prop. 5.14. The non-abelian  $\infty$ -sheaf hypercohomology (Def. 5.15) of  $\underline{A}_{\tau_{\text{diff}}}$  over  $\mathcal{X}$  coincides with the  $\tau_{\text{diff}}$ -twisted differential non-abelian cohomology of  $\mathcal{X}$  (Def. 5.13):*

$$\pi_0 \mathbb{R}\Gamma_{\mathcal{X}}(\underline{A}_{\tau_{\text{diff}}}) \simeq \widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}, A). \quad (284)$$

non-abelian  $\infty$ -sheaf hypercohomology
twisted differential non-abelian cohomology



*Proof.* As in [SS20b, Remark 3.34]. □

It is useful to decompose this construction of twisted differential cohomology via  $\infty$ -sheaf hypercohomology again as a homotopy pullback of corresponding  $\infty$ -sheaves representing plain twisted cohomology and plain twisted differential forms:

**Remark 5.17** (Homotopy pullback of  $\infty$ -sheaves representing twisted differential cohomology). Given a differential twist  $\tau_{\text{diff}}$  (277) on some  $\mathcal{X} \in \text{Ho}(\text{SmoothStacks}_{\infty})$  (357) with components  $(\tau, \tau_{\text{dR}}, L_{\mathbb{R}}\tau)$  (Example A.27),

(i) Consider the pullback stacks over  $\mathcal{X}$  in the following diagram

$$\begin{array}{ccc}
 \mathbb{R}\tau^* \text{Disc}(A//G) & \xrightarrow{\quad} & \text{Disc}(A//G) \\
 \downarrow & \searrow & \downarrow \rho \\
 \mathbb{R}(L_{\mathbb{R}}\tau)^* \flat \exp(I_{BG}(A//G)) & \xrightarrow{\quad} & \flat \exp(I_{BG}(A//G)) \\
 \downarrow & \searrow & \downarrow \\
 \mathcal{X} & \xrightarrow{\quad \tau \quad} & \text{Disc}(BG) \\
 \downarrow & \searrow & \downarrow \\
 \mathbb{R}\tau_{\text{dR}}^* \Omega_{\text{dR}}(-; I_{BG}(A//G))_{\text{flat}} & \xrightarrow{\quad} & \Omega_{\text{dR}}(-; I_{BG}(A//G))_{\text{flat}} \\
 \downarrow & \searrow & \downarrow \\
 \mathcal{X} & \xrightarrow{\quad L_{\mathbb{R}}\tau \quad} & \flat \exp(IBG) \\
 \downarrow & \searrow & \downarrow \\
 \mathcal{X} & \xrightarrow{\quad \tau_{\text{dR}} \quad} & \Omega_{\text{dR}}(-; IBG)
 \end{array}$$

Here the right hand side is (275) and all front-facing squares are homotopy pullbacks (Def. A.24).

(ii) By commutativity of homotopy limits over each other, these form a homotopy pullback square as on the right of the following diagram, which gives, under the derived right adjoint  $\mathbb{R}\text{LcllCnstnt}$  (282) a homotopy pullback diagram of  $\infty$ -sheaves of sections as shown on the left:

$$\begin{array}{ccc}
 \underline{A}_{\tau_{\text{diff}}} & \xrightarrow{\quad} & \underline{\Omega}(-; IA)_{\text{flat}\tau_{\text{dR}}} \\
 \downarrow & \text{(hpb)} & \downarrow \\
 \underline{A}_{\tau} & \xrightarrow{\quad} & \flat \exp(IA)_{L_{\mathbb{R}}\tau} \\
 & & \text{:= } \mathbb{R}\text{LcllCnstnt} \left( \begin{array}{ccc} \mathbb{R}\tau_{\text{diff}}^*(A//G)_{\text{diff}} & \xrightarrow{\quad} & \mathbb{R}\tau_{\text{dR}}^* \Omega_{\text{dR}}(-; I_{BG}(A//G)) \\ \downarrow & \text{(hpb)} & \downarrow \\ \mathbb{R}\tau^*(A//G) & \xrightarrow{\quad} & \mathbb{R}(L_{\mathbb{R}}\tau)^* \flat \exp(I(A//G)) \end{array} \right) \in \text{Ho}(\mathbf{\acute{E}t}_{\mathcal{X}}). \quad (285)
 \end{array}$$

Here the top left item  $\underline{A}_{\tau_{\text{diff}}}$  from (283) is the  $\infty$ -sheaf whose global sections give the  $\tau_{\text{diff}}$ -twisted differential cohomology, by Lemma 5.16.

In differential enhancement of Prop. 2.38 and in twisted enhancement of Example 4.34, we have:

**Example 5.18** (Twisted differential generalized cohomology). Let  $\mathcal{X} = X$  be a smooth manifold (Example A.46) and  $R$  be a suitable ring spectrum, and let

$$\begin{array}{ccc}
 E_0 & \xrightarrow{\quad} & (R_0) // \text{GL}_R(1) \\
 \mathbb{R}\tau^* \rho_R \downarrow & \text{(hpb)} & \downarrow \rho_R \\
 X & \xrightarrow{\quad \tau \quad} & B\text{GL}_R(1)
 \end{array}$$

be a twist for twisted generalized  $R$ -cohomology over  $X$  (51), as in Lemma 2.38.

(i) Then the corresponding homotopy pullback diagram (285), which exhibits, by Lemma 5.16, twisted differential non-abelian cohomology (Def. 5.13) with coefficients in  $E_0$  as  $\infty$ -sheaf hypercohomology (Def. 5.15), is the image under  $\mathbb{R}\Omega_X^{\infty}$  of the homotopy pullback diagram of sheaves of spectra considered in [BN14, Def. 4.11], shown on the right below, for canonical/minimal differential refinement as in Example 4.34:

$$\begin{array}{ccc}
\underline{R_0}_{\tau_{\text{diff}}} & \longrightarrow & \underline{\Omega(-; \mathbb{I}R_0)}_{\text{flat } \tau_{\text{dR}}} \\
\downarrow & \text{(hpb)} & \downarrow \\
\underline{R_0}_{\tau} & \longrightarrow & \underline{\flat \exp(\mathbb{I}R_0)}_{L_{\mathbb{R}} \tau}
\end{array}
\cong \mathbb{R}\Omega_X^\infty \left( \begin{array}{ccc}
\text{Diff}(E) & \longrightarrow & H\mathcal{M}_{\leq 0} \\
\downarrow & \text{(hpb)} & \downarrow \\
\text{Disc}(E) & \longrightarrow & H\mathcal{M}
\end{array} \right)$$

This is the twisted/parametrized analog of the relation (239).

(ii) Accordingly, the twisted differential generalized  $R$ -cohomology according to [BN14, Def. 4.13] is subsumed by twisted differential non-abelian cohomology, via Lemma 5.16.

In differential enhancement of Prop. 5.5 and in twisted generalization of Example 4.36 we have:

**Example 5.19** (Twisted Chern character in twisted differential K-theory). Consider again the local coefficient bundle

$$\begin{array}{ccc}
\text{KU}_0 & \longrightarrow & \text{KU}_0 // \text{BU}(1) \\
& & \downarrow \rho \\
& & \text{B}^2\text{U}(1)
\end{array}$$

for complex topological K-theory (Example 2.36). By Example 5.18, the twisted differential non-abelian cohomology theory (Def. 5.13) induced from these local coefficients is twisted differential K-theory, as discussed in [CMW09] for torsion twists (review in [BS12, §7]). By the diagram (4) of cohomology operations on twisted differential cohomology, one may regard the corresponding twisted curvature map (280)

$$\widehat{K}^{\tau_{\text{diff}}}(\mathcal{X}) \xrightarrow{(F_{\text{KU}_0}^{\tau_{\text{dR}}})_*} \Omega_{\text{dR}}^{\tau_{\text{dR}}}(\mathcal{X}; \mathbb{I}\text{KU}_0)_{\text{flat}}$$

(with values in flat  $\tau_{\text{dR}} \simeq H_3$ -twisted differential forms, by Example 3.94) as an incarnation of the Chern character map on twisted differential K-theory. This is the perspective taken in [CMW09, p. 2][Pa18] for torsion twists, and in [BN14, p. 6] for general twists.

However, in the spirit of the Cheeger-Simons homomorphism (4.3), any lift of a cohomology operation (here: rationalization) to differential cohomology should be enhanced all the way to a secondary cohomology operation (Def. 4.42, now to be generalized to a *twisted* secondary cohomology operation, Def. 5.21 below) whose codomain is itself a (twisted) differential cohomology theory. The twisted Chern character enhanced to a secondary cohomology operation this way is Example 5.22 below, following the perspective taken in [GS19a, §3.2][GS19c, §2.3].

**Secondary twisted non-abelian cohomology operations.** We introduce the twisted generalization of secondary non-abelian cohomology operations (Def. 5.21 below). This requires the following twisted analog of the technical condition in Def. 4.41:

**Definition 5.20** (Twisted absolute minimal model). For

$$\begin{array}{ccc}
A_1 // G_1 & \xrightarrow{c_t} & A_2 // G_2 \\
\rho_1 \downarrow & & \rho_2 \downarrow \\
BG_1 & \xrightarrow{c_b} & BG_2
\end{array} \in \text{SimplicialSets}$$

a transformation (55) between local coefficient bundles (31), and for  $c_b$  an absolute minimal model (Def. 4.41) of the map  $c_b$  between spaces of twists, hence with induced transformation (254)

$$\begin{array}{ccc}
\text{Disc}(BG_1) & \xrightarrow{\text{Disc}(c_b)} & \text{Disc}(BG_2) \\
\searrow \text{ch}_{BG_1} & & \searrow \text{ch}_{BG_2} \\
\flat \exp(\mathbb{I}BG_1) & \xrightarrow{(c_b)_*} & \flat \exp(\mathbb{I}BG_2)
\end{array}$$

between the differential character maps (Def. 4.32) on the spaces of twists, we say that a corresponding *twisted absolute minimal model* is a lift of  $c_b$  to a morphism

$$\mathfrak{L}_{BG_1}(A_1 // G_1) \dashrightarrow^{c_t} \mathfrak{L}_{BG_1}(A_1 // G_1) \quad (286)$$

between the relative rational Whitehead  $L_\infty$ -algebras of the local coefficient bundles (Prop. 3.70) which (i) yields a transformation

$$\begin{array}{ccc} \text{Disc}(A_1 // G_1) & \xrightarrow{\text{Disc}(c_t)} & \text{Disc}(A_2 // G_2) \\ \searrow \text{ch}_{A_1 // G_1}^{BG_1} & & \searrow \text{ch}_{A_2 // G_2}^{BG_2} \\ \mathfrak{bexp}(\mathfrak{L}_{BG_1}(A_1 // G_1)) & \dashrightarrow_{(c_t)_*} & \mathfrak{bexp}(\mathfrak{L}_{BG_2}(A_2 // G_2)) \end{array}$$

of the twisted differential characters (275) (thus being an “absolute minimal model for  $c_t$  relative to  $c_b$ ”), (ii) compatible with the transformation of the differential characters on the twisting space, in that the following cube commutes:

$$\begin{array}{ccccc} \text{Disc}(A_1 // G_1) & \xrightarrow{\text{Disc}(c_t)} & \text{Disc}(A_2 // G_2) & & \\ \downarrow \rho_1 & \searrow \text{ch}_{A_1 // G_1}^{BG_1} & \downarrow \rho_2 & \searrow \text{ch}_{A_2 // G_2}^{BG_2} & \\ \mathfrak{bexp}(\mathfrak{L}_{BG_1}(A_1 // G_1)) & \xrightarrow{(c_t)_*} & \mathfrak{bexp}(\mathfrak{L}_{BG_2}(A_2 // G_2)) & & \\ \downarrow \text{atlas} & \swarrow \text{Disc}(c_b) & \downarrow \text{atlas} & \swarrow \text{atlas} & \\ \text{Disc}(BG_1) & \xrightarrow{\text{Disc}(c_b)} & \text{Disc}(BG_2) & & \\ \downarrow \text{ch}_{BG_1} & \swarrow \text{atlas} & \downarrow \text{ch}_{BG_2} & \swarrow \text{atlas} & \\ \mathfrak{bexp}(\mathfrak{L}BG_1) & \xrightarrow{(c_b)_*} & \mathfrak{bexp}(\mathfrak{L}BG_2) & & \\ \downarrow \text{atlas} & \swarrow \text{atlas} & \downarrow \text{atlas} & \swarrow \text{atlas} & \\ \Omega(-; \mathfrak{L}BG_1)_{\text{flat}} & \xrightarrow{(c_b)_*} & \Omega(-; \mathfrak{L}BG_2)_{\text{flat}} & & \end{array} \quad (287)$$

At the level of dgc-algebras, the condition that  $c_t$  (286) is a twisted absolute minimal model for the transformation of local coefficient bundles means equivalently that it makes the following cube commute:

$$\begin{array}{ccccc} \Omega_{\text{PLdR}}^\bullet(A_1 // G_1) & \xleftarrow{\Omega_{\text{PLdR}}^\bullet(c_t)} & \Omega_{\text{PLdR}}^\bullet(A_2 // G_2) & & \\ \uparrow \Omega_{\text{PLdR}}^\bullet(\rho_1) & \swarrow \text{min}_{P_{A_1 // G_1}}^{BG_1} & \uparrow \Omega_{\text{PLdR}}^\bullet(\rho_2) & \swarrow \text{min}_{P_{A_1 // G_1}}^{BG_1} & \\ \Omega_{\text{PLdR}}^\bullet(BG_1) & \xleftarrow{\Omega_{\text{PLdR}}^\bullet(c_t)} & \Omega_{\text{PLdR}}^\bullet(BG_2) & & \\ \downarrow \text{CE}(\iota_{\rho_1}) & \swarrow \text{CE}(c_t) & \downarrow \text{CE}(\iota_{\rho_2}) & \swarrow \text{CE}(c_t) & \\ \text{CE}(\mathfrak{L}BG_1) & \xleftarrow{c_b} & \text{CE}(\mathfrak{L}BG_2) & & \end{array} \quad (288)$$

In differential enhancement of Def. 2.41 and in twisted generalization of Def. 4.42, we set:

**Definition 5.21** (Twisted secondary non-abelian cohomology operations). Let

$$\begin{array}{ccc} A_1 // G_1 & \xrightarrow{c_t} & A_2 // G_2 \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ BG_1 & \xrightarrow{c_b} & BG_2 \end{array} \in \text{SimplicialSets}$$

be a transformation (55) between local coefficient bundles (31), together with an absolute minimal model  $c_b$  (Def. 4.41) for the base map, and a compatible twisted absolute minimal model  $c_t$  (Def. 5.20) for the total map. Then forming stage-wise homotopy pullbacks (Def. A.24) in the required commuting cube (287) yields a transformation of corresponding differential coefficient bundles (276):

$$\begin{array}{ccc} (A_1 // G_1)_{\text{diff}} & \xrightarrow{(c_t)_{\text{diff}}} & (A_2 // G_2)_{\text{diff}} \\ (\rho_1)_{\text{diff}} \downarrow & & \downarrow (\rho_2)_{\text{diff}} \\ (BG_1)_{\text{diff}} & \xrightarrow{(c_b)_{\text{diff}}} & (BG_2)_{\text{diff}} \end{array} \in \text{PSh}(\text{CartesianSpaces}, \text{SimplicialSets}). \quad (289)$$

This yields, in turn, a natural transformation of twisted differential non-abelian cohomology sets (Def. 5.13), hence a *twisted secondary non-abelian cohomology operation*, by pasting composition, hence by right derived base change (Example A.19) along  $(\rho_1)_{\text{diff}}$  followed by composition with  $(c_t)_{\text{diff}}$  regarded as a morphism in the slice (Example A.10) over  $(BG_1)_{\text{diff}}$ :

$$\widehat{H}^{\tau_{\text{diff}}}(\mathcal{X}; A_1) \xrightarrow{((c_t)_{\text{diff}} \circ (-)) \circ ((\rho_1)_{\text{diff}})_*} \widehat{H}^{(c_b)_{\text{diff}} \circ \tau_{\text{diff}}}(\mathcal{X}; A_2).$$

In differential enhancement of Prop. 5.5, we have:

**Example 5.22** (Twisted differential character on twisted differential K-theory). Consider the rationalization (Def. 3.55) over the actual *rational numbers* (see Remark 3.51) of the local coefficient bundle (47) for degree-3 twisted complex topological K-theory (Example 2.36).

(i) This is captured by the diagram

$$\begin{array}{ccc} \text{KU}_0 // \text{BU}(1) & \xrightarrow{\eta_{\text{KU}_0 // \text{BU}(1)}^{\mathbb{Q}}} & L_{\mathbb{Q}}(\text{KU}_0 // \text{BU}(1)) \\ \rho \downarrow & & \downarrow L_{\mathbb{R}} \rho \\ \text{B}^2\text{U}(1) & \xrightarrow{\eta_{\text{B}^2\text{U}(1)}^{\mathbb{Q}}} & L_{\mathbb{Q}}(\text{B}^2\text{U}(1)) \end{array} \quad (290)$$

regarded as a transformation of local coefficient bundles from twisted K-theory to twisted even-periodic rational cohomology:

$$L_{\mathbb{Q}}\text{KU}_0 \simeq \Omega^{\infty} \left( \underbrace{\bigoplus_k \Sigma^{2k} H\mathbb{Q}}_{=: H_{\text{per}}\mathbb{Q}} \right).$$

(ii) Since rationalization is idempotent (113), which here means that  $L_{\mathbb{R}} \circ L_{\mathbb{Q}} \simeq L_{\mathbb{R}}$ , in this situation an absolute minimal model (Def. 4.41) of the base map  $c_b = \eta_{\text{B}^2\text{U}(1)}^{\mathbb{R}}$  and a twisted absolute minimal model (Def. 5.20) of the total map  $c_t = \eta_{\text{KU}_0 // \text{BU}(1)}^{\mathbb{R}}$  exist and are given, respectively, simply by the identity morphisms

$$c_b := \text{id}_{\text{B}^2\text{U}(1)} \quad \text{and} \quad c_t := \text{id}_{\text{B}^2\text{U}(1)}(\text{KU}_0 // \text{BU}(1)).$$

(iii) Therefore, the induced twisted secondary cohomology operation Def. 5.21 exists, and is for each differential twist  $\tau_{\text{diff}}$  a transformation

$$\widehat{K}^{\tau_{\text{diff}}}(\mathcal{X}) \xrightarrow{\text{ch}_{\text{diff}}^{\tau_{\text{diff}}} := (\eta_{\text{KU}_0 // \text{BU}(1)}^{\mathbb{R}})_{\text{diff}}} \widehat{H}_{\text{per}}\mathbb{Q}^{L_{\mathbb{Q}}\tau_{\text{diff}}}(\mathcal{X}) \quad (291)$$

from twisted differential K-theory to twisted differential periodic rational cohomology theory.

(iv) This is the twisted differential Chern character map on twisted differential complex K-theory as conceived in [GS19a, §3.2][GS19c, Prop. 4]. The analogous statement holds for the twisted differential Pontrjagin character (as in Example 4.11) on twisted differential real K-theory [GS19d, Thm. 12].

(v) Notice that this construction is close to but more structured than the plain curvature map on twisted differential K-theory (Example 5.19): If we considered the transformation of local coefficients as in (290) but for rationalization  $L_{\mathbb{R}}$  over the real numbers (Remark 3.51), then the induced twisted secondary cohomology operation would be equivalent to the twisted curvature map. Instead, (291) refines the plain curvature map to a twisted secondary operation that retains information about rational periods.

### 5.3 Twisted character on twisted differential Cohomotopy

We discuss here (Example 5.23 below) the twisted non-abelian character map on J-twisted Cohomotopy (Example 2.40) in degree 4, and on Twistorial Cohomotopy (Example 2.44). We highlight the induced charge quantization (Prop. 5.24 below) and comment on the relevance to high energy physics (Remark 5.29).

These twisted non-abelian cohomotopical character maps have been introduced and analyzed in [FSS19b] and [FSS20]. The general theory of non-abelian characters developed here shows how these cohomotopical characters are cousins both of generalized abelian characters such as the Chern character on twisted higher K-theory (§5.1), notably of the character on topological modular forms (by Example 4.13, and Remark 4.14) as well as of non-abelian characters such as the Chern-Weil homomorphism (§4.2) and the Cheeger-Simons homomorphism (§4.3).

#### Cohomotopical character maps.

**Example 5.23** (Character map on J-twisted Cohomotopy and on Twistorial Cohomotopy [FSS19b, Prop. 3.20] [FSS20, Prop. 3.9]). Let  $X$  be an 8-dimensional smooth spin manifold equipped with tangential  $\mathrm{Sp}(2)$ -structure  $\tau$  (59). Then the twisted non-abelian character maps (Def. 5.4) on J-twisted Cohomotopy (Example 2.40) in degree 4, and on Twistorial Cohomotopy (Example 2.44) are of the following form:

$$\begin{array}{ccc}
 \begin{array}{c} \mathbb{C}P^3 // \mathrm{Sp}(2) \\ \downarrow \text{Borel-equivariantized} \\ \text{twistor fibration} \\ t_{\mathbb{H}} // \mathrm{Sp}(2) \\ \downarrow \\ S^4 // \mathrm{Sp}(2) \end{array} & \begin{array}{c} \text{Twistorial} \\ \text{Cohomotopy} \\ \mathcal{S}^\tau(X) := H^\tau(X; \mathbb{C}P^3) \\ \downarrow \text{cohomology operation} \\ \text{along twistor fibration} \\ (t_{\mathbb{H}})_* \\ \pi^\tau(X) := H^\tau(X; S^4) \\ \text{J-twisted} \\ \text{4-Cohomotopy} \end{array} & \begin{array}{c} \text{character map on} \\ \text{Twistorial Cohomotopy} \\ \xrightarrow{\mathrm{ch}_{\mathbb{C}P^3}^\tau} H_{\mathrm{dR}}^{\tau} (X; \mathbb{C}P^3) \\ \downarrow (U_{\mathbb{H}})_* \\ H_{\mathrm{dR}}^{\tau} (X; \mathbb{S}^4) \\ \text{character map in} \\ \text{J-twisted Cohomotopy} \end{array} = \left\{ \begin{array}{l} H_3, \quad dH_3 = G_4 - \frac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ F_2, \quad dF_2 = 0, \\ 2G_7, \quad d2G_7 = -(G_4 - \frac{1}{4}p_1(\nabla)) \wedge (G_4 + \frac{1}{4}p_1(\nabla)) - \chi_8(\nabla), \\ G_4 \quad d \quad G_4 = 0 \end{array} \right\} / \sim \\
 & & \begin{array}{c} \begin{array}{cccc} H_3 & F_2 & 2G_7 & G_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 2G_7 & G_4 \end{array} \\ \downarrow \\ \begin{array}{l} d2G_7 = -(G_4 - \frac{1}{4}p_1(\nabla)) \wedge (G_4 + \frac{1}{4}p_1(\nabla)) - \chi_8(\nabla), \\ d \quad G_4 = 0, \end{array} \end{array}
 \end{array}$$

Here:

- (i) The twisted non-abelian de Rham cohomology targets on the right are as shown, by Example 3.96. (In particular the twisted curvature forms in the first line are relative to  $\mathbb{S}^4$ .)
- (ii) The vertical twisted non-abelian cohomology operation (Def. 2.41) on the left is induced from the Borel-equivariantized twistor fibration (61), and that on the right from its associated morphism of rational Whitehead  $L_\infty$ -algebras (Prop. 3.70).

**Proposition 5.24** (Charge-quantization in J-twisted Cohomotopy [FSS19b, Prop. 3.13][FSS20, Cor. 3.11]). Consider the twisted non-abelian character maps (Def. 5.4) in J-twisted Cohomotopy and in Twistorial Cohomotopy from Example 5.23.

(i) A necessary condition for a flat  $\mathrm{Sp}(2)$ -twisted  $\mathbb{S}^4$ -valued differential form datum  $(G_4, G_7)$  to lift through the J-twisted cohomotopical character map (i.e. to be in its image) is that the de Rham class of  $G_4$ , when shifted by the fourth fraction of the Pontrjagin form, is in the image, under the de Rham homomorphism (Example 4.9), of an integral class:

$$[G_4 - \frac{1}{4}p_1(\nabla)] \in H^4(X; \mathbb{Z}) \longrightarrow H_{\mathrm{dR}}^4(X). \quad (292)$$

(ii) A necessary condition for a flat  $\mathrm{Sp}(2)$ -twisted  $\mathbb{C}P^3$ -valued differential form datum  $(G_4, G_7, F_2, H_3)$  to lift through the character map in Twistorial Cohomotopy is that the de Rham class of  $G_4$  shifted by the fourth fraction of the Pontrjagin form is in the image, under the de Rham homomorphism (Example 4.9), of an integral class, and as such equal to the  $[F_2]$  cup-square:

$$[G_4 - \frac{1}{4}p_1(\nabla)] = [F_2 \wedge F_2] \in H^4(X; \mathbb{Z}) \longrightarrow H_{\mathrm{dR}}^4(X). \quad (293)$$

## Twisted differential Cohomotopy theory.

**Definition 5.25** (Differential twists for twistorial Cohomotopy). Let  $X^8$  be an 8-dimensional smooth spin manifold equipped with tangential  $\mathrm{Sp}(2)$ -structure  $\tau$  (59). By (41) in Example 2.33, by (57) in Example 2.42, and by (11) in Example 2.3, we have

$$[\tau] \in H^{\mathrm{tr}}(X; \mathrm{O}(n)/\mathrm{Sp}(2)) \xrightarrow{(Bi)_*} H(X^8; \mathrm{BSp}(2)) \simeq \mathrm{Sp}(2)\mathrm{Bundles}(X)_{/\sim}. \quad (294)$$

This gives, in particular, the class of a smooth principal  $\mathrm{Sp}(2)$ -bundle  $P \rightarrow X$  to which the tangent bundle  $TX$  is associated. With (212), we may choose an  $\mathrm{Sp}(2)$ -connection  $\nabla$  on  $P$ , and, by Prop. 4.37, this connection has a class  $[\tau_{\mathrm{diff}}]$  in differential non-abelian cohomology (Def. 4.33) with coefficient in  $BG$ :

$$\begin{array}{c} H(X^8; \mathrm{BSp}(2)) \simeq \mathrm{Sp}(2)\mathrm{Bundles}(X^8)_{/\sim} \longleftarrow \mathrm{Sp}(2)\mathrm{Connections}(X^8)_{/\sim} \dashrightarrow \widehat{H}(X^8; \mathrm{BSp}(2)) \\ [\tau] \longleftarrow [P] \longleftarrow [\nabla] \longrightarrow [\tau_{\mathrm{diff}}]. \end{array}$$

Any such  $\tau_{\mathrm{diff}}$  serves as a *differential twist* (277) for *twistorial Cohomotopy* in the following.

In twisted generalization of Example 4.38, we have:

**Example 5.26** (Differential twistorial Cohomotopy). Let  $X^8$  be a spin 8-manifold equipped with tangential  $\mathrm{Sp}(2)$ -structure  $\tau$  (59), and with a corresponding differential twist  $\tau_{\mathrm{diff}}$  (Def. 5.25).

(i) Consider the local coefficient bundle (54),  $S^4 \rightarrow S^4 // \mathrm{Sp}(2) \xrightarrow{J_{\mathrm{CP}^3}} \mathrm{BSp}(2)$ , for *J-twisted 4-Cohomotopy* (Example 2.40) pulled back along  $\mathrm{BSp}(2) \xrightarrow{\simeq} \mathrm{BSpin}(5) \rightarrow \mathrm{BO}(5)$ . This induces, via Def. 5.13, a twisted differential non-abelian cohomology theory  $\widehat{\mathcal{T}}^{\tau_{\mathrm{diff}}}(-)$ , which we call *J-twisted differential 4-Cohomotopy*, whose value on manifolds  $\mathcal{X} = X^8 \times \mathbb{R}^k$  sits in a cohomology operation diagram (4) of this form:

$$\begin{array}{ccc} \begin{array}{c} \text{differential} \\ \text{J-twisted} \\ \text{4-Cohomotopy} \end{array} \widehat{\pi}^{\tau_{\mathrm{diff}}^4}(\mathcal{X}) & \xrightarrow[\text{J-twisted}]{\text{cohomotopical} \\ \text{curvature}} F_{S^4}^{\tau_{\mathrm{dR}}^4} & \left\{ \begin{array}{l} 2G_7, \\ G_4 \end{array} \in \Omega_{\mathrm{dR}}^{\bullet}(\mathcal{X}) \mid \begin{array}{l} d2G_7 = -(G_4 - \frac{1}{4}p_1(\nabla)) \wedge (G_4 + \frac{1}{4}p_1(\nabla)) - \mathcal{X}_8(\nabla), \\ dG_4 = 0 \end{array} \right\} \\ \downarrow & & \downarrow \\ \begin{array}{c} \text{J-twisted} \\ \text{4-Cohomotopy} \\ \text{(Example 2.40)} \end{array} \pi^{\tau^4}(\mathcal{X}) & \xrightarrow[\text{character map}]{\text{ch}_{S^4}^{\tau^4}} & \begin{array}{c} \text{J-twisted} \\ \text{de Rham cohomology} \\ \text{(Def 3.98)} \end{array} H_{\mathrm{dR}}^{\tau^4}(\mathcal{X}; \mathbb{S}^4). \end{array} \quad (295)$$

(Example 5.23)

(ii) Consider the local coefficient bundle (61)  $\mathbb{C}P^3 \rightarrow \mathbb{C}P^3 // \mathrm{Sp}(2) \xrightarrow{J_{\mathrm{CP}^3}} \mathrm{BSp}(2)$  for *twistorial Cohomotopy* (Def. 2.44). This induces, via Def. 5.13, a twisted differential non-abelian cohomology theory  $\widehat{\mathcal{T}}^{\tau_{\mathrm{diff}}}(-)$ , which we call *differential twistorial Cohomotopy*, whose value on manifolds  $\mathcal{X} = X^8 \times \mathbb{R}^k$  sits in a cohomology operation diagram (4) of this form:

$$\begin{array}{ccc} \begin{array}{c} \text{differential} \\ \text{twistorial} \\ \text{Cohomotopy} \end{array} \widehat{\mathcal{T}}^{\tau_{\mathrm{diff}}}(\mathcal{X}) & \xrightarrow[\text{twistorial}]{\text{curvature}} F_{\mathbb{C}P^3}^{\tau_{\mathrm{dR}}} & \left\{ \begin{array}{l} H_3, \\ F_2, \\ 2G_7, \\ G_4 \end{array} \in \Omega_{\mathrm{dR}}^{\bullet}(\mathcal{X}) \mid \begin{array}{l} dH_3 = G_4 - \frac{1}{4}p_1(\nabla) - F_2 \wedge F_2, \\ dF_2 = 0, \\ d2G_7 = -(G_4 - \frac{1}{4}p_1(\nabla)) \wedge (G_4 + \frac{1}{4}p_1(\nabla)) - \mathcal{X}_8(\nabla), \\ dG_4 = 0 \end{array} \right\} \\ \downarrow & & \downarrow \\ \begin{array}{c} \text{twistorial} \\ \text{Cohomotopy} \\ \text{(Example 2.44)} \end{array} \mathcal{T}^{\tau}(\mathcal{X}) & \xrightarrow[\text{character map}]{\text{ch}_{\mathbb{C}P^3}^{\tau}} & \begin{array}{c} \text{twistorial} \\ \text{de Rham cohomology} \\ \text{(Def 3.98)} \end{array} H_{\mathrm{dR}}^{\tau}(\mathcal{X}; \mathbb{C}P^3). \end{array} \quad (296)$$

(Example 5.23)

**Proposition 5.27** (Twisted secondary cohomology operation induced by twistor fibration). *The defining twisted non-abelian cohomology operation (62) from twistorial Cohomotopy (Example 2.44) to J-twisted 4-Cohomotopy (Example 2.40), induced by the  $\mathrm{Sp}(2)$ -equivariantized twistor fibration  $t_{\mathbb{H}} // \mathrm{Sp}(2)$  (61) refines to a twisted secondary cohomology operation (Def. 5.21) from differential twistorial Cohomotopy to differential J-twisted Cohomotopy (Example 5.26):*

$$\begin{array}{ccc}
\begin{array}{l} \text{differential} \\ \text{twistorial} \\ \text{Cohomotopy} \end{array} & \widehat{\mathcal{T}}^{\tau_{\mathrm{diff}}}(\mathcal{X}) & \xrightarrow{c_{\mathbb{C}P^3}^{\tau}} \mathcal{T}^{\tau}(\mathcal{X}) \\
\downarrow & \downarrow \text{along } \mathrm{Sp}(2)\text{-equivariantized} \\
\begin{array}{l} \text{twisted secondary} \\ \text{cohomology operation} \end{array} & ((t_{\mathbb{H}} // \mathrm{Sp}(2))_{\mathrm{diff}})_* & \downarrow \text{twisted primary} \\
& & \text{cohomology operation} \\
\begin{array}{l} \text{differential} \\ \text{J-twisted} \\ \text{4-Cohomotopy} \end{array} & \widehat{\pi}^{\tau^4_{\mathrm{diff}}}(\mathcal{X}) & \xrightarrow{c_{S^4}^{\tau^4}} \pi^{\tau^4}(\mathcal{X})
\end{array}$$

*Proof.* By Def. 5.21, we need to show that we have a twisted absolute minimal model (Def. 5.20) for the  $\mathrm{Sp}(2)$ -equivariantized twistor fibration (61). By (288) this means that we can find a morphism

$$\mathfrak{I}_{B\mathrm{Sp}(2)}(\mathbb{C}P^3 // \mathrm{Sp}(2)) - \frac{t_{\mathbb{H}} // \mathrm{ISp}(2)}{-} \gg \mathfrak{I}_{B\mathrm{Sp}(2)}S^4 // \mathrm{Sp}(2) \quad (297)$$

between the relative Whitehead  $L_{\infty}$ -algebras (Prop. 3.70) of the two local coefficient bundles, which makes the following cube of transformations of derived PL-de Rham adjunction units commute:

$$\begin{array}{ccc}
\exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(\mathbb{C}P^3 // \mathrm{Sp}(2)) & \xrightarrow{\exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(t_{\mathbb{H}} // \mathrm{Sp}(2))} & \exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(S^4 // \mathrm{Sp}(2)) \\
\downarrow \exp \left( p_{\mathbb{C}P^3 // \mathrm{Sp}(2)}^{\min_{\mathrm{Sp}(2)}} \right) & & \downarrow \exp \left( p_{S^4 // \mathrm{Sp}(2)}^{\min_{B\mathrm{Sp}(2)}} \right) \\
\exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(J_{\mathbb{C}P^3}) & \xrightarrow{\exp \circ \mathrm{CE}(\mathfrak{I}_{B\mathrm{Sp}(2)}(\mathbb{C}P^3 // \mathrm{Sp}(2)))} & \exp \circ \mathrm{CE}(\mathfrak{I}(S^4 // \mathrm{Sp}(2))) \\
\downarrow \exp \left( p_{B\mathrm{Sp}(2)}^{\min} \right) & & \downarrow \exp \left( p_{B\mathrm{Sp}(2)}^{\min} \right) \\
\exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(B\mathrm{Sp}(2)) & \xrightarrow{\quad \quad \quad} & \exp \circ \Omega_{\mathrm{PLdR}}^{\bullet}(B\mathrm{Sp}(2)) \\
\downarrow \exp \left( p_{B\mathrm{Sp}(2)}^{\min} \right) & & \downarrow \exp \left( p_{B\mathrm{Sp}(2)}^{\min} \right) \\
\exp \circ \mathrm{CE}(\mathfrak{I}B\mathrm{Sp}(2)) & \xrightarrow{\quad \quad \quad} & \exp \circ \mathrm{CE}(\mathfrak{I}B\mathrm{Sp}(2))
\end{array}$$

But, from Example 3.96, we see that the total object of the relative Whitehead  $L_{\infty}$ -algebra of  $\mathbb{C}P^3 // \mathrm{Sp}(2)$  relative to  $\mathfrak{I}B\mathrm{Sp}(2)$  coincides with that relative to  $\mathfrak{I}_{B\mathrm{Sp}(2)}S^4 // \mathrm{Sp}(2)$ . Therefore, we may take the twisted absolute minimal model (297) to be equal to the relative Whitehead  $L_{\infty}$ -algebra projection (the top arrow in Example 3.96). This makes the required top front square commute by the commuting triangle in Prop. 3.49.  $\square$

**Remark 5.28** (Lifting against the twisted differential twistor fibration).

In terms of differential moduli  $\infty$ -stacks (275), the result of Prop. 5.27 with Example 5.26 says that lifting a twisted differential Cohomotopy cocycle  $\widehat{C}_3$  with 4-flux density  $G_4$  against the twisted differential refinement (289) of the equivariantized twistor fibration (61) to a differential twistorial Cohomotopy cocycle  $(\widehat{C}_3, \widehat{C}_2, \widehat{C}_1)$  involves, on twisted curvature forms (280) the appearance of a 2-flux density  $F_2$  and of a 3-form  $H_3$  such that  $dH_3 = G_4 - \frac{1}{4}p_1(\nabla) - F_2 \wedge F_2$ .

$$\begin{array}{ccc}
\mathbb{R}^{1,1} \times X^8 - \frac{(\widehat{C}_3, \widehat{B}_2, \widehat{A}_1)}{-} \gg (\mathbb{C}P^3 // \mathrm{Sp}(2))_{\mathrm{diff}} & & \\
\downarrow & & \downarrow \text{twisted differential} \\
\mathbb{R}^{2,1} \times X^8 \xrightarrow[\text{C-field}]{\widehat{C}_3} (S^4 // \mathrm{Sp}(2))_{\mathrm{diff}} & & \text{twistor fibration}
\end{array}$$



**Remark 5.29** (M-theory fields and Hypothesis H). In summary, we have found:

(i) A cocycle  $\widehat{C}_3$  in J-twisted differential 4-Cohomotopy (Example 5.26) has as curvature/character forms (280):

- (a) a closed 4-form  $G_4$ , hence a 4-flux density,
- (b) a non-closed 7-form  $G_7$ ,

satisfying the following Bianchi identities (Example 5.23) and integrality conditions (Prop. 5.24):

$$\begin{array}{ccc} \text{differential} & & \text{flat twisted cohomotopical} \\ \text{J-twisted 4-Cohomotopy} & \xrightarrow{\text{curvature (non-abelian character form representative)}} & \text{differential forms} \\ \widehat{\pi}^{\text{diff}}(\mathcal{X}) & \longrightarrow & \Omega_{\text{dR}}^{\text{tr}}(X; \mathbb{I}\mathbb{S}^4)_{\text{flat}} \end{array} \quad (298)$$

$$(\widehat{C}_3) \quad \longmapsto \quad \left( \begin{array}{c|l} G_4, & \text{shifted C-field flux quantization} \\ 2G_7 & [G_4 - \frac{1}{4}p_1(\omega)] \in H^4(X; \mathbb{Z}) \\ & d G_4 = 0 \\ & d 2G_7 = -(G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) - 24I_8(\omega) \\ & \text{C-field tadpole cancellation \& M5 Hopf WZ term level quantization} \end{array} \right)$$

(ii) Lifting this cocycle through the twisted differential twistor fibration (Prop. 5.27) to a cocycle  $(\widehat{C}_3, \widehat{B}_2, \widehat{A}_1)$  in differential twistorial Cohomotopy (Example 5.26) involves (Remark 5.28) adjoining to the 4-flux density  $G_4$ :

- (c) a closed 2-form curvature  $F_2$ , hence a 2-flux density,
- (d) a non-closed 3-form  $H_3$ ,

such that these curvature/character forms satisfy the following Bianchi identities (Example 5.23) and integrality conditions (Prop. 5.24):

$$\begin{array}{ccc} \text{differential} & & \text{flat twistorial} \\ \text{twistorial Cohomotopy} & \xrightarrow{\text{curvature (non-abelian character form representative)}} & \text{differential forms} \\ \widehat{\mathcal{T}}^{\text{diff}}(\mathcal{X}) & \longrightarrow & \Omega_{\text{dR}}^{\text{tr}}(X; \mathbb{I}\mathbb{C}P^3)_{\text{flat}} \end{array} \quad (299)$$

$$(\widehat{C}_3, \widehat{B}_2, \widehat{A}_1) \quad \longmapsto \quad \left( \begin{array}{c|l} H_3, & dH_3 = G_4 - \frac{1}{4}p_1(\omega) - F_2 \wedge F_2, \\ & \text{Hořava-Witten Green-Schwarz mechanism} \\ & [G_4 - \frac{1}{4}p_1(\omega)] = [F_2 \wedge F_2] \in H^4(X; \mathbb{Z}) \\ G_4, F_2, & d G_4 = 0, \quad d F_2 = 0, \\ 2G_7 & d 2G_7 = -(G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) - 24I_8(\omega) \\ & \text{C-field tadpole cancellation \& M5 Hopf WZ term level quantization} \end{array} \right)$$

(iii) With these cohomotopical curvature/character forms interpreted as flux densities, this is the Bianchi identities and charge quantization expected in M-theory on the supergravity C-field ( $\widehat{C}_3$ ), the heterotic B-field ( $\widehat{B}_2$ ) and the heterotic  $S(U(1)^2) \subset E_8$  gauge field ( $\widehat{A}_1$ ), with the following prominent features:

(a) **The charge quantization:**

- (1)  $[G_4 - \frac{1}{4}p_1(\nabla)] \in H^4(\mathcal{X}; \mathbb{Z})$  is expected for the C-field in the M-theory bulk (see [FSS19b, Table 1] for pointers); (300)
- (2)  $[G_4 - \frac{1}{4}p_1(\nabla)] = [F_2 \wedge F_2] \in H^4(\mathcal{X}; \mathbb{Z})$  is expected on heterotic boundaries (see [FSS20, §1] for pointers).

(b) **The quadratic functions:**

- (1)  $G_4 \mapsto (G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) + 24I_8(\omega)$  constitute the Hopf Wess-Zumino term (see [FSS19c][SS20a]) (301)
- (2)  $F_2 \mapsto F_2 \wedge F_2$  constitute the 2nd Chern class of an  $S(U(1)^2) \subset E_8$  bundle (see [FSS20, (7)]).

(iv) These are necessary, not yet sufficient constraints on cohomotopical lifts. The full constraints may be computed by a Postnikov tower analysis [GS20] and turn out to coincide with a list of further expected conditions in M-theory (see [FSS19b, Table 1]).

All this suggests the *Hypothesis H* [Sa13][FSS19b][FSS19c][SS19a][SS19b][BMSS19][SS20a][FSS20] that the elusive cohomology theory which controls M-theory in analogy to how K-theory controls string theory is: twisted differential non-abelian Cohomotopy theory.

**Cohomotopical character into K-theory.** We may regard the *secondary non-abelian Hurewicz/Boardman homomorphism* (Example 4.43) from differential 4-Cohomotopy (Example 4.38) to differential K-theory (Example 4.36), as a non-abelian but K-theory valued character, lifting the target of the cohomotopical character (Example 5.23) from rational cohomology to K-theory (compare [BSS19, Fig. 1]):

$$\begin{array}{ccc}
 \text{differential Cohomotopy} & & \text{differential K-theory} \\
 \widehat{\tau}^4(\mathcal{X}) & \xrightarrow{\beta_{\text{diff}}^4} & \widehat{\text{KU}}^0(\mathcal{X}) \\
 \text{differential non-abelian Boardman homomorphism (Example 4.43)} & & \text{differential Chern character (Example (5.22))} \\
 \text{charge-quantization in M-theory} & \longleftarrow & \text{charge-quantization in string theory} \\
 & & \text{differential rational cohomology} \\
 & & \widehat{H}_{\text{per}\mathbb{Q}}^0(\mathcal{X})
 \end{array} \quad (302)$$

(i) Lifting through this differential Boardman homomorphism induces *secondary charge quantization conditions* on K-theory, analogous to (2) but invisible even in generalized cohomology, instead now coming from non-abelian cohomology theory.

(ii) In the plain version (302) (i.e. disregarding twisting and equivariant enhancement) the effect of  $\beta_{\text{diff}}^4$  on curvature forms (234) is (by Example 4.43) to forget the quadratic function (301) on  $G_4$  and to inject what remains as the 4-form curvature component  $F_4$  in differential K-theory:

$$\begin{array}{ccc}
 \text{differential Cohomotopy} & & \text{differential K-theory} \\
 \widehat{\tau}^4(\mathcal{X}) & \xrightarrow{\beta_{\text{diff}}^4} & \widehat{\text{KU}}^0(\mathcal{X}) \\
 \text{secondary non-abelian Boardman homomorphism} & & \text{curvatures/flux densities} \\
 (F_{S^4})_* \downarrow & & \downarrow (F_{\text{KU}_0})_* \\
 \left\{ \begin{array}{l} 2G_7, \\ G_4 \end{array} \middle| \begin{array}{l} d2G_7 = -G_4 \wedge G_4 \\ dG_4 = 0 \end{array} \right\} & \xrightarrow{\begin{array}{l} G_4 \mapsto F_4 \\ G_7 \mapsto 0 \end{array}} & \left\{ (F_{2k}) \mid dF_{2k} = 0 \right\}.
 \end{array} \quad (303)$$

This is a ‘cohomotopical enhancement’ of the reduction in [DMW02] of  $E_8$  bundles in M-theory to the K-theory of type IIA string theory, now characterized by higher Postnikov stages of the Boardman homomorphism [GS20]. The remaining RR-flux components in  $\{F_{2k}\}$  are also found in the cohomotopical character, through cohomological double dimensional reduction formulated in parametrized homotopy theory: this is discussed in detail in [BMSS19].

(iii) The twisted generalization of the non-abelian Boardman homomorphism in (302) and (303) is more subtle, since the degree-3 twist of K-theory does not arise from the J-twist of Cohomotopy, but arises, together with the further RR-flux components, from  $S^1$ -equivariantization/double dimensional reduction of Cohomotopy [FSS16a][BMSS19], reproducing the reduction of  $E_8$  bundles from M-theory to type IIA in [MaSa04].

**Outlook: Equivariant enhancement.** The twisted non-abelian character theory presented here enhances further to *equivariant* non-abelian cohomology on orbi-orientifolds, by combining it with the techniques developed in [HSS18][SS20b] (essentially: parametrizing the construction here over the orbit category of the equivariance group). The resulting *character map in equivariant non-abelian cohomology* is discussed in [SS20c, §2,3]. For example, the equivariant enhancement of the cohomotopical character into K-theory (302), lifting the RR-fields in equivariant K-theory through the equivariantization enhancement of the Boardman homomorphism on the left of (302), enforces [SS19a][BSS19] ‘tadpole cancellation’ conditions expected in string theory at orbifold singularities.

## A Model category theory

For ease of reference and to highlight some less widely used aspects needed in the main text, we record basics of homotopy theory via model category theory [Qu67] (review in [Ho99][Hir02][Lu09, A.2]) and of homotopy topos theory [Re10] via model categories of simplicial presheaves [Bro73][Ja87][Du01] (review in [Du98][Lu09, §A.3.3][Ja15]).

**Topology.** By

$$\text{TopologicalSpaces} \in \text{Categories}$$

we denote a *convenient* [St67] (in particular: cartesian closed) category of topological spaces such as compactly-generated spaces [Stri09] or  $\Delta$ -generated spaces [Du03], equivalently numerically-generated spaces [SYH10], or D-topological spaces [SS20a, Prop. 2.4].

**Categories.** Let  $\mathcal{C}$  be a category.

(i) For  $X, A \in \mathcal{C}$  a pair of objects, we write

$$\mathcal{C}(X, A) := \text{Hom}_{\mathcal{C}}(X, A)$$

for the set of *morphisms* from  $X$  to  $A$ .

(ii) For  $\mathcal{C}, \mathcal{D}$  two categories, we denote a pair of *adjoint functors* between them by

$$\mathcal{D} \begin{array}{c} \xleftarrow{L} \\ \perp \\ \xrightarrow{R} \end{array} \mathcal{C} \quad \Leftrightarrow \quad \mathcal{D}(L(-), -) \simeq \mathcal{C}(-, R(-)). \quad (304)$$

(iii) A Cartesian square in  $\mathcal{C}$  we indicate by *pullback* notation  $f^*(-)$  and/or by the symbol “(pb)”:

$$\begin{array}{ccc} f^*A & \xrightarrow{f^*p} & A \\ \downarrow & \text{(pb)} & \downarrow p \\ B_1 & \xrightarrow{f} & B_2. \end{array} \quad (305)$$

(iv) Dually, a co-Cartesian square in  $\mathcal{C}$  we indicate by *pushout* notation  $f_*(-)$  and/or by the symbol “(po)”:

$$\begin{array}{ccc} X & \xrightarrow{q} & A \\ f \downarrow & \text{(po)} & \downarrow q_*f \\ Y & \longrightarrow & f_*A. \end{array} \quad (306)$$

**Model categories.**

**Definition A.1** (Weak equivalences). A *category with weak equivalences* is a category  $\mathcal{C}$  equipped with a sub-class  $W \subset \text{Mor}(\mathcal{C})$  of its morphisms, to be called the class of *weak equivalences*, such that

(i)  $W$  contains the class of isomorphisms;

(ii)  $W$  satisfies the cancellation property (“2-out-of-3”): if in any commuting triangle in  $\mathcal{C}$

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{g \circ f} & Z \end{array} \quad (307)$$

two morphisms are in  $W$ , then so is the third.

**Definition A.2** (Weak factorization system). A *weak factorization system* in a category  $\mathcal{C}$  is a pair of sub-classes of morphisms  $\text{Proj}, \text{Inj} \subset \text{Mor}(\mathcal{C})$  such that

(i) every morphisms  $X \xrightarrow{f} Y$  in  $\mathcal{C}$  may be factored through a morphism in  $\text{Proj}$  followed by one in  $\text{Inj}$ :

$$f : X \xrightarrow{\in \text{Proj}} Z \xrightarrow{\in \text{Inj}} Y \quad (308)$$

(ii) For every commuting square in  $\mathcal{C}$  with left morphism in  $\text{Proj}$  and right morphism in  $\text{Inj}$ , there exists a lift, namely a dashed morphism

$$\begin{array}{ccc} X & \longrightarrow & A \\ \in \text{Proj} \downarrow & \dashrightarrow \exists & \downarrow \in \text{Inj} \\ Y & \longrightarrow & B \end{array} \quad (309)$$

making the resulting triangles commute.

(iii) Given  $\text{Inj}$  (resp.  $\text{Proj}$ ), the class  $\text{Proj}$  (resp.  $\text{Inj}$ ) is the largest class for which (309) holds.

**Definition A.3** (Model category [Jo08, Def. E.1.2][Rie09]). A *model category* is a category  $\mathbf{C}$  that has all small limits and colimits, equipped with three sub-classes of its class of morphisms, to be denoted

$W$  – *weak equivalences*

$\text{Fib}$  – *fibrations*

$\text{Cof}$  – *cofibrations*

such that

(i) The class  $W$  makes  $\mathbf{C}$  a category with weak equivalences (Def. A.1);

(ii) The pairs  $(\text{Fib}, \text{Cof} \cap W)$  and  $(\text{Fib} \cap W, \text{Cof})$  are weak factorization systems (Def. A.2).

**Remark A.4** (Minimal assumptions). By item (iii) in Def. A.2 a model category structure is specified already by the classes  $W$  and  $\text{Fib}$ , or alternatively by the classes  $W$  and  $\text{Cof}$ . Moreover, it follows from Def. A.3 that also the class  $W$  is stable under retracts [Jo08, Prop. E.1.3][Rie09, Lemma 2.4]: Given a commuting diagram in the model category  $\mathbf{C}$  of the form on the left here

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & X \\ f \downarrow & & \downarrow \in W & & \downarrow f \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \end{array} \Rightarrow f \in W \quad (310)$$

with the middle morphism a weak equivalence, then also  $f$  is a weak equivalence.

**Definition A.5** (Proper model category). A model category  $\mathbf{C}$  Def. A.3 is called

(i) *right proper*, if pullback along fibrations preserves weak equivalences:

$$\begin{array}{ccc} X & \xrightarrow{p^*f} & A \\ \downarrow & \text{(pb)} & \downarrow p \in \text{Fib} \\ Y & \xrightarrow{f \in W} & B \end{array} \Rightarrow p^*f \in W \quad (311)$$

(ii) *left proper*, if pushout along cofibrations preserves weak equivalences, hence if the opposite model category (Example A.9) is right proper.

**Notation A.6** (Fibrant and cofibrant objects). Let  $\mathbf{C}$  be a model category (Def. A.3)

(i) We write  $*$   $\in \mathbf{C}$  for the terminal object and  $\emptyset \in \mathbf{C}$  for the initial object.

(ii) An object  $X \in \mathbf{C}$  is called:

(a) *fibrant* if the unique morphism to the terminal object is a fibration,  $X \xrightarrow{\in \text{Fib}} *$ ;

(b) *cofibrant* if the unique morphism from the initial object is a cofibration,  $\emptyset \xrightarrow{\in \text{Cof}} X$ .

We write  $\mathbf{C}_{\text{fib}}, \mathbf{C}^{\text{cof}}, \mathbf{C}_{\text{fib}}^{\text{cof}} \subset \mathbf{C}$  for the full subcategories on, respectively, fibrant objects, or cofibrant objects or objects that are both fibrant and cofibrant.

(iii) Given an object  $X \in \mathbf{C}$

(a) A *fibrant replacement* is a factorization (308) of the terminal morphism as

$$X \xrightarrow[\in \text{Cof} \cap W]{j_X} PX \xrightarrow[\in \text{Fib}]{q_X} * . \quad (312)$$

(b) A *cofibrant replacement* is a factorization (308) of the initial morphism as

$$\emptyset \xrightarrow[\in \text{Cof}]{i_X} QX \xrightarrow[\in \text{Fib} \cap W]{p_X} X . \quad (313)$$

**Example A.7** (Classical model structure on topological spaces [Qu67, §II.3][Hir15]). The category of  $\text{TopologicalSpaces}$  carries a model category structure whose

(i)  $W$  – weak equivalence are the weak homotopy equivalences;

(ii)  $\text{Fib}$  – fibrations are the Serre fibrations.

We denote this model category by

$$\text{TopologicalSpaces}_{\text{Qu}} \in \text{ModelCategories} .$$

**Example A.8** (Classical model structure on simplicial sets [Qu67, §II.3][GJ99]). The category of  $\text{SimplicialSets}$  carries a model category structure whose

- (i)  $W$  – weak equivalence are those whose geometric realization is a weak homotopy equivalence;
- (ii)  $\text{Cof}$  – cofibrations are the monomorphisms (degreewise injections).

We denote this model category by

$$\text{SimplicialSets}_{\text{Qu}} \in \text{ModelCategories} .$$

**Example A.9** (Opposite model category [Hir02, §7.1.8]). If  $\mathbf{C}$  is a model category (Def. A.3) then the opposite underlying category becomes a model category  $\mathbf{C}^{\text{op}}$  with the same weak equivalences (up to reversal) and with fibrations (resp. cofibrations) the cofibrations (resp. fibrations) of  $\mathbf{C}$ , up to reversal.

**Example A.10** (Slice model structure [Hir02, §7.6.4]). Let  $\mathbf{C}$  be a model category (Def. A.3)

(i) For  $X \in \mathbf{C}$  any object, the slice category  $\mathbf{C}^{/X}$ , whose objects are morphisms to  $X$  and whose morphisms are commuting triangles in  $\mathbf{C}$  over  $X$

$$\mathbf{C}^{/X}(a, b) := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow a & \swarrow b \\ & X & \end{array} \right\}$$

becomes itself a model category, whose weak equivalence, fibrations and cofibrations are those morphisms whose underlying morphisms  $f$  are such in  $\mathbf{C}$ .

(ii) Dually there is the coslice model category  $\mathbf{C}^{X/} := ((\mathbf{C}^{\text{op}})^{/X})^{\text{op}}$ , being the opposite model category (Example A.9) of the slice category of the opposite of  $\mathbf{C}$ :

$$\mathbf{C}^{X/}(a, b) := \left\{ \begin{array}{ccc} & X & \\ a \swarrow & & \searrow b \\ A & \xrightarrow{f} & B \end{array} \right\}$$

### Homotopy categories.

**Definition A.11** (Cylinder objects and Path space objects [Qu67, Def. I.4]). Let  $\mathbf{C}$  be a model category (Def. A.3).

(a) With  $A \in \mathbf{C}_{\text{fib}}$  a fibrant object (Notation A.6), a *path space object* for  $A$  is a factorization of the diagonal morphism  $\Delta_A$  through a weak equivalence followed by a fibration:

$$A \xrightarrow{\in W} \text{Paths}(A) \xrightarrow{(p_0, p_1) \in \text{Fib}} A \times A . \quad (314)$$

$$\Delta_A$$

(b) With  $X \in \mathbf{C}_{\text{cof}}$  a cofibrant object (Notation A.6), a *cylinder object* for  $X$  is a factorization of the co-diagonal morphism  $\nabla_X$  through cofibration followed by a weak equivalence:

$$X \sqcup X \xrightarrow{(i_0, i_1) \in \text{Cof}} \text{Cyl}(X) \xrightarrow{\in W} X . \quad (315)$$

$$\nabla_X$$

**Example A.12** (Standard cylinder object in simplicial sets). For  $X \in \text{SimplicialSets}_{\text{Q}}$  (Example A.8) a cylinder object (Def. A.11) is evidently given by Cartesian product  $X \times \Delta[1]$  with the 1-simplex, with  $(i_0, i_1)$  being the two endpoint inclusions.

**Definition A.13** (Homotopy). Let  $\mathbf{C}$  be a model category (Def. A.3),  $X \in \mathbf{C}^{\text{cof}}$  a cofibrant object,  $A \in \mathbf{C}_{\text{fib}}$  a fibrant object (Notation A.6). Then a *homotopy* between a pair of morphisms  $f, g \in \mathbf{C}(X, A)$ , to be denoted

$$\phi : f \Rightarrow g \quad \text{or} \quad \begin{array}{ccc} & f & \\ X & \begin{array}{c} \curvearrowright \\ \Downarrow \phi \\ \curvearrowleft \end{array} & A \\ & g & \end{array}$$

is a morphism  $\phi_l \in \mathbf{C}(\text{Cyl}(X), A)$  out of a cylinder object for  $X$  or a morphism  $\phi_r \in \mathbf{C}(X, \text{Paths}(A))$  to a path space object for  $A$   $\text{Paths}(A)$  (Def. A.11) which make either of these diagrams commute:

$$\begin{array}{ccc}
\begin{array}{ccc}
X & & \\
i_0 \downarrow & \searrow f & \\
\text{Cyl}(X) & \xrightarrow{\phi_l} & A, \\
i_1 \uparrow & \nearrow g & \\
X & & 
\end{array}
& &
\begin{array}{ccc}
& & A \\
& \nearrow f & \uparrow p_0 \\
X & \xrightarrow{\phi_r} & \text{Paths}(A) \\
& \searrow g & \downarrow p_1 \\
& & A
\end{array}
\end{array}$$

**Proposition A.14** (Homotopy classes). *Let  $\mathbf{C}$  be a model category,  $X \in \mathbf{C}^{\text{cof}}$  and  $A \in \mathbf{C}_{\text{fib}}$  (Notation A.6). Then homotopy (Def. A.13) is an equivalence relation  $\sim$  on the hom-set  $\mathbf{C}(X, A)$ . We write*

$$\mathbf{C}(X, A)_{/\sim} \in \text{Sets} \quad (316)$$

for the corresponding set of homotopy classes of morphisms from  $X$  to  $A$ .

**Definition A.15** (Homotopy category of a model category). For  $\mathbf{C}$  a model category (Def. A.3),

(i) we write

$$\text{Ho}(\mathbf{C}) := (\mathbf{C}_{\text{fib}}^{\text{cof}})_{/\sim_r} \in \text{Categories} \quad (317)$$

for the category whose objects are those objects of  $\mathbf{C}$  that are both fibrant and cofibrant (Notation A.6), and whose morphisms are the right homotopy classes of morphisms in  $\mathbf{C}$  (Def. 316):

$$X, A \in \mathbf{C}_{\text{fib}}^{\text{cof}} \Rightarrow \text{Ho}(\mathbf{C})(X, A) := \mathbf{C}(X, A)_{/\sim_r}$$

and composition of morphisms is induced from composition of representatives in  $\mathbf{C}$ .

(ii) Given a choice of fibrant replacement  $P$  and of cofibrant replacement  $Q$  for each object of  $\mathbf{C}$  (Notation A.6) we obtain a functor

$$\mathbf{C} \xrightarrow{\gamma_{\mathbf{C}}} \text{Ho}(\mathbf{C}), \quad (318)$$

which (a) sends any object  $X \in \mathbf{C}$  to  $PQX$  and sends (b) any morphism  $X \xrightarrow{f} A$  to the right homotopy class (316) of any lift (309)  $PQf$  obtained from any lift  $Qf$  in the following diagrams:

$$\begin{array}{ccc}
\emptyset \longrightarrow QY & & QX \xrightarrow{j_{QR} \circ Qf} PQY \\
i_X \downarrow \quad \dashrightarrow Qf \quad \downarrow p_Y & \rightsquigarrow & j_{QX} \downarrow \quad \dashrightarrow PQf \quad \downarrow q_{QR} \\
QX \xrightarrow{f \circ p_X} Y & & PQX \longrightarrow *
\end{array}$$

**Proposition A.16** (Homotopy category is localization). *For a model category  $\mathbf{C}$  (Def. A.3) the functor  $\mathbf{C} \xrightarrow{\gamma_{\mathbf{C}}} \text{Ho}(\mathbf{C})$  (318) from Def. A.15 exhibits the homotopy category as the localization of the model category at its class of weak equivalences:  $\gamma_{\mathbf{C}}$  sends all weak equivalences in  $\mathbf{C}$  to isomorphisms, and is the universal functor with this property.*

The restriction to fibrant-and-cofibrant objects in Def. A.15 is convenient for defining composition of morphisms, but for computing hom-sets in the homotopy category it is sufficient that the domain object is cofibrant, and the codomain fibrant:

**Proposition A.17** ([Qu67, §I.1 Cor. 7]). *Let  $\mathcal{C}$  be a model category (Def. A.3). For  $X \in \mathcal{C}^{\text{cof}}$  a cofibrant object and  $A \in \mathcal{C}_{\text{fib}}$  a fibrant object, any choice of fibrant replacement  $PX$  and cofibrant replacement  $QA$  (Notation A.6) induces a bijection between the set of homotopy classes (Def. A.13) and the hom-set in the homotopy category (Def. A.15) between  $X$  and  $A$ :*

$$\mathcal{C}(X, A)_{/\sim_r} \xrightarrow[\simeq]{\mathcal{C}(j_X, p_A)} \text{Ho}(\mathbf{C})(X, A).$$

### Quillen adjunctions.

**Definition A.18** (Quillen adjunction). Let  $\mathbf{D}, \mathbf{C}$  be model categories (Def. A.3). Then a pair of adjoint functors  $(L \dashv R)$  (304) between their underlying categories is called a *Quillen adjunction*, to be denoted

$$\mathbf{D} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow[\perp_{\text{Qu}}]{R} \end{array} \mathbf{C} \quad (319)$$

if the following equivalent conditions hold:

- $L$  preserves Cof, and  $R$  preserves Fib;
- $L$  preserves Cof and  $\text{Cof} \cap W$ ;
- $R$  preserves Fib and  $\text{Fib} \cap W$ .

**Example A.19** (Base change Quillen adjunction). Let  $\mathbf{C}$  be a model category (Def. A.3),  $B_1, B_2 \in \mathbf{C}_{\text{fib}}$  a pair of fibrant objects (Notation A.6) and

$$B_1 \xrightarrow{f} B_2 \in \mathbf{C} \quad (320)$$

a morphism. Then we have a Quillen adjunction (Def. A.18)

$$\mathbf{C}/B_2 \begin{array}{c} \xleftarrow{f_!} \\ \perp_{\text{Qu}} \\ \xrightarrow{f^*} \end{array} \mathbf{C}/B_1$$

between the slice model categories (Example A.10), where:

(i) The left adjoint functor  $f_!$  is given by postcomposition in  $\mathbf{C}$  with  $f$  (320):

$$f_* : \begin{array}{ccc} X & \xrightarrow{c} & A \\ & \searrow \tau & \swarrow \rho \\ & B_1 & \end{array} \mapsto \begin{array}{ccc} X & \xrightarrow{c} & A \\ & \searrow \tau & \swarrow \rho \\ & B_1 & \\ & & \downarrow f \\ & & B_2 \end{array} \quad (321)$$

(ii) The right adjoint functor  $f^*$  is given by pullback (305) along  $f$  (320).

That these functors indeed form an adjunction  $f_! \dashv f^*$  follows from the defining universal property of the pullback (305):

$$\begin{array}{ccc} \mathbf{C}/B_2(f_*\tau, \rho) & \simeq & \mathbf{C}/B_1(\tau, f^*\rho) \\ \begin{array}{ccc} X & \overset{c}{\dashrightarrow} & A \\ & \searrow \tau & \downarrow \rho \\ & B_1 & \downarrow f \\ & & B_2 \end{array} & \leftrightarrow & \begin{array}{ccc} X & \overset{\tilde{c}}{\dashrightarrow} & f^*A \longrightarrow A \\ & \searrow \tau & \downarrow \rho \\ & B_1 & \downarrow f \\ & & B_2 \end{array} \end{array} \quad (322)$$

That this adjunction is a Quillen adjunction (Def. A.18) follows since  $f_!$  (321) evidently preserves each of  $W$  and Cof (even Fib) separately, by Example A.10.

**Lemma A.20** (Ken Brown's lemma [Ho99, Lemma 1.1.12][Bro73]). *Given a Quillen adjunction  $L \dashv R$  (Def. A.18),*

- (i) *the right Quillen functor  $R$  preserves all weak equivalences between fibrant objects.*
- (ii) *the left Quillen functor  $L$  preserves all weak equivalences between cofibrant objects.*

**Proposition A.21** (Derived functors). *Given a Quillen adjunction  $(L \dashv_{\text{Qu}} R)$  (Def. A.18), there are adjoint functors  $\mathbb{L}L \dashv \mathbb{R}R$  (304) between the homotopy categories (Def. A.15)*

$$\text{Ho}(\mathbf{D}) \begin{array}{c} \xleftarrow{\mathbb{L}L} \\ \perp \\ \xrightarrow{\mathbb{R}R} \end{array} \text{Ho}(\mathbf{C}) \quad (323)$$

whose composites with the localization functors (318) make the following squares commute up to natural isomorphism:

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{R} & \mathbf{C} \\ \gamma_D \downarrow & \simeq & \downarrow \gamma_C \\ \text{Ho}(\mathbf{D}) & \xrightarrow{\mathbb{R}R} & \text{Ho}(\mathbf{C}) \end{array} \quad \begin{array}{ccc} \mathbf{D} & \xleftarrow{L} & \mathbf{C} \\ \gamma_D \downarrow & \simeq & \downarrow \gamma_C \\ \text{Ho}(\mathbf{D}) & \xleftarrow{\mathbb{L}L} & \text{Ho}(\mathbf{C}) \end{array}$$

These are unique up to natural isomorphism, and are called the left and right derived functors of  $L$  and  $R$ , respectively.

**Example A.22** (Derived functors via (co-)fibrant replacement). It is convenient to leave the localization functors  $\gamma$  (318) notationally implicit, and understand objects of  $\mathbf{C}$  as objects of  $\text{Ho}(\mathbf{C})$ , via  $\gamma$ . Then:

- (i) The value of a left derived functor  $\mathbb{L}L$  (Prop. A.21) on an object  $c \in \mathbf{C}$  is equivalently the value of  $L$  on a cofibrant replacement  $Qc$  (313):

$$\mathbb{L}L(c) \simeq L(Qc) \in \text{Ho}(\mathbf{D}). \quad (324)$$

(ii) The value of a right derived functor  $\mathbb{R}R$  (Prop. A.21) on an object  $d \in \mathbf{D}$  is equivalently the value of  $R$  on a cofibrant replacement  $Pd$  (312):

$$\mathbb{R}R(d) \simeq R(Pd) \in \text{Ho}(\mathbf{C}). \quad (325)$$

(iii) The derived unit  $\mathbb{D}\eta$  of the derived adjunction (323), is, on any cofibrant object  $c \in \mathbf{C}^{\text{cof}}$ , given by

$$\mathbb{D}\eta_c : c \xrightarrow{\eta_c} R(L(c)) \xrightarrow{R(j_{L(c)})} R(PL(c)) \in \text{Ho}(\mathbf{C}) \quad (326)$$

where  $L(c) \xrightarrow{j_{L(c)}} PL(c)$  is any fibrant replacement (312).

(iv) The derived co-unit  $\mathbb{D}\varepsilon$  of the derived adjunction (323), is, on any fibrant object  $d \in \mathbf{D}_{\text{fib}}$ , given by

$$\mathbb{D}\varepsilon_d : L(QR(d)) \xrightarrow{L(p_{R(d)})} L(R(d)) \xrightarrow{\varepsilon_d} d \in \text{Ho}(\mathbf{D}) \quad (327)$$

where  $QR(d) \xrightarrow{p_{R(d)}} R(d)$  is any cofibrant replacement (313).

### Homotopy fibers and homotopy pullback.

**Definition A.23** (Homotopy fiber). Let  $\mathbf{C}$  be a model category (Def. A.3).

(i) For  $A \xrightarrow{p} B$  a morphism in  $\mathbf{C}$  with  $B \in \mathbf{C}_{\text{fib}} \subset \mathbf{C}$  a fibrant object (Notation A.6), and for  $* \xrightarrow{b} B$  a morphism from the terminal object (a ‘‘point in  $B$ ’’), the *homotopy fiber* of  $p$  over  $b$  is the image in the homotopy category (318) of the ordinary fiber over  $b$ , i.e. the pullback (305) along  $b$  in  $\mathbf{C}$ , of any fibration  $\tilde{p}$  weakly equivalent to  $p$ :

$$\text{hofib}_b(p) \longrightarrow A \quad \downarrow \rho \quad B \quad := \quad \gamma_{\mathbf{C}} \left( \begin{array}{ccc} \text{fib}_b(\tilde{p}) & \longrightarrow & \tilde{A} \longleftarrow \in W & A \\ \downarrow & \text{(pb)} & \tilde{p} \downarrow \in \text{Fib} & \downarrow \rho \\ * & \xrightarrow{b} & B \end{array} \right) \in \text{Ho}(\mathbf{C}). \quad (328)$$

This is well-defined in that  $\text{hofib}_b(p) \in \text{Ho}(\mathbf{C})$  depends on the choice of fibration replacement  $\tilde{p}$  only up to isomorphism in the homotopy category.

(ii) Dually, homotopy co-fibers are homotopy fibers in the opposite model category (Def. A.9).

More generally:

**Definition A.24** (Homotopy pullback). Given a model category  $\mathbf{C}$  (Def. A.3) and a pair of coincident morphisms

$$\begin{array}{ccc} & A & \\ & \downarrow \rho & \\ X & \xrightarrow{\tau} & B \end{array}$$

between fibrant objects, the *homotopy pullback* of  $\rho$  along  $\tau$  (or *homotopy fiber product of  $\rho$  with  $\tau$* ) is the image of  $\rho$ , regarded as an object in the homotopy category (Def. A.15) of the slice model category (Example A.10) under the right derived functor (Prop. A.21) of the right base change functor along  $\tau$  (Example A.19):

$$\text{Ho}(\mathbf{C}^B) \ni \begin{array}{ccc} A & & \mathbb{R}\tau^*A \\ \downarrow \rho & \xrightarrow{\text{homotopy pullback}} & \downarrow \mathbb{R}\tau^*\rho \\ B & & X \end{array} := \text{Ho}(\mathbf{C}^X), \quad (329)$$

By (322) the derived adjunction counit (327) on (329) gives a commuting square in (318) the homotopy category of  $\mathbf{C}$

$$\begin{array}{ccc} \mathbb{R}\tau^*A & \longrightarrow & A \\ \mathbb{R}\tau^*\rho \downarrow & \text{(hpb)} & \downarrow \rho \\ X & \xrightarrow{\tau} & B \end{array} := \gamma_{\mathbf{C}} \left( \begin{array}{ccc} \tau^*\tilde{A} & \longrightarrow & \tilde{A} \longleftarrow \in W & A \\ \downarrow & \text{(pb)} & \tilde{p} \downarrow \in \text{Fib} & \downarrow \rho \\ X & \xrightarrow{\tau} & B \end{array} \right) \in \text{Ho}(\mathbf{C}). \quad (330)$$

This square in the homotopy category, together with its pre-image pullback square in the model category, is the *homotopy pullback square* of  $\rho$  along  $\tau$ .



**Example A.25** (Homotopy fiber is homotopy pullback to the point). Homotopy fibers (Def. A.23) are the homotopy pullbacks (Def. A.24) to the terminal object, by (325).

**Lemma A.26** (Factorization lemma [Bro73, p. 431]). Let  $\mathbf{C}$  be a model category (Def. A.3) and  $A \xrightarrow{\rho} B \in \mathbf{C}_{\text{fib}}$  a morphism between fibrant objects. Then for  $\text{Paths}(B)$  a path space object for  $B$  (Def. A.11) the vertical composite in the following diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\in W} & p_1^* A & \xrightarrow{\in W} & A \\
 & \searrow \rho & \downarrow & \text{(pb)} & \downarrow \rho \\
 & & \text{Paths}(B) & \xrightarrow{p_1} & B \\
 & & \downarrow p_0 & & \\
 & & B & & 
 \end{array} \tag{331}$$

is a fibration, and in fact a fibration resolution of  $\rho$ , in that it factors  $\rho$  through a weak equivalence.

**Example A.27** (Homotopy pullback via triples). Given a model category  $\mathbf{C}$  (Def. A.3) and a pair of coincident morphisms

$$\begin{array}{c}
 A \\
 \downarrow \rho \\
 X \xrightarrow{\tau} B
 \end{array}$$

between fibrant objects, Lemma A.26 says that the corresponding homotopy pullback (Def. A.24) is computed by the following diagram

$$\begin{array}{ccc}
 \mathbb{R}\tau^* A & \xrightarrow{\quad} & A \\
 \downarrow & \searrow \phi & \downarrow \rho \\
 & & \text{Paths}(B) \xrightarrow{p_0} B \\
 & & \downarrow p_1 \\
 X & \xrightarrow{\tau} & B
 \end{array}
 =
 \begin{array}{ccc}
 \tau^*(p_0 \circ p_1^* \rho) & \xrightarrow{\quad} & p_1^* A \xrightarrow{\quad} A \\
 \downarrow & \text{(pb)} & \downarrow \text{(pb)} \downarrow \rho \\
 & & \text{Paths}(B) \xrightarrow{p_0} B \\
 & & \downarrow p_1 \\
 X & \xrightarrow{\tau} & B
 \end{array}$$

Here the right hand side exhibits the left hand side as a limit cone. This means that the homotopy pullback  $\mathbb{R}\tau^* A$  is universally characterized by the fact that morphisms into it are triples  $(f, g, \phi)$ , consisting of a pair of morphisms  $f, g$  to  $A, X$ , respectively, and a right homotopy  $\phi$  (Def. A.13) between their composites with  $\rho$  and  $\tau$ , respectively:

$$\mathbf{C}(-; \mathbb{R}\tau^* A) \simeq \left\{ (f, g, \phi) \left| \begin{array}{ccc} & \overset{f}{\dashrightarrow} & A \\ \downarrow & \phi & \downarrow \rho \\ g & \dashrightarrow & B \\ \downarrow & & \downarrow \tau \\ X & \xrightarrow{\tau} & B \end{array} \right. \right\} \tag{332}$$

### Quillen equivalences.

**Lemma A.28** (Conditions characterizing Quillen equivalences). Given a Quillen adjunction  $L \dashv_{\text{Qu}} R$  (Def. A.18), the following conditions are equivalent:

- The left derived functor (Prop. A.21) is an equivalence of homotopy categories (Def. A.15)  $\text{Ho}(\mathcal{D}) \xleftarrow[\simeq]{\mathbb{L}L} \text{Ho}(\mathcal{C})$ .
- The right derived functor (Prop. A.21) is an equivalence of homotopy categories (Def. A.15)  $\text{Ho}(\mathcal{D}) \xrightarrow[\simeq]{\mathbb{R}R} \text{Ho}(\mathcal{C})$ .
- Both of the following two conditions hold:
  - (i) The derived adjunction unit  $\mathbb{D}\eta$  (326) is a natural isomorphism, hence (326) is a weak equivalence in  $\mathbf{C}$ ;
  - (ii) The derived adjunction counit  $\mathbb{D}\epsilon$  (327) is a natural isomorphism, hence (327) is a weak equivalence in  $\mathbf{D}$ .

- For  $c \in \mathbf{C}^{\text{cof}}$  and  $d \in \mathbf{D}_{\text{fib}}$ , a morphism out of  $L(c)$  is a weak equivalence precisely if its adjunct into  $R(d)$  is:

$$L(c) \xrightarrow[\in \mathbf{W}]{f} d \quad \Leftrightarrow \quad c \xrightarrow[\in \mathbf{W}]{\tilde{f}} R(d). \quad (333)$$

**Definition A.29** (Quillen equivalence). If the equivalent conditions from Lemma A.28 are met, a Quillen adjunction  $L \dashv_{\text{Qu}} R$  (Def. A.18) is called a *Quillen equivalence*, which we denote as follows:

$$\mathcal{D} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow[\simeq_{\text{Qu}}]{} \\ \xrightarrow{R} \end{array} \mathcal{C}.$$

Hence:

**Proposition A.30** (Derived equivalence of homotopy categories). *The derived adjunction (Prop. A.21) of a Quillen equivalence (Def. A.29) is an adjoint equivalence of homotopy categories (Def. A.15):*

$$\text{Ho}(\mathbf{D}) \begin{array}{c} \xleftarrow{\mathbb{L}L} \\ \xrightarrow[\simeq]{} \\ \xrightarrow{\mathbb{R}R} \end{array} \text{Ho}(\mathbf{C}). \quad (334)$$

**Lemma A.31** (Quillen equivalence when left adjoint creates weak equivalences [EI19, Lemma 3.3]). *Let  $L \dashv_{\text{Qu}} R$  be a Quillen adjunction (Def. A.18) such that the left adjoint functor  $L$  creates weak equivalences, in that for all morphisms  $f$  in  $\mathbf{C}$  we have*

$$f \in \mathbf{W}_{\mathbf{C}} \quad \Leftrightarrow \quad L(f) \in \mathbf{W}_{\mathbf{D}}. \quad (335)$$

*Then  $L \dashv_{\text{Qu}} R$  is a Quillen equivalence (Def. A.29) precisely if the adjunction co-unit  $\varepsilon_d$  is a weak equivalence on all fibrant objects  $d \in \mathbf{C}_{\text{fib}}$ .*

*Proof.* By Lemma A.28, it is sufficient to check that the (i) derived unit and (ii) derived counit of the adjunction are weak equivalences precisely if the ordinary counit is a weak equivalence.

(ii) For the derived counit (327)

$$\mathbb{D}\varepsilon_c : L(QR(d)) \xrightarrow[\in \mathbf{W}]{L(p_{R(d)})} L(R(d)) \xrightarrow{\varepsilon_d} d$$

we have that  $p_{R(d)}$  is a weak equivalence (313), and since  $L$  preserves this, by assumption, so is  $L(p_{R(d)})$ . Therefore  $\mathbb{D}\varepsilon_d$  is a weak equivalence precisely if  $\varepsilon_d$  is, by 2-out-of-3 (307).

(i) For the derived unit (326)

$$c \xrightarrow{\eta_c} R(L(c)) \xrightarrow{R(j_{L(c)})} R(PL(c))$$

consider the composite of its image under  $L$  with the adjunction counit, as shown in the middle row of the following diagram:

$$L(c) \begin{array}{c} \xrightarrow{L(\mathbb{D}\eta_c)} \\ \xrightarrow{L(\eta_c)} \\ \xrightarrow{j_{L(c)} \in \mathbf{W}} \end{array} L \circ R(L(c)) \xrightarrow{L \circ R(j_{L(c)})} L \circ R(PL(c)) \xrightarrow{\varepsilon_{PL(c)}} PL(c).$$

By the formula for adjoints, this composite equals the adjunct of the derived adjunction unit, hence  $j_{L(c)}$ , as shown by the bottom morphism, which is a weak equivalence (312). Now, since  $L$  creates weak equivalences by assumption,  $L(\mathbb{D}\eta_c)$  is a weak equivalence precisely if  $\mathbb{D}\eta_c$  is a weak equivalence. Therefore it follows, again by 2-out-of-3 (307), that this is the case precisely if the adjunction counit  $\varepsilon$  is a weak equivalence on the fibrant object  $PL(c)$ .  $\square$

**Proposition A.32** (Base change along weak equivalence in right proper model category). *Let  $\mathbf{C}$  be a right proper model category (Def. A.5). Then its base change Quillen adjunction (Example A.19) along any weak equivalence*

$$B_1 \xrightarrow[\in \mathbf{W}]{f} B_2 \quad \in \mathbf{C}$$

*is a Quillen equivalence (Def. A.29):*

$$\mathbf{C}/B_2 \begin{array}{c} \xleftarrow{f_!} \\ \xrightarrow[\simeq_{\text{Qu}}]{} \\ \xrightarrow{f^*} \end{array} \mathbf{C}/B_1.$$

*Proof.* Observe that  $B_2 \xrightarrow{\text{id}} B_2$  is the terminal object of  $\mathbf{C}/B_2$ , so that the fibrant objects of  $\mathbf{C}/B_2$  correspond to the fibrations in  $\mathbf{C}$  over  $B_2$ . Therefore, the condition (333) says that for  $f_! \dashv f^*$  to be a Quillen equivalence it is sufficient that in (322)  $c$  is a weak equivalence precisely if  $\tilde{c}$  is, assuming that  $\rho$  is a fibration:

$$\begin{array}{ccc}
 X & \overset{c}{\dashrightarrow} & A \\
 \searrow \tau & & \downarrow \rho \in \text{Fib} \\
 & & B_1 \\
 & & \searrow f \in W \\
 & & B_2
 \end{array}
 \quad \leftrightarrow \quad
 \begin{array}{ccccc}
 X & \overset{\tilde{c}}{\dashrightarrow} & f^*A & \xrightarrow{\rho^*f \in W} & A \\
 \searrow \tau & & \downarrow f^*\rho & \text{(pb)} & \downarrow \rho \in \text{Fib} \\
 & & B_1 & \searrow f \in W & B_2
 \end{array}
 \quad (336)$$

But under this assumption, right-properness implies that  $\rho^*f$  is a weak equivalence (311), so that the statement follows by 2-out-of-3 (307).

*Alternative Proof.* The conclusion also follows with Lemma A.31: The left adjoint functor  $L = f_!$  clearly creates weak equivalences (335) (by the nature of the slice model structure, Example A.10), so that Lemma A.31 asserts that we have a Quillen equivalence as soon as the ordinary adjunction counit is a weak equivalence on all fibrant objects. By (322), the adjunction counit on a fibration  $\rho \in \text{Fib}$  is the dashed morphism  $\rho^*f$  in the following diagram on the right:

$$\begin{array}{ccc}
 f^*A & \overset{\text{id}}{\dashrightarrow} & f^*A & \xrightarrow{\rho^*f \in W} & A \\
 \searrow f^*\rho & & \downarrow f^*\rho & \text{(pb)} & \downarrow \rho \in \text{Fib} \\
 & & B_1 & \searrow f \in W & B_2
 \end{array}
 \quad \leftrightarrow \quad
 \begin{array}{ccc}
 f^*A & \overset{\rho^*f \in W}{\dashrightarrow} & A \\
 \searrow f^*\rho & & \downarrow \rho \in \text{Fib} \\
 & & B_1 \\
 & & \searrow f \in W \\
 & & B_2
 \end{array}
 \quad (337)$$

And hence this is a weak equivalence, again by right-properness.  $\square$

**Example A.33** (Quillen equivalence between topological spaces and simplicial sets [Qu67]). Forming simplicial sets constitutes a Quillen equivalence (Def. A.29)

$$\begin{array}{ccc}
 \text{TopologicalSpaces}_{\text{Qu}} & \begin{array}{c} \xleftarrow{\text{geometric realization}} \\ \xrightarrow{\text{Sing}} \end{array} & \text{SimplicialSets}_{\text{Qu}} \\
 & \begin{array}{c} \xleftarrow{|\!-\!|} \\ \xrightarrow{\simeq_{\text{Qu}}} \end{array} & \\
 & \text{singular simplicial complex} & 
 \end{array}
 \quad (338)$$

between the classical model structure on topological spaces (Example A.7) and the classical model structure on simplicial sets (Example A.8).

**Example A.34** (Classical homotopy category). By Prop. A.30 and the derived adjunction (Prop. A.21) of the  $|\!-\!| \dashv \mathbb{R}\text{Sing}$ -adjunction (Example A.33) is an equivalence between the homotopy categories (Def. A.15) of the classical model category of topological spaces (Example A.7) and the classical model category of simplicial sets (Example A.8):

$$\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}}) \begin{array}{c} \xleftarrow{\mathbb{L}|\!-\!|} \\ \xrightarrow{\mathbb{R}\text{Sing}} \end{array} \text{Ho}(\text{SimplicialSets}_{\text{Qu}}). \quad (339)$$

Either of these is the *classical homotopy category*. We refer to its objects as *homotopy types*, to be distinguished from the actual topological spaces or simplicial set that represent them.

### Cell complexes.

**Proposition A.35** (Skeleta and truncation [May67, §II.8][DK84, §1.2 (vi)]). *For each  $n \in \mathbb{N}$  there is a pair of adjoint functors*

$$\text{SimplicialSets} \begin{array}{c} \xleftarrow{\text{sk}} \\ \xrightarrow{\text{cosk}} \end{array} \text{SimplicialSets}, \quad (340)$$

where  $\text{sk}_n(S)$  is the simplicial sub-set generated by the simplices in  $S$  of dimension  $\leq n$  (hence including only all their degenerate higher simplices), and where

$$\text{cosk}_n(S) : [k] \mapsto \text{SimplicialSets}(\text{sk}_n(\Delta[k]), S).$$

Here  $\text{cosk}_{n+1}$  preserves fibrant objects of the classical model structure (Example A.8) and models  $n$ -truncation, in that:

$$\pi_k|\text{cosk}_{n+1}(S)| = 0 \quad \text{for } k \geq n+1$$

and there are natural fibrations  $S \xrightarrow{p_n} \text{cosk}_n(S)$  such that

$$\pi_k|S| \xrightarrow[\simeq]{\pi_k|p_n|} \pi_k|\text{cosk}_{n+1}(S)| \quad \text{for } k \leq n.$$

For  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  we write

$$A(n) := |\text{cosk}_{n+1}(\text{Sing}(A))| \quad (341)$$

We say that  $A$  is  $n$ -truncated if it is equivalent to its  $n$ -truncation:

$$A \text{ is } n\text{-truncated} \iff A \simeq A(n). \quad (342)$$

for its  $n$ -truncation.

**Example A.36** (Simplicial sets are weakly equivalent to singular simplicial sets of their realization). For  $S \in \text{SimplicialSets}$ , the unit of the adjunction (338) is a weak equivalence:

$$S \xrightarrow[\in \mathbb{W}]{\eta_S} \text{Sing}(|S|). \quad (343)$$

Notice that, a priori, the characterization of Quillen equivalences (Lemma A.28) only says, with Example A.33, that the derived adjunction unit, hence the composite

$$S \xrightarrow{\eta_S} \text{Sing}(|S|) \xrightarrow{\text{Sing}(j_{|S|})} \text{Sing}(P|S|)$$

is a weak equivalence, where  $j_{|S|}$  is a Kan fibrant replacement for  $|S|$ . But since all topological spaces are fibrant (Example A.7), the above simpler condition follows.

**Example A.37** (Homotopy types of manifolds via triangulations). For  $X \in \text{TopologicalSpaces}$  equipped with the structure of an  $n$ -manifold, there exists a *triangulation* of  $X$ , namely an  $n$ -skeletal simplicial set (Prop. A.35)

$$\text{Tr}(X) \in \text{SimplicialSets}, \quad \text{sk}_n(\text{Tr}(X)) = \text{Tr}(X) \quad (344)$$

equipped with a homeomorphism to  $X$  out of its geometric realization (338)

$$|\text{Tr}(X)| \xrightarrow[\text{homeo}]{p} X, \quad (345)$$

Since the inclusion

$$\text{Tr}(X) \hookrightarrow \text{Sing}(|\text{Tr}(X)|) \xrightarrow[\in \text{Iso}]{\text{Sing}(p)} \text{Sing}(X), \quad (346)$$

is a weak equivalence (by Example A.36), the triangulation represents the homotopy type (339) of the manifold.

**Proposition A.38** (Homotopy classes of maps out of  $n$ -manifolds). *Let  $X \in \text{TopologicalSpaces}$  admit the structure of an  $n$ -manifold. Then for any  $A \in \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})$  (Example A.34) the homotopy classes of maps  $X \rightarrow A$  are in natural bijection to the homotopy classes into the  $n$ -truncation (341) of  $A$ :*

$$\text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, A) \simeq \text{Ho}(\text{TopologicalSpaces}_{\text{Qu}})(X, A(n)) \quad (347)$$

*Proof.* Consider the following sequence of natural isomorphisms

$$\begin{aligned}
& \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})(X, A) \\
& \simeq \mathrm{Ho}(\mathrm{SimplicialSets}_{\mathrm{Qu}})(\mathrm{Sing}(X), \mathrm{Sing}(A)) \\
& \simeq \mathrm{Ho}(\mathrm{SimplicialSets}_{\mathrm{Qu}})(\mathrm{Tr}(X), \mathrm{Sing}(A)) \\
& \simeq \mathrm{SimplicialSets}(\mathrm{Tr}(X), \mathrm{Sing}(A)) / \mathrm{SimplicialSets}(\mathrm{Tr}(X) \times \Delta[1], \mathrm{Sing}(A)) \\
& \simeq \mathrm{SimplicialSets}(\mathrm{sk}_{n+1}(\mathrm{Tr}(X)), \mathrm{Sing}(A)) / \mathrm{SimplicialSets}(\mathrm{sk}_{n+1}(\mathrm{Tr}(X) \times \Delta[1]), \mathrm{Sing}(A)) \\
& \simeq \mathrm{SimplicialSets}(\mathrm{Tr}(X), \mathrm{cosk}_{n+1}(\mathrm{Sing}(A))) / \mathrm{SimplicialSets}(\mathrm{Tr}(X) \times \Delta[1], \mathrm{cosk}_{n+1}(\mathrm{Sing}(A))) \\
& \simeq \mathrm{Ho}(\mathrm{SimplicialSets}_{\mathrm{Qu}})(\mathrm{Tr}(X), \mathrm{cosk}_{n+1}(\mathrm{Sing}(A))) \\
& \simeq \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})(|\mathrm{Tr}(X)|, |\mathrm{cosk}_{n+1}(\mathrm{Sing}(A))|) \\
& \simeq \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})(X, A(n)).
\end{aligned}$$

Here the first step is (A.34), using, with Example A.22, that all topological spaces are fibrant and all simplicial sets cofibrant. The second step uses (346). The third step uses Example A.12 with Prop. A.17 (observing that  $\mathrm{Sing}(A)$  is fibrant as  $A$  is and  $\mathrm{Sing}$  is right Quillen) to express the morphisms in the homotopy category as equivalence classes of simplicial maps under the relation that identifies those pairs of maps that extend to a map on the cylinder  $\mathrm{Tr}(X) \times \Delta[1]$ . The fourth step observes that with  $\mathrm{Tr}(X)$  being  $n$ -skeletal (344), its cylinder is  $(n+1)$ -skeletal. The fifth step is thus the  $\mathrm{sk}_{n+1} \dashv \mathrm{cosk}_{n+1}$ -adjunction isomorphism (340). The sixth step applies again Prop. A.17, using that  $\mathrm{cosk}_{n+1}$  preserves fibrancy (Prop. A.35). The seventh step is the reverse of the first step, with the same argument on (co-)fibrancy. The last step uses (345) in the first argument and (341) in the second. The composite of these isomorphisms is the desired (347).  $\square$

**Proposition A.39** (Postnikov tower [GJ99, Cor. 3.7]). *Let  $X \in \mathrm{Ho}(\mathrm{TopologicalSpaces}_{\mathrm{Qu}})$  (Example A.34). If  $X$  is connected, then its sequence of  $n$ -truncations (341) forms a system of fibrations with homotopy fibers (Def. A.23) the Eilenberg-MacLane spaces (10) of the homotopy group in the given degree:*

$$\begin{array}{ccc}
& & \vdots \\
& & \downarrow \\
K(\pi_3(X), 3) & \xrightarrow{\text{hfib}(p_3^X)} & X(3) \\
& & \downarrow p_3^X \\
K(\pi_2(X), 2) & \xrightarrow{\text{hfib}(p_2^X)} & X(2) \\
& & \downarrow p_2^X \\
K(\pi_1(X), 1) & \xrightarrow{\text{hfib}(p_1^X)} & X(1) \\
& & \downarrow p_1^X \\
& & X(0).
\end{array}$$

If  $X$  is not connected then this applies to each of its connected components.

### Stable model categories.

**Example A.40** (Looping/suspension-adjunction). On the category of pointed topological spaces, equipped with the coslice model structure under the point (Example A.10) of the classical model structure (Example A.7), the operation of forming based loop spaces  $\Omega X := \text{Maps}^{*/}(S^1, X)$  is the right adjoint in a Quillen adjunction (Def. A.18)

$$\text{TopologicalSpaces}_{\text{Qu}}^{*/} \begin{array}{c} \xleftarrow{\Sigma} \\ \xrightarrow{\Omega} \end{array} \text{TopologicalSpaces}_{\text{Qu}}^{*/} \quad (348)$$

whose left adjoint is the reduced suspension operation  $\Sigma X := S^1 \wedge X := (S^1 \times X)/(S^1 \times \{*_X\} \sqcup \{*_S\} \times X)$ .

**Example A.41** (Stable model category of sequential spectra [BF78][GJ99, §X.4]). There exists a model category (Def. A.3)  $\text{SequentialSpectra}_{\text{BF}}$  whose objects are sequences

$$E := \left\{ E_n \in \text{TopologicalSpaces}, \Sigma E_n \xrightarrow{\sigma_n} E_{n+1} \right\}_{n \in \mathbb{N}}$$

of topological spaces  $E_n$  and continuous function  $\sigma_n$  from their suspension  $\Sigma E_n$  (Example A.40) to the next space in the sequences; and whose morphisms  $E \xrightarrow{f} F$  are sequences of component maps  $E_n \xrightarrow{f_n} F_n$  that commute with the  $\sigma$ s. Moreover:

$W$  – *weak equivalences* are the morphisms that induce isomorphisms on all *stable homotopy groups*  $\pi_\bullet(X) := \varinjlim_n \pi_{\bullet+k}(X_k)$  (where the colimit is formed using the  $\sigma$ 's);

$\text{Cof}$  – *cofibrations* are those morphisms  $E \xrightarrow{f} F$  such that the maps

$$E_0 \xrightarrow{f_0} F_0 \quad \text{and} \quad \forall_{n \in \mathbb{N}} E_{n+1} \sqcup_{\Sigma E_n} \Sigma F_n \xrightarrow{(f_{n+1}, \sigma_n^F)} F_{n+1} \in \text{Cof}$$

are cofibrations in the classical model structure on topological spaces (Example A.7).

$\text{Fib}$  – *Fibrant objects* are the  $\Omega$ -spectra, namely those sequences of spaces  $\{E_n\}$  for which the  $\Sigma \dashv \Omega$ -adjunct (348) of each  $\sigma_n$  is a weak equivalence:

$$\left\{ E_n \in \text{TopologicalSpaces}_{\text{Qu}}^{*/}, E_n \xrightarrow{\tilde{\sigma}_n} \Omega E_{n+1} \right\}_{n \in \mathbb{N}} \quad (349)$$

**Example A.42** (Derived stabilization adjunction). The suspension/looping Quillen adjunction on pointed spaces (Example A.40) extends to a commuting diagram of Quillen adjunctions (Def. A.18) to and on the stable model category of spectra (Example A.41)

$$\begin{array}{ccc}
\text{TopologicalSpaces}_{\text{Qu}}^{*/} & \begin{array}{c} \xleftarrow{\Sigma} \\ \xrightarrow{\Omega} \end{array} & \text{TopologicalSpaces}_{\text{Qu}}^{*/} \\
\Sigma^\infty \downarrow \dashv_{\text{Qu}} \uparrow \Omega^\infty & & \Sigma^\infty \downarrow \dashv_{\text{Qu}} \uparrow \Omega^\infty \\
\text{SequentialSpectra}_{\text{BF}} & \begin{array}{c} \xleftarrow{\Sigma} \\ \xrightarrow{\Omega} \end{array} & \text{SequentialSpectra}_{\text{BF}}.
\end{array} \quad (350)$$

such that the bottom adjunction is a Quillen equivalence (Def. A.29), hence such that under passage to derived adjunctions (Prop. A.21)

$$\begin{array}{ccc}
\mathrm{Ho}\left(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{*/}\right) & \begin{array}{c} \xleftarrow{\mathbb{L}\Sigma} \\ \xrightarrow[\mathbb{R}\Omega]{\perp} \end{array} & \mathrm{Ho}\left(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{*/}\right) \\
\mathbb{L}\Sigma^\infty \downarrow \uparrow \mathbb{R}\Omega^\infty & & \mathbb{L}\Sigma^\infty \downarrow \uparrow \mathbb{R}\Omega^\infty \\
\mathrm{Ho}\left(\mathrm{SequentialSpectra}_{\mathrm{BF}}\right) & \begin{array}{c} \xleftarrow{\mathbb{L}\Sigma} \\ \xrightarrow[\mathbb{R}\Omega]{\simeq} \end{array} & \mathrm{Ho}\left(\mathrm{SequentialSpectra}_{\mathrm{BF}}\right)
\end{array} \tag{351}$$

the bottom adjunction is an equivalence, thus exhibiting the homotopy category of spectra as being *stable* under looping/suspension.

We say that

- (i)  $\mathrm{Ho}\left(\mathrm{SequentialSpectra}_{\mathrm{BF}}\right)$  is the *stable homotopy category* of spectra;
- (ii) the vertical adjunction  $(\mathbb{L}\Sigma^\infty \dashv \mathbb{R}\Omega^\infty)$  is the *stabilization adjunction* between homotopy types (339) and spectra.
- (iii) the images of  $\Sigma^\infty$  are the *suspension spectra*.
- (iv) For  $E \in \mathrm{Ho}\left(\mathrm{SequentialSpectra}_{\mathrm{BF}}\right)$  and  $n \in \mathbb{N}$  we write (for brevity and in view of (349))

$$E_n := \mathbb{R}\Omega^\infty((\mathbb{L}\Sigma)^n E) \in \mathrm{Ho}\left(\mathrm{TopologicalSpaces}_{\mathrm{Qu}}^{*/}\right) \tag{352}$$

for the homotopy type of the  $n$ th component space of the spectrum.

### Smooth $\infty$ -stacks.

**Definition A.43** (Simplicial presheaves over Cartesian spaces). We write

$$(i) \quad \mathrm{CartesianSpaces} := \left\{ \mathbb{R}^{n_1} \xrightarrow{\text{smooth}} \mathbb{R}^{n_2} \right\}_{n_i \in \mathbb{N}} \tag{353}$$

for the category whose objects are the Cartesian spaces  $\mathbb{R}^n$ , for  $n \in \mathbb{N}$ , and whose morphisms are the *smooth* functions between these (hence the full subcategory of  $\mathrm{SmoothManifolds}$  on the Cartesian spaces).

$$(ii) \quad \mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets}) := \mathrm{Functors}(\mathrm{CartesianSpaces}^{\mathrm{op}}, \mathrm{SimplicialSets}) \tag{354}$$

for the category of functors from the opposite of  $\mathrm{CartesianSpaces}$  (353) to  $\mathrm{SimplicialSets}$ ,

**Example A.44** (Model structure on simplicial presheaves over Cartesian spaces [Du98][Du01][FSS10, §A]). The category of simplicial presheaves over Cartesian spaces (Prop. 354) carries the following model category structures (Def. A.3):

- (i) The global projective model structure

$$\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}} \tag{355}$$

whose

$\mathbf{W}$  – *weak equivalences* are the morphisms which over each  $\mathbb{R}^n$  are weak equivalence in  $\mathrm{SimplicialSets}_{\mathrm{Qu}}$  (Example A.8),

$\mathbf{Fib}$  – *fibrations* are the morphisms which over each  $\mathbb{R}^n$  are fibrations in  $\mathrm{SimplicialSets}_{\mathrm{Qu}}$  (Example A.8),

- (ii) The local projective model structure

$$\mathrm{SmoothStack}_\infty := \mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}, \mathrm{loc}} \tag{356}$$

whose:

$\mathbf{W}$  – *weak equivalences* are the morphisms whose stalk at  $0 \in \mathbb{R}^n$  is a weak equivalence in  $\mathrm{SimplicialSets}_{\mathrm{Qu}}$  (Example A.8), for all  $n \in \mathbb{N}$ ;

$\mathbf{Cof}$  – *cofibrations* are the morphisms with the left lifting property (309) against the class of morphisms which over each  $\mathbb{R}^n$  are in  $\mathbf{Fib} \cap \mathbf{W}$  of  $\mathrm{SimplicialSets}_{\mathrm{Qu}}$ .

**Definition A.45** (Homotopy category of smooth  $\infty$ -stacks). We write

$$\mathrm{Ho}(\mathrm{SmoothStacks}_\infty) := \mathrm{Ho}\left(\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}, \mathrm{loc}}\right) \quad (357)$$

for the homotopy category (Def. A.15) of the local projective model category of simplicial presheaves over CartesianSpaces (Example A.44). We say that the objects of  $\mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$  (357) are *smooth  $\infty$ -stacks*.

For exposition of smooth  $\infty$ -stack theory see [FSS12b, §2][FSS13a][SS20a, §1]. In particular, notice:

**Example A.46** (Smooth manifolds as smooth  $\infty$ -stacks). For  $X \in \mathrm{SmoothManifolds}$  it is incarnated as a smooth  $\infty$ -stack (Def. A.45) by the rule

$$\mathcal{X} = \left( \mathbb{R}^n \mapsto (\Delta[k] \mapsto \mathrm{SmoothManifolds}(\mathbb{R}^n, X)) \right). \quad (358)$$

This construction constitutes to a full embedding

$$\mathrm{SmoothManifolds} \hookrightarrow \mathrm{Ho}(\mathrm{SmoothStacks}_\infty)$$

of smooth manifolds into smooth  $\infty$ -stacks.

**Lemma A.47** ( $\infty$ -Stackification preserves finite homotopy limits). *The identity functors constitute a Quillen adjunction (Def. A.18) between the local and the global projective model categories of Example A.44:*

$$\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}, \mathrm{loc}} \begin{array}{c} \xleftarrow{\mathrm{id}} \\ \xrightarrow{\perp_{\mathrm{Qu}}} \\ \xrightarrow{\mathrm{id}} \end{array} \mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}}.$$

Moreover, this is such that the derived left adjoint functor (Prop. A.21)

$$L^{\mathrm{loc}} : \mathrm{Ho}\left(\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}}\right) \xrightarrow{\mathbb{L}\mathrm{id}} \mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \quad (359)$$

(the  $\infty$ -stackification functor) preserves homotopy pullbacks (Def. A.24).

**Proposition A.48** (Shape Quillen adjunction [Sch13, Prop. 4.4.8][SS20a, Example 3.18]). *We have a Quillen adjunction (Def. A.18)*

$$\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{SimplicialSets})_{\mathrm{proj}, \mathrm{loc}} \begin{array}{c} \xrightarrow{\mathrm{Shp}} \\ \xrightarrow{\perp_{\mathrm{Qu}}} \\ \xleftarrow{\mathrm{Disc}} \end{array} \mathrm{SimplicialSets}_{\mathrm{Qu}}$$

between the projective local model structure on simplicial presheaves over CartesianSpaces (Example A.44) and the classical model structure on simplicial sets (Example A.8), hence a derived adjunction (Prop. A.21) between homotopy category of  $\infty$ -stacks (Def. A.45) and the classical homotopy category (Example A.34)

$$\mathrm{Ho}(\mathrm{SmoothStacks}_\infty) \begin{array}{c} \xrightarrow{\mathbb{L}\mathrm{Shp}} \\ \xrightarrow{\perp_{\mathrm{Qu}}} \\ \xleftarrow{\mathbb{R}\mathrm{Disc}} \end{array} \mathrm{Ho}(\mathrm{SimplicialSets}_{\mathrm{Qu}})$$

whose (underived) right adjoint sends a simplicial set to the presheaf which is constant on that simplicial set:

$$\mathrm{Disc}(S) := \mathrm{const}(S) : (\mathbb{R}^n \mapsto S). \quad (360)$$

## Homological algebra.

**Example A.49** (Projective model structure on connective chain complexes [Qu67, §II.4 (5.)]). The category  $\mathrm{ChainComplexes}_\mathbb{Z}^{\geq 0}$  of connective chain complexes of abelian groups (i.e. concentrated in non-negative degrees with differential of degree -1) carries a model category structure (Def. A.3) whose

W – weak equivalences are the quasi-isomorphisms (those inducing isomorphisms on all chain homology groups)

Fib – fibrations are the positive-degree wise surjections

Cof – cofibrations are the morphisms with degreewise injective kernels.

We write  $(\mathrm{ChainComplexes}_\mathbb{Z}^{\geq 0})_{\mathrm{proj}}$  for this model category.

More generally:

**Example A.50** (Projective model structure on presheaves of connective chain complexes [Ja03, p. 7]). The category of presheaves of connective chain complexes over CartesianSpaces (353) carries the structure of a model category whose weak equivalences and fibrations are objectwise those of  $(\mathrm{ChainComplexes}_\mathbb{Z}^{\geq 0})_{\mathrm{proj}}$  (Example A.49).

We write  $\mathrm{PSh}(\mathrm{CartesianSpaces}, \mathrm{ChainComplexes}_\mathbb{Z}^{\geq 0})_{\mathrm{proj}}$  for this model category.



**Proposition A.51** (Dold-Kan correspondence [Do58, Thm 1.9][Kan58][GJ99, §III.2][SSh03a, §2.1]). *Given  $A_\bullet \in \text{SimplicialAbelianGroups}$ , its normalized chain complex is the connective chain complex of abelian groups (Example A.49) which in degree  $n \in \mathbb{N}$  is the quotient of  $A_n$  by the degenerate cells and whose differential is the alternating sum of the face maps:*

$$N(A)_\bullet := \left\{ N(A)_n := A_n / \sigma(A_{n+1}), \partial_n := \sum_{i=0}^n (-1)^i d_i : N(A)_n \rightarrow N(A)_{n-1} \right\}_{n \in \mathbb{N}} \in \text{ChainComplexes}_{\mathbb{Z}}^{\geq 0}. \quad (361)$$

(i) *This construction constitutes an adjoint equivalence of categories*

$$\text{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \begin{array}{c} \xleftarrow{N} \\ \xrightarrow{\simeq} \end{array} \text{SimplicialAbelianGroups} \quad (362)$$

(ii) *such that simplicial homotopy groups of  $A \in \text{SimplicialAbelianGroups} \rightarrow \text{SimplicialSet}$  are identified with chain homology groups of the normalized chain complex ([GJ99, Cor. III.2.5]):*

$$\pi_\bullet(A) \simeq H_\bullet(NA). \quad (363)$$

**Example A.52** (Model structure on simplicial abelian groups [Qu69, §III.2][SSh03a, §4.1]). The category of  $\text{SimplicialAbelianGroups}$  carries a model category structure (Def. A.3) whose

W – weak equivalences are the morphisms which are weak equivalences as morphisms in  $\text{SimplicialSets}_{\text{Qu}}$  (Example A.8)

Fib – fibrations are the morphisms which are fibrations as morphisms in  $\text{SimplicialSets}_{\text{Qu}}$  (Example A.8)

In other words, this is the transferred model structure along the free/forgetful adjunction, which thus becomes a Quillen adjunction (Def. A.18):

$$\text{SimplicialAbelianGroup}_{\text{proj}} \begin{array}{c} \xleftarrow{\mathbb{Z}[-]} \\ \xrightarrow{\perp_{\text{Qu}}} \end{array} \text{SimplicialSets}_{\text{Qu}}. \quad (364)$$

**Proposition A.53** (Dold-Kan Quillen equivalence [SSh03a, §4.1][Ja03, Lemma 1.5]). *With respect to the projective model structure on connective chain complexes (Example A.49) and the projective model structure on simplicial abelian groups (Example A.52) the Dold-Kan correspondence (Prop. A.51) is a Quillen equivalence (Def. A.29):*

$$\left( \text{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \right)_{\text{proj}} \begin{array}{c} \xleftarrow{N} \\ \xrightarrow{\simeq_{\text{Qu}}} \end{array} \text{SimplicialAbelianGroups}_{\text{proj}}, \quad (365)$$

where both functors preserve all three classes of morphisms, Fib, Cof and W, separately.

**Example A.54** (Dold-Kan construction [FSS10, §3.2.3][FSS12b, §2.4]). i) We write DK for the total right adjoint in the composite of the free Quillen adjunction (364) and the Dold-Kan equivalence (365):

$$\left( \text{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \right)_{\text{proj}} \begin{array}{c} \xleftarrow{N} \\ \xrightarrow{\simeq_{\text{Qu}}} \end{array} \text{SimplicialAbelianGroups}_{\text{proj}} \begin{array}{c} \xleftarrow{\mathbb{Z}[-]} \\ \xrightarrow{\perp_{\text{Qu}}} \end{array} \text{SimplicialSets}_{\text{Qu}}. \quad (366)$$

DK

ii) This extends to a right Quillen functor on global projective model categories of presheaves (Example A.44, Example A.50). whose right derived functor (Prop. A.21)  $\mathbb{R}DK$  composed with the  $\infty$ -stackification functor (359) is thus of the form

$$\text{Ho} \left( \text{PSh}(\text{CartesianSpaces}, \left( \text{ChainComplexes}_{\mathbb{Z}}^{\geq 0} \right)_{\text{proj}}) \right) \begin{array}{c} \xrightarrow{\text{derived Dold-Kan construction } \mathbb{R}DK} \\ \searrow \text{\scriptsize } \infty\text{-stackified Dold-Kan construction} \end{array} \text{Ho} \left( \text{PSh}(\text{CartesianSpaces}, \text{SimplicialSets})_{\text{proj}} \right) \begin{array}{c} \downarrow L^{\text{loc}} \infty\text{-stackification} \\ \text{Ho}(\text{SmoothStacks}_{\infty}) \end{array}$$

and preserves homotopy pullbacks (by Lemma A.47).

**Example A.55** (Projective model structure on unbounded chain complexes [Ho99, Thm. 2.3.11]). The category  $\text{ChainComplexes}_z$  of unbounded chain complexes of abelian groups carries a model category structure (Def. A.3) whose:

W – weak equivalences are the quasi-isomorphisms;

Fib – fibrations are the degreewise surjections.

We write  $(\text{ChainComplexes}_z)_{\text{proj}}$  for this model category.

**Proposition A.56** (Stable Dold-Kan construction). *The Dold-Kan construction (Def. A.54) lifts along the stabilization adjunction (Example A.42) from connective to unbounded chain complexes (Example A.55), such as to make the following diagram commute:*

$$\begin{array}{ccccc}
& & \xrightarrow{\text{Dold-Kan correspondence}} & & \\
& & \mathbb{R}\text{DK} & & \\
\text{Ho}\left(\left(\text{ChainComplexes}_z^{\geq 0}\right)_{\text{proj}}\right) & \xrightarrow{\simeq} & \text{Ho}\left(\text{SimplicialAbelianGroups}_{\text{proj}}\right) & \longrightarrow & \text{Ho}\left(\text{SimplicialSets}_{\text{Qu}}\right) \\
\downarrow \uparrow \mathbb{R}\Omega^\infty & & & & \downarrow \uparrow \mathbb{R}\Omega^\infty \\
\text{Ho}\left(\left(\text{ChainComplexes}_z\right)_{\text{proj}}\right) & \xrightarrow{\simeq} & \text{Ho}\left((H\mathbb{Z})\text{ModuleSpectra}\right) & \longrightarrow & \text{Ho}\left(\text{SequentialSpectra}_{\text{BF}}\right) \\
& & \xrightarrow{\text{stable Dold-Kan construction}} & & \\
& & \mathbb{R}\text{DK}_{\text{st}} & & 
\end{array} \quad (367)$$

Here the right adjoint on chain complexes is the homological truncation from below:

$$\mathbb{R}\Omega^\infty\left(\dots \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0 \xrightarrow{\partial_{-1}} V_{-1} \xrightarrow{\partial_{-2}} \dots\right) = \left(\dots \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} \ker(\partial_{-1})\right). \quad (368)$$

*Proof.* (i) It is clear from inspection that the assignment (368) is right adjoint to the inclusion of connective chain complexes, so that we have a pair of adjoint functors

$$\left(\text{ChainComplexes}_z\right)_{\text{proj}} \begin{array}{c} \longleftarrow \\ \xrightarrow{\perp_{\text{Qu}}} \\ \xrightarrow{\Omega^\infty} \end{array} \left(\text{ChainComplexes}_z^{\geq 0}\right)_{\text{proj}}. \quad (369)$$

Moreover, it is immediate that this is a Quillen adjunction (Def. A.18) between the projective model structure on connective chain complexes (Example A.49) and that on unbounded chain complexes (Example A.55):  $\Omega^\infty$  clearly preserves fibrations (using that those between connective chain complexes need to be surjective only in positive degrees!) and clearly preserves all weak equivalences. Finally, since all chain complexes in the projective model structure are fibrant, we have that with  $\Omega^\infty$  also  $\mathbb{R}\Omega^\infty$  is given by (368), via Example A.22.

(ii) A Quillen adjunction of the form

$$\left(\text{ChainComplexes}_z\right)_{\text{proj}} \begin{array}{c} \longleftarrow \\ \xrightarrow{\perp_{\simeq}} \\ \xrightarrow{H} \\ \xrightarrow{\text{DK}_{\text{st}}} \end{array} (H\mathbb{Z})\text{ModuleSpectra} \begin{array}{c} \longleftarrow \\ \xrightarrow{\perp_{\text{Qu}}} \end{array} \text{SequentialSpectra}_{\text{BF}} \quad (370)$$

is established in [SSh01, §B.1], where

(a) the first step is a Quillen equivalence (Def. A.29) between the projective model structure on unbounded chain complexes (Example A.55) and a model category of module spectra over the Eilenberg-MacLane spectrum  $H\mathbb{Z}$  [SSh01, §B.1.11];

(b) the second step is a Quillen adjunction [SSh01, p. 37, item ii)] to the Bousfield-Friedlander model structure (Example A.41) whose right adjoint assigns underlying sequential spectra; such that

(c) the composite right adjoint  $\text{DK}_{\text{st}}$  (370) further composed with  $\Omega^\infty$  on spectra (350) equals the composite of  $\Omega^\infty$  on chain complexes (369) with the unstable Dold-Kan construction (366):

$$\Omega^\infty \circ \text{DK}_{\text{st}} \simeq \text{DK} \circ \Omega^\infty$$

(by immediate inspection of the construction in [SSh01, p. 38-39]).

(iii) By uniqueness of adjoints, this implies that the Quillen adjunction of the stable Dold-Kan construction (370) is intertwined by the Quillen adjunctions involving  $\Omega^\infty$  with the Quillen adjunction of the unstable Dold-Kan construction (366), and hence the commuting diagram of derived functors (A.56) follows (Prop. A.21).  $\square$

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